Z Transform

Z transform of a sequence U_n $(n=0,1,2 \dots)$ $(U_n = 0 \text{ if } n<0)$ is denoted by

$$Z(U_n) = \sum_{n=0}^{\infty} U_n z^{-n} = U(z)$$

(I) $Z(a^n) = \sum_{n=0}^{\infty} a^n z^{-n} = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \dots$

$$= \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}$$

(II) $Z(1) = \frac{z}{z-1}$

(III) $Z[(-1)^n] = \frac{z}{z+1}$

(IV) $U(n) = \begin{cases} 0 & n<0 \\ 1 & n \geq 0 \end{cases}$

$$Z(U_n) = \sum_{n=0}^{\infty} U(n) z^{-n} = \frac{z}{z-1}$$

(V) $\delta(n) = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$

$$Z(\delta(n)) = \sum_{n=0}^{\infty} \delta(n) z^{-n} = 1$$

(1) $Z(n^p) = -z \frac{d}{dz} Z(n^{p-1})$

$$Z(n^p) = \sum_{n=0}^{\infty} n^p z^{-n}$$

$$Z(n^{p-1}) = \sum_{n=0}^{\infty} n^{p-1} z^{-n} \quad -(1)$$

$$\frac{d}{dz} Z(n^{p-1}) = \sum_{n=0}^{\infty} n^{p-1}(-n) z^{-n-1} = -z^{-1}\sum_{n=0}^{\infty} n^{p} z^{-n} = -z^{-1} Z(n^{p})$$

$$\Rightarrow \boxed{Z(n^{p}) = -z\frac{d}{dz} Z(n^{p-1})}$$

(1) $Z(n) = -z\dfrac{d}{dz} Z(1)$

$$= -z\frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= -z\left[\frac{(z-1).1 - z(1)}{(z-1)^2}\right]$$

$$\boxed{Z(n) = \frac{z}{(z-1)^2}}$$

$$Z(n^2) = -z\frac{d}{dz} Z(n) = -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right)$$

$$= -z\left[\frac{(z-1)^2 - 2(z-1)z}{(z-1)^4}\right]$$

$$= \cancel{-z}\left[\cancel{z^2} - 2\cancel{z} + \cancel{1}\right.$$

$$= \frac{-z(z-1-2z)}{(z-1)^3}$$

$$= \frac{z(z+1)}{(z-1)^3} = \frac{z^2+z}{(z-1)^3}$$

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(1) $Z[a_1 U_n + a_2 V_n] = a_1 U(z) + a_2 V(z)$

$= a_1 \sum_{n=0}^{\infty} U_n z^{-n} + a_2 \sum_{n=0}^{\infty} V_n z^{-n}$

(2) $Z(U_n) = U(z)$

$Z(a^{-n} U_n) = U(az)$

$Z(a^n U_n) = U\left(\frac{z}{a}\right)$

$Z[a^{-n} U_n] = \sum_{n=0}^{\infty} a^{-n} U_n z^{-n} = \sum_{n=0}^{\infty} U_n (az)^{-n} = U(az)$

$Z(\cos n\theta)$

$Z(\sin n\theta)$

$Z(e^{-in\theta}) = Z((e^{-i\theta})^n) = \frac{z}{z - e^{-i\theta}} \times \frac{(z - e^{i\theta})}{(z - e^{i\theta})}$

$= \frac{z(z - e^{i\theta})}{z^2 + 1 - z(e^{i\theta} + e^{-i\theta})}$

$Z(\cos n\theta - i\sin n\theta) = \frac{z(z - e^{i\theta})}{z^2 + 1 - 2z\cos\theta} = \frac{z(z - \cos\theta - i\sin\theta)}{z^2 + 1 - 2z\cos\theta}$

* $Z(\cos n\theta) = \frac{z(z - \cos\theta)}{z^2 + 1 - 2z\cos\theta}$

* $Z(\sin n\theta) = \frac{z\sin\theta}{z^2 + 1 - 2z\cos\theta}$

$Z(2^n \cos n\theta) = \frac{z/2\,(z/2 - \cos\theta)}{z^2/4 + 1 - 2\,z/2\,\cos\theta}$

$= \frac{z(z - 2\cos\theta)}{z^2 + 4 - 4z\cos\theta}$

$$Z\left(\frac{1}{\lfloor n}\right) = \sum_{n=0}^{\infty} \frac{1}{\lfloor n} z^{-n} = 1 + \frac{1}{\lfloor 1}\frac{1}{z} + \frac{1}{\lfloor 2}\frac{1}{z^2} + \frac{1}{\lfloor 3}\frac{1}{z^3} ----$$

$$= e^{1/z}$$

$$Z\left(\frac{2^n}{\lfloor n}\right) = e^{2/z}$$

$$Z\left[\frac{1}{2^n \lfloor n}\right] = e^{1/2z}$$

(1) $Z(U_n) = U(Z)$

$$Z(U_{n-k}) = \sum_{n=0}^{\infty} U_{n-k}\, z^{-n}$$

$$= U_{-k} + U_{-k+1} z^{-1} + --- U_0 z^{-k} + U_1 z^{-k-1} + ---$$

$$= U_0 z^{-k} + U_1 z^{-k-1} + ---$$

$$= z^{-k}\left[U_0 + U_1 z^{-1} + U_2 z^{-2} + ---\right]$$

$$= z^{-k} \sum_{n=0}^{\infty} U_n z^{-n}$$

$$= z^{-k}\, U(z)$$

* ~~$z(U_n$~~ $Z(U_n) = U(z)$

$$Z(U_{n+k}) = \sum_{n=0}^{\infty} U_{n+k} z^{-n} = z^k \sum_{n=0}^{\infty} U_{n+k} z^{-(n+k)}$$

$$= z^k\left[U_k z^{-k} + U_{k+1} z^{-k-1} + U_{k+2} z^{-k-2} + ---\right]$$

$$= z^k\left[\sum_{n=0}^{\infty} U_n z^{-n} - U_0 - U_1 z^{-1} - U_2 z^{-2} --- U_{k-1} z^{-(k-1)}\right]$$

$$= z^k\left[U(z) - U_0 - U_1 z^{-1} - U_2 z^{-2} --- U_{k-1} z^{-(k-1)}\right]$$

(#) $$Z\left(\frac{1}{\lfloor n+1}\right) = z^{1}\left[e^{1/z} - 1\right]$$

$$= z\left(e^{1/z} - 1\right)$$

$$Z\left(\frac{1}{(n+2)!}\right) = z^2\left[e^{1/z} - 1 - \frac{1}{z}\right]$$

$$= z\left[z e^{1/z} - z - 1\right]$$

* Multiplication by n

If $Z(U_n) = U(z)$

$$Z(nU_n) = -z\frac{d}{dz}U(z)$$

$$\boxed{Z(n^m U_n) = (-z)^m \frac{d^m}{dz^m} U(z)}$$

$$Z(nU_n) = \sum_{n=0}^{\infty} n U_n z^{-n}$$

$$U(z) = \sum_{n=0}^{\infty} U_n z^{-n}$$

$$\frac{dU(z)}{dz} = \sum_{n=0}^{\infty} U_n(-n)z^{-n-1}$$

$$= -z^{-1}\sum_{n=0}^{\infty} n U_n z^{-n}$$

$$= -z^{-1} Z(nU_n)$$

$$\boxed{Z(nU_n) = -z\frac{d}{dz}U(z)}$$

Q. $Z[n^2 e^{n\theta}]$

$$Z[e^{n\theta}] = \frac{z}{z - e^{\theta}}$$

$$Z[n^2 e^{n\theta}] = (-z)^2 \frac{d^2}{dz^2}\left(\frac{z}{z - e^{\theta}}\right)$$

$$= z^2 \frac{d}{dz}\left[\frac{z - e^{\theta} - z}{(z-e^{\theta})^2}\right]$$

$$= z^2 \frac{d}{dz}\frac{-e^{\theta}}{(z-e^{\theta})^2} = -e^{\theta} z^2 \frac{(-2)}{(z-e^{\theta})^3}$$

