Set

Set:- A set is a collection of distinct objects. Thus we use the notation $\{a, b, c\}$ to denote the set which is the collection of objects a, b and c. The objects in a set are also called the elements or the members of the set. We usually also give names to sets. For example, we write $S = \{a, b, c\}$ to mean that the set named S is the collection of objects a, b, and c. Consequently, we can refer to the set S as well as to the set $\{a, b, c\}$. As another example, we may have

B.Tech Second yr. CS Branch = {Ramesh, Mohan... Deepak}

The name of the set {Ramesh, Mohan... Deepak} is rather long. The reader probably would want to suggest alternative names such as S or CS. However, there is nothing wrong conceptually with having a long name.

We use the notation $a \in S$ to mean that a is an element in the set S. In that case, we also say that S contains the element a.

We use the notation $d \notin S$ to mean that d is not an element in the set S.

Methods of Representing a Set :-

(i) **Roster method or Listing method or Tabular Method** :- In this method a set is described by listing all its element, separating them by comma and enclosing them within brackets.

Ex:- If A is the set of even natural nos less than 8.

$A = \{2, 4, 6\}$

(ii) Set Builder Method :- A set is described by means of some property which is shared by all the elements of the set.

Ex:- If A is the set of all even positive integers ~~that~~ not larger than 10 then,

A = { x : x is the set of all even positive integers not larger than 10 }.

Type of sets :-

(i) Singleton Set : If a set consists of only one element it is called a singleton set.

For example : {1}, {a}, {b} etc.

(ii) Finite Set : A set consisting of natural no.s of objects, i.e. in which the no. of elements is finite, is called finite set.

For example:- A = {5, 7, 9, 11}, B = {2, 3, 5}

Since A contains 4 elements and B contains 3 elements, so both are finite sets.

(iii) Infinite set : If no. of elements in a set is infinite, the set is called infinite set.

For example :- set of natural no.s

N = { 1, 2, 3, ---- } is an infinite set

(iv) Equal Set :- Two sets A and B consisting of the same elements are called equal set.

For example- A = {1, 3, 5}

B = {1, 3, 5}

So Here A = B

or symbolically:

$$A - B = \{x \mid x \in A ;\ x \notin B\}$$

Similarly

$$B - A = \{x \mid x \in B ;\ x \notin A\}$$

Example: Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$ then find $A-B$ and $B-A$

Solⁿ: $A - B = \{1, 3\}$

$B - A = \{6, 8\}$

Symmetric difference of two sets P and Q, denoted by $P \oplus Q$, is the set containing exactly all the elements that are in P or in Q but not in both. In other words, $P \oplus Q$ is the set $(P \cup Q) - (P \cap Q)$. For example

$$\{a,b\} \oplus \{a,c\} = \{b,c\}$$

$$\{a,b\} \oplus \phi = \{a,b\}$$

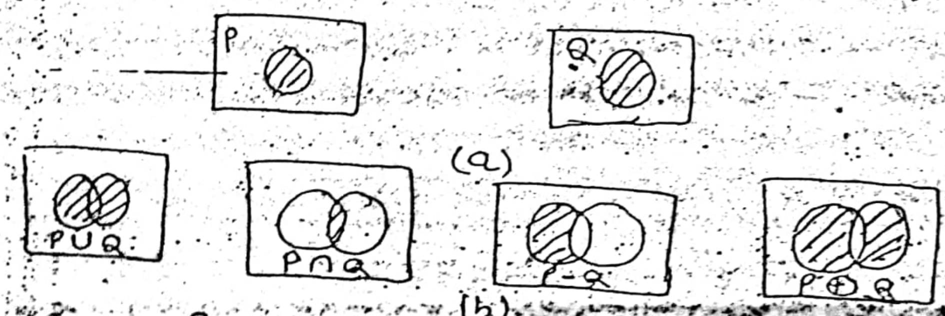
$$\{a,b\} \oplus \{a,b\} = \phi$$

Power set:- The power set of a set A, denoted by P(A), is the set contains exactly all the subsets of A. ~~There~~ ~~P(A)~~ If A has n elements in it, then P(A) has 2^n elements.

~~$P(\{a,b\}) = \{\{\}, \{a\}, \{b\}, \{a,b\}\}$~~ Ex: If $A = \{a,b\}$ then $P(A) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

Note ~~that for any set A, $\{\phi\} \in P(A)$ as well as $\{\phi\} \subseteq P(A)$~~. Ex: The empty set ϕ has only subset, therefore $P(\phi) = \{\phi\}$

Venn diagrams:- Set obtained from combinations of given sets can be represented pictorially. If we let P and Q be the sets represented by the cross hatched area in Fig 1a, then the cross hatched area in fig. 1b represent the sets $P \cup Q$, $P \cap Q$, $P-Q$ and $P \oplus Q$. These diagram are known as Venn diagrams.



(a)

(b)

Fig 1

(V) Pair Set : A set having two elements is called ...

For example :

$$\{1,3\}, \{0,5\}, \{10,12\}, \text{ etc.}$$

(vi) Empty set: If a set consists of no elements, it is called the empty set or null set or Void set and is represented by ϕ.

Here a point is to be noted that ϕ is a null set but $\{\phi\}$ is Singleton set.

Combination of sets – The union of two sets P and Q denoted by $P \cup Q$, is the set, whose elements are exactly the elements in either P or Q (or both). For example,

$$\{a,b\} \cup \{c,d\} = \{a,b,c,d\}$$

$$\{a,b\} \cup \{a,c\} = \{a,b,c\}$$

$$\{a,b\} \cup \phi = \{a,b\}$$

$$\{a,b\} \cup \{\{a,b\}\} = \{a,b,\{a,b\}\}$$

In symbols: $P \cup Q = \{x : x \in P \text{ or } x \in Q\}$

Ex: Let $P = \{2,3,4\}$ and $Q = \{1,5\}$ then $P \cup Q = \{1,2,3,4,5\}$

Intersection of two sets P and Q denoted by $P \cap Q$ is the set whose elements are exactly those elements that are in both P and Q. For example

$$\{a,b\} \cap \{a,c\} = \{a\}$$

$$\{a,b\} \cap \{c,d\} = \phi$$

$$\{a,b\} \cap \phi = \phi$$

In Symbols: $P \cap Q = \{x : x \in P \text{ and } x \in Q\}$

Ex: Let $P = \{2,3,5,8\}$ and $Q = \{5,9,10,3\}$ then $P \cap Q = \{5,3\}$

The difference of two sets P and Q, denoted $P - Q$, is the set containing exactly those elements in P that are not in Q. For example,

$$\{a,b,c\} - \{a\} = \{b,c\}$$

$$\{a,b,c\} - \{a,d\} = \{b,c\}$$

If $X = \{x : x = 2n+1, n \in Z\}$ and $Y = \{x : x = 2n, n \in Z\}$
then
$X \cup Y = \{x : x$ is an integer, $x \in Z\} = Z$

Disjoint Sets: Two sets X and Y are said to be disjoint if $X \cap Y = \phi$

Ex- $A = \{1, 3, 4, 5\}$, $B = \{2, 6, 7\}$

Here $\boxed{A \cap B = \phi}$

Cardinal no. of a Set:- The no. of elements in the set is called cardinal no. of a set or order of a set, denoted by $n(A)$

Ex- $A = \{1, 2, 3, 5, 8\}$

$n(A) = 5$

Theorem 1: Every set is a subset of itself.

Proof: Let X be any set. Then each element of X is clearly in X itself.

Hence $X \subseteq X$

Theorem 2- The Empty set is a subset of every set.

Proof:- Let X be any set and ϕ be the empty set. In order to show that $\phi \subseteq X$, we must show that every element of ϕ is an element of X also. But ϕ contains no element, so every element of ϕ is in X. Hence $\phi \subseteq X$.

Laws of Algebra of sets

1. **Idempotent Laws**
   a) $X \cup X = X$ b) $X \cap X = X$

2. **Identity Laws**
   a) $X \cup \phi = X$ b) $X \cap U = X$
   c) $X \cap \phi = \phi$ d) $X \cup U = U$

3. **Commutative Laws**
   a) $X \cup Y = Y \cup X$ b) $X \cap Y = Y \cap X$

(iv) Associative Laws:

a) $(X \cup Y) \cup Z = X \cup (Y \cup Z)$

b) $(X \cap (Y \cap Z)) = (X \cap Y) \cap Z$

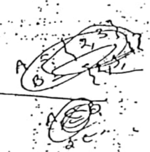
(V) De-Morgan's Laws:

a) $(X \cup Y)' = X' \cap Y'$ b) $(X \cap Y)' = X' \cup Y'$

(VI) Distributive laws -

a) $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$

b) $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$



Q:- Prove that

$X \cap (Y \cap Z) = (X \cap Y) \cap Z$

Proof. Let x be an arbitrary element of ~~$(x \cap Y) \cap Z$ Then~~ $X \cap (Y \cap Z)$.

~~$x \in (x \cap Y) \cap Z$~~ then

~~$\Rightarrow x$~~

$x \in X \cap (Y \cap Z)$

$\Rightarrow x \in X$ and $x \in (Y \cap Z)$

$\Rightarrow x \in X$ and $(x \in Y$ and $x \in Z)$

$\Rightarrow (x \in X$ and $x \in Y)$ and $x \in Z$

$\Rightarrow x \in (X \cap Y)$ and $x \in Z$

$\Rightarrow x \in (X \cap Y) \cap Z$

$\therefore X \cap (Y \cap Z) \subseteq (X \cap Y) \cap Z$

Similarly $(X \cap Y) \cap Z \subseteq X \cap (Y \cap Z)$

Hence, $X \cap (Y \cap Z) = (X \cap Y) \cap Z$

Partition of a set X:-

Let X be any set such that

$X = \bigcup_{i=1}^{n} A_i$

that is A_is are subsets of X.

A partition of a set X is a collection $\{A_i\}$ of nonempty subsets of X such that

(a) Each x in X belongs to one of the A_i.

