模态逻辑笔记：从入门到入土

Notes on Modal Logic: *from Zero to Hero*

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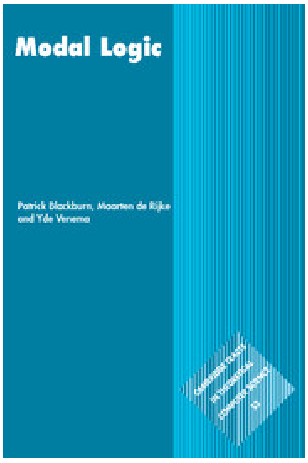
Textbook: the Blue Book (2001)



the Blue Book

Recommended reading: Davey & Priestley, *Introduction to Lattices and Order*, CUP 2nd edition, 2002.

Other references:

1. Blackburn *et al.* *Handbook of modal logic* (2007)
2. van Benthem, *Modal Logic for Open Minds* (2010)
3. The lectures by Yanjing Wang (2023): https://wangyanjing.com/advanced-modal-logic/
4.
5. 文学锋, 模态逻辑教程 (2021)

陈锦盛老师教授的方法论：

文章的一般结构：

Definition....

⋮

Example....

⋮

Proposition...

⋮

Lemma...

⋮

Theorem...

⋮

Corollary...

⋮

中间的内容一般是说明性的或者是过渡，但有时候这些内容反而会影响对概念的理解。

Some slogans of modal logic:

- *Modal languages are simple yet expressive language for talking about relational structures.*
- *Modal languages provide an internal, local perspective on relation structures.*
- *Modal languages are not isolated formal systems.*
- *Bisimulations are to modal logic what partial isomorphisms are to first order logic.* []
- *Truth equals membership.* (canonical model, truth lemma)
- *Orthodox modal logics are the direct modal correspondents of standard algebraic equational axiomatizations for varieties of boolean algebras with operators.*

Table 1: 框架和代数操作

K is a class of frames	
$\mathrm{S}_f\mathsf{K}$	generated subframes
$\mathrm{H}_f\mathsf{K}$	bounded morphic images
$\mathrm{P}_f\mathsf{K}$	disjoint unions
$\mathrm{I}_f\mathsf{K}$	isomorphic copies

K is a class of algebras	
$\mathrm{S}\mathsf{K}$	subalgebras
$\mathrm{H}\mathsf{K}$	homomorphic images
$\mathrm{P}\mathsf{K}$	isomorphic copies of products
$\mathrm{Cm}\mathsf{K}$	the class of all complex algebras

Contents

Chapter 1

Basic Concepts

1.1 Relational structures

定义 1.1 (relational structures)**.** A **relational structure** is a tuple $\mathfrak{F} = (W, R_i)_{i \in I}$, where $W \neq \emptyset$ and $R_i \subseteq W^n$ is a n-ary relation on W for each $i \in I \neq \emptyset$ and $n \in \mathbb{N}$. $\dashv$

Note:

1. R_i can with arbitrary arity.
2. $\mathfrak{F}$ contains at least one relation since $I \neq \emptyset$.

There are many examples for relational structure (W, R):

- *strict partial order*: irreflexive + transitive
- *linear order* (*total order*): irreflexive + transitive + trichotomy
- *partial order*: transitive + reflexive + antisymmetric
- etc.

定义 1.2 (reflexive closure, transitive closure)**.** For any *binary* relation R on a non-empty set W,

- R^+, the **reflexive closure** of R is the smallest transitive relation on W that contains R.
- R^*, the **reflexive transitive closure** of R is the smallest reflexive and transitive relation on W containing R.

$\dashv$

命题 1.3. For any binary relation R on W:

1. $R^+ = \bigcap\{R' \subseteq W \mid R' \text{ is transitive } \& \ R \subseteq R'\}$
2. $R^* = \bigcap\{R' \subseteq W \mid R' \text{ is transitive and reflexive } \& \ R \subseteq R'\}$
3. $R^+uv \ \Leftrightarrow$ there is a sequence $u = w_0, w_1, \ldots, w_n = v$ $(n > 0)$ such that Rw_iw_{i+1} for each $i < n$. (R^+uv means that v is reachable from u in a finite number of R-steps)

4. $R^*uv \Leftrightarrow u = v$ or there is a sequence $u = w_0, w_1, \ldots, w_n = v$ $(n > 0)$ such that Rw_iw_{i+1} for each $i < n$.

$\dashv$

Proof. 内容... ■

1.2 Modal languages

定义 1.4 (Basic modal language)**.** Given a set of *countable* number of propositional variables Prop and an unary modal operator $\Diamond$. The **basic modal language** $\mathcal{L}_\Diamond$ is given by following BNF rule:

$$\mathcal{L}_\Diamond \ni \varphi ::= p \mid \bot \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Diamond\varphi$$

where $p \in \mathsf{Prop}$. $\dashv$

NB: Because the bottom $\bot \notin \mathsf{Prop}$, hence if $\mathsf{Prop} = \emptyset$ then $\mathcal{L}_\Diamond \neq \emptyset$.

定义 1.5 (Modal similarity type)**.** A **modal similarity type** is a pair $\tau = (O, \rho)$ where O is a non-empty set of modal operators and $\rho\colon O \to \mathbb{N}$ assigns to each modal operator a finite *arity*. $\dashv$

定义 1.6 (Modal language under τ)**.** Given a modal similarity type τ and Prop, the **model language** $\mathcal{L}_{(\tau,\mathsf{Prop})}$ is defined by following BNF rule:

$$\mathcal{L}_{(\tau,\mathsf{Prop})} \ni \varphi ::= p \mid \bot \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \triangle(\varphi_1, \ldots, \varphi_{\rho(\triangle)})$$

where $p \in \mathsf{Prop}$ and $\triangle \in \tau$. $\dashv$

Dual operators (*nabla*):

$$\nabla(\varphi_1, \ldots, \varphi_n) \coloneqq \neg\triangle(\neg\varphi_1, \ldots, \neg\varphi_n)$$

注记 1.7.

1. the name of *similarity type* is from *universal algebra*.

2. τ 说明了一个语言的模态词有哪些以及这些模态词的元数.

$\dashv$

定义 1.8 (Substitution)**.** Given a modal language $\mathcal{L}_{(\tau,\mathsf{Prop})}$, a **substitution** is a function $\sigma\colon \mathsf{Prop} \to \mathcal{L}_{(\tau,\mathsf{Prop})}$. We can extend a substitution by $(\cdot)^\sigma\colon \mathcal{L}_{(\tau,\mathsf{Prop})} \to \mathcal{L}_{(\tau,\mathsf{Prop})}$ which recursively given by:

$$\begin{array}{rcl} p^\sigma & = & \sigma(p) \\ \bot^\sigma & = & \bot \\ (\neg\varphi)^\sigma & = & \neg\varphi^\sigma \\ (\varphi \vee \psi) & = & \varphi^\sigma \vee \psi^\sigma \\ (\triangle(\varphi_1, \ldots, \varphi_n))^\sigma & = & \triangle(\varphi_1^\sigma, \ldots, \varphi_n^\sigma) \end{array}$$

Saying that χ is a **substitution instance** of φ if there is some substitution σ such that $\chi = \varphi^\sigma$. $\dashv$

1.3 Models and Frames

When talking about model/frame we often say that, a model/frame for *which language*.

1.3.1 Models and frames for basic language $\mathcal{L}_\Diamond$

定义 1.9 (Models and frames for $\mathcal{L}_\Diamond$)**.** A **frame** for $\mathcal{L}_\Diamond$ is a pair $\mathfrak{F} = (W, R)$ where $W \neq \emptyset$ and $R \subseteq W \times W$.

A **model** for $\mathcal{L}_\Diamond$ is structure $\mathfrak{M} = (W, R, V)$, where (W, R) is a frame and V , called a **valuation**, is a map: $\mathsf{Prop} \to \wp(W)$.

Given a model $\mathfrak{M} = (\mathfrak{F}, V)$, we say that $\mathfrak{M}$ is *based on* $\mathfrak{F}$, and $\mathfrak{F}$ is the frame *underlying* $\mathfrak{M}$. ⊣

注记 1.10. A benefit of the definition of V is that, a model can be viewed as a *first-order structure* (or a relational structure) in a natural way, namely

$$\mathfrak{M} = (W, R, V(p), V(q), V(r), \dots)$$

where $V(p)$ is an unary relation on W, i.e., a *predicate*, so is for $V(q), V(r), \dots$

But there are many other ways to define valuation, maybe not equivalent. ⊣

定义 1.11 (Satisfiability)**.** For any model $\mathfrak{M} = (W, R, V)$ and $w \in W$, a formula φ **satisfied** in $(\mathfrak{M}, w)$, notation $\mathfrak{M}, w \Vdash \varphi$, recursively define as follows:

$$\begin{array}{lll}
\mathfrak{M}, w \Vdash p & :\Leftrightarrow & w \in V(p) \qquad p \in \mathsf{Prop} \\
\mathfrak{M}, w \Vdash \bot & & never \\
\mathfrak{M}, w \Vdash \neg\varphi & :\Leftrightarrow & \mathfrak{M}, w \nVdash \varphi \\
\mathfrak{M}, w \Vdash \varphi \vee \psi & :\Leftrightarrow & \mathfrak{M}, w \Vdash \varphi \textit{ or } \mathfrak{M}, w \Vdash \psi \\
\mathfrak{M}, w \Vdash \Diamond\varphi & :\Leftrightarrow & \exists v \in W, Rwv, \mathfrak{M}, v \Vdash \varphi
\end{array}$$

A formula φ is **satisfiable** if there is a model $\mathfrak{M}$ and some state w in $\mathfrak{M}$ such that $\mathfrak{M}, w \Vdash \varphi$. ⊣

定义 1.12 (Truth set)**.** Given a model $\mathfrak{M} = (W, R, V)$, the **truth set** of φ in $\mathfrak{M}$ is given by:

$$[\![\varphi]\!]_\mathfrak{M} := \{w \in W \mid \mathfrak{M}, w \Vdash \varphi\}$$

⊣

命题 1.13. Given a model $\mathfrak{M} = (W, R, V)$, then

$$[\![p]\!]_\mathfrak{M} = V(p) \qquad [\![\bot]\!]_\mathfrak{M} = \emptyset \qquad [\![\neg\varphi]\!]_\mathfrak{M} = W \setminus [\![\varphi]\!]_\mathfrak{M} \qquad [\![\varphi \vee \psi]\!]_\mathfrak{M} = [\![\varphi]\!]_\mathfrak{M} \cup [\![\psi]\!]_\mathfrak{M}$$

$$[\![\Diamond\varphi]\!]_\mathfrak{M} = \{w \in W \mid \exists v, Rwv, v \in [\![\varphi]\!]_\mathfrak{M}\}$$

$$[\![\Box\varphi]\!]_\mathfrak{M} = \{w \in W \mid \forall v, Rwv \ \Rightarrow \ v \in [\![\varphi]\!]_\mathfrak{M}\}$$

⊣

1.3.2 For more general language

$$\begin{array}{lll}
\mathfrak{M}, w \Vdash \triangle(\varphi_1, \dots, \varphi_n) & :\Leftrightarrow & \exists v_1, \dots, v_n \in W, (w, v_1, \dots, v_n) \in R_\triangle, \forall i \in \{1, 2, \dots, n\}, \mathfrak{M}, v_i \Vdash \varphi_i \\
\mathfrak{M}, w \Vdash \nabla(\varphi_1, \dots, \varphi_n) & :\Leftrightarrow & \forall v_1, \dots, v_n \in W, (w, v_1, \dots, v_n) \in R_\triangle \ \Rightarrow \ \exists i \in \{1, 2, \dots, n\}, \mathfrak{M}, v_i \Vdash \varphi_i \\
\mathfrak{M}, w \Vdash \bigcirc & :\Leftrightarrow & w \in R_\bigcirc
\end{array}$$

where $\bigcirc$ is a *nullary modality*.

注记 1.14. The graded modality $\Diamond^{\geq n}$ is a good example to understood this general definition.

1.3.3 Validity

定义 1.15 (Validity and Logic)**.** There are different validities on different levels.

1. $\mathfrak{F}, w \Vdash \varphi$: $\quad \forall V \in \wp(W)^{\mathsf{Prop}}$[1]$, (\mathfrak{F}, V), w \Vdash \varphi$.
2. $\mathfrak{F} \Vdash \varphi$: $\quad \forall w \in W, (\mathfrak{F}, w) \Vdash \varphi$.
3. $\mathsf{F} \Vdash \varphi$: $\quad \forall \mathfrak{F} \in \mathsf{F}, \mathfrak{F} \Vdash \varphi$.
4. $\Vdash \varphi$: $\quad \forall \mathfrak{F}, \mathfrak{F} \Vdash \varphi$.

The set of all valid formulae in a class of frame F is called the **logic of** F, notation Λ_{F}, that is $\Lambda_{\mathsf{F}} := \{\varphi \mid \mathsf{F} \Vdash \varphi\}$. $\dashv$

1.4 General Frames (skip)

1.5 Modal Consequence Relation

1.5.1 local

定义 1.16 (Local semantic consequence)**.** Let S be a class of models or frames, for any formula φ and set of formulae Σ. We say φ is a **local semantic consequence** of Σ over S, notation $\Sigma \Vdash_{\mathsf{S}} \varphi$, if for all models $\mathfrak{M}$ in S and all states w in $\mathfrak{M}$: $\mathfrak{M}, w \Vdash \Sigma \;\Rightarrow\; \mathfrak{M}, w \Vdash \varphi$. $\dashv$

1.5.2 global

定义 1.17 (Global semantic consequence)**.** Let S be a class of models or frames, for any formula φ and set of formulae Σ. We say φ is a **gocal semantic consequence** of Σ over S, notation $\Sigma \Vdash^g_{\mathsf{S}} \varphi$, if for all structure $\mathfrak{G}$ in S ($\mathfrak{G}$ could be a model or a frame): $\mathfrak{G} \Vdash \Sigma \;\Rightarrow\; \mathfrak{G} \Vdash \varphi$. $\dashv$

1.6 Normal Modal Logics

定义 1.18 (Axiom system **K**)**.** The axiom system **K** is containing following axioms and rules:

- Axioms
  1. **PC**: all propositional tautologies;
  2. K: $\Box(p \to q) \to (\Box p \to \Box q)$ (also known as *distribution axiom*)
  3. Dual: $\Diamond p \leftrightarrow \neg\Box\neg p$
- Rules
  1. MP: $\varphi \to \psi, \varphi \;/\; \psi$
  2. Sub: $\varphi \;/\; \varphi^\sigma$ where σ is a substitution

[1] For any set A, B, $B^A := \{f \mid f\colon A \to B\}$.

3. $\text{Gen}_\Box$: $\varphi \ / \ \Box\varphi$

A **K-proof** is a finite sequence of formulae $\varphi_1, \dots, \varphi_n$, for each φ_i ($1 \leq i \leq n$), either φ_i is an axiom of **K**, or φ_i is obtained by one or more earlier formulae in the sequence by applying a rule of **K**.

If $\varphi_1, \dots, \varphi_n$ is a **K**-proof and $\varphi = \varphi_n$, then we say that φ is **K**-provable, notation $\vdash_{\mathbf{K}} \varphi$, and say φ is a **theorem** of **K**. ⊣

注记 1.19. There are some comments on the three rules:

1. MP:
   (a) MP preserves *validity*: $\Vdash \varphi \to \psi, \Vdash \varphi \ \Rightarrow \ \Vdash \psi$
   (b) MP preserves *satisfiability*: $\mathfrak{M}, w \Vdash \varphi \to \psi, \mathfrak{M}, w \Vdash \varphi \ \Rightarrow \ \mathfrak{M}, w \Vdash \psi$
   (c) MP preserves *global truth*: $\mathfrak{M} \Vdash \varphi \to \psi, \mathfrak{M} \Vdash \varphi \ \Rightarrow \ \mathfrak{M} \Vdash \psi$
2. Sub:
   (a) Sub preserves *validity*: $\Vdash \varphi \ \Rightarrow \ \Vdash \varphi^\sigma$
   (b) Sub not preserve *satisfiability*
   (c) Sub not preserve *global truth*
3. $\text{Gen}_\Box$
   (a) $\text{Gen}_\Box$ preserves *validity*: $\Vdash \varphi \ \Rightarrow \ \Vdash \Box\varphi$
   (b) $\text{Gen}_\Box$ not preserve *satisfiability*
   (c) $\text{Gen}_\Box$ preserves *global truth*: $\mathfrak{M} \Vdash \varphi \ \Rightarrow \ \mathfrak{M} \Vdash \Box\varphi$

In a word:

	preserves *validity*	preserve *satisfiability*	preserves *global truth*
MP	√	√	√
Sub	√	×	×
Gen	√	×	√

Hence (MP) is our best friend that we can trust him in all levels.

定义 1.20 (Normal modal logics)**.** A **normal modal logic** Λ is a set of formulae that contains all tautologies, K-axiom, Dual-axiom and is closed under MP, Sub and $\text{Gen}_\Box$. The smallest normal modal logic is called **K**. ⊣

命题 1.21. Let F be a class of frames, then $\Lambda_{\mathsf{F}} := \{\varphi \mid \mathsf{F} \Vdash \varphi\}$ is a normal modal logic. ⊣

Proof. See exercise: 1.6.7 . ■

S5 was introduced before C.I. Lewis by H. McColl (1906).

定义 1.22 (finitely axiomatization)**.** If $L = \mathbf{K} \oplus \Sigma$ and Σ is finite, the we call L **finitely axiomatizble**. ⊣

定义 1.23 (Kripke completeness)**.** For any syntax logic L, if there is some class of frames $\mathfrak{F}$ such that L is sound and complete w.r.t $\mathfrak{F}$ (L is *characterized* by F), then we call L **Kripke complete**. $\dashv$

Note that: a Kripke complete logic L can be characterized by different classes of frames (we shall see many examples in what follows). If L is Kripke complete then it is clearly determined by the class $\mathsf{Fr}L$ of all frames for L, i.e., $L = \mathsf{LogFr}L$.

$\mathsf{Fr}L := \{\mathfrak{F} \mid \mathfrak{F} \Vdash L\}$

quasi-order = transitive + reflexive. R^* is the smallest quasi-order on W to contain R.

Fr**S5** is the class of all frames with equivalence accessibility relations. But note that **S5** is also determined by the class of all *universal frames* which is a proper subclass of Fr**S5**.

定理 1.24. **GL** is Kripke complete. Fr**GL** is the class of all *Noetherian strict partial orders*. $\dashv$

A binary relation R is called **Noetherian** if there is no infinite strictly ascending chain of points in W.

$GL = K \oplus (4) \oplus$ Lo0b axiom Lo0b axiom:$\Box(\Box p \to p) \to \Box p$

Due to Makinson (1971), is that there are precisely two maximal (with respect to $\subseteq$) *consistent* modal logics

$$Verum = K4 \oplus \Diamond p$$

$$Triv = K4 \oplus \Box p \leftrightarrow p$$

according to Makinson's theorem, at least one of the frames • or o is a frame for every consistent modal logic.

虽然模态公式不能定义反自反的框架类，但一些规则可以，如 irreflexivity rules:

$$\frac{\neg(p \to \Diamond_i p) \to \phi}{\phi}$$

where $p \notin \phi$.(Gabbay 1981a, Marx and Venema 1997).

1.7 Selected exercises for Ch.1

1.1.1

1.1.2

1.1.3

1.3.1(合同引理) Show that when evaluating a formula ϕ in a model, the only relevant information in the valuation is the assignments it makes to the propositional letters actually occurring in ϕ. More precisely, let $\mathfrak{F}$ be a frame, and V and V' be two valuations on $\mathfrak{F}$ such that$V(p) = V'(p)$ for all proposition letters p in ϕ. Show that $(\mathfrak{F}, V) \Vdash \phi$ iff $(\mathfrak{F}, V') \Vdash \phi$. Work in the basic modal language.

