* Smooth surface (Def^n)

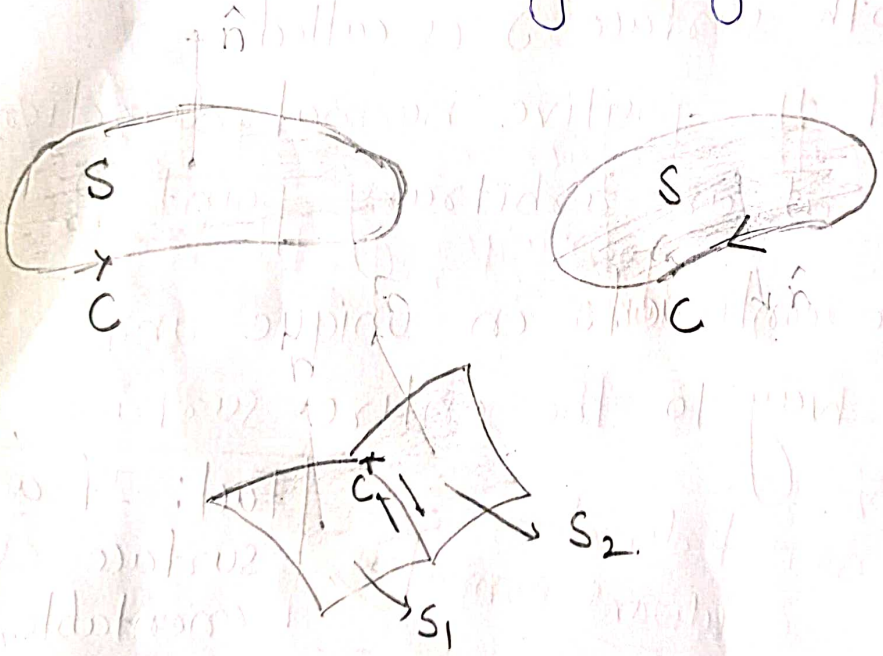
A surface that has unique tangent plane at every point of the surface is said to be a smooth surface.

OR

A surface S is called a smooth surface if it's surface normal depends continuously on the points of S.

* Piecewise smooth (Def^n)

S is called piecewise smooth if it consists of finitely many smooth portions.



** Orientation of Surfaces :- (Depend on choice of unit normal vector field on S)

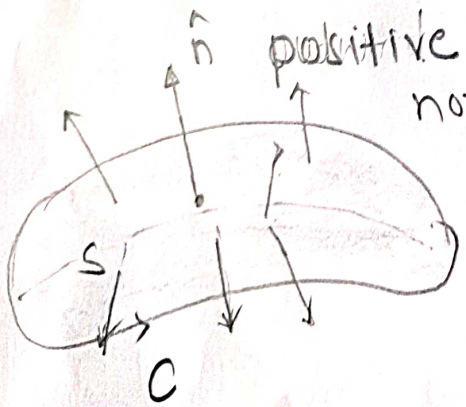
- The value of the integral depends on the choice of the normal vector $\hat{n}$. (Instead of $\hat{n}$ we could choose $-\hat{n}$.)

- If we change the orientation of S, this means that we replace $\hat{n}$ with $-\hat{n}$.

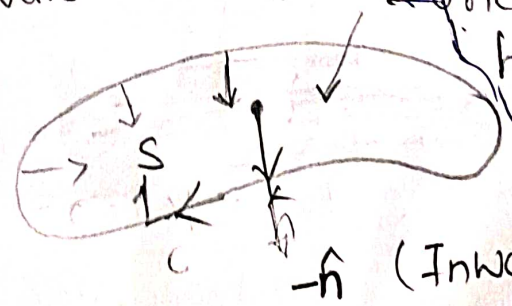
- $-\hat{n}$ is calculated from $-\vec{N} = \vec{r}_v \times \vec{r}_u$
  $\hookrightarrow = \frac{-\vec{N}}{|-\vec{N}|}$

→ Orientation of Smooth Surfaces

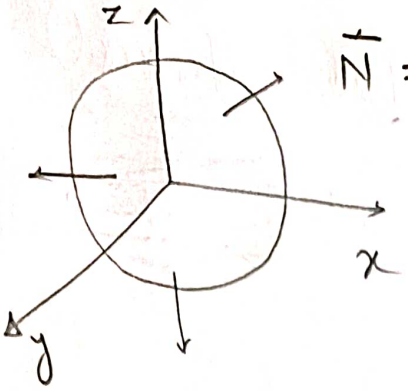
A smooth surface S is called orientable if the positive normal direction when given at an arbitrary point P_0 of S can be continued in unique and continuous way to the entire surface.



