

Week 5: Neural Networks

Matthew Caldwell

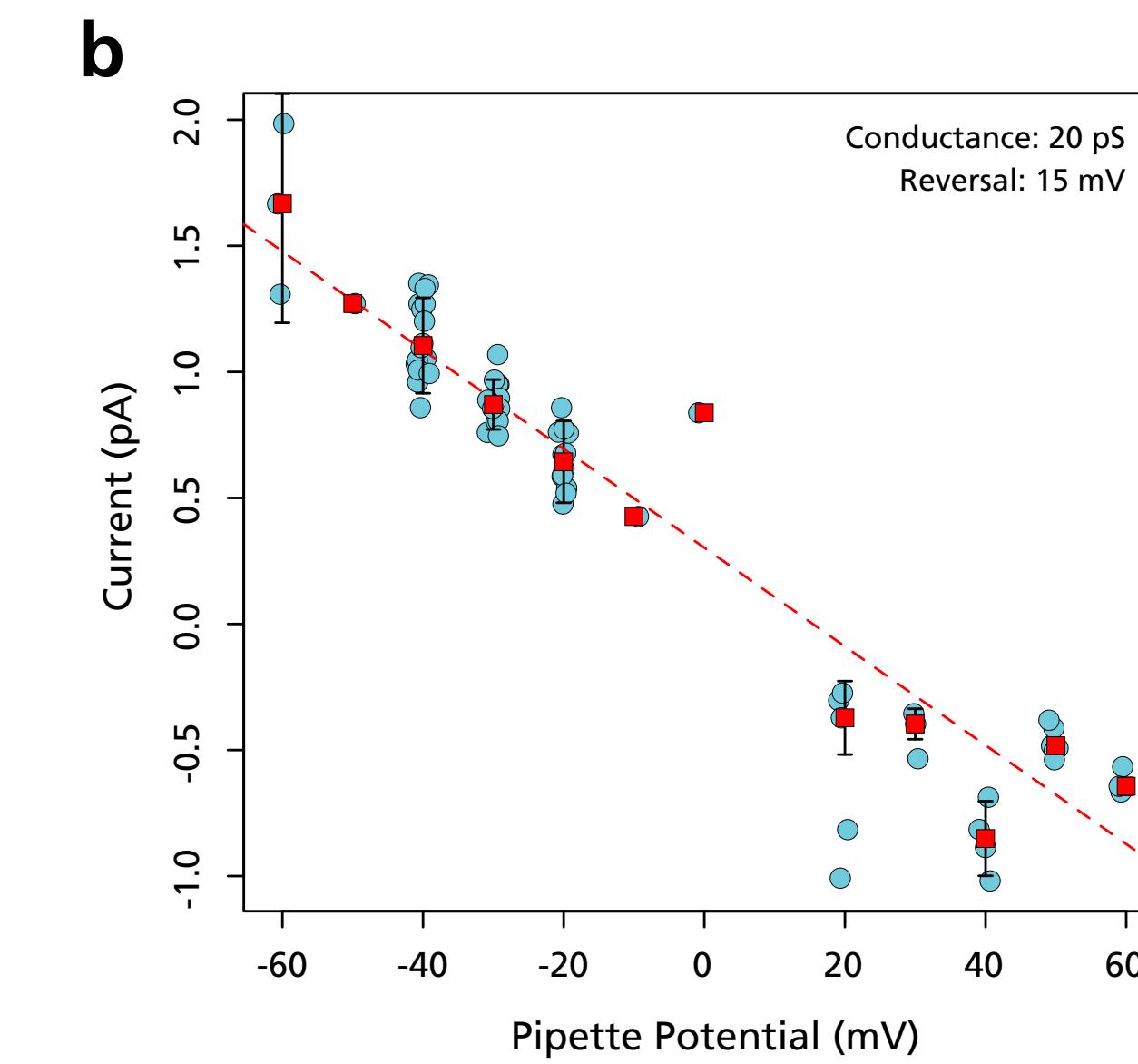
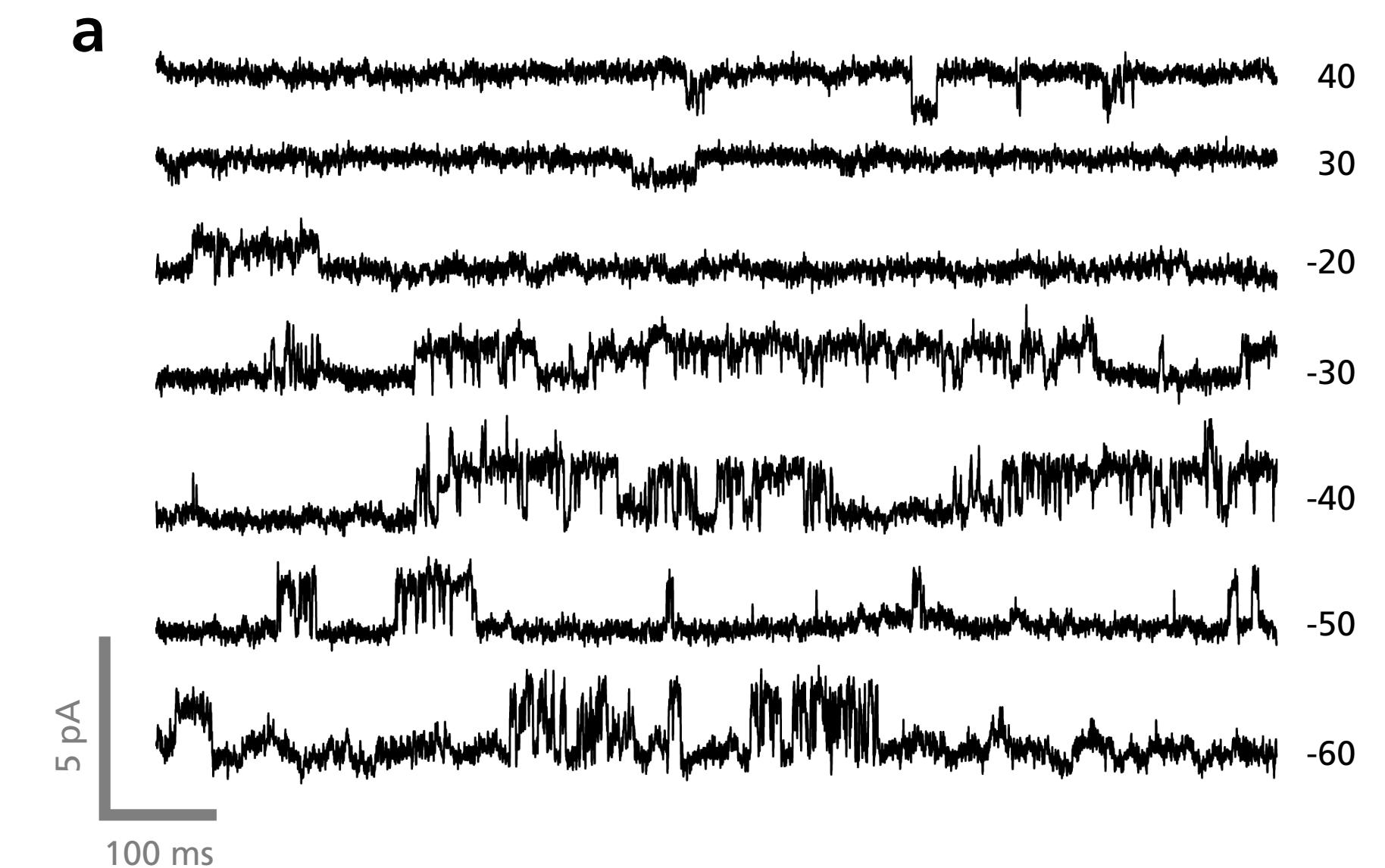
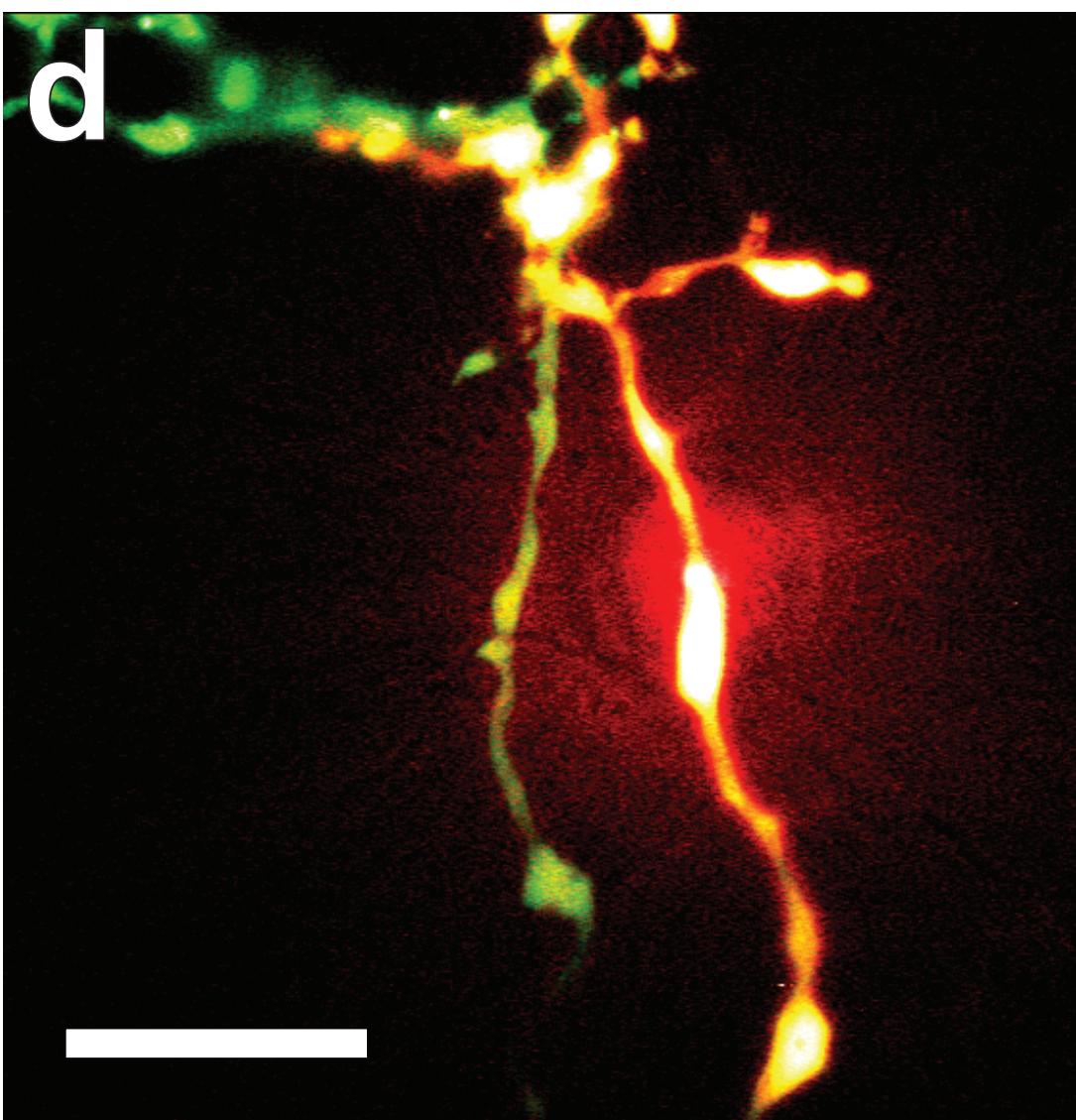
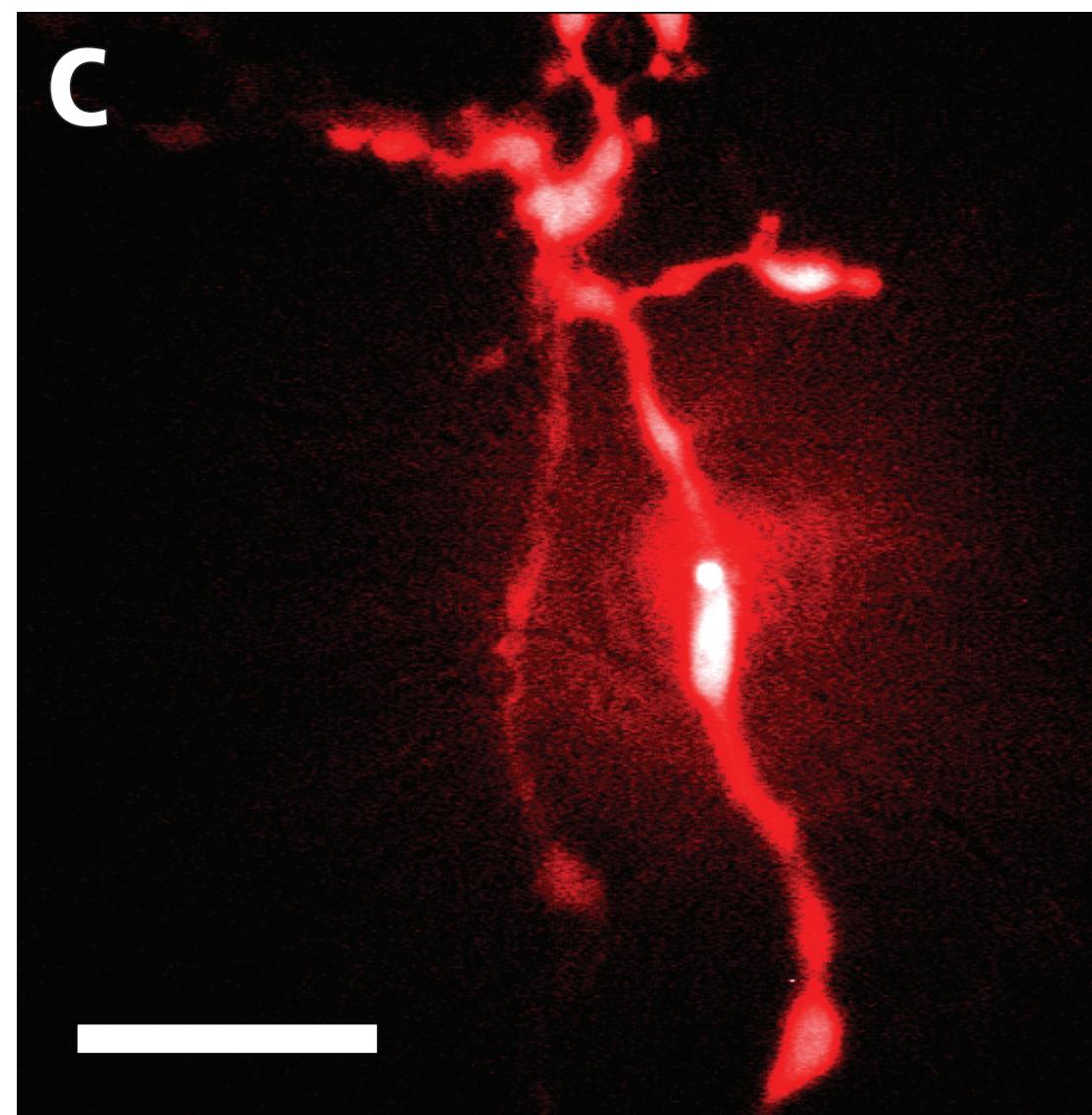
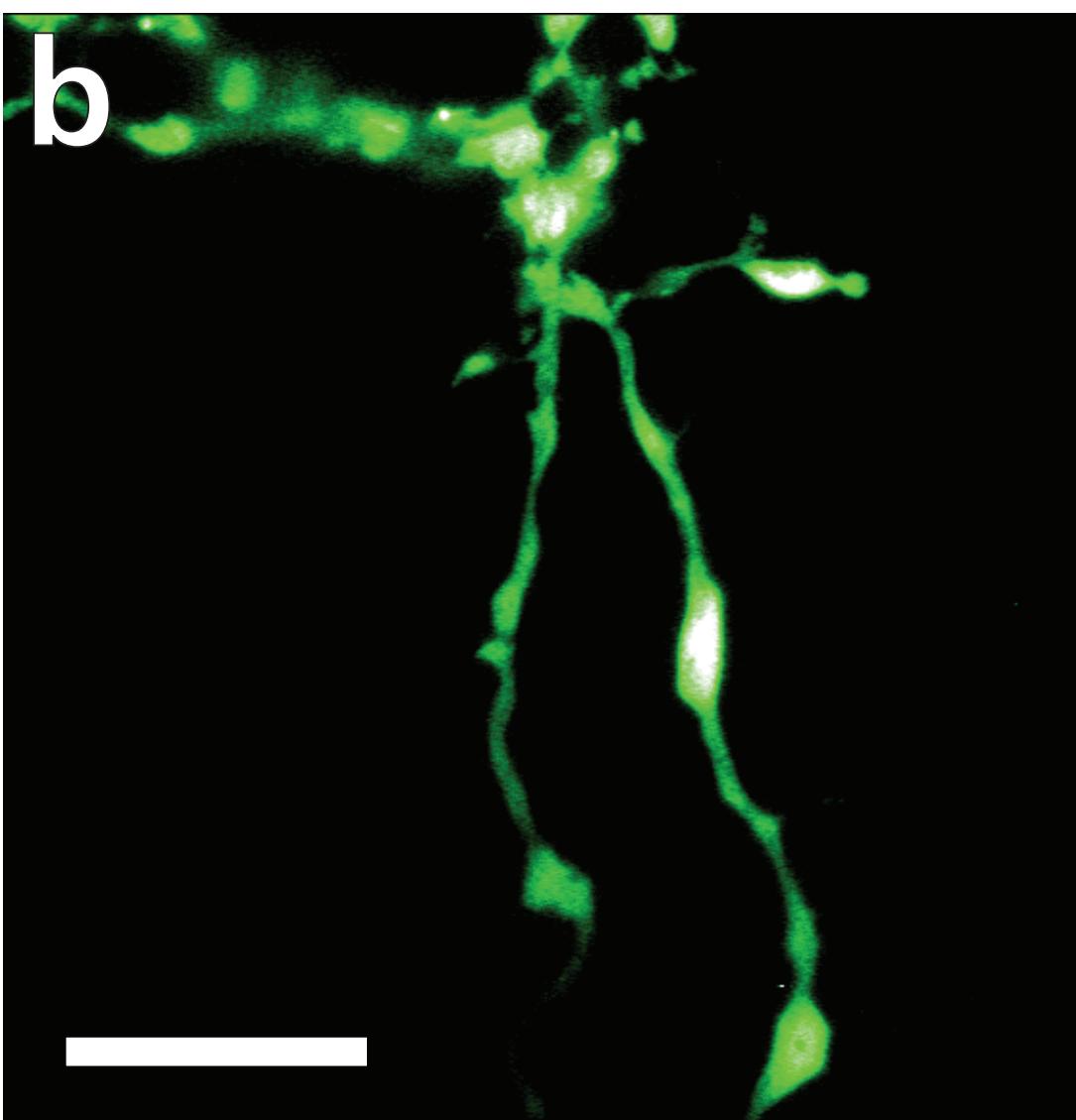
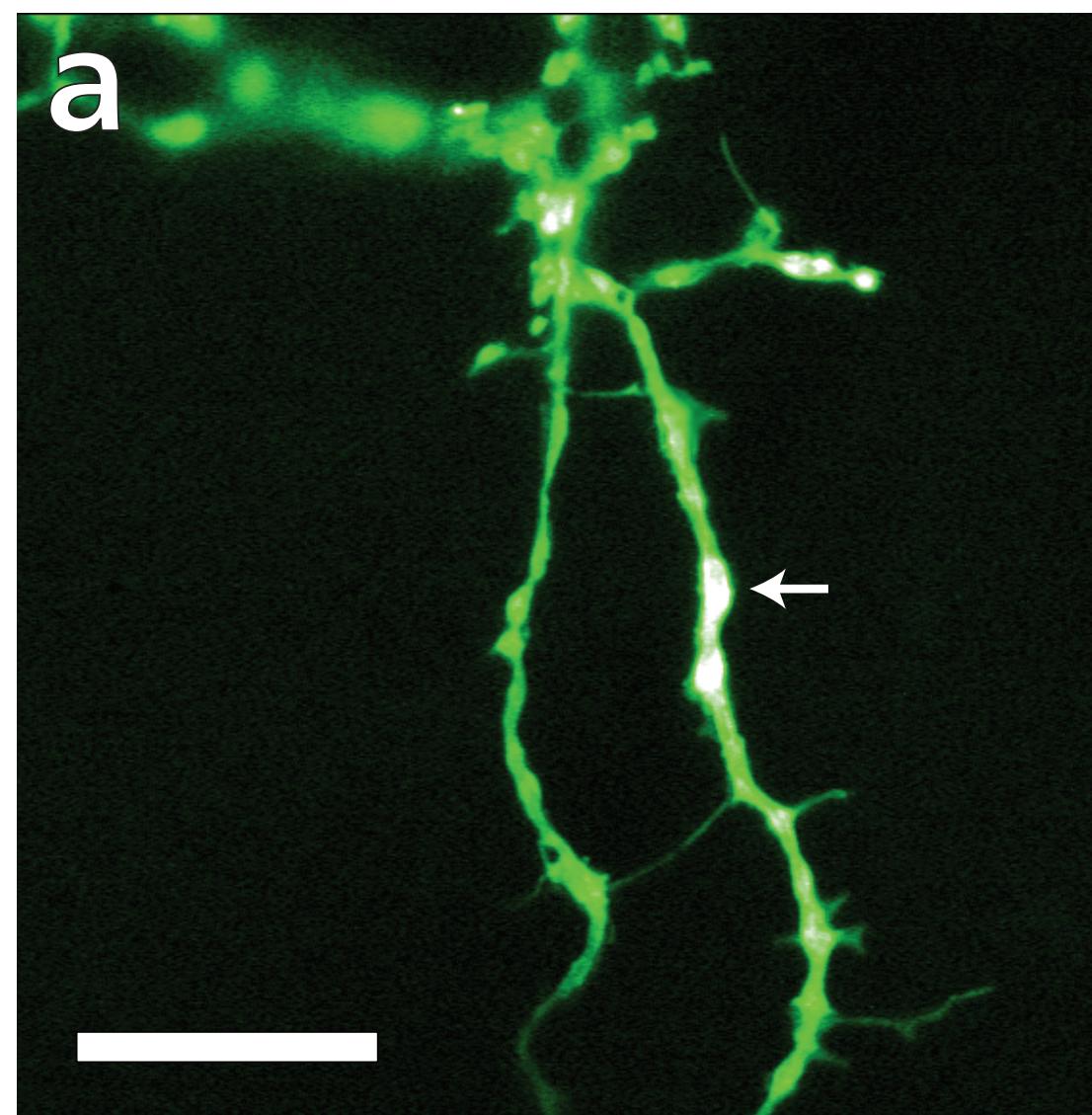
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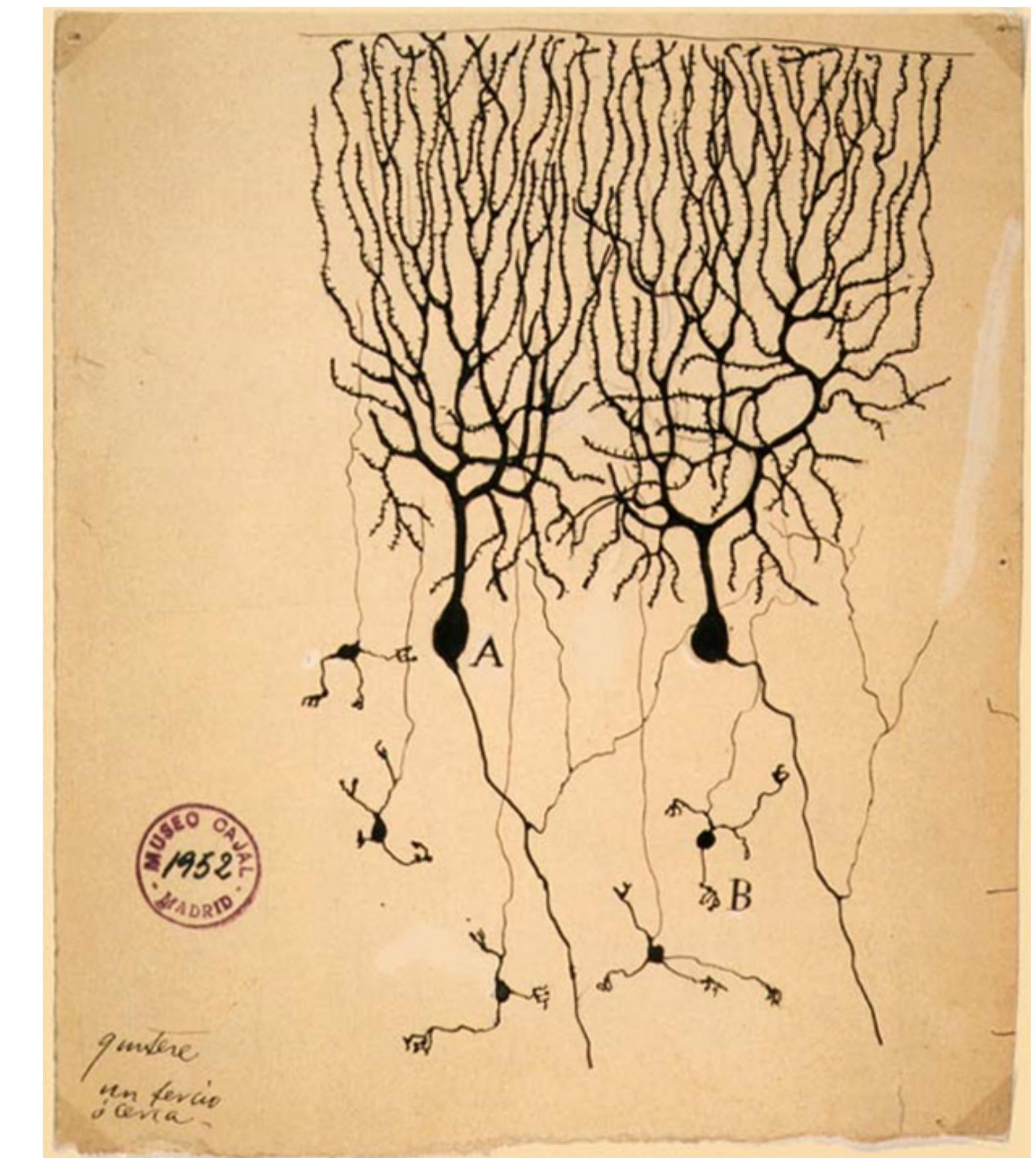
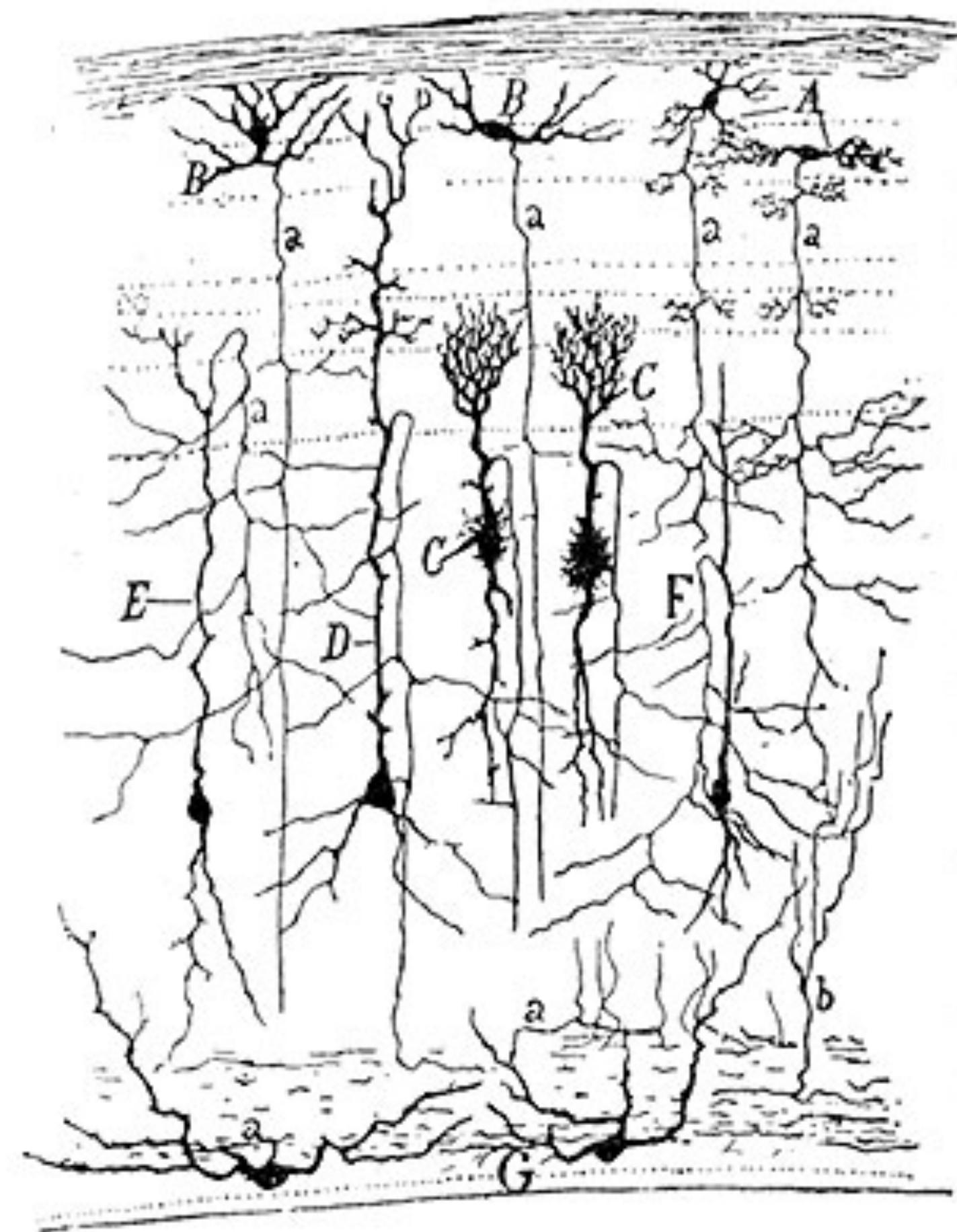
Admin

