CS

*. Types of signals:

- Deterministic
- Random.

Those signals whose value at any instant is determined by its analytical or graphical representation are deterministic signals.

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Random signals: The signals which are unpredictable in nature.

Causes:—

1. Partial ignorance of generation mechanism
2. Complexity in calculating the full analysis.

*. Random Variable →

Usually the outcome of random experiment is not convenient for mathematical analysis
Eg: Getting a head/tail on a coin.
Hence, random variable describes the process of assigning a no. or value to the outcome of random experiment
Eg: Getting head = 0
Getting tail = 1.
Random variable is a rule that maps the sample space point into real nos.
Random variables are always represented by capital values (X, Y, Z, etc) and value assigned is represented by small letter (x, y, z, etc).

Random Variable
↓ Continous Random Variable (CRV)
↓ Discrete Random Variable (DRV)

Continous Random Var → The random variable that can assume any value over a range of real values
eg: Temperature at any instant.
Discrete Random Var → The random variable that can assume only discrete values
eg: Rolling a dice.

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*. **PRV :**

Suppose 3 tosses of a coin and no. of head represents RV (X)

$X \rightarrow$ No. of Heads

$S.S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

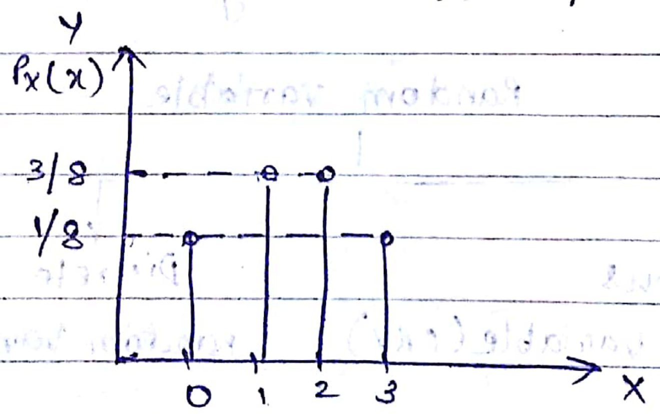
$X = 3 \quad 2 \quad 2 \quad 2 \quad 1 \quad 1 \quad 1 \quad 0$

1. Probability Distribution :→ It represents the probability that a random variable assumes a specific value

$$P_X(x) = P(X=x)$$

$$\boxed{\sum_{i=1}^{n} P_X(x_i) = 1.}$$

$$P(X=0) = P(X=3) = 1/8.$$
$$P(X=1) = P(X=2) = 3/8.$$



2. For any two RVs →.
Suppose X and Y are two RVs.
Joint probability distribution:

$$P_{XY}(x_i, y_j) = P(X=x_i; Y=y_j)$$

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and $$\boxed{\sum_{i=1}^{n}\sum_{j=1}^{m} P_{XY}(x_i, y_j) = 1.}$$

for two independent RVs :

$$P_{XY}(x_i, y_j) = P_X(x_i) \cdot P_Y(y_j)$$

$$\left[P(A \cap B) = P(A)\,P(B) \right]$$

3. Conditional Probability :→

$$P_{X/Y}(x_i/y_j) = P(X = x_i)$$
when $Y = y_j$ has already occurred.

or

$$P_{Y/X}(y_j/x_i) = P(Y = y_j)$$
when $X = x_i$ has already occurred.

such that

$$\sum_{x=i}^{n} P_{X/Y}(x_i/y_j) = 1.$$

or $$\sum_{y=j}^{n} P_{Y/X}(y_j/x_i) = 1.$$

4. Cummulative Distribution function (CDF) →

It is represented by $F_X(x) = P(X \leq x)$

It gives the probability that a random variable assumes a value less than or equal to a specific value.

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*. CDF (Commulative Distributive function) :-

$$F_X(x) = P(X \leq x)$$

It gives the probability that a R V 'X' is having a value less than or equal to 'x'

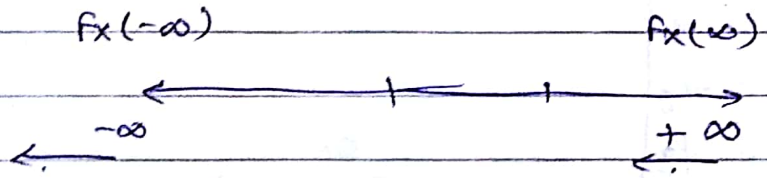
Properties :-

1) $F_X(x) \geq 0$

It is always non-negative.

2) $F_X(-\infty) = 0$

$F_X(\infty) = 1$.



3) It is always a non-decreasing function.

For $x_2 \geq x_1$

$F_X(x_2) \geq F_Y(x_1)$

Or

$$F_X(x_2) = P(X \leq x_2)$$
$$= P[(X \leq x_1) \cup (x_1 < X \leq x_2)]$$
$$= F_X(x_1) + \text{a positive value}$$