(b) ~~The sets~~ of $\{A_i\}$ are mutually disjoints that is, if $A_i \neq A_j$ for $i \neq j$ then

$A_i \cap A_j = \phi$

Ex: Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,
$A_1 = \{1, 3, 5\}$, $A_2 = \{2, 4, 6, 8\}$, $A_3 = \{7, 9\}$ and $A_4 = \{10\}$ then $P = \{A_1, A_2, A_3, A_4\}$ is a partition of X, because

(i) No element of P, which is a subset of X is a null set

(ii) $X = A_1 \cup A_2 \cup A_3 \cup A_4$

(iii) All pairs of elements of P are disjoint sets

Ordered Pair :-

Representation of two objects within a small bracket is called and ordered pair if the position of each object is specified.

For example - (x, y)

Two ordered pairs (x,y) and (a,b) are equal if and only if $x = a$ and $y = b$.

Cartesian Product

Let X and Y be two sets. The cartesian product set of X and Y consists of all ordered pairs (x,y) where $x \in X$ and $y \in Y$. It is denoted by $X \times Y$.

$$X \times Y = \{(x,y) : x \in X \text{ and } y \in Y\}$$

Ex :- Let $X = \{1, 2, 3\}$ and $B = \{a, b\}$

then $X \times Y = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

Ex: Show that $n[P[P[P(\phi)]]] = 4$

Soln:

$P(\phi) = \{\phi\}$

$P[P(\phi)] = \{\phi, \{\phi\}\}$

$P[P[P(\phi)]] = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$

Hence ~~Hence~~ $n[P[P[P(\phi)]]] = 4$

Relations

Cartesian Product: Consider two arbitrary sets A and B. The set of all ordered pairs (a,b) where $a \in A$ and $b \in B$ is called the Cartesian product of A and B.

$$A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$$

We can also write A^2 instead of $A \times A$.

Ex.1: Let $A = \{1,2\}$ and $B = \{a,b,c\}$. Then

$$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)\}$$

$$A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$$

Relations: Let A and B be sets. A binary relation or simply relation from A to B is a subset of $A \times B$.

Suppose R is a relation from A to B. Then R is a set of ordered pairs where each first element comes from A and each second element comes from B. We use the notation aRb to denote that $(a,b) \in R$ and $a \not R b$ to denote that $(a,b) \notin R$.

Domain of a relation R: is the set of all first elements of the ordered pairs which belong to R.

Range of a relation R: is the set of all second elements of the ordered pair which belong to R.

Ex.2: Let $A = \{2,3\}$, $B = \{a,b\}$

$$R = \{ (2,b), (3,a), (3,b) \}$$

Domain: $\{2,3\}$

Range: $\{a,b\}$

Inverse Relation: Let R be any relation from a set A to set B. The [illegible] of R, denoted by R^{-1}, is the relation [illegible] B to A. That is

$$R^{-1} = \{(b,a) : (a,b) \in R\}$$

For Example, the inverse of the relation

$R = \{(1,y), (1,z), (3,y)\}$ from $A = \{1,2,3\}$ to $B = \{x, y, z\}$ follows:

$$R^{-1} = \{(y,1), (z,1), (y,3)\}$$

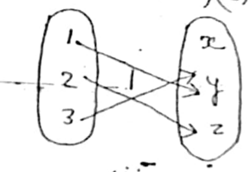
Clearly, if R is any relation, then $\boxed{(R^{-1})^{-1} = R}$

Pictorial Representation of Relation:

Consider $R = \{(1,y), (1,z), (3,y)\}$

	x	y	z
1	0	1	1
2	0	0	0
3	0	1	0

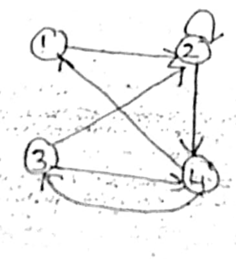
(i)



(ii)

Directed Graphs of Relations on Sets.

Consider $R = \{(1,2), (2,2), (2,4), (3,2), (3,4), (4,1), (4,3)\}$



(iii)

Example: If R and S are two equivalance relations on set A, then show that $R \cap S$ is also an equivalence relation.

Sol^n. To show that $R \cap S$ is an equivalence relation we need to prove that $R \cap S$ is reflexive, symmetric and transitive.

(i) **Reflexive:** Since R and S are equivalence relations defined on set A.

$\Rightarrow \forall a \in A, (a,a) \in R$ and $(a,a) \in S$

$\Rightarrow (a,a) \in R \cap S$

i.e. $\forall a \in A, (a,a) \in R \cap S$

Thus $R \cap S$ is reflexive.

(ii) **Symmetric:** Let $(a,b) \in R \cap S$

$\Rightarrow (a,b) \in R$ and $(a,b) \in S$

Since R and S both are equivalence relations

If $(a,b) \in R \Rightarrow (b,a) \in R$

and if $(a,b) \in S \Rightarrow (b,a) \in S$

Now $(b,a) \in R$ and $(b,a) \in S$

$\Rightarrow (b,a) \in R \cap S$

So, if $(a,b) \in R \cap S \Rightarrow (b,a) \in R \cap S$

(iii) **Transitive:** Let $(a,b) \in R \cap S$ and $(b,c) \in R \cap S$

$\Rightarrow (a,b) \in R$ and $(a,b) \in S$ and $(b,c) \in R$ and $(b,c) \in S$

Since, both R and S are equivalence relations

so if $(a,b) \in R, (b,c) \in R$

$\Rightarrow (a,c) \in R$

and if $(a,b) \in S, (b,c) \in S$

$\Rightarrow (a,c) \in S$

$\Rightarrow (a,c) \in R \cap S$

so if $(a,b) \in R \cap S$ and $(b,c) \in R \cap S$

$\Rightarrow (a,c) \in R \cap S$

Hence, $(R \cap S)$ is transitive.

~~Therefore $(R \cap S)$ is reflexive~~ symmetric and transitive.

Example:

Let $X = \{1, 2, 3, 4\}$

$R = \{(1,2), (2,3), (1,3), (3,4)\}$. What is the symmetric closure of R?

Sol^n:

$R^{-1} = \{(2,1), (3,2), (3,1), (4,3)\}$

Symmetric closure of $R_1 = (R \cup R^{-1})$

$= \{(1,2) (2,3) (1,3), (3,4), (2,1) (3,2), (3,1), (4,3)\}$

(iii) **Transitive closure**

obtained by repeatedly adding (a,c) to R for each (a,b) and (b,c) in R.

Example: Let $A = \{1,2,3,4\}$ and a relation R (on A) $= \{(1,2), (2,3), (3,3)\}$ clearly R is not a transitive relation because R contain $(1,2)$, $(2,3)$ but not $(1,3)$. Hence for transitive closure we add the element $(1,3)$ and get the transitive closure of R is given by

$S = \{(1,2), (2,3), (3,3), (1,3)\}$

Equivalence Relations:

A binary relation R defined on set A is called an equivalence relation if

a) R is reflexive i.e. $\forall a \in A$, aRa

b) R is symmetric i.e. if $(a,b) \in R \Rightarrow (b,a) \in R$

c) R is transitive, i.e. if $(a,b) \in R \Rightarrow (b,c) \in R$ then $(a,c) \in R$.

(IV) Asymmetric Relation : A relation R defined on a set A is asymmetric if whenever aRb, then $b \not R a$.

Example : Let $A = \{a, b, c\}$ and $R = \{(a,b), (b,c)\}$ be relation on A. clearly R is asymmetric.

(V) Irreflexive Relation . A relation R in a set X is irreflexive if, for every $x \in X$, $(x,x) \notin R$.

Example : Let $X = \{1, 2, 3\}$

$R = \{(2,1), (1,2), (3,2), (2,3), (1,3)\}$

clearly R is irreflexive.

(VI) Transitive Relation : A relation R defined on a Set X is transitive if whenever $(x,y) \in R$ and $(y,z) \in R$, then $(x,z) \in R$.

Example : Let $X = \{1, 2, 3\}$

$R = \{(1,1), (1,2), (2,2), (2,3), (2,1)\}$

$S = \{(1,2), (2,3), (1,3), (3,3), (2,1)\}$

Then R is transitive, but S is not transitive. (1,2)

Since 2S1 and 2S2 but $(2,2) \notin S$

(VII) Identity Relation

Let A be a non empty set and I_A be a relation in A defined such that

$I_A = \{(x,y) : x, y \in A \text{ and } x = y\}$

then I_A is called and Identity relation in A. I_A shall be a set of all ordered pairs (x,y) belonging to $A \times A$ for which $x = y$

An identity relation on A is always reflexive, but converse in not true, e.g.

$\{(a,a),(b,b),(c,c),(a,b)\}$ is reflexive but not an identity relation on $A = \{a,b,c\}$

(VIII) Universal Relation

A relation R in a set A is called universal relation if $R = A \times A$.

For example if $A = \{1,2,3\}$, then

$R = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$ is a universal relation

Closure operations on Relations

For any property X, the X closure of a set A is defined as the smallest superset of A that has the given property.

(I) Reflexive closure:

Let R be a relation on A. The reflexive closure of R is obtained by adding (a,a) to R, for each $a \in A$.

Example: Let $X = \{2,4\}$

$R = \{(2,2)(2,4)\}$ be a relation on X

Find the reflexive closure of R.

$R = \{(2,2)(2,4)\} \cup (4,4)$

$= \{(2,2)(2,4)(4,4)\}$

(II) Symmetric closure:

Symmetric closure R_1 of a relation R is obtained by adding (b,a) to R for each (a,b) in R

$R = (R \cup R^{-1})$

Properties of Relations (Types of Relations)

(i) **Reflexive**: A relation R on a set A is reflexive if aRa for every $a \in A$, that is, if $(a,a) \in R$ for every $a \in A$.

Example: Let $A = \{1, 2, 3\}$

$R = \{(1,1), (1,2), (2,2), (2,3), (3,3)\}$

A relation R on A is reflexive because it contains the ~~four~~ three pair $(1,1), (2,2), (3,3)$.

(ii) **Symmetric**: A relation R on a set A is symmetric if whenever $(x,y) \in R$ then $(y,x) \in R$.

Example: Let $X = \{1, 2, 3\}$

$R = \{(1,1) (1,2) (2,1) (2,3) (3,2)\}$

(iii) **Anti-symmetric**: A relation R on a set X is antisymmetric if, whenever $(x,y) \in R$ and $(y,x) \in R$ then $x = y$.

Thus, R is anti symmetric if we have never both aRb and bRa except when $a = b$

EX: Let $X = \{1, 2, 3\}$

$R_1 = \{(1,3), (3,1), (2,3)\}$ is neither symmetric nor antisymmetric.

