Digital Image Processing (CSE 478)
Lecture18: Filter Banks and Wavelets

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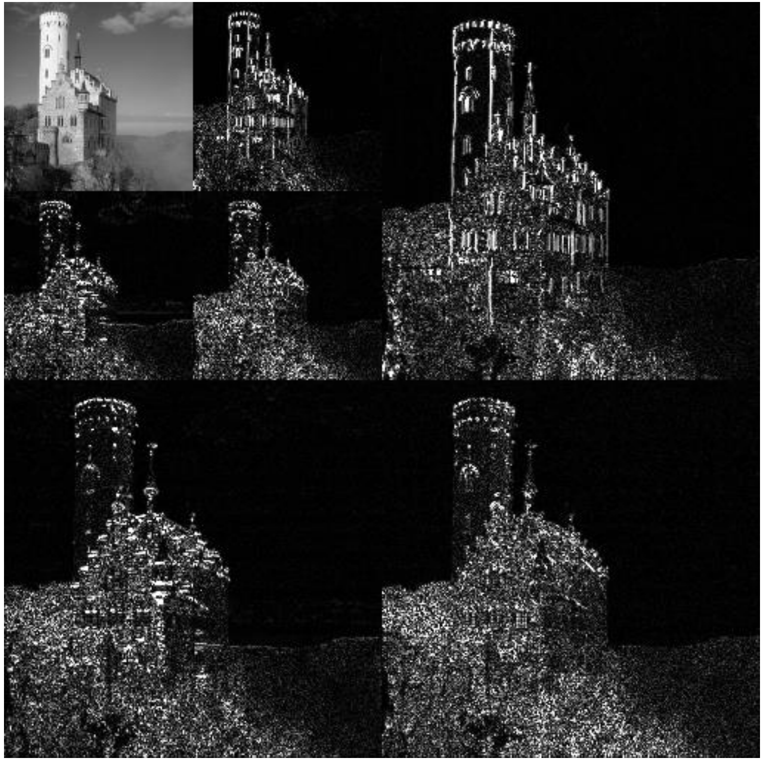
Today's Lecture

- 2D DWT
- Multi scale DWT

Multi resolution processing

- Wavelet is an approach for Multi Resolution Processing



General one-stage two-channel filter bank transform

- Analysis filters l_a, h_a need not be the 2-point averaging/difference filters
- Any pair of low/high pass finite impulse response filters will do

Analysis filter bank:

- $x \rightarrow \mathbf{X} = (\mathbf{X}_{\boldsymbol{l}}, \mathbf{X}_{\boldsymbol{h}})$
- Where, $\mathbf{X}_{\boldsymbol{l}} = D(x * l_a)$; $\mathbf{X}_{\boldsymbol{h}} = D(x * h_a)$

Synthesis filter bank:

- $x' = l_s * U(\mathbf{X}_{\boldsymbol{l}}) + h_s * U(\mathbf{X}_{\boldsymbol{h}})$

Any analysis/synthesis filters could be used as long as x' is perfect reconstruction of x (delay of m indices is allowed)

Discrete Wavelet Transform (DWT)

Example: $\mathrm{x} = (a, b, c, d); l_a = \left(\frac{1}{2}, \frac{1}{2}, 0{,}0\right)$ and $h_a = \left(\frac{1}{2}, -\frac{1}{2}, 0{,}0\right)$

$$\mathbf{x} * l_a = \frac{1}{2}(a + d, b + a, c + b, d + c), \text{ ext. periodically}$$
$$\mathbf{x} * \mathbf{h}_a = \frac{1}{2}(a - d, b - a, c - b, d - c), \text{ ext. periodically}$$

- Truncating and Downsampling

$$\mathbf{X}_l = \frac{1}{2}(a + d, c + b) \quad \mathbf{X}_h = \frac{1}{2}(a - d, c - b)$$

- The DWT of x is

$$\mathbf{X} = \frac{1}{2}(a + d, c + b, a - d, c - b)$$

Discrete Wavelet Transform (DWT)

Matrix view of DWT

$$\mathbf{X} = W_4^a \mathbf{x},$$

$$\mathbf{X} = \frac{1}{2}(a + d, c + b, a - d, c - b)$$

$$W_4^a = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}$$

Inverse Discrete Wavelet Transform (IDWT)

Example: $\mathbf{X} = (A, B, C, D); l_s = (1,0,0,1)$ and $h_s = (1,0,0,-1)$

- Up-sampling: $U(\mathbf{X}_l) = (A, 0, B, 0),\ U(\mathbf{X}_h) = (C, 0, D, 0)$

- Convolving with synthesis filters

$$
\begin{aligned}
U(\mathbf{X}_l) * l_s &= (A, B, B, A) \\
U(\mathbf{X}_h) * \mathbf{h}_s &= (C, -D, D, -C)
\end{aligned}
$$

- IDFT of $\mathbf{X}$ is

$$
\mathrm{x} = (A + C, B - D, B + D, A - C)
$$

Inverse Discrete Wavelet Transform (IDWT)

Matrix view of IDWT

$$\mathbf{x} = W_4^s \mathbf{X},$$

$$\mathbf{x} = (A + C, B - D, B + D, A - C)$$

$$W_4^s = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

Verify

$$W_4^s \cdot W_4^a = I$$

2D-DWT

- Let A be a M×N grayscale image
- The one stage, 2D-DWT is the linear mapping given by:

$$\mathcal{W}_a^1(A) = W_M^a A (W_N^a)^T$$

- W_M and W_N are M×M and N×N analysis matrices determined by (l_a, h_a)

Transform each column of A and then transform each row of resulting matrix

2D-DWT

- The inverse one stage transform is given by:

$$\mathcal{W}_s^1(\hat{A}) = W_M^s \hat{A} (W_N^s)^T$$

- W_M and W_N are M×M and N×N synthesis matrices determined by (l_s, h_s)

2D-DWT

- Transform each column of A and then transform each row of resulting matrix

$$\mathcal{W}_a^1(A) = W_M^a A (W_N^a)^T$$

- Four components

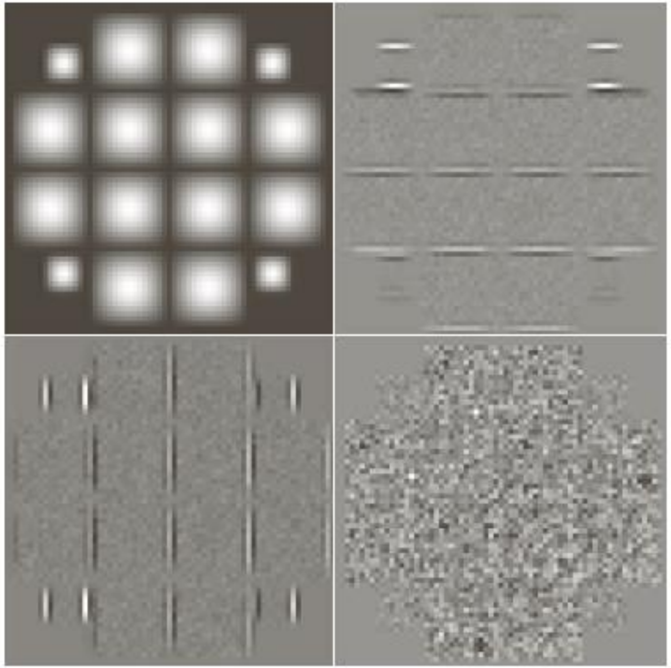
	Low	High
Low	*LL* (approximation)	*HL* (horizontal details)
High	*LH* (vertical details)	*HH* (diagonal details)