- No labs, lectures or office hour next week

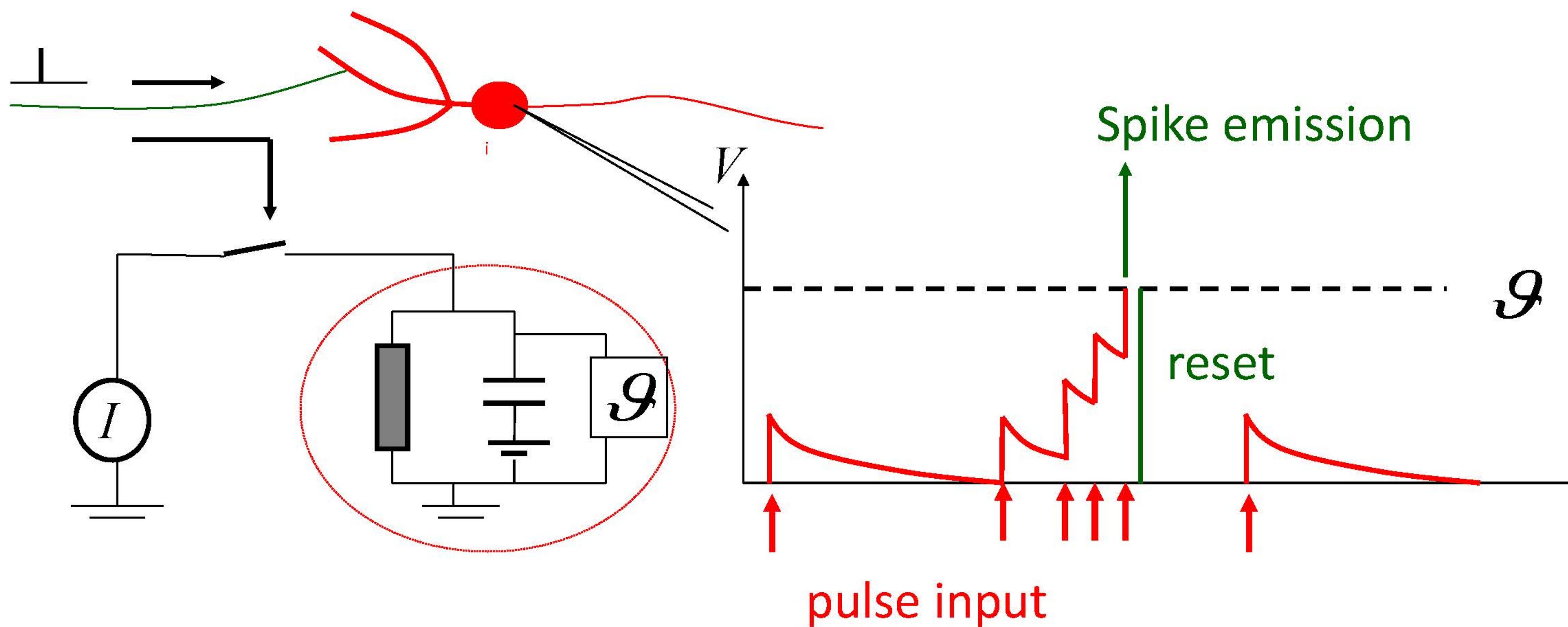
Week 5 Recap

Safety in Numbers





Leaky Integrate-and-Fire Model



$$\text{fire} = \begin{cases} 1 & \text{if } \mathbf{x} \cdot \mathbf{w} \geq \text{threshold} \\ 0 & \text{otherwise} \end{cases}$$

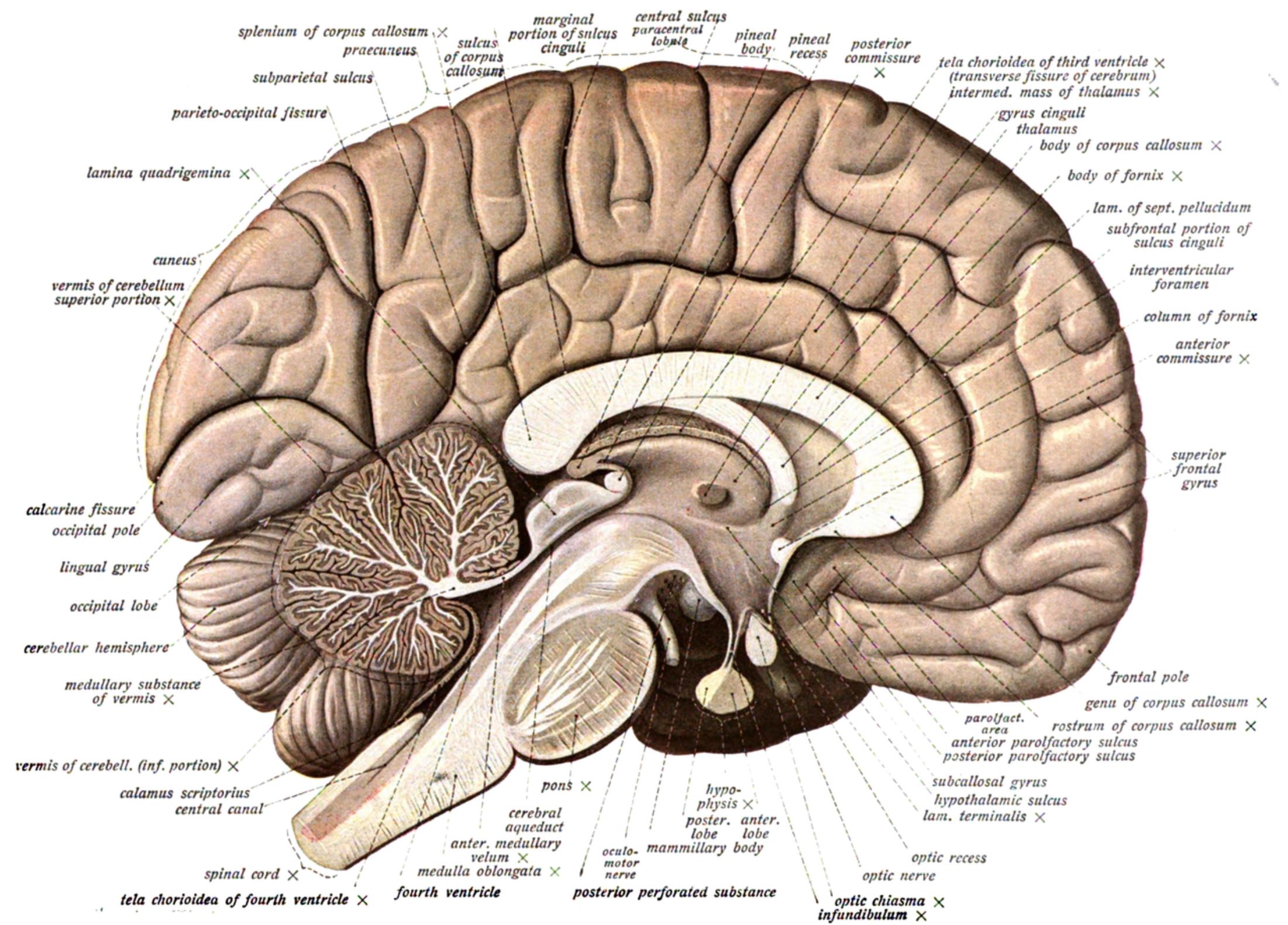
$$\text{fire} = \begin{cases} 1 & \text{if } \mathbf{x} \cdot \mathbf{w} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{fire} = \begin{cases} 1 & \text{if } \mathbf{x} \cdot \mathbf{w} \geq 0 \\ 0 & \text{otherwise} \end{cases} \neq$$

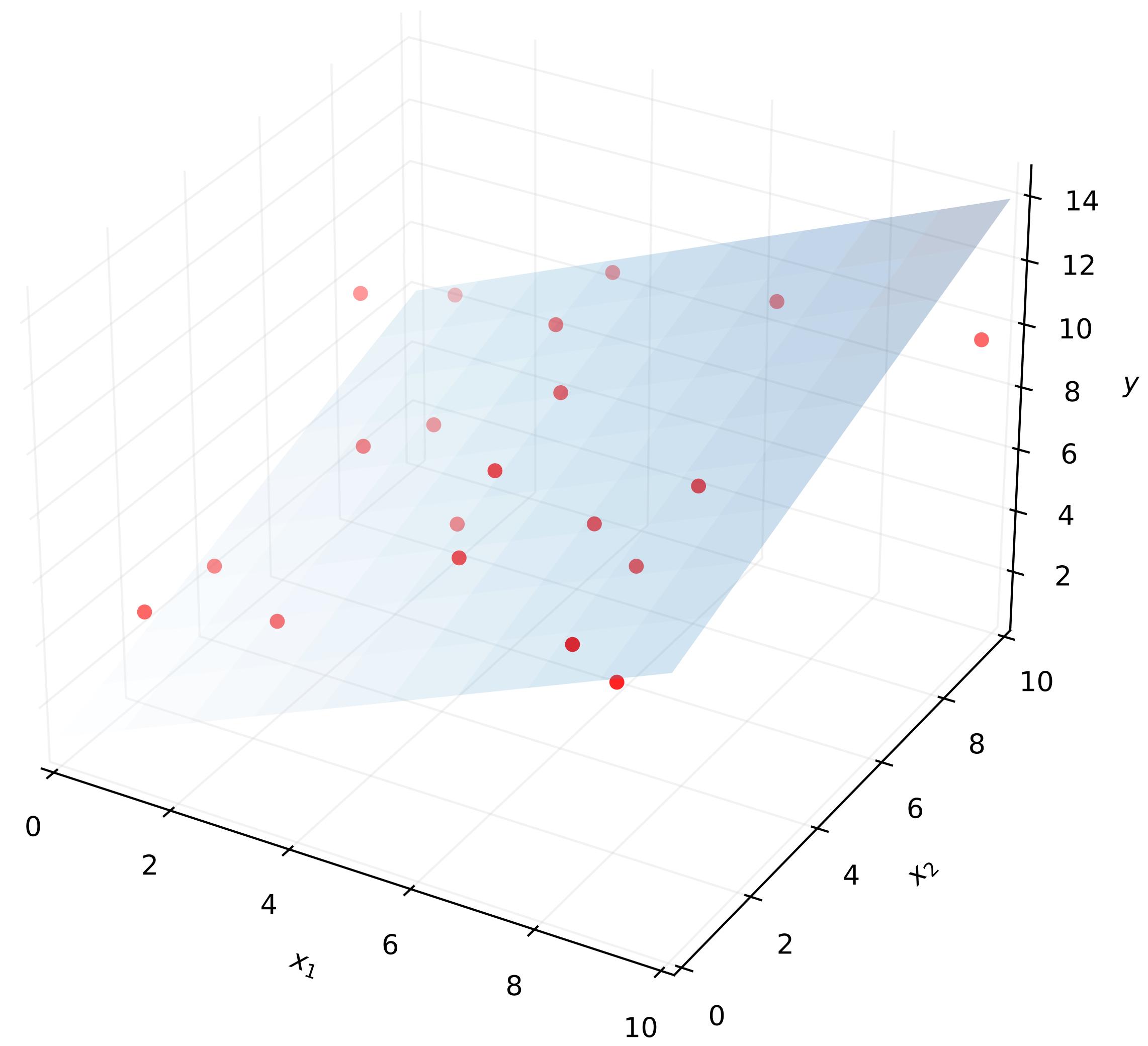
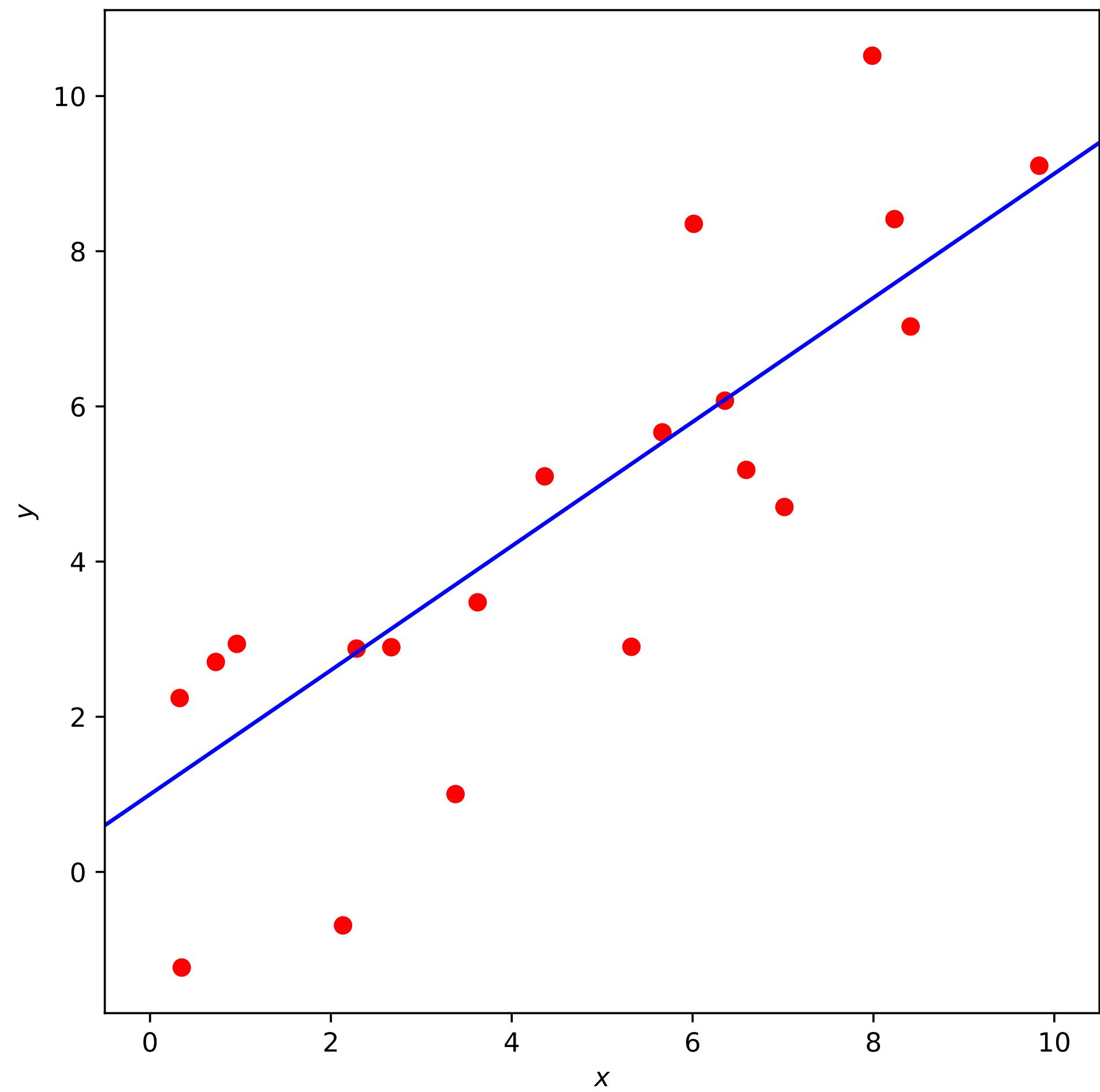