$R_2 = \{(1,1) (2,2)\}$ is both symmetric and antisymmetric.

Composition of Relations

Let R be a relation from X to Y and S be a relation from Y to Z. The composite relation of R and S is a relation from X to Z defined by

$$(x,y) \in R \text{ and } (y,z) \in S \Rightarrow (x,z) \in R \circ S$$

The operation "o" in $R \circ S$ is called "composition of relations".

$$x \xrightarrow{R} y \xrightarrow{S} z \qquad x \xrightarrow{R \circ S} y$$

For example,

– Let $R = \{(1,2), (2,3), \cancel{(3,4)}, (2,2)\}$

$S = \{(2,3), (4,1), (4,3), (2,1)\}$

Then $R \circ S = \{(1,3), (1,1), (3,1), (3,3), (2,3), (2,1)\}$

Example: Let $R = \{(1,1), (2,1), (3,2), (4,3)\}$.

Show that $R^4 = R^3$

Sol^n: $R^2 = R \circ R$

$= \{(1,1), (2,1), (3,1), (4,2)\}$

$R^3 = R^2 \circ R$

$= \{(1,1), (2,1), (3,1), (4,1)\}$

$R^4 = R^3 \circ R$

$= \{(1,1), (2,1), (3,1), (4,1)\}$

Therefore

$R^4 = R^3$

Example: Let R be an equivalence relation on set A, then prove that R^{-1} is also an equivalence relation on set A.

Sol^n. To prove that R^{-1} is an equivalence relation, we need to prove that R^{-1} is reflexive, symmetric and transitive.

(i) **Reflexive:** We know that R is an equivalence relation, then

$\forall a \in A,\ (a,a) \in R$

if $(a,a) \in R \Rightarrow (a,a) \in R^{-1}$ (by definition of inverse)

$\forall a \in A,\ (a,a) \in R^{-1}$.

Hence, R^{-1} is a reflexive relation.

(ii) **Symmetric:**

Let $(a,b) \in R$ for $a,b \in A$

Now $(a,b) \in R \Rightarrow (b,a) \in R$ as R is symmetric

By the definition of inverse

If $(a,b) \in R \Rightarrow (b,a) \in R^{-1}$

also $(b,a) \in R \Rightarrow (a,b) \in R^{-1}$

Thus if $(b,a) \in R^{-1} \Rightarrow (a,b) \in R^{-1}$ for $a,b \in A$

Hence, R is a symmetric relation.

(iii) **Transitive:** Let $(a,b) \in R$ and $(b,c) \in R$

~~$\Rightarrow (b,c) \in R$~~ $\Rightarrow (b,a) \in R^{-1}$ and $(c,b) \in R^{-1}$ (by definition of inverse)

By transitive property of R.

If $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$

If $(a,c) \in R \Rightarrow (c,a) \in R^{-1}$

Thus $(c,b) \in R^{-1}$ and $(b,a) \in R^{-1}$

$\Rightarrow (c,a) \in R^{-1}$

Hence R^{-1} is transitive.

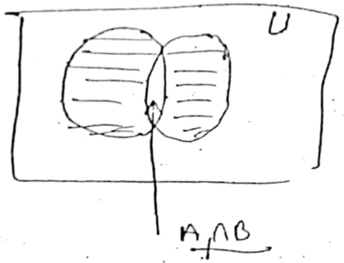
So, R^{-1} is an equivalence relation because it is reflexive, symmetric and transitive.

Inclusion – Exclusion Principle

Consider two finite set A and B with $n(A)$ and $n(B)$ elements. Then the cardinality of the union of two sets A and B is given by

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Proof: We will prove this with the help of Venn diagram.



Here we can conclude that the no. of elements in $(A \cup B)$ includes the no. of elements of $(A \cap B)$ twice, once in $n(A)$ and another in $n(B)$.

Therefore if we subtract $n(A \cap B)$ from the sum of $n(A)$ and $n(B)$ we get the elements in $(A \cup B)$:

i.e. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Note: If A & B are disjoint

i.e. $A \cap B = \phi$

then $n(A \cup B) = n(A) + n(B)$

Def^n: Let an i a

Def^n: Let f be a function from the set A to the set B and let S be a subset of A. The image of S is the subset of B that consists of the images of the element of S. We denote the image of S by $f(S)$, so that

$$f(S) = \{ f(s) \mid s \in S \}$$

Example:

Let $A = \{a, b, c, d, e\}$ and $B = \{1,2,3,4\}$ with $f(a) = 2$, $f(b) = 1$, $f(c) = 4$, $f(d) = 1$ and $f(e) = 1$. The image of the subset $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$

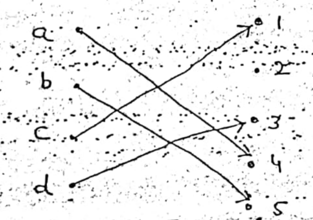
Types of function

One to one Function (Injective)

Some functions have distinct images at distinct members of their domain. These functions are said to be one to one.

Example: Determine whether the function f from $\{a, b, c, d\}$ to $\{1,2,3,4,5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$ and $f(d) = 3$ is one to one.

Sol^n - The function f is one to one since f takes on different values at the four elements of its domain.



Fig(c): A one to one function

Onto or surjective function

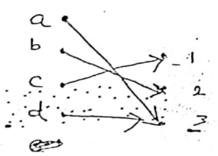
A function f from A to B is called onto or surjective, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

A function f is called a surjection ~~are called~~ if it is onto.

That is, every element of the co-domain is the image of some element of the domain. Function this property are called onto functions.

Example :- Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$ and $f(d) = 3$. Is f an onto function?

Sol^n. Since all three elements of the codomain are images of elements in the domain, we see that f is onto.



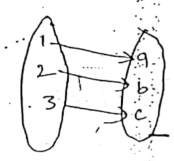


Fig (d) : An onto function , Fig (e) : Bijection function

Bijection function (one to one onto) function

The function f is a one-to-one correspondence or a bijection, if it is both one to one and onto.

Ex- Let f be function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f is a bijection?

Sol^n- The function f is one to one and onto. It is one to one since the function takes on distinct values. It is onto since all four elements of the codomain are images of elements in the domain. Hence, f is a bijection.

Function

In many instances we assign to each element of a set a particular element of a second set. For example Suppose that each element (Student) in a discrete mathmatics class is assigned a letter grade from the set {A, B, C, D, F}. And suppose that the grades are A for Ramesh, C for Shyam, B for Mohan, A for Arvind and F for Vikas.

This assignment of grades is illustrated in Fig 1.

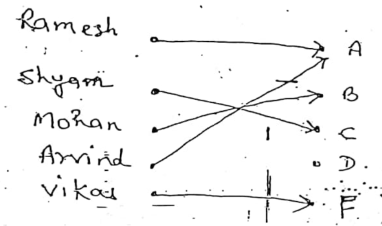


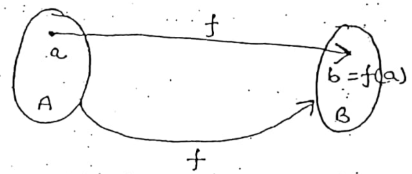
Fig: a: Assignment of Grades in a Discrete mathmatics class. (In terms of relation)

This assignment is an example of a function.

Definition:- Let A and B be sets. A function f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write

$$f : A \rightarrow B$$

Scanned by CamScanner



Fig(b): The function f maps A to B

Def. If f is a function from A to B, we say that A is the domain of f and B is the codomain of f. If $f(a)=b$, we say that b is the image of a and a is a preimage of b. The range of f is the set of all images of elements of A. Also if f is a function from A to B we say that f maps A to B.

Def: Sum and ~~Product~~ functions

Let f_1 and f_2 be functions from A to R. Then f_1+f_2 and $f_1 f_2$ are also functions from A to R defined by

$$(f_1+f_2)(x) = f_1(x)+f_2(x)$$
$$(f_1 f_2)(x) = f_1(x) f_2(x)$$

Ex- Let f_1 and f_2 be functions from R to R such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions f_1+f_2 and $f_1 f_2$?

Soln:-

$$(f_1+f_2)(x) = f_1(x)+f_2(x) = x^2+(x-x^2) = x.$$

and

$$(f_1 f_2)(x) = x^2(x-x^2) = x^3-x^4$$

Equal Functions :

Consider two functions f and h from a set X to a set Y. The functions f and h are called equal functions if and only if $f(a) = h(a)$ for every $a \in X$.

The function f and h are called unequal functions if there exist at least one element $a \in X$ such that $f(a) \neq h(a)$

Example :- Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$. Consider the function $f : X \to Y$, $g : X \to Y$ and $h : X \to Y$ such that.

$$f = \{(1,a), (2,a), (3,c)\}$$
$$g = \{(1,b), (2,a), (3,c)\}$$
$$h = \{(1,a), (2,a), (3,c)\}$$

Determine which functions are equal and which are unequal.

Solⁿ :- The functions f and h are equal functions. The functions f and g and g and h are unequal functions.

Identity Functions :

Consider any set A. Let the function $f : A \to A$. The function f is called the identity function if each element of set A has image on itself.

i.e. $f(a) = a \ \forall \ a \in A$

It is denoted by I

For example :

Consider, $A = \{1, 2, 3, 4, 5\}$ and $f : A \to A$ such that

$$f = \{(1,1), (2,2), (3,3), (4,4), (5,5)\}$$

The function f is an identity function as each element of A is mapped onto itself. The function f is one to one and onto.

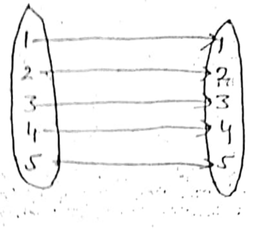


Fig: Identity Function

Inverse (or Invertible) Function:

A function $f: X \to Y$ is invertible if and only if it is a bijective function.

Let $f: X \to Y$ is a bijective function. Then, the function $f^{-1}: Y \to X$ which associates to each element $y \in Y$, the element $x (= f^{-1}(y)) \in X$, whose f-image was $y \in Y$, is called the inverse of the function $f: X \to Y$.

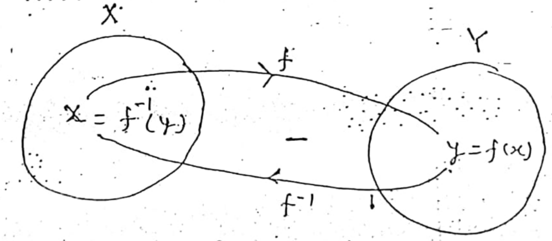


Fig: Inverse Function

The inverse function for f exists if f^{-1} is a function from Y to X.