Proof. Let $\mathfrak{F} = (W, R)$, V and V' are two valuations as mentioned above, we firstly prove the following lemma by induction on ϕ:

$$(*) \quad \forall w \in W : (\mathfrak{F}, V), w \Vdash \phi \;\Leftrightarrow\; (\mathfrak{F}, V'), w \Vdash \phi.$$

Base case

- If ϕ is a propositional letter p, then for all $w \in W$

$$\begin{aligned}(\mathfrak{F}, V), w \Vdash p \quad &\Leftrightarrow \quad w \in V(p), && \text{(by definition)}\\ &\Leftrightarrow \quad w \in V'(p), && \text{(by assumption)}\\ &\Leftrightarrow \quad (\mathfrak{F}, V'), w \Vdash p. && \text{(by definition)}\end{aligned}$$

- If $\phi = \bot$, then for all $w \in W$, $(\mathfrak{F}, V), w \Vdash \phi \;\Leftrightarrow\; (\mathfrak{F}, V'), w \Vdash \phi$ trivially.

Induction step:
If ϕ is of the form $\neg\psi$ or $\psi \vee \chi$, this is easily done. The crucial case is the form $\Diamond\psi$.

$$\begin{aligned}(\mathfrak{F}, V), w \Vdash \Diamond\psi \quad &\Leftrightarrow \quad \exists v, Rwv, (\mathfrak{F}, V), v \Vdash \psi, && \text{(by definition)}\\ &\Leftrightarrow \quad \exists v, Rwv, (\mathfrak{F}, V'), v \Vdash \psi, && \text{(by induction hypothesis)}\\ &\Leftrightarrow \quad (\mathfrak{F}, V'), w \Vdash \Diamond\psi. && \text{(by definition)}\end{aligned}$$

Then the desired proposition

$$(\mathfrak{F}, V) \Vdash \phi \;\Leftrightarrow\; (\mathfrak{F}, V') \Vdash \phi$$

is just a corollary of $(*)$. ■

1.3.4 Show that every formula that has the form of a propositional tautology is valid. Further, show that $\Box(p \to q) \to (\Box p \to \Box q)$ is valid.

Proof.
(1) (we only work in the basic modal language here)
Firstly, we give a formal definition for what is a formula has the form of a propositional tautology.

> **Definition: Modal tautologies**
>
> ---
>
> A modal formula ϕ is called a *modal tautology* (shouldn't be confused with *proposition tautology*), if $\phi = \alpha^\sigma$ where σ is a substitution, α is a formula of propositional logic and α is a proposition tautology.

In effect, therefore, we have to show that:

$(\circledast)$ Every modal tautology is valid.

Our proof strategy is listed as follows:
(i) Firstly, we choice a propositional calculus PC and show all axioms (or axiom schemes) of PC are modal valid.
(ii) Then, we show MP preserves validity
(iii) Consequently, we know that all theorems of PC are valid since (i) and (ii)
(iv) Therefore all proposition tautologies are valid by the Soundness and Completeness of propositional logic.
(v) Show that substitution (Sub) preserves validity.
(vi) Finally, since all modal tautology can obtained by a proposition tautology and a substitution, then by (iv) and (v), every modal tautology is valid.

We show (i) only here, and the proof of (ii) and (v) can be find in the latter proof of soundness for **K**.

The following propositional calculus is from p28 in A.G. Hamilton, *Logic for mathematicians*, Cambridge University Press 1978.

Propositional Calculus PC (three axiom schemes and one rule)	
$(L1)$	$\varphi \to (\psi \to \varphi)$
$(L2)$	$\varphi \to (\psi \to \chi) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
$(L3)$	$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$
(MP)	$\dfrac{\varphi \to \psi, \varphi}{\psi}$

Then we show those three axiom schemes are modal valid.

- If $(L1)$ is not modal valid,

  then $M, w \nVdash \varphi \to (\psi \to \varphi)$ for some model M and some w in M.

  hence $M, w \Vdash \varphi$ and $M, w \nVdash \psi \to \varphi$.

  But the latter means that $M, w \Vdash \psi$ and $M, w \nVdash \varphi$.

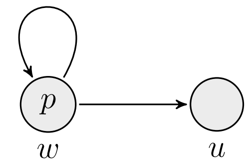
  Contradiction!

- The validity for $(L2)$ and $(L3)$ is similar, we won't repeat it again.

(2)
Following we show that $\Box(p \to q) \to (\Box p \to \Box q)$ is valid.
Take any frame $\mathfrak{F}$ and any state w in $\mathfrak{F}$, and let V be a valuation on $\mathfrak{F}$.
We have to show that if $(\mathfrak{F}, V), w \Vdash \Box(p \to q)$ and $(\mathfrak{F}, V), w \Vdash \Box p$, then $(\mathfrak{F}, V), w \Vdash \Box q$.
So assume that $(\mathfrak{F}, V), w \Vdash \Box(p \to q)$ and $(\mathfrak{F}, V), w \Vdash \Box p$.
Then, by definition for any state v such that Rwv we have $(\mathfrak{F}, V), v \Vdash p \to q$ and $(\mathfrak{F}, V), v \Vdash p$, hence $(\mathfrak{F}, V), v \Vdash q$.
But since Rwv and v is an arbitrary state,
then by definition we have $(\mathfrak{F}, V), w \Vdash \Box q$. ■

1.3.5 Show that every formula of the following formulas is not valid by constructing a frame $\mathfrak{F} = (W, R)$ that refutes it.
(a) $\Box\bot$ (b) $\Diamond p \to \Box p$ (c) $p \to \Box\Diamond p$ (d) $\Diamond\Box p \to \Box\Diamond p$.
Find, for each of these formulas, a non-empty class of frames on which it is valid.

Proof. Let's consider following frame $\mathfrak{F}$, then we show that this frame refutes all above formulas.
Let $\mathfrak{F} = (W, R)$ where $W = \{w, u\}$ and $R = \{(w, w), (w, u)\}$,
we visualize $\mathfrak{F}$ (with a valuation) as follows:



Now we define a valuation V on $\mathfrak{F}$ by

$$V(q) = \begin{cases} \{w\} & q = p \\ \emptyset & q \neq p \end{cases}$$

We use $w \Vdash \varphi$ instead of $(\mathfrak{F}, V), w \Vdash \varphi$ for convenience. Then we know:

$w \Vdash \Diamond p$ since Rww and $w \Vdash p$;

$w \nVdash \Box p$ since Rwv but $u \nVdash p$;

$w \nVdash \Box\Diamond p$ since Rwu but u has no successors, which means $u \nVdash \Diamond p$;

$w \Vdash \Diamond\Box p$ since Rwu and v is a '*dead end*', that is $u \Vdash \Box p$.

Then,

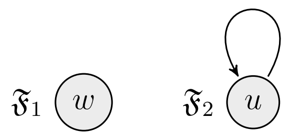
(a) $w \nVdash \Box\bot$ since Rwu but $u \nVdash \bot$;

(b) $w \nVdash \Diamond p \to \Box p$ since $w \Vdash \Diamond p$ but $w \nVdash \Box p$

(c) $w \nVdash p \to \Box\Diamond p$ since $w \Vdash p$ but $w \nVdash \Box\Diamond p$

(d) $w \nVdash \Diamond\Box p \to \Box\Diamond p$ since $w \Vdash \Diamond\Box p$ but $w \nVdash \Box\Diamond p$

Considering two classes of frames F_1 and F_2, where $\mathsf{F}_1 = \{\mathfrak{F}_1\}$ and $\mathsf{F}_2 = \{\mathfrak{F}_2\}$,



It is easy to check that,

(a) is valid in F_1; (b), (c) and (d) is valid in F_2. ■

1.6.7 Let F be a class of frames. Show that Λ_{F} is a normal modal logic.

Proof. Because all tautologies is valid on any frame, so is for the axioms K and Dual, then we only need to show that Λ_{F} is closed under MP, Sub and Nec.

(1) MP: if $\phi, \phi \to \psi \in \Lambda_{\mathsf{F}}$, then take any model $\mathfrak{M}$ from F and any state w in $\mathfrak{M}$ we have $\mathfrak{M}, w \models \phi$ and $\mathfrak{M}, w \models \phi \to \psi$, hence $\mathfrak{M}, w \models \psi$, because $\mathfrak{M}$ and w are arbitrary from F, then ψ is valid on F, that is $\psi \in \Lambda_{\mathsf{F}}$.

★ (2) Sub: we need a lemma here:

lemma: Suppose $M = (W, R, V)$ is a model, and $\phi^\sigma = \phi[\psi_1/p_1, \cdots, \psi_n/p_n]$ is the substitution instance of ϕ under substitution σ. Define $M' = (W, R, V')$ by $V'(p_i) = \{w \in W \mid M, w \Vdash \psi_i\}$. Then for any $w \in W$:

$$M, w \Vdash \phi^\sigma \;\Leftrightarrow\; M', w \Vdash \phi.$$

Assume $\phi \in \Lambda_{\mathsf{F}}$, that is, $\mathsf{F} \Vdash \phi$, but $\phi^\theta \notin \Lambda_{\mathsf{F}}$ for some substitution θ, i.e $\mathsf{F} \nVdash \phi^\theta$. Then for some model $M = (W, R, V)$ from F and some $w \in W$ we have $M, w \nVdash \phi^\theta$, hence $M', w \nVdash \phi$ by above lemma, but this is contradicts to ϕ is valid in F. Therefore, if $\phi \in \Lambda_{\mathsf{F}}$ then $\phi^\theta \in \Lambda_{\mathsf{F}}$ for any substitution θ.

(3) Nec: suppose $\phi \in \Lambda_{\mathsf{F}}$ but $\Box\phi \notin \Lambda_{\mathsf{F}}$, then there are a frame $F = (W, R)$ from F, a valuation V and a state $w \in W$ such that $(F, V), w \nVdash \Box\phi$. Hence there must be a state $u \in W$ for which Rwu and $(F, V), u \Vdash \neg\phi$, but this contradicts with ϕ is valid on F. Therefore $\Box\phi \in \Lambda_{\mathsf{F}}$ ■

Show that **K** *is sound with respect to the class of all frames*.

Proof. We already known that:

(1) <u>All axioms of **K** are valid</u>.

(all tautologies are valid and the K-axiom is valid (see exercise 1.3.4, p27), moreover the Dual-axiom is valid (see the discussion in paragraph 5 of p34))

(2) Furthermore, we assume that <u>all rules of **K** are preserve validity</u>, we will give a proof in the last.

Then to show **K** is *sound*, it is sufficient to show that all **K**-provable formulas are valid.
Suppose φ is **K**-provable for any formula φ.
By induction on n, the length of proof for φ.
Base case:

- If $n = 1$, then by the definition of **K**-proof, that means φ is an axiom of **K**,
  but all axioms of **K** are valid,
  hence φ is valid.

Induction step: Suppose φ has a proof of length $n > 1$.

- If φ is an axiom of **K**, then φ is valid as same as base case.

- If φ is obtained by MP from previous formulas $\chi \to \varphi$ and χ,
  by induction hypothesis, $\chi \to \varphi$ and χ are valid,
  and MP preserves validity,
  hence φ is valid.

- If φ is obtained by Sub or Gen$_\Box$ from χ,
  by inductive hypothesis, χ is valid,
  moreover Sub and Gen$_\Box$ both preserve validity,
  therefore φ is valid.

In the end, we will show that *modus ponens* (MP), *uniform substitution* (Sub) and *Generalization* (Gen$_\Box$) are preserve validity.

(a) For MP.

That is to show: if $\phi \to \psi$ and ψ are valid, then so is ψ.

Suppose $\Vdash \phi, \Vdash \phi \to \psi$,
Then $M, w \models \phi$ and $M, w \models \phi \to \psi$ for some model M and some w in M.
Hence $M, w \models \psi$ by the definition.

Therefore $\Vdash \psi$ because M and w are arbitrary.

(b) For Gen$_\Box$.

That is to show: if ϕ is valid, then so is $\Box\phi$.

Assume $\Vdash \phi$. To show $\Vdash \Box\phi$, let $M = (W, R, V)$ be any model and $w \in W$.

For any $u \in W$, if Rwu then $M, u \Vdash \phi$ since ϕ is valid, and hence $M, u \Vdash \Box\phi$ by the definition.

Since M and w are arbitrary, then $\Vdash \Box\varphi$.

(c) For Sub.

That is to show: if ϕ is valid, then so is ϕ^σ for any substitution σ.

First we need a lemma:

Lemma 2: Suppose ϕ only contains $p_1, \ldots, p_n$ as its propositional letters, and ϕ^σ is the substitution instance of ϕ under substitution σ, where $\sigma(p_i) = \psi_i$ for each $1 \leq i \leq n$.

For any models $M = (W, R, V)$, define $M' = (W, R, V')$ by $V'(p_i) = \{w \in W \mid M, w \Vdash \psi_i\}$. Then for any $w \in W$: $M, w \Vdash \phi^\sigma \;\Leftrightarrow\; M', w \Vdash \phi$.

Proof for this Lemma 2

By induction on ϕ.

Base case:

· If $\phi = p$, then $p_i^\sigma = \psi_i$.
Hence $M, w \Vdash \psi_i \;\Leftrightarrow\; M', w \Vdash p_i$ by the definition of V'.

· If $\phi = \bot$, then $\bot^\sigma = \bot$.
Both $M, w \nVdash \bot$ and $M', w \nVdash \bot$.

Induction step

· If ϕ is of the form $\neg\psi$ or $\psi \vee \chi$, this is easily done. The more crucial case is the form $\Diamond\psi$.

· if $\phi = \Diamond\psi$, then

$$\begin{aligned}
M, w \Vdash (\Diamond\psi)^\sigma \quad &\Leftrightarrow \quad M, w \Vdash \Diamond\psi^\sigma \\
&\Leftrightarrow \quad M, u \Vdash \psi^\sigma \qquad \text{for some } u \text{ such that } Rwu \\
&\Leftrightarrow \quad M', u \Vdash \psi \qquad \text{by inductive hypothesis} \\
&\Leftrightarrow \quad M', w \Vdash \Diamond\psi \qquad \text{since } Rwu
\end{aligned}$$

Therefore we complete the induction proof of above lemma.

Then, assume ϕ is valid,
but ϕ^σ is invalid for some substitution σ, such that $\sigma(p_i) = \psi_i$.
Hence $M, w \nVdash \phi^\sigma$ for some model $M = (W, R, V)$ and some $w \in W$ since ϕ^σ is invalid,
hence we have $M', w \nVdash \phi$ by above **lemma 2**,
but this contradicts with that ϕ is valid.
Therefore, if ϕ is valid, then so is ϕ^σ for any substitution σ. ■

Explains what the normal modal logic K is, and what does it mean to call K *sound* and *complete*.

Answer:

What is K?

K is known as the smallest normal modal logic, it means K is a kind of logic. But what is logic? In mathematics, a logic is regarded as a set of formulas, and a formula is just a element of a language, hence we start by looking at what the language is, or more precisely, what the modal language is.

A modal language consist of some materials. these materials are called *signature*, which includes :

- a countable set of propositional variables: Prop;
- three boolean conectives: $\bot, \neg, \vee$;
- a modal operator: $\Diamond$;
- finally, two guys who are often neglected: (and).

Modal language is a palace built of these materials, mathematically defined as (by BNF):

$$\mathcal{L} \ni \varphi ::= \bot \mid p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Diamond\varphi.$$

where $p \in \mathsf{Prop}$. We often need some abbreviations, such as $\Box\varphi := \neg\Diamond\neg\varphi$, $(\varphi \wedge \psi) := \neg(\neg\varphi \vee \neg\psi)$, etc.

Note that $\bot$ as a primitive symbol here has an additional purpose, that is, when $\mathsf{Prop} = \emptyset$, the existence of $\bot$ ensures that our modal language is not empty.

Let's go back to K, as mentioned above K is just a set of formulas of $\mathcal{L}$, but the price to pay for **K** to be a logic is that it must satisfy some conditions, so that it does not appear to be a pack of nonsense.

These conditions have two, (1) it must contain some formulas called *axioms*, and (2) it must be closed under some *rules*. We list the axioms and rules as follows:

axioms and rules		
axioms		
	PC	all propositional tautologies
	K	$\Box(p \to q) \to (\Box p \to \Box q)$
	Dual	$\Diamond p \leftrightarrow \neg\Box\neg p$
rules		
	Modus ponens (MP)	given $\varphi \to \psi$ and φ, prove ψ.
	Substitution (Sub)	given φ, prove φ^σ, where σ is a substitution function.
	Generalization (Gen)	given φ, prove $\Box\varphi$.

In fact, if a set of formulas contains all above axioms and is closed under all these rules, then we celled this set is a *normal modal logic*. In this case, K is the smallest normal modal logic.

Soundness and Completeness

We call a formula is *valid*, notation $\Vdash \varphi$, if it is true in any state of any model. Let $\mathbf{L} := \{\varphi \mid \Vdash \varphi\}$, that is $\mathbf{L}$ is the set of all valid formulas.

Since **K** is just a set of formulas, and intuitively, all axioms of **K** are valid and all its rules preserve validity. Hence we want to know the relationship between **K** and **L**.

If $\mathbf{K} \subseteq \mathbf{L}$, then we call $\mathbf{K}$ is *sound*.
If $\mathbf{K} \supseteq \mathbf{L}$, then we call $\mathbf{K}$ is *complete*.

There is another way to describe soundness and completeness.
Say φ is a *theorem* of $\mathbf{K}$, notation $\vdash_{\mathbf{K}} \varphi$, if there is a finite sequence of formulas $\psi_1, \ldots, \psi_n$ such that:

- $\psi_n = \varphi$;
- for all ψ_k $(1 \leq k \leq n)$,
  - ψ_k is an axiom of $\mathbf{K}$; or
  - ψ_k is follows from $\psi_1, \ldots, \psi_{k-1}$ by applying a rule of $\mathbf{K}$.

In this case, for any formula φ:
$\mathbf{K}$ is *sound*, if $\vdash_{\mathbf{K}} \varphi$ implies $\Vdash \varphi$;
$\mathbf{K}$ is *complete* if $\Vdash \varphi$ implies $\vdash_{\mathbf{K}} \varphi$.

Chapter 2

Modal model theory

2.1 Three kinds of model constructions

2.1.1 Disjoint unions

2.1.2 Generated submodels

定义 2.1 (subframes, generated subframes)**.** A frame $\mathfrak{F}' = (W', R')$ is a **subframe** of $\mathfrak{F} = (W, R)$, if $W' \subseteq W$ and $R' = R \cap (W' \times W)$ (that is R' is the restriction of R to W').

A subframe $\mathfrak{G} = (W', R')$ of $\mathfrak{F} = (W, R)$ is called a **generated subframe** of $\mathfrak{F}$, if W' is *upward closed* in $\mathfrak{F}$ ($\forall x \in W' \forall y \in W : xRy \Rightarrow y \in W'$). ⊣

定义 2.2 (rooted frame/model)**.** A frame $\mathfrak{F} = W, R$ is called **rooted** if there is a $w_0 \in W$ such that $W = \{w \mid w_0 R^* w\}$. A models $\mathfrak{M} = (\mathfrak{F}, R, V)$ is *rooted* if $\mathfrak{F}$ is rooted.

Such w_0 is called a **root** of $\mathfrak{F}$ ($\mathfrak{M}$). Maybe there are many roots in a frame or model (considering a symmetry frame). ⊣

注记 2.3.