Fact: If a surface is orientable, it has 2 orientations



Two possible orientation of S an orientation of C. (curve)



$\vec{N} = \vec{r}_u \times \vec{r}_v$ (parametric representation)

or

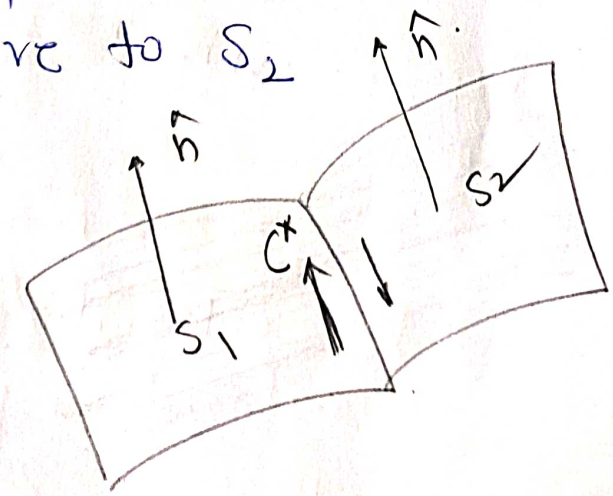
$\vec{n} = \nabla f$ (cartesian representation)

OIR

If the normal vector taken to closed (boundry of a solid) and bounded surfaces remain to be outward normal through out then such surfaces are said to be orientable surfaces

⇒ **Orientation of Piecewise Smooth Surfaces**

A piecewise smooth surfaces is called orientable if we can orient each smooth piece of S so that along each curve C^* which is common boundary of two pieces S_1 and S_2 the positive direction of C^* relative to S_1 is opposite to the direction of C^* relative to S_2



Nonorientable Surfaces

Ex Möbious strips

Note ① A smooth surface is orientable if there is a continuous field of unit normal vectors.

Surface integrals of vector-valued functions :-

We now investigate surface integrals over vector fields. The idea is an extension of our work on flux over closed curves in the plane. Thus we will require <u>orientation</u> and <u>normal vectors</u> to surfaces

An orientation of a surface S is a continuously varying choice of unit vector $\hat{n}(P)$ at each point P on the surface.

To define a surface integral, we take a surface S, given by a parametric representation

$$\vec{r}(u,v) = [x(u,v), y(u,v), z(u,v)]$$

$$= x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$

where $\vec{r}(u,v)$ varies over a region R in the uv-plane.

We assume S to be piecewise smooth. (i.e S consists of finitely many smooth portions). So that S has a normal vector.

$\vec{N} = \vec{r}_u \times \vec{r}_v$ and unit normal vector $\hat{n} = \frac{1}{|\vec{N}|}\vec{N}$

at every point (except perhaps for some edges)

For a given <u>vector function $\vec{F}$</u> we can now define surface integral over S by

$$\boxed{\iint_S \vec{F} \cdot \hat{n}\, dA = \iint_R \vec{F}(\vec{r}(u,v)) \cdot \vec{N}(u,v)\, du\, dv} \quad (*)$$

The '<u>flux</u>' of $\vec{F}$ over the <u>surface S</u> is the <u>surface integral</u> of the normal component of $\vec{F}$ over S.

(or)

This is called the flux integral and measures the flow $\vec{F}$ across the surface S.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S (\vec{F} \cdot \hat{n})\, dS$$

$$= \iint_R \vec{F}(\vec{r}(u,v)) \cdot \frac{\vec{N}(u,v)}{|\vec{N}(u,v)|} |\vec{N}(u,v)|\, du\, dv$$

$$= \iint_R \vec{F}(\vec{r}(u,v)) \cdot \vec{N}(u,v)\, du\, dv.$$

How to solve the surface integral when vector field and paramatrized surface is given -- ? → Ans $\boxed{(*)}$

Representation $\underline{z = f(x,y)}$

Show that $z = f(x,y)$ or $g(x,y,z) = z - f(x,y) = 0$

Can be written

$$\vec{r}(u,v) = [u, v, f(u,v)]$$

and

$$\vec{N} = \text{grad } g = [-f_u, -f_v, 1] \quad \text{— (1)}$$

$\rightarrow$ Let $x = u$, $y = v$ then $z = f(u,v)$.