Initial Value Theorem

If $Z(U_n) = U(z)$

$$U_0 = \lim_{z\to\infty} U(z)$$

$$Z(U_n) = U(z) = \sum_{n=0}^{\infty} U_n z^{-n} = U_0 + U_1 z^{-1} + U_2 z^{-2} + \cdots$$

$$U(z) = U_0 + U_1 z^{-1} + U_2 z^{-2} + \cdots \quad -(\text{I})$$

$$\lim_{z\to\infty} U(z) = U_0 \quad -(\text{II})$$

~~Multiply~~ Multiplying eq^n (I) by z

$$z U(z) = U_0 z + U_1 + U_2 z^{-1} + \cdots$$

$$z\left[U(z) - U_0\right] = U_1 + U_2 z^{-1} + \cdots$$

$$\lim_{z\to\infty} z\left[U(z) - U_0\right] = U_1$$

Similarly

$$U_2 = \lim_{z\to\infty} z^2\left(U(z) - U_0 - \frac{U_1}{z}\right)$$

Final Value Theorem

$$Z(U_n) = U(z)$$

$$\lim_{n\to\infty}(U_n) = \lim_{z\to 1}(z-1)\,U(z)$$

Q. Find $Z(u_{n+2})$ if $Z(u_n) = \frac{z}{z-1} + \frac{z}{z^2+1}$

Sol^n $Z(u_{n+2}) = z^2\left[Z(u_n) - u_0 - u_1 z^{-1}\right]$

$$u_0 = \lim_{z\to\infty} U(z)$$

$$= \lim_{z\to\infty}\left[\frac{z}{z-1} + \frac{z}{z^2+1}\right]$$

$$= \cancel{lt}\ 1$$

$$u_1 = \lim_{z\to\infty} z\left[U(z) - u_0\right]$$

$$= \lim_{z\to\infty} z\left[\frac{z}{z-1} + \frac{z}{z^2+1} - 1\right]$$

$$= \lim_{z\to\infty}\left[\frac{z^2}{z-1} + \frac{z^2}{z^2+1} - z\right]$$

$$= \cancel{\lim_{z\to\infty}\left[\frac{z^2 - z + 1}{z-1} + \frac{z^2}{z^2+1}\right]} \quad \cancel{\lim_{z\to\infty}\left[\frac{1}{\frac{1}{z} - \frac{1}{z^2}}\right]}$$

$$= \cancel{1} \lim_{z\to\infty}\left[\frac{z}{z-1} + \frac{z^2}{z^2+1}\right] = 2$$

$$Z(u_{n+2}) = z^2\left[\frac{z}{z-1} + \frac{z}{z^2+1} - 1 - \frac{2}{z}\right]$$

(i) $Z(u_n) = U(z) = \frac{2z^2 + 5z + 14}{(z-1)^4}$ Find u_2, u_3

(ii) $Z((n+1)^2)$

(iii) $Z(e^{-an}\sin\theta)$

(iv) $Z\left[2^n + 5\sin\frac{n\pi}{4} - 3a^4\right]$

Z^{-1} Transform

We know that

$$Z[U_n] = U(z)$$

$$Z^{-1}[U(z)] = U_n$$

- $$Z(n) = \frac{z}{(z-1)^2}$$

$$\Rightarrow Z^{-1}\left[\frac{z}{(z-1)^2}\right] = n$$

- $$Z(a^n) = \frac{z}{z-a}$$

$$\Rightarrow Z^{-1}\left[\frac{z}{z-a}\right] = a^n$$

(I) $$Z^{-1}\left[\frac{1}{z-a}\right] = a^{n-1}$$

(II) $$Z^{-1}\left[\frac{1}{z+a}\right] = (-a)^{n-1}$$

(III) $$Z^{-1}\left[\frac{z}{z-a}\right] = a^n$$

(IV) $$Z^{-1}\left[\frac{z}{z+a}\right] = (-a)^n$$

(V) $$Z^{-1}\left[\frac{1}{(z-a)^2}\right] = (n+1)a^{n-2}$$

(VI) $$Z^{-1}\left[\frac{1}{(z+a)^2}\right] = (n+1)(-a)^{n-2}$$

(VII) $$Z^{-1}\left[\frac{1}{(z-a)^3}\right] = \frac{1}{2}(n+1)(n+2)a^{n-3}$$

(VIII) $$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = (n+1)a^n$$

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* Convolution

If

$$Z(u_n) = U(z) = \sum_{n=0}^{\infty} u_n z^{-n} = u_0 + u_1/z + u_2/z^2 + ---$$

$$Z(v_n) = V(z) = \sum_{n=0}^{\infty} v_n z^{-n} = v_0 + v_1/z + v_2/z^2 + ---$$

$$Z^{-1}[U(z).V(z)] = \sum_{m=0}^{n} u_m v_{n-m} = u_n * v_n$$

$$U(z).V(z) = \left(u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} ---\right)\left(v_0 + \frac{v_1}{z} + \frac{v_2}{z^2} ---\right)$$

$$= (u_0v_0) + (u_0v_1 + v_0u_1)z^{-1} + (u_0v_2 + u_1v_1 + u_2v_0)z^{-2} + ---$$

$$--- + (u_0v_n + u_1v_{n-1} + --- u_nv_0)z^{-n} + ---$$

$$= \sum_{n=0}^{\infty} (u_0v_n + u_1v_{n-1} + --- u_nv_0)z^{-n}$$

$$= Z(u_0v_n + u_1v_{n-1} + --- u_nv_0)$$

$$Z^{-1}[U(z).V(z)] = u_0v_n + u_1v_{n-1} + --- + u_nv_0$$

$$= \sum_{m=0}^{n} u_m v_{n-m}$$

Q Show that $Z^{-1}\left[\frac{1}{(z-a)^2}\right] = (n+1)a^{n-2}$

Solⁿ- $Z^{-1}\left[\frac{1}{(z-a)^2}\right] = Z^{-1}\left[\frac{1}{z-a} \cdot \frac{1}{z-a}\right]$

$$= \sum_{m=0}^{n} a^{m-1} a^{n-m-1} = \sum_{m=0}^{n} a^{n-2}$$

$$= a^{n-2} \sum_{m=0}^{n} 1$$

$$= a^{n-2}[1 + 1 + --- 1]$$

$$= a^{n-2}(n+1)$$

Q. $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = \frac{a^{n+1} - b^{n+1}}{a-b}$ $a > b$

Solⁿ- $Z^{-1}\left[\frac{z}{z-a} \cdot \frac{z}{z-b}\right] = \sum_{m=0}^{n} a^m b^{n-m}$

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$$= b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m$$

$$= b^n \left[\frac{1-(a/b)^{n+1}}{1-a/b}\right]$$

$$= \frac{b^n}{b^{n+1}} \left[\frac{b^{n+1}-a^{n+1}}{\frac{b-a}{b}}\right]$$

$$= \frac{b^{n+1}-a^{n+1}}{b-a}$$

$$= \frac{a^{n+1}-b^{n+1}}{a-b}$$

Q. Find $Z^{-1}\left[\left(\frac{z}{z-a}\right)^3\right]$ using convolution theorem and hence find $\left(\frac{z}{z-1}\right)^3$.