- 不交并的逆是生成子模型的特例，即对所有 $i \in I, \mathfrak{M}_i \rightarrowtail \biguplus_{i \in I} \mathfrak{M}_i$
-

2.1.3 Bounded morphisms (P-morphisms)

定义 2.4 (Bounded morphisms)**.** Let $\mathfrak{M}_1 = (W_1, R_1, V_1)$ and $\mathfrak{M}_2 = (W_2, R_2, V_2)$ be two modal models. A function $f : W_1 \to W_2$ is a **bounded morphism** from $\mathfrak{M}_1$ to $\mathfrak{M}_2$, if f satisfies:

(1) for any propositional variable p: $\mathfrak{M}_1, w_1 \Vdash p \Leftrightarrow \mathfrak{M}_2, f(w_1) \Vdash p$;
($w_1 \in V_1(p) \Leftrightarrow f(w_1) \in V_2(p)$, in other words)

(2) if $(w_1, u_1) \in R_1$ then $(f(w_1), f(u_1)) \in R_2$;

(3) if $(w_2, u_2) \in R_2$ and $\exists w_1 \in W_1$ such that $f(w_1) = w_2$, then $\exists u_1 \in W_1$ such that $(w_1, u_1) \in R_1$ and $f(u_1) = u_2$.

If there is a *surjective* (onto) bounded morphism from $\mathfrak{M}_1$ to $\mathfrak{M}_2$, then we call $\mathfrak{M}_2$ is a **bounded morphic image** of $\mathfrak{M}_1$, notation $\mathfrak{M}_1 \twoheadrightarrow \mathfrak{M}_2$. ⊣

注记 2.5.

- the clauses (1),(2) ensures that a bounded morphism is a *homomorphism*.
- bounded morphisms is also called *p-morphisms* or *zigzag morphisms* (due to van Benthem).
- 之所以会要求一个受限射是**满射**，一个重要的原因是，只有是满射的情况下，在框架层次，受限射保持有效性，即

  if f is a *surjective* bounded morphism from $\mathfrak{F}$ to $\mathfrak{G}$, then $\mathfrak{F} \Vdash \varphi \;\Rightarrow\; \mathfrak{G} \Vdash \varphi$.

  cf. p **【】 ??**

$\dashv$

命题 2.6 (modal invariance under bounded morphisms)**.** Let $\mathfrak{M}_1 = (W_1, R_1, V_1)$ and $\mathfrak{M}_2 = (W_2, R_2, V_2)$ be two modal models. If f is a bounded morphism from $\mathfrak{M}_1$ to $\mathfrak{M}_2$, then for any $w_1 \in W_1$ and for any formula φ, we have :

$$\mathfrak{M}_1, w_1 \Vdash \varphi \;\Leftrightarrow\; \mathfrak{M}_2, f(w_1) \Vdash \varphi.$$

That is, *modal satisfaction is invariant under bounded morphisms*. $\dashv$

Proof. Let $\mathfrak{M}_1, \mathfrak{M}_2$ and f be as mentioned above. By induction on φ.

Base case:

If $\varphi = p$, the by clause (1) of the definition of bounded morphism, the proposition is deserved.

If $\varphi = \bot$, both $\mathfrak{M}_1, w_1 \nVdash \bot$ and $\mathfrak{M}_2, f(w_1) \nVdash \bot$.

Induction step:

If $\varphi = \neg\psi$, then

$$\begin{aligned} \mathfrak{M}_1, w_1 \Vdash \neg\psi &\Leftrightarrow \mathfrak{M}_1, w_1 \nVdash \psi \\ &\Leftrightarrow \mathfrak{M}_2, f(w_1) \nVdash \psi \qquad \text{(induction hypothesis)} \\ &\Leftrightarrow \mathfrak{M}_2, f(w_1) \Vdash \neg\psi. \end{aligned}$$

If $\varphi = \psi \vee \chi$, then

$$\begin{aligned} \mathfrak{M}_1, w_1 \Vdash \psi \vee \chi &\Leftrightarrow \mathfrak{M}_1, w_1 \Vdash \psi \text{ or } \mathfrak{M}_1, w_1 \Vdash \chi \\ &\Leftrightarrow \mathfrak{M}_2, f(w_1) \Vdash \psi \text{ or } \mathfrak{M}_2, f(w_1) \Vdash \chi \qquad \text{(induction hypothesis)} \\ &\Leftrightarrow \mathfrak{M}_2, f(w_1) \Vdash \psi \vee \chi. \end{aligned}$$

If $\varphi = \Diamond\psi$, then

- if $\mathfrak{M}_1, w_1 \Vdash \Diamond\varphi$, then $\exists u_1 \in W_1, (w_1, u_1) \in R_1$ and $\mathfrak{M}_1, u_1 \Vdash \psi$.
  By induction hypothesis, $\mathfrak{M}_2, f(u_1) \Vdash \psi$.
  By the clause (2) of the definition of f, $(f(w_1), f(u_1)) \in R_2$,
  hence $\mathfrak{M}_2, f(w_1) \Vdash \Diamond\psi$.
- if $\mathfrak{M}_2, f(w_1) \Vdash \Diamond\psi$,
  then $\exists u_2 \in W_2, (f(w_1), u_2) \in R_2$ and $\mathfrak{M}_2, u_2 \Vdash \psi$.
  By the clause (3) of the definition of f, $\exists u_1 \in W_1$ such that $(w_1, u_1) \in R_1$ and $f(u_1) = u_2$.
  By induction hypothesis, $\mathfrak{M}_1, u_1 \Vdash \psi$ since $u_2 = f(u_1)$.
  Therefore $\mathfrak{M}_1, w_1 \Vdash \Diamond\psi$. ∎

Tree model property

Following is a application of bounded morphism. We will show that if a formula is satisfiable, the it satisfied by a tree-like model. The strategy is that:

1. Suppose φ is satisfiable, that is $\mathfrak{M}, w \Vdash \varphi$ for some model $\mathfrak{M}$ and state w;
2. Let $\mathfrak{M}'$ be the *submodel generated* by w, by invariance, $\mathfrak{M}', w \Vdash \varphi$;
3. From $\mathfrak{M}'$ (a *rooted*-model) to generate a tree-like model $\mathfrak{T}$.
4. Use bounded morphism show that the tree-construction preserves modal satisfaction. ($\mathfrak{T} \twoheadrightarrow \mathfrak{M}'$)
5. Then φ is satisfied in the tree-like model $\mathfrak{T}$.

The key steps are (3) and (4).

命题 2.7 (Tree model property)**.** For any *rooted-model* $\mathfrak{M} = (W, R, V)$, there is a *tree-like model* $\mathfrak{T}$ such that $\mathfrak{T} \twoheadrightarrow \mathfrak{M}$, that is, there is a *surjective bounded morphism* f from $\mathfrak{T}$ to $\mathfrak{M}$. $\dashv$

Proof. Let w be the root of $\mathfrak{M}$. Define $\mathfrak{T} = (W', R', V')$ as follows (the **unraveling** of $\mathfrak{M}$).

1. $W' := \{(w, u_1, \dots, u_n) \mid$ there is a path $wRu_1R\cdots Ru_n$ in $\mathfrak{M},\ n \geq 0\}$[2]
2. $(w, u_1, \dots, u_n)R'\bar{x} \ \Leftrightarrow\ \exists v \in W, Ru_nv$ and $\bar{x} = (w, u_1, \dots, u_n, v)$
3. $(w, u_1, \dots, u_n) \in V'(p) \ \Leftrightarrow\ u_n \in V(p)$

Define a function $f\colon W' \to W$ (use $f(w, u_1, \dots, u_n)$ instead of $f((w, u_1, \dots, u_n))$ for convenience) by

$$f(w, u_1, \dots, u_n) := u_n.$$

Following we show f is bounded morphism and surjective.

For bounded morphism:

- By the definition of V', that $(w, u_1, \dots, u_n) \in V'(p)$ iff $f(w, u_1, \dots, u_n) = u_n \in V(p)$;
- We have to show that if $(w, u_1, \dots, u_n)R'(w, v_1, \dots, v_m)$, then $f(w, u_1, \dots, u_n)Rf(w, v_1, \dots, v_m)$.
  Suppose $(w, u_1, \dots, u_n)R'(w, v_1, \dots, v_m)$,
  By the definition of R',
  we have Ru_nv_m,
  moreover, $f(w, u_1, \dots, u_n) = u_n, f(w, v_1, \dots, v_m) = v_m$ by the definition of f.
  Hence $f(w, u_1, \dots, u_n)Rf(w, v_1, \dots, v_m)$.
- We have to show that if Ru_nv_m and $\exists(w, u_1, \dots, u_n) \in W'$ such that $f(w, u_1, \dots, u_n) = u_n$, then $\exists(w, v_1, \dots, v_m) \in W'$ such that $(w, u_1, \dots, u_n)R'(w, v_1, \dots, v_m)$ and $f(w, v_1, \dots, v_m) = v_m$.

  Assume Ru_nv_m and $\exists(w, u_1, \dots, u_n) \in W'$ such that $f(w, u_1, \dots, u_n) = u_n$,
  then by the definition, there is a path $(w, u_1, \dots, u_n, v_m)$ in $\mathfrak{M}$.
  Hence $(w, u_1, \dots, u_n, v_m) \in W'$. By the definition of R' and f, we have
  $(w, u_1, \dots, u_n)R'(w, u_1, \dots, u_n, v_m)$ and $f(w, u_1, \dots, u_n, v_m) = v_m$.

[2] or $W' := \{(w, u_1, \dots, u_n) \mid (w, u_i) \in R^*, 0 \leq i \leq n, n \geq 0\}$ where R^* is the transitive and reflexive closure of R.

For subjective:

we have to show that for all $u \in W$, there is $(w, u_1, \ldots, u_n) \in W'$ such that $f(w, u_1, \ldots, u_n) = u$.

Let u be any state in $\mathfrak{M}$, since $\mathfrak{M}$ is *rooted*,

which means that there is a path from root w to u in $\mathfrak{M}$.

Suppose this path is $(w, u_1, \ldots, u_n)$ where $u_n = u$,

then $(w, u_1, \ldots, u_n) \in W'$,

hence $f(w, u_1, \ldots, u_n) = u_n = u$. ■

Now suppose φ is satisfiable, that is $\mathfrak{M}, w \Vdash \varphi$ for some model $\mathfrak{M}$ and state w in $\mathfrak{M}$. Let $\mathfrak{M}'$ be the submodel generated by w, then $\mathfrak{M}', w \Vdash \varphi$ and $\mathfrak{M}'$ is a *rooted* model. Moreover we can form a tree-like model $\mathfrak{T}$ as just above. Therefore, *any satisfiable formula is satisfiable in a tree-like model.*

Following figure is an example for unraveling which from Gabbay et al. *Many-dimensional Modal Logics*, 2003, p23.

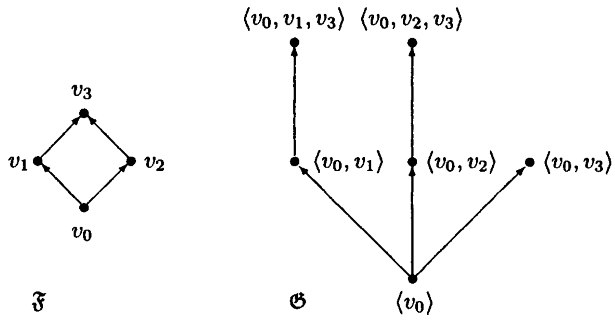


Figure 2.1: an example of unraveling

The frame $\mathfrak{T}$ is called the **unraveling** of $\mathfrak{F}$, two properties of $\mathfrak{T}$ make the unraveling construction important in modal logic.

1. $f\colon \langle w_0, w_1, \ldots, w_n\rangle \mapsto w_n$ is a surjective bounded morphism (as we already mentioned above)

2. $\mathfrak{T}$ has a rather special form known as an *intransitive tree*.

An intransitive frame is clearly *irreflexive*.

An immediate consequence of this is that **K** is characterized by the class of intransitive trees.

命题 2.8. If ϕ is satisfiable in a frame, then it is also satisfiable in a *finite* intransitive tree of $depth \leq md(\phi)$. $\dashv$

2.2 Bisimulation

Slogan: bisimulations are to modal logic what partial isomorphisms are to first order logic.

定义 2.9 (Bisimulation)**.** Given two model $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$.

A non-empty binary relation $Z \subseteq W \times W'$ is called a **bisimulation** between $\mathfrak{M}$ and $\mathfrak{M}'$, notation $Z : \mathfrak{M}' \leftrightarroweq \mathfrak{M}'$ (or $Z : \mathfrak{M} \rightleftarrows \mathfrak{M}'$), if the following conditions are satisfied:

- *atom condition*: $wZw' \;\Rightarrow\; w \Vdash p \;\Leftrightarrow\; w' \Vdash p$ for all $p \in \mathsf{Prop}$;

- *zig* (*forth condition*): wZw' and $Rwu \;\Rightarrow\; \exists u' \in W'$ s.t. $R'w'u'$ and uZu';

- *zag* (*back condition*): wZw' and $R'w'u' \Rightarrow \exists u \in W$ s.t. Rwu and uZu';

If $(w, w') \in Z$, then we say w and w' are **bisimilar**, notation $w \leftrightarroweq w'$.
If there is a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$, we write $\mathfrak{M} \leftrightarroweq \mathfrak{M}'$. ⊣

注记 2.10.

1. bisimulation v.s bisimilar
2. bisimulation is coinductive definition.
3. bisimulation is a relation, whereas bounded morphism is a function.
4. the *empty relation* $\emptyset$ is a bisimulation (vacuously)

⊣

Disjoint unions, *generated submodels*, *isomorphisms*, and *bounded morphisms*, are all bisimulations:

命题 2.11. Let τ be a modal similarity type, and let $\mathfrak{M}$, $\mathfrak{M}'$ and $\mathfrak{M}_i$ $(i \in I)$ be τ-models. ⊣

(i) If $\mathfrak{M} \cong \mathfrak{M}'$, then $\mathfrak{M} \leftrightarroweq \mathfrak{M}'$.

(ii) For every $i \in I$ and every w in $\mathfrak{M}_i$, $\mathfrak{M}_i, w \leftrightarroweq \biguplus_i \mathfrak{M}_i, w$.

(iii) If $\mathfrak{M}' \rightarrowtail \mathfrak{M}$, then $\mathfrak{M}', w \leftrightarroweq \mathfrak{M}, w$ for all w in $\mathfrak{M}'$.

(iv) If $f : \mathfrak{M} \twoheadrightarrow \mathfrak{M}'$, then $\mathfrak{M}, w \leftrightarroweq \mathfrak{M}', f(w)$ for all w in $\mathfrak{M}$.

Proof. See here. ■

定理 2.12 (Invariant under bisimulation)**.** Modal formulas are invariant under bisimulation. That is
$\mathfrak{M}, w \leftrightarroweq \mathfrak{M}', w' \Rightarrow \mathfrak{M}, w \leftrightsquigarrow \mathfrak{M}', w'$. ⊣

Proof. Suppose $\mathfrak{M}, w \leftrightarroweq \mathfrak{M}', w'$, it suffices to show that $\mathfrak{M}, w \Vdash \varphi \Leftrightarrow \mathfrak{M}', w' \Vdash \varphi$ for any formula φ. By induction on φ.

■

2.2.1 Hennessy-Milner Theorem

定义 2.13 (Image finite model)**.** Let $\mathfrak{M} = (W, R, V)$ be a model,
$\mathfrak{M}$ is **image-finite** if $\forall w \in W, \{u \mid Rwu\}$ is finite. ⊣

NB:
$\mathfrak{M}$ is finite $\Rightarrow$ $\mathfrak{M}$ is image-finite. But $\mathfrak{M}$ is image-finite $\not\Rightarrow$ $\mathfrak{M}$ is finite
Every <u>finite</u> structure and every <u>deterministic</u> structure is image finite.

定理 2.14 (Hennessy-Milner Theorem). If $\mathfrak{M}$ and $\mathfrak{M}'$ are two image-finite models. Then

$$w \leftrightarroweq w' \Leftrightarrow w \leftrightsquigarrow w'.$$

⊣

Proof. $\Rightarrow$ trivially by Theorem 2.12.

$\Leftarrow$ （**Basic idea**: *the relation of modal equivalence itself is a bisimulation.* ）

We show that the relation $\leftrightsquigarrow$ itself is a bisimulation.

- For *atom condition*: immediately by modal equivalence.
- For *forth condition*:

  Assume $w \leftrightsquigarrow w'$ and Rwv,

  and suppose for the sake of contradiction that there is no $v' \in W'$ such that $R'w'v'$ and $v \leftrightsquigarrow v'$.

  Let

  $$w'\!\uparrow = \{v' \in W' \mid R'w'v'\}^{3},$$

  then clearly

  – $w'\!\uparrow$ is non-empty, otherwise, $w' \Vdash \Box\bot$ which contradicts $w \leftrightsquigarrow w'$ since w has a successor v.

  – $w'\!\uparrow$ is finite, since $\mathfrak{M}'$ is image-finite.

  Rewrite $w'\!\uparrow$ as $\{v'_1, \ldots, v'_n\}$ since it is finite.

  By assumption, for each $v'_i \in w'\!\uparrow$ we have $v \not\leftrightsquigarrow v'_i$.

  Hence for any $1 \leq i \leq n$, there exists a formula ψ_i such that $v \Vdash \psi_i$ but $v'_i \nVdash \psi_i$.

  Let

  $$\psi = \psi_1 \wedge \cdots \wedge \psi_n$$

  then $v \Vdash \psi$ but $v'_i \nVdash \psi$ for all $v'_i \in w'\!\uparrow$.

  Since Rwv and $R'w'v'_i$, it follows that

  $$\mathfrak{M}, w \Vdash \Diamond\psi \qquad \text{but} \qquad \mathfrak{M}', w' \nVdash \Diamond\psi$$

  which contradicts with $w \leftrightsquigarrow w'$.

  Consequently, there is a $v' \in W'$ such that $R'w'v'$ and $v \leftrightsquigarrow v'$.

- For *back condition*:

  Similar with the forth condition. ■

Comments:

1. It is crucial that $w'\!\uparrow$ is finite which based on $\mathfrak{M}'$ is image-finite.

2.

[3] $w'\!\uparrow$ called the **upset** of w'.

2.3 Bisimulation games

2.3.1 game

2.3.2 title

2.4 Finite model property (*fmp*)

If a modal formula is satisfiable on an arbitrary model, then it is satisfiable on a finite model.

定义 2.15 (Finite model property (*fmp*))**.** Let M be a class of models.

Say a set Δ of formulas has the **finite model property** w.r.t M, if for all $\varphi \in \Delta$, φ is satisfiable in some model in M, then φ is satisfiable in a finite model in M. $\dashv$

- modal language has fmp means that: modal language lack the expressive strength to force the existence of *infinite model*;
- but there is some first-order formulas which can only be satisfied on infinite model (反自反 + 传递 + 持续)

Two methods for building fmp ofr modal logic: (1) selecting a finite modal ; (2) via filtration (to define a quotient structure).

2.5 *fmp* via selection (finite-tree-model property)

2.5.1 n-bisimilarity

定义 2.16 (Modal degree)**.** Define $deg\colon \mathcal{L}_{\Diamond} \to \mathbb{N}$ as follows:

$$\begin{array}{rcl} deg(p) & = & deg(\bot) = 0 \\ deg(\neg\varphi) & = & deg(\varphi) \\ deg(\varphi \vee \psi) & = & max\{deg(\varphi), deg(\psi)\} \\ deg(\Diamond\varphi) & = & deg(\varphi) + 1 \end{array}$$

$deg(\varphi)$ is called the **modal degree** (or **modal depth**) of formula φ.