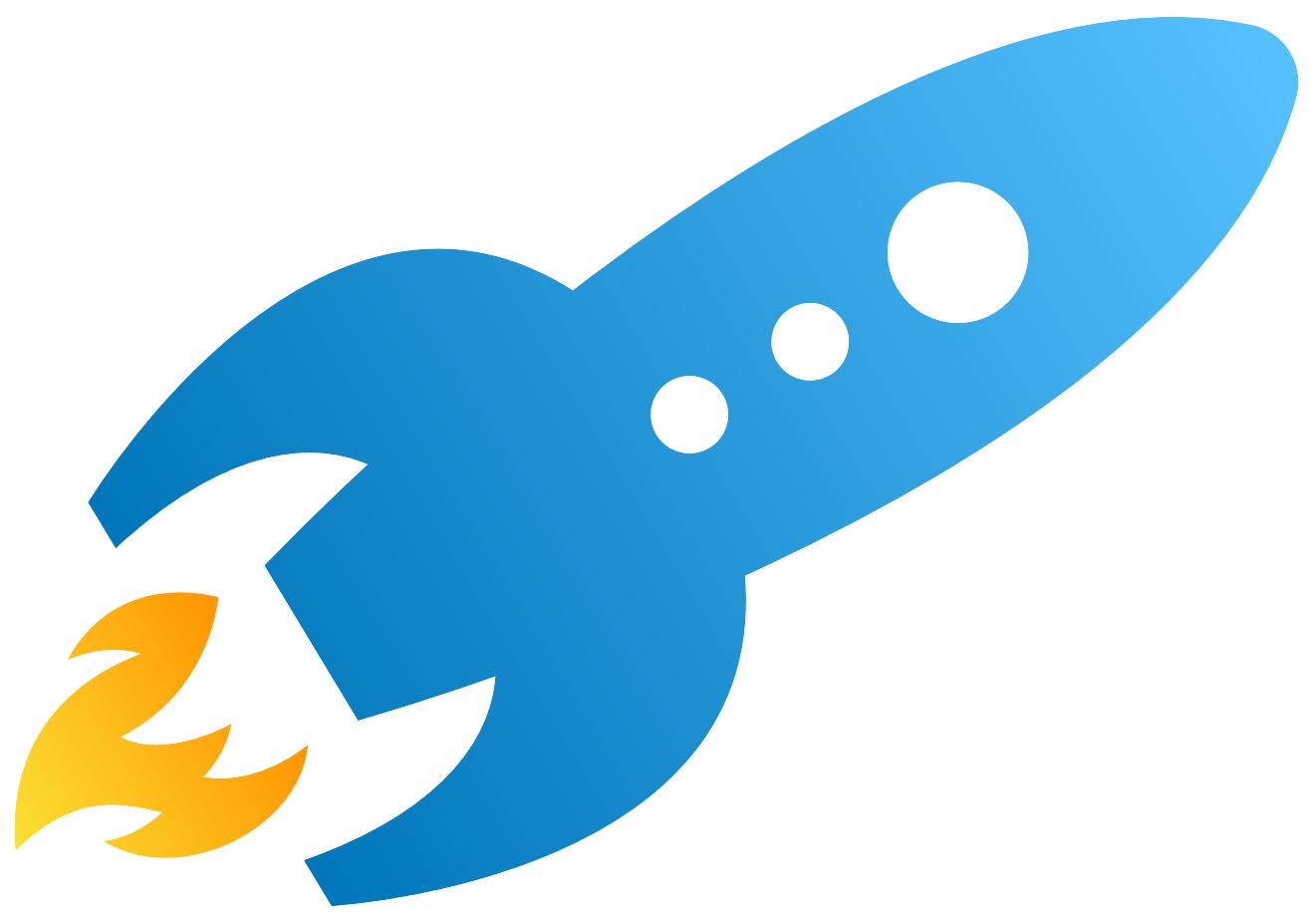
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$$\text{fire} = \begin{cases} 1 & \text{if } \mathbf{x} \cdot \mathbf{w} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

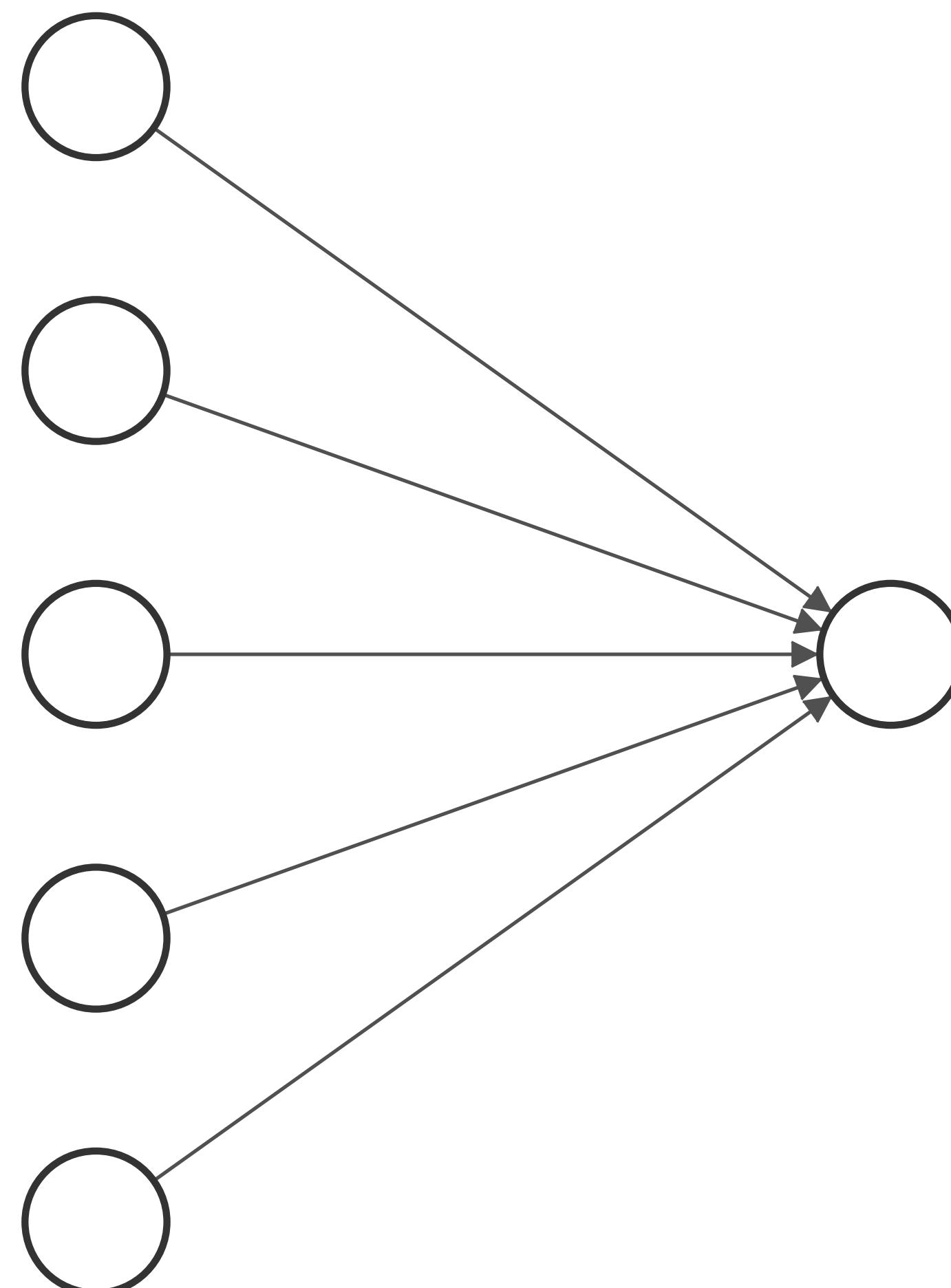


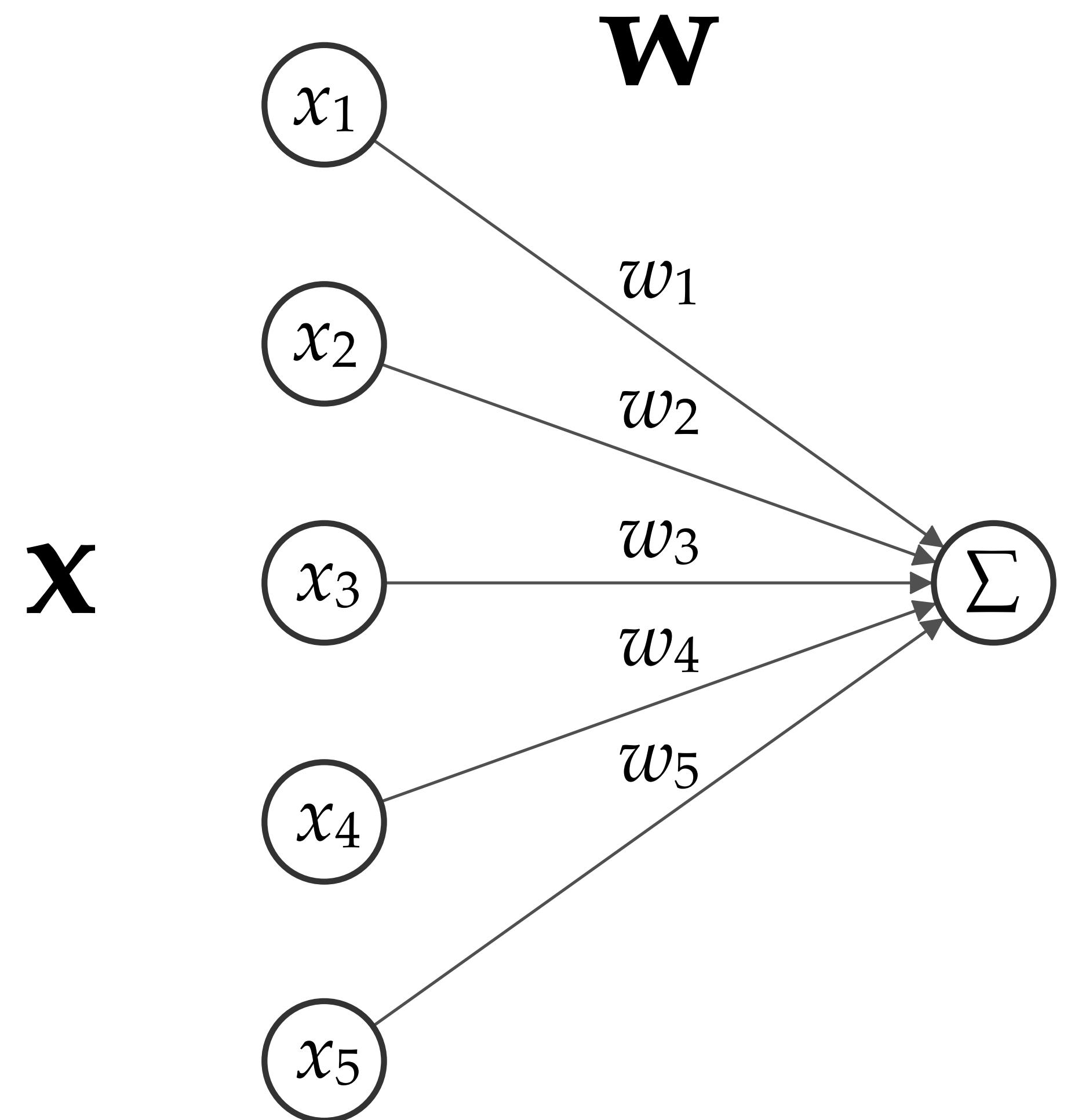




$$\mathbf{x} \cdot \mathbf{w} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{bmatrix}$$

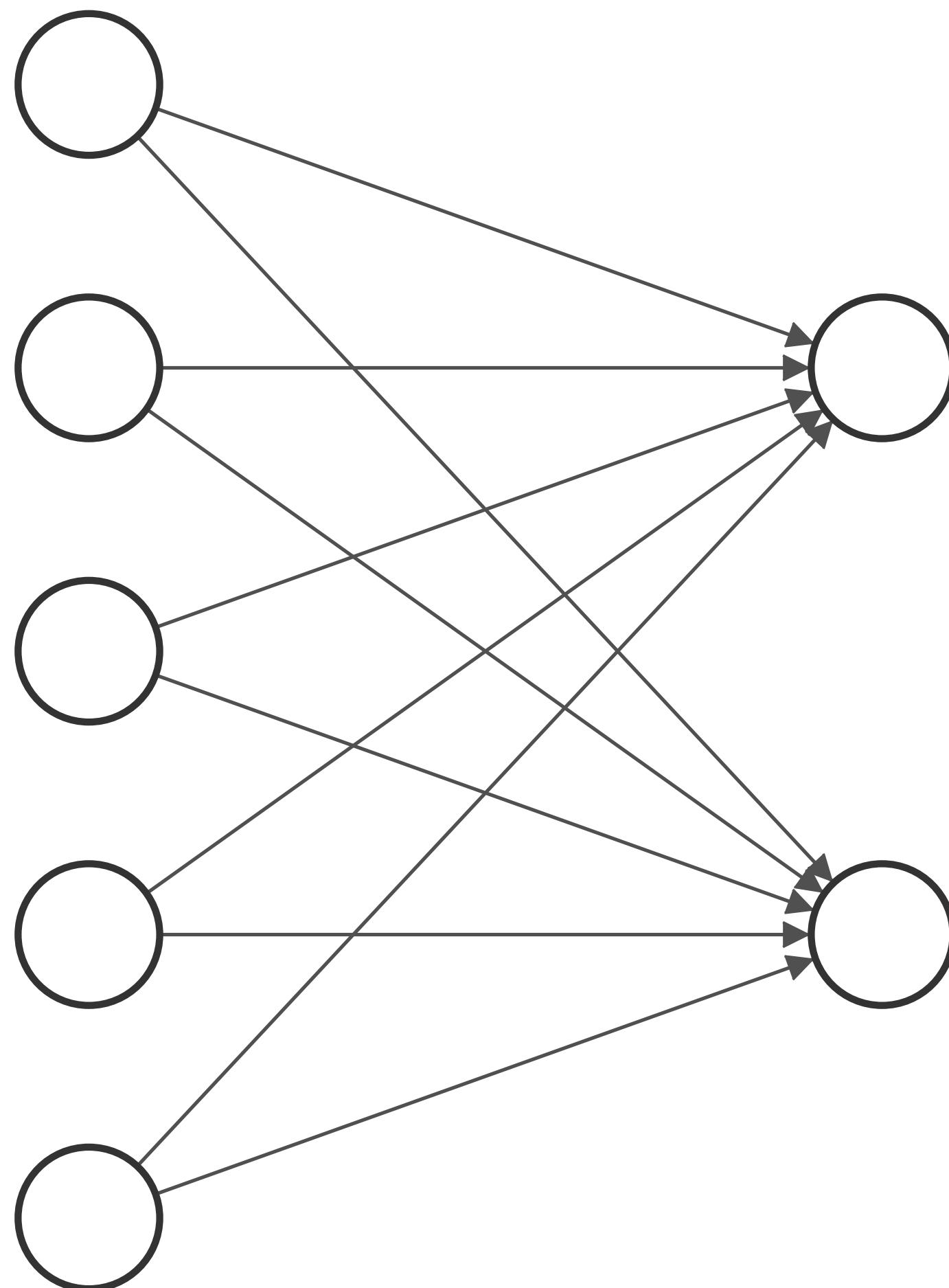
$$\mathbf{x}^\top \mathbf{w} = [x_1 \quad x_2 \quad \dots \quad x_d] \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{bmatrix}$$