2D-DWT

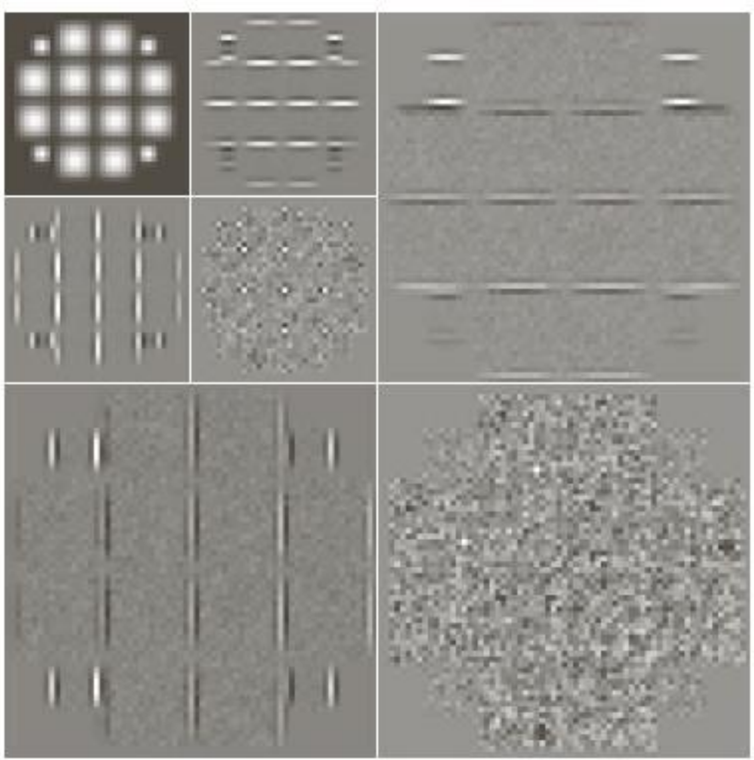
```
im=rgb2gray(imread('color_pencil.jpg'));
[LL,LH,HL,HH]=dwt2(im,'haar');

figure, subplot(2,2,1);imshow(LL,[]);title('LL band of image');
subplot(2,2,2);imshow(LH,[]);title('LH band of image');
subplot(2,2,3);imshow(HL,[]);title('HL band of image');
subplot(2,2,4);imshow(HH,[]);title('HH band of image');
```