Obviously $deg(\varphi \wedge \psi) = deg(\varphi \vee \psi)$ and $deg(\Box\varphi) = def(\Diamond\varphi)$. $\dashv$

一个公式的模态度是该公式中模态词嵌套的最大层数，而不是模态词的个数。

引理 2.17 (Finiteness lemma)**.** Suppose our language with finite modalities and finite proposition letters, then

1. 在语言 ML_n 中，只有有穷多个互不等价的公式。
2. For all n and any state w in a model $\mathfrak{M}$, $\exists\psi$ such that

$$w \Vdash \psi \;\Leftrightarrow\; w \Vdash \Gamma^n = \{\varphi \mid w \Vdash \varphi \;\&\; deg(\varphi) \leq n\}$$

$\dashv$

Proof. 1.

当 $n=0$，且 $|\mathsf{Prop}|=m$，则只有 2^{2^m} 个互不等价的命题公式。

2. 由（1）可知， ■

定义 2.18 (n-bisimulations)**.** Let $\mathfrak{M}=(W,R,V)$ and $\mathfrak{M}'=(W',R',V')$ be two models, we say w and w' are n-**bisimilar**, notation $w \leftrightarrows_n w'$, if there exists a sequence of $(n+1)$ binary relations $Z_n \subseteq \cdots \subseteq Z_0$ satisfy (for $k \leq n-1$):

1. wZ_nw';
2. $vZ_0v' \ \Rightarrow\ v$ and v' agree on all propositional variables;
3. $vZ_{k+1}v'$ and $Rvu \ \Rightarrow\ \exists u' \in W'$: $R'v'u'$ and uZ_ku';
4. $vZ_{k+1}v'$ and $R'v'u' \ \Rightarrow\ \exists u \in W$: Rvu and uZ_ku';

If $w \leftrightarrows_n w'$, then intuitionally w and w' bisimulate up to depth n.

$w \leftrightarrows w' \ \Rightarrow\ w \leftrightarrows_n w'$ for all n, but the converse need not hold. ⊣

定义 2.19 (n-bisimulation (def. 2))**.** 【】【】【】 ⊣

命题 2.20 (n-bisimilarity and modal equivalence)**.** Let $\mathfrak{M}$ and $\mathfrak{M}'$ be models for a modal language with <u>finite modalities</u> and <u>finite proposition letters</u> (*finite conditions*), then for every w in $\mathfrak{M}$ and w' in $\mathfrak{M}'$:

$$w \leftrightarrows_n w' \ \Leftrightarrow\ w \leftrightsquigarrow_n w'$$

where $w \leftrightsquigarrow_n w'$ iff w and w' agree on all modal formulas of degree at most n. ⊣

Proof.

proof outline
$\Rightarrow$ By induction on n (that is a *double induction proof*) $\Leftarrow$ Similarly to the proof in Hennessy-Milner Theorem.

$\Rightarrow$ By induction on n.

Base case: $n=0$.

Suppose $w \leftrightarrows_0 w'$, then we show that $w \leftrightsquigarrow_0 w'$.

By induction on formula φ with $deg(\varphi) = 0$.

Base case:

(1) if $\varphi = p$, since $w \leftrightarrows_0 w'$, then w and w' agree on all propositional variables.

(2) if $\varphi = \bot$, both w and w' refutes $\bot$.

Induction hypothesis (IH_1): for any subformula χ of φ: $w \Vdash \chi \;\Leftrightarrow\; w' \Vdash \chi$.

Induction step:

(1) if $\varphi = \neg\psi$, then

$$\begin{aligned} w \Vdash \neg\psi \quad &\Leftrightarrow \quad w \nVdash \psi \\ &\Leftrightarrow \quad w' \nVdash \psi \qquad (\text{ by } \mathrm{IH}_1 \text{ }) \\ &\Leftrightarrow \quad w' \Vdash \neg\psi. \end{aligned}$$

(2) if $\varphi = \psi \vee \chi$, then

$$\begin{aligned} w \Vdash \psi \vee \chi \quad &\Leftrightarrow \quad w \Vdash \psi \text{ or } w \Vdash \chi \\ &\Leftrightarrow \quad w' \Vdash \psi \text{ or } w' \Vdash \chi \qquad (\text{by } \mathrm{IH}_1) \\ &\Leftrightarrow \quad w' \Vdash \psi \vee \chi. \end{aligned}$$

(It's not going to be that $\varphi = \Diamond\psi$ since $deg(\varphi) = 0$ while $deg(\Diamond\psi) \geq 1$)

Induction hypothesis (IH): *If* $n = k$, *then* $w \leftrightarrows_n w'$ *implies* $w \leftrightsquigarrow_n w'$.

Induction step: $n = k + 1$.

Suppose $w \leftrightarrows_{k+1} w'$, then $w \leftrightarrows_k w'$ by the definition.

Following we show that $w \leftrightsquigarrow_{k+1} w'$.

By induction on formula φ with where $deg(\varphi) \leq k+1$.

Base case:

(1) If $\varphi = \psi$ with $deg(\psi) \leq k$. Since $w \leftrightarrows_k w'$ then by **IH** we have $w \leftrightsquigarrow_k w'$, that is, $w \Vdash \psi \;\Leftrightarrow\; w' \Vdash \psi$ for $deg(\psi) \leq k$.

Induction step:

(1) Boolean cases are trivial.

(2) if $\varphi = \Diamond\psi$ and $deg(\psi) \leq k$.

- Suppose $w \Vdash \Diamond\psi$,
  then $\exists u$, Rwu and $u \Vdash \psi$.
  Since $w \leftrightarrows_{k+1} w'$ and Rwu.
  then $\exists u'$, $R'w'u'$ and $u \leftrightarrows_k u'$ by definition.
  From $u \leftrightarrows_k u'$ and by **IH** we have $u \leftrightsquigarrow_k u'$.
  Then $u' \Vdash \psi$ since $u \Vdash \psi$ and $deg(\psi) \leq k$.

  Hence $w' \Vdash \Diamond\psi$ since $R'w'u'$ and $u' \Vdash \psi$.

- Suppose $w' \Vdash \Diamond\psi$, then by a similar argument we have $w \Vdash \Diamond\psi$.

  (一些评论：在证明 $w \Vdash \Diamond\varphi$ 时，我们只用到了最外层归纳证明的归纳假设 **IH**，而没有用到第二层归纳证明中的归纳假设。这看似是错误的，实则不然。在有多层嵌套的归纳证明中，较里层归纳步由于可用的前提比较多，会存在用不到该层次的归纳假设而只需要最外层的归纳假设的情况。这是可接受的，因为只是前提增加了但我们不用该前提而已。但是如果在单层的归纳证明中，归纳步没有用到归纳假设往往说明该证明有错误。要注意“单层归纳证明”和“嵌套归纳证明”这二者的区别。)

Therefore $w \leftrightsquigarrow_{k+1} w'$.

By the above induction proofs, we know that, if $w \leftrightarrows_n$ then $w \leftrightsquigarrow_n w'$.

$\Leftarrow$

Suppose $w \leftrightsquigarrow_n w'$, we have to show that there exists a sequence of binary relations satisfy those conditions in the definition of n-bisimulation.

Following we prove that $\leftrightsquigarrow_n, \leftrightsquigarrow_{n-1}, \ldots, \leftrightsquigarrow_0$ are the relations which we need.

Obviously $\leftrightsquigarrow_n \subseteq \leftrightsquigarrow_{n-1} \subseteq \cdots \subseteq \leftrightsquigarrow_0$.

(i) $w \leftrightsquigarrow_n w'$ by assumption.

(ii) If $v \leftrightsquigarrow_0 v'$, then v and v' agree on all formulas φ with $deg(\varphi) \leq 0$, they agree on all proposition letters obviously.

(iii) If $v \leftrightsquigarrow_{k+1} v'$ and Rvu (where $k \leq n-1$).

Further suppose there is no u' in $\mathfrak{M}'$ s.t. $R'v'u'$ and $u \leftrightsquigarrow_k u'$. i.e., $\forall u', R'v'u' \;\Rightarrow\; u \not\leftrightsquigarrow_k u'$.

Let $v'{\uparrow} = \{u' \mid R'w'u'\}$.

$v'{\uparrow} \neq \emptyset$, otherwise $v' \Vdash \Box\bot$ and hence $v \Vdash \Box\bot$ by $v \leftrightsquigarrow_{k+1} v'$, but this contradicts with Rvu.

By Lemma 2.17 (the Finiteness Lemma), there is ψ with $deg(\psi) \leq k$ such that

$$u \Vdash \psi \Leftrightarrow u \Vdash \Gamma^k = \{\varphi \mid u \Vdash \varphi \,\&\, deg(\varphi) \leq k\}.$$

For any $u' \in v'{\uparrow}$ we have $u \not\leftrightsquigarrow_k u'$, hence $u' \nVdash \psi$, consequently

$$v \Vdash \Diamond\psi \quad \text{but} \quad v' \nVdash \Diamond\psi.$$

But that contradicts with $v \leftrightsquigarrow_{k+1} v'$ since $deg(\Diamond\psi) \leq k+1$.

Therefore, there is a u' in $\mathfrak{M}'$ such that $R'v'u'$ and $u \leftrightsquigarrow_k u'$.

(iv) Suppose $v \leftrightsquigarrow_{k+1} v'$ and $R'v'u'$ (where $k \leq n-1$).

The argument is analogue with above one.

■

2.5.2 finite-tree-property

定义 2.21 (the **Height** of the *rooted* modals)**.** Given a *rooted* model $\mathfrak{M} = (W, R, V)$ with root w. The **height** of states in $\mathfrak{M}$ is defined by induction.

The **height** of the root w is 0 (only root with height 0); the states of **height** $n+1$ are those *immediate successors* of elements of height n that have not yet been assigned a height smaller than $n+1$.

The **height** of a rooted model $\mathfrak{M}$ is the *maximum* n such that there is a state of height n in $\mathfrak{M}$, if such a maximum exists; otherwise the **height** of $\mathfrak{M}$ is *infinite*.

For $k \in \omega$, the **restriction** of a rooted model $\mathfrak{M}$ to k, notation $\mathfrak{M} \upharpoonright k$ is defined as the submodel containing only states whose height is at most k. Formally, $\mathfrak{M} \upharpoonright k = (W_k, R_k, V_k)$, where $W_k = \{v \mid height(v) \leq k\}$, $R_k = R \cap (W_k \times W_k)$, and $V_k(p) = V(p) \cap W_k$ for each $p \in \mathsf{Prop}$. ⊣

注记 2.22.

- For any rooted model $\mathfrak{M}$ and any $k \in \omega$, $\mathfrak{M} \upharpoonright k$ is well-defined since the root w satisfies $height(w) = 0 \leq k$, which means that the domain of $\mathfrak{M} \upharpoonright k$ is non-empty.
- Generally $\mathfrak{M} \upharpoonright k$ is not a generated submodel of $\mathfrak{M}$.
- $\mathfrak{M} \upharpoonright k$ contains all states that can be reached from the root in at most k steps along the accessibility relation R.

⊣

引理 2.23. Let $\mathfrak{M}$ be a rooted model, $k \in \omega$, then for any state w in $\mathfrak{M} \upharpoonright k$,

$$\mathfrak{M} \upharpoonright k, w \leftrightarroweq_l \mathfrak{M}, w$$

where $l = k - height(w)$. ⊣

Proof. Suppose $\mathfrak{M} = (W, R, V)$, then $\mathfrak{M} \upharpoonright k = (W', R', V')$, where $W' = \{v \in W \mid height(v) \leq k\}$, R' and V' are obtained by restricting the R and V to W'.

Let $Z = \{(v, v) \mid v \in W'\}$ and $Z_l = Z_{l-1} = \cdots = Z_0 = Z$.

Clearly $Z \subseteq W' \times W$ and $Z_l \subseteq Z_{l-1} \subseteq \cdots \subseteq Z_0$.

(i) wZ_lw since $Z_n = Z$ is the identity relation on W'.

(ii) If vZ_0v, of course they agree on all proposition letters.

(iii) If $vZ_{i+1}v$ and $R'vu$ (where $0 \leq i \leq l-1$),

then Rvu and uZ_iu since $R' \subseteq R$.

(iv) If $vZ_{i+1}v$ and Rvu (where $0 \leq i \leq l-1$),

then we have $height(u) \leq k$.

Otherwise, suppose $height(u) > k$, then $height(v) = k$ since $v \in W'$ and u is an immediate successor of v.

In this moment, since $vZ_{i+1}v$, and $l = k - height(v) = 0$, then $i + 1 \leq 0$, i.e., $i \leq -1$, that contradicts with $i \geq 0$.

Therefore $u \in W'$ since $height(u) \leq k$.

Hence Rvu implies $R'vu$, obviously uZ_iu.

$vZ_{i+1}v$ means vZ_0v,
while at this moment $l = i + 1 = k - height(v) = 0$, hence $i = -1$, contradicts with $i \geq 0$.
Therefore, by the definition of n-bisimilarity, we have $\mathfrak{M} \restriction k, w \leftrightarrows_l \mathfrak{M}, w$. ■

Together Proposition 2.20 and above lemma:

Every satisfiable modal formula can be satisfied on a model of finite *height*. But this model may be *infinitely branching*, hence we have to discard unwanted branches to obtain a really desired finite model.

定理 2.24 (*fmp* via **Selection / Finite Tree Model Property**). For any formula φ, if φ is satisfiable, then φ is satisfiable on a *finite* model. ⊣

Proof. 【砍树 + 裁枝】

Given a formula φ, suppose it is satisfiable at a pointed modal $\mathfrak{M}_1, w_1$.

By *tree model property* (Proposition 2.7), there exists a tree-like model $\mathfrak{M}_2$ with root w_2 such that $\mathfrak{M}_2, w_2 \Vdash \varphi$.

Let $k = deg(\varphi)$. Clearly $\mathfrak{M}_2$ is a rooted model, let $\mathfrak{M}_2 \restriction k$ be the restriction of $\mathfrak{M}_2$ to k, then $\mathfrak{M}_2, w_2 \leftrightarrows_k \mathfrak{M}_2 \restriction k, w_2$ by Lemma 2.23 (notice that w_2 is the root and $height(w) = 0$).

According to Proposition 2.20, we have $\mathfrak{M}_2 \restriction k, w_2 \Vdash \varphi$.

【砍树完成】

Suppose $\mathfrak{M}_2 \restriction k = (W, R, V)$, define $\mathfrak{M}_4 = (W', R', V')$ by

$$\begin{aligned} W' &:= S_0 \cup S_1 \cup \cdots \cup S_k \\ R' &:= R \cap (W' \times W') \\ V'(p) &:= V(p) \cap W' \quad \text{for any proposition letter } p \end{aligned}$$

where $S_0, S_1, \ldots, S_k$ are recursively defined as follows:

$$S_0 = \{w_2\}$$

For any $v \in S_n$ $(0 \leq n \leq k-1)$, let $\Gamma_v := \{\psi \mid v \Vdash \psi \text{ and } deg(\psi) \leq k - n\}$.

By Proposition 2.17 (the Finiteness Lemma), we can partition Γ_v into finitely many equivalence classes.

That is $\Gamma_v = [\psi_1] \cup \cdots \cup [\psi_m]$, where $[\psi_i] = \{\theta \mid \Vdash \theta \leftrightarrow \psi_i\}$.

Let $\Gamma'_v = \{\psi_1, \psi_2, \ldots, \psi_m\}$, i.e., Γ'_v is the set of representative elements for each $[\psi_i]$.

For each $\psi_i \in \Gamma'_v$:

- if $\psi_i = \Diamond\chi$, let $\psi_i^\circ = \{u \mid Rvu \text{ and } u \Vdash \chi\}$ (ψ_i° may be infinite in this case).
- if $\psi_i \neq \Diamond\chi$, let $\psi_i^\circ = \emptyset$.

Since Γ'_v is finite, hence we can get an finite sequence of sets

$$\psi_1^\circ, \psi_2^\circ, \ldots, \psi_m^\circ$$

for each $\psi_i^\circ \neq \emptyset$ we select an element from ψ_i°; otherwise, if $\psi_i^\circ = \emptyset$ then we ignore it [4].

Then let $\vec{v}$ be the set of all selected states

(notice that according the way for selecting an element from ψ_i°, we can obtain different $\vec{v}$), and obviously $\vec{v}$ is finite.

Now we define S_{n+1} as:

$$S_{n+1} = \bigcup_{v \in S_n} \vec{v} \qquad (0 \leq n \leq k-1)$$

Claim: $\mathfrak{M}_4, w_2 \; \leftrightarroweq_k \mathfrak{M}_2 \upharpoonright k, w_2$

---

proof for this claim:

◀

Therefore $\mathfrak{M}_4, w_2 \Vdash \varphi$. In addition, $\mathfrak{M}_4$ is finite since each S_i is finite from its construction process. Consequently, $\mathfrak{M}_4$ is the desired finite model for φ. ■

例 2.25. Let $\mathfrak{N} = (\omega, <, V)$, where ω is the set of natural numbers, $<$ is the .... , and $V(p) = \omega$. Clearly $\mathfrak{N}, 0 \Vdash \Box\Diamond\Box p$, and $deg(\varphi) = 3$.

The unravelling of $(\mathfrak{N}, 0)$ is an infinite tree with infinite depth and infinite branches. ⊣

2.6 *fmp* via filtration

2.6.1 filtration

Why we need filtration to prove *fmp*?

- When considering some class of frames, for instance the reflexive frames, the unravelling of these frames will no longer reflexive. Hence we need some operations to reduce our model but maintain the desired properties, that is what filtration is good at.

定义 2.26 (Subformula closure)**.** A set of formulas Σ is **closed under subformulas** (or **subformula closed**) if $subf(\Sigma) = \Sigma$. ⊣

[4]Here we don't presuppose the Axiom of Choice, since the number of sets from which to choose the elements is finite.

- Prop is subformula closed;
- ML, the basic modal language, is subformula closed;
- $\{p, q, \Diamond(p \vee q), p \vee q\}$ is subformula closed;
- $subf(\varphi)$ is subformula closed, moreover, is *finite*;
- ...

定义 2.27 (Filtration). Let $\mathfrak{M} = (W, R, V)$ be a model and Σ a subformula closed set of formulas. Let $\leftrightsquigarrow_\Sigma \subseteq W \times W$ be a relation on W given by:

$$w \leftrightsquigarrow_\Sigma v \iff \forall \varphi \in \Sigma : (w \Vdash \varphi \iff v \Vdash \varphi).$$

Note that $\leftrightsquigarrow_\Sigma$ is an equivalence relation, let $|w|_\Sigma$ be the equivalence class of w w.r.t $\leftrightsquigarrow_\Sigma$, or simply $|w|$ if no confusion will arise.

The mapping $w \mapsto |w|$ is called the **natural map**.

Let $W_\Sigma = \{|W|_\Sigma \mid w \in W\}$. $\mathfrak{M}^f_\Sigma = (W^f, R^f, V^f)$ is any model such that:

1. $W^f = W_\Sigma$.
2. $Rwv \Rightarrow R^f|w||v|$.
3. $R^f|w||v| \Rightarrow \forall \Diamond\varphi \in \Sigma(v \Vdash \varphi \Rightarrow w \Vdash \Diamond\varphi)$
4. $V^f(p) = \{|w| \mid w \Vdash p\}$

$\mathfrak{M}^f_\Sigma$ is called a **filtration** of $\mathfrak{M}$ through Σ. ⊣

注记 2.28.

- All filtrations have the same set of worlds W_Σ and the same valuation V^f. Different filtrations have different relations R^f.
- item (2) show that the *natural map* is a homomorphism from $\mathfrak{M}$ to its arbitrary filtration. 实际上，$\mathfrak{M}^f$ 是 $\mathfrak{M}$ 的同态像，因而可以保留 $\mathfrak{M}$ 的一部分结构特征。
- filtration 的定义不能保障这样得到的结构就是一个模型，或者也不能就认定这样的结构一定存在。这些都需要额外的证明。(数学概念特别要注意"存在性"和"唯一性"这两点)
- item (3) is pretty similar to the *canonical relation*, but lack '$\forall \Diamond\varphi \in \Sigma$', see page**??**.