For example:-

consider $X = \{1, 2, 3\}$, $Y = \{k, l, m\}$ and $f: X \to Y$ such that

$f = \{(1, k), (2, m), (3, l)\}$ as in fig.

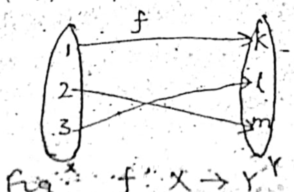
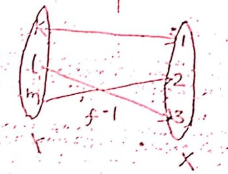


Fig: $f: X \to Y$

The inverse function of f is shown in fig:



Composition function:-

Consider function $f: A \to B$ and $g: B \to C$; that is where the codomain of f is the domain of g. Then we may define a new function from A to C, called the composition of f and g and written $g \circ f$, as follows:

$$(g \circ f)(a) = g(f(a))$$

That is, to find composition of f and g, first find the image of a under f and then find the image of f(a) under g.

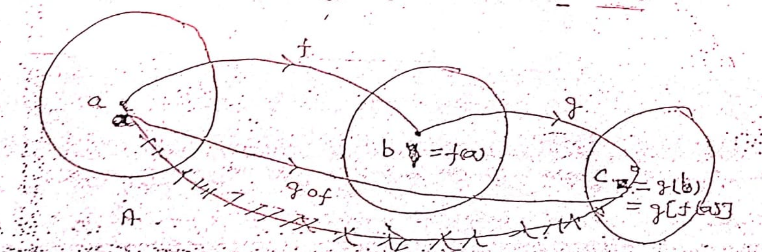


fig: Composition of function.

Example:- Let $X = \{1,2,3\}$, $Y = \{a,b\}$ and $Z = \{5,6,7\}$. Consider the functions $f = \{(1,a),(2,a)(3,b)\}$ and $g = \{(a,5),(b,7)\}$ as in figs 1 and figs 2. Find the composition of $g \circ f$.

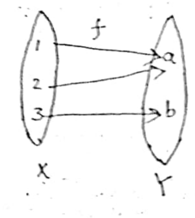


Fig (1)

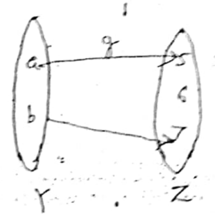


Fig (2)

Sol^n: The composition function $g \circ f$ is shown in fig (3):

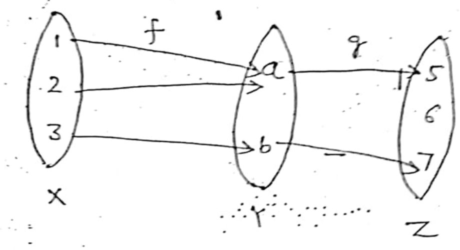


Fig (3)

$(g \circ f)(1) = g[f(1)] = g(a) = 5$

$(g \circ f)(2) = g[f(2)] = g(a) = 5$

$(g \circ f)(3) = g[f(3)] = g(b) = 7$

Q:- Let $f, g : R \to R$ defined by $f(x) = 2x+3$ and $g(x) = x^2$. Find $(g \circ f)(x)$ and $(f \circ g)(x)$.

Sol^n:

$f(x) = 2x+3$

Then $(g \circ f)(x) = g(f(x))$

$= g(2x+3)$

$= (2x+3)^2$

$g(x) = x^2$

$(f \circ g)(x) = f(g(x))$

$= f(x^2)$

$= 2(x)^2 + 3$

$= 2x^2 + 3$

Books
1) Discrete mathematics
Seymour Lipschutz 2nd Ed. (TMH)
2) Discrete mathematics
Dr. Anjana Gupta (Katson)
3) Rosen

Propositional Logic

Logic is the study of reasoning.

Propositions : A proposition (or statement) is a declarative sentence which is either universaly true or universaly false but not both.

Consider the sentence " Blood is red". When you read this sentence, we can immediately decide whether it is true or false. In mathematical logic we call such sentence as statements or propositions. It is denoted by lower case letters, p, q, etc. If the proposition is universally true we give it a truth value T (or 1) otherwise F (or 0).

Example 1 :

(I) Paris in France.
(ii) $1+1=2$
(iii) What is your name ?
(iv) Watch the movie.

In above example first sentence and second sentence is universally true therefore is a proposition. But the 3rd, 4th sentences are not even the declarative sentences and therefore none of them are propositions.

Compound Propositions: Statements can be combined by logical connectives to obtain compound statements.

Example: Tajmahal is white and is in Agra.

Logical Connectives or Sentence Connectives

(i) Conjunction: Symbol ($\wedge$)

Any two propositions can be (combined by) "and" to form a compound proposition called conjunction. Symbolically

$p \wedge q$

read "p and q" denotes the conjunction of p and q

NOTE: If p and q are true, then $p \wedge q$ is true, otherwise $p \wedge q$ is false.

Truth Table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example: Let p: Tajmahal is white.

q: Tajmahal is in Agra.

$p \wedge q$: Tajmahal is white and is in Agra.

The truth value of $p \wedge q$ is T.

(ii) Disjunction ($\vee$) ; ($p \vee q$)

read "p or q", denotes the disjunction of p and q.

NOTE: If p and q are false, then $p \vee q$ is false, otherwise $p \vee q$ is true.

Truth Table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example:

(a) Paris is in France or 2+2=4

(b) Paris is in France or 2+2=5

(c) Paris is in England or 2+2=4

(d) Paris is in England or 2+2=5

only the last statement (d) is false. Each of other statements is true since at least one of its substatement is true.

(iii) Negation ($\neg p$)

Symbolically : $\neg p$, $\sim p$

read "not p", denotes the negation of p.

Note: If p is true then $\neg p$ is false, and if p is false, then $\neg p$ is true.

Scanned by CamScanner

Truth Table

p	$\neg p$
T	F
F	T

Example: Let p: Ram is a good boy.
then $\neg p$: Ram is not a good boy.

(iv) Conditional statement (If then)
or Implication

Let p and q be two propositions, we define a proposition, "If p then q" denoted by —

$p \to q$ [p implies q]

Which is ~~true~~ false if p is true and q is false.

Truth Table

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Ex:— p: If he is rich

q: He is unhappy.

$p \to q$: If he is rich then he is unhappy.

Another Example

If I am hungry then I will ~~eat~~ eat

Scanned by CamScanner

(V.) Biconditional statement ($\leftrightarrow$)
or (If and only if)

$p \Leftrightarrow q$

read as " p if and only if q"
Which is true if both p and q are true or if both p and q are false.

Ex:
(a): You will pass the exam if and only if you will work hard

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

p: You will pass the exam.
q: You will work hard.

Symbolically we can write,

$p \Leftrightarrow q$

Converse, Contrapositive, and Inverse

Conditional statement $p \to q$
In particular, there are three related conditional statements occur so often that they have special names.

Converse of $p \to q$: $q \to p$

Inverse of $p \to q$: $\neg p \to \neg q$

Contrapositive of $p \to q$: $\neg q \to \neg p$

Table

p	q	$p \to q$	$\neg p$	$\neg q$	Inverse $\neg p \to \neg q$	Converse $q \to p$	$\neg q \to \neg p$
T	T	T	F	F	T	T	T
T	F	F	F	T	T	T	F
F	T	T	T	F	F	F	T
F	F	T	T	T	T	T	T

Contingency: is neither a tautology nor a contradiction, for instance $p \vee q$ is a contingency.

Satisfiable: If there is at least one truth assignment to its variables that makes it true.

For instance p = true and q = false makes $p \vee q$ true so $p \vee q$ is satisfiable.

p	q	$p \vee q$	$p \vee q \to p$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

Normal Form: By comparing truth tables, one determine whether two logical expressions P and Q are equivalent. But the process is very tedious when the no. of variables increases. A better method is to transform the expressions P and Q to some standard forms of expression P' and Q' such that a simple comparison of P' and Q' shows whether

$P \equiv Q$, The standards forms are called normal forms or canonical forms.

There are two types of normal forms, disjunctive normal forms and conjunctive normal forms. It will be convient to use the words product and sum in place of the logical connectives conjunction and disjunction.

(I) Disjunctive Normal Forms:

Elementary Product: In a logical expression a product of the variables and their negations is called an elementary product.

For Example $p \wedge \neg q$, $\neg p \wedge \neg q$, $\neg p \wedge q$ are elementary products.

Elementary sum: A sum of the variables and their negations is called an elementary sum.

For Examples $p \vee q$, $p \vee \neg q$, $\neg p \vee \neg q$ are elementary sums.

A logical expression is said to be in DNF if it is the sum of elementary products.

For Example: $p \vee (q \wedge r)$ and $p \vee (\neg q \wedge r)$ are in DNF, $p \wedge (q \vee r)$ is not in DNF.

Procedure to obtain a DNF of a given logical expression:

Steps

(1) Remove all $\rightarrow$ and $\Leftrightarrow$ by an equivalent expression containing the connectives $\wedge$, $\vee$, $\neg$ only

(ii) Eliminate $\neg$ before sums and products by using the double negation or by using De Morgan's laws.

(iii) Apply the distributive law until a sum of elementary product is obtained.

<u>Example (1)</u>: Obtain the DNF of the following.

a) $p \wedge (p \to q)$

b) ~~$p \wedge (\ldots)$~~ $p \vee (\neg p \to (q \vee (q \to \neg r)))$

<u>Soln</u> a) $p \wedge (p \to q) \equiv p \wedge (\neg p \vee q)$ $\boxed{\text{Using } p \to q \equiv \neg p \vee q}$

$\equiv (p \wedge \neg p) \vee (p \wedge q)$

(Using Distributive Law)

Which is the required DNF.

b) ~~...~~

$p \vee (\neg p \Rightarrow (q \vee (q \Rightarrow \neg r))) \equiv p \vee (\neg p \Rightarrow (q \vee (\neg q \vee \neg r)))$

$\equiv p \vee (p \vee (q \vee (\neg q \vee \neg r)))$

$\equiv p \vee p \vee q \vee \neg q \vee \neg r$

$\equiv p \vee q \vee \neg q \vee \neg r$

Which is the required DNF

Logic in Proof

Arguments: Given a certain set of proposition (i.e statements), we are required to derive other propositions by logical reasoning. The given set of propositions is called premises (or Hypothesis) and the proposition derived from this set is called Conclusion.