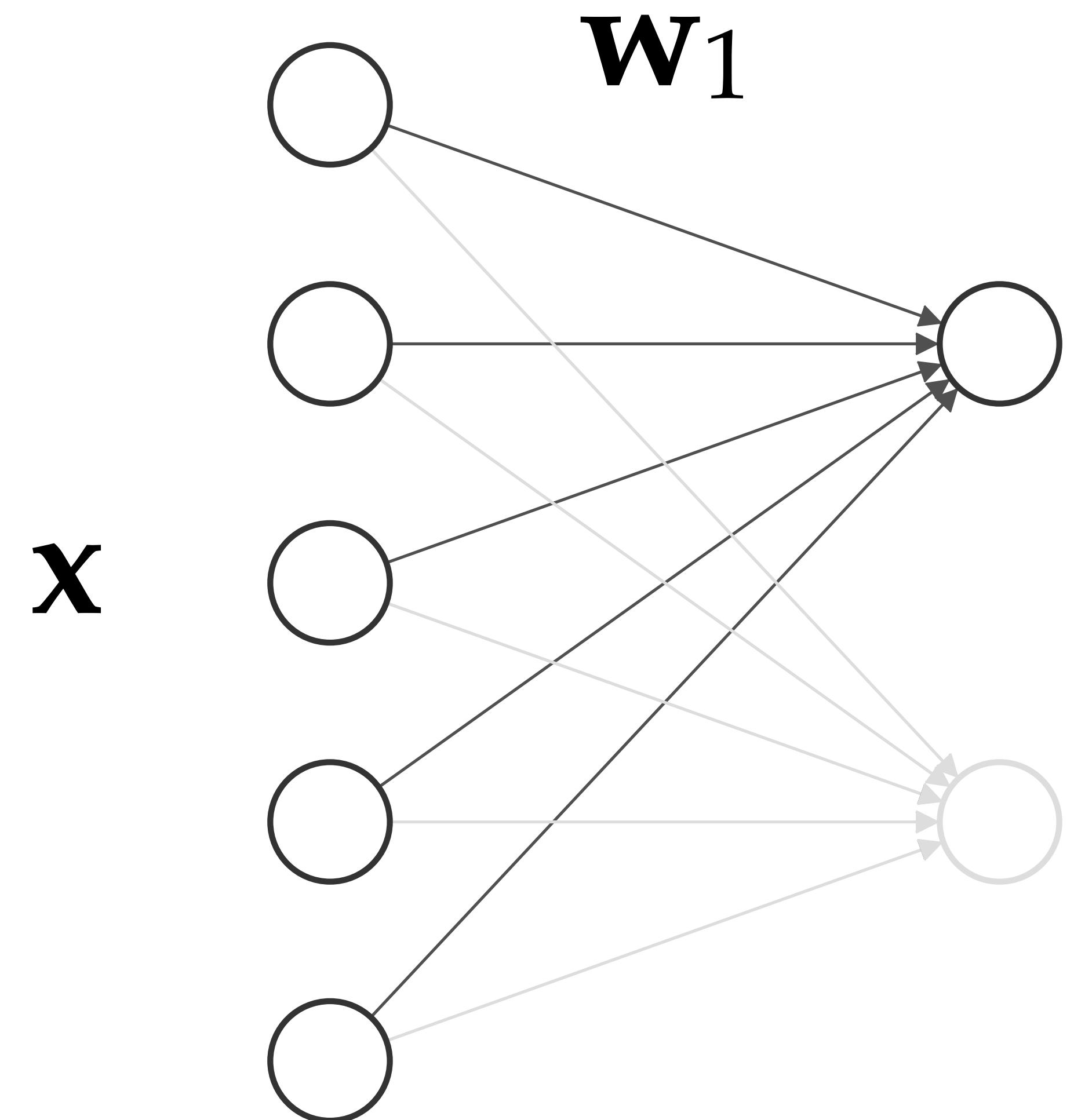


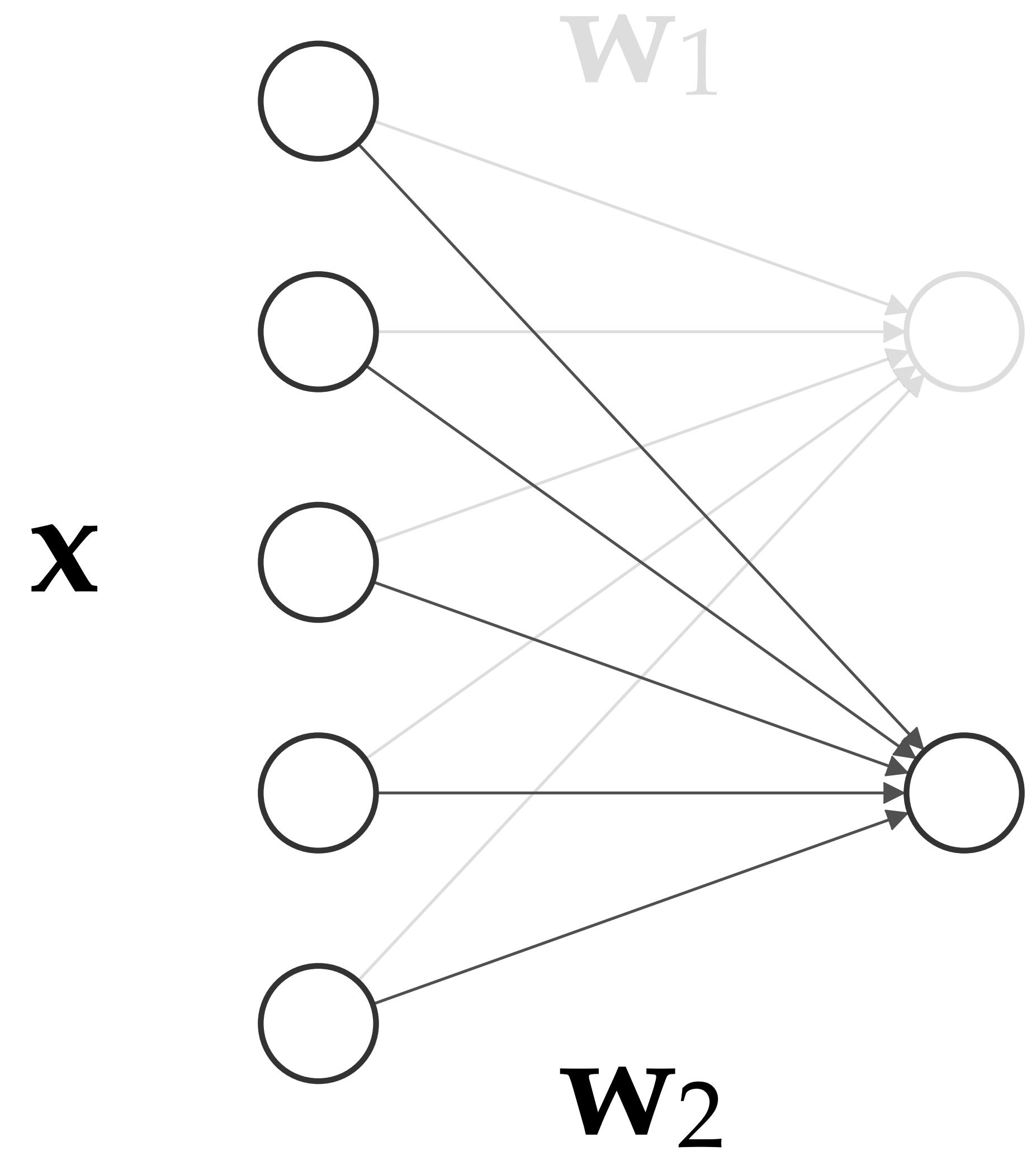


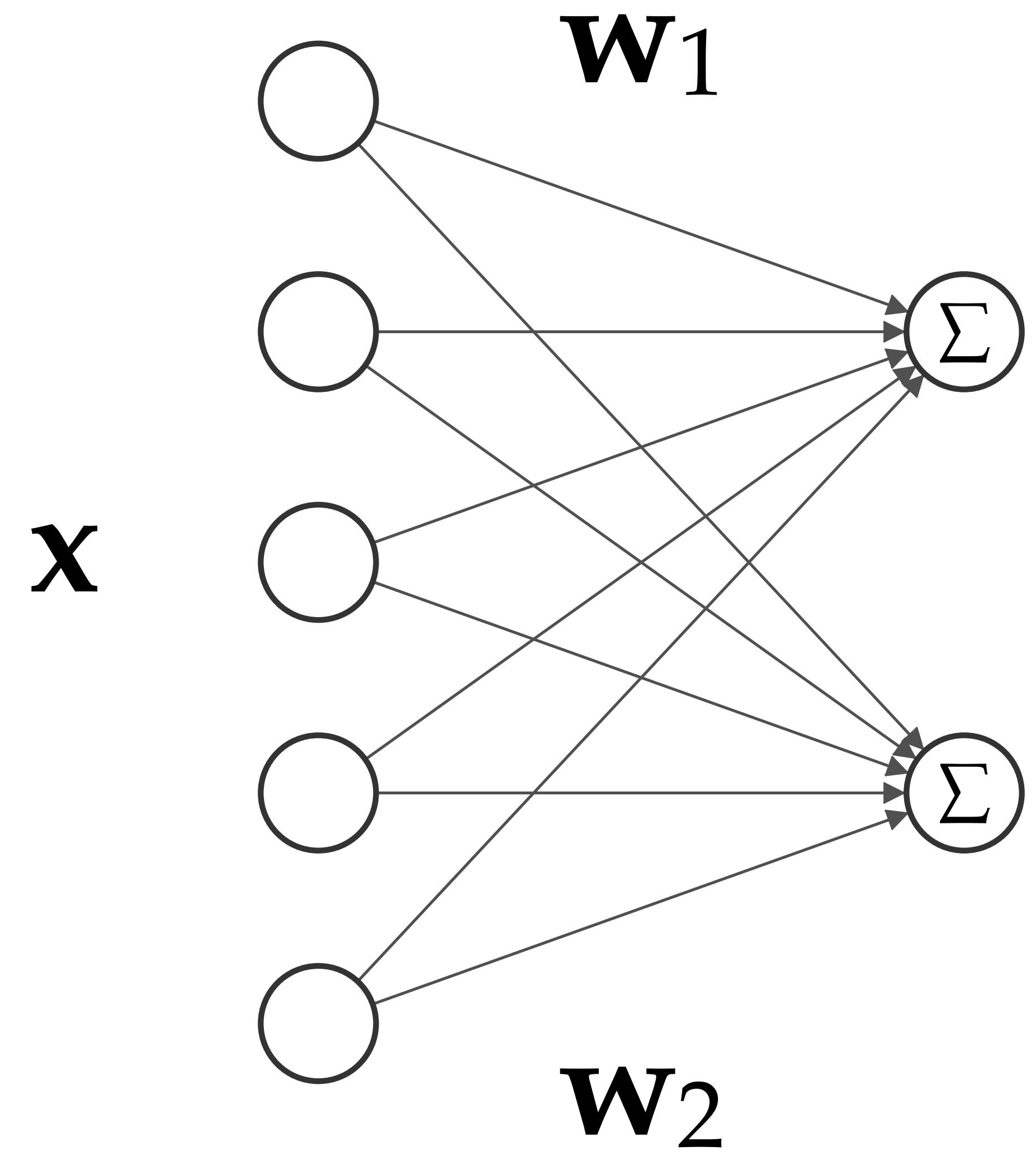
$$\mathbf{x}^\top \mathbf{w} = [x_1 \quad x_2 \quad \dots \quad x_d] \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{bmatrix}$$

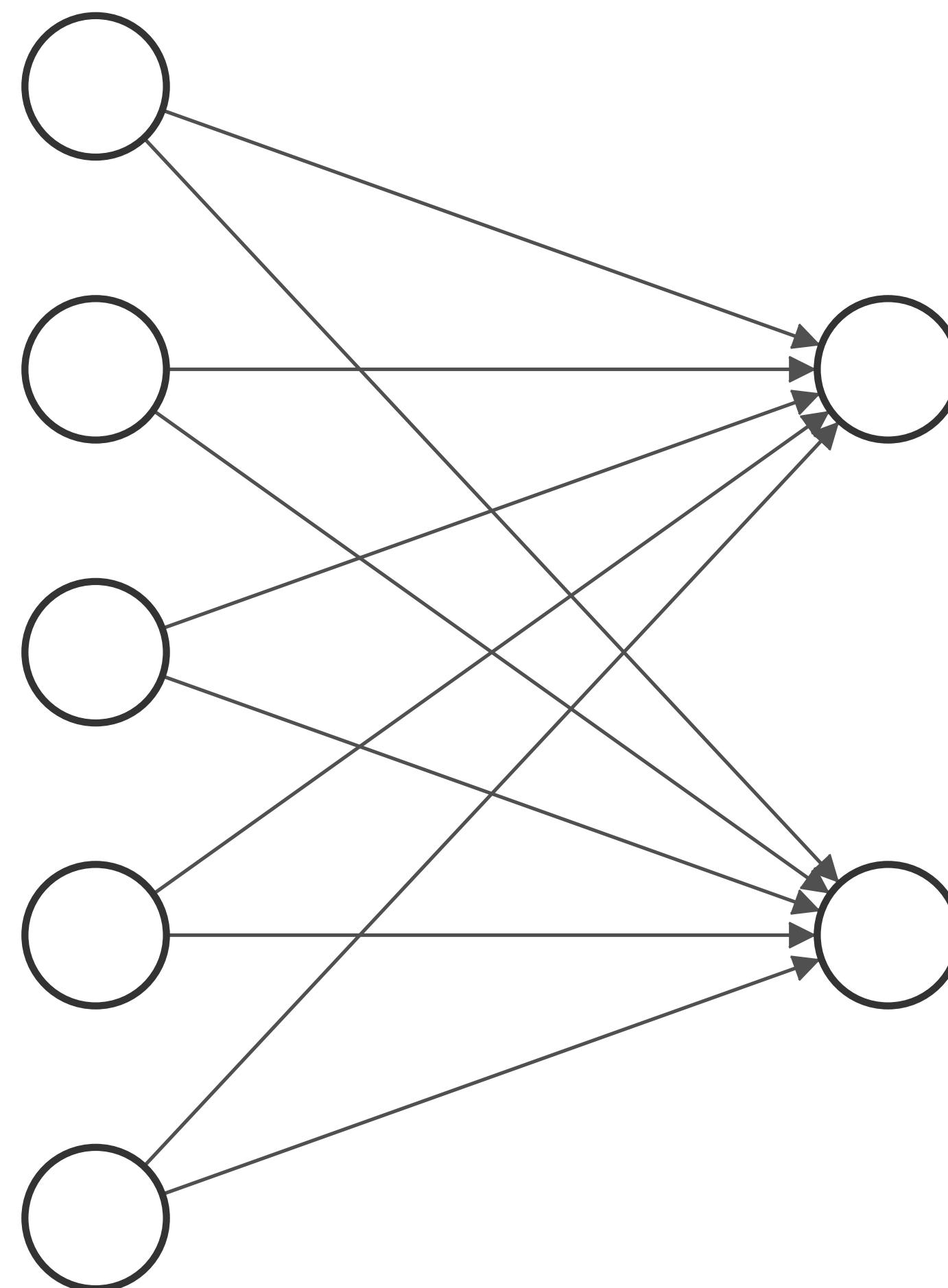
x









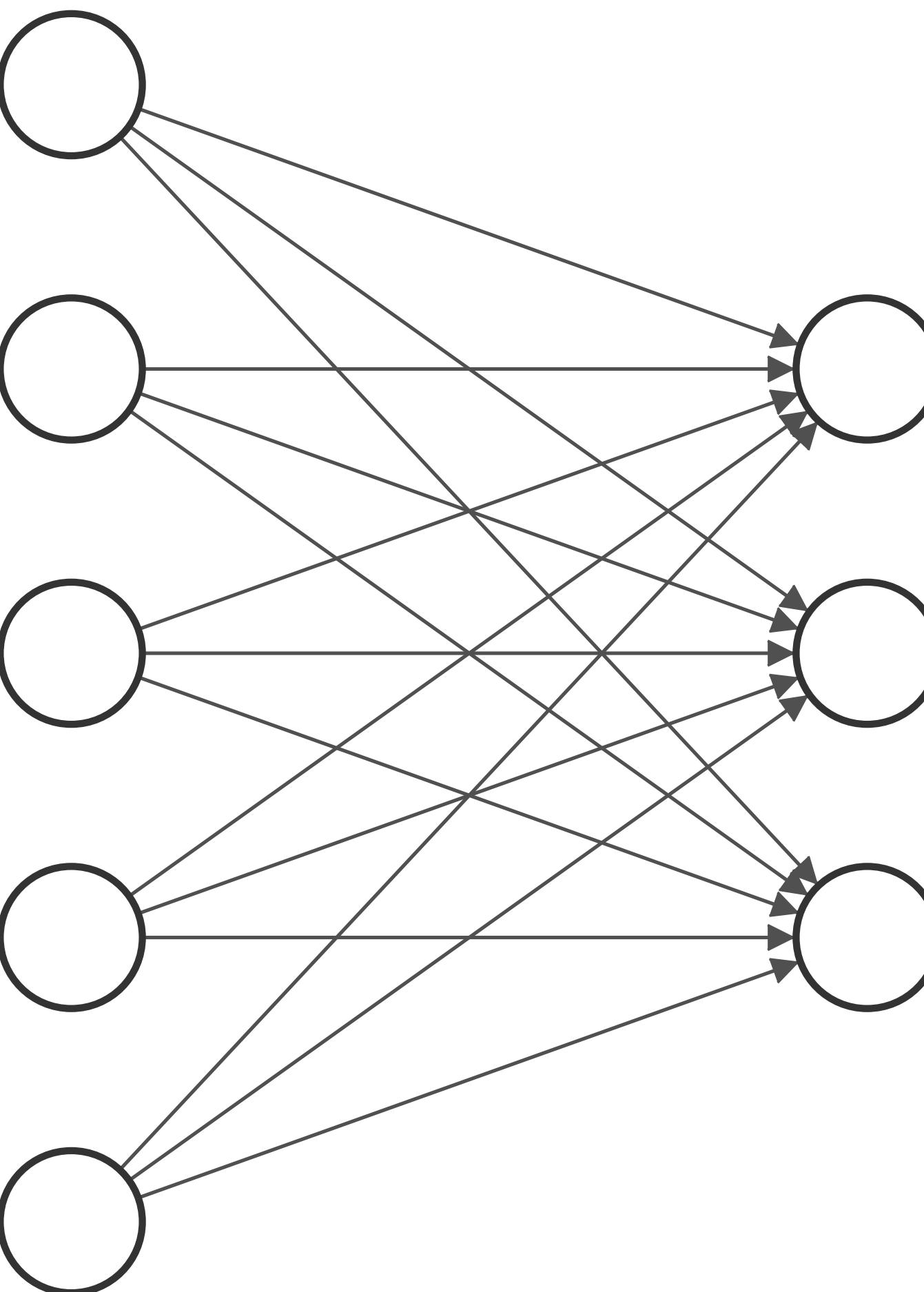


$$\mathbf{x}^\top \mathbf{w} = [x_1 \quad x_2 \quad \dots \quad x_d] \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{bmatrix}$$

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{bmatrix}$$

$$\mathbf{W} = [\mathbf{w}_1 \quad \mathbf{w}_2] = \begin{bmatrix} w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \\ \vdots & \vdots \\ w_{d,1} & w_{d,2} \end{bmatrix}$$

$$\begin{aligned}
\mathbf{x}^\top \mathbf{W} &= [x_1 \quad x_2 \quad \dots \quad x_d] \begin{bmatrix} w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \\ \vdots & \vdots \\ w_{d,1} & w_{d,2} \end{bmatrix} \\
&= [\mathbf{x} \cdot \mathbf{w}_1 \quad \mathbf{x} \cdot \mathbf{w}_2]
\end{aligned}$$

d  k

$$\begin{aligned}
\mathbf{x}^\top \mathbf{W} &= [x_1 \quad x_2 \quad \dots \quad x_d] \begin{bmatrix} w_{1,1} & w_{1,2} & \dots & w_{1,k} \\ w_{2,1} & w_{2,2} & \dots & w_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ w_{d,1} & w_{d,2} & \dots & w_{d,k} \end{bmatrix} \\
&= [\mathbf{x} \cdot \mathbf{w}_1 \quad \mathbf{x} \cdot \mathbf{w}_2 \quad \dots \quad \mathbf{x} \cdot \mathbf{w}_k]
\end{aligned}$$

↓ Feature dimensions ↓

$$x = \begin{bmatrix} x_1^\top \\ x_2^\top \\ \vdots \\ x_n^\top \end{bmatrix} = \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,d} \\ x_{1,1} & x_{1,2} & \cdots & x_{1,d} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,d} \end{bmatrix}$$

↑ Samples ↑

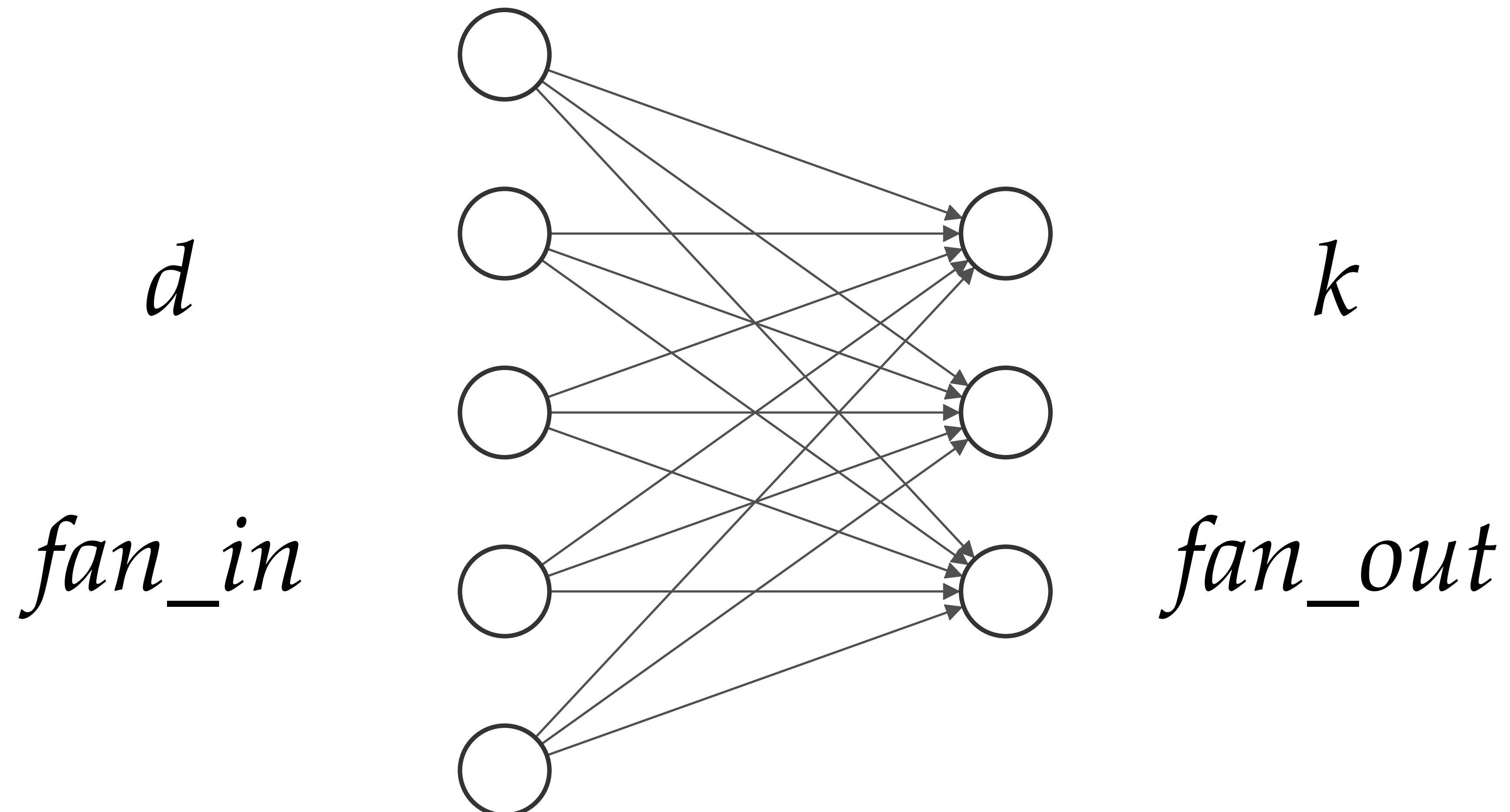
$$\begin{aligned}
\mathbf{xw} &= \begin{bmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,d} \\ x_{2,1} & x_{2,2} & \dots & x_{2,d} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & \dots & x_{n,d} \end{bmatrix} \begin{bmatrix} w_{1,1} & w_{1,2} & \dots & w_{1,k} \\ w_{2,1} & w_{2,2} & \dots & w_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ w_{d,1} & w_{d,2} & \dots & w_{d,k} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{x}_1 \cdot \mathbf{w}_1 & \mathbf{x}_1 \cdot \mathbf{w}_2 & \dots & \mathbf{x}_1 \cdot \mathbf{w}_k \\ \mathbf{x}_2 \cdot \mathbf{w}_1 & \mathbf{x}_2 \cdot \mathbf{w}_2 & \dots & \mathbf{x}_2 \cdot \mathbf{w}_k \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_n \cdot \mathbf{w}_1 & \mathbf{x}_n \cdot \mathbf{w}_2 & \dots & \mathbf{x}_n \cdot \mathbf{w}_k \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\mathbf{XW} &= \begin{bmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,d} \\ x_{2,1} & x_{2,2} & \dots & x_{2,d} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & \dots & x_{n,d} \end{bmatrix} \begin{bmatrix} w_{1,1} & w_{1,2} & \dots & w_{1,k} \\ w_{2,1} & w_{2,2} & \dots & w_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ w_{d,1} & w_{d,2} & \dots & w_{d,k} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{x}_1 \cdot \mathbf{w}_1 & \mathbf{x}_1 \cdot \mathbf{w}_2 & \dots & \mathbf{x}_1 \cdot \mathbf{w}_k \\ \mathbf{x}_2 \cdot \mathbf{w}_1 & \mathbf{x}_2 \cdot \mathbf{w}_2 & \dots & \mathbf{x}_2 \cdot \mathbf{w}_k \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{x}_n \cdot \mathbf{w}_1 & \mathbf{x}_n \cdot \mathbf{w}_2 & \dots & \mathbf{x}_n \cdot \mathbf{w}_k \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{y}}_1^\top \\ \hat{\mathbf{y}}_2^\top \\ \vdots \\ \hat{\mathbf{y}}_n^\top \end{bmatrix} = \hat{\mathbf{Y}}
\end{aligned}$$