⊣

命题 2.29 (Filtrations are finite). Let Σ be a *finite* subformula closed set of formulas. For any model $\mathfrak{M}$, if $\mathfrak{M}^f_\Sigma$ is a filtration of $\mathfrak{M}$ through Σ, then $\mathfrak{M}^f_\Sigma$ contains at most $2^{|\Sigma|}$ states. ⊣

Proof. The domain $W^f = \{|w|_\Sigma \mid w \in W\}$ of $\mathfrak{M}^f_\Sigma$ is a set of equivalence classes w.r.t $\leftrightsquigarrow_\Sigma$.

Define a function $g\colon W^f \to \wp(\Sigma)$ by

$$g(|w|_\Sigma) = \{\varphi \in \Sigma \mid w \Vdash \varphi\}$$

It is easy to check that g is well-defined and injective.

Hence $|W^f| \leq |\wp(\Sigma)| = 2^{|\Sigma|}$. ■

定理 2.30 (Filtration Theorem). Let $\mathfrak{M}^f = (W_\Sigma, R^f, V^f)$ be any filtration of model $\mathfrak{M}$ through a subformula closed set Σ. Then for any $\varphi \in \Sigma$ and any w in $\mathfrak{M}$,

$$\mathfrak{M}, w \Vdash \varphi \;\Leftrightarrow\; \mathfrak{M}^f, |w| \Vdash \varphi.$$

$\dashv$

Proof. By induction on φ.

Base case:

- If $\varphi = p$, then $w \Vdash p \;\Leftrightarrow\; |w| \in V^f(p)$ (by the definite of V^f) $\Leftrightarrow\; \mathfrak{M}^f, |w| \Vdash p$.
- If $\varphi = \bot$, neither $\mathfrak{M}, w \Vdash \bot$ nor $\mathfrak{M}^f, |w| \Vdash \bot$.

Fact: Σ is subformula closed allows us to apply the **inductive hypothesis**.

Induction step:

- The boolean cases are straightforward.
- If $\varphi = \Diamond\psi$,
  - Suppose $\mathfrak{M}, w \Vdash \Diamond\psi$, then there is u such that Rwu and $\mathfrak{M}, u \Vdash \psi$.
    As $\mathfrak{M}^f$ is a filtration, $R^f|w||u|$ since Rwu by the clause (ii) in the Definition of filtration.
    As Σ is subformula closed, $\psi \in \Sigma$, thus by **IH**, $\mathfrak{M}^f, |u| \Vdash \psi$.
    Hence $\mathfrak{M}^f, |w| \Vdash \Diamond\psi$ by $R^f|w||u|$.
  - Suppose $\mathfrak{M}^f, |w| \Vdash \Diamond\psi$, then there is $|u|$ such that $R^f|w||u|$ and $\mathfrak{M}^f, |u| \Vdash \psi$.
    As $\psi \in \Sigma$, by **IH**, $\mathfrak{M}, u \Vdash \psi$.
    By the clause (iii) in the Definition of filtration, $\mathfrak{M}, w \Vdash \Diamond\psi$.

(Observe that clause (2) and (3) of the Definition of filtration are designed to make the modal case of the induction step go through in the proof above. Σ 的子公式封闭性在上述证明中是关键的) ■

定理 2.31 (*fmp* via Filtration). If a formula φ is satisfiable, then it is satisfiable on a finite model. Indeed, it is satisfiable on a finite model containing at most $2^{|sf(\varphi)|}$. $\dashv$

Proof. Suppose φ is satisfiable on a model $\mathfrak{M}$. Take any filtration of $\mathfrak{M}$ through $sf(\varphi)$ (which is finite and subformula closed), then φ is satisfiable in this filtration from the **Filtration Theorem**.

The bound on the size of this filtration is by Proposition 2.29. ■

定理 2.32 (strong finite model property 小模型性质)**.** If a modal formula φ is satisfiable, then it is satisfiable in a finite model with the size bounded by 2^n, where n is the length of φ. $\dashv$

Proof. By the above theorem with the fact: the number of subformulas of a formula φ is less or equal than the length of φ. ■

如下引理说明，filtration 确实是存在的。

引理 2.33 (**Smallest** and **Largest** filtration)**.** Let $\mathfrak{M}$ be any model, Σ any subformula closed set of formulas, W_Σ the set of equivalence classes induced by $\leftrightsquigarrow_\Sigma$, and V^f the standard valuation on W_Σ. Define R^s and R^l as follows:

$$\begin{array}{lcl} R^s|w||v| & \Leftrightarrow & \exists w' \in |w|, \exists v' \in |v| : Rw'v' \\ R^l|w||v| & \Leftrightarrow & \forall \Diamond\varphi \in \Sigma : \mathfrak{M}, v \Vdash \varphi \ \Rightarrow \ \mathfrak{M}, w \Vdash \Diamond\varphi. \end{array}$$

Then both (W_Σ, R^s, V^f) and (W_Σ, R^l, V^f) are filtrations of $\mathfrak{M}$ through Σ.
Furthermore, if (W_Σ, R^f, V^f) is a filtration of $\mathfrak{M}$ through Σ, then $R^s \subseteq R^f \subseteq R^f$. ⊣

Proof. To show that (W_Σ, R^s, V^f) is a filtration:
It suffices to show that R^s fulfills clauses (ii) and (iii) of the Definition of filtration.

- For (ii): Suppose Rwv, since $w \in |w|$ and $v \in |v|$, then $R^s|w||v|$ by definition.
- For (iii): Suppose $R^s|w||v|$, and further suppose that $\Diamond\varphi \in \Sigma$ and $\mathfrak{M}, v \Vdash \varphi$.
  As $R^s|w||v|$, there exists $w' \in |w|$ and $v' \in |v|$ such that $Rw'v'$.
  As $\varphi \in \Sigma$ and $\mathfrak{M}, v \Vdash \varphi$, then $\mathfrak{M}, v' \Vdash \varphi$ since $v \leftrightsquigarrow_\Sigma v'$.
  But $Rw'v'$, so $\mathfrak{M}, w' \Vdash \Diamond\varphi$.
  In addition, $\Diamond\varphi \in \Sigma$, thus as $w' \leftrightsquigarrow_\Sigma w$ it follows that $\mathfrak{M}, w \Vdash \Diamond\varphi$.

To show that (W_Σ, R^l, V^f) is a filtration:
It suffices to show that R^l fulfills clauses (ii) and (iii) of the Definition of filtration.

- For (ii): Suppose Rwv, and further suppose that $\Diamond\varphi \in \Sigma$ and $\mathfrak{M}, v \Vdash \varphi$.
  It follows that $\mathfrak{M}, w \Vdash \Diamond\varphi$ since Rwv.
  Hence $R^l|w||v|$ by definition.
- For (iii): Immediately from the definition of R^l.

To show that $R^s \subseteq R^f$.
For any w and v, suppose $R^s|w||v|$, it suffices to show that $R^f|w||v|$.

To show that $R^f \subseteq R^l$.
For any w and v, suppose $R^f|w||v|$, it suffices to show that $R^l|w||v|$. ■

2.6.2 filtration and properties of relation

Seriality and Reflexivity

Transitive filtration:

引理 2.34 (Transitive filtration)**.** Let $\mathfrak{M}$ be a model, Σ a subformula closed set of formulas, and W_Σ the set of equivalence classes induced on $\mathfrak{M}$ by $\leftrightsquigarrow_\Sigma$. Let R^t be the binary relation on W_Σ defined by

$$R^t|w||v| \ \Leftrightarrow \ \forall\varphi : (\Diamond\varphi \in \Sigma, \mathfrak{M}, v \Vdash \varphi \vee \Diamond\varphi \ \Rightarrow \ \mathfrak{M}, w \Vdash \Diamond\varphi).$$

If R is transitive then (W_Σ, R^t, V^f) is a filtration and R^t is transitive. ⊣

Proof. Suppose R is transitive.

For (W_Σ, R^t, V^f) is a filtration, it suffices to show that R^t satisfies the clause (ii) and (iii) in the definition of filtration.

1. Suppose Rwv, we have to show that $R^t|w||v|$.

   By definition, assume for any $\Diamond\varphi \in \Sigma$, $\mathfrak{M}, v \Vdash \varphi \vee \Diamond\varphi$, we only need to show $\mathfrak{M}, w \Vdash \Diamond\varphi$.

   Since Rwv, thus $\mathfrak{M}, w \Vdash \Diamond(\varphi \vee \Diamond\varphi)$.

   Then $\mathfrak{M}, w \Vdash \Diamond\varphi \vee \Diamond\Diamond\varphi$ since the formula $\Diamond(\varphi \vee \Diamond\varphi) \leftrightarrow (\Diamond\varphi \vee \Diamond\Diamond\varphi)$ is valid.

   There are two cases:

   (a) If $\mathfrak{M}, w \Vdash \Diamond\varphi$, then we done!

   (b) If $\mathfrak{M}, w \Vdash \Diamond\Diamond\varphi$,
   note that R is transitive, it is easy to check that $\Diamond\Diamond\varphi \to \Diamond\varphi$ is valid on $\mathfrak{M}$.
   Consequently, $\mathfrak{M}, w \Vdash \Diamond\varphi$

2. Suppose $R^t|w||v|$, we have to show that for all $\Diamond\varphi \in \Sigma$: if $\mathfrak{M}, v \Vdash \varphi$ then $\mathfrak{M}, w \Vdash \Diamond\varphi$.

   Further suppose for all $\Diamond\varphi \in \Sigma$, $\mathfrak{M}, v \Vdash \varphi$.
   Then $\mathfrak{M}, v \Vdash \varphi \vee \Diamond\varphi$ by our semantics.
   Hence by the definition of R^t, $\mathfrak{M}, w \Vdash \Diamond\varphi$.

For the Transitivity for R^t , suppose $R^t|w||v|$ and $R^t|v||u|$, we need to show that $R^t|w||u|$.
By definition, from $R^t|w||v|$ and $R^t|v||u|$ we have (for any $\Diamond\varphi \in \Sigma$):

(i) $\mathfrak{M}, v \Vdash \varphi \vee \Diamond\varphi \;\Rightarrow\; \mathfrak{M}, w \Vdash \Diamond\varphi$.

(ii) $\mathfrak{M}, u \Vdash \varphi \vee \Diamond\varphi \;\Rightarrow\; \mathfrak{M}, v \Vdash \Diamond\varphi$.

In order to show $R^t|w||u|$, it suffices to show that $\forall\Diamond\varphi \in \Sigma, \mathfrak{M}, u \Vdash \varphi \vee \Diamond\varphi \;\Rightarrow\; \mathfrak{M}, w \Vdash \varphi$.
Further assume $\mathfrak{M}, u \Vdash \varphi \vee \Diamond\varphi$ for any $\Diamond\varphi \in \Sigma$,
then by (ii), $\mathfrak{M}, v \Vdash \Diamond\varphi$, hence $\mathfrak{M}, v \Vdash \varphi \vee \Diamond\varphi$.
It follows that $\mathfrak{M}, w \Vdash \Diamond\varphi$ by (i). ■

Table 2.1: $\mathfrak{M}$ and its filtration $\mathfrak{M}^f$

$\mathfrak{M}$ 性质	$R^f|w||v|$ 的定义	(这里还不知道填什么)
持续性 Seriality		
自反性 Reflexivity		
传递性 Transitivity	$\forall\varphi : (\Diamond\varphi \in \Sigma, \mathfrak{M}, v \Vdash \varphi \vee \Diamond\varphi \;\Rightarrow\; \mathfrak{M}, w \Vdash \Diamond\varphi)$.	

2.7 Filters and Ultrafilters, M-saturation, Ultrafilter Extension

ultrafilter extension is a way of building modally-saturated models.

M-saturation : a general notation of *image-finiteness*; a kind of *compactness* property.

2.7.1 Filter and Ultrafilter

定义 2.35 (Filters and Ultrafilters). Let $W \neq \emptyset$. A **filter** $\mathcal{F}$ over W is a set $\mathcal{F} \subseteq \wp(W)$ such that:

1. (含全集) $W \in \mathcal{F}$.
2. (交集封闭) $X, Y \in \mathcal{F} \;\Rightarrow\; X \cap Y \in \mathcal{F}$. 【类似 $\wedge$ 封闭】
3. (扩集封闭) $X \in \mathcal{F}, X \subseteq Y \subseteq W \;\Rightarrow\; Y \in \mathcal{F}$. 【类似 $\rightarrow$ 封闭】

If a filter $\mathcal{F} \neq \wp(W)$, then $\mathcal{F}$ is a **proper filter**.

An **ultrafilter** over W is a *proper filter* U such that either $X \in U$ or $W \setminus X \in U$ for any $X \subseteq W$. Formally, $\forall X \in \wp(W) : (X \in U \;\Leftrightarrow\; W \setminus X \notin U)$. ⊣

命题 2.36 (Some properties of Ultrafilter)**.** Let u be an ultrafilter over W, for any $X, Y \subseteq W$:

1. $X \cup Y \in u \;\Rightarrow\; X \in u$ or $Y \in u$.
2. $X \cap Y \in u \;\Rightarrow\; X \in u$ & $Y \in u$.
3. $\mathcal{F}$ is a proper filter $\;\Leftrightarrow\; \emptyset \notin \mathcal{F}$.
4. a filter is an ultrafilter $\;\Leftrightarrow\;$ it is a proper and has no proper extensions.
   (ultrafilter = *maximal proper filter*)

⊣

Proof.

1. '$X \in u$ or $Y \in u$', iff, '$X \notin u \;\Rightarrow\; Y \in u$'.
Suppose $X \cup Y \in u$ and $X \notin w$.
Then $\overline{X} \in U$, hence $\overline{X} \cap (X \cup Y) \in u$ since u is a filter.
It follows that $Y \in u$ since $\overline{X} \cap (X \cup Y) \subseteq Y$.

2. Trivially since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$.

3.

4. ■

Intuition

a subset of W can be viewed as the extension of a formula which holds exactly on the states in this subset. From this point of view:

- a **filter** is a set of formulas which is closed under $\wedge$ and $\rightarrow$;
- a **proper filter** is a consistent set;
- an **ultrafilter** is a maximal consistent set.

定义 2.37 (Principal Ultrafilter)**.** Let $W \neq \emptyset$. Given an element $w \in W$, the **principal ultrafilter** (主超滤) π_w is the set $\{X \subseteq W \mid w \in X\}$. ⊣

命题 2.38.

1. π_w is an ultrafilter.
2. any ultrafilter over a finite set is a principal ultrafilter. ⊣

Proof. 内容... ■

A non-principal ultrafilter (if exists) contains only infinite subsets and all the co-finite subsets of W. It also means that there is no non-principle ultrafilter over a finite W. 只有无穷集上才有「非主超滤」

定义 2.39 (Generated filter)**.** Let $W \neq \emptyset$, $E \subseteq \wp(W)$. The **filter generated by** E is the intersection $\mathcal{F}_E$ of the collection of all filters over W which include E, that is,

$$\mathcal{F}_E := \bigcap\{\mathcal{F}' \mid E \subseteq \mathcal{F}' \ \&\ \mathcal{F}' \text{ is a filter over } W\}.$$

⊣

命题 2.40.

1. $\mathcal{F}_E$ 是包含 E 的最小滤子.
2. π_w is a principal ultraflter $\Leftrightarrow$ π_w is the filter generated by then singleton set $\{w\}$. ⊣

Proof. 内容... ■

定义 2.41 (Finite intersection property). A family of set E has the **finite intersection property** if the intersection of any finite number of elements of E is non-empty.

Formally, for any $n \in \mathbb{N}$ and each $S_i \in E$ $(0 \leq i \leq n)$, the $S_0 \cap S_1 \cap \cdots \cap S_n \neq \emptyset$. ⊣

引理 2.42 (Finite Intersection Lemma). For any family of set E, if $E = \mathcal{A} \cup \mathcal{B}$, and

1. $\mathcal{A}$ is closed under intersection;
2. $\mathcal{B}$ is closed under intersection;
3. for any $X \in \mathcal{A}$ and any $Y \in \mathcal{B}$, $X \cap Y \neq \emptyset$;

then E has finite intersection property. ⊣

Proof. By the condition (3), both $\mathcal{A}$ and $\mathcal{B}$ don't contain $\emptyset$.

Suppose for the sake of contradiction that E doesn't has finite intersection property, then $S_0 \cap \cdots \cap S_n = \emptyset$ for some n and each $S_i \in E$ $(0 \leq i \leq n)$. There are three cases:

- all $S_i \in \mathcal{A}$, then $S_0 \cap \cdots \cap S_n = \emptyset \in \mathcal{A}$ since $\mathcal{A}$ is closed under intersection. It contradicts with that $\emptyset \notin \mathcal{A}$.
- all $S_i \in \mathcal{B}$, then $S_0 \cap \cdots \cap S_n = \emptyset \in \mathcal{B}$ since $\mathcal{B}$ is closed under intersection. Again, it contradicts with that $\emptyset \notin \mathcal{B}$.
- In $S_0, S_1, \ldots, S_n$, some comes from $\mathcal{A}$ and others from $\mathcal{B}$, since both $\mathcal{A}$ and $\mathcal{B}$ are closed under intersection, then $S_0 \cap \cdots \cap S_n = A \cap B$ where $A \in \mathcal{A}$, $B \in \mathcal{B}$. Hence $S_0 \cap \cdots \cap S_n = A \cap B \neq \emptyset$ by assumption. Contradiction!

Therefore we complete the proof of this lemma. ■

命题 2.43. 若 $E \subseteq \wp(W)$ 具有有穷交性质，则 E 可以扩充为 W 上的真滤子。 ⊣

Proof. 设 $E \subseteq \wp(W)$ 具有有穷交性质。由于 $E \subseteq \mathcal{F}_E$ 且 $\mathcal{F}_E$ 是滤子，只需证明 $\mathcal{F}_E$ 是真的即可。由有穷交的定义可知，$\mathcal{F}_E$ 中的元素均不为空集，即 $\emptyset \notin \mathcal{F}_E$，则据定义 $\mathcal{F}_E$ 是真滤子。 ■

Following theorem show that any proper filter can be extended to an ultrafilter, just similar to the *Lindenbaum Lemma*.

Zorn Lemma (a version of the **axiom of choice**)

若非空 偏序 P 中的每条链（全序子集）都有上界，则 P 有极大元。

- 偏序
- 链
- 上界
- 极大元

定理 2.44 (Ultrafilter Theorem)**.** For $W \neq \emptyset$, any proper filter over W can be extended to an ultrafilter over W. ⊣

Proof. (A non-constructive proof via Zorn Lemma)

Let $\mathcal{F}$ be a filter over W and

$$P := \{\mathcal{F}' \supseteq \mathcal{F} \mid \mathcal{F}' \text{ is a proper filter over } W\}$$

It suffices to show than every chains of $(P, \subseteq)$ has upper bound, by **Zorn Lemma**, P has a maximum element u, that is u is a maximal proper filter, in other words, u in an ultrafilter, moreover $\mathcal{F} \subseteq u$.

Let C is any chain of $(P, \subseteq)$, then $\bigcup C$ is a proper filter containing $\mathcal{F}$, hence C has a upper bound $\bigcup C$. ■

推论 2.45. Any non-empty subset $E \subseteq \wp(W)$ with the finite intersection property can be extended to an ultrafilter over W. ⊣

Proof. Since $E \subseteq \wp(W)$ with the finite intersection property, then E can be extended to a proper filter, by Ultrafilter Theorem, E can be extended to an ultrafilter. ■

Therefore, to construct an ultrafilter from a non-empty set $E \subseteq \wp(W)$, we just need to verify whether E has finite intersection property.

To build a non-principle ultrafilter over an infinite set W, we can start from the proper filter of all the con-finite subsets of W, and apply the ultrafilter theorem.