$\therefore$, $F_X(x_2) \geq F_X(x_1)$

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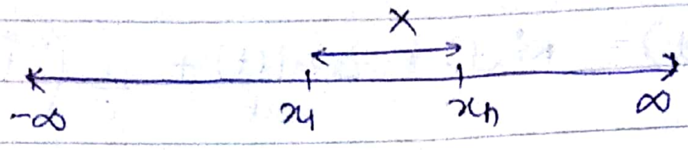
Suppose a R.V 'x' lies b/w $x_1 \leq x_n$



$$F_x(x_1) = 0$$
$$F_x(x_n) = 1$$

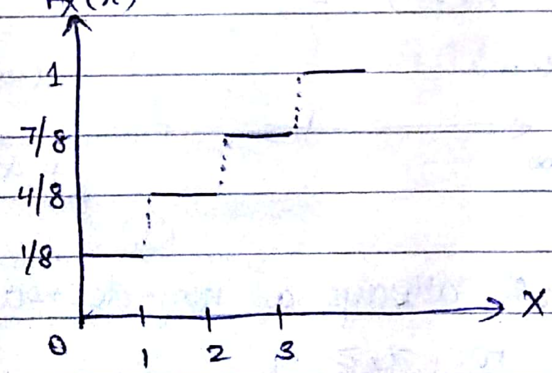
In general $F_x(x_i) = \sum_{i=1}^{n} P_x(x_i)$

eg: for a toss of 3 coins getting a head $\Rightarrow$ R.V (X)

TTT (0)   HHH (3)   Total = 8 SS

$X \Rightarrow$	0	1	2	3
$P_x(x) =$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
$F_x(x) =$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{7}{8}$	1

CDF



*. PDF (Probability Distribution function):-
Represented by '$f_x(x)$'.
Defined as the derivative of CDF.

$$\therefore, \; f_x(x) = \frac{d}{dx} F_x(x)$$

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As $F_X(x) = P(X \leq x)$

or

$= P(-\infty < X \leq x)$

$\therefore \quad F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

Area under curve.

Properties:-

1) It is always non-negative.

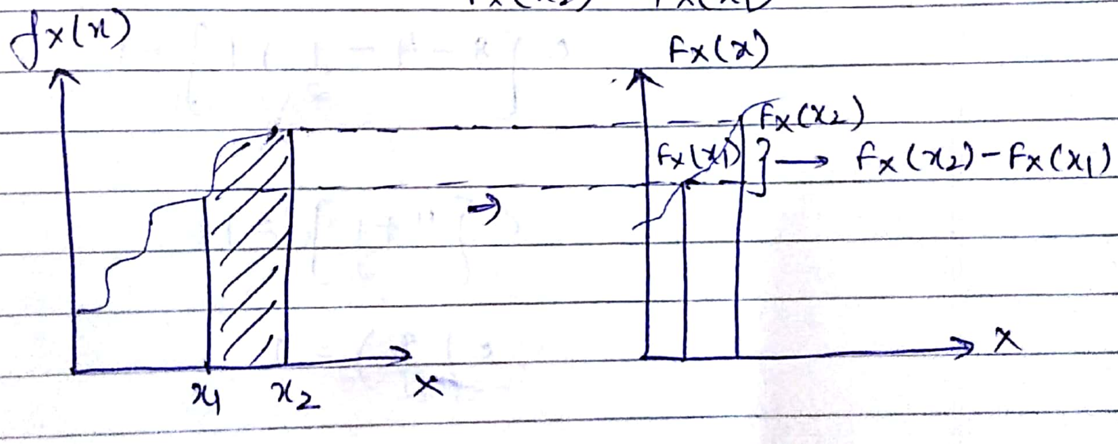
$f_X(x) \geq 0.$

As $F_X(x)$ (CDF) is a monotonically increasing function so its derivative will always be positive.

2) If R.V. 'x' lies between x_1 and x_2 then

$\Rightarrow \int_{x_1}^{x_2} f_X(x)\,dx \Rightarrow P(x_1 \leq X \leq x_2)$

$= F_X(x_2) - F_X(x_1)$



3) For entire range

$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$= F_X(\infty) - F_X(-\infty)$

$= 1 - 0$

$= 1$

Be Positive...

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Dt.
Pg. B+

Q Find a constant 'c' such that PDF is given by

$$f_X(x) = \begin{cases} c(x-1) & 1<x<4 \\ 0 & \text{otherwise} \end{cases}$$

sol

$$\int_1^4 f_X(x)\,dx = 1$$

$$\int_1^4 c(x-1)\,dx = 1$$

$$c\int_1^4 (x-1)\,dx = 1$$

$$c\left[\frac{x^2}{2} - x\right]_1^4 = 1$$

$$c\left[8-4-\frac{1}{2}+1\right] = 1$$

$$c\left[4+\frac{1}{2}\right] = 1$$

$$c\left(\frac{9}{2}\right) = 1$$

$$c = \frac{2}{9}$$

Q PDF is given by $f_X(x) = ae^{-b|x|}$ where 'x' is R.V. lying b/w $-\infty < X < \infty$.

a) Find relation b/w a & b.

b) Find CDF

c) Probability that output is b/w 1 & 2.

Be Positive...