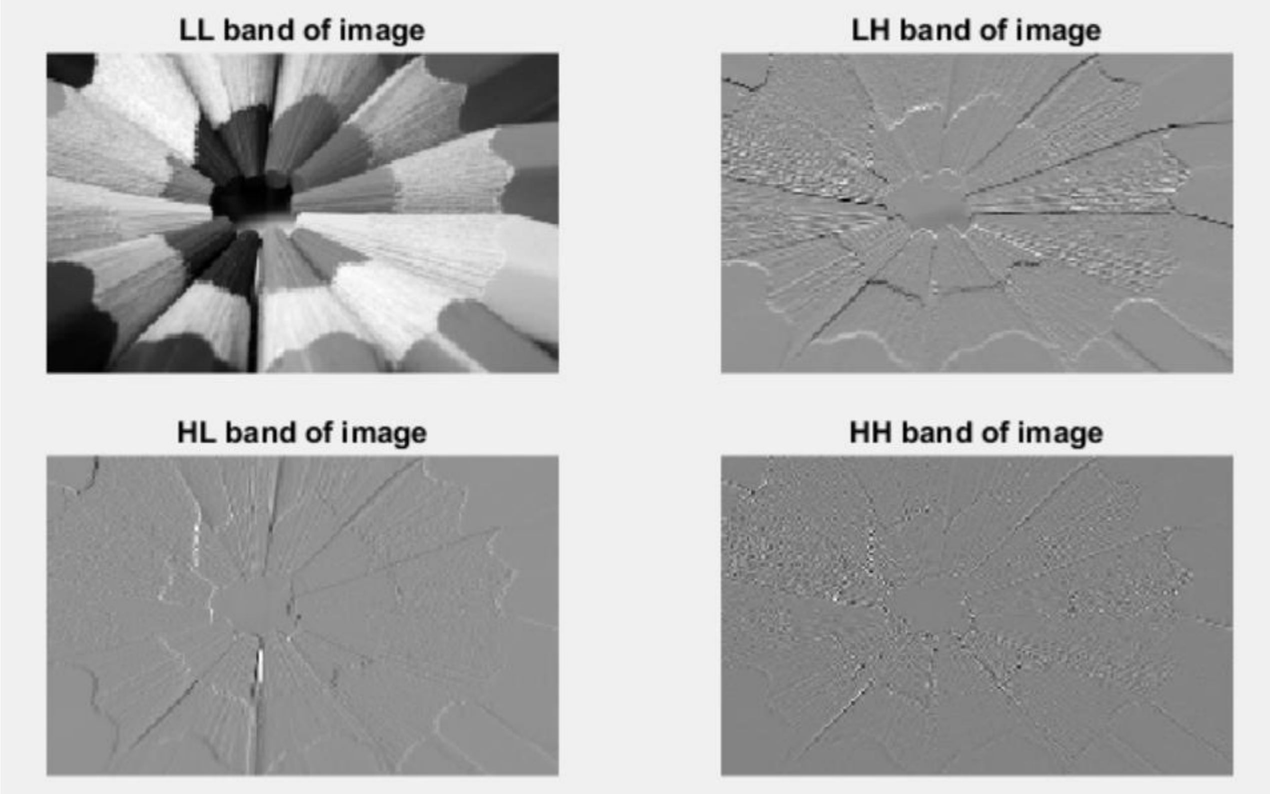
2D-DWT



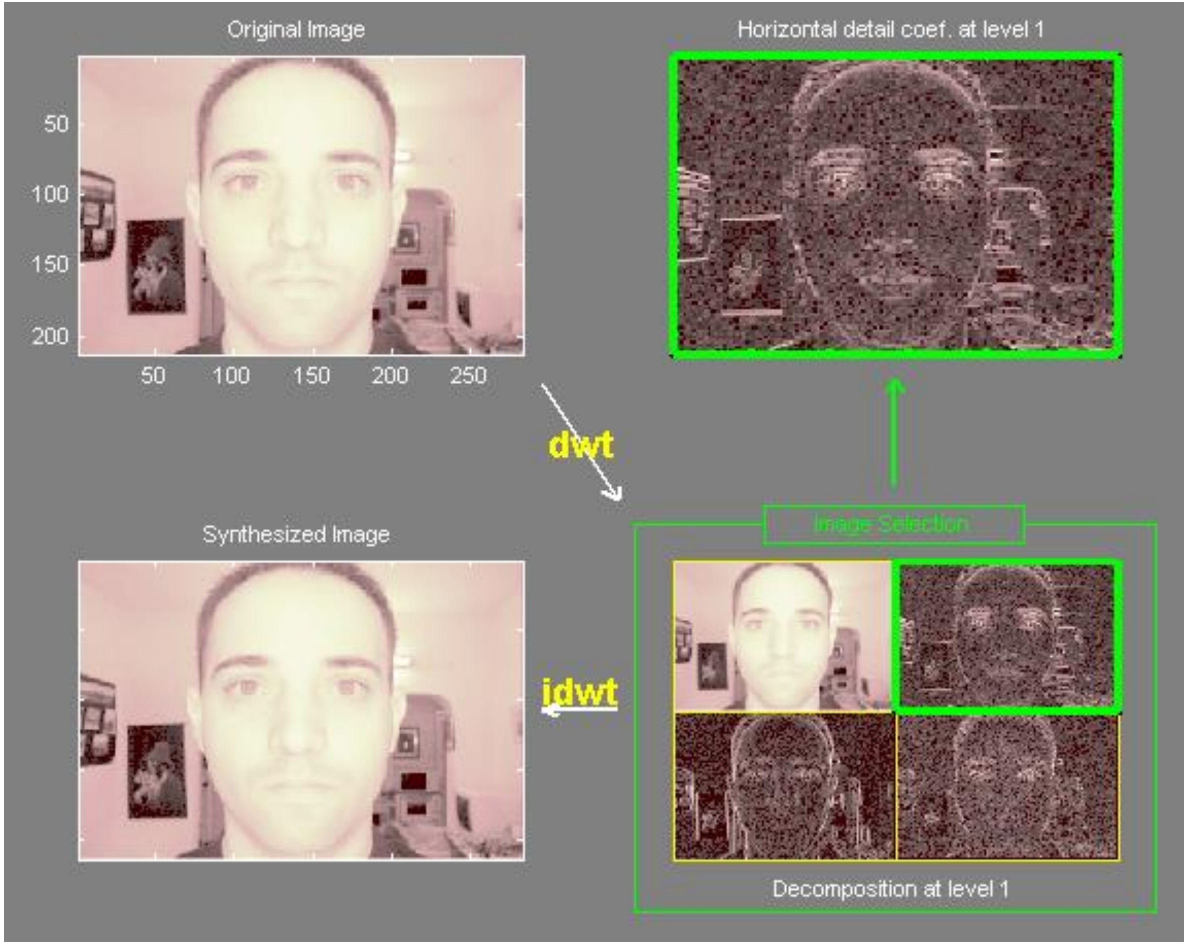
2D-DWT



2D-DWT

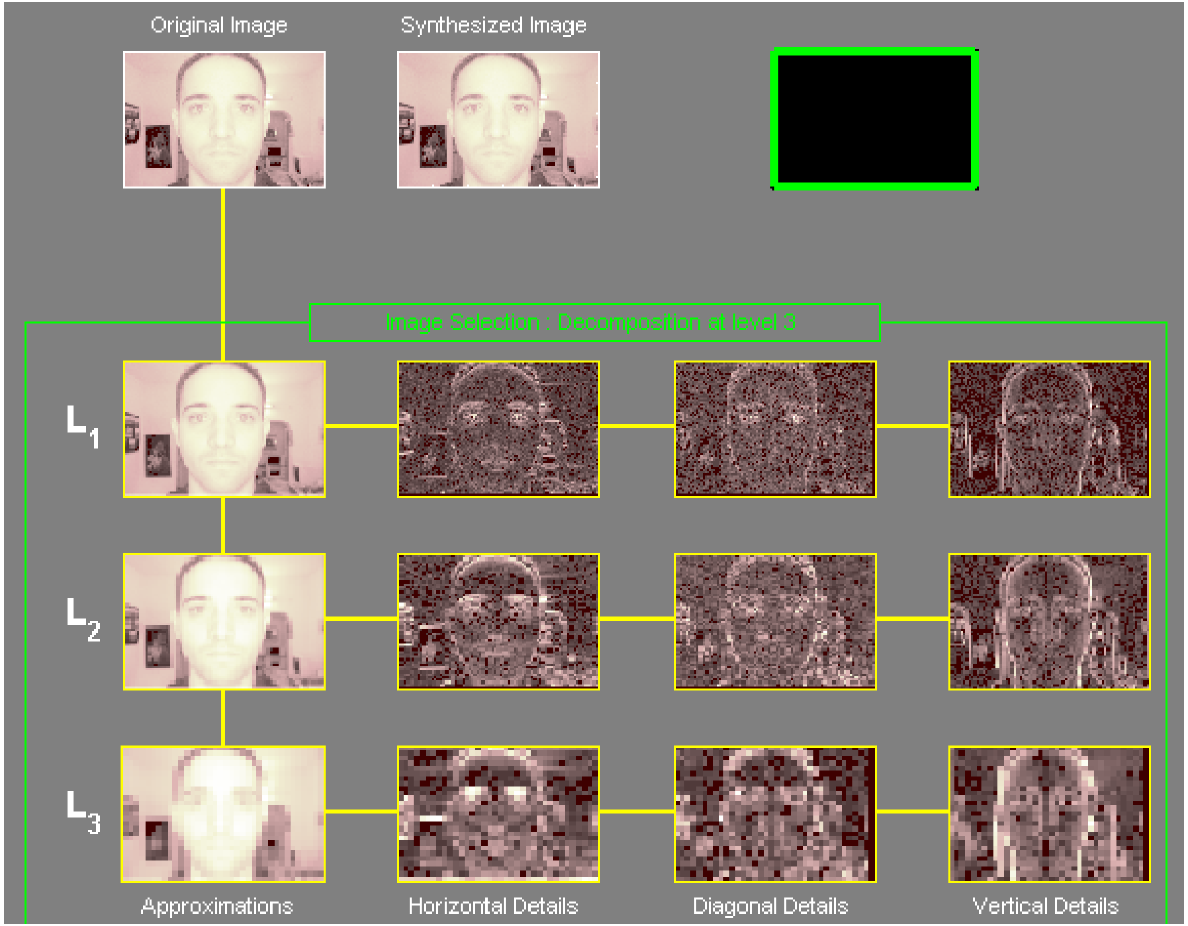

LL band of image
LH band of image
HL band of image
HH band of image

2D-DWT



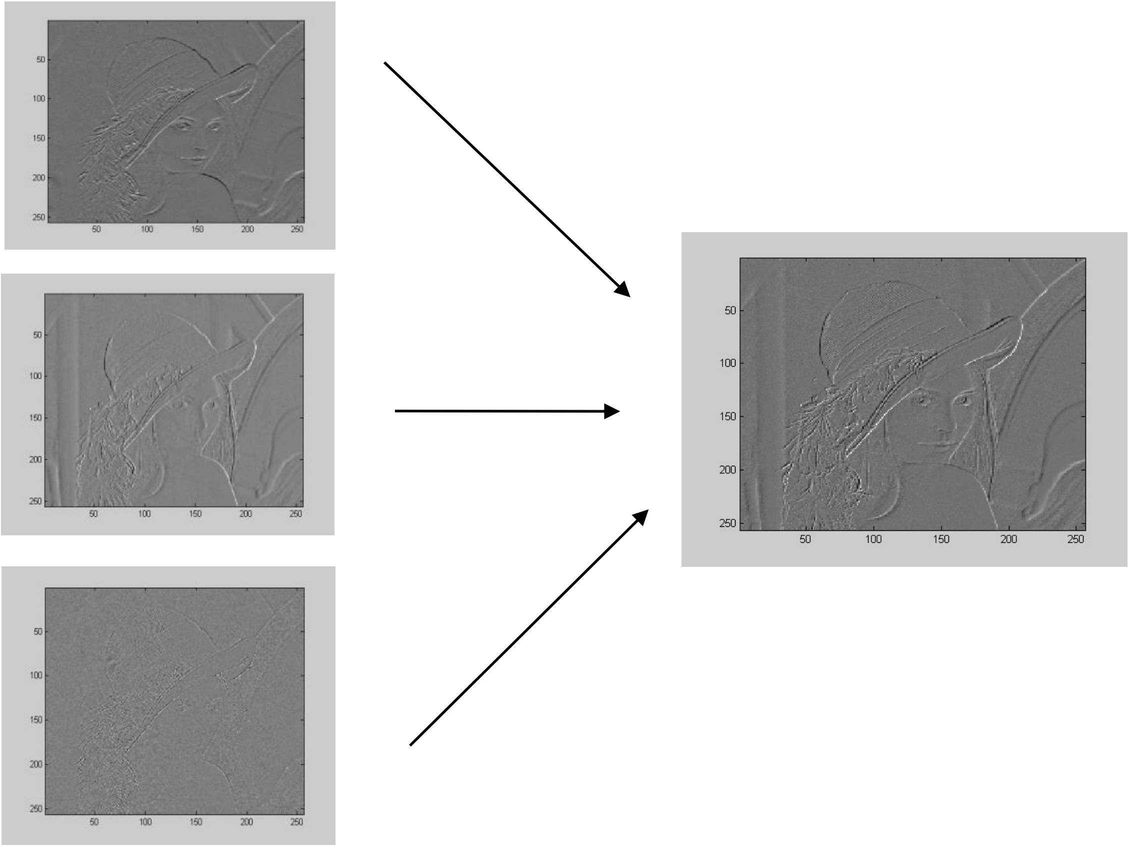
Courtesy: Michigan state university

2D-DWT