Representation of an Argument:

An argument $p_1, p_2, \ldots, p_n \vdash q$ is written as

$$\left.\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \end{array}\right\} \text{premises}$$
$$\overline{\quad q \text{ (conclusion)} \quad}$$

~~Valid~~ Fallacy Argument: An argument which is not valid is said to be a fallacy or invalid Argument

Valid Argument: The argument $p_1, p_2 \cdots p_n \vdash q$ is said to be valid if and only if the statement

$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$ is tautology.

Rules of Inference

	Rule of Inference	Tautology	Name of Rule
1.	$\dfrac{p}{\therefore\ p \lor q}$	$p \rightarrow (p \lor q)$	Addition
2.	$\dfrac{p \land q}{\therefore\ p}$	$(p \land q) \rightarrow p$	Simplification
3.	$\dfrac{p \quad p \rightarrow q}{\therefore\ q}$	$[p \land (p \rightarrow q)] \rightarrow q$	Modus ponens (Rule of detachment)
4.	$\dfrac{\neg q \quad p \rightarrow q}{\therefore\ \neg p}$	$[\neg q \land (p \rightarrow q)] \rightarrow \neg p$	Modus tollens
5.	$\dfrac{p \rightarrow q \quad q \rightarrow r}{\therefore\ p \rightarrow r}$	$[(p \rightarrow q) \land (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Hypothetical syllogism (Chain Rule)
6.	$\dfrac{p \lor q \quad \neg p}{\therefore\ q}$	$[(p \lor q) \land (\neg p)] \rightarrow q$	Disjunctive syllogism

Q1: Consider the following argument:

"It is below freezing now. Therefore, it is either below freezing or raining now."

Test the validity of the above statement.

Solⁿ: 1st method:

Let p : It is below freezing now.

q : It is raining now.

therefore $p \rightarrow p \vee q$

Here "Addition is the rule of inference. Therefore above argument is valid. Ans

2ⁿ method: We can also prove by truth table

p	q	$p \vee q$	$p \rightarrow p \vee q$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

In above truth table last column contains all True value. Therefor above argument is valid

Q2: Show that the argument $p, p \rightarrow q \vdash q$ is valid

Solⁿ: 1st method: Without using truth table.

consider the following sequence of argument

p (premise)

$p \rightarrow q$ (premise)

$\therefore q$ (conclusion by rule of detachment)

Hence, by modus pones (Rule of detachment) the argument is valid.

2nd method: By using truth table

we shall construct truth table for the

$$[p \wedge (p \to q)] \to q$$

p	q	$p \to q$	$[p \wedge (p \to q)]$	$(p \wedge (p \to q)) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since last column contains only True values hence the given argument is valid.

Q3. Represent the argument

If it rains today, then we will not have a party today.

If we do not have party today, then we will have a party tommorrow.

Therefore, if it rains today, then we will have a party tomorrow.

Symbolically and determine whether argument is valid.

Sol^n: If we let

p: It's raining today.

q: we will not have a party today.

r: We will have a party tomorrow.

The argument is of the form

$$\begin{array}{c} p \to q \\ q \to r \\ \hline \therefore\ p \to r \end{array}$$

Hence, the argument is a hypothetical syllogism and this argument is valid.

Q4. Test the validity of the argument :
If 8 is even then 2 does not divide 9. Either 7 is not prime or 2 divides 9. But 7 is prime, therefore, 8 is odd.

Sol^n. Let p : 8 is even

q : 2 divides 9

r : 7 is prime

The given argument, in symbolic form, may be written as

$p \to \neg q$ (a premise)

$\neg r \vee q$ (")

r (")

$\therefore$ $\neg p$ (conclusion)

The given argument will be valid if the statement $[(p \to \neg q) \wedge (\neg r \vee q) \wedge r] \to (\neg p)$ is a tautology.

We construct the truth table _

p	q	r	$\neg p$	$\neg q$	$\neg r$	$p \to \neg q$	$\neg r \vee q$	$(p \to \neg q) \wedge (\neg r \vee q) \wedge r$ $= f$ (say)	$f \to (\neg p)$
T	T	T	F	F	F	F	T	F	T
T	T	F	F	F	T	F	T	F	T
T	F	T	F	T	F	T	F	F	T
T	F	F	F	T	T	T	T	F	T
F	T	T	T	F	F	T	T	T	T
F	T	F	T	F	T	T	T	F	T
F	F	T	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T	F	T

Since last column only contains True values hence, the argument is valid.

Minterms:

Let p and q be two statement variables. All possible formulas which consists of conjunction of p (or its negations) and q (or its negation). None of the formula should contain both variable and its negation. These formulas are called minterms. No two minterms are equivalent.

For n variables $p_1, p_2, \ldots, p_n$ there are 2^n such minterms. In particular for 2 variables say p and q there are 2^2 such formulas

$p \wedge q$, $p \wedge \neg q$, $\neg p \wedge q$ and $\neg p \wedge \neg q$.

$p \wedge \neg p$ is not a minterm.

One can find the minterms by using truth table. Each row in the truth table will give one minterm by using conjunction between the variables.

For Example: consider the table having two variables

x	y	minterms
T	T	$x \wedge y$
T	F	$x \wedge \neg y$
F	T	$\neg x \wedge y$
F	F	$\neg x \wedge \neg y$

In above table there are 2^n rows for n no. of variables; these rows gives each minterms.

Principal Disjunctive Normal Form (PDNF)

For a given formula, an equivalent formula consisting of disjunction of minterms ~~of~~ only is known as principal DNF or "Sum of product canonical form".

Example: Write PDNF of the following:

a) $(\neg p \vee \neg q) \rightarrow (p \leftrightarrow \neg q)$

$\equiv \sim(p \wedge q) \rightarrow (p \wedge \neg q) \vee (\neg p \wedge q)$

we know that

$\sim(p \wedge q) \equiv (\neg p \vee \neg q)$

and

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

and

$p \rightarrow q \equiv \neg p \vee q$

$\equiv \sim\sim(p \wedge q) \vee [(p \wedge \neg q) \vee (\neg p \wedge q)]$

$= (p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge q)$

Disjunction of minterms therefore it is required PDNF.

b) $q \wedge (p \vee \neg q)$

$\equiv (q \wedge p) \vee (q \wedge \neg q)$

$\equiv (q \wedge p) \vee F$

$\equiv (q \wedge p)$ is the required form.

Conjunctive Normal Form (CNF)

A formula which is equivalent to a given formula which consists of a product of elementary sum is called conjunctive normal form of the given formula. For a given formula conjunctive normal form is not unique.

Example: Obtain the conjunctive normal form of

(a) $p \wedge (p \to q)$

(b) $\neg(p \vee q) \leftrightarrow (p \wedge q)$

Sol (a):

$p \wedge (p \to q)$

$p \wedge (\neg p \vee q)$

Which is required conjunction normal form.

(b) $\sim(p \vee q) \leftrightarrow (p \wedge q)$

$\equiv [\sim(p \vee q) \to (p \wedge q)] \wedge [(p \wedge q) \to \sim(p \vee q)]$

$\equiv [(p \vee q) \vee (p \wedge q)] \wedge [\sim(p \wedge q) \vee \sim(p \vee q)]$

[By $p \to q \equiv \neg p \vee q$]

$\equiv [(p \vee q) \vee (p \wedge q)] \wedge [(\neg p \vee \neg q) \vee (\neg p \wedge \neg q)]$

(By Demorgan's Laws)

$\equiv [(p \vee q \vee p) \wedge (p \vee q \vee q) \wedge (\neg p \vee \neg q \vee \neg p) \wedge (\neg p \vee \neg q \vee \neg q)$

Which is required CNF

Scanned by CamScanner

Tautology

Some propositions contain only T in the last column of their truth tables. Such propositions are called tautology.

Ex: (a) $p \vee \neg p$

(b) $p \leftrightarrow p$

(c) $p \rightarrow p \vee q$

Consider truth table of $p \vee \neg p$

Truth Table

p	$\neg p$	$p \vee \neg p$
F	T	T
T	F	T

← Tautology

In the above truth table last column contains all True value. Therefore $p \vee \neg p$ is tautology.

Q 1: Verify that the proposition $p \vee \neg(p \wedge q)$ is tautology.

Solⁿ:

Truth Table

p	q	$(p \wedge q)$	$\neg(p \wedge q)$	$p \vee \neg(p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

← Tautology

Since the truth value of $p \vee \neg(p \wedge q)$ is T for all values of p and q, the proposition is tautology.

Contradiction

Some propositions contain ~~ab~~ only F in last column of their truth ~~values~~ tables. Such proposition are called contradiction.

Example

(a) $p \wedge \neg p$

(b) $(p \vee q) \wedge (\neg p) \wedge (\neg q)$

Truth Table for $p \wedge \neg p$

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

← Contradiction

Q1: Prove that $p \leftrightarrow (\neg p)$ is a contradiction.

Q2: Prove that $(p \vee q) \wedge (\neg p) \wedge (\neg q)$ is a contradiction

Logical Equivalence

Two statements (or propositions) are called logically equivalent if the truth values of both the statements are always identical.

Example

(a) Show that $p \to q \equiv (\neg q) \to (\neg p)$ are logically equivalent.

p	q	$\neg p$	$\neg q$	$\neg q \to \neg p$	$p \to q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Both column are identical. Therefore it is logically equivalent.

(b) Show that $\neg(p \to q) \equiv (p \wedge \neg q)$

Laws of the algebra of propositions

Idempotent Laws

(1a) $p \vee p \equiv p$ (1b) $p \wedge p \equiv p$

Associative laws

(2a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$ (2b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

Commutative Laws

(3a) $p \vee q \equiv q \vee p$

(3b) $p \wedge q \equiv q \wedge p$

Distributive Laws

(4a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(4b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Identity Laws

(5a) $\cancel{p \vee F}$ $p \wedge T \equiv p$ (5b) $p \wedge F \equiv F$

(6a) $p \vee T \equiv T$ (6b) $p \vee F \equiv p$

Complement Laws

(7a) $p \vee \neg p \equiv T$ (7b) $p \wedge \neg p \equiv F$

(8a) $\neg T \equiv F$, (8b) $\neg F \equiv T$

Involution Law

(9) $\neg\neg p \equiv p$

De Morgan's Laws

(10a) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

(10b) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

Negation

Example ①: Suppose p = Dhoni is a good player of Cricket.

then $\neg p$ = Dhoni is not a good player of Cricket.