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Dt.
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sol. a) $\int_{-\infty}^{\infty} ae^{-b|x|} dx = \int_{-\infty}^{0} ae^{-b(-x)} dx + \int_{0}^{\infty} ae^{-bx} dx = 1$

$\boxed{b = 2a.}$

b)

CDF

$\Rightarrow F_X(x) = \int_{-\infty}^{x} ae^{-b|x|} dx$

Case I: $x<0$	Case II: $x>0$.
$\int_{-\infty}^{x} ae^{-b(-x)} dx$	$\int_{-\infty}^{0} ae^{-bx} dx + \int_{0}^{x} ae^{-bx} dx$
$\Rightarrow \frac{1}{2} e^{bx}$	$\Rightarrow 1 - \frac{1}{2} e^{-bx}$
using $(b=2a)$.	

c) $\int_{1}^{2} f_X(x) dx$

$\Rightarrow \cancel{\int_{1}^{2} f_X(x)}$

$\Rightarrow \int_{1}^{2} ae^{-b|x|} dx$

$\Rightarrow \int_{1}^{2} ae^{-bx} dx.$

$a\int_{1}^{2} e^{-bx} dx$

$\cancel{ab}\left[-\frac{a}{b} e^{-bx}\right]_{1}^{2} = -\frac{a}{b} e^{-2b} + \frac{a}{b} e^{-b}$

Be Positive...

$$\frac{1}{2}e^{-b} - \frac{1}{2}e^{-2b}$$

$$= \frac{1}{2}\left(e^{-b} - e^{-2b}\right)$$

*. **Joint CDF →**

For a single RV 'x'

CDF $F_X(x) = P(X \leq x)$

For any two RV X & Y

Joint CDF, $F_{XY}(x,y) = P(X \leq x, Y \leq y)$

Properties:

a) It is always non-negative

b) It gives the probability of a joint sample space.

B+

c) It is Always a non-decreasing func. of both X & Y.

* Joint PDF →

For a single RV 'x'

PDF $f_X(x) = \frac{d}{dx} F_X(x)$

For any two RV X & Y

Joint PDF $f_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} [F_{XY}(x,y)]$

As $F_{XY}(x,y) = P(-\infty < X \le x, -\infty < Y \le y)$

$\therefore F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(x,y)\, dx\, dy$.

Properties :

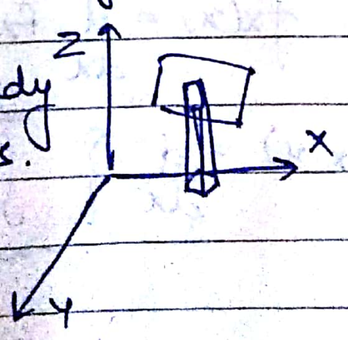
a) It is always non-negative.
i.e. $f_{XY}(x,y) \ge 0$.

b) For a finite interval
$X \in (x_1, x_2)$
$Y \in (y_1, y_2)$

$\Rightarrow \underbrace{\int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{XY}(x,y)\, dx\, dy}$

It gives volume bounded by surface $f_{XY}(x,y)$

This function already contains the z axis.



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c) for entire range:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1.$$

As, $$F_{XY}(x,y) = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{XY}(x,y)\,dx\,dy$$

If X & Y IRV then

$$F_{XY}(x,y) = \int_{x_1}^{x_2} f_X(x)\,dx \int_{y_1}^{y_2} f_Y(y)\,dy$$

$$= F_X(x) \cdot F_Y(y)$$

*. Marginal PDF →

As, $F_{XY}(x,y) = P(X \le x, Y \le y)$

For a single RV 'x'

$\cancel{F_{XY}(x,y) = P(X \le x, -\infty < Y < \infty)}$

$$F_X(x) = P(X \le x, -\infty < Y < \infty)$$

$$= \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy$$

$$f_X(x) = \frac{d}{dx} F_X(x)$$

$$f_X(x) = \frac{d}{dx}\left\{\int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy\right]$$

Dt.
Pg. B+

$$\boxed{f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy}$$ Marginal Density/PDF of x

or

$$\boxed{f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx}$$ Marginal Density/PDF of y.

Q for a joint PDF

$$f_{XY}(x,y) = \frac{1}{4} e^{-|x|-|y|}$$

a) Are X & Y IRV

b) find the probability that

$x \le 1$

$y \le 0$.

sol a)

$$f_{XY}(x,y) = \frac{1}{4} e^{-|x|-|y|}$$

$$= \frac{1}{4} e^{-|x|} \cdot e^{-|y|}$$

$$= \left(\frac{1}{2} e^{-|x|}\right) \cdot \left(\frac{1}{2} e^{-|y|}\right)$$

$$= f_X(x) \cdot f_Y(y)$$

IRV.

b)

$$\int_{-\infty}^{1} \frac{1}{2} e^{-|x|}\,dx \cdot \int_{-\infty}^{0} \frac{1}{2} e^{-|y|}\,dy$$

$$= \frac{1}{4} \int_{-\infty}^{1} e^{-|x|}\,dx \int_{-\infty}^{0} e^{-|y|}\,dy$$

$$= \frac{1}{4} \int_{-\infty}^{0} e^{x} dx + \int_{0}^{1} e^{-x} dx \left[\int_{-\infty}^{0} e^{y} dy \right]$$

$$= \frac{1}{4} \left[e^{x} \Big|_{-\infty}^{0} + -e^{-x} \Big|_{0}^{1} \right] \left[e^{y} \Big|_{-\infty}^{0} \right]$$

$$\frac{1}{4}$$

*. Mean / Average / Expectation : →

$\bar{X}$ or μ_X or $E[X]$

a) DRV : Suppose RV 'x' lies within $\{x_1, x_2 - - - - x_n\}$

$$E[X] = \sum_{i=1}^{n} x_i P_X(x_i)$$

b) CRV : Suppose RV 'x' is continous in nature

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

Special Cases :

a) If there is a function of random variable

$$g(X) = X^n$$

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx.$$

$$= \int_{-\infty}^{\infty} x^n f_X(x) dx.$$

Mean $E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \mu$

$Var(X) = E[X^2] - \mu^2$

$SD = \sqrt{Var(x)}$

Q. for a given RV 'x'

$$f_X(x) = \begin{cases} 1/2\pi, & 0 \le x \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

find 1. mean

2. Var

3. SD.

Mean $=$ $E[X]$

$$\int_0^{2\pi} x \times \frac{1}{2\pi}\, dx = \left[\frac{1}{2\pi}\frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\frac{4\pi^2}{2} - 0 = \pi$$

Be Positive...