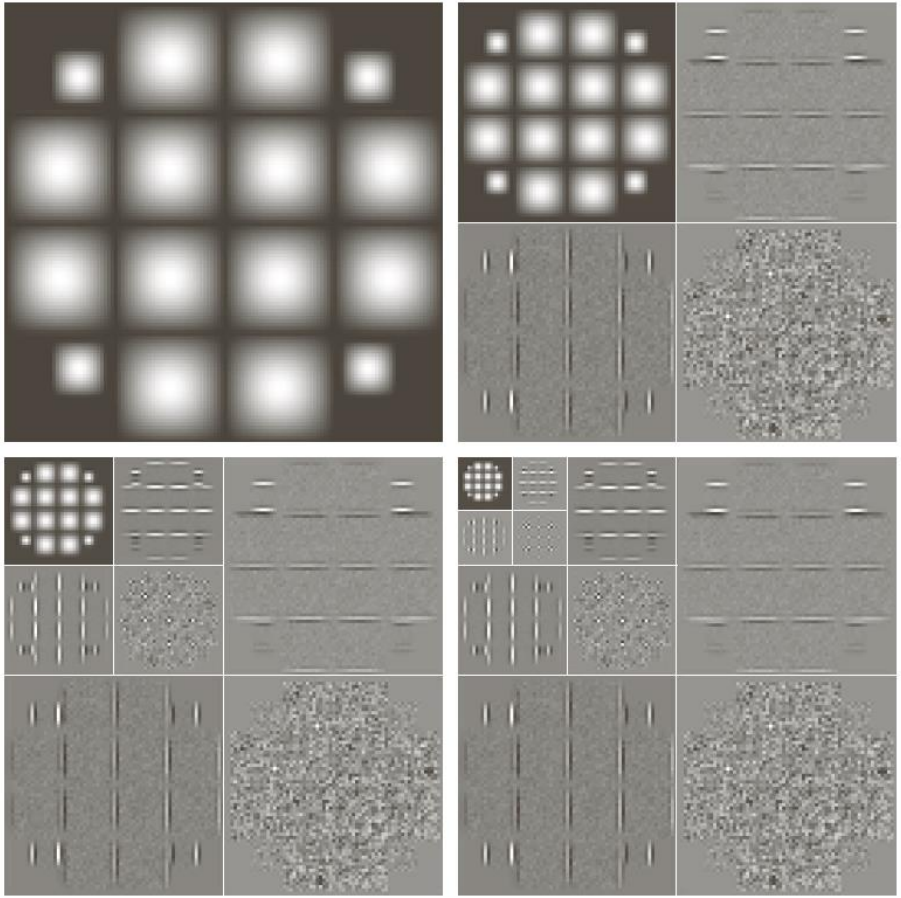


Courtesy: Michigan state university

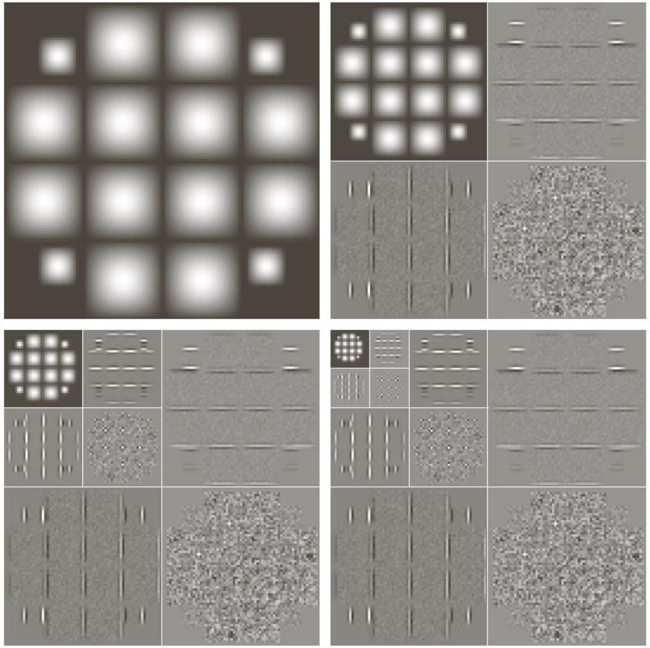
DWT applications: edge detection

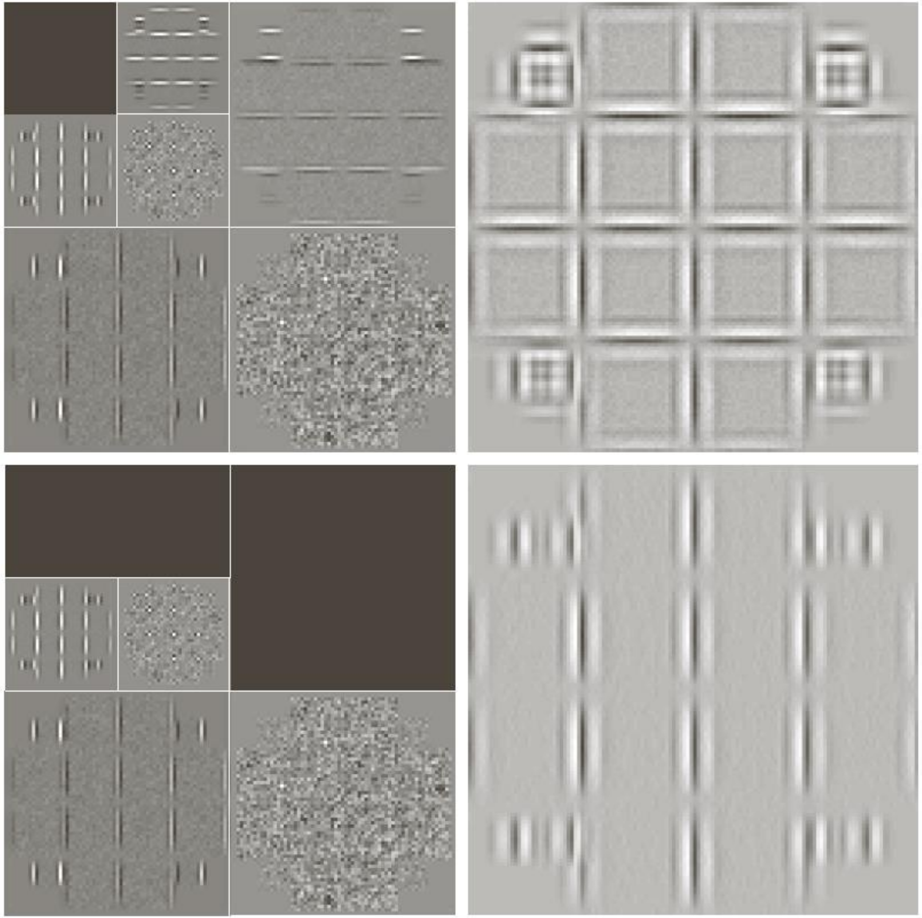


DWT applications: edge detection

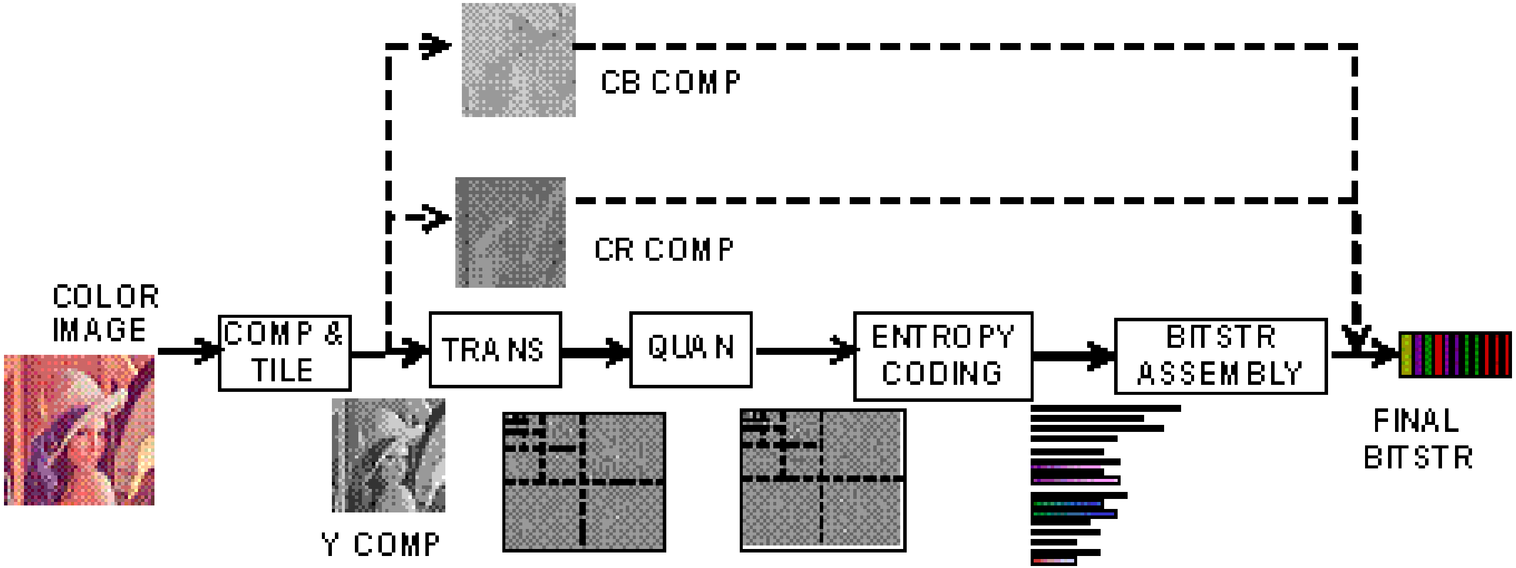


DWT applications: edge detection

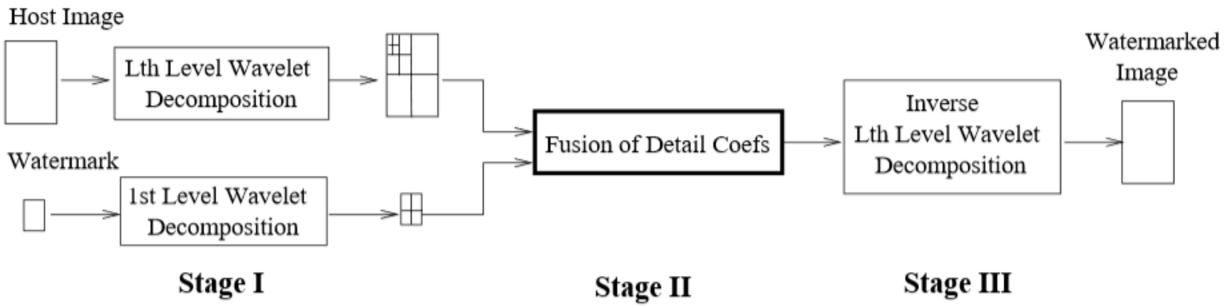