. . . .

定义 2.46 (Product of the FOL-Models)**.** 内容... ⊣

定理 2.47 (Łoś's Theorem)**.** 内容... ⊣

. . . .

例 2.48. 一些滤、超滤、生成滤、主超滤的例子：

- Clearly $\wp(W)$ is a filter over W.
- If S is an infinite set, then $\{X \mid X \subseteq S, X \text{ is co-finite}\} = \{X \mid S \setminus X \text{ is finite }\}$ is a proper filter over S.

  (A subset of an infinite set is *co-finite* if its complement is finite)
- an alternative definition of **ultrafilter** (as a *maximal proper filters*):

$\dashv$

2.7.2 Hennessy-Milner classes and M-saturation

定义 2.49 (Hennessy-Milner Classes (HM-p))**.** A class of models K is a **Hennessy-Milner class**, or has the **Hennessy-Milner property** (HM-P) , if for every two models $\mathfrak{M}$ and $\mathfrak{M}'$ in K and any two states w, w' of $\mathfrak{M}$ and $\mathfrak{M}'$ respectively:
$\mathfrak{M}, w \leftrightsquigarrow \mathfrak{M}', w' \;\Rightarrow\; \mathfrak{M}, w \;\underline{\leftrightarrow}\; \mathfrak{M}', w'$. $\dashv$

定义 2.50 (M-saturation)**.** Let $\mathfrak{M} = (W, R, V)$ be a model, $X \subseteq W$ and Σ is a set of formulas.

Σ is **satisfiable** in X if $\exists x \in X$ such that $\mathfrak{M}, x \Vdash \Sigma$.

Σ is **finitely satisfiable** in X if every finite subset of Σ is satisfiable in X. ($\forall\Gamma \subseteq \Sigma^*, \exists x \in X : \mathfrak{M}, x \Vdash \Gamma$, where Σ^* is the set of finite subsets of Σ)

A $\mathfrak{M}$ is **m-saturated** if for every state w in $\mathfrak{M}$ and every set Σ of formulas:

$$\Sigma \text{ is finitely satisfiable in } R[w] \;\Rightarrow\; \Sigma \text{ is satisfiable in } R[w].$$

where $R[w]$ is the set of successors of w. $\dashv$

命题 2.51. The class of m-saturated model has the Hennessy-Milner property. $\dashv$

Proof. 书上 p93 页的证明还有问题。

Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M} = (W', R', V')$ be two m-saturated models.

It suffices to prove that :

$$\mathfrak{M}, w \leftrightsquigarrow \mathfrak{M}', w' \;\Rightarrow\; w \;\underline{\leftrightarrow}\; w',$$

for any two state w, w' of $\mathfrak{M}$ and $\mathfrak{M}'$ respectively.

Suppose $\mathfrak{M}, w \leftrightsquigarrow \mathfrak{M}', w'$, then for $\leftrightsquigarrow$ is a bisimulation,

Atom condition: trivially.

Forth condition: Assume $u \;\underline{\leftrightarrow}\; u'$ and Ruv.

Let $\Sigma = \{\varphi \mid v \Vdash \varphi\}$,

for any finite subset Δ of Σ we have $v \Vdash \Delta$, hence $w \Vdash \Diamond \bigwedge \Delta$.

Then $w' \Vdash \Diamond \bigwedge \Delta$ since $w \leftrightsquigarrow w'$.

It follow that w' has s successor v_Δ such that $v \Vdash \Delta$

which means Σ is finitely satisfiable in $R'[w']$.

By m-saturation, Σ is satisfiable in a successor v' of w', that is $v' \Vdash \Sigma'$.

Following we show that $\Sigma = \{\varphi \mid v' \Vdash \varphi\}$
if $\exists \psi \notin \Sigma$ but $v' \Vdash \psi$.
then $w' \Vdash \Diamond\psi$, hence $w \Vdash \Diamond\psi$ by $w \leftrightsquigarrow w'$.
It follows that
?????????
Back condition:

■

2.7.3 Ultrafilter extension

超滤扩张可以看作是典范模型的纯语义版本，它用超滤替代典范模型中的极大一致集。[文 2021]

☛ Not all models are m-saturated. How to turn a model into an m-saturated one?

☛ We need to add some successors such that every finitely satisfiable set of formulas is satisfiable in one of the successors.

☛ How to do it? 在原来的模型上加点，使用「超滤扩张」

定义 2.52 (Ultrafilter extension)**.** Given a model $\mathfrak{M} = (W, R, V)$, its **ultrafilter extension** is $\mathfrak{M}^{\text{ue}} = (W^{\text{ue}}, R^{\text{ue}}, V^{\text{ue}})$ where:

- $W^{\text{ue}} = \{u \mid u \text{ is an ultrafilter over } W\}$
- $R^{\text{ue}}uv :\Leftrightarrow \forall X \subseteq W : X \in v \Rightarrow m_R(X) \in u$
- $V^{\text{ue}}(p) = \{u \mid V(p) \in u\}$

where $m_R(X) = \{w \mid \exists v \in X, Rwv\}$. ⊣

$R^{\text{ue}}uu'$ 的另一个等价定义（以 □ 为初始符号）:

$$R^{\text{ue}}uu' :\Leftrightarrow \forall Y \subseteq W : l_R(Y) \in u \Rightarrow Y \in u'$$

where $l_R(X) = \{w \mid \forall v \in W : Rwv \Rightarrow v \in X\}$.

- $m_R(X)$: the set of points that 'can see' a state in X;
- $l_R(X)$: the set of points that 'only see' a state in X;

. . . .

☞ The **intuition** of $R^{\text{ue}}uu'$ （和典范关系、最大 filtration 如出一辙）:
- $\forall \varphi : \varphi \in u' \Rightarrow \Diamond\varphi \in u$;
- $\forall \varphi : \Box\varphi \in u \Rightarrow \varphi \in u'$
超滤扩张和典范模型有很大的相似性。二者关系见下表 【】
【】
【】

. . . .

$$Rwv \Leftrightarrow R^{\text{ue}}\pi_w\pi_v$$

$(*)$

Suppose Rwv, and for all X, $X \in \pi_v$ (i.e $v \in X$ by the definition of principal ultrafilter).
Since $m_R(X) = \{u \mid \exists v \in X, Ruv\}$ and $v \in X, Rwv$, hence $w \in m_R(X)$, that is $m_R(X) \in \pi_w$.

Suppose $\forall X : v \in X \;\Rightarrow\; w \in m_R(X)$. Clearly $v \in \{v\}$ clearly, hence $w \in m_R(\{v\})$, by definition, Rwv.

$$\begin{aligned}
Rwv \quad & \Leftrightarrow \quad \forall X \subseteq W : v \in X \;\Rightarrow\; w \in m_R(X) && \text{by } (*) \\
& \Leftrightarrow \quad \forall X \subseteq W : X \in \pi_v \;\Rightarrow\; m_R(X) \in \pi_w && \text{by the definition of } \pi_v, \pi_w \\
& \Leftrightarrow \quad R^{\mathrm{ue}}\pi_w\pi_v && \text{by the definition of } R^{\mathrm{ue}}
\end{aligned}$$

Therefore, the submodel of $\mathfrak{M}^{\mathrm{ue}}$ obtained by restricting to the principal ultrafilters is an isomorphic copy of $\mathfrak{M}$.

The extra worlds in $\mathfrak{M}^{\mathrm{ue}}$ are non-principle ultrafilters. By Ultrafilter theorem and its corollary, such non-principle ultrafilters exists if W is finite. This justifies the name: ultrafilter **extension**.

超滤扩张只对无穷模型才有实质意义。因为在对无穷模型做超滤扩张的时候确实添加上了额外的点，即那些「非主超滤」。

我们期待得到如下结果：

$$\mathfrak{M}, w \leftrightsquigarrow \mathfrak{M}^{\mathrm{ue}}, \pi_w \qquad (\circledast)$$

It is a bit hard to prove $(\circledast)$ directly since the induction hypothesis would be only about principal ultrafilters in $\mathfrak{M}^{\mathrm{ue}}$, but clearly a principal ultrafilter π_w may have a successor which is a non-principal ultrafilter given that $\mathfrak{M}$ is infinite. Thus we prove the following more general result first:

(虽然要证明的结论更强，但同时能用的归纳假设也就更强。)

定理 2.53 (Truth Lemma of Ultrafilter Extensions). Given a model $\mathfrak{M} = (W, R, V)$, then for any formula φ and any ultrafilter u in $\mathfrak{M}^{\mathrm{ue}}$:

$$\mathfrak{M}^{\mathrm{ue}}, u \Vdash \varphi \;\Leftrightarrow\; V(\varphi) \in u$$

where $V(\varphi) = \{w \in W \mid \mathfrak{M}, w \Vdash \varphi\}$ is the truth set of φ in model $\mathfrak{M}$. $\dashv$

Proof. By induction on φ. (to handle the Boolean cases, we need to use some properties of ultrafilters. hence they are not trivial)

Base case

- $\varphi = \bot$, since $V(\bot) = \emptyset$, then neither $V(\bot) \in u$ (since u is an ultrafilter) nor $u \Vdash \bot$.

- $\varphi = p$,

$$\begin{aligned}
V(p) \in u \quad & \Leftrightarrow \quad u \in V^{\mathrm{ue}}(p) && \text{by the definition of } V^{\mathrm{ue}} \\
& \Leftrightarrow \quad u \Vdash p && \text{by the semantics}
\end{aligned}$$

Induction hypothesis (IH): for any $u \in W^{\mathrm{ue}}$ and any formulas ψ whose length is less then φ, $V(\psi) \in u \;\Leftrightarrow\; \mathfrak{M}^{\mathrm{ue}}, u \Vdash \psi$.

Induction step

Boolean cases:

- $\varphi = \neg\psi$,

$$\begin{array}{rcll} V(\neg\psi) \in u & \Leftrightarrow & W \setminus V(\psi) \in u & \text{since } V(\neg\psi) = W \setminus V(\psi) \\ & \Leftrightarrow & V(\psi) \notin u & \text{since } u \text{ is an ultrafilter} \\ & \Leftrightarrow & u \nVdash \psi & \text{by } (\mathbf{IH}) \\ & \Leftrightarrow & u \Vdash \neg\psi. & \end{array}$$

- $\varphi = \psi \vee \chi$

$$\begin{array}{rcll} V(\psi \vee \chi) \in u & \Leftrightarrow & V(\psi) \cup V(\chi) \in u & \text{since } V(\psi \vee \chi) = V(\psi) \cup V(\chi) \\ & \Leftrightarrow & V(\psi) \in u \text{ or } V(\chi) \in u & \text{by the properties of ultrafilter} \\ & \Leftrightarrow & u \Vdash \psi \text{ or } u \Vdash \chi & \text{by } (\mathbf{IH}) \\ & \Leftrightarrow & u \Vdash \psi \vee \chi. & \end{array}$$

The non-trivial case of modal formula.

For $\varphi = \Diamond\psi$

From left-to-right:

$$\begin{array}{rcll} \mathfrak{M}^{\mathrm{ue}}, u \Vdash \Diamond\phi & \Leftrightarrow & \exists v, R^{\mathrm{ue}}uv, \mathfrak{M}^{\mathrm{ue}}, v \Vdash \psi & \\ & \Leftrightarrow & \exists v, R^{\mathrm{ue}}uv, V(\psi) \in v & \text{by } (\mathbf{IH}) \\ & \Rightarrow & m_R(V(\psi)) \in u & \text{by the definition of } R^{\mathrm{ue}} \\ & \Rightarrow & V(\Diamond\psi) \in u & \text{by } m_R(V(\psi)) = V(\Diamond\psi). \end{array}$$

From right-to-left (和完全性证明中 存在引理 那里的构造方法类似)

Suppose $V(\Diamond\psi) \in u$. We have to find an ultrafilter v such that $R^{\mathrm{ue}}uv$ and $V(\psi) \in v$. Then $\mathfrak{M}^{\mathrm{ue}}, v \Vdash \psi$ by $(\mathbf{IH})$, it follows that $\mathfrak{M}^{\mathrm{ue}}, u \Vdash \Diamond\psi$.

Following we will show that how to find that v.

Let

$$\mathcal{B} := \{Y \subseteq W \mid l_R(Y) \in u\}$$
$$v_0' := \mathcal{B} \cup \{V(\psi)\}$$

If v_0' has the *finite intersection property*, by **Ultrafilter Theorem**, v' can be extended to an ultrafilter v. Moreover, $R^{\mathrm{ue}}uv$ since $\{Y \subseteq W \mid l_R(Y) \in u\} \subseteq v$.

But by **Finite Intersection Lemma** (Lemma 2.42), it suffices to show that

- $\mathcal{B}$ is closed under intersection;
- $\{V(\psi)\}$ is closed under intersection;
- for any $Y \in \mathcal{B}$, $\mathcal{B} \cap V(\psi) \neq \emptyset$.

Hence $\mathcal{B}$ is closed under intersection.

$\{V(\psi)\}$ is closed under intersection is trivial, since it is a singleton set.

. . . .

☛ If $\Box$ is then primary modality symbol, for the case of $\varphi = \Box\psi$:

■

借由上面的这个定理的帮助，我们可以轻松地就证明 (⁎).

定理 2.54. $\mathfrak{M}, w \rightsquigarrow \mathfrak{M}^{\mathrm{ue}}, \pi_w$ for any pointed model $\mathfrak{M}, w$. ⊣

Proof. For any formula φ:

$$\begin{array}{llll} \mathfrak{M}, w \Vdash \varphi & \Leftrightarrow & w \in V(\varphi) & \\ & \Leftrightarrow & V(\varphi) \in \pi_w & \text{by the definition of } \pi_w \text{ (recall that } \pi_w := \{Y \subseteq W \mid w \in Y\}) \\ & \Leftrightarrow & \pi_w \Vdash \varphi & \text{by previous Theorem 2.53} \quad \blacksquare \end{array}$$

下面的定理说明了我们的目标：一个模型可以通过超滤扩张变成一个 m-saturated 模型。

定理 2.55. Given a model $\mathfrak{M}$, then $\mathfrak{M}^{ue}$ is m-saturated. ⊣

Proof. Let $\mathfrak{M} = (W, R, V)$ be a model and $\mathfrak{M}^{ue} = (W^{ue}, R^{ue}, V^{ue})$ its ultrafilter extension. For $\mathfrak{M}^{ue}$ is m-saturated, it suffices to show that for any ultrafilter u in $\mathfrak{M}^{ue}$ and any set of formulas Σ, if Σ is finitely satisfiable in $R[u]$, then Σ is satisfiable in $R[u]$. (where $R[u]$ is the set of successors of u)

Suppose $u \in W^{ue}$ and Σ is a set of formulas which is finitely satisfiable in $R[u]$, we have to show that Σ is satisfiable in $R[u]$, that is, there is an ultrafilter v such that $R^{ue}uv$ and $v \Vdash \Sigma$. Hence we are going to construct an ultrafilter which satisfies these two conditions.

Define

$$\mathcal{A} := \{V(\varphi) \mid \varphi \in \Sigma'\}$$
$$\mathcal{B} := \{Y \mid l_R(Y) \in u\}$$

where Σ' is the set of finite conjunctions of formulas in Σ, that is, $\Sigma' := \{\psi_0 \wedge \cdots \wedge \psi_n \mid n \geq 0, \psi_k \in \Sigma \text{ for each } 0 \leq k \leq n\}$, clearly $\Sigma \subseteq \Sigma'$.

If $\mathcal{A} \cup \mathcal{B}$ has the *finite intersection property*, then by **Ultrafilter Theorem**, $\mathcal{A} \cup \mathcal{B}$ can be extended to an ultrafilter v. Moreover, (i) $R^{ue}uv$ since $\{Y \mid l_R(Y) \in u\} \subseteq v$; and (ii) $V(\varphi) \in v$ for all $\varphi \in \Sigma$, by previous Theorem 2.53, $v \Vdash \Sigma$.

Thus, it is sufficient to show that $\mathcal{A} \cup \mathcal{B}$ has the *finite intersection property*, but by **Finite Intersection Lemma** (see p.38), we only need to show that

1. $\mathcal{A}$ is closes under intersections;
2. $\mathcal{B}$ is closes under intersections;
3. for any $A \in \mathcal{B}$ and any $B \in \mathcal{B}$, $A \cap B \neq \emptyset$.
   (证明这点的时候就需要 m-saturated 的前提条件了)

For any $V(\varphi_i), V(\varphi_j) \in \mathcal{A}$. $V(\varphi_i) \cap V(\varphi_j) = V(\varphi_i \wedge \varphi_j)$. $\varphi_i \wedge \varphi_j \in \Sigma'$ since $\varphi_i, \varphi_j \in \Sigma'$, it implies that $V(\varphi_i) \cap V(\varphi_j) \in \mathcal{A}$. Hence $\mathcal{A}$ is closed under intersections;

It is already be proved that $\mathcal{B}$ is closed under intersections in Theorem 2.53.

Take an arbitrary $\varphi \in \Sigma'$ and an arbitrary $Y \subseteq W$ such that $l_R(Y) \in u$, we need to show $V(\varphi) \cap Y \neq \emptyset$. Since $\varphi \in \Sigma'$, then by assumption that Σ is finitely satisfiable in $R[u]$, there is a successor w of u such that $w \Vdash \varphi$, in other words, $V(\varphi) \in w$ (by Theorem ??). In addition, $l_R(Y) \in u$ with $R^{ue}uw$ implies $Y \in w$. Hence $V(\varphi) \cap Y \in w$ since w is an ultrafilter which is closed under intersections, and $V(\varphi) \cap Y$ cannot be identical to the empty set. ■

定理 2.56 (Bisimilarity-somewhere-else). For any pointed models $\mathfrak{M}, w$ and $\mathfrak{N}, v$,

$$\mathfrak{M}, w \leftrightsquigarrow \mathfrak{N}, v \quad \Rightarrow \quad \mathfrak{M}^{ue}, \pi_w \leftrightarrows \mathfrak{N}^{ue}, \pi_v$$

⊣

$$
\begin{array}{ccc}
\mathfrak{M}, w & \equiv_{\mathsf{ML}} & \mathfrak{N}, v \\
\equiv_{\mathsf{ML}} & & \equiv_{\mathsf{ML}} \\
\mathfrak{M}^{\mathrm{ue}}, \pi_w & \equiv_{\mathsf{ML}} & \mathfrak{N}^{\mathrm{ue}}, \pi_v \\
 & \shortparallel & \\
\text{m-saturated} & \underline{\leftrightarrow} & \text{m-saturated}
\end{array}
$$

Figure 2.2: a detour argument

Proof. By a detour (曲线救国):

Note that the class of m-saturated models has *H-M property*. ■

2.8 中场休息：来看一些例子吧

之前所引入的概念可能过于抽象了，因此单独开辟一节来具体看一些例子。在本节中，之前所出现的比较重要的概念都将轮番出场。

2.8.1 $\mathbb{N}$ 上的滤和超滤

$\mathbb{N}$ 的子集有很多，如

$$A = \{0, 1, 2, 3, 4\}$$
$$B = \{0, 2, 4, 6, \ldots, 2n, \ldots\}$$

但 $\mathbb{N}$ 上的每一个子集（即 $\mathbb{N}$ 上的每一个一元关系）都对应于自然数的某种性质。比如 A 对应性质"小于 5"，B 对应性质"是偶数"。

$\mathbb{N}$ 的任何子集都是某种方程的解集（或视作其特征函数的定义域），如对于任意 $N \in \mathbb{N}$，定义（特征函数）f_N 如下

$$f_N(x) = \begin{cases} 1 & x \in N \\ 0 & x \in \mathbb{N} \setminus N \end{cases}$$

则 N 是方程

$$f_N(x) = 1$$

的解集。而空集是矛盾方程的解集，$\mathbb{N}$ 本身是恒等式的解集（如 $x = x$）。

一个 $\mathbb{N}$ 上的滤子本质上就是自然数某些性质的相容组合。相容是什么意思？如"是偶数"和"小于 5"是相容的性质，但"是偶数"和"是奇数"就是不相容的性质。

换句话说，$\mathbb{N}$ 上的滤子相当于某种相容方程组。

a

2.9 The standard translation (skip)

Standard translation which embeds modal languages into the first-order language (without equality $\equiv$).