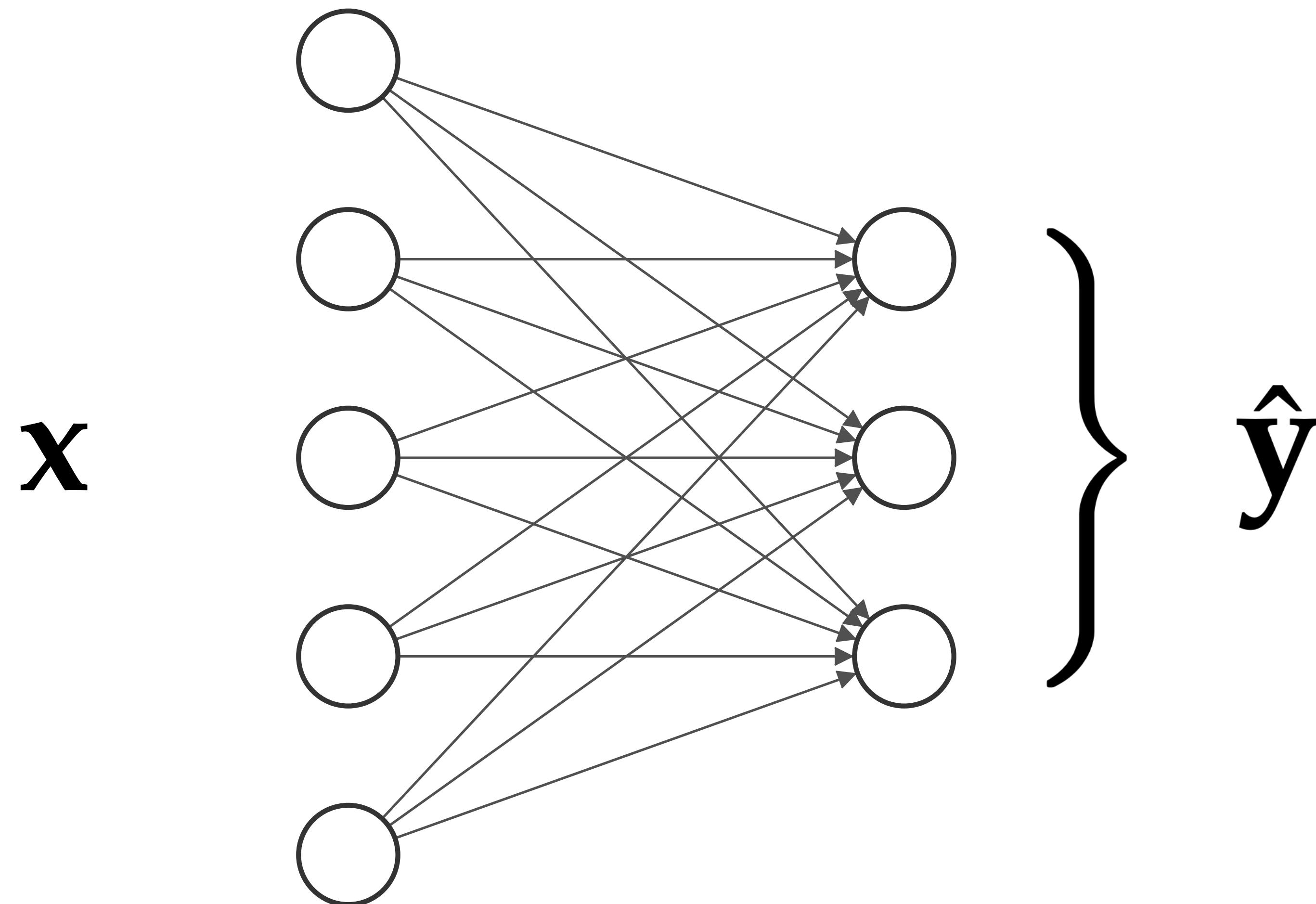
$$\mathbf{X} \in \mathbb{R}^{n \times d}$$

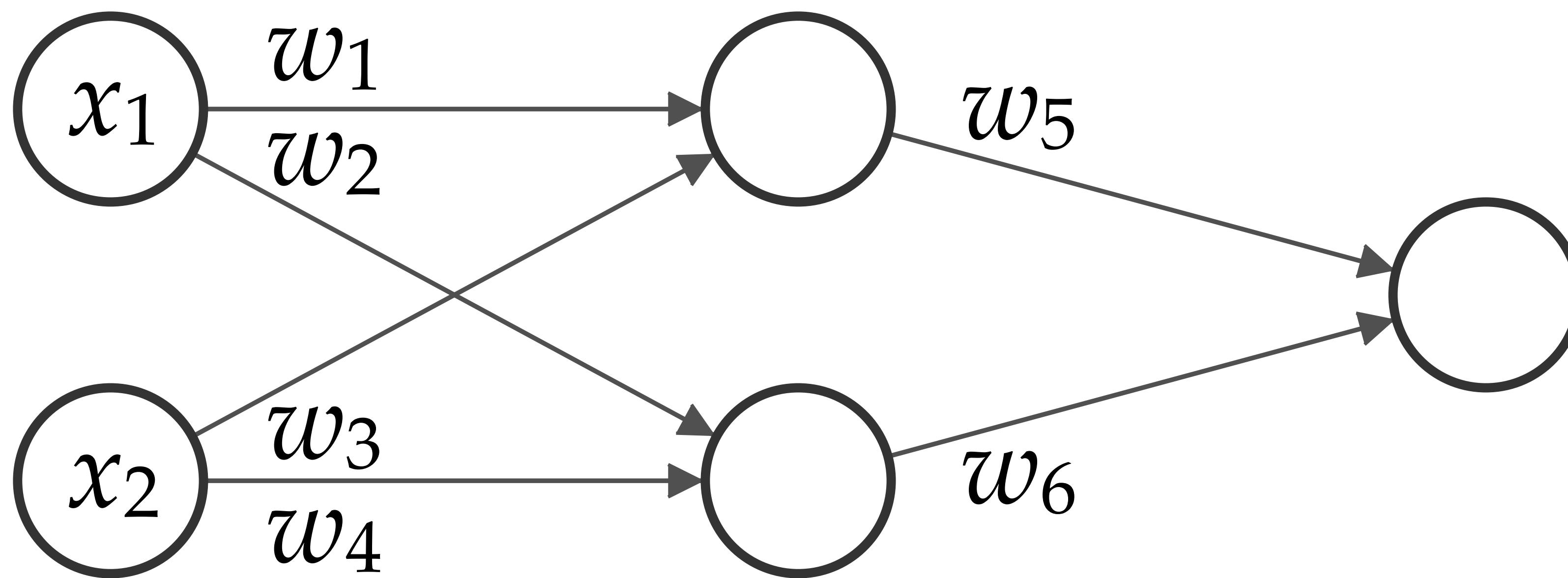
$$\mathbf{W} \in \mathbb{R}^{d \times k}$$

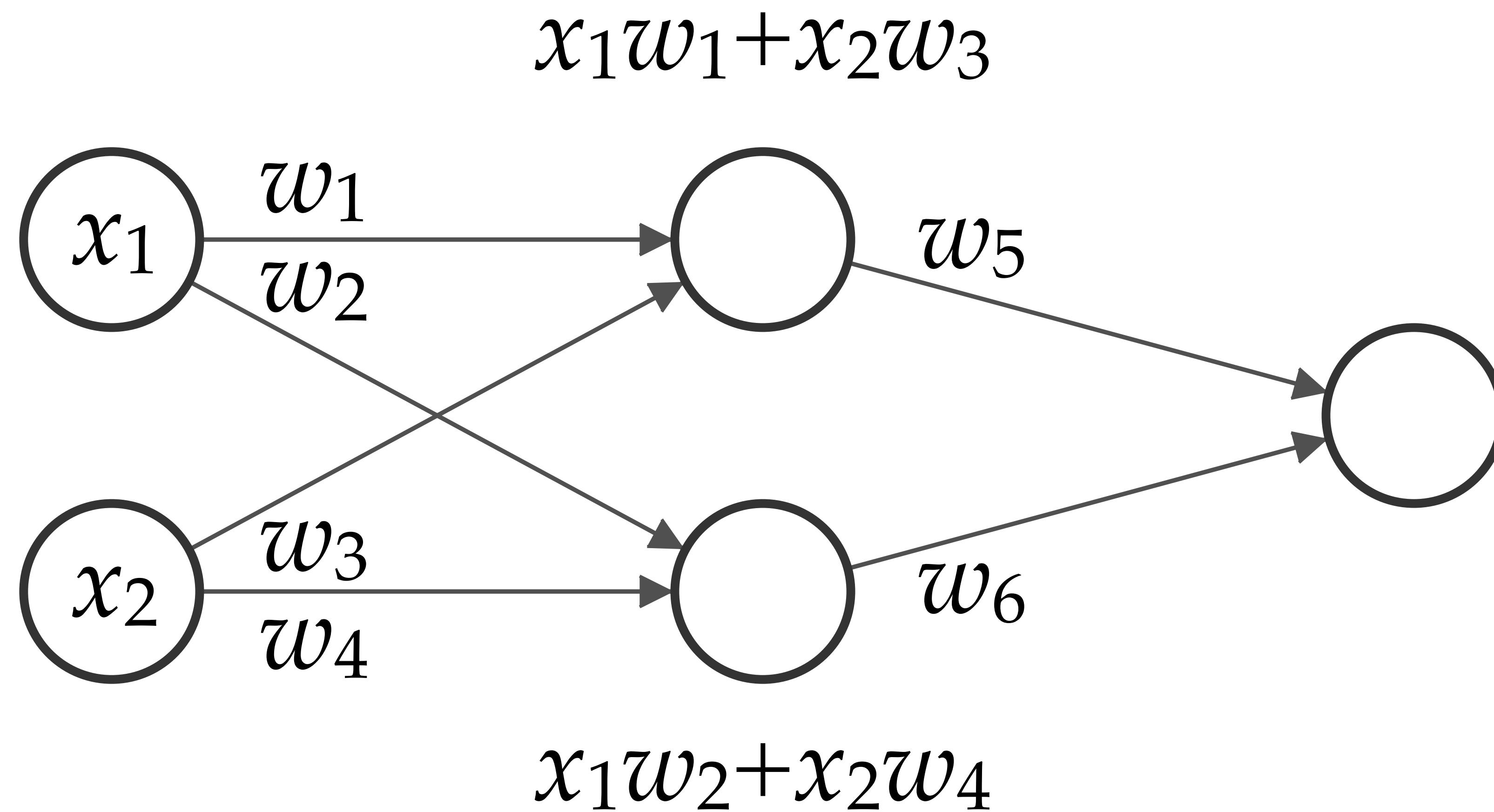
$$\hat{\mathbf{Y}} \in \mathbb{R}^{n \times k}$$

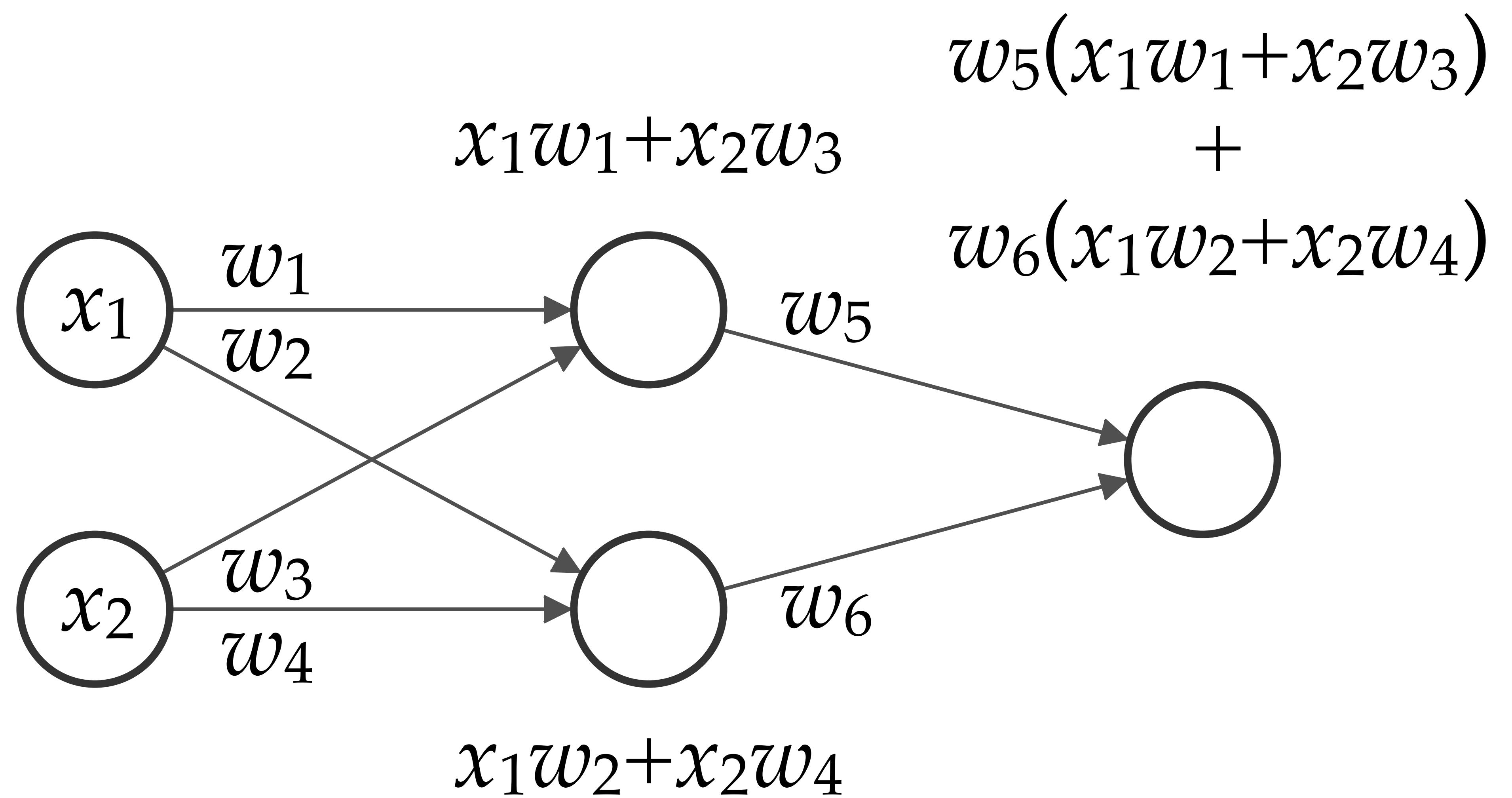


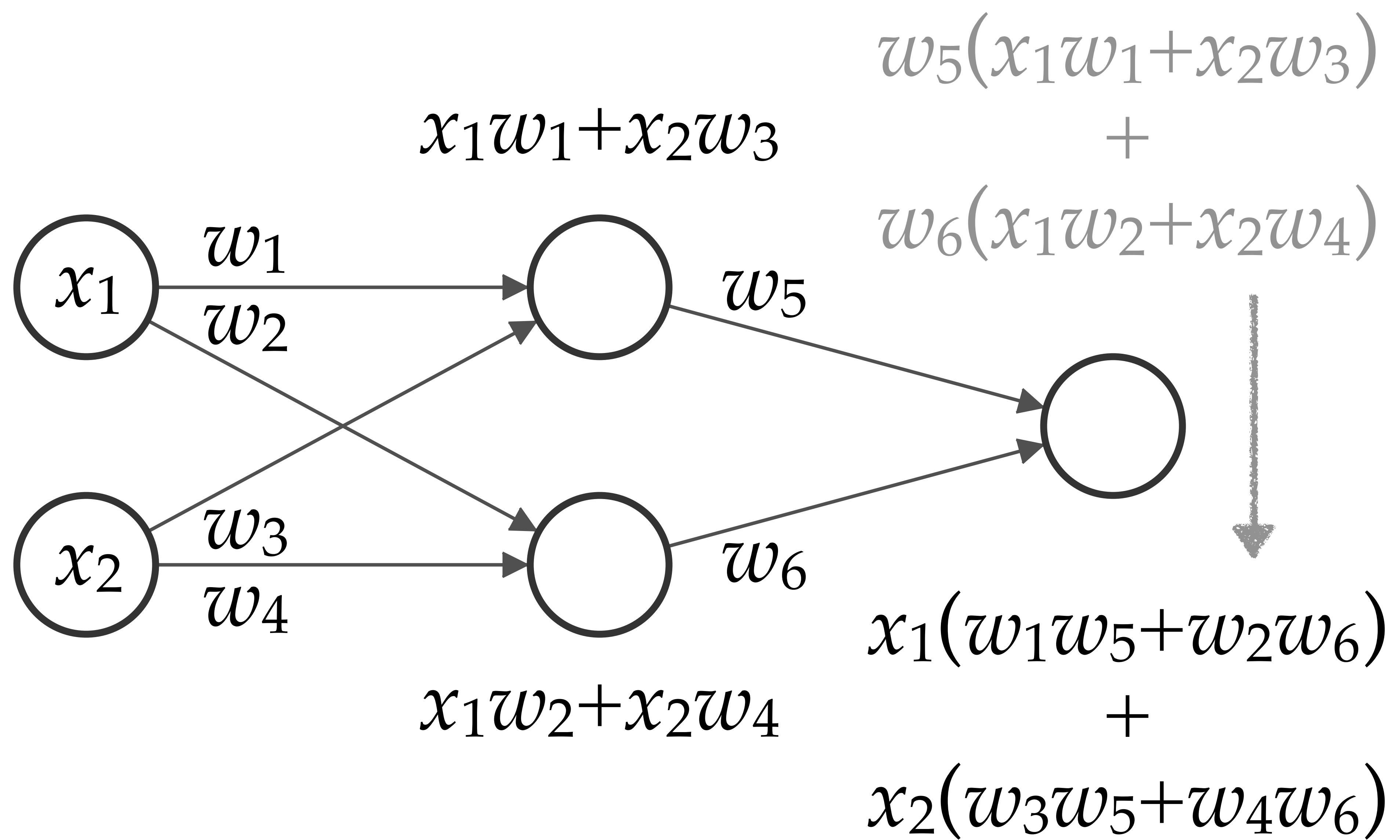


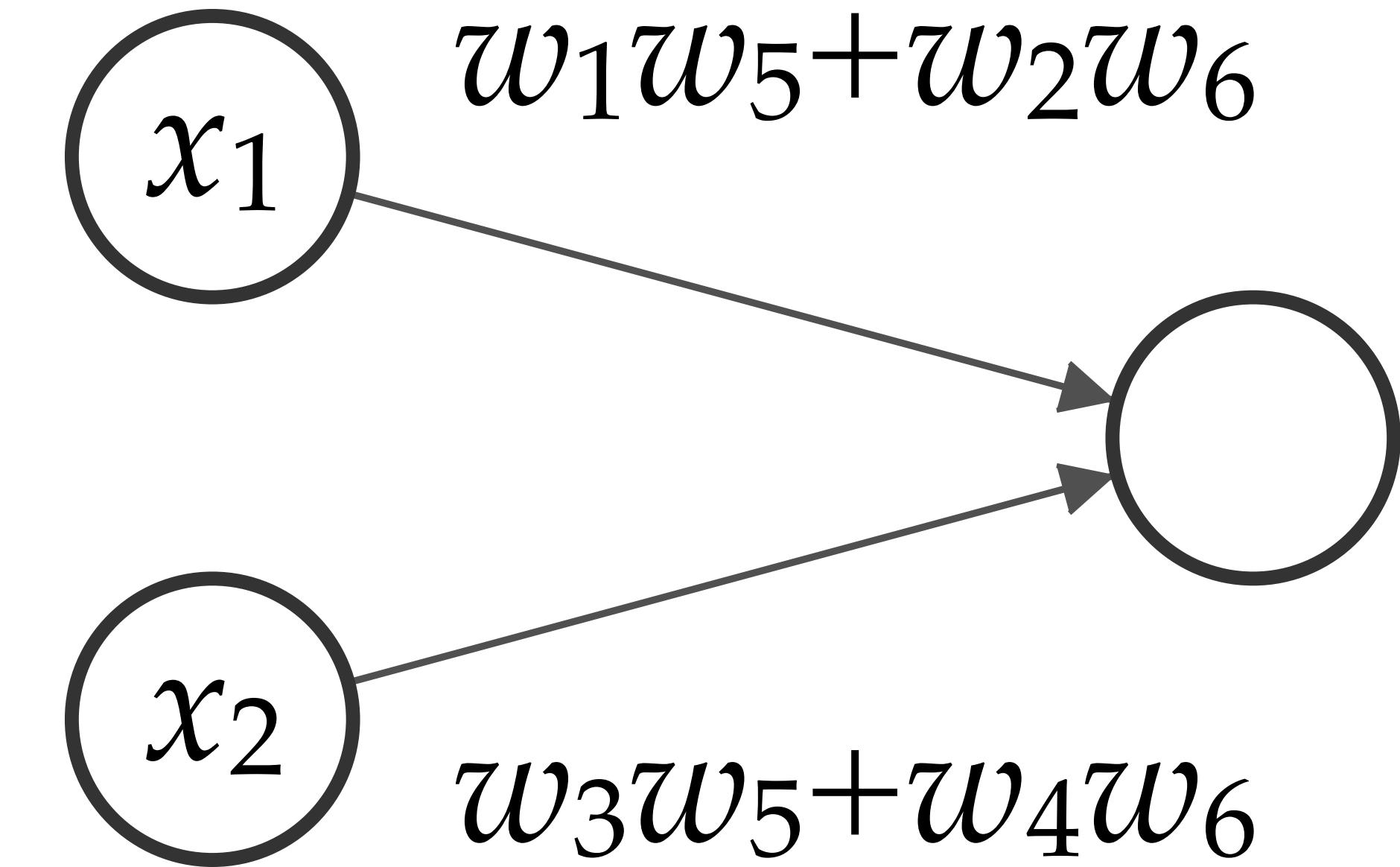












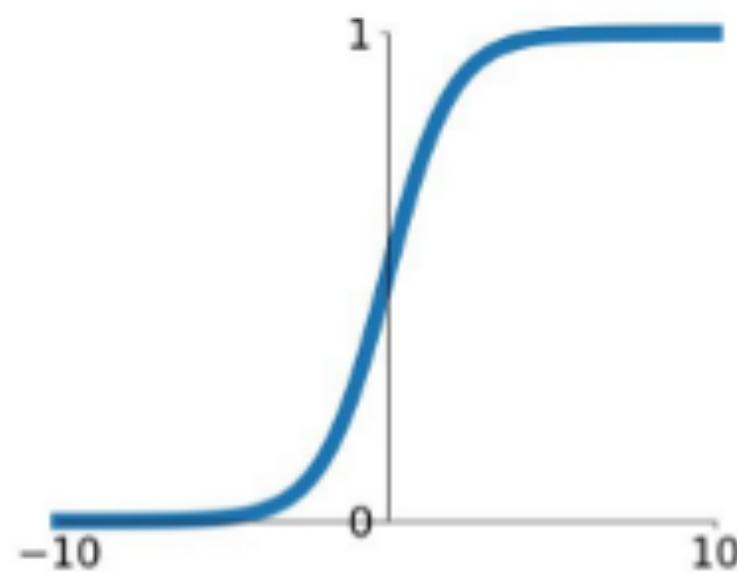
$$\begin{aligned}
\hat{y} &= (((((xW_1)W_2)\dots)W_k) \\
&= x(W_1W_2\dots W_k) \\
&\equiv xW
\end{aligned}$$

$$\text{fire} = \begin{cases} 1 & \text{if } \mathbf{x} \cdot \mathbf{w} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{y} = a_k(\dots a_2(a_1(xW_1)W_2)\dots W_k)$$