DWT applications: compression



DWT applications: watermarking



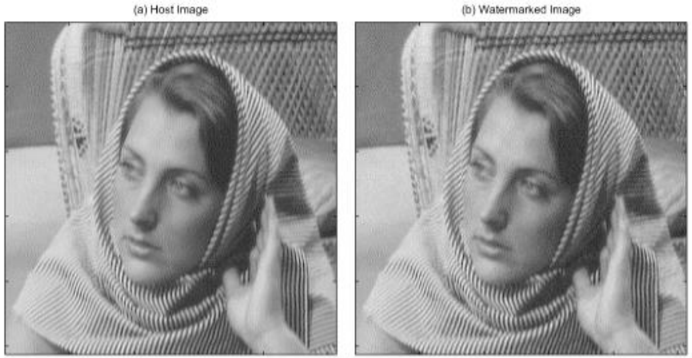


Figure 2: (a) Host Image (left), (b) Watermarked Image (right).

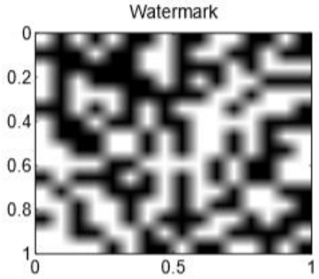
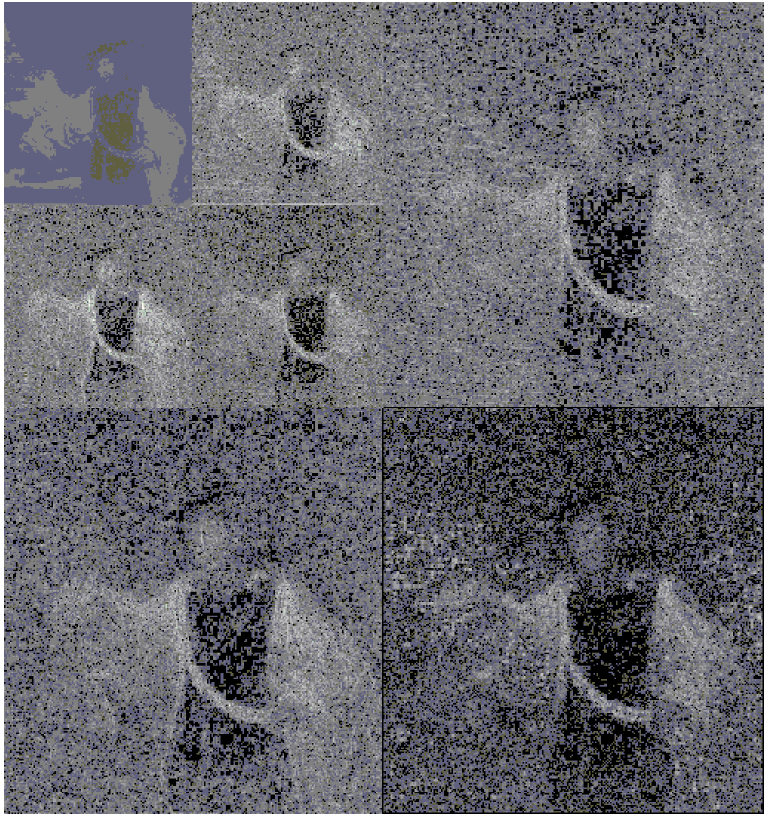


Figure 3: The 256 bit embedded watermark.