Every Kripke model $\mathfrak{M} = (\mathfrak{F}, V)$ can be regarded as a first-order structure.

Every first-order structure of the form$I = (D, R^I, P_0^I, \dots)$ can be considered as a Kripke model.

$$\varphi \in \mathbf{K} \;\Leftrightarrow\; ST_x(\varphi) \in \mathbf{QCl}$$

对于 **S5** 中的公式，此时标准翻译是一个从模态公式集到所有只有一个变元的一阶公式集的双射。is one-one and onto the set of all one-variable first-order formulas. 因此：

the logic **S5** can be regarded as the one-variable fragment of classical first-order logic (Wajsberg 1933).

2.10 Correspondence theory 0: Expressive power

Logical languages can express properties of mathematical structures (models).

The standard notions for comparing how much logical languages can say about models are

(1) **distinguishing power**: can a language tell the difference between two models? and

(2) **expressive power**: which classes of models can be defined by a formula of the language?

Distinguishing power vs. Expressive power

Let $L_1 = (\mathcal{L}_1, \mathcal{C}, \models)$ and $L_2 = (\mathcal{L}_2, \mathcal{C}, \Vdash)$ be two logics defined on the same class of models $\mathcal{C}$.

We say logic L_2 is **at least as distinguishing as** logic L_1, notation $L_1 \preccurlyeq_d L_2$, iff

$$\forall (\mathfrak{M}, w), (\mathfrak{N}, v) \in \mathcal{C} : (\mathfrak{M}, w) \parallel_{\mathcal{L}_1} (\mathfrak{N}, v) \ \Rightarrow \ (\mathfrak{M}, w) \parallel_{\mathcal{L}_2} (\mathfrak{N}, v)$$

where $(\mathfrak{M}, w) \parallel_{\mathcal{L}} (\mathfrak{N}, v)$ means that $(\mathfrak{M}, w) \not\equiv_{\mathcal{L}} (\mathfrak{N}, v)$.

Note that $L_1 \preccurlyeq_d L_2$ iff $\equiv_{\mathcal{L}_2} \subseteq \equiv_{\mathcal{L}_1}$.

We say logic L_2 is **at least as expressive as** logic L_1, notation $L_1 \preccurlyeq_e L_2$, iff

$$\forall \varphi_1 \in \mathcal{L}_1, \exists \varphi_2 \in \mathcal{L}_2 : \forall (\mathfrak{M}, w) \in \mathcal{C} : \mathfrak{M}, w \models \varphi_1 \ \Leftrightarrow \ \mathfrak{M}, w \Vdash \varphi_2.$$

Say L_1 and L_2 are **equally distinguishing** , $L_1 \approx_d L_2$, if $L_1 \preccurlyeq_d L_2$ and $L_2 \preccurlyeq_d L_1$.

L_1 and L_2 are **equally expressive** , $L_1 \approx_e L_2$, if $L_1 \preccurlyeq_e L_2$ and $L_2 \preccurlyeq_e L_1$.

It is not hard to prove that: $L_1 \preccurlyeq_e L_2 \Rightarrow L_1 \preccurlyeq_d L_2$, but the converse may fail. For example, compare the propositional logic and its syntactic fragment with proposition letters only, they have the same distinguishing power bur different expressive power.

Therefore, showing that there exists a pair of models that on logic can distinguish but the other one cannot is sufficient to demonstrate that these two logics have different expressive.

We have seen such idea before:

If a property can 'distinguish' two models that are modally-equivalent (or bisimilar) then this property cannot be expressed by a modal formula.

【On the other hand, in general, by showing that there is no such a pair, we cannot prove immediately that the two logics have the same expressive power. However, there are cases where the comparison of expressive power can be reduced to the comparison of distinguishing power. Modal logic is also such an example when compared to FOL. 】

2.11 Standard translation: modal logic as a fragment of FOL

Kripke models $(W, \{R_\triangledown\}_{\triangledown \in \tau}, V)$ can be viewed as first-order structures:

- W is the domain;
- $R_\triangledown$ is the interpretation for a relation symbol;

- $V(p)$ is the interpretation of a predicate for each p.

Thus we can use first-order formulas (with one free variable) to express the meaning of modal formulas.

定义 2.57 (Corresponding First-Order Language)**.** Fix a Prop. Given a modal similarity τ, the **corresponding first-order language** (with **equality** $\equiv$) $\mathcal{L}^{\tau}_{\text{FOL}}$ of modal language $\mathcal{L}^{\tau}_{\text{ML}}$ has

- infinitely many unary P corresponding to $p \in \mathsf{Prop}$; and
- $n+1$-ary relation symbols R_{∇} for each $\nabla \in \tau$ with $\rho(\nabla) = n$.

Then the formula in $\mathcal{L}^{\tau}_{\text{FOL}}$ is given by:

$$\mathcal{L}^{\tau}_{\text{FOL}} \ni \phi ::= Px \mid x \equiv x \mid R_{\nabla}x \dots x \mid \neg\phi \mid (\phi \wedge \phi) \mid \forall x \phi.$$

We write $\phi(x)$ for a first-order formula with one free variable x. ⊣

【以后用变体的 φ 表示模态公式，正体的 ϕ 表示一阶公式。
$\Vdash$ 表示模态中的语义关系，而 $\models$ 表示经典逻辑（命题 + 一阶）中的语义关系】

定义 2.58 (Standard translation)**.** $ST_x \colon \mathcal{L}^{\tau}_{\text{ML}} \to \mathcal{L}^{\tau}_{\text{FOL}}$ ⊣

定理 2.59 (Local and Global Correspondence on *Models*)**.**

$$\begin{array}{lrcl} (1) & \mathfrak{M}, w \Vdash \varphi & \Leftrightarrow & \mathfrak{M} \models ST_x(\varphi)[x \mapsto w] \\ (2) & \mathfrak{M} \Vdash \varphi & \Leftrightarrow & \mathfrak{M} \models \forall x ST_x(\varphi) \end{array}$$

⊣

Proof. By induction on φ. ■

2.12 Heritages from FOL

主要继承那些比较 universe 的性质。

2.12.1 Compactness

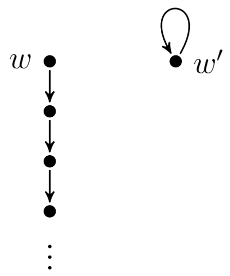
定理 2.60 (Compactness Theorem). Every finitely satisfiable set of modal formulas is satisfiable. ⊣

2.12.2 Löwenhenim-Skolem Theorem

定理 2.61 (Löwenhenim-Skolem Theorem). If a set of modal formulas is satisfiable in at least one infinite model, then it is satisfiable in models of every infinite cardinality (assuming the modal language is countable) ⊣

By ST, any modal formula is equivalent to a formula $\alpha(x)$ is the corresponding first-order language with on free variable. Clearly the converse is not true.

For example, Rxx



$w \leftrightarroweq w'$, 但是右边的模型自反而左边的不自反，因此自反性在基本的模态语言中不可表达。

Thus, ML is a proper fragment of $\mathcal{L}^{\tau}_{FOL}$

Some firs-order properties can not be express in ML.

Table 2.2: Some firs-order properties can not be express in ML.

name	first-order formula
自反性	ffffff
反自反	dddd

☞ Question: Which $\alpha(x)$ is equivalent to a modal formula? In other word, we would like to "characterize" the fragment of $\mathcal{L}^{\tau}_{FOL}$ which corresponds to ML.

☞ An answer: van Benthem characterization theorem.

2.13 van Benthem Characterization theorem: Characterizing modal logic in FOL

2.13.1 Ultraproducts

定义 2.62 (Ultraproducts of Set 超积)**.** 内容... ⊣

定义 2.63 (Ultraproducts of Models)**.** 内容... ⊣

命题 2.64 (Ultraproducts Invariant)**.** 内容... ⊣

2.13.2 van Benthem Characterization Theorem

2.13.3 Model Definability

2.14 Rosen's characterization theorem: first exploration of the Finite Modal Model Theory

2.14.1 Ehrenfeucht-Fraïssé games (EF-games)

bisimulation game 的一阶对应物

定理 2.65 (Rosen's Theorem)**.** 内容... $\dashv$

2.15 Definability of models class

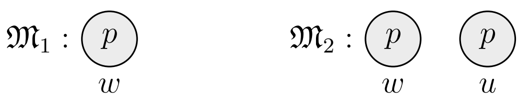
2.16 Selected exercises for Ch.2

homework No.4 (2023,03,15)

2.1.1 Suppose we wanted an operator D with the following satisfaction definition: for any model $\mathfrak{M}$ and any formula ϕ, $\mathfrak{M}, w \Vdash D\phi$ iff there is a $u \neq w$ such that $\mathfrak{M}, u \Vdash \phi$. This operator is called the *difference operator* and we will discuss it further in Section 7.1. Is the difference operator definable in the basic modal language?

Proof. Suppose for the sake of contradiction that D is definable in the basic modal language.

Then there is an expression $\alpha(p)$ containing only symbols from the basic modal language,
such that for any model $\mathfrak{M}$,
we have $\mathfrak{M}, w \Vdash \alpha(p) \;\Leftrightarrow\; \mathfrak{M}, w \Vdash Dp$.
Considering following two models

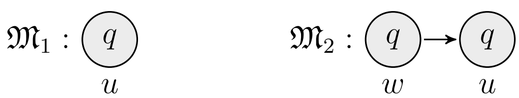


Then $\mathfrak{M}_1, w \nVdash \alpha(p)$ but $\mathfrak{M}_2, w \Vdash \alpha(p)$ by the semantics of D.
Note that $\mathfrak{M}_1$ is a generated submodel of $\mathfrak{M}_2$ (generated by $\{w\}$),
hence $\mathfrak{M}_1, w \Vdash \alpha(p)$ by *modal satisfaction is invariant under generated submodel.*
Contradiction!
Therefore difference operator is not definable in the basic modal language. ■

2.1.2 Use generated submodels to show that the backward looking modality (that is,the P of the basic temporal language) cannot be defined in terms of the forward looking operator $\Diamond$.

Proof. Suppose for the sake of contradiction that D is definable in terms of operator $\Diamond$.

Then we could find an expression $\alpha(p)$ containing only symbols from the basic modal language,
such that for any model $\mathfrak{M}$,
we have $\mathfrak{M}, w \Vdash \alpha(q) \;\Leftrightarrow\; \mathfrak{M}, w \Vdash Pq$.
Considering following two models



Then $\mathfrak{M}_1, u \not\Vdash \alpha(p)$ but $\mathfrak{M}_2, u \Vdash \alpha(p)$ by the semantics of P.
Note that $\mathfrak{M}_1$ is a generated submodel of $\mathfrak{M}_2$ (generated by $\{u\}$),
hence $\mathfrak{M}_1, u \Vdash \alpha(p)$ by *modal satisfaction is invariant under generated submodel.*
Contradiction!
Therefore P is not definable in terms of operator $\Diamond$. ■

2.1.4 Show that the mapping f defined in the proof of Proposition 2.15 is indeed a surjective bounded morphism.

Proof. Following we show that f is a bounded morphism and surjective.
(Note that we use $f(w, u_1, \ldots, u_n)$ instead of $f((w, u_1, \ldots, u_n))$ for convenience)

For bounded morphism:

1. By the definition of V', that $(w, u_1, \ldots, u_n) \in V'(p)$ iff $u_n = f(w, u_1, \ldots, u_n) \in V(p)$;

2. We have to show that if $(w, u_1, \ldots, u_n)R'(w, v_1, \ldots, v_m)$, then $f(w, u_1, \ldots, u_n)Rf(w, v_1, \ldots, v_m)$.
   Suppose $(w, u_1, \ldots, u_n)R'(w, v_1, \ldots, v_m)$,
   By the definition of R',
   we have Ru_nv_m,
   moreover, $f(w, u_1, \ldots, u_n) = u_n, f(w, v_1, \ldots, v_m) = v_m$ by the definition of f.
   Hence $f(w, u_1, \ldots, u_n)Rf(w, v_1, \ldots, v_m)$.

3. We have to show that if $f(w, u_1, \ldots, u_n)Rv_m$ then $\exists(w, v_1, \ldots, v_m) \in W'$ such that $(w, u_1, \ldots, u_n)R'(w, v_1, \ldots, v_m)$ and $f(w, v_1, \ldots, v_m) = v_m$.

   Assume $f(w, u_1, \ldots, u_n)Rv_m$,
   then by the definition, there is a path $(w, u_1, \ldots, u_n, v_m)$ in $\mathfrak{M}$.
   Hence $(w, u_1, \ldots, u_n, v_m) \in W'$. By the definition of R' and f, we have $(w, u_1, \ldots, u_n)R'(w, u_1, \ldots, u_n, v_m)$ and $f(w, u_1, \ldots, u_n, v_m) = v_m$.

For subjective:
we have to show that
for all $u \in W$, there is $(w, u_1, \ldots, u_n) \in W'$ such that $f(w, u_1, \ldots, u_n) = u$.
Let u be any state in $\mathfrak{M}$, note that $\mathfrak{M}$ is *rooted*,
which means that there is a path from the root w to u in $\mathfrak{M}$.
Suppose this path is $(w, u_1, \ldots, u_n)$ where $u_n = u$,
then $(w, u_1, \ldots, u_n) \in W'$ by the construction of unraveling,
hence $f(w, u_1, \ldots, u_n) = u_n = u$. ■

Proposition 2.19 Let τ be a modal similarity type, and let $\mathfrak{M}$, $\mathfrak{M}'$ and $\mathfrak{M}_i$ ($i \in I$) be τ-models.

(i) If $\mathfrak{M} \cong \mathfrak{M}'$, then $\mathfrak{M} \leftrightarroweq \mathfrak{M}'$.

(ii) For every $i \in I$ and every w in $\mathfrak{M}_i$, $\mathfrak{M}_i, w \leftrightarroweq \biguplus_i \mathfrak{M}_i, w$.

(iii) If $\mathfrak{M}' \rightarrowtail \mathfrak{M}$, then $\mathfrak{M}', w \leftrightarroweq \mathfrak{M}, w$ for all w in $\mathfrak{M}'$.

(iv) If $f : \mathfrak{M} \twoheadrightarrow \mathfrak{M}'$, then $\mathfrak{M}, w \leftrightarroweq \mathfrak{M}', f(w)$ for all w in $\mathfrak{M}$.

Proof. We are only working in the basic modal language here.

(i)

Suppose $\mathfrak{M} = (W, R, V), \mathfrak{M}' = (W', R', V)$ and $\mathfrak{M} \cong \mathfrak{M}'$,
which means that there is a isomorphism f from $\mathfrak{M}$ into $\mathfrak{M}'$.
Define a binary relation $Z \subseteq W \times W'$ by

$$(w, w') \in Z \;\Leftrightarrow\; f(w) = w'$$

Following show that Z is a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$.

1. For *atom condition*:
   if wZw', which means $f(w) = w'$, then w and w' satisfy the same propositional letters since f is a isomorphism.

2. For *forth condition*:
   if wZw' and Rwv.
   Since f is a isomorphism, then $R'f(w)f(v)$ by Rwv.
   Moreover, $vZf(v)$ and $f(w) = w'$ by definition of Z.
   That is there exists $f(v)$ in $\mathfrak{M}'$ such that $vZf(v)$ and $R'w'f(v)$.

3. For *back condition*:
   if wZw' and $R'w'v'$,
   then $f(w) = w'$ by the definition of Z.
   Moreover, there is a v in $\mathfrak{M}$ such that Rwv and $f(v) = v'$ since f is a isomorphism.
   Therefore, there exists v in $\mathfrak{M}$ such that vZv' and Rwv.

Hence $\mathfrak{M} \leftrightarroweq \mathfrak{M}'$ since there is a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$.

(ii)

It has been proven in p.66 of the Blue book.

(iii)

Suppose $\mathfrak{M} = (W, R, V), \mathfrak{M}' = (W', R', V)$ and $\mathfrak{M}' \rightarrowtail \mathfrak{M}$,
which means that $\mathfrak{M}'$ is a generated submodel of $\mathfrak{M}$.
Let $Z \coloneqq \{(w, w) \mid w \in W'\}$.
Following show that Z is a bisimulation.

1. For *atom condition*: trivially.

2. For *forth condition*:
   if $w'Zw$ and $R'w'v'$,
   then $w' = w$ and $v'Zv'$ by the definition of Z.
   Let $v = v'$, and $R'w'v'$ implies Rwv since $\mathfrak{M}'$ is a submodel of $\mathfrak{M}'$.
   That is there exists v in $\mathfrak{M}$ such that $v'Zv$ and Rwv.

3. For *back condition*:
   if $w'Zw$ and Rwv,
   then $w' = w$ and vZv by the definition of Z.
   Moreover, v is in $\mathfrak{M}'$ by the definition of generated submodel.
   Let $v' = v$, and Rwv implies $R'w'v'$ since $\mathfrak{M}'$ is a submodel of $\mathfrak{M}'$.
   That is there exists v' in $\mathfrak{M}'$ such that $v'Zv$ and $R'w'v'$.

By the definition of Z, then for all w in $\mathfrak{M}'$ we have $\mathfrak{M}', w \;\underline{\leftrightarrow}\; \mathfrak{M}, w$.

(iv)

Suppose $\mathfrak{M} = (W, R, V), \mathfrak{M}' = (W', R', V)$ and $f : \mathfrak{M} \twoheadrightarrow \mathfrak{M}'$,
which means that $\mathfrak{M}'$ is a bounded morphic image of $\mathfrak{M}$ w.r.t. f.
Let $Z := \{(w, f(w)) \mid w \in W\}$
Following show that Z is a bisimulation.

1. For *atom condition*:
   trivially by the definition of bounded morphism.

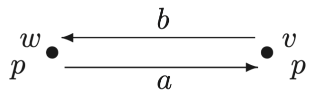
2. For *forth condition*:
   if $wZf(w)$ and Rwv,
   then $R'f(w)f(v)$ by Rwv,
   and $vZf(v)$ by the definition of Z.

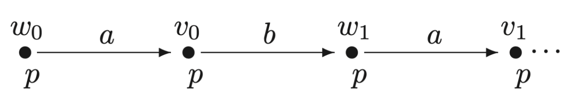
   That is there exists $f(v)$ in $\mathfrak{M}'$ such that $vZf(v)$ and $R'f(w)f(v)$.

3. For *back condition*:
   if $wZf(w)$ and $R'f(w)v'$,
   by the *back condition* of bounded morphism,
   there exists v such that Rwv, and $f(v) = v'$,
   that is vZv' since $f(v) = v'$. ■

homework No.5 (2023,03,22)

2.2.1 (p71) Consider a modal similarity type with two diamonds $\langle a \rangle$ and $\langle b \rangle$, and with $\Phi = \{p\}$. Show that the following two models are bisimilar:





Proof. Let $\mathfrak{M}_1 = (W_1, R_a, R_b, V_1)$ be the left model, and $\mathfrak{M}_2 = (W_2, R'_a, R'_b, V_2)$ the right model.

It suffices to show that $\mathfrak{M}_2 \twoheadrightarrow \mathfrak{M}_1$, viz., $\mathfrak{M}_1$ is a bounded morphic image of $\mathfrak{M}_2$. Then by **Proposition 2.19 (iv)**, $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are bisimilar.