B+

$$E[X^2] = \int_0^{2\pi} x^2 \times \frac{1}{2\pi}\, dx$$

$$= \left[\frac{1}{2\pi} \frac{x^3}{3} \right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \frac{8\pi^3}{3} - 0$$

$$= \frac{4\pi^2}{3}$$

$$\text{Var}(X) = E[X^2] - \mu^2 \qquad (\mu = \text{Mean})$$

$$= \frac{4\pi^2}{3} - \pi^2$$

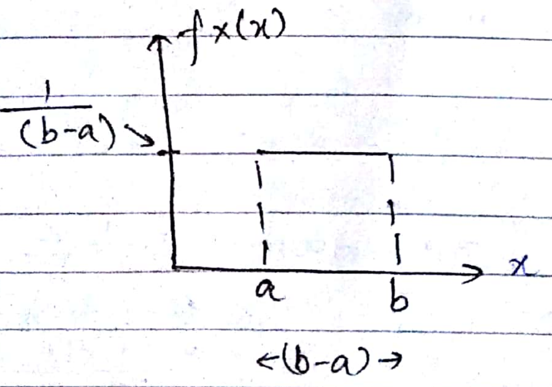
$$= \frac{4\pi^2 - 3\pi^2}{3}$$

$$= \frac{\pi^2}{3}$$

$$SD = \sqrt{\frac{\pi^2}{3}} = \frac{\pi}{\sqrt{3}}$$

*. **UDF (Uniform Distribution function) :**

A R.V 'x' is uniformly distributed within range (a,b)



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$$\therefore f_x(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise.} \end{cases}$$

find mean & variance?

$$\text{Mean} = \int_a^b \frac{x}{b-a}\,dx$$

$$= \left[\frac{1}{b-a}\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{b-a}\frac{b^2}{2} - \frac{1}{b-a}\frac{a^2}{2}.$$

$$= \frac{1}{2(b-a)}(b^2-a^2).$$

$$= \frac{1}{2(\cancel{b-a})}\cancel{(b-a)}(b+a)$$

$$= \frac{a+b}{2}.$$

$$E[x^2] = \int_a^b x^2 \frac{1}{b-a}\,dx$$

$$= \left[\frac{1}{b-a}\frac{x^3}{3}\right]_a^b$$

$$= \frac{1}{b-a}\frac{b^3}{3} - \frac{1}{b-a}\frac{a^3}{3}$$

$$\frac{1}{3(b-a)}(b^3-a^3)$$

$$\frac{1}{3(\cancel{b}-a)}$$

Be Positive...

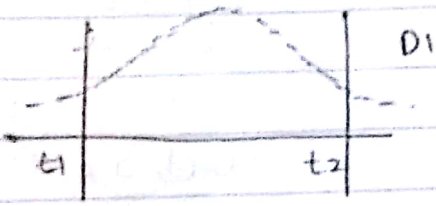
$$Var(x) = \frac{1}{3(b-a)}(b^3-a^3) - \left(\frac{a+b}{2}\right)^2$$

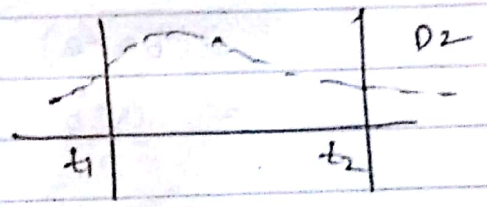
$$= \frac{(a-b)^2}{12}.$$

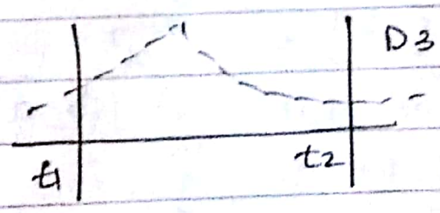
*. **Random/Stochastic Process:**

A random process is a collection of infinite no of random variables which are generally dependent and is represented as a function of time.

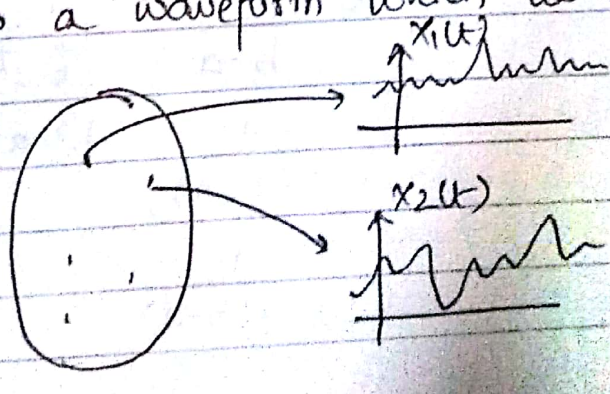
Eg: Measurement of temperature of a city b/w a particular time interval.