Sol^n- $Z^{-1}\left[\left(\frac{z}{z-a}\right)^3\right] = Z^{-1}\left[\left(\frac{z}{z-a}\right)^2 \frac{z}{z-a}\right]$

$$Z^{-1}\left[\left(\frac{z}{z-a}\right)^2 \frac{z}{z-a}\right] = \sum_{m=0}^{n} (m+1)a^m a^{n-m}$$

$$= a^n \sum_{m=0}^{n} (m+1)$$

$$= a^n \left[\sum_{m=0}^{n} m + \sum_{m=0}^{n} 1\right]$$

$$= a^n \left[\frac{n(n+1)}{2} + (n+1)\right]$$

$$= a^n (n+1)\left[\frac{n+2}{2}\right]$$

$$Z^{-1}\left[\frac{z}{z-a}\right] = a^n$$

$$Z^{-1}\left[\frac{z}{z-a} \cdot \frac{z}{z-a}\right] = \sum_{m=0}^{n} a^m a^{n-m}$$

$$= \sum_{m=0}^{n} a^n$$

$$= (n+1)a^n$$

Application of z transform

Q. Solve the difference eqⁿ $u_{n+2} + 2u_{n+1} + u_n = n$ —①

$u_0 = u_1 = 0$

using z transform

Solⁿ z transform on both sides

$$Z(u_{n+2}) + 2Z(u_{n+1}) + Z(u_n) = Z(n)$$

$$\Rightarrow z^2\left[U(z) - u_0 - \frac{u_1}{z}\right] + 2z\left[U(z) - u_0\right] + U(z) = \frac{z}{(z-1)^2}$$

$$\cancel{z^2 + 2z)}\ (z^2 + 2z + 1)\,U(z) = \frac{z}{(z-1)^2}$$

$$U(z) = \frac{z}{(z-1)^2(z^2+2z+1)}$$

$$= \frac{z}{(z-1)^2(z+1)^2}$$

$$u_n = Z^{-1}\left[\overbrace{\frac{z}{(z-1)^2}}^{V(z)} \cdot \overbrace{\frac{1}{(z+1)^2}}^{Y(z)}\right]$$

$$= \sum_{m=0}^{n} (m+1)(-a)^{m-2}(n-m)$$

$$= \sum_{m=0}^{n} (m+1)(n-m)(-1)^m \qquad a=1$$

Q. Solve the difference eqⁿ using z transform

$u_{n+2} - 4u_{n+1} + 3u_n = 5^n$

Solⁿ z transform on both sides

$$z^2\left[U(z) - u_0 - \frac{u_1}{z}\right] - 4z\left[U(z) - u_0\right] + 3U(z) = \frac{z}{z-5}$$

$$(z^2 - 4z + 3)\,U(z) - u_0 z^2 - z(u_1 - 4u_0) = \frac{z}{z-5}$$

$$U(z) = \frac{z}{(z-5)(z-3)(z-1)} + \frac{u_0 z^2}{(z-3)(z-1)} + \frac{z(u_1 - 4u_0)}{(z-3)(z-1)}$$

$$u_n = Z^{-1}\left[\frac{z}{(z-3)(z-1)(z-5)}\right] + u_0 Z^{-1}\left[\frac{z^2}{(z-3)(z-1)}\right] + (u_1 - 4u_0) Z^{-1}\left[\frac{z}{(z-1)(z-3)}\right]$$

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Numerical Method

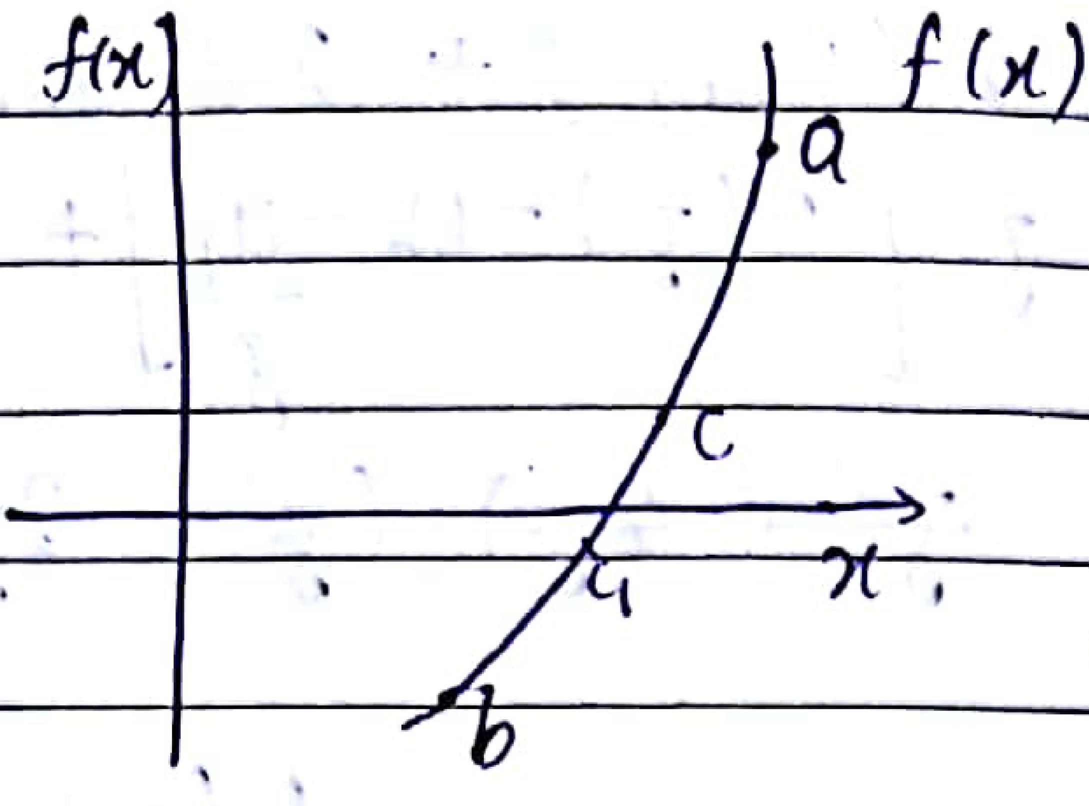
Bisection method for finding the algebraic and transidental method:

$f(x) = 0$

i) a, b

ii) $f(a).f(b) < 0$

iii) $c = \frac{a+b}{2}$, $f(c) > 0, < 0, =$



(1) $f(c) > 0$

$\Downarrow$ b, c

$c_1 = \frac{b+c}{2}$, $f(c_1)$

Q. Find the roots of $x^2 - 5x + 4 = 0$ using bisection method

Sol^n (1) $a = 0$, $b = 2$

$f(a) > 0$, $f(b) < 0$

(II) $f(0).f(2) < 0$

at least one root lies b/w $(0, 2)$

$c = \frac{0+2}{2} = 1$

$f(1) = 0$

Q. Find the roots of $f(x) = x^3 - 4x - 9$ upto decimal to 2nd place.

Sol^n - I $f(2) < 0$ $\quad$ $f(3) > 0$

$f(2) = -9$ $\quad$ $f(3) = 6$

Roots lies b/w 2 and 3

$c = \frac{2+3}{2} = 2.5$

$f(2.5) = -3.375 < 0$

II $(2.5, 3)$

$c_1 = \frac{2.5+3}{2} = 2.75$

$f(2.75) = 0.796 > 0$

III $(2.5, 2.75)$

$c_2 = \frac{2.5 + 2.75}{2} = 2.625$

$f(2.625) = -1.412 < 0$

IV $(2.625, 2.75)$

$c_3 = \cancel{0}\ \frac{2.625 + 2.75}{2} = 2.6875$

$f(2.6875) = -0.339 < 0$

V $(2.6875, 2.75)$

$(2.6875, 2.75)$

$c_4 = \frac{2.6875 + 2.75}{2} = 2.7187$

$f(2.7187) > 0$ $\quad$ (0.22)

VI $(2.6875 \quad 2.7187)$

$c_5 = \frac{2.6875 + 2.7187}{2} = 2.7031$

$f(2.7031) = -0.0615 < 0$

VII $(2.7031, 2.7187)$

$\cancel{f(}$ $c_6 = \frac{2.7031 + 2.7031}{2} = 2.7109$

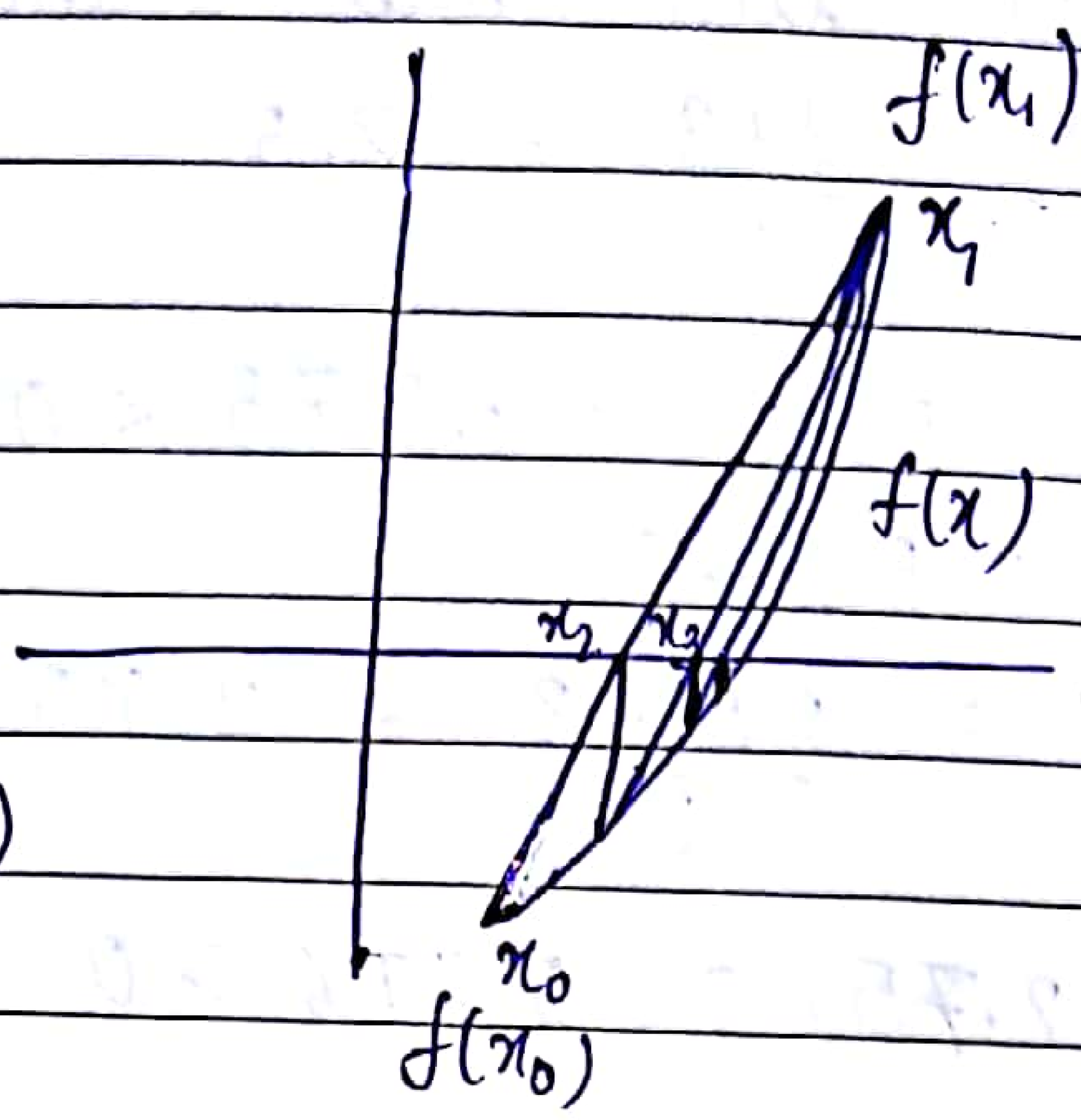
$f(2.7109) = 0.07 > 0$

$(2.7031, 2.7109)$

Regula falshi Method

① x_0, x_1

$f(x_0)\, f(x_1) < 0$



$$\frac{x - x_0}{x_1 - x_0} = \frac{f(x) - f(x_0)}{f(x_1) - f(x_0)}$$

$$x - x_0 = \frac{(x_1 - x_0)(f(x) - f(x_0))}{f(x_1) - f(x_0)}$$

$$x = x_0 + \frac{(x_1 - x_0)(f(x) - f(x_0))}{f(x_1) - f(x_0)}$$

$$x_2 = x_0 + \frac{(x_1 - x_0)(f(x_2) - f(x_0))}{f(x_1) - f(x_0)}$$

$$x_2 = x_0 + \frac{(x_1 - x_0)(-f(x_0))}{f(x_1) - f(x_0)}$$

$$\boxed{x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}}$$

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$$x_4 = \frac{x_3 f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)}$$