Let f be a map from $\mathfrak{M}_2$ to $\mathfrak{M}_1$ given by

$$f(w_i) = w$$
$$f(v_i) = v$$

where $i \geq 0$. It follows that f is a surjective bounded morphism from $\mathfrak{M}_2$ to $\mathfrak{M}_1$.

For bounded morphism:

1. Obviously $f(x)$ and x satisfy the same proposition letters for any state x in $\mathfrak{M}_2$.

2. Suppose $R'_a w_i v_i$ and $R'_b v_i w_{i+1}$ for all $i \geq 0$,
   it follows that $f(w_i) = f(w_{i+1}) = w$, $f(v_i) = v$ by the definition of f,
   Hence $R_a wv$ and $R_b vw$ which holds in $\mathfrak{M}_1$.

3. Suppose $R_a f(w_i) v$, then $f(v_i) = v$ and $R'_a w_i v_i$;
   Suppose $R_b f(v_i) w$, then $f(w_{i+1}) = w$ and $R'_b v_i w_{i+1}$.

For f is surjective, it's trivial since $f(w_0) = w$ and $f(v_0) = v$. ■

Proposition 2.31 (p.75)

Proof. $\Leftarrow$

Suppose $w \leftrightsquigarrow_n w'$, then we have to show that $w \leftrightarroweq_n w'$.

It suffices to show that there exists a sequence of binary relations satisfy those conditions of the definition for n-bisimulation.

Following we show that $\leftrightsquigarrow_n, \leftrightsquigarrow_{n-1}, \ldots, \leftrightsquigarrow_0$ are the relations what we need.

Obviously $\leftrightsquigarrow_n \subseteq \leftrightsquigarrow_{n-1} \subseteq \cdots \subseteq \leftrightsquigarrow_0$ by the definition of $\leftrightsquigarrow_n$.

1. $w \leftrightsquigarrow_n w'$ by assumption.

2. If $v \leftrightsquigarrow_0 v'$, then v and v' agree on all formulas φ with $deg(\varphi) \leq 0$
   obviously v and v' agree on all proposition letters.

3. If $v \leftrightsquigarrow_{i+1} v'$ and Rvu (where $i \leq n-1$).
   Then we need to find a u' in $\mathfrak{M}'$ such that $R'v'u'$ and $u \leftrightsquigarrow_i u'$.

   Let $\Gamma = \{\psi \mid u \Vdash \psi \text{ and } deg(\psi) \leq i\}$

   Define a relation $\sim$ on Γ by
   $$\phi \sim \theta \Leftrightarrow \Vdash \phi \leftrightarrow \theta$$
   it is easy to check that $\sim$ is a equivalence relation.

   Then the numbers of equivalence classes under $\sim$ is finite by Proposition 2.29-(i) p74,

   say $[\psi_1], [\psi_2], \ldots, [\psi_n]$ are those equivalence classes.

   Let $\psi = \psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_n$.

Then $u \Vdash \psi$ and $deg(\psi) \leq i$ obviously.

hence $v \Vdash \Diamond\psi$ since Rvu.

By $v \leftrightsquigarrow_{i+1} v'$ we have $v' \Vdash \Diamond\psi$ since $deg(\Diamond\psi) \leq i+1$.

Following $\exists u', R'v'u'$ and $u' \Vdash \psi$.

by the construction of ψ and modulo modal equivalence,

we have $u' \Vdash \Gamma$.

then $u \leftrightsquigarrow_i u'$ in other words.

4. If $v \leftrightsquigarrow_{i+1} v'$ and $R'v'u'$ (where $i \leq n-1$).
   Then we need to find u in $\mathfrak{M}$ such that Rvu and $u \leftrightsquigarrow_i u'$.

   We can find that u in a similar way above. ■

2.17 Modal Model Theory: A Summary

Key words: bisimulation $\leftrightarrows$, modal equivalence $\leftrightsquigarrow$, game, n-bisimulation $\leftrightarrows_n$, fmp, selection, filtration $\mathfrak{M}^f = (W^f, R^f, V^f)$, filter and ultrafilter, ultrafilter extension $\mathfrak{M}^{\mathrm{ue}} = (W^{\mathrm{ue}}, R^{\mathrm{ue}}, V^{\mathrm{ue}})$, ultraproduct,

Some model construction methods:

- bounded morphism, disjoint union, generated submodel, unravelling (tree)
- bisimulation contraction
- filtration
- ultrafilter extension, ultraproduct, ultrapower, ultrafilter union,

Important result:

- van Benthem characterization theorem
- Rosen characterization theorem

This is DuoDuo 🐈 Happy Modal Logic!

Chapter 3

Frame theory

有效性本质上是个二阶概念。

3.1 Frame Definability

定义 3.1 (Validity)**.** 内容... ⊣

定义 3.2 (Definability)**.** 内容... ⊣

一个模态公式所定义的框架类是唯一的吗????

定义 3.3 (Frame Languages)**.** 内容... ⊣

定义 3.4 (Frame Correspondence)**.** 内容... ⊣

命题 3.5 (Some frame correspondence results)**.** ddddd

Table 3.1: some frame correspondence results

properties	modal formula	dual formula
$\forall x Rxx$	(T) $p \to \Diamond p$	$\Box p \to p$
	(4) $\Diamond\Diamond p \to \Diamond p$	$\Box p \to \Box\Box p$
	(5) $\Diamond p \to \Box\Diamond p$	$\Diamond\Box p \to \Box p$

ddddd ⊣

Proof. 内容... ■

3.2 Frame Definability and Second-Order Logic

命题 3.6 (Frame Correspondence)**.**

$$\begin{aligned} \mathfrak{F}, w \Vdash \varphi \quad &\Leftrightarrow \quad \mathfrak{F} \models \\ \mathfrak{F} \Vdash \varphi \quad &\Leftrightarrow \quad \mathfrak{F} \models \end{aligned}$$

⊣

Proof. 内容... ■

3.3 Definable and Undefinable Properties

3.4 Finite Frame (skip)

3.5 Automatic First-Order Correspondence

3.6 Sahlqvist Theory

3.7 Advanced Frame Theory

Chapter 4

Completeness

4.1 Normal Modal Logics

4.2 Canonical Models

4.3 Completeness via Canonicity

4.4 Incompleteness

4.5 Step by Step

Chapter 5

Algebra Semantics

5.1 Universal algebras

定义 5.1 (Similarity type)**.** 内容... ⊣

定义 5.2 (Algebras)**.** 内容... ⊣

定义 5.3 (Homomorphisms、Homomorphic image、Isomorphism)**.** 内容... ⊣

定义 5.4 (Subalgebras)**.** 内容... ⊣

定义 5.5 (Product algebras)**.** 内容... ⊣

定义 5.6 (Varieties)**.** $\mathbb{V}\mathsf{C}$ ⊣

定理 5.7 (Bffo... theorem)**.**

$$\mathbb{V}\mathsf{C} = \mathbf{HSP}\mathsf{C}$$

⊣

定义 5.8 (Congruences)**.** 内容... ⊣

定义 5.9 (Quotient algebras)**.** 内容... ⊣

命题 5.10 (Homomorphism and Congruences)**.** 内容... ⊣

5.2 Algebraic model theory

定义 5.11 (Algebra language)**.** The set $Ter(\mathcal{T}, X)$ of **terms** is given by following rule:

$$Ter(\mathcal{T}, X) \ni t ::= x \mid f(t_1, \ldots, t_{\rho(f)})$$

where $x \in X, f \in \mathcal{T}$ and $t_1, \ldots, t_{\rho(f)} \in Ter(\mathcal{T}, X)$.

A **equation** is a pair of terms, notation $s \approx t$. ⊣

定义 5.12 (Term algebras)**.** 内容... ⊣

定义 5.13. 内容... ⊣

5.3 Boolean algebras & Propositional logic

5.3.1 Boolean algebras

5.3.2 Lindenbaum-Tarski algebras

5.3.3 Stone's Theorem

5.3.4 Completeness of PL via algebra

5.4 Algebraic semantics for Modal Logics

Chapter 6

Hybrid Logic

This chapter cf:

- the Sec.-7.3 in the Blue Book ;
- ....
- ...

the sates of model are not directly reflected in **syntax**.

Some slogans:

•

• States are first class citizens.

6.1 So many hybrid languages

6.2 Basic hybrid language $\mathcal{L}_@$

定义 6.1 (The basic hybrid language $\mathcal{L}_@$)**.**

$$\mathcal{L}_@ \ni ::= i \mid p \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid @_i\varphi,$$

where $i \in \mathsf{NOM}, p \in \mathsf{Prop}$. $\dashv$

For semantics, given a frame $\mathfrak{F} = (W, R)$, a valuation is $V :$ $\mathsf{Prop} \cup \mathsf{NOM}$ such that $V(i)$ is singleton for each $i \in \mathsf{NOM}$.

we have:

$$\begin{array}{lcl} \mathfrak{M}, w \Vdash i & \Leftrightarrow & w \in V(i) \quad (or \ \ \Leftrightarrow \ \ i^V = w) \\ \mathfrak{M}, w \Vdash @_i\varphi & \Leftrightarrow & \mathfrak{M}, i^V \Vdash \varphi \end{array}$$

6.2.1 Expressivity

All pure formulas define first-order frame conditions.

6.3 Axiom system $\mathbf{K}_@$ and $\mathbf{K}_@^+$

Following are some theorem:

不同的文献对最小的正规混合逻辑有不同的公理化：

Table 6.1: expressivity of $\mathcal{L}_@$

properties	definable in	the basic modal language
reflexivity	$\forall x Rxx$	$i \to \Diamond i$
symmetry	$\forall xy(Rxy \to Ryx)$	$i \to \Box\Diamond i$
transitivity		$\Diamond\Diamond i \to \Diamond i$
density		$\Diamond i \to \Diamond\Diamond i$
determinism		$\Diamond i \to \Box i$
properties	not definable in	the basic modal language
irreflexivity		$i \to \neg\Diamond i$
asymmetry		$i \to \neg\Diamond\Diamond i$
antisymmetry		$i \to \Box(\Diamond i \to i)$
intransitivity		$\Diamond\Diamond i \to \neg\Diamond i$
universality		$\Diamond i$
trichotomy		$@_j\Diamond i \vee @_j i \vee @_i\Diamond j$
at most 2 states		$@_i(\neg j \wedge \neg k) \to @_j k$

Table 6.2: axiom system $\mathbf{K}_@$ and $\mathbf{K}_@^+$ (I Love Hybrid Logic :)

$\mathbf{K}_@$		comments
(PC)	all tautologies	
(K)	$\Box(p \to q) \to (\Box p \to \Box q)$	
(Dual)	$\Diamond p \leftrightarrow \neg\Box\neg p$	
(MP)	$\varphi \to \psi, \varphi \ / \ \psi$	
(Gen$_\Box$)	$\varphi \ / \ \Box\varphi$	
(Gen$_@$)	$\varphi \ / \ @_i\varphi$	
(Sub-sorted)		
(K$_@$)	$@_i(p \to q) \to (@_i p \to @_i q)$	
(Self-dual)	$@_i p \leftrightarrow \neg @_i \neg\varphi$	
(Intro)	$i \wedge p \to @_i p$	
(Ref)	$@_i i$	
(Sym)	$@_i j \leftrightarrow @_j i$	
(Nom)	$@_i j \wedge @_j p \to @_i p$	particularly (Tran): $@_i j \wedge @_j k \to @_i k$
(Agree)	$@_j @_i p \leftrightarrow @_i p$	
(Back)	$\Diamond @_i p \to @_i p$	the interaction between $\Diamond$-$@$
extra rules for $\mathbf{K}_@^+$		
(Name)	$j \to \theta \ / \ \theta$	in both, $j \neq i$ and $j \notin \varphi \wedge \theta$.
(Paste)	$@_i\Diamond j \wedge @_j\varphi \to \theta \ / \ @_i\Diamond\varphi \to \theta$	

(Elim)	$(i \wedge @_i p) \to p$	
(Bridge)	$\Diamond i \wedge @_i p \to \Diamond p$	another $@$-$\Diamond$ interaction principle

(NAME′) $\dfrac{@_j\varphi}{\varphi}$ (this version of NAME rule also works)

引理 6.2 (Lemma 7.22). Let $\mathfrak{M} = (\mathfrak{F}, V)$ be a named model and ϕ a pure formula. Suppose that for all pure instances ψ of ϕ, $\mathfrak{M} \Vdash \phi^\sigma$. Then $\mathfrak{F} \Vdash \phi$. $\dashv$

Proof. Suppose for the sake of contradiction that $\mathfrak{F} \nVdash \phi$. Then $(\mathfrak{F}, V'), w \nVdash \phi$ for some valuation V' and state w. Let $i_1, \ldots, i_n$ be the nominals occurring in ϕ. Let $j_1, \ldots, j_n$ be nominals such that $V(j_1) = V'(i_1)$, ...,$V(j_n) = V'(i_n)$, such nominals exist since all states in $\mathfrak{F}$ were named under V. Since $(\mathfrak{F}, V'), w \nVdash \phi$, then we have that $(\mathfrak{F}, V), w \nVdash \phi[j_1/i_1, \ldots, j_n/i_n]$ (by Lemma 【】), but that contradicts with $\mathfrak{M} \Vdash \phi^\sigma$ for any substitution σ. We conclude that $\mathfrak{F} \Vdash \phi$. ■

6.4 完全性

在语言 $\mathcal{L}_{\mathcal{H}(@,\downarrow)}$ 中，所有 Sahlqvist 公式能定义的框架类，纯公式都能定义。因此在语言 $\mathcal{L}_{\mathcal{H}(@,\downarrow)}$ 中，纯完全性包括了 *Sahlqvist* 完全性。

但是在语言 $\mathcal{L}_{\mathcal{H}(@)}$ 中，纯完全性和 *Sahlqvist* 完全性不一样，因为在 $\mathcal{L}_{\mathcal{H}(@)}$ 中存在 Sahlqvist 公式可定义的框架类，但纯公式不能定义，如

$$(\text{CR}) \qquad \Diamond\Box p \to \Box\Diamond p$$

是 Sahlqvist 公式但不是纯公式。

然而如果在 $\mathcal{L}_{\mathcal{H}(@)}$ 中添加逆模态算子（如基本时态语言），则每个 Sahlqvist 公式都可以转化为纯公式，此时纯完全性也就可以蕴含 *Sahlqvist* 完全性。

定理 6.3 (Pure completeness)**.**

1. Let Σ be any set of pure formulas of $\mathcal{L}_{\mathcal{H}(@)}$. Then $\mathsf{K}_{\mathcal{H}(@)} + \Sigma$ is strongly complete for the class of frames defined by Σ.
2. Let Σ be any set of pure formulas of $\mathcal{L}_{\mathcal{H}(@,\downarrow)}$. Then $\mathsf{K}_{\mathcal{H}(@,\downarrow)} + \Sigma$ is strongly complete for the class of frames defined by Σ.

$\dashv$

定理 6.4 (Sahlqvist completeness)**.** Let Σ be any set of Sahlqvist formulas in $\mathcal{L}_{\mathcal{H}(@)}$. Then $\mathsf{K}_{\mathcal{H}(@)} + \Sigma$ is strongly complete for the class of frames defined by Σ. $\dashv$

6.4.1 The proof of pure completeness

引理 6.5. Let $\mathfrak{M} = (\mathfrak{F}, V)$ be a named model and φ a pure formula. If $\mathfrak{M} \Vdash \varphi^\sigma$ for any substitution σ, then $\mathfrak{F} \Vdash \varphi$.

For any any substitution σ, $\mathfrak{M} \Vdash \varphi^\sigma$ iff $\mathfrak{F} \Vdash \varphi$. $\dashv$

Proof. By induction on φ.

If $\varphi = i$. For any substitution σ, $\mathfrak{M} \Vdash i^\sigma$ is impossible, that is vacuous truth.

If $\varphi = \neg\psi$ and ψ is a pure formula.

If $\varphi = \psi \wedge chi$ and ψ, χ are pure formulas.

If $\varphi = \Diamond\psi$ and ψ is a pure formula.

If $\varphi = @_i\psi$ and ψ is a pure formula. ■

The lemma 6.5 says that for *named models* and *pure formulas* the gap between *truth* in a model and *validity* in a frame is non-existent.

定义 6.6 (named set and)**.** 内容... $\dashv$

引理 6.7 (Lindenbaum lemma)**.** Every $\mathsf{K}_{\mathcal{H}(@)} + \Sigma$-consistent set Γ can be extended to a maximal $\mathsf{K}_{\mathcal{H}(@)} + \Sigma$-consistent set Γ^+ such that

1. $\exists i \in \Gamma^+$ and i is a nominal; (Γ^+ 有名字)
2. If $@_i\Diamond\varphi \in \Gamma^+$, then there is a nominal j such that $@_i\Diamond j \in \Gamma^+$ and $@_j\varphi \in \Gamma^+$. ($\Diamond$ 饱和)

$\dashv$

6.5 可判定性和计算复杂度

一些可判定性和计算复杂度结果：

1. 在所有框架类上，$\mathcal{L}_{\mathcal{H}(@)}$ 的 SAT 问题是 PSPACE-complete。
2.
3.

混合语言的表达力谱系:

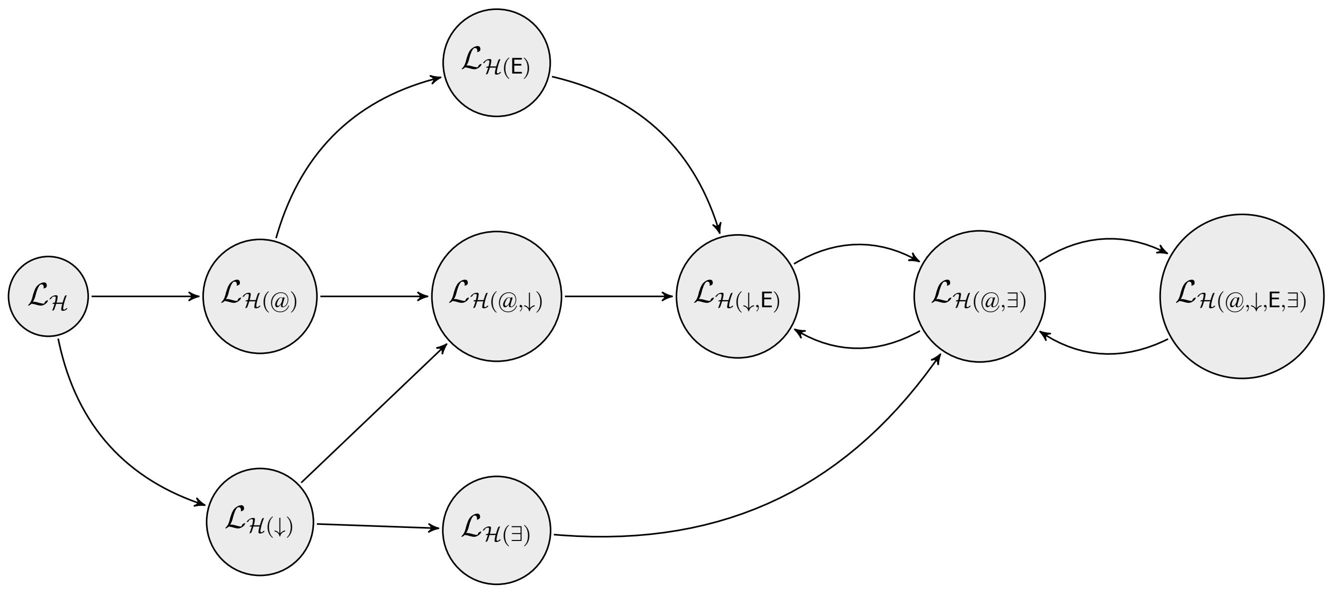


Figure 6.1: 混合语言表达力谱系

Bibliography