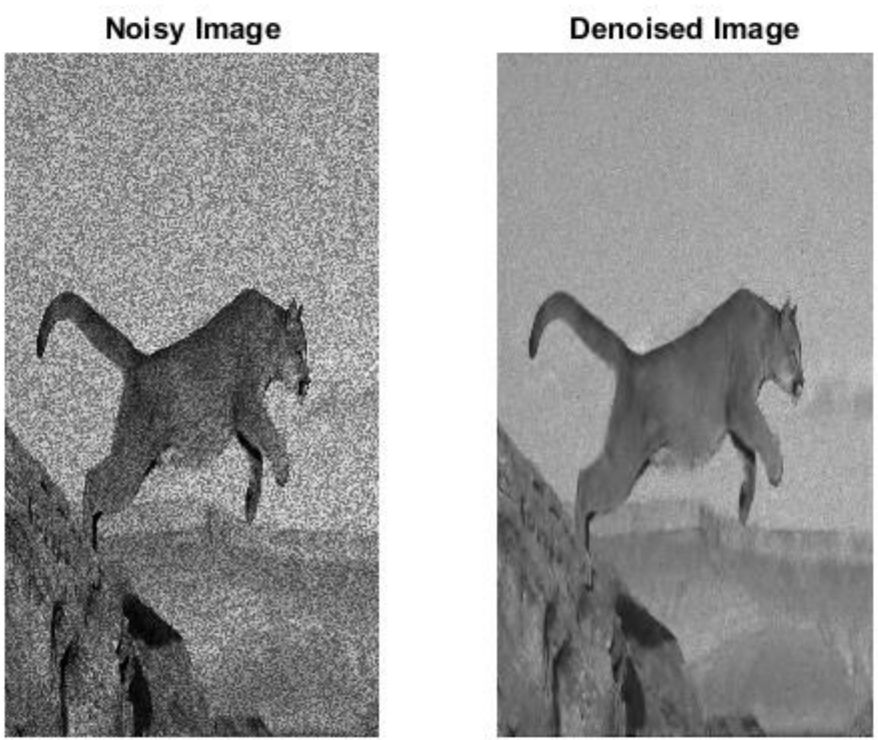
Courtesy: Kudur et al. ICIP 97

DWT applications: analysis

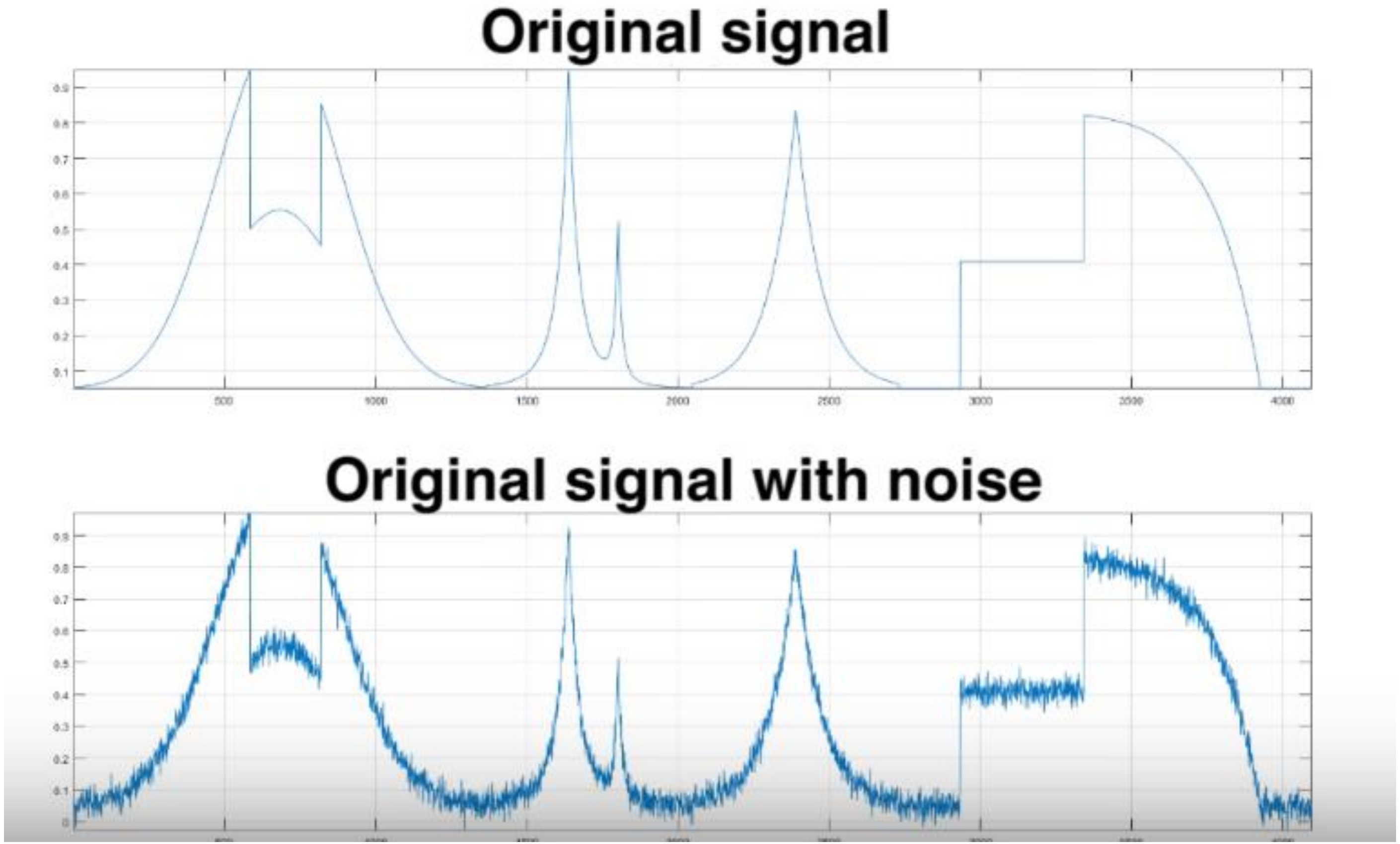


Courtesy: art spy

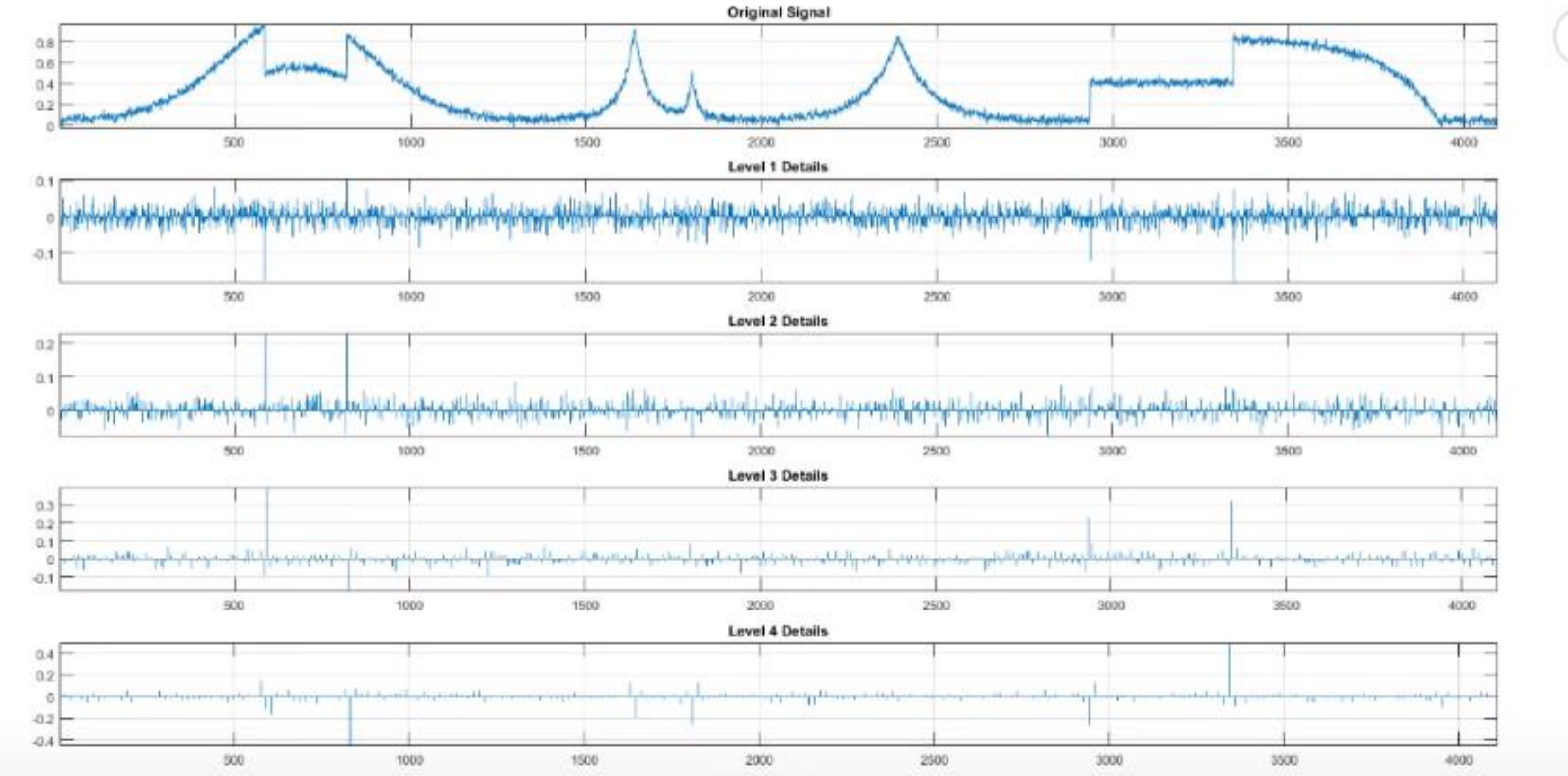
DWT applications: denoising



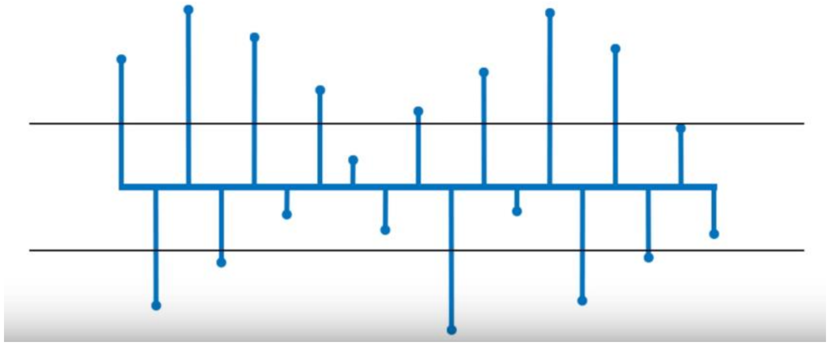
DWT applications: denoising

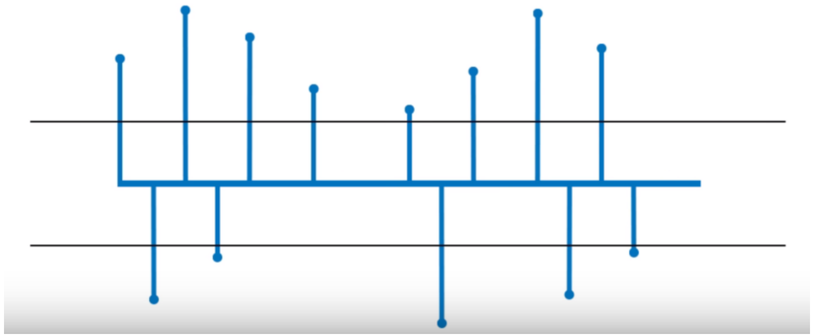


DWT applications: denoising

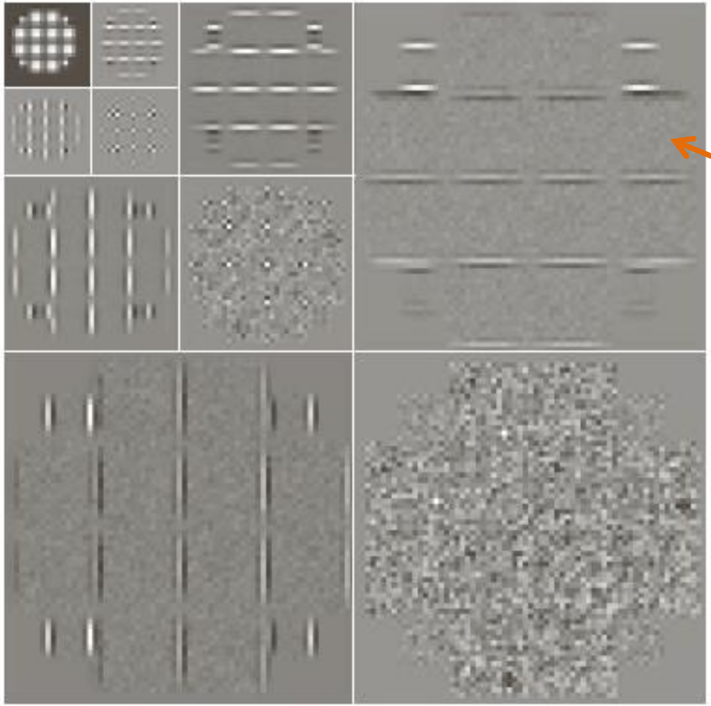


DWT applications: denoising





Re look: Multi scale-DWT



Multi scale-DWT

- Suppose we reconstruct the signal from $\mathbf{X}_l$ only

$$\mathrm{IDWT}(\mathbf{X}_\ell||\mathbf{0}) = \alpha_1(\mathbf{x}) \in \mathbb{C}^N$$

- $\alpha_1(\mathrm{x})$ is the stage one approximation of x

- Signal reconstruction from detail coefficients:

$$\mathrm{IDWT}(\mathbf{0}||\mathbf{X}_h) = \delta_1(\mathbf{x}) \in \mathbb{C}^N$$

- $\delta_1(\mathrm{x})$ is the stage one approximation of x