Example ②: Find the negation of the propositions.

(a) It is cold

(b) Today is Sunday.

(c) 3 is an odd integer and 8 is an even integer.

Sol^n: (a) It is not cold

(b) Today is not Sunday.

(c) 3 is not an odd integer and 8 is not an even integer.

Example ③: Let p = It is cold.

q = It is raining

(a) $\neg p$ (b) $p \wedge q$ (c) $p \vee q$ (d) $q \leftrightarrow p$

(e) $q \vee \sim p$ (f) $\sim p \vee \sim q$ (g) $p \leftrightarrow \sim q$

(h) $\sim\sim q$ (i) $(p \wedge \neg q) \rightarrow p$ (j) $(p \wedge \sim q) \rightarrow q$

Write above propositions in English sentence form

Sol^n

(a) It is not cold.

(b) It is cold and raining.

(c) It is cold or It is raining.

(d) It is raining if and only if it is cold.

(e) It is raining or it is not cold.

(f) It is not cold or It is not raining.

(g) It is not cold if and only if it is not raining.

Scanned by CamScanner

(h) It is not true that it is not raining.

(i) If it is cold and not raining, then it is cold.

(j) If it is raining and not cold, then it is raining.

Kinds of Conditional

Conditional	Name of Kind
$p \to q$	Direct Implication
$q \to p$	converse "
$\neg p \to \neg q$	Inverse "
$\neg q \to \neg p$	Contrapositive "

Example (4): The contrapositive of statement is given as

"If $x < 2$ then $x+4 < 6$

Write the Converse and Inverse.

Sol^n

$\neg q = x < 2$ –

$\neg p = x+4 < 6$

Converse: If $x > 2$ then $x+4 > 6$

Inverse: If $x+4 > 6$ then $x > 2$

Q1 Determ... are t...

(a) $(p \wedge ($... $(p \to q$...

Q1 Determine whether the following statements are tautology or not.

(a) $(p \wedge (p \wedge q)) \rightarrow q$ (Tautology)

(b) $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$ (Tautology)

(c) $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$ (Tautology)

(d) $(p \wedge \neg q) \vee (q \wedge \neg p)$ (Not tautology)

Predicate Calculus (Propositional Function)

Consider two propositions "Ram is a student" and "Shyam is a student". As propositions, there is no relation between them, but we know they have something in common. Both Ram and Shyam share the property of being a student.

We can replace the two propositions by a single statement "x is a student". By replacing x by Ram or Shyam (or any other name), we get many propositions.

The common feature expressed by "is a student" is called a predicate.

Sentences involving predicates describing the property of objects are denoted by $P(x)$, where P denotes the predicate and x is a variable denoting any object.

For example $P(x)$ can denote "x is a student". In this sentence, x is a variable and P denotes the predicate "is a student".

The sentence "x is the father of y" also involves a predicate "is the father of y". Here predicate describes the relation between two persons. We can write this sentence as $F(x, y)$.

Similarly, $2x + 3y = 4z$ can be described by $S(x, y, z)$.

$S(2, 0, 1)$ is the proposition

$2 \cdot 2 + 3 \cdot 0 = 4 \cdot 1$ (whose truth value is T)

$S(1,1,1)$ is the proposition

$2 \cdot 1 + 3 \cdot 1 = 4 \cdot 1$ (whose truth value is F)

Universal and Existential Quantifier

Quantifiers are words that ~~referto~~ to quantities such as "some" or "all"

The phrase "for all" (denoted by $\forall$) is called the universal quantifier.

For Example, consider the sentence "All human beings are mortal."

Let $P(x)$ denotes "x is mortal."

Then the above sentence can be written as

$$(\forall x \in S)\, P(x) \quad \text{or} \quad \forall x\, P(x) \quad \text{—} ①$$

where S denotes the set of all human beings.

$\forall x$ represent each of the following phrases, since they have essentially the same

for all x

for every x

for each x.

The sentence ① is called universal statement.

Truth Values for $\forall x\, P(x)$:

$\forall x\, P(x)$ is true if $P(x)$ is true for every x in U.

$\forall x\, P(x)$ is false, if and only if, $P(x)$ is false for at least one x in U.

For example: consider "~~There exist x such that~~ For all x, $x^2 = (-x)^2$"

We can write above sentence

$\forall x\, P(x)$

where $P(x)$ is $x^2 = (-x)^2$

The phrase "there exists" (denoted by $\exists$) is called the existential quantifier.

<u>For Example:</u>

The statement "for some x, $x^2-6x+9=0$" can be written as

$$\exists x\ (x^2-6x+9=0) \text{ or } \exists x\, P(x) \text{ or } \quad (2')$$

This is a true statement since $(\exists x \in N)\, P(x)$

$$\{x : x^2-6x+9=0\} = \{2,3\} \neq \phi$$

$\exists x$ represents each of the following phrases.

There exists an x

There is an x

For some x

There is at least one x

The statement (2') is called an existential statement.

<u>Truth Values for $\exists x\, P(x)$</u>

$\exists x\, P(x)$ is true if $P(x)$ is true for at least one x in U.

$\exists x\, P(x)$ is false, for every x in U.

<u>Example:</u> Let Z, the set of integer, be the universe of discourse (simply universe) and consider the statements

$$(\forall x \in Z),\ x^2 = x$$

$$(\exists x \in Z),\ x^2 = x$$

Find the truth values of each of the statements.

<u>Soln:</u>

Let $P(x)$ be proposition $x^2 = x$.

Then $\forall x\, P(x)$ is false because $P(3)$ i.e. $3^2=3$ is false and $\exists P(x)$ is true, because at least one proposition $P(x)$ is true: infact exactly two of them are true, namely $P(0)$ and $P(1)$.

Now consider the statements

There is a student in this class who has taken a course in DMS. The statement can be written as

$$\exists x\, Q(x)$$

Where $Q(x)$ is the statement "x has taken a course in DMS."

It negation read as

"It is not the case that there ... ."

This is equivalent to

"All students in this class has not taken DMS.

i.e. $\forall x\, \neg Q(x)$

This example illustrates the equivalence

$$\sim \exists Q(x) \equiv \forall x \sim Q(x)$$

Thus, the negation of an existential statement is logically equivalent to a universal statement.

We list these facts as follows.

Statement		Negation	
All true	$\forall x F(x)$	$\exists x [\sim F(x)]$	At least one false
At least one false	$\exists x [\sim F(x)]$	$\forall x F(x)$	All true
All false	$\forall x [\sim F(x)]$	$\exists x F(x)$	At least one true
At least one true	$\exists x F(x)$	$\forall x [\sim F(x)]$	All false

Ex:- Negate the statement

For all real no.s x, if $x > 3$ then $x^2 > 9$

Soln- Let $P(x)$ & $Q(x)$ denote

"$x > 3$" and "$x^2 > 9$".

Then the given statement can be written as $\forall x\, (P(x) \rightarrow Q(x))$

which is equivalent to

$\forall x\ (\neg P(x) \lor Q(x))$

The negation of the statement is

$\exists x(P(x) \land \neg Q(x))$

There exists a real no. x such that $x > 3$ & $x^2 \leq 9$

Q:- Translate each of the statement into symbols, using quantifiers, variables and predicate symbols.

Let $P(x)$: x can speak Tamil

and $Q(x)$: x knows the language C++

a) There is a student who can speak Tamil & who knows C++

b) " " " " " " " but doesn't knows C++

c) Every student either can speak Tamil or knows C++.

d) No student can speak Tamil or knows C++.

Ans.
a) $\exists x\ (P(x) \land Q(x))$
b) $\exists x(P(x) \land \neg Q(x))$
c) $\forall x\ (P(x) \lor Q(x))$
d) $\forall x\ \neg(P(x) \lor Q(x))$

	p	Q	$p \to q$	$\neg p \lor q$
F	T	T	T	T
F	T	F	F	T
T	F	T	T	T
T	F	F	T	T

$\neg p$	p	Q	$p \to q$	$\neg p \lor q$
F	T	T	T	T
F	T	F	F	T
T	F	T	T	T
T	F	F	T	T

One Resolution
One Literal
ementary
and a

One Resolution Principle

One Literal Rule: Examine whether there is a complementary pair of a literal in C_1 (for example, P) and a literal ($\neg P$) in C_2 then to delete the pair from C_1 and C_2 to obtain clause C_3 which is Q.

Consider the following clauses

C_1 : P

C_2 : $\neg P \lor Q$

Using the one literal Rule, from C_1 and C_2 we can obtain a clause C_3 : Q.

Extending the above rule and applying it to any pair of clauses we have a resolution rule:

For any two clauses C_1 and C_2, if there is a literal L_1 in C_1 that is complementary to a literal L_2 in C_2, then delete L_1 and L_2 from C_1 and C_2 respectively and construct the disjunction of the remaining clauses. The constructed clause is a resolvent of C_1 and C_2.

Example 1: Consider the following clauses.

C_1 : $P \lor R$

C_2 : $\neg P \lor Q$

Clause C_1 has the literal P which is complementary to $\neg P$ in C_2. Therefore, by deleting P and $\neg P$ from C_1 and C_2 respectively and constructing the disjunction of the remaining clauses R and Q we obtain a resolvent

$R \lor Q$.

Example 2: Consider the clauses

C_1: $\neg P \vee Q$

C_2: $\neg P \vee R$

Since there is no literal in C_1 that is complementary to any literal in C_2, there is no resolvent of C_1 and C_2.

Example

Example 3: Consider the clauses

C_1: $\neg P \vee Q \vee R$

C_2: $\neg Q \vee S$

Then resolvent of C_1 and C_2 is $\neg P \vee R \vee S$

Definition: Given a set S of clause, a resolution of C from S is a finite sequence $C_1, C_2, \ldots C_k$ of clauses such that each C_i either is a clause of S or a resolvent of clauses preceding C_i and $C_k = C$. A deduction of $\square$ from S is called proof (refutation) of S.