For a random process, the outcome of experiment is mapped into a waveform which is a function of time.



a) CDF:

For a RV 'X'

$$F_X(x) = P(X \le x)$$

For a R. process

$$F_X(x;t) = P(X(t) \le x)$$

--- 1st order

2nd order

$$\cancel{F_{X_1X_2}}\,F_X(x_1,x_2;t_1,t_2)$$
$$= P(X(t_1) \le x_1;\ X(t_2) \le x_2)$$

n^{th} order

$$F_X(x_1,x_2,\ldots x_n;\ t_1,t_2,\ldots t_n)$$
$$= P[X(t_1) \le x_1;\ X(t_2) \le x_2 \ldots X(t_n) \le x_n]$$

b) PDF:

for a RV 'X'

$$f_X(x) = \frac{d}{dx} F_X(x)$$

For a R. process

$$f_X(x;t) = \frac{d}{dx} F_X(x;t) \rightarrow 1^{ST} \text{ order}$$

2^{nd} order

$$f_X(x_1,x_2;t_1,t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_X[x_1,x_2;t_1,t_2]$$

n^{th} order

$$f_X(x_1,x_2,\ldots x_n;t_1,t_2 \ldots t_n)$$
$$= \frac{\partial^n}{\partial x_1 \partial x_2 \ldots \partial x_n} F_X[x_1,x_2,\ldots x_n;\ t_1,t_2,\ldots t_n]$$

c) Mean:

For a RV 'x'

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

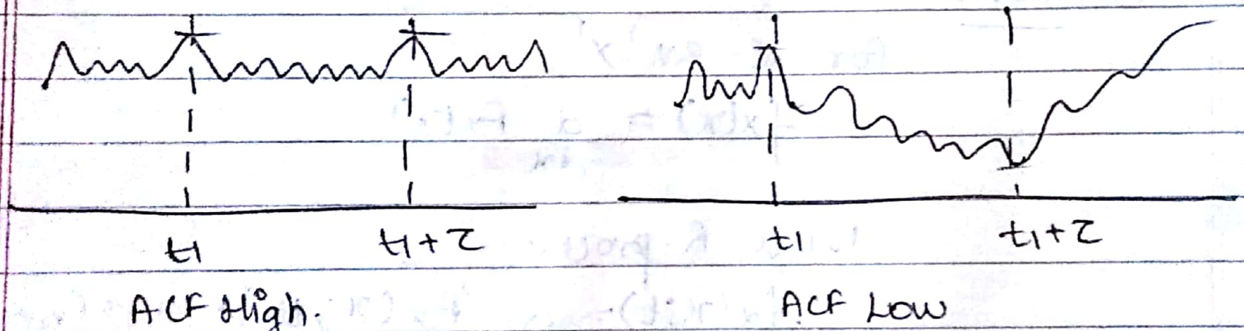
For a R. process

$$E[X(t)] = \int_{-\infty}^{\infty} x f_X(x;t)\, dx$$

d) Auto correlation function:

It gives the measurement of similarity of amplitudes at two different instants

At t_1 and $t_2 = t_1 + \tau$



$$R_X(t_1, t_2) = E[X(t_1), X(t_2)]$$
$$= E[X_1 X_2]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2; t_1, t_2)\, dx_1\, dx_2$$

*. Stationary Random Process:

A random process is said to be stationary if its statistical characteristics do not change with shift of time origin.

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a) PDF of the random process must be same at t_1 and t_2

$$\therefore f_X(x; t_1) = f_X(x; t_2)$$

b) Mean of the random process must be same at t_1 and t_2

$$\therefore \int_{-\infty}^{\infty} x f_X(x, t_1)\, dx = \int_{-\infty}^{\infty} x f_X(x, t_2)\, dx$$

c) Auto correlation function must depend on time difference

$$t_2 - t_1 = \tau$$

$$R_X(t_1, t_2) = R_X(\tau)$$

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CS

Random Process

$$\text{Random Process} \rightarrow \begin{cases} \text{strict sense stationary (SSS)} \\ \text{Wide sense stationary (WSS)} \end{cases}$$

* Wide sense Stationary process: These random processes are not stationary in the strict sense but mean and autocorrelation function do not change with shift of time origin.

As a strict sense stationary process should start at $t = -\infty$ and stops at $t = +\infty$. but no such process is possible practically hence a process may appear stationary over a specific period of time. So, it is called wide sense stationary.

Q. for a R. Process $A\cos(\omega t + \phi)$ where 'ϕ' is RV which lies b/w $(0, 2\pi)$

$$f_\phi(\phi) = \begin{cases} \frac{1}{2\pi}, & 0 \le \phi \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

Show that WSS

Sol 1) Mean $= \int_0^{2\pi} A\cos(\omega t + \phi) \cdot \frac{1}{2\pi} \, d\phi$

$$= \frac{A}{2\pi} \int_0^{2\pi} \cos(\omega t + \phi) \, d\phi$$

$$\Downarrow$$

$$0.$$

Mean is independent.