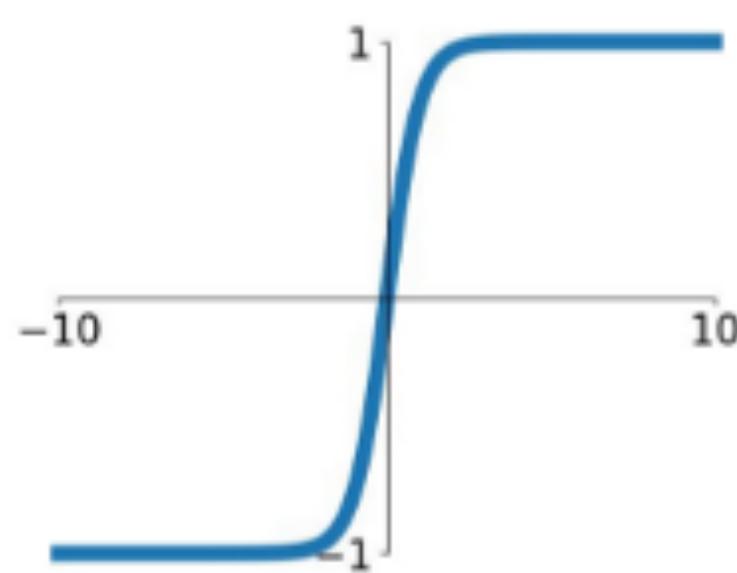
Sigmoid

$$\sigma(x) = \frac{1}{1+e^{-x}}$$



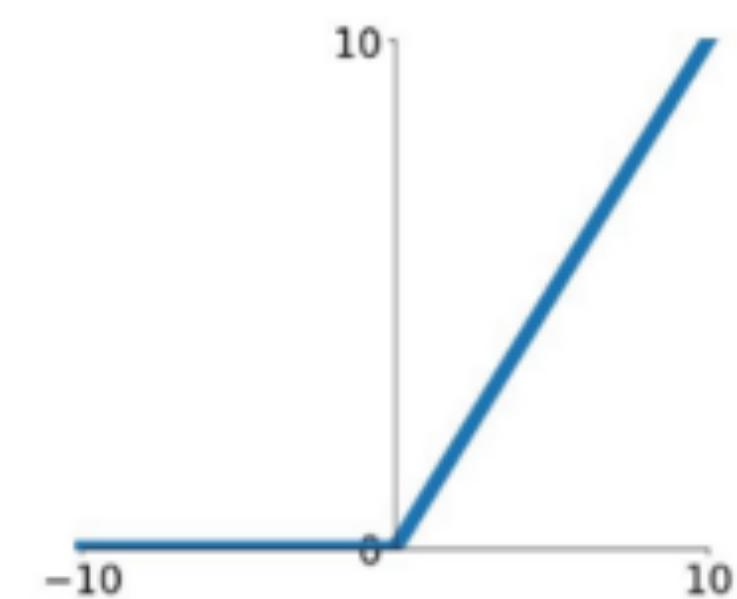
tanh

$$\tanh(x)$$



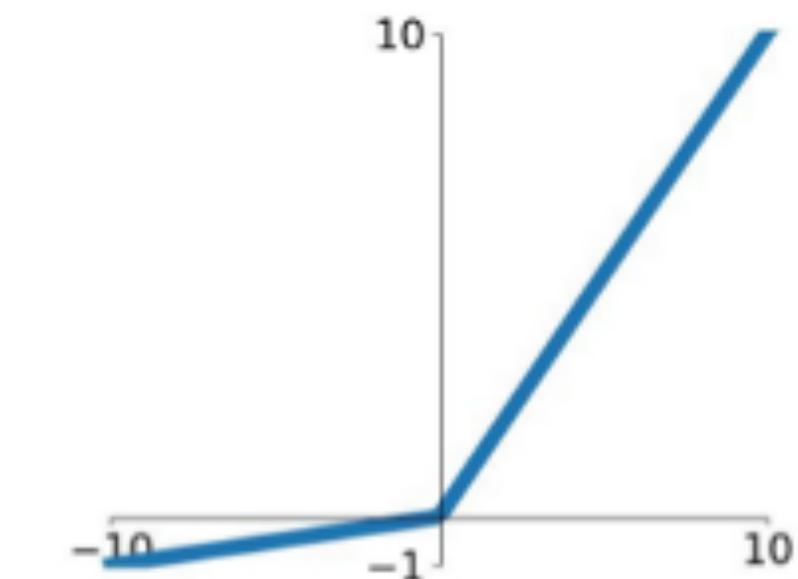
ReLU

$$\max(0, x)$$



Leaky ReLU

$$\max(0.1x, x)$$

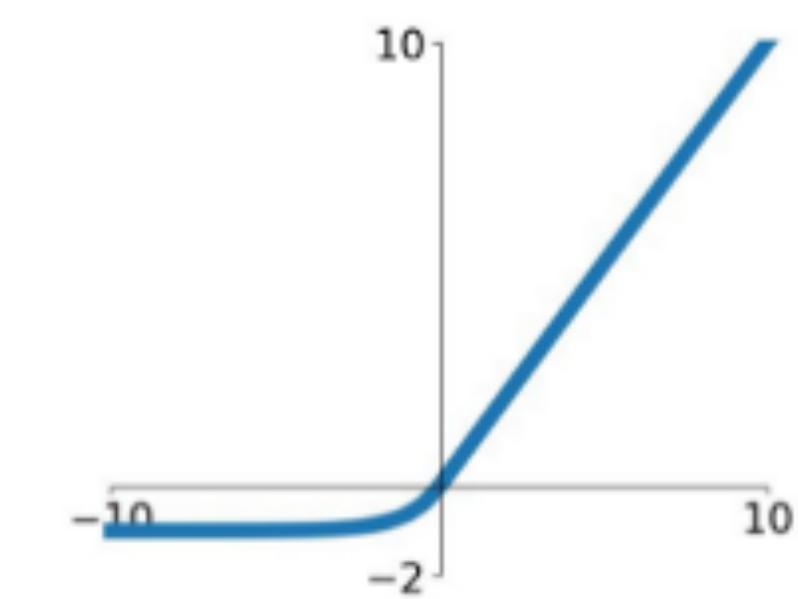


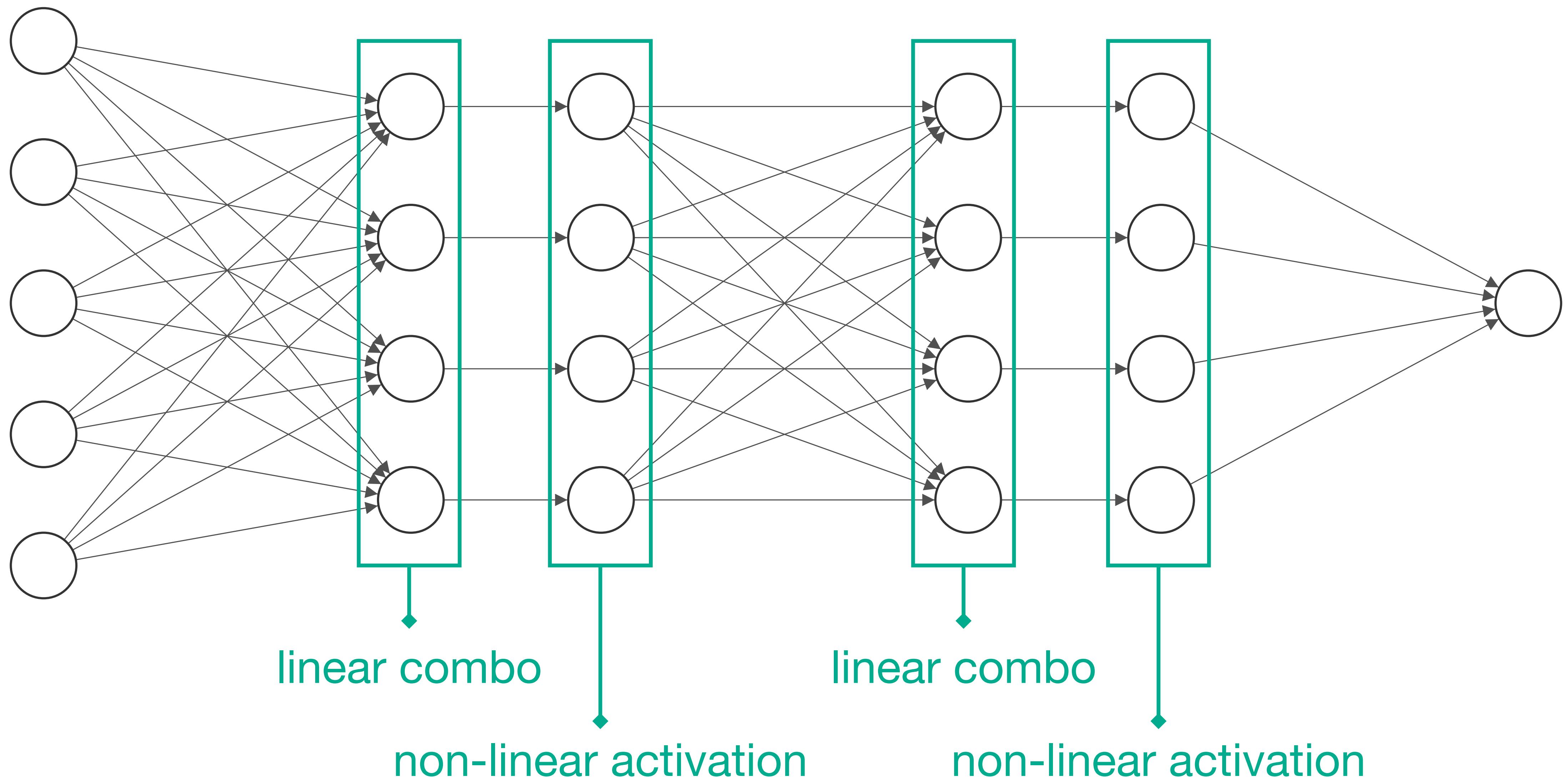
Maxout

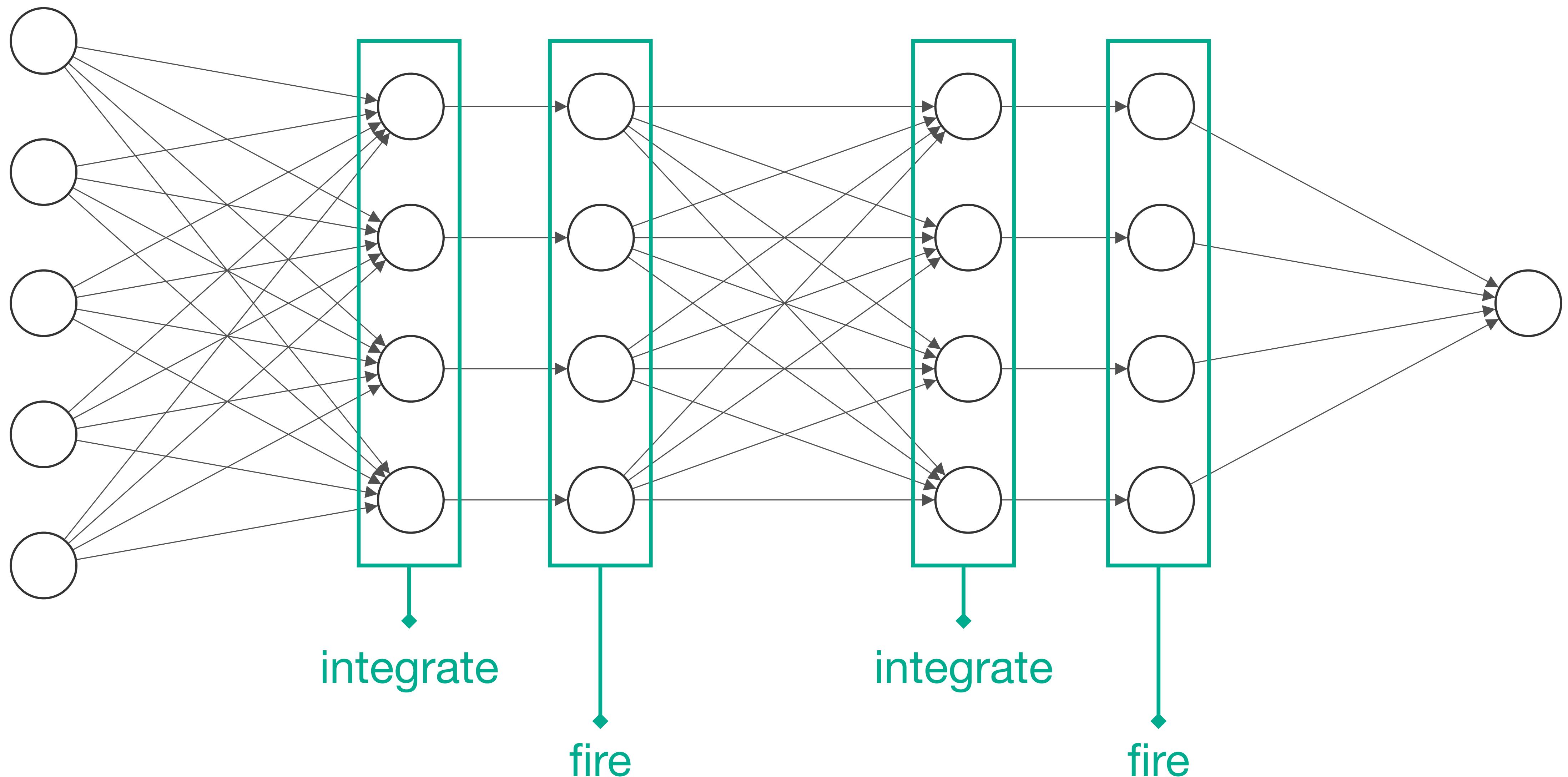
$$\max(w_1^T x + b_1, w_2^T x + b_2)$$

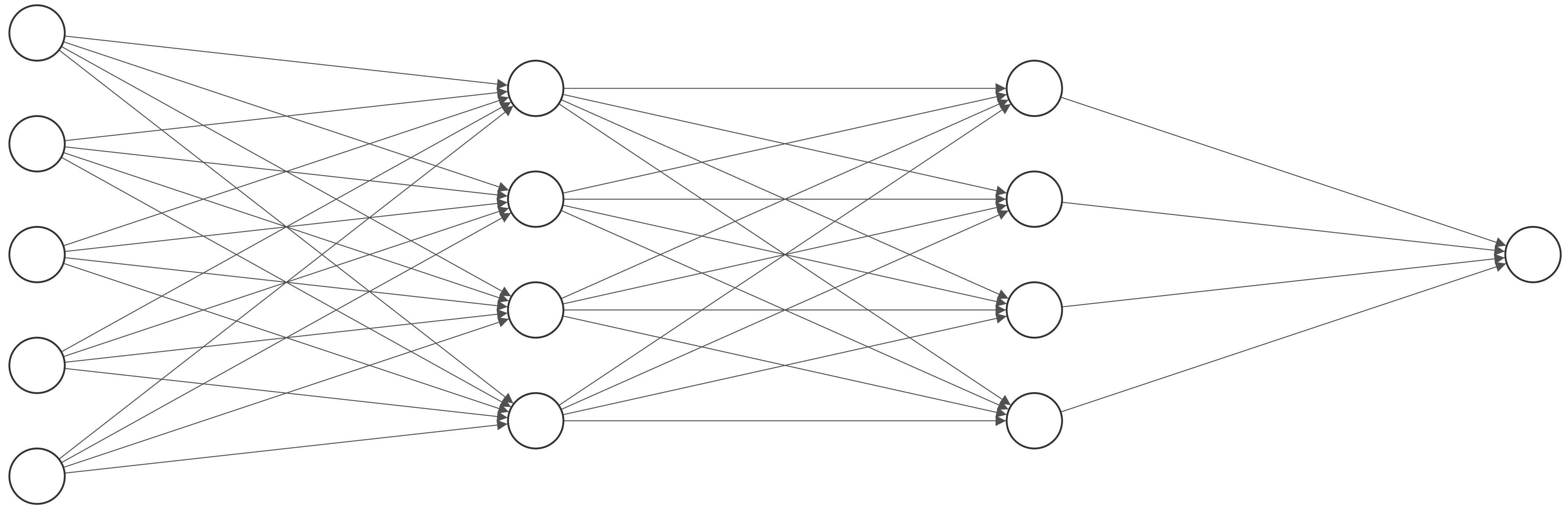
ELU

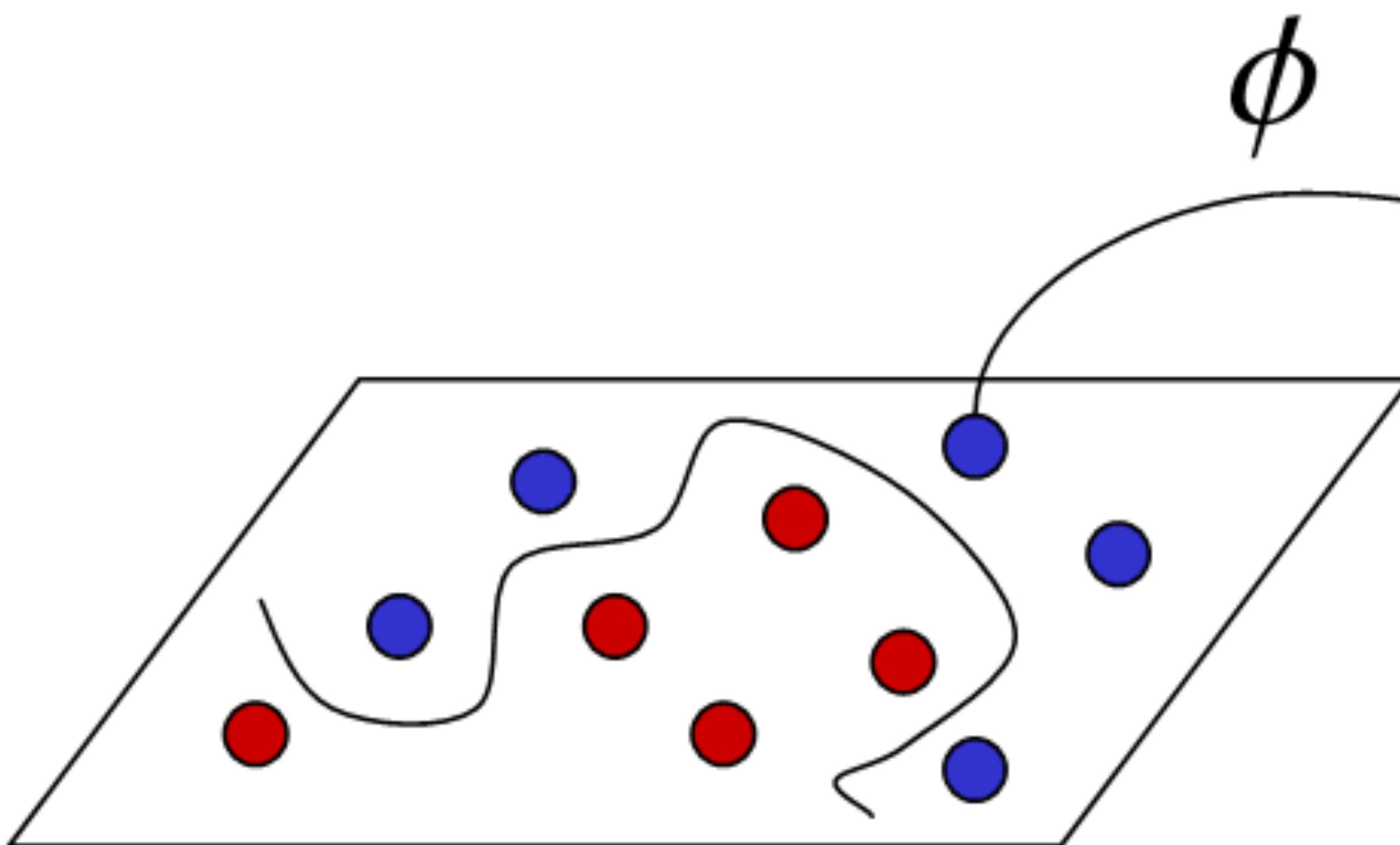
$$\begin{cases} x & x \geq 0 \\ \alpha(e^x - 1) & x < 0 \end{cases}$$