Q. Find the solution of $f(x) = x^3 - 2x - 5$.

Soln- $f(2) = -1, \quad f(3) = 16$

$x_0 = 2, \quad x_1 = 3$

$f(2) \cdot f(3) < 0$

$$x_2 = \frac{2f(3) - 3f(2)}{f(3) - f(2)}$$

$$= \frac{2(16) - 3(-1)}{16 + 1}$$

$$= \frac{32 + 3}{17} = \frac{35}{17} = 2.0588$$

$f(2.0588) = -0.3908 < 0$

(II) $x_3 = \frac{(2.0588) \times 16 - 3(-0.3908)}{16 + 0.3908}$

$= 2.0813$

(III) $x_4 = 2.0862$

$x_5 = 2.0915$

$x_6 = 2.0934$

$x_7 = 2.0941$

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Newton Raphson Method

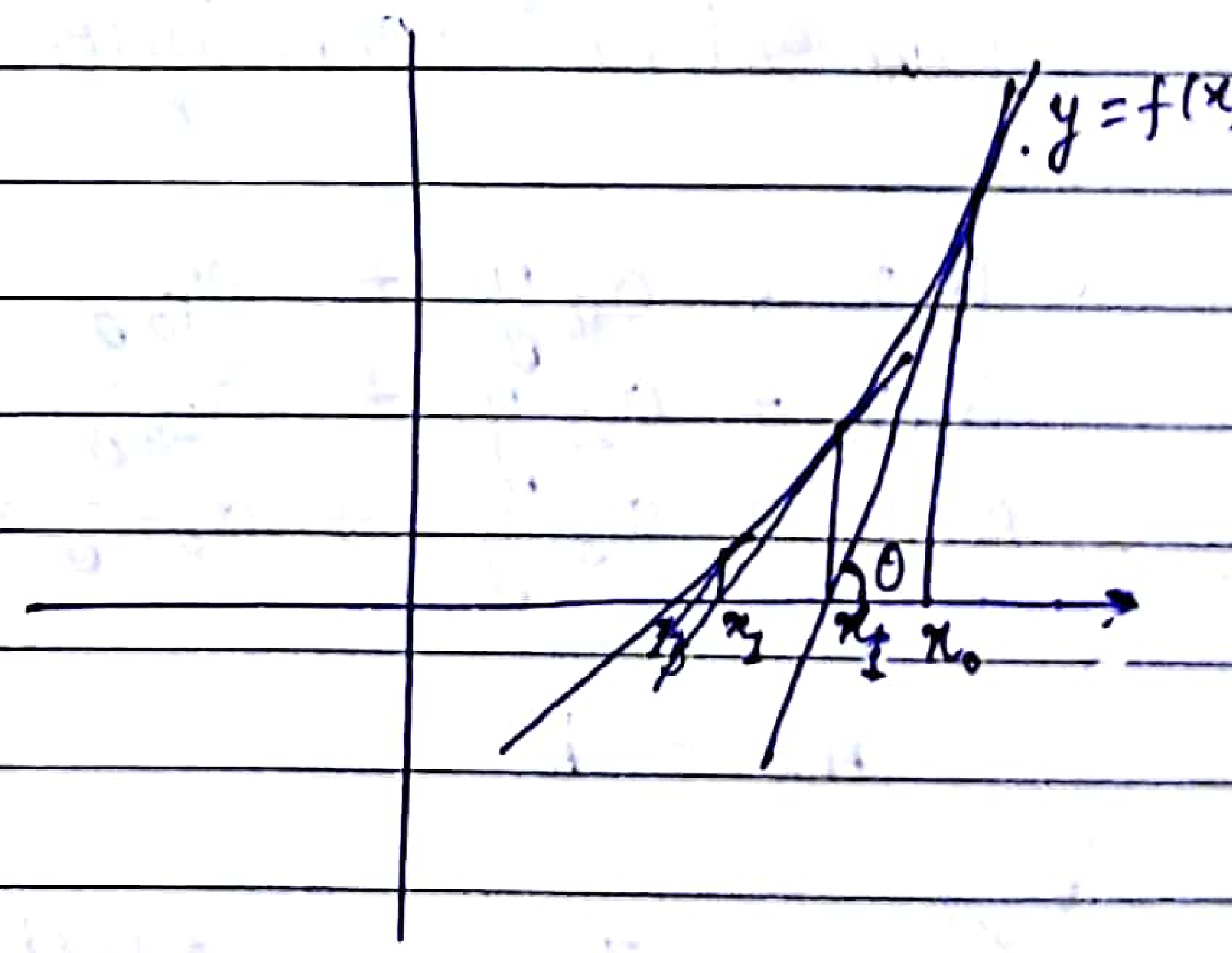
$$\tan\theta = \frac{f(x_0)}{x_0 - x_1} = f'(x_0)$$

$$\Rightarrow \frac{f(x_0)}{f'(x_0)} = x_0 - x_1$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

$$\boxed{x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}}$$

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$



Rate of conversion is quadratic

Fails in case $f'(x_{n-1}) = 0$

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Q. Find the roots of the fn $f(x) = x^3 - 5x + 1$ upto decimal to 2 places.

Solⁿ- Taking $x_0 = 2$ $\qquad f'(x) = 3x^2 - 5$

$$~~x_0 = x~~ \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{(-1)}{7}$$

$$= \frac{14+1}{7} = \frac{15}{7} = 2.1428$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.1428 - \frac{0.124}{8.764} = 2.127$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.127 - \frac{(-0.012)}{8.57}$$

$$= 2.128$$

\# Solution of linear system of equation using numerical technique

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

$$AX = b$$

Gauss Jacobis Method

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

$$x_2 = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} y_1 - \frac{a_{13}}{a_{11}} z_1$$

$$y_2 = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_2 - \frac{a_{23}}{a_{22}} z_1$$

$$z_2 = \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_2 - \frac{a_{32}}{a_{33}} y_2$$

Q. $4x + y + 3z = 17$

$x + 5y + z = 14$

$2x - y + 8z = 12$

Sol^n- Gauss Jacobis

$$x = \frac{17}{4} - \frac{1}{4} y - \frac{3}{4} z$$

$$y = \frac{14}{5} - \frac{1}{5} x - \frac{1}{5} z$$

$$z = \frac{12}{8} - \frac{2x}{8} + \frac{1}{8} y$$

$x = 4.25 - 0.25y - 0.75z$

$y = 2.8 - 0.2x - 0.2z$

$z = 1.5 - 0.25x + 0.125y$

Taking $x_0 = 0$, $y_0 = 0$, $z_0 = 0$.

$x_1 = 4.25$

$y_1 = 2.8$

$z_1 = 1.5$

$x_2 = 4.25 - 0.25 \times 2.8 - 0.75 \times 1.5 = 2.425$

$y_2 = 2.8 - 0.2 \times 4.25 - 0.2 \times 1.5 = 1.65$

$z_2 = 1.5 - 0.25 \times 4.25 + (0.125)(2.8) = 0.7875$

$x_3 = 4.25 - 0.25 \times 1.65 - 0.75 \times \cancel{1.5}\, 0.7875 = 3.2468$

$y_3 = 2.8 - 0.2 \times 2.425 - 0.2 \times 0.7875 = 2.1575$

$z_3 = 1.5 - (0.25) \times 2.242 + 0.125 \times 1.65 = 1.1457$

Finite differences and interpolation

$$x : \quad x_0 \quad x \quad x_0+h \quad x_0+2h \quad x_0+3h \quad x_0+4h \ldots$$
$$f(x) = y : \quad y_0 \quad f(x) \quad y_1 \quad y_2 \quad y_3 \quad y_4$$

$x \downarrow f(x) \downarrow$ interpolation

(i) $\Delta y_n = y_{n+1} - y_n$

$\Delta^2 y_n = \Delta(\Delta y_n) = \Delta(y_{n+1} - y_n) = y_{n+2} - y_{n+1} - y_{n+1} + y_n$

$= y_{n+2} - 2y_{n+1} + y_n$

$\Delta^3 y_n = y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n$

$\Delta^k y_n = {}^kC_0 y_{n+k} - {}^kC_1 y_{n+k-1} + {}^kC_2 y_{n+k-2} + \cdots + {}^kC_k y_n$