2) Auto correlation function :

$$E[x_1(t)\, x_2(t)]$$

$$= \int_0^{2\pi} \left(\frac{1}{2\pi}\right)^2 A\cos(\omega t_1+\phi)\, A\cos(\omega t_2+\phi)\, d\phi$$

$$= \left(\frac{A}{2\pi}\right)^2 \int_0^{2\pi} \left\{ \cos[\omega(t_1+t_2)+\phi] + \cos[\omega(t_1-t_2)] \right\} d\phi$$

$$= \left(\frac{A}{2\pi}\right)^2 \int_0^{2\pi} \cos[\omega(t_1+t_2)+\phi]\, d\phi \Rightarrow 0$$

$$+ \left(\frac{A}{2\pi}\right)^2 \int_0^{2\pi} \cos[\omega(t_1-t_2)]\, d\phi$$

$$= \cancel{fn}\ fn(t_1-t_2)$$

OR.

$$E[A\cos(\omega t_1+\phi) \cdot A\cos(\omega t_2+\phi)]$$

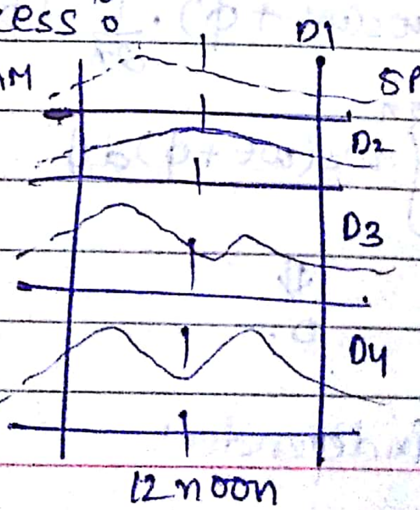
$$E[\underbrace{A^2(\cos(\omega(t_1+t_2)+\phi)}_{0} + \cos(\omega(t_1-t_2))]$$

$$= f(t_1-t_2)$$

Depends upon diff. of time.

$\Rightarrow$ It is WSS.

*. Ergodic Process :



Avg
- Ensemble Avg
- Time Avg.

Ensemble Average :→ Average is taken for collection of w.f for a fixed time.

$$\widetilde{X(t)} = \int_{-\infty}^{\infty} x \, f_X(x;t) \, dx$$

Time Average :→ Average is taken for fixed single waveform over entire period.

$$\overline{X(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) \, dt.$$

A random process is ergodic if ensemble avg. is equal to time avg. and this property is called ergodicity.

After this NOTES.

③ PDF — First order PDF

$$f_X(x;t) = \frac{\partial}{\partial x}\left[F_X(x;t)\right] \quad ⑦$$

Second order PDF

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2}\left[F_X(x_1, x_2; t_1, t_2)\right]$$

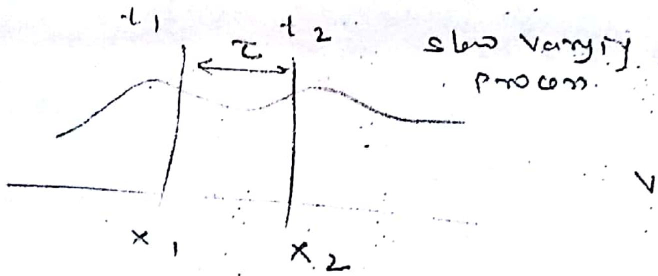
nth order PDF

$$f_X(x_1, x_2 \ldots x_n; t_1, t_2 \ldots t_n) = \frac{\partial^n}{\partial x_1 \partial x_2 \ldots \partial x_n}\left[F_X(x_1, x_2 \ldots x_n; t_1, t_2 \ldots t_n)\right]$$

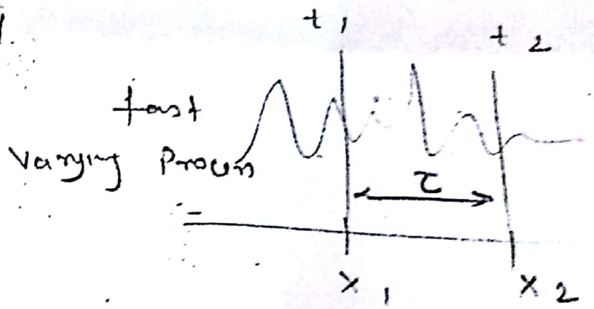
④ **Mean**

$$E[X(t)] = \int_{-\infty}^{\infty} x\, f_X(x;t)\, dx$$

⑤ **Auto Correlation function** → It gives the measure of similarity of amplitudes at two given instants t_1 & $t_2 = t_1 + \tau$.



(strong correlation)



(weak correlation)

$$R_X(t_1, t_2) = E[X(t_1)\,X(t_2)] = E[X_1 X_2]$$

$$\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2\, f_X(x_1, x_2; t_1, t_2)\, dx_1\, dx_2$$

Stationary Random Process → A Random Process whose statistical characteristics don't change with a shift of time origin

(i) PDF of X at t_1 & t_2 must be same

$$\therefore f_X(x;t_1) = f_X(x;t_2) = f_X(x)$$

(ii) Mean of X at t_1 & $t_2 = t_1 + \tau$ must be same.

$$\left.\begin{aligned} \mu_X(t_1) &= \int_{-\infty}^{\infty} x f_X(x; t_1)\, dx \\ \mu_X(t_2) &= \int_{-\infty}^{\infty} x f_X(x; t_2)\, dx \end{aligned}\right\} \text{Same or Constant}$$

(iii) Auto Correlation function must depend on time difference

$\tau = t_2 - t_1$

$\therefore \quad R_X(t_1, t_2) = R_X(t_2 - t_1) = R_X(\tau)$

so $\quad R_X(\tau) = E[X(t)\, X(t+\tau)]$

Wide Sense (weakly) Stationary Process :- These process are not stationary in strict sense but mean & Auto Correlation function don't change with a shift of time origin.

$\because$ A stationary process in strict sense should have statistics independent of time so such process should start at $t = -\infty$ & stops at $t = \infty$ but as no such type of process is possible practically so the process may appear stationary over a specific period of time. It is called WSS.