- Stage 1 representation of signal: $\boxed{\mathrm{x} = \alpha_1(\mathrm{x}) + \delta_1(\mathrm{x})}$

Multi scale-DWT

- At each stage, the sequence of detail representations $\delta_1(\mathrm{x}), \delta_2(\mathrm{x}), \ldots \ldots, \delta_{m-1}(\mathrm{x})$ is extended by one term, $\delta_m(\mathrm{x})$

$$
\begin{array}{ll}
\text{stage 1:} & \mathbf{x} = \alpha_1(\mathbf{x}) + \delta_1(\mathbf{x}) \\
\text{stage 2:} & \mathbf{x} = \alpha_2(\mathbf{x}) + \delta_2(\mathbf{x}) + \delta_1(\mathbf{x}) \\
\text{stage 3:} & \mathbf{x} = \alpha_3(\mathbf{x}) + \delta_3(\mathbf{x}) + \delta_2(\mathbf{x}) + \delta_1(\mathbf{x}) \\
 & \vdots \\
\text{stage m:} & \mathbf{x} = \alpha_m(\mathbf{x}) + \delta_m(\mathbf{x}) + \delta_{m-1}(\mathbf{x}) + \cdots + \delta_1(\mathbf{x})
\end{array}
$$

Multiscale representation of a signal

Let me invert the notations!