$(E-1)^k y_n = {}^kC_0 E^k y_n - {}^kC_1 E^{k-1} y_n + {}^kC_2 E^{k-2} y_n \cdots {}^kC_k E^{k-k} y_n$

$= {}^kC_0 y_{n+k} - {}^kC_1 y_{n+k-1} + \cdots$

(ii) $\nabla y_n = y_n - y_{n-1}$

$\nabla^2 y_n = \nabla(\nabla y_n) = \nabla(y_n - y_{n-1})$

$= y_n - y_{n-1} - y_{n-1} + y_{n-2}$

$= y_n - 2y_{n-1} + y_{n-2}$

Forward difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

Backward difference table:-

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
		∇y_1			
x_1	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		∇y_3		$\nabla^3 y_4$	
x_3	y_3		$\nabla^2 y_4$		
		∇y_4			
x_4	y_4				

Center operator

$$\delta y_n = \left(y_{n+\frac{1}{2}} - y_{n-\frac{1}{2}}\right)$$

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
x_0	y_0				
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			
x_4	y_4				

Average operator

$$\mu y_n = \frac{1}{2}\left(y_{n+\frac{1}{2}} + y_{n-\frac{1}{2}}\right)$$

(I) $\Delta = E - 1$

(II) $\nabla = 1 - E^{-1}$

(III) $\delta = E^{1/2} - E^{-1/2}$

(IV) $\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right]$

(V) $\Delta = E\nabla = \nabla E = \delta E^{1/2}$

(VI) $E = e^{hD}$, $D = \frac{d}{dx}$ $h \rightarrow$ Increment

(I) $\Delta y_n = y_{n+1} - y_n = Ey_n - y_n = (E-1)y_n$

$\Rightarrow \Delta = E - 1$

(II) $\nabla y_n = y_n - y_{n-1} = y_n - E^{-1}y_n$

$= (1 - E^{-1})y_n$

$\Rightarrow \nabla = 1 - E^{-1}$

(III) $\delta y_n = y_{n+1/2} - y_{n-1/2} = E^{1/2}y_n - E^{-1/2}y_n = (E^{1/2} - E^{-1/2})y_n$

$\Rightarrow \delta = (E^{1/2} - E^{-1/2})$

(V) $E\nabla y_n = E(\nabla y_n) = E(y_n - y_{n-1}) = y_{n+1} - y_n = \Delta y_n$

$E\nabla = \Delta$

(VI) $Ef(x) = f(x+h) = f(x) + \frac{h}{1!}f'(x) + \frac{h^2}{2!}f''(x) + \cdots$

$= f(x) + \frac{h}{1!}Df(x) + \frac{h^2}{2!}D^2f(x) + \cdots$

$= \left(1 + \frac{h}{1!}D + \frac{h^2}{2!}D^2 + \cdots\right)f(x)$

$\Rightarrow E = \left(1 + \frac{hD}{1!} + \frac{h^2}{2!}D^2 + \cdots\right)$

$E = e^{hD}$

(i) $hD = \log(1+\Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$

(ii) $(E^{1/2} + E^{-1/2})(1+\Delta)^{1/2} = 2+\Delta$

(iii) $\Delta = \frac{1}{2}\delta^2 + \delta(\sqrt{1+\delta^2/4})$

(iv) $\Delta^3 y_2 = \nabla^3 y_5$

Differences of a Polynomials

$f(x) = ax^n + bx^{n-1} + cx^{n-2} + \cdots + l$

$\Delta f(x) = f(x+h) - f(x)$

$= a(x+h)^n + b(x+h)^{n-1} + \cdots - (ax^n + bx^{n-1} \cdots + l)$

$= a[(x+h)^n - x^n] + b[(x+h)^{n-1} - x^{n-1}] + \cdots$

$= ahnx^{n-1} + (\ )x^{n-2} + (\ )x^{n-3} \cdots$

$\Delta^2 f(x) = ahn(n-1)x^{n-2} + (\ )x^{n-3} + (\ )x^{n-4} \cdots$

$\boxed{\Delta^n f(x) = a\,n!\,h^n}$

Ex: $\Delta^{10}[(1-ax)(1-bx^2)(1-cx^3)(1-dx^4)] = \Delta^{10}[abcd\,x^{10} + (\ )x^9 + (\ )x^8 \cdots]$

Increment of unity

Soln: $\Delta^{10} f(x) = abcd\,10!$

Estimation of missing term in the given table

If n entries are given then the data can be represented by a polynomial f(x) of degree n-1. By the fundamental theorem of difference calculus:

~~$(n-1)\Delta f(x)$~~ = $\boxed{\begin{array}{l}\Delta^{n-1} f(x) = \text{constant} \\ \Delta^n f(x) = 0\end{array}}$

Q. Find the missing term in the given table:

x	y
0	1
1	3
2	9
3	?
4	81

~~$\Delta^4 f(x) = 0$~~

Since 4 entries are given

$\therefore$ the data can be represented by polynomial of degree 3

$\therefore \Delta^4 f(x) = 0$

$(E-1)^4 f(x) = 0$

$[E^4 - 4E^3 + 6E^2 - 4E + 1] f(x) = 0$

$E^4 f(x) - 4E^3 f(x) + 6E^2 f(x) - 4E f(x) + f(x) = 0$

~~Put $x=0$~~ Put $x=0$

$E^4 f(0) - 4E^3 f(0) + 6E^2 f(0) - 4E f(0) + f(0) = 0$

$\Rightarrow f(4) - 4f(3) + 6f(2) - 4f(1) + f(0) = 0$

$81 - 4x + 6 \times 9 - 4 \times 3 + 1 = 0$

$4x = 81 + 54 - 12 + 1$

$x = 31$

24/10/17

Newton Forward Interpolation formula

$x: \quad x_0 \quad x_0+h \quad x_0+2h \quad x_0+3h \quad - - - - \quad x_0+nh = x_n$

$f(x) = y: \quad y_0 \quad y_1 \quad y_2 \quad y_3 \quad - - - - \quad y_n$

$x = x_0 + uh$

$x \rightarrow$ step size

$uh \rightarrow$ any real no.

$f(x) = y_u = f(x_0 + uh) = E^u f(x_0) = E^u y_0 = (1+\Delta)^u y_0$

$\nabla = 1 - E^{-1}$

$E^{-1} = 1 - \nabla \Rightarrow E = (1-\nabla)^{-1}$

$E^{u} = (1-\nabla)^{-u}$

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$$= \left(1 + \frac{u}{1!}\Delta + \frac{u(u-1)}{2!}\Delta^2 + \frac{u(u-1)(u-2)}{3!}\Delta^3 + \dots\right) y_0$$

$$f(x) = y_u = \left(y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots\right.$$

Newton backward Interpolation Formula

$x = x_n + uh \qquad u \rightarrow -ve$

$$f(x) = f(x_n + uh) = E^{u} f(x_n) = E^{u} y_n = (1-\nabla)^{-u} y_n$$

$$= \left(1 + \frac{u\nabla}{1!} + \frac{u(u+1)\nabla^2}{2!} + \frac{u(u+1)(u+2)\nabla^3}{3!} + \dots\right) y_n$$

$$= y_n + \frac{u\nabla y_n}{1!} + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \dots$$