Ergodic Process

Average → Ensemble Avg
Average → Time Avg

Ensemble Avg :- when Avg is taken over collection of waveform for a fixed time

$$\overline{x(t)} = \int_{-\infty}^{\infty} x f_X(x,t)\, dx$$

Time Avg :- when Avg is taken over entire time period for fixed single waveform

$$\widetilde{x(t)} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t)\, dt$$

A Random process is ergodic if Time avg are equal to ensemble avg

$\therefore \overline{x(t)} = \widetilde{x(t)}$ This property is called ergodicity

Properties of Auto Correlation function (ACF)

for a stationary process $x(t)$, the ACF is given as $R_X(\tau) = E[X(t)X(t+\tau)]$

1) $R_X(\tau)$ is an even function of τ

$$R_X(\tau) = R_X(-\tau)$$

Put $\tau \to -\tau$

$R_X(-\tau) = E[X(t)X(t-\tau)] = R_X(\tau)$ proved

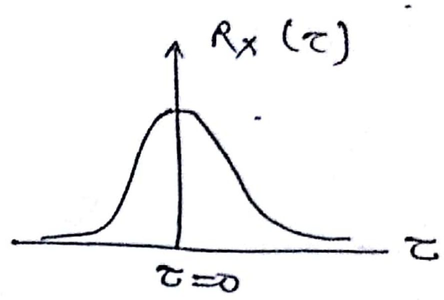
(2) Mean square value of Process can be obtained putty $\tau = 0$ in $R_X(\tau)$

As $R_X(\tau) = E[X(t)\, X(t+\tau)]$

put $\tau = 0$ $\quad \therefore \; R_X(\tau) = E[X^2(t)]$

(3) ACF has max. magnitude at $\tau = 0$

i.e $|R_X(\tau)| = R_X(0)$



Cross correlation function $\rightarrow$ For two Random processes $X(t)$ & $Y(t)$ with ACF $R_X(t_1, t_2)$ & $R_Y(t_1, t_2)$

then cross correlation function is given by

$$R_{XY}(t_1, t_2) = E[X(t_1)\, Y(t_2)]$$

If $X(t)$ & $Y(t)$ are wide sense stating (WSS) process then $R_{XY}(t_1, t_2)$ can be written as $R_{XY}(\tau)$

where $\tau = t_2 - t_1$

such that $R_{XY}(\tau) = R_{YX}(-\tau)$

Covariance $\rightarrow$ Covariance of two RV X & Y is given as $\mathrm{cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$

(2)

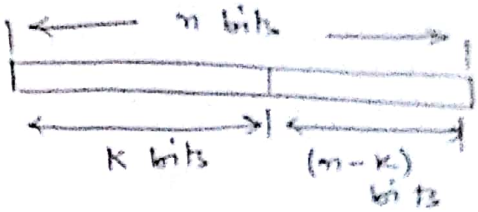
Binomial Distribution

It is used for repeated trials of an experiment

Total n bits

K correct bits

$n-K$ in error bits



Probability of getting a correct bit $\Rightarrow p$

" " " incorrect bit $\Rightarrow 1-p$

Probability of getting 'K' correct bits out of n bits $\Rightarrow {}^{n}C_{K}\, p^{K} (1-p)^{n-K}$

$$\therefore \quad P(X=k) = {}^{n}C_{K}\, p^{k} (1-p)^{n-k}$$

where $\quad {}^{n}C_{K} = \dfrac{n!}{K!\,(n-K)!} \quad$ (binomial coeff)

1) CDF $\quad F_X(x) = \sum_{k=0}^{n} {}^{n}C_{K}\, p^{k} (1-p)^{n-k}$

2) mean $\quad E[X] = np$

3) Variance $\quad Var = np(1-p)$

Q: If 8 digit binary words are transmitted over a noisy channel, with a per-digit error prob. of 0.01. Calculate the prob. that 3 digits out of 8 are in error. Also find mean & variance.

Sol:- $\quad n = 8$ $\quad P(X=K) = {}^{8}C_{3} (0.01)^{3} (0.99)^{5}$

$K = 3$ $\quad \Rightarrow 5.32 \times 10^{-5}$

$P = 0.01$

Mean $= np = 8 \times 0.01 = 0.08$

$Var = np(1-p)$

$= 0.0792$

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Poisson Distribution → when 'n' is very large & Prob 'p' is very small then binomial distribution is approximated by Poisson distribution.

$$P(X=k) = \frac{m^k e^{-m}}{k!} \quad \text{where } m = np \text{ (mean)}$$

$$\Rightarrow \frac{(np)^k e^{-np}}{k!}.$$

1) Mean $= np$
2) Var $= np$
3) S.D $= \sqrt{np}$

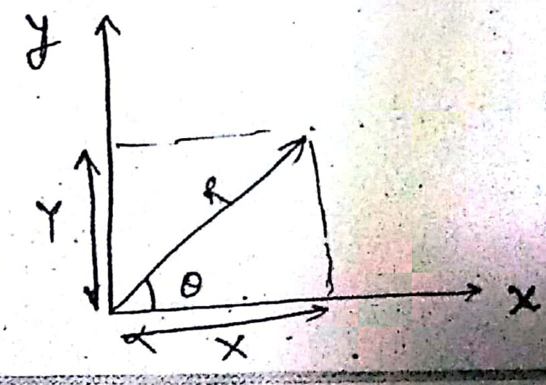
Q Suppose 10,000 digits are transmitted over a noisy channel having error probability per digit equal to 5×10^{-5}. Find prob of getting two digits in errors.