Example 4: Consider the set

1. $\neg P \vee Q$
2. $\neg Q$
3. P

From (1) and (2), we obtain a resolvent

4. $\neg P$

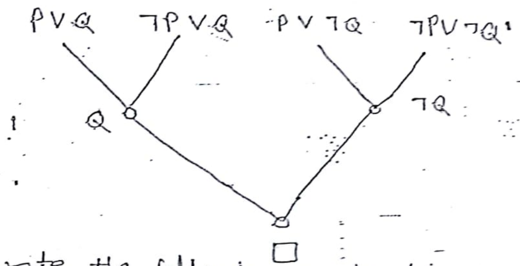
from (4) and (3), we obtain resolvent

5. $\square$ (Empty clause)

Hence, the proof.

Example 5 : For the set

1. $P \vee Q$
2. $\neg P \vee Q$
3. $P \vee \neg Q$
4. $\neg P \vee \neg Q$



We generate the following resolvent

5. Q from (1) and (2)
6. $\neg Q$ from (3) and (4)
7. $\Box$ from (5) and (6).

Example

$S = \{\neg P \vee \neg Q \vee R,\ P \vee R,\ Q \vee R,\ \neg R\}$. Show that set S is unsatisfiable.

Solⁿ

1. $\neg P \vee \neg Q \vee R$
2. $P \vee R$
3. $Q \vee R$
4. $\neg R$
5. $\neg Q \vee R$ [From (1) and (2)]
6. $\neg P \vee R$ [From (1) and (3)]
7. $\neg P \vee \neg Q$ [From (1) and (4)]
8. P [From (2) and (4)]
9. Q [From (3) and (4)]
10. $\neg Q \vee R$ [From (1) and (8)]
11. $\neg P \vee R$ [From (1) and (9)]
12. R [From (2) and (6)]
13. $\neg Q$ [From (4) and (5)]
14. $\neg P$ [From (4) and (6)]
15. $\square$ [From (4) and (12)]

Example

F_1 : $P \to (\neg Q \vee (R \vee S))$

F_2 : P

F_3 : $\neg S$

F_4 : $\neg Q$

Show that F_4 is a logical consequence of F_1, F_2 and F_3.

Solⁿ:

1. $\neg P \vee \neg Q \vee R$ [From F_1]
2. $\neg P \vee \neg Q \vee S$ [From F_1]
3. P [From F_2]
4. $\neg S$ [From F_3]
5. Q <u>Negation</u> of conclusion

Using resolution,

6. $\neg Q \vee S$ A resolvent of (3) and (2)
7. S A resolvent of (6) and (5)
8. $\square$ A " " (7) and (4)

Ex

We have four statements

1. $P \to S$ 2. $S \to U$ 3. P 4*. U

Show that (4*) follows from (1), (2), (3)

Solⁿ: We have

1. $\neg P \vee S$
2. $\neg S \vee U$
3. P
4. U

We prove that U is a logical consequence of (1), (2), (3) by refutation (proof).

1. $\neg P \vee S$
2. $\neg S \vee U$
3. P
4. $\neg U$ Negation of conclusion
5. S [From (1) and (3)]
6. U [From (2) and (5)]
7. $\square$ [From (6) and (4)]

Different Methods of Proof

(1) Direct Proof

Ex(1) Give the direct proof of the statement.

"The product of two odd integers is odd"

Sol^n. Let x and y be two odd integers then

$x = 2n+1$ for $n \in Z$

$y = 2m+1$ for $m \in Z$

$$xy = (2n+1)(2m+1)$$

$$= 4mn + 2n + 2m + 1$$

$$= 2(mn+n+m) + 1$$

$$= 2p+1 \quad (p \in Z)$$

is an odd integer.

Therefore the result follows.

Ex(2): Give a direct proof of the statement.

"The square of an even integer is an even integer".

Sol^n: Let x be an even integer then for some $n \in Z$

$$x = 2n$$

Squaring on both sides

$$x^2 = (2n)^2 = 4n^2 = 2(2n)^2$$

$$= 2p \quad \text{where } p = 2n^2 \in Z$$

$$= \text{Even}$$

the result follows.

Indirect Proof

Proofs that are not direct are called indirect. The two main types of indirect proof both use the negation of the conclusion, so they are often suitable when that negation is easy to state. The first type of proof is contrapositive proof. The second type of indirect proof is known as proof by contradiction.

(a) Proof by contrapositive: The method of contrapositive says $p \to q$ is logically equivalent to its contrapositive ($\neg q \to \neg p$)

Ex (1): Prove that statement "If $3n+2$ is odd, then n is odd".

Solⁿ. Let p : $3n+2$ is odd

q : n is odd

We need to prove $p \Rightarrow q$.

∴ Consider the contrapositive

$\neg q \to \neg p$

$\neg q$: n is even

$\Rightarrow n = 2K \quad (K \in \mathbb{Z})$

Consider

$$3n+2 = 3(2K)+2$$
$$= 6K+2$$
$$= 2(3K+1)$$
$$= \text{Even}$$
$$\Rightarrow 3n+2 \text{ is even}$$
$$\Rightarrow \neg p \text{ holds}$$

Therefore by method of contrapositive the result holds.

Proof by Contradiction : In this type of proof, we assume the opposite of what we trying to prove and a logical contradiction.

Ex: Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Sol^n: Suppose that $\sqrt{2}$ is rational, there exist integers a and b such that $\sqrt{2} = a/b$ where a and b have no common factors.

squaring both sides we get

$$2 = \frac{a^2}{b^2}$$

$$\Rightarrow 2b^2 = a^2$$

Hence a^2 is a multiple of 2, and therefore even.

$\Rightarrow$ a is even.

Hence $a = 2k$ for some integer k.

then $2b^2 = (2k)^2$

and therefore $b^2 = 2k^2$

Thus b^2 is even

$\Rightarrow$ b is even

But now a and b have a common factor of 2, which is contradiction to the statement that a and b have no common factors,

Hence our initial assumption that $\sqrt{2}$ is rational is false. Thus $\sqrt{2}$ is irrational.

Mathematical Induction:

Mathematical induction is a technique by which one can prove mathematical statements involving positive integers.

There are three steps to proof using the principle of mathematical induction.

Step 1. (Inductive base): verify that $P(1)$ is true.

Step 2. (Inductive hypothesis): Assume that $P(k)$ is true for an arbitrary value of k.

Step 3. (Inductive step): verify that $P(k+1)$ is true on the basis of the inductive hypothesis

Q1.- Show that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}, \; n \geq 1$$ by mathematical induction

Sol^n.- Let $P(n)$ be the given statement, then

1. Inductive base: for $n = 1$, we have

$$1^2 = \frac{1(1+1)(2+1)}{6} = \frac{1(2)(3)}{6} = 1$$

i.e. L.H.S = R.H.S

so, $P(1)$ is true.

2. Inductive Hypothesis:

Assume $P(k)$ is true i.e.

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

3. Inductive step: Now we want to show that $P(k+1)$ is true. i.e.

$$1^2 + 2^2 + 3^2 + \cdots (k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\text{L.H.S.} = 1^2 + 2^2 + 3^2 + \cdots (k+1)^2$$

$$= \underbrace{1^2 + 2^2 + 3^2 + \cdots + k^2}_{} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= (K+1)\left[\frac{K(2K+1)}{6} + (K+1)\right]$$

$$= (K+1)\left[\frac{2K^2 + K + 6K + 6}{6}\right]$$

$$= \frac{(K+1)\,(2K^2+7K+6)}{6}$$

$$= \frac{(K+1)\,(K+2)\,(2K+3)}{6}$$

$= R.H.S$

That is $P(K+1)$ is true whenever $P(K)$ is true.

Mathematical induction reduce the proof to a finite no. of steps and guarantee that there is no positive n for which the statement fails to be determined

Q: Show that if n is a positive integer then

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Solⁿ: Let $P(n)$ be the proposition that the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

We must do two things to prove that $P(n)$ is true for $n = 1, 2, 3, \ldots$ Namely we must show that $P(1)$ is true and that the conditional statement $P(k)$ implies $P(k+1)$ is true for $k = 1, 2, 3, \ldots$

Basis Step:

$P(1)$ is true, because $1 = \frac{1(1+1)}{2} = 1$

L.H.S = R.H.S

Hypothesis
Inductive ~~step~~: We assume that $P(k)$ holds for an arbitrary positive integer k. That is we assume that

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

Inductive step: Under this assumption it must be shown that $P(k+1)$ is true, namely that

$$1+2+\cdots+k+(k+1) = \frac{(k+1)[(k+1)+1]}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

is also true, when we add $k+1$ to both sides of the eqn in $P(k)$, we obtain

$$1+2+\ldots k+(k+1) = \frac{k(k+1)}{2}+(k+1)$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

this last equation shows that $P(k+1)$ is true under the assumption that $P(k)$ is true. This completes the inductive step.

Sol^n: Assuming that $\sqrt{3}$ is rational.

This means that there exists +ive integers x and y

Such that $\sqrt{3} = \frac{x}{y}$.

$x = \sqrt{3}y$

$\Rightarrow x^2 = 3y^2$

$\Rightarrow 3 | x^2$ (3 divides x^2)

$\Rightarrow 3 | x$ (" " x) —— (1)

$\frac{x^2}{3} = y^2$

Therefore, by def^n, $x = 3p$ for some $p \in \mathbb{Z}$

$x^2 = 9p^2$

But $x^2 = 3y^2$ also

$9p^2 = 3y^2$

$\Rightarrow 3p^2 = y^2$

$\Rightarrow 3 | y^2$ (3 divides y^2)

$\Rightarrow 3 | y$ —— (2)

(1) and (2) implies that 3 divides both x and y which contradicts our assumption.

Therefore we conclude that our assumption is wrong, that $\sqrt{3}$ is rational.

So $\sqrt{3}$ is irrational.

Counting Principle:

Sum Rule: Suppose some event E can occur in m ways and second event F can occur in n ways, and suppose both events cannot occur simultaneously. Then E or F can occur in $m+n$ ways.

More generally, suppose an event E_1 can occur in n_1 ways, a second event E_2 can occur in n_2 ways, a third event E_3 can occur in n_3 ways ... and suppose no two of the events can occur at the same time. Then one of the events can occur in $n_1+n_2+n_3+\ldots$ ways.