Then parametric representation for

$$g(x,y,z) = z - f(x,y) = 0$$

is $\vec{r}(u,v) = [u, v, f(u,v)]$

Now $\vec{r}_u = [1, 0, f_u]$, $\vec{r}_v = [0, 1, f_v]$

$$\therefore \vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix} = -f_u \hat{i} - f_v \hat{j} + \hat{k}$$

Thus $\vec{N} = [-f_u, -f_v, 1]$ — LHS of (1)

But $\text{grad } g = [g_x, g_y, g_z]$

$$= [-f_x, -f_y, 1]$$

But $x = u$ and $y = v$

$\therefore \text{grad } g = [-f_u, -f_v, 1]$. $\longrightarrow$ RHS of (1)

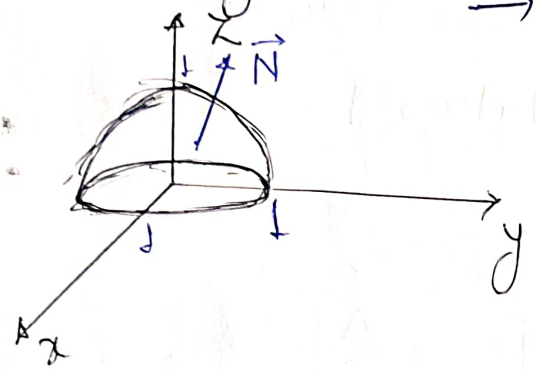
$\therefore \vec{N} = \text{grad } g = [-f_u, -f_v, 1]$.

Example :-

Calculate the flux of

$$\vec{F} = z\hat{i} + yz\hat{j} + 2x\hat{k}$$

across the upper hemisphere of the unit sphere oriented with outward-pointing normals.



→ Given data

$$\vec{F} = z\hat{i} + yz\hat{j} + 2x\hat{k}$$

S : upper hemisphere.

$$z = \sqrt{1-x^2-y^2} \Rightarrow z = f(x,y)$$

$$\therefore g(x,y,z) = z - \sqrt{1-x^2-y^2}$$

$$\text{i.e } g(x,y,z) = z - f(x,y)$$

Let $\vec{r}(x,y) = (x, y, f(x,y))$

$$= (x, y, \sqrt{1-x^2-y^2})$$

With $R := \{x^2+y^2 \le 1\}$ → (To decide the limits of x and y see region in xy-plane) or (R is projection of S into xy-plane)

Produce normal vector

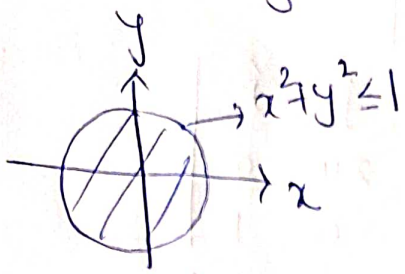
$$\vec{N} = \nabla g = [-f_x, -f_y, 1]$$

$$= \left[\frac{x}{\sqrt{1-(x^2+y^2)}}, \frac{y}{\sqrt{1-(x^2+y^2)}}, 1\right]$$

Also $\vec{F}(\vec{r}(x,y)) = [\sqrt{1-(x^2+y^2)}, y\sqrt{1-(x^2+y^2)}, 2x]$

thus $\vec{F}(\vec{r}(x,y)) \cdot \vec{N}(x,y) = x + y^2 + 2x = 2x + y^2$

$$\therefore I = \iint_S \vec{F}\cdot\hat{n}\,dA = \iint_R \vec{F}(\vec{r}(x,y))\cdot\vec{N}(x,y)\,dx\,dy$$



$$= \iint_R 3x+y^2\,dA$$

$$= \int_0^{2\pi}\int_0^1 (3r\cos\theta + r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\cos\theta\,d\theta \int_0^1 3r^2\,dr + \int_0^{2\pi}\sin^2\theta\,d\theta\int_0^1 r^3\,dr$$

$$= 0 + \pi\cdot\frac{1}{4} = \frac{\pi}{4}$$

$$\left(\because \int_0^{2\pi}\cos\theta\,d\theta = 0,\ \int_0^{2\pi}\sin^2\theta\,d\theta = \pi\right).$$

In this eg $S: z = f(x,y)$ and $\vec{F} \to$ vector field

<u>Example</u> Compute the flux of water through the parabolic cylinder $S: y = x^2,\ 0\le x\le 2,\ 0\le z\le 3$. if the velocity vector is $\vec{V} = \vec{F} = [3z^2, 6, 6xz]$, speed being measure in meters/sec.