Sol:- $n = 10000$
$k = 2$
$p = 5 \times 10^{-5}$
$\Big\}\ np = 0.5$

$$P(X=k) = \frac{(0.5)^2 \cdot e^{-0.5}}{2} = 0.0758$$

Rayleigh's distribution → used for continuous random R. Variable variables. Let X & Y be two independent such that $m_x = m_y = m$

& $\sigma_x = \sigma_y = \sigma$,



$R = \sqrt{X^2 + Y^2}$

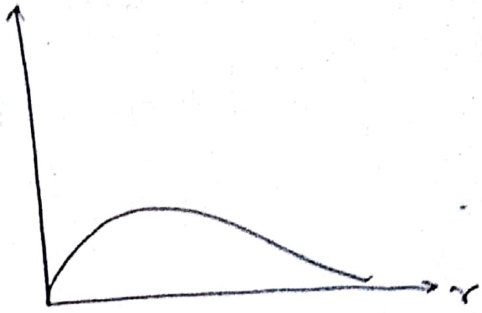
$\theta = \tan^{-1}[Y/X]$

we R & ϕ are Rayleigh random variables.

PDF is given by

$$f_R(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} \quad \text{for } r \geq 0$$

$$= 0 \quad \text{for } r < 0$$



<u>Mean</u> $m = \sigma\sqrt{\pi/2}$

<u>Variance</u> $\text{Var} = \left[2 - \pi/2\right]\sigma^2$

Q - Consider a random process $X(t) = A\cos(\omega t + \phi)$
A & ω are deterministic & distributed over $[-\pi, \pi]$ show that it is wide sense stationary. (WSS)

Sol: - Find mean

$$E[X(t)] = \int_{-\infty}^{\infty} A\cos(\omega t + \phi)\, f_\phi(\phi)\, d\phi$$

As $f_\phi(\phi) = \begin{cases} \frac{1}{2\pi} & -\pi \leq \phi \leq \pi \\ 0 & \text{otherwise} \end{cases}$

$$= \frac{A}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \phi)\, d\phi = 0$$

Find Auto Correla function

$$= E[X(t)\,X(t+\tau)]$$

$$= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \phi)\cos[\omega(t+\tau) + \phi]\, d\phi$$

$$= \frac{A^2}{2}\cos\omega\tau \quad \text{— a function of } \tau$$

So it is WSS

Q The joint pdf of X & Y is given by

$$f_{XY}(x, y) = xy\, e^{-(x^2+y^2)/2} u(x) u(y)$$

a) find marginal pdf $f_X(x)$ & $f_Y(y)$

b) Are X & Y independent

sol:- $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy$

$$= \int_0^{x} xy\, e^{-(x^2+y^2)/2} u(x)\, dy$$

$$= x e^{-x^2/2} u(x) \underbrace{\int_0^{\infty} y e^{-y^2/2}\, dy}_{1} = x e^{-x^2/2} u(x)$$

similarly $f_Y(y) = y e^{-y^2/2} u(y)$

b) As $f_{XY}(x, y) = f_X(x) \cdot f_Y(y)$

so they are independent.

For $n = 2$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$ --- Mean Square value of RX

<u>Gaussian / Normal Distribution</u> → It is most important continuous prob. distribution ∵ most of natural phenomenon can be characterized by R.V with normal distribution

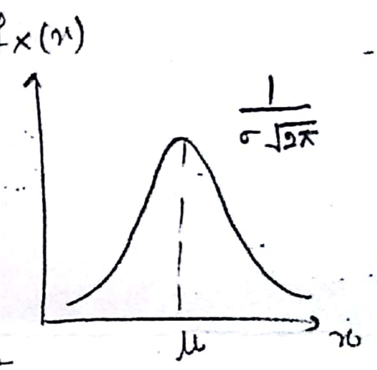
PDF $$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

$\mu \rightarrow$ Mean

$\sigma^2 \rightarrow$ Variance

at $x = 0$ $\quad f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$

at $x = \mu$ $\quad f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}$



CDF $$F_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(x-\mu)^2/2\sigma^2}\,dx$$

Let $\frac{\mu - x}{\sigma\sqrt{2}} = z \Rightarrow +dx = -\sigma\sqrt{2}\,dz$

$x \rightarrow -\infty \quad ; \quad z \rightarrow \infty$

$x \rightarrow x \quad ; \quad z \rightarrow \frac{\mu - x}{\sigma\sqrt{2}}$

$$F_X(x) = \frac{1}{\sqrt{\pi}} \int_{\infty}^{z} \frac{1}{\sigma\sqrt{2}} e^{-z^2} \times (-\sigma\sqrt{2}\,dz)$$

$$= \frac{1}{\sqrt{\pi}} \int_{z}^{\infty} e^{-z^2}\,dz$$

$\therefore$ CDF $= \frac{1}{2}\,\mathrm{erfc}(z)$

CDF $= \frac{1}{2}\,\mathrm{erfc}\left(\frac{\mu - x}{\sigma\sqrt{2}}\right)$

It is related to Error fn

$$\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2}\,dt$$

$$\mathrm{erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^{\infty} e^{-t^2}\,dt$$

$$\mathrm{erf}(t) + \mathrm{erfc}(t) = 1$$