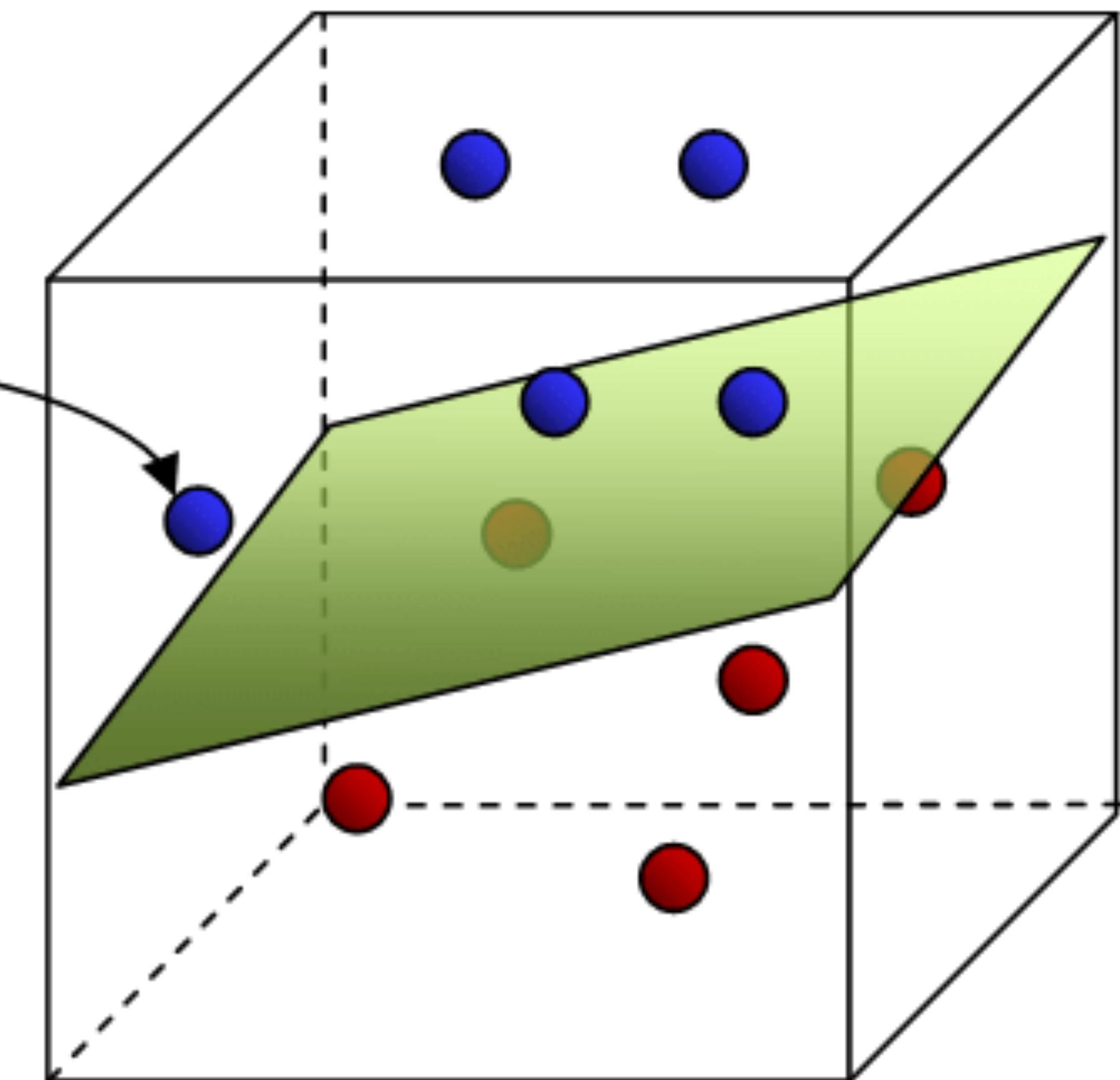




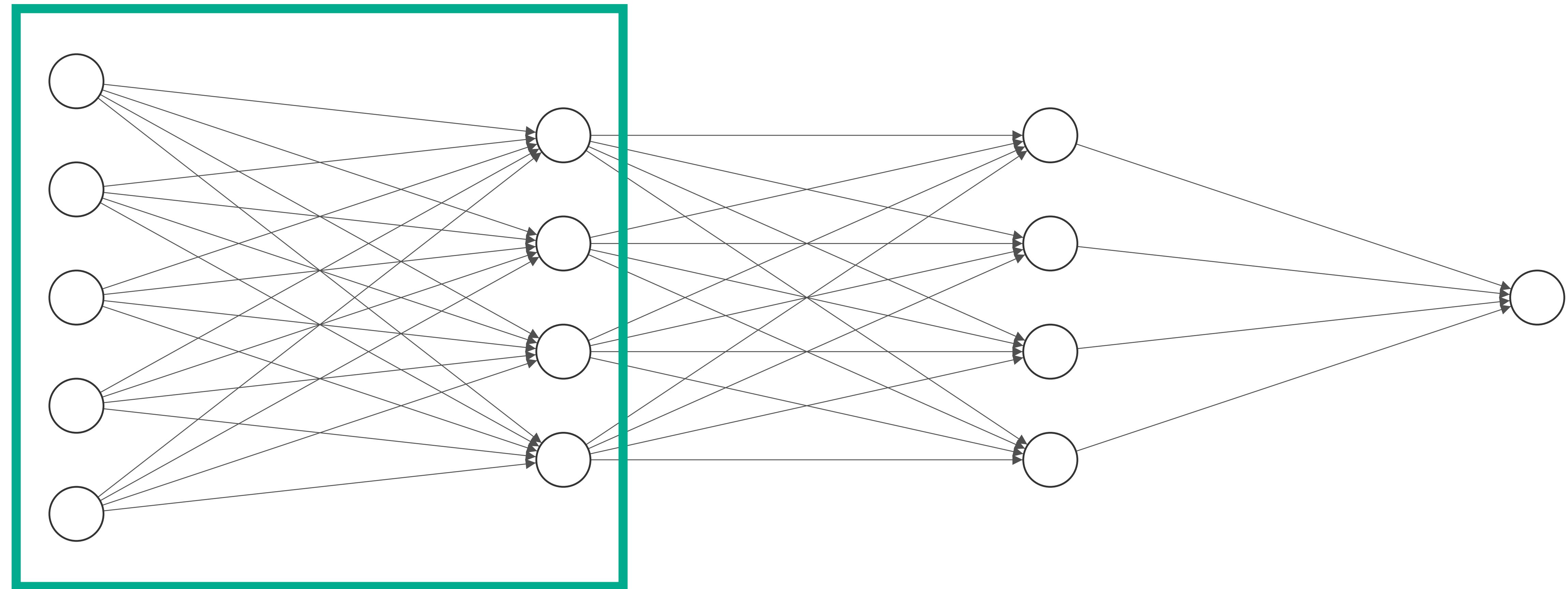


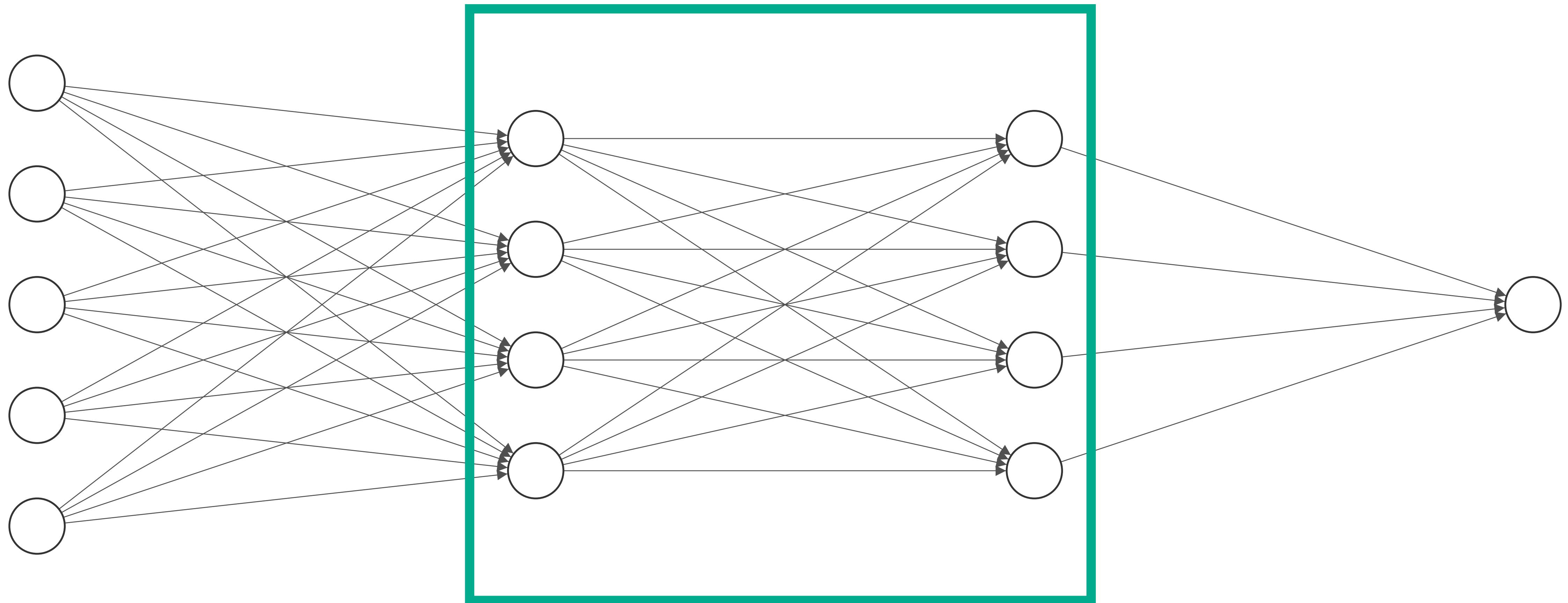


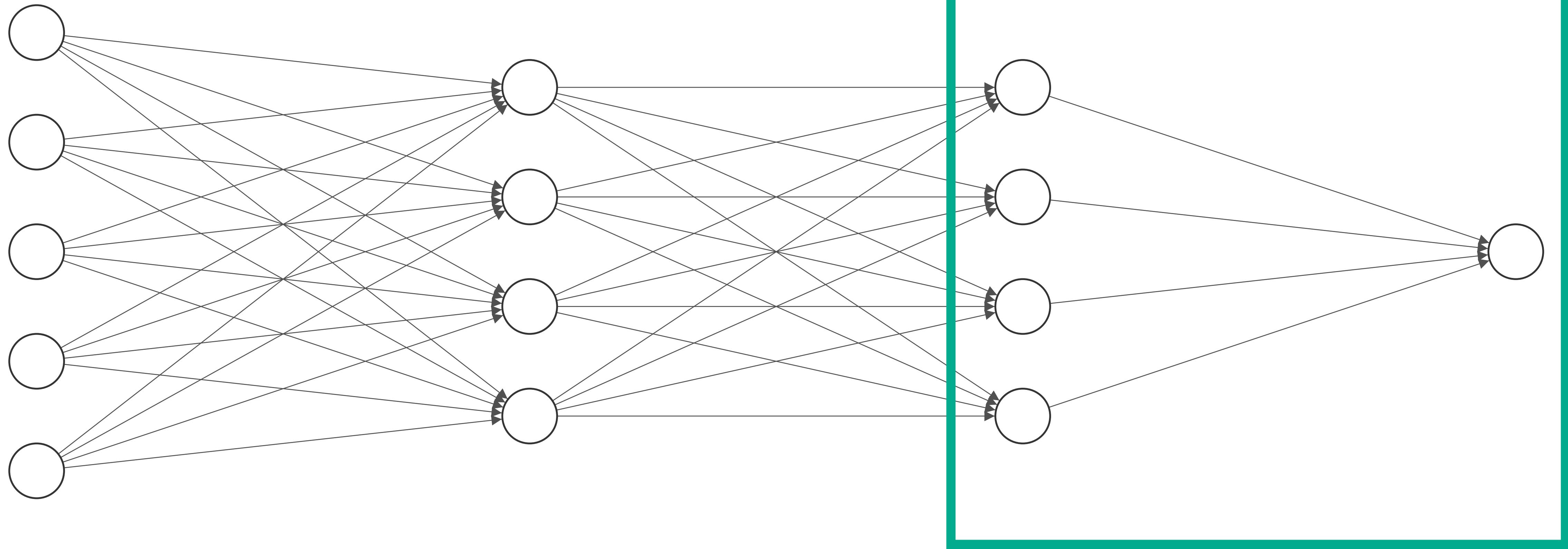
a) Input Space

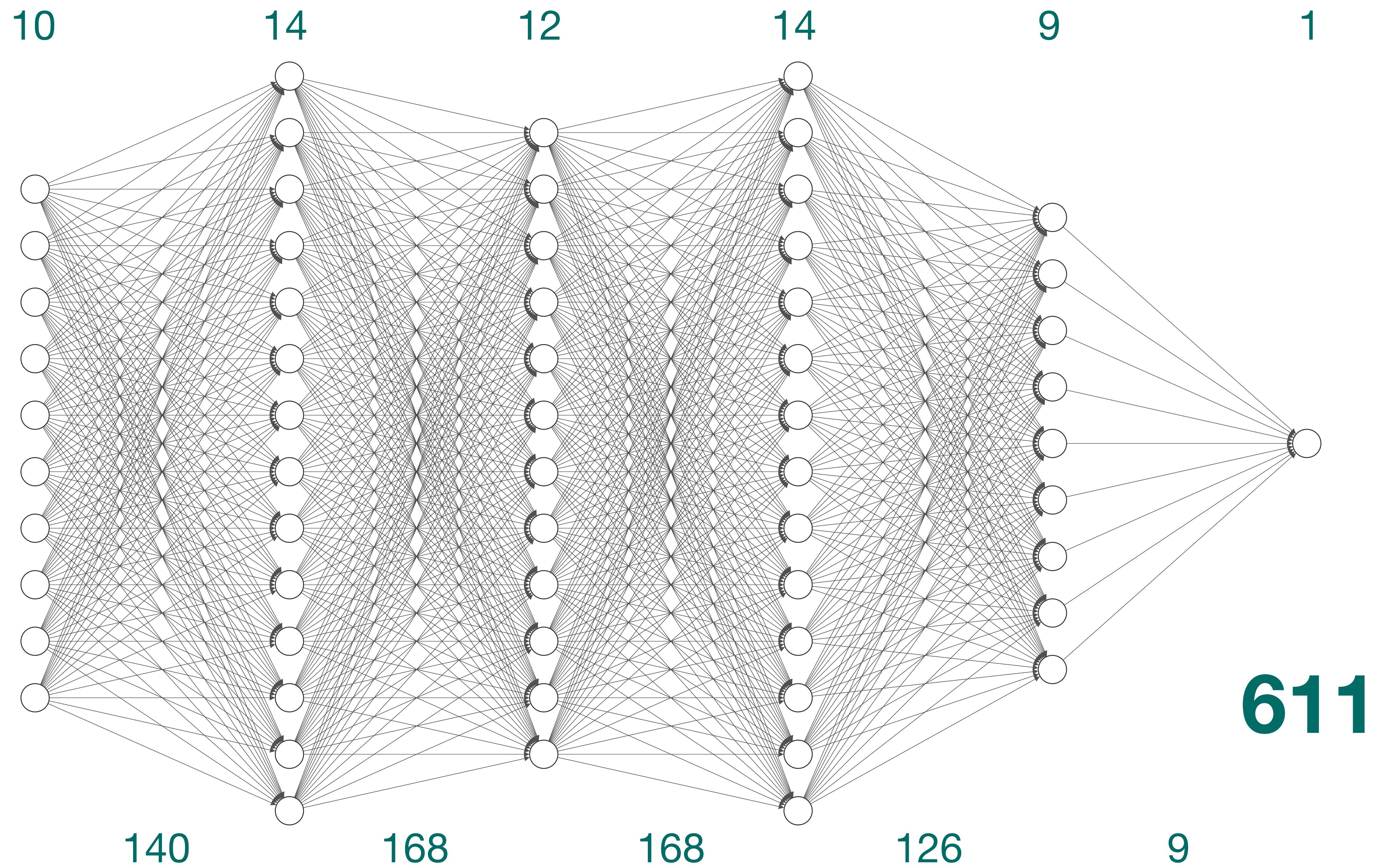


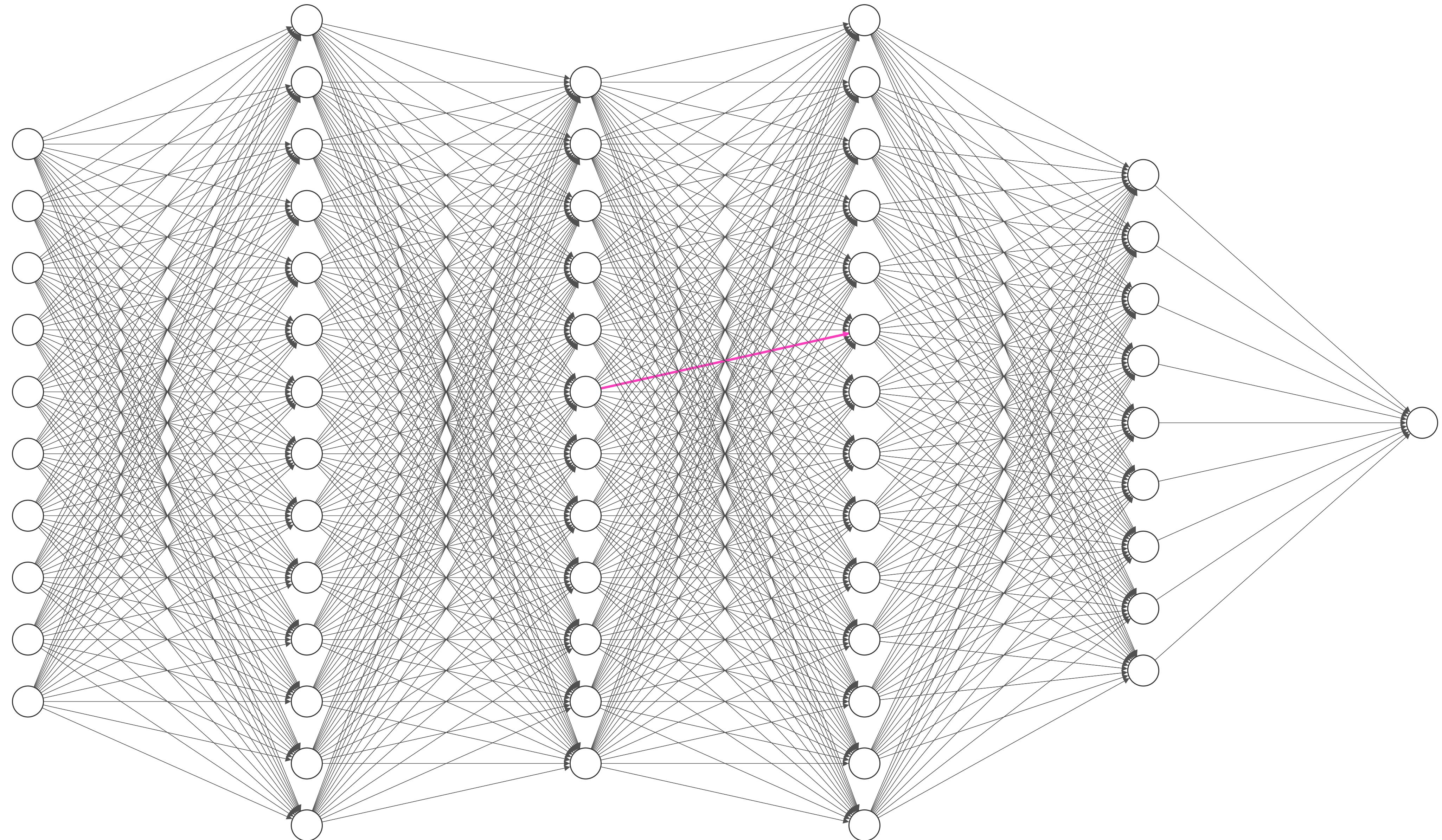
b) Feature Space











analytic vs numerical

analytic optimisation

$$\mathbf{w}^* = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y}$$

numerical optimisation

$$\mathbf{w} \leftarrow \mathbf{w} - \alpha \boxed{\nabla_{\mathbf{w}} L(\mathbf{f}, \mathbf{X}, \mathbf{y}, \mathbf{w})}$$