Example: Suppose there are 8 male professors and 5 female professors teaching a calculus class. A student can choose a calculus professor in $8+5 = 13$ ways.

Product Rule:

If an event can occur in m ways and second event in n ways, and if the no. of ways the second event occurs, does not depend upon the occurrence of the first event, then the two events can occur simultaneously in mn ways.

In general if E_i $(i = 1, 2, \ldots k)$ are k events and if E_1 can occur in n_1 ways, E_2 can occur n_2 ways (no matter how E_1 occurs), E_3 can occur in n_3 ways (no matter how E_1 and E_2 occurs), ... E_k occurs in n_k ways (no matter how $k-1$ events occurs), then the k events can occur simultaneously in $n_1 \times n_2 \times n_3 \times \ldots \times n_k$ ways

Example: Three person enter into car, where are 5 seats. In how many ways can they take up their seats.

Sol^n: The 1st ~~first~~ person has a choice of seats and can sit in any one of those 5 seats. So there are 5 ways of occupying the first seat. ∴ The 2nd person has a choice of 4 seats. Similarly the IIIrd person has a choice of 3 seats.

Hence the required no. of ways in which all the three persons can seat is $5 \times 4 \times 3 = 60$

Permutation: (Arrangements of objects)

A permutation is an arrangement of a no. of objects in some definite order taken some or all at a time.

The total no. of permutation of n distinct objects taken r at a time is denoted by ^nP_r or $P(n,r)$.

Theorem: The no. of r-permutations of a set of n (distinct) elements is given by

$$P(n,r) = \frac{\lfloor n}{\lfloor n-r}$$

Question 1:- Find the no. of 2-permutations of the elements of the set $\{a,b,c\}$.

Sol^n.-

$$P(3,2) = 6$$

The various 2-permutations in question 1 are ab, ac, ba, bc, ca and cb. They can be obtained using tree diagram in figure 1.

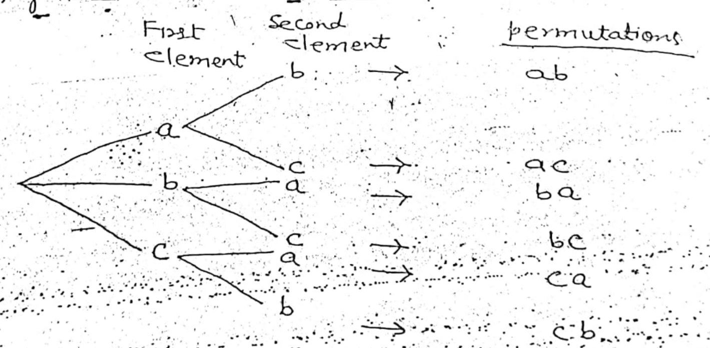


Fig. 1

Theorem 2: Prove that the no. of permutations of n things taken all at a time is $\underline{|n}$.

Proof: We know that

$$^nP_n = \frac{\underline{|n}}{\underline{|n-n}}$$

$$= \frac{\underline{|n}}{\underline{|0}}$$

$$= \frac{\underline{|n}}{1}$$

$$= \underline{|n}$$

Hence proved

Question 2: How many 6 digit no.s can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, if no digit is repeated

Sol^n: There are 8 numbers.

$\therefore$ Total no. of 6 digit numbers $= {}^8P_6$

$$= \frac{\underline{|8}}{\underline{|8-6}} = \frac{\underline{|8}}{\underline{|2}} = \frac{8\times7\times6\times5\times4\times3\times\cancel{\underline{|2}}}{\cancel{\underline{|2}}}$$

$$= 22560$$

Circular Permuatations:

A circular permutation is an arrangements of objects around a ~~circle~~ circle. The no. of permutation of n objects in a circle can be found by keeping one element fixed and arranging remaining $(n-1)$ elements which can be done in $(n-1)!$ ways.

Theorem: The no. of cyclic permutations of n (distinct) items is $(n-1)!$.

Proof: To avoid duplicates, let us assign a fixed position to the first item a_1 around the circle. Now $n-1$ positions are left. So the second item can a_2 can be placed in any one of the $n-1$ positions. Now $n-2$ positions are left. Therefore, the third item a_3 can be placed in any of the $n-2$ positions. Continue like this until all items have been placed. Thus by the multiplication principle, the no. of circular permutations is

$1 \cdot (n-1)(n-2) \cdots 2 \cdot 1 = (n-1)!$

Example: Find the no. of different ways five ~~Zen~~ Zinnias can be planted in a circle.

Solⁿ: No. of circular permutations of five items $= (5-1)! = 24$

Permutations with Repetitions:

Consider the word REFERENCE. If we swap the second E with the fourth E in the word, we do not get a new word. How we can compute the no. of permutations in such cases?

Example: Find the no. of different arrangements of the letters of the word REFERENCE.

Solution: The word REFERENCE contains nine letters. If they were all distinct, the answer would be $9! = 362,880$. But, since duplicate letter exist, the answer is indeed much less.

Let N denote the no of different words. We shall find the value of N in an indirect way.

The word REFERENCE contains two R's and E's; the remaining letters are distinct. Think of the two R's as two distinct letters, R_1 and R_2 and the four E's as four distinct letters, E_1 through E_4. The letters R_1 and R_2 can be arranged in $\underline{|2}$ ways and the four E's in $\underline{|4}$ ways

Therefore if all the letters were distinct, th would be a total of $\lfloor 2 \lfloor 4 \, N$ different words.

thus $\lfloor 2 \lfloor 4 \, N = \lfloor 9$; so

$$N = \frac{\lfloor 9}{\lfloor 2 \lfloor 4}$$

$$= 7560$$

Theorem: The no. of permutations of n items of which n_1 items are of one type, n_2 are of a second type, ..., and n_k are of a kth type, is $\lfloor n / (\lfloor n_1 \lfloor n_2 \cdots \lfloor n_k)$.

Example: Determine the no. of permutations of the word PROGRAMMING.

Sol^n: There are 11 letters in the word PROGRAMMING out of which G's and M's and R's are two each.

The total no. of permutation is

$$= \frac{\lfloor 11}{\lfloor 2 \lfloor 2 \lfloor 2}$$

$$= 4989600$$

Example: There are 4 blue, 3 Red and 2 black pen in a box. These are drawn one by one. Determine all the different permutations.

Sol^n: There are total 9 pens in the box out of which 4 are blue, 3 are red and 2 are black.

$\therefore$ The total no. of permutations is

$$= \frac{\lfloor 9}{\lfloor 4 \times \lfloor 3 \times \lfloor 2}$$

$$= 1080$$

Scanned by CamScanner

Combination:

~~Recall~~ A permutation is an ordered arrangement of elements in a set. sometimes, however, the order of elements is irrelevent, only their membership is important. we will investigate such unordered arrangements in this section.

For example, a committee such as A = { costa, shea, weiss, Hall, chen } is just a set and the order in which the names are listed is immaterial. suppose we would like to form a subcommittee of A consisting of three members. Three such subcommittees are: {costa, shea, weiss}, {costa, shea, Hall} and { costa, shea, chen }.

Each is a combination of the five elements taken three at a time, or a 3-combination.

Combination: An r-combination of a set n elements $0 \le r \le n$, is a subset containing r elements.

The no. of r combinations of a set with n elements is denoted by $C(n,r)$ or $\binom{n}{r}$. The no. of combinations is also called the binomial coefficient.

Theorem: The no. of r-combinations of a set of n elements is given by

$$C(n,r) = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}} \qquad 0 \le r \le n$$

Example: A collection of 10 electric bulbs contain 3 defective ones.

(a). In how many ways can a sample of four bulbs be selected?

(b). In how many ways can a sample of four bulbs be selected which contain 2 good bulbs and 2 defective ones?

Soln:

(a) The four bulbs can be selected out of 10 bulbs in

$$^{10}C_4 = \frac{\lfloor 10}{\lfloor 4 \lfloor 6}$$

$$= 210 \text{ ways}$$

(b) 2 good bulbs can be selected out of 7 good bulbs in $^7C_2$ ways and 2 defective bulbs can be selected out of 3 defective bulbs in $^3C_2$ ways.

Thus, the no. of ways in which a sample of four bulbs containing two good bulbs and two defective bulbs can be selected is

$$^7C_2 \times {}^3C_2 = \frac{\lfloor 7}{\lfloor 2 \lfloor 5} \times \frac{\lfloor 3}{\lfloor 2 \lfloor 1}$$

$$= \frac{7\times 6}{2} \times 3$$

$$= 63$$

Example: In how many ways can a committee of 5 teachers and 4 students be chosen from 8 teachers and 14 students.

Soln: committee can be formed in

$$^8C_5 \times {}^{14}C_4 = \frac{\lfloor 8}{\lfloor 5 \lfloor 3} \times \frac{\lfloor 14}{\lfloor 4 \lfloor 10}$$

$$= \frac{8\times 7\times 6}{3\times 2} \times \frac{14\times 13\times 12\times 11}{4\times 3\times 2}$$

$$= 56056$$

Combination with Repetitions:

Example:- Find the no. of 3-combinations of the set $S = \{a, b\}$.

Soln: S contains $n = 2$ elements. Since each combination must contain three elements $r = 3$. Since $r > n$, the elements of each combination must be repeated.

A combination may contain three a's, two a's and one b, one a and two b's or three b's. Using the set notation the 3-combination are $\{a, a, a\}$, $\{a, a, b\}$, $\{a, b, b\}$ and $\{b, b, b\}$.

So there are four 3-combination of a set of two elements.

Theorem: The no. of r-combinations with repetitions from a set of n elements is $C(n+r-1, r)$.

The Pigeonhole Principle: If n pigeonholes ~~principle~~ are occupied by $n+1$ or more pigeons, then at least one pigeonhole is occupied by more than one pigon.

Example: Suppose a department contains 13 professors. Then two of the professors (objects) were born in the same month (pigeonholes).

Generalized Pigeonhole Principle:

If n pigeonholes are occupied by $kn+1$ or more pigeons, where k is a positive integer, then at least one pigeonhole is occupied by $k+1$ or more pigeons.

Example: Find the no. of students in a class to be sure that ~~sure~~ three of them are born in the same month.

Solⁿ: Here the $n = 12$ months are pigeon holes and $k+1 = 3$ or $k = 2$.

Hence among any $kn+1 = 25$ students (pigeons), three of them are born in the same month.