Find the value of $x = 42$

x:	20	30	40	50	60	70	80
y:	21	28	27	32	40	43	45

Sol^n –

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
20	21	$\Delta y_0 = 7$	$\Delta^2 y_0 = -8$	$\Delta^3 y_0 = 14$	$\Delta^4 y_0 = -17$	$\Delta^5 y_0 = 12$	$\Delta^6 y_0 = 5$
30	28	$\Delta y_1 = -1$	$\Delta^2 y_1 = 6$	$\Delta^3 y_1 = -3$	$\Delta^4 y_1 = -5$	$\Delta^5 y_1 = 17$	
40	27	$\Delta y_2 = 5$	$\Delta^2 y_2 = 3$	$\Delta^3 y_2 = -8$	$\Delta^4 y_1 = 12$		
50	32	$\Delta y_3 = 8$	$\Delta^2 y_3 = 5$	$\Delta^3 y_3 = 4$			
60	40	$\Delta y_4 = 3$	$\Delta^2 y_4 = -1$				
70	43	$\Delta y_5 = 2$					
80	45						

$x = 42$

$x_0 = 20, \; h = 10$

$x = x_0 + uh$

$42 = 20 + u \times 10$

$u = 2.2$

$$f(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}$$

$$= 21 + \frac{2.2}{1!} \times 7 + \frac{2.2(2.2-1)}{2}(-8) + \frac{2.2(2.2-1)(2.2-2)}{6}(14)$$

$$\frac{2.2(2.2-1)(2.2-2)(2.2-3)}{24}(-17) + \cdots$$

Q. Find the no. of students who obtained marks b/w 70 to 75

Marks:	30-40	40-50	50-60	60-70	70-80
No. of students:	31	42	51	35	31

Sol^n-

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
40	y_0 31				
		$\nabla y_1 = 42$			
50	y_1 73		$\nabla^2 y_2 = 9$		
		$\nabla y_2 = 51$		$\nabla^3 y_3 = -25$	
60	y_2 124		$\nabla^2 y_3 = -16$		$\nabla^4 y_4 = 37$
		$\nabla y_3 = 35$		$\nabla^3 y_4 = 12$	
70	y_3 159		$\nabla^2 y_4 = -4$		
$x=75$					
		$\nabla y_4 = 31$			
80	y_4 190				

$$f(x) = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \cdots$$

$x = x_n + uh$

$75 = 80 + u \times 10$

$u = \frac{-1}{2}$

$$f(x) = 190 + \left(\frac{-1}{2}\right)(31) + \frac{\left(\frac{-1}{2}\right)\left(\frac{1}{2}\right)}{2} \times (-4) + \frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{3}{2}\right)(12)}{6}$$

$$+ \frac{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)37}{24}$$

$f(75) = 172.81 \simeq 173$

No. of students who obtained marks b/w 70 - 75
= 173 - 159
= 14

Data Interpolation Formula

Sterling's Centre Interpolation Formula

$$f(x) = y_0 + \frac{u}{2!}\frac{(\Delta y_0 + \Delta y_{-1})}{2} + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2}$$
$$+ \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_0-4h	y_{-4}						
		Δy_{-4}	$\Delta^2 y_{-4}$				
x_0-3h	y_{-3}						
		Δy_{-3}		$\Delta^3 y_{-4}$	$\Delta^4 y_{-4}$		
x_0-2h	y_{-2}		$\Delta^2 y_{-3}$			$\Delta^5 y_{-4}$	
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_0-h	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$			
x_0	y_0		$\Delta^2 y_{-1}$			$\Delta^5 y_{-3}$	$\Delta^6 y_{-4}$
		Δy_0		$\Delta^3 y_{-1}$	$\Delta^4 y_{-2}$		
x_0+h	y_1		$\Delta^2 y_0$				$\Delta^6 y_{-3}$
x_0+2h	y_2	Δy_1		$\Delta^3 y_0$	$\Delta^4 y_{-1}$	$\Delta^5 y_{-2}$	
			$\Delta^2 y_1$		$\Delta^4 y_0$	$\Delta^5 y_{-1}$	$\Delta^6 y_{-2}$
x_0+3h	y_3	Δy_2		$\Delta^3 y_1$			
			$\Delta^2 y_2$				
x_0+4h	y_4	Δy_3					

Q. Find the value of ~~f(x) = at~~ f(x) at x = 11

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
2	21.857			
		-0.832		
6	21.025		-0.061	
		-0.893		-0.033
$x_0 =$ 10	20.132		-0.094	0.026
		-0.987		-0.007
14	19.145		-0.101	
		-1.088		
18	18.057			

$x = x_0 + uh$

$11 = 10 + u \times 4$

$u = 0.25$

$$y_{11} = 20.132 + \left(\frac{1}{4}\right)\left(\frac{-0.893 - 0.987}{2}\right) + \frac{\left(\frac{1}{16}\right)}{2}(0.094) +$$

$$\frac{\frac{1}{4}\left(\frac{1}{16} - 1\right)}{6}\left(\frac{-0.033 - 0.007}{2}\right)$$

$$+ \frac{\frac{1}{16}\left(\frac{1}{16} - 1\right)(0.026)}{24}$$

31/10/17

Divided Difference

$x : x_0 \quad x_1 \quad x_2 \quad -----$

$y : y_0 \quad y_1 \quad y_2 \quad -----$

$$\Delta y = [x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_0 - y_1}{x_0 - x_1}$$

$$\Delta^2 y = [x_0, x_1, x_2] = \frac{[x_0, x_1] - [x_1, x_2]}{x_0 - x_2}$$

$$\Delta^3 y = [x_0, x_1, x_2, x_3] = \frac{[x_0, x_1, x_2] - [x_1, x_2, x_3]}{x_0 - x_3}$$

Q. Find the value of $\Delta^3 y$ or $[a,b,c,d]$ if $f(x) = 1/x$

x	$f(x)=y$	Δy	$\Delta^2 y$	$\Delta^3 y$
a	$\frac{1}{a}$			
		$\frac{\frac{1}{a}-\frac{1}{b}}{a-b}=\frac{-1}{ab}$		
b	$\frac{1}{b}$		$\frac{-\frac{1}{ab}-\left(-\frac{1}{bc}\right)}{a-c}=\frac{1}{abc}$	
		$\frac{\frac{1}{b}-\frac{1}{c}}{b-c}=\frac{-1}{bc}$		$\frac{\frac{1}{abc}-\frac{1}{bcd}}{a-d}=\frac{-1}{abcd}$
c	$\frac{1}{c}$		$\frac{-\frac{1}{bc}-\left(-\frac{1}{cd}\right)}{b-d}=\frac{1}{bcd}$	
		$\frac{\frac{1}{c}-\frac{1}{d}}{c-d}=\frac{-1}{cd}$		
d	$\frac{1}{d}$			

Newton's divided differences for unequal space

$f(x) = \; x : x_0, \overset{x}{\wedge} x_1, \text{-----} x_n$

$\qquad y : y_0, y_1, \text{-----} y_n$

$$f(x) = y = y_0 + (x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2)[x_0, x_1, x_2, x_3] \text{-----}$$

$$= y_0 + (x-x_0)\Delta x_0 + (x-x_0)(x-x_1)\Delta^2 x_0 + \text{--}$$

Langrange's interpolation formula for unequal space

$$f(x) = \frac{(x-x_1)(x-x_2)\text{---}(x-x_n)}{(x_0-x_1)(x_0-x_2)\text{----}(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)\text{---}(x-x_n)}{(x_1-x_0)(x_1-x_2)\text{----}(x_1-x_n)} y_1 + \text{----} \frac{(x-x_0)(x-x_1)\text{---}(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\text{---}(x_n-x_{n-1})} y_n$$