Multi scale- 1D Haar Wavelet transform

l_a

1/2	1/2

h_a

-1/2	1/2

9	7	3	5	6	10	2	6

Approx α_2

8	4	8	4

1	-1	-2	-2

Detail coeff δ_2

Approx α_1

6	6

2	2

Detail coeff δ_1

Approx α_0

6

0

Detail coeff δ_0

Multi scale- 1D Haar Wavelet transform

l_a

1/2	1/2

h_a

-1/2	1/2

9	7	3	5	6	10	2	6

Wavelet transform only contains these pixels

1	-1	-2	-2

Detail coeff δ_2

2	2

Detail coeff δ_1

Approx α_0

6

0

Detail coeff δ_0

Multi scale- 1D Haar Wavelet transform

l_a

1/2	1/2

h_a

-1/2	1/2

9	7	3	5	6	10	2	6

Wavelet transform only contains these pixels

1	-1	-2	-2

Detail coeff δ_2

2	2

Detail coeff δ_1

Approx α_0

6

0

Detail coeff δ_0

X (3 stage) =

6	0	2	2	1	-1	-2	-2

Multi scale- 1D Haar Wavelet transform

$$\begin{array}{c} \alpha \\ \delta_0 \\ \delta_1 \\ \\ \\ \delta_2 \\ \\ \\ \end{array} \begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Multi scale- 1D Haar Wavelet transform

9	7	3	5	6	10	2	6

8	4	8	4

1	-1	-2	-2

$$
\begin{array}{c|c|}
\alpha_0 & 6 \\
\delta_0 & 0 \\
\delta_1 & 2 \\
 & 2 \\
\delta_2 & 1 \\
 & -1 \\
 & -2 \\
 & -2
\end{array}
=
\begin{array}{c}
 \\
 \\
\frac{1}{2} \\
 \\
 \\
 \\
 \\
 \\
\end{array}
\begin{bmatrix}
 & & & & & & & \\
 & & & & & & & \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\cdot \frac{1}{2}
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6
\end{bmatrix}
$$

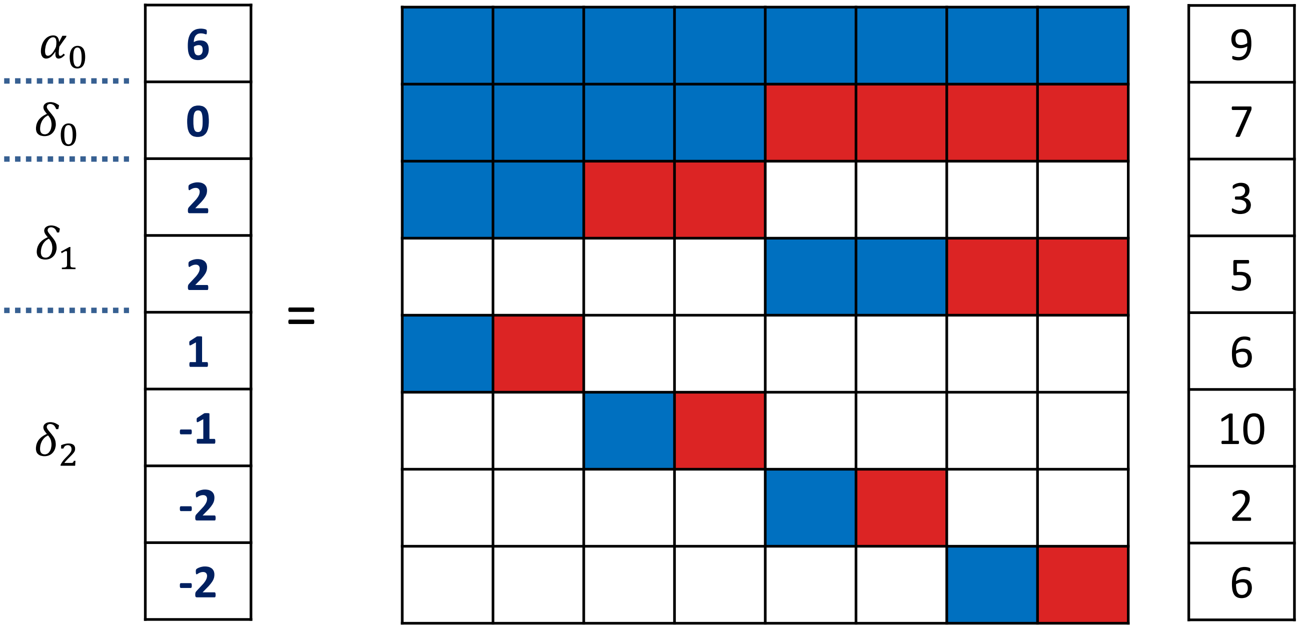
Multi scale- 1D Haar Wavelet transform

$$
\begin{array}{c}
x_0 \\ D_0 \\ D_1 \\ \\ D_2 \\ \\ \\ \\
\end{array}
\begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix}
=
\begin{array}{c}
\\ \\ \frac{1}{4} \\ \\ \\ \frac{1}{2} \\ \\ \\
\end{array}
\begin{bmatrix}
 & & & & & & & \\
 & & & & & & & \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}
\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}
$$

Multi scale- 1D Haar Wavelet transform

$$
\begin{matrix} \alpha_0 \\ \delta_0 \\ \delta_1 \\ \\ \\ \delta_2 \\ \\ \\ \end{matrix}
\begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix}
=
\begin{matrix} \frac{1}{8} \\ \\ \frac{1}{4} \\ \\ \\ \frac{1}{2} \\ \\ \\ \end{matrix}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}
\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}
$$

Multi scale- 1D Haar Wavelet transform



Multi scale- Haar Basis

$$\psi(x) \equiv \begin{cases} 1 & 0 \le x < \frac{1}{2} \\ -1 & \frac{1}{2} < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\psi_{jk}(x) \equiv \psi\left(2^j x - k\right)$$

j is non negative integer and $0 \le k \le 2^j - 1$

Multi scale- 1D Haar Wavelet transform

- Two stage transform

$$\mathcal{W}_2^a = \begin{pmatrix} W_{N/2}^a & 0 \\ 0 & \mathbf{I}_{N/2} \end{pmatrix} W_N^a$$

Multi scale- 1D Haar Wavelet transform

- An r stage DWT is obtained by iteration:

$$\mathcal{W}_r^a = \begin{pmatrix} W^a_{N/2^{r-1}} & 0 \\ 0 & \mathbf{I}_{N(1-1/2^{r-1})} \end{pmatrix} \cdots \begin{pmatrix} W^a_{N/2} & 0 \\ 0 & \mathbf{I}_{N/2} \end{pmatrix} W^a_N$$

- The inverse DWT is governed by the matrix

$$\mathcal{W}_r^s = W^s_N \begin{pmatrix} W^s_{N/2} & 0 \\ 0 & \mathbf{I}_{N/2} \end{pmatrix} \cdots \begin{pmatrix} W^s_{N/2^{r-1}} & 0 \\ 0 & \mathbf{I}_{N(1-1/2^{r-1})} \end{pmatrix}$$