$\rightarrow$ Writing $x = u,\ z = v$, we have $y = x^2 = u^2$.

$$\vec{r}(u,v) = [u, u^2, v],\quad 0\le u\le 2,\ 0\le v\le 3.$$

$$\vec{N} = \vec{r}_u \times \vec{r}_v, \quad \vec{r}_u = [1, 2u, 0]$$

$$\vec{r}_v = [0, 0, 1]$$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2u\hat{i} - 1\hat{j} + 0\hat{k}$$

$$\therefore \vec{N} = [2u, -1, 0]$$

Also $\vec{F}(\vec{r}(u,v)) = [3v^2, 6, 6uv]$.

$$\vec{F}(\vec{r}(u,v)) \cdot \vec{N} = [3v^2, 6, 6uv] \cdot [2u, -1, 0]$$

$$= 6uv^2 - 6$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, dA = \int_0^3 \int_0^2 (6uv^2 - 6)\, du\, dv$$

$$= \int_0^3 \left(\cancel{6}^{3} \frac{u^2}{\cancel{2}} v^2 - 6u\right)_0^2 dv$$

$$= \int_0^3 \left[3(2)^2 v^2 - 6(2) - 0\right] dv$$

$$= \int_0^3 (12v^2 - 12)\, dv$$

$$= \left[12\frac{v^3}{3} - 12v\right]_0^3$$

$$= \cancel{12}^{4} \cdot \frac{(3)^3}{\cancel{3}} - 12(3)$$

$$= 108 - 36$$

$$= 72 \left[m^3/sec\right].$$

Flux integrals $\int_S \vec{F}\cdot\hat{n}\,dA$

Evaluate these integrals for the following data. Indicate the kind of surface.

(1) $\vec{F} = [2x, 5y, 0]$, $S: \vec{r} = [u, v, 4u+3v]$

$0 \le u \le 1$, $-8 \le v \le 8$

$\rightarrow$ Here $\vec{r} = [u, v, 4u+3v]$

$x = u$
$y = v$
$z = 4x+3v$

$\therefore$ $z = f(x,y) \rightarrow z = 4x+3v$ – surface. Plane.

$\vec{r}_u = [1, 0, 4]$, $\vec{r}_v = [0, 1, 3]$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 4 \\ 0 & 1 & 3 \end{vmatrix}$$

$= -4\hat{i} - 3\hat{j} + \hat{k}$

$\therefore \vec{N} = [-4, -3, 1]$

or

$g(x,y,z) = z - [4x+3v]$

$\vec{N} = \text{grad}\, g = [-4, -3, 1]$

Now $\vec{F}(\vec{r}(u,v)) = [2u, 5v, 0]$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,dA = \iint_R \vec{F}(\vec{r}(u,v))\cdot\vec{N}(u,v)\,du\,dv$$