Q. Find $f(9)$ from the following table:

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
5	150				
		121			
7	392		24		
		265		1	
11	1452		32		0
		457		1	
13	2366		42		
		709			
17	5202				

Using Newton's divided difference formula

Sol^n- $f(x) = y_0 + (x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2] + (x-x_0)(x-x_1)(x-x_2)[x_0, x_1, x_2, x_3] + (x-x_0)(x-x_1)(x-x_2)(x-x_3)[x_0, x_1, x_2, x_3, \ldots]$

$= 150 + 4 \times 121 + 4 \times 2 \times 24 + 4 \times 2 \times (-2) \times 1$

$= 810$

Q. Find the value of $f(12)$ using Langrange's interpola formula.

	x	$f(x)$
x_0	10	$120 = y_0$
x_1	13	$128 = y_1$
x_2	15	$132 = y_2$
x_3	17	$135 = y_3$

$$f(12) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$x = x_0 + hu$

$dx = h\,du \Rightarrow \frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \cdot \frac{\partial u}{\partial x} \Rightarrow \frac{\partial y}{\partial x} = \frac{1}{h}\frac{\partial y}{\partial u}$

$$= \frac{(-1)(-3)(-5)}{(-3)(-5)(-7)} \times 120 + \frac{(2)(-3)(-5)}{(3)(-2)(-4)} \times 128 + \frac{(2)(-1)(-5)}{(5)(2)(-2)} \times 132$$

$$+ \frac{(2)(-1)(-3)}{(7)(4)(2)} \times 135 =$$

* Differentiation of differences

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + ---$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + ---\right]$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + ---\right]$$

$$y = f(x) = y_0 + (x-x_0)[x_0, x_1] + (x-x_0)(x-x_1)[x_0, x_1, x_2] + ----$$

$y = f(x)$

$$\left(\frac{dy}{dx}\right)_{x=2} = \left(\frac{df(x)}{dx}\right)_{x=2}$$

From the following data find $\left(\frac{dy}{dx}\right)_{x=2}$

	x_0	x_1	x_2
x :	0	2	3
y :	2	-2	-1

Solⁿ:-
$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

$$= \frac{(x-2)(x-3)(2)}{6} + \frac{(x)(x-3)}{-2}(-2) + \frac{x(x-2)}{3}(-1)$$

$$= \frac{(x-2)(x-3)}{3} + \frac{x(x-3) - x(x-2)}{3}$$

$$= \frac{x^2-5x+6+3x^2-9x-x^2+2x}{3}$$

$$y = \frac{3x^2-12x+6}{3}$$

$$= x^2-4x+2$$

$$\frac{dy}{dx} = 2x-4$$

$$\left.\frac{dy}{dx}\right]_{x=2} = 0$$

7/10/17

Maxima / Minima

$x : x_0 \quad x_1 \quad x_2 \; \cdots \; x_n$

$y : y_0 \quad y_1 \quad y_2 \; \cdots \; y_n$

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots = 0$$

$$\frac{dy}{dx} = \frac{1}{h}\frac{dy}{du}$$

$f(u) = 0$

$u = ?$

$x = x_0 + uh$

$$\frac{dy}{dx} = 0 \Rightarrow \frac{1}{h}\frac{dy}{du} = 0$$

$$\frac{dy}{du} = 0$$

Q. x : 0 1 2 3 4 5

y : 22 26 28 29 31 32

Find maxima & minima from the table.

Sol^n:-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	22					
		4				
1	26		-2			
		2		1		
2	28		-1		1	
		1		2		-5
3	29		1		-4	
		2		-2		
4	31		-1			
		1				
5	32					

$$\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{du} = 0$$

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 +$$

$$\frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \frac{u(u-1)(u-2)(u-3)(u-4)}{5!}\Delta^5 y_0$$

$$\frac{dy}{du} = 0$$

$\Rightarrow u = n$ --

$x = x_0 + uh$

$x = 0 + nh$

Numerical Integration

Newton Cotes Quadrature formula

1. ~~Trap~~ Trapizaidal formula [from NCQ] [n=1]
2. Simpson's 1/3 formula [from NCQ] [n=2]
3. Simpson's 3/8 formula [from NCQ] [n=3]

$$I = \int_a^b f(x)\,dx$$

$x_0 \quad x_1 \;----\; x_n \qquad\qquad a = x_0 \quad b = x_n$

$y_0 \quad y_1 \;----\; y_n$

(1) $$\boxed{I = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + y_4 --- y_{n-1})\right]}$$

$$h = \frac{b-a}{n}$$

(2) $$I = \frac{h}{3}\left[(y_0 + y_n) + 2(y_2 + y_4 + y_6 ---) + 4(y_1 + y_3 + y_5 ---)\right]$$

$$h = \frac{b-a}{n}$$

(3) $$I = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + y_8 ---) + 2(y_3 + y_6 + ----)\right]$$

$$h = \frac{b-a}{n}$$

Q. $\int_1^7 (x^3 + 1)\,dx$

Sol^n-

x:	1	2	3	4	5	6	7
y:	2	9	28	65	126	217	344

classmate

Date
Page

Sol^n $I = \frac{1}{2}[(2+344)+2(9+28+65+126+217)] = \cancel{962}$ 618

$I = \frac{1}{3}[(2+344)+2(28+126)\cancel{+344})+4(9+65+217)]$

$= 612.66$

$I = \frac{3}{8}[(2+344)+3(9+28+126+217)+2(65)]$

$= 606$

$\int_1^7 (x^3+1)dx$

$\left[\frac{x^4}{4}+x\right]_1^7$

$\left(\frac{7^4}{4}+7\right) - \left(\frac{1}{4}+1\right)$

$\frac{2429}{4} - 5 = 606$

Numerical Differentiation

Taylor Series Method
Euler's Method
Modified Euler's Method
RK Method of 4th order
Pickard's method

$\cancel{y}f(x) = f(x_0) + \frac{(x-x_0)}{1!}f'(x_0) + \frac{(x-x_0)^2}{2!}f''(x_0) + \cdots$

$y(x) = y_0 + \frac{(x-x_0)}{1!}y_0' + \frac{x-x_0}{2!}y_0'' + \cdots$

~~If~~ $y' = f(x,y)$

$y'_0 = f(x_0, y_0)$

$y'' = f'(x,y)$

$y''_0 = f'(x_0, y_0)$

Q. Find the solⁿ of

$y(0.25)$ & $y(0.5)$ using Taylor series method to solve $y' = x^2 + y^2$ —(I) ~~+y~~ $y(0) = 1$ —(II)

Solⁿ- $y(0.25) = 1 + \frac{(0.25-0)}{1!} \times 1 + \frac{(0.25-0)^2}{2!} \times 2 + \frac{(0.25-0)^3}{3!} \times$

$= 1.25 + 0.0625$

$y' = x^2 + y^2$

$y'' = 2x + 2yy' = 2$

$y''' = 2 + 2y'^2 + 2yy'' = 8$