$$= \int_{-8}^{8}\int_{0}^{1} [2u, 5v, 0]\cdot[-4, -3, 1]\,du\,dv$$

$$= \int_{-8}^{8}\int_{0}^{1} (-8u - 15v)\, du\, dv$$

$$= \int_{-8}^{8} \left[-8\frac{u^2}{2} - 15vu \right]_0^1 dv$$

$$= \int_{-8}^{8} \left[-\frac{8(1)^2}{2} - 15v(1) - (0) \right] dv$$

$$= \int_{-8}^{8} (-4 - 15v)\, dv$$

$$= \left[-4v - 15\frac{v^2}{2} \right]_{-8}^{8}$$

$$= -4(8) - 15\frac{(8)^2}{2} - \left[-4(-8) - 15\frac{(-8)^2}{2} \right]$$

$$= -4(8) - \cancel{15\frac{(8)^2}{2}} - 4(8) + \cancel{15\frac{(8)^2}{2}}$$

$$= -32 - 32$$

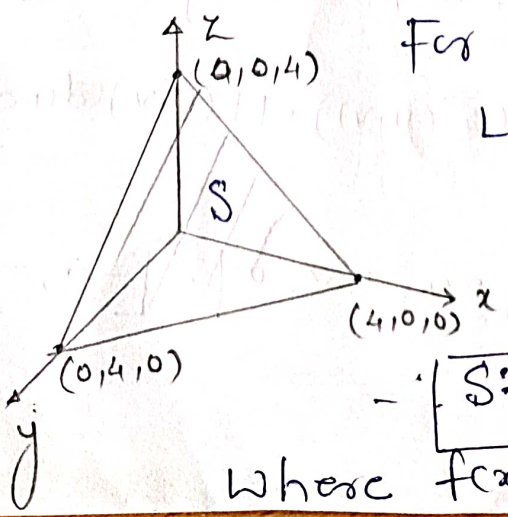
$$= \underline{\underline{-64}}$$

<u>Example (2)</u> $\vec{F} = [x^2, y^2, z^2]$

$S: x+y+z=4 \quad , x \geq 0, \; y \geq 0, \; z \geq 0$

$\rightarrow$ Here $\vec{F} = [x^2, y^2, z^2]$

$S: x+y+z=4 \; , \quad x \geq 0, \; y \geq 0, \; z \geq 0$



For parametric representation

Let $u = x$, $y = v \rightarrow$ xy-plane $\Downarrow$ uv-plane

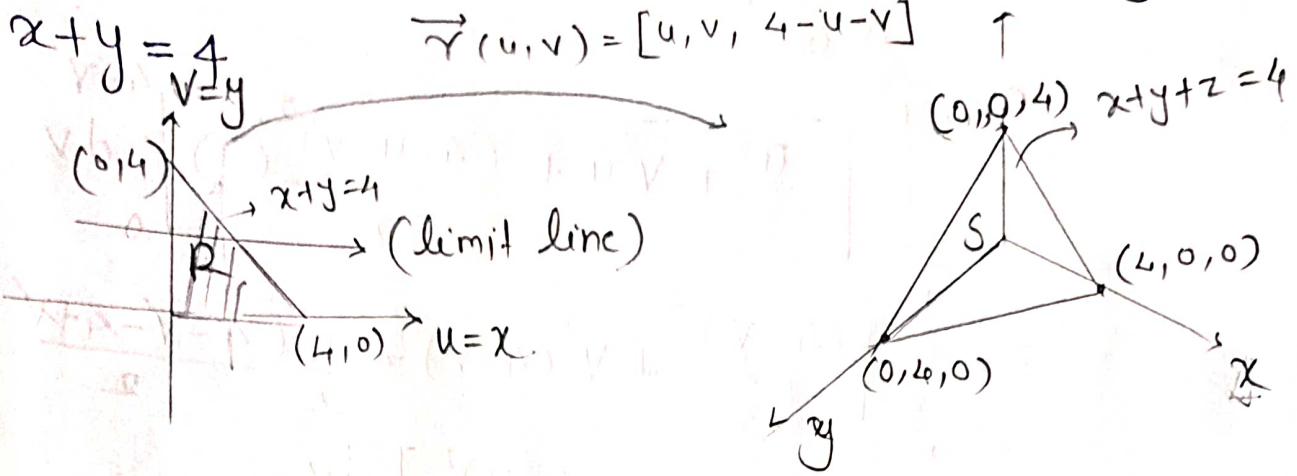
$z = 4 - x - y \Rightarrow z = f(x,y)$

$\therefore z = 4 - u - v$

$\boxed{S: g(x,y,z) = z - f(x,y)}$

Where $f(x,y) = 4 - u - v$

Thus $\vec{r}(u,v) = [u, v, 4-u-v]$, To obtain the limits of u and v, take a projection of S in xy-plane, so you have tringle bdd by two co-ordinate axies and the straight line



To find the limits of u and v look R-region in xy-plane (i.e uv-plane), $0 \le \underline{u=x} \le 4-v$ and $0 \le v \le 4$ (i.e $0 \le y \le 4$)

Here $g(x,y,z) = z-(4-x-y)$

$\vec{N} = \nabla g = [1, 1, 1]$

OR

$\vec{N} = \vec{r}_u \times \vec{r}_v$

$\vec{r}_u = [1, 0, -1]$

$\vec{r}_v = [0, 1, -1]$

$\therefore \vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}$

$= \hat{i} + \hat{j} + \hat{k}$

$= [1, 1, 1]$

Also

$\vec{F}(\vec{r}(u,v)) = [u^2, v^2, (4-u-v)^2]$

and $\vec{F}(\vec{r}(u,v)) \cdot \vec{N} = [u^2, v^2, (4-u-v)^2] \cdot [1, 1, 1]$

$= u^2 + v^2 + (4-u-v)^2$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,dA = \iint_R \left[u^2+v^2+(4-u-v)^2\right] du\,dv$$

$$= \int_0^1\int_0^{4-v} \left[u^2+v^2+(4-u-v)^2\right] du\,dv$$

$$= \int_0^1 \left[\frac{u^3}{3}+v^2u+\frac{(4-u-v)^2}{2}(-1)\right]_0^{4-v} dv$$

$$= \int_0^1 \left[\frac{(4-v)^3}{3}+v^2(4-v)-\frac{1}{2}\cancel{(4-v-4+v)}^{2} + \frac{(4-v)^2}{2}\right] dv$$

$$= \int_0^1 \left[\frac{(4-v)^3}{3}+v^2 4-v^3+\frac{(4-v)^2}{2}\right] dv \rightarrow \boxed{\text{do it}}$$