$$\nabla_{\mathbf{w}} L = \begin{bmatrix} \frac{\partial L}{\partial w_1} \\ \frac{\partial L}{\partial w_2} \\ \vdots \\ \frac{\partial L}{\partial w_m} \end{bmatrix}$$

$$\frac{\partial L}{\partial w_i} = \lim_{\delta \rightarrow 0} \frac{L(..., w_i + \delta, ...) - L(..., w_i, ...)}{\delta}$$

numerical gradient

$$\frac{\partial L}{\partial w_i} \approx \frac{L(..., w_i + \delta, ...) - L(..., w_i, ...)}{\delta}$$

analytic gradient

$$\begin{aligned}
\frac{\partial}{\partial w_j} l(\mathbf{w}) &= \frac{\partial}{\partial w_j} \left[y \log(\sigma(\mathbf{x} \cdot \mathbf{w})) + (1 - y) \log(1 - \sigma(\mathbf{x} \cdot \mathbf{w})) \right] \\
&= \left[y \frac{1}{\sigma(\mathbf{x} \cdot \mathbf{w})} \frac{\partial}{\partial w_j} \sigma(\mathbf{x} \cdot \mathbf{w}) \right] + \left[(1 - y) \frac{1}{1 - \sigma(\mathbf{x} \cdot \mathbf{w})} \left(-\frac{\partial}{\partial w_j} \sigma(\mathbf{x} \cdot \mathbf{w}) \right) \right] \\
&= \left[y \frac{1}{\sigma(\mathbf{x} \cdot \mathbf{w})} \frac{\partial}{\partial w_j} \sigma(\mathbf{x} \cdot \mathbf{w}) \right] - \left[(1 - y) \frac{1}{1 - \sigma(\mathbf{x} \cdot \mathbf{w})} \frac{\partial}{\partial w_j} \sigma(\mathbf{x} \cdot \mathbf{w}) \right] \\
&= \left[y \frac{1}{\sigma(\mathbf{x} \cdot \mathbf{w})} - (1 - y) \frac{1}{1 - \sigma(\mathbf{x} \cdot \mathbf{w})} \right] \frac{\partial}{\partial w_j} \sigma(\mathbf{x} \cdot \mathbf{w}) \\
&= \left[y \frac{1}{\sigma(\mathbf{x} \cdot \mathbf{w})} - (1 - y) \frac{1}{1 - \sigma(\mathbf{x} \cdot \mathbf{w})} \right] \sigma(\mathbf{x} \cdot \mathbf{w})(1 - \sigma(\mathbf{x} \cdot \mathbf{w})) \frac{\partial}{\partial w_j} (\mathbf{x} \cdot \mathbf{w}) \\
&= (y(1 - \sigma(\mathbf{x} \cdot \mathbf{w})) - (1 - y)\sigma(\mathbf{x} \cdot \mathbf{w})) \cdot x_j \\
&= (y - y\sigma(\mathbf{x} \cdot \mathbf{w}) - \sigma(\mathbf{x} \cdot \mathbf{w}) + y\sigma(\mathbf{x} \cdot \mathbf{w})) \cdot x_j \\
&= (y - \sigma(\mathbf{x} \cdot \mathbf{w})) \cdot x_j \\
&= (y - \hat{y})x_j
\end{aligned}$$

the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$(f \circ g)' = (f' \circ g) \times g'$$

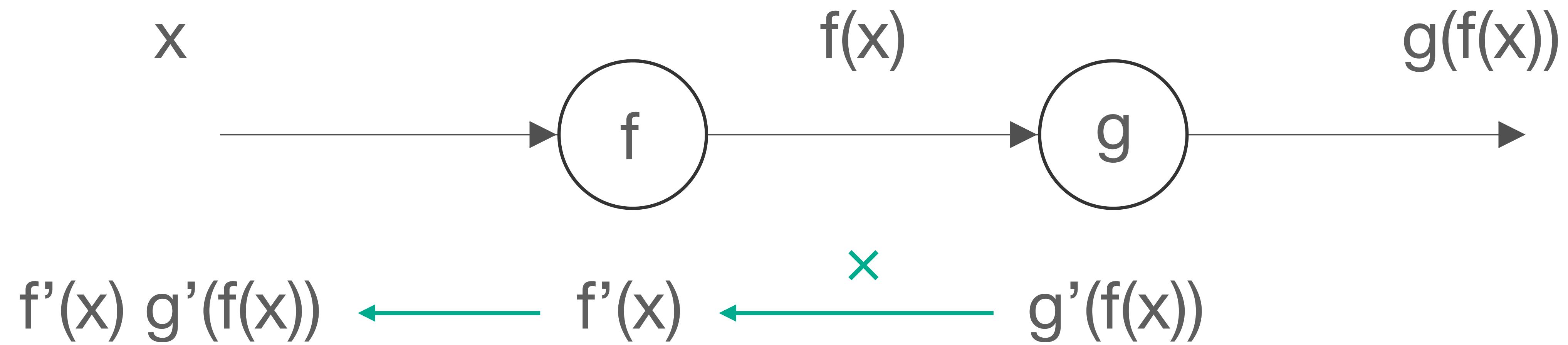
the chain rule

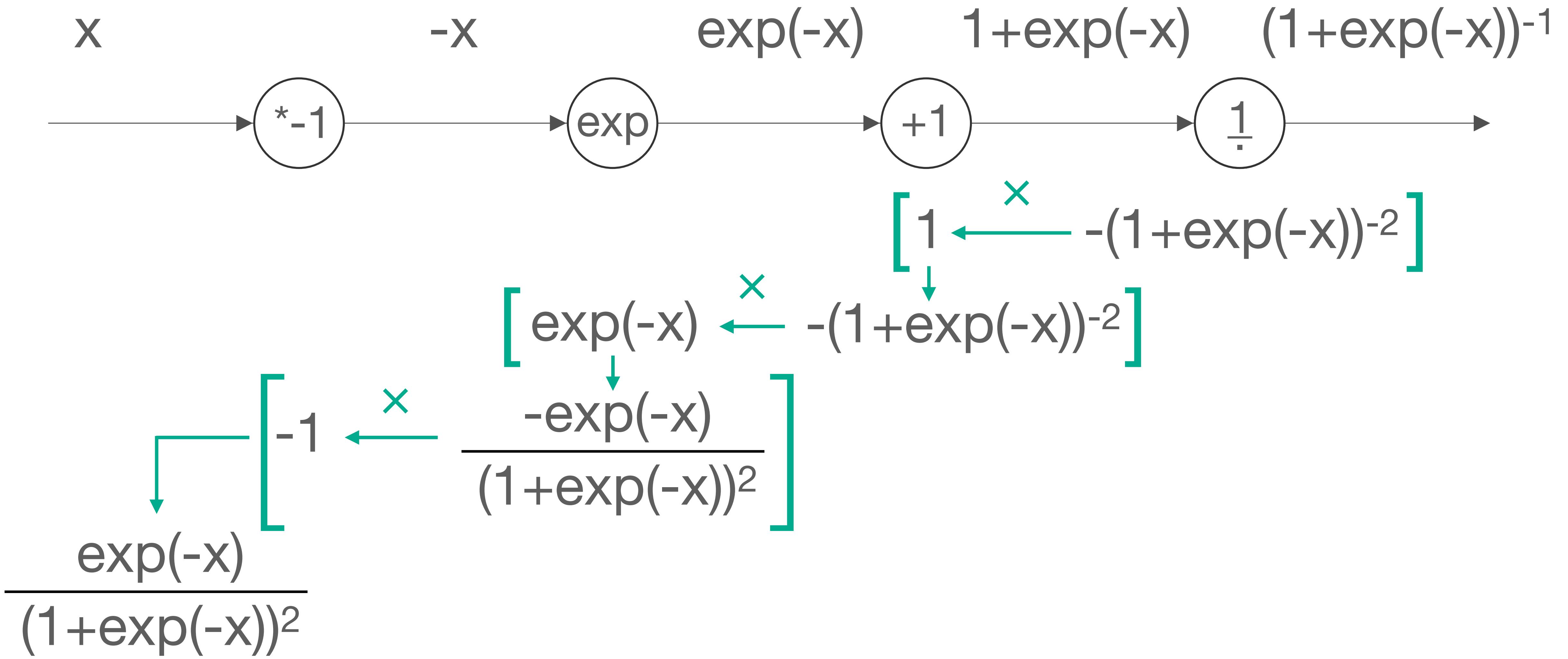
gradients multiply

$$[f(g(x))]' = f'(g(x)) \times g'(x)$$

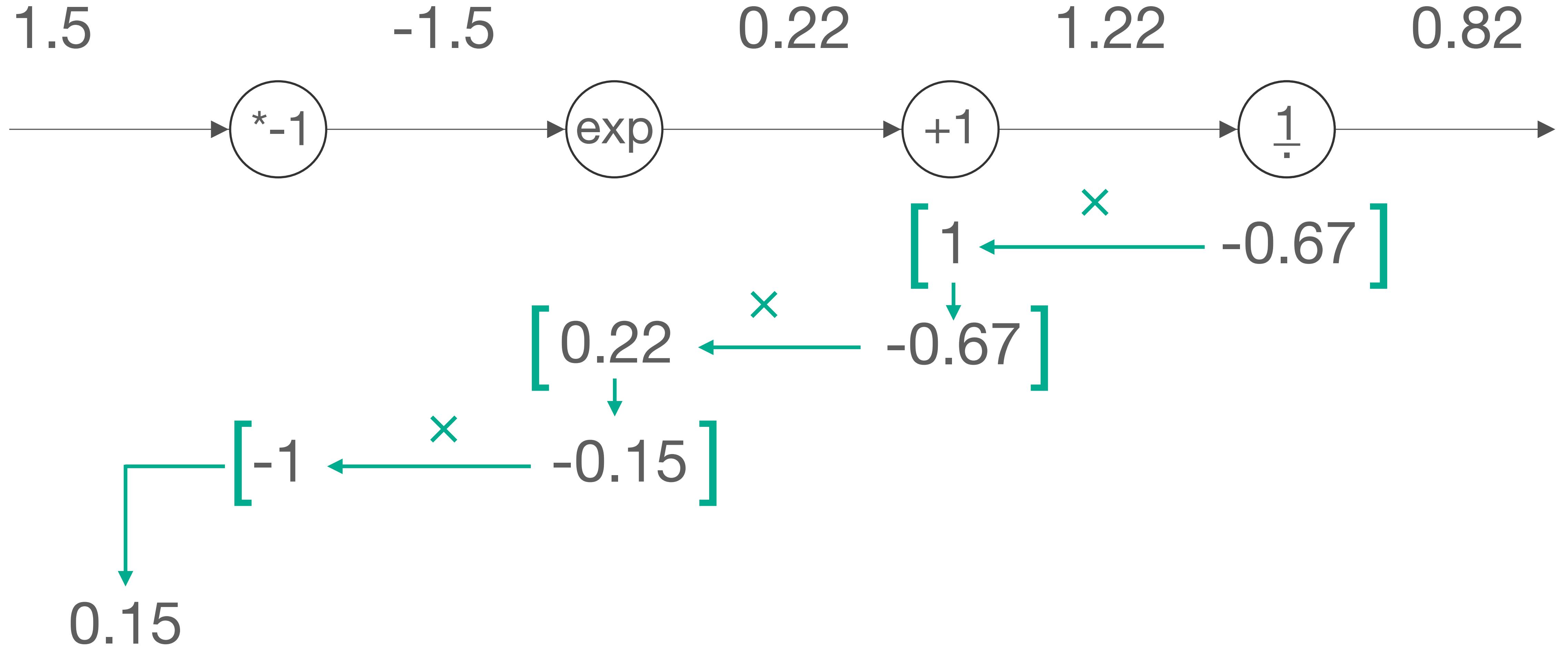
The diagram illustrates the application of the chain rule to the composite function $[f(g(x))]$. The expression is shown in black text. A blue arrow originates from the inner function $g(x)$ within the brackets and points to the term $g'(x)$ in the derivative. An orange arrow originates from the outer function $f(g(x))$ and points to the term $f'(g(x))$ in the derivative.

$$\left[f(g(h(x))) \right]' = f'(g(h(x))) \times g'(h(x)) \times h'(x)$$





gradients are just numbers!



```
out1 = f1(x)
out2 = f2(out1)
out3 = f3(out2)
out4 = f4(out3)
loss = f5(out4)
```

```
dloss_out4 = df5(out4)
dloss_out3 = df4(out3) * dloss_out4
dloss_out2 = df3(out2) * dloss_out3
dloss_out1 = df2(out1) * dloss_out2
dloss_x = df1(x) * dloss_out1
```

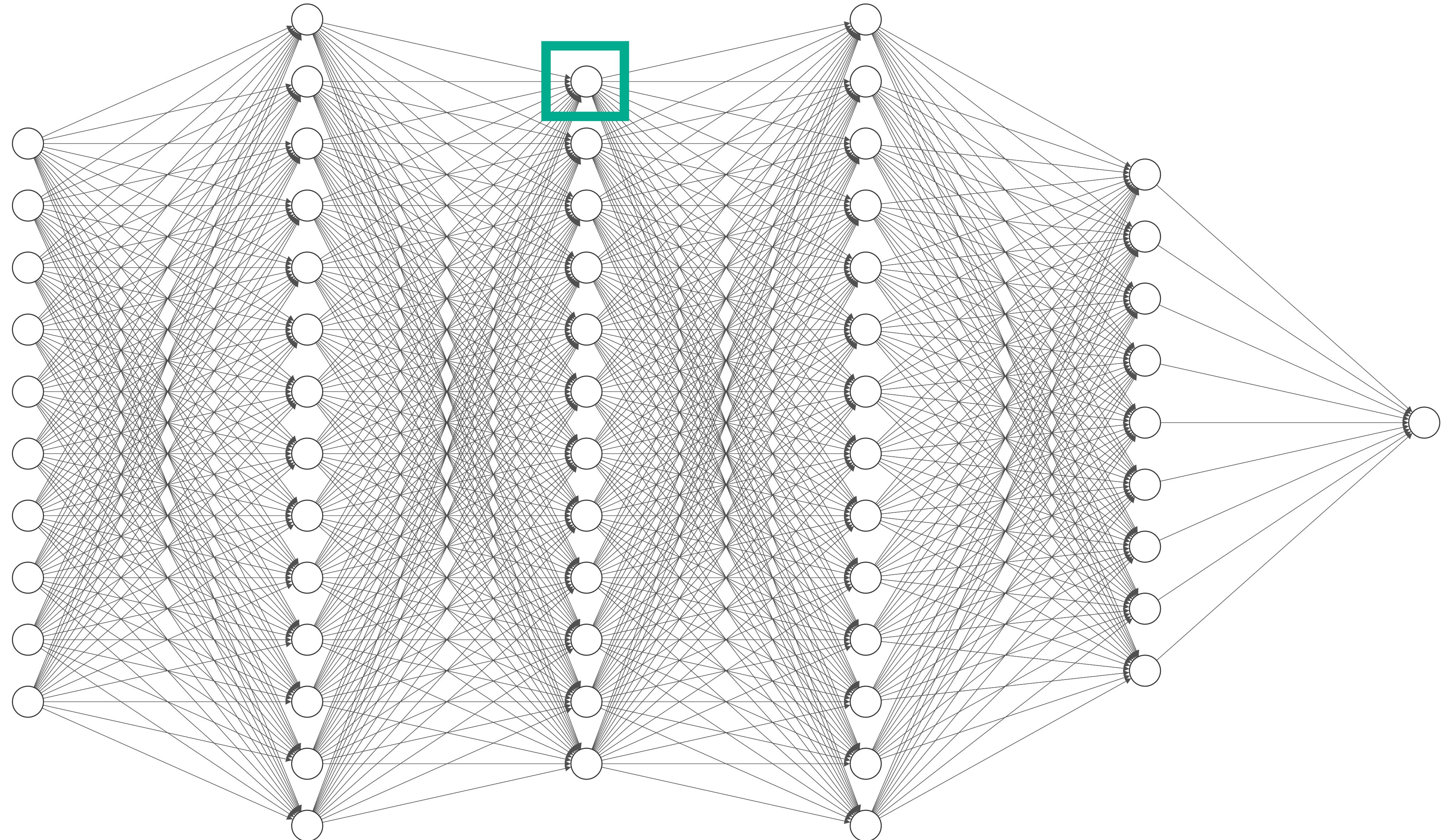
```
out1 = f1(x)
out2 = f2(out1)
out3 = f3(out2)
out4 = f4(out3)
loss = f5(out4)
```

local gradient

downstream gradient

```
dloss_out4 = df5(out4)
dloss_out3 = df4(out3) * dloss_out4
dloss_out2 = df3(out2) * dloss_out3
dloss_out1 = df2(out1) * dloss_out2
dloss_x = df1(x) * dloss_out1
```

analytic gradients, numerically
backpropagation



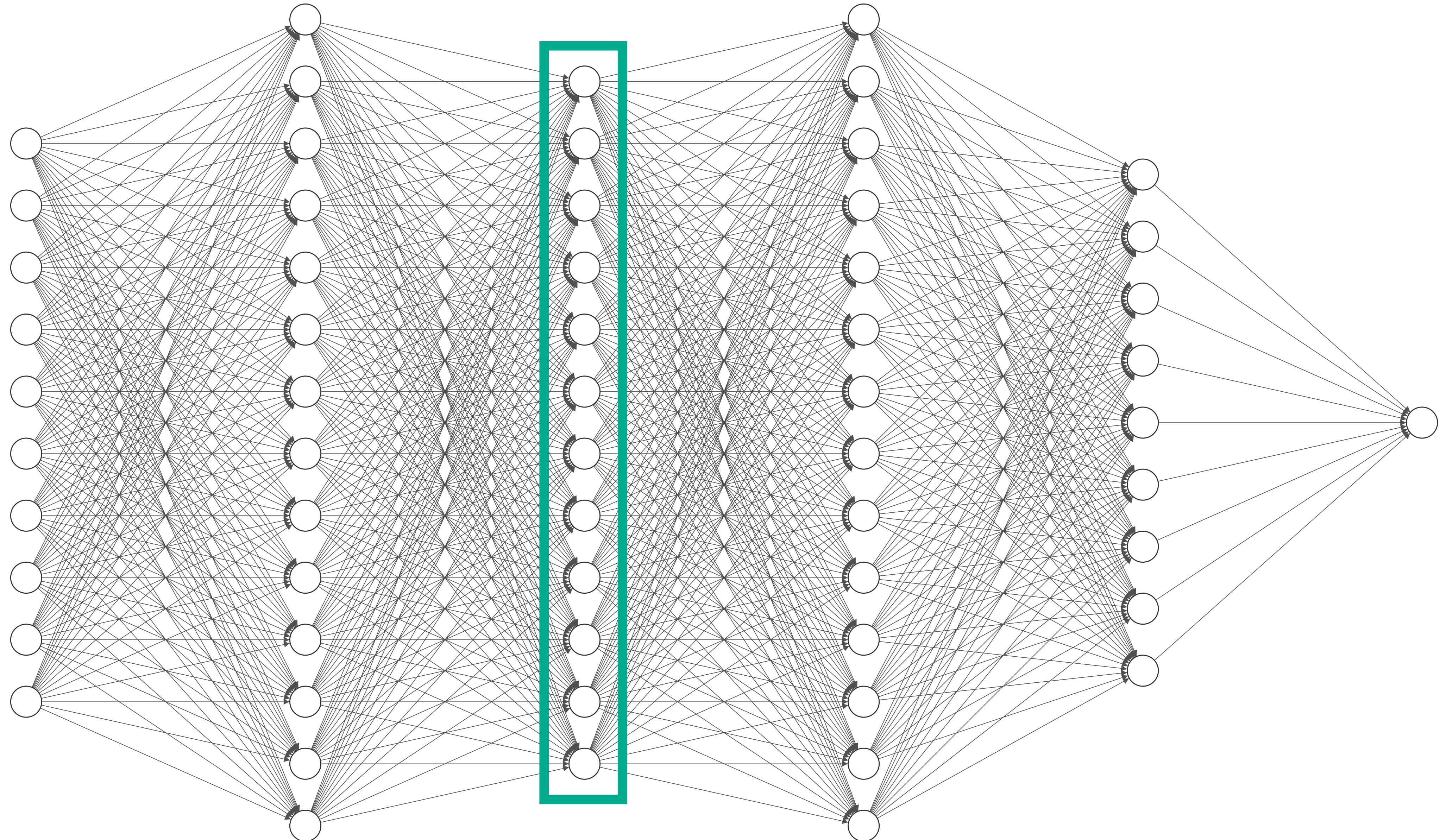
$$z = \mathbf{x} \cdot \mathbf{w}$$

$$\nabla_{\mathbf{x}} z = \mathbf{w}$$

$$\nabla_{\mathbf{w}} z = \mathbf{x}$$

$$a(\mathbf{x}) = [f(x_1), f(x_2), \dots, f(x_d)]$$

$$\nabla_{\mathbf{x}} a = [f'(x_1), f'(x_2), \dots, f'(x_d)]$$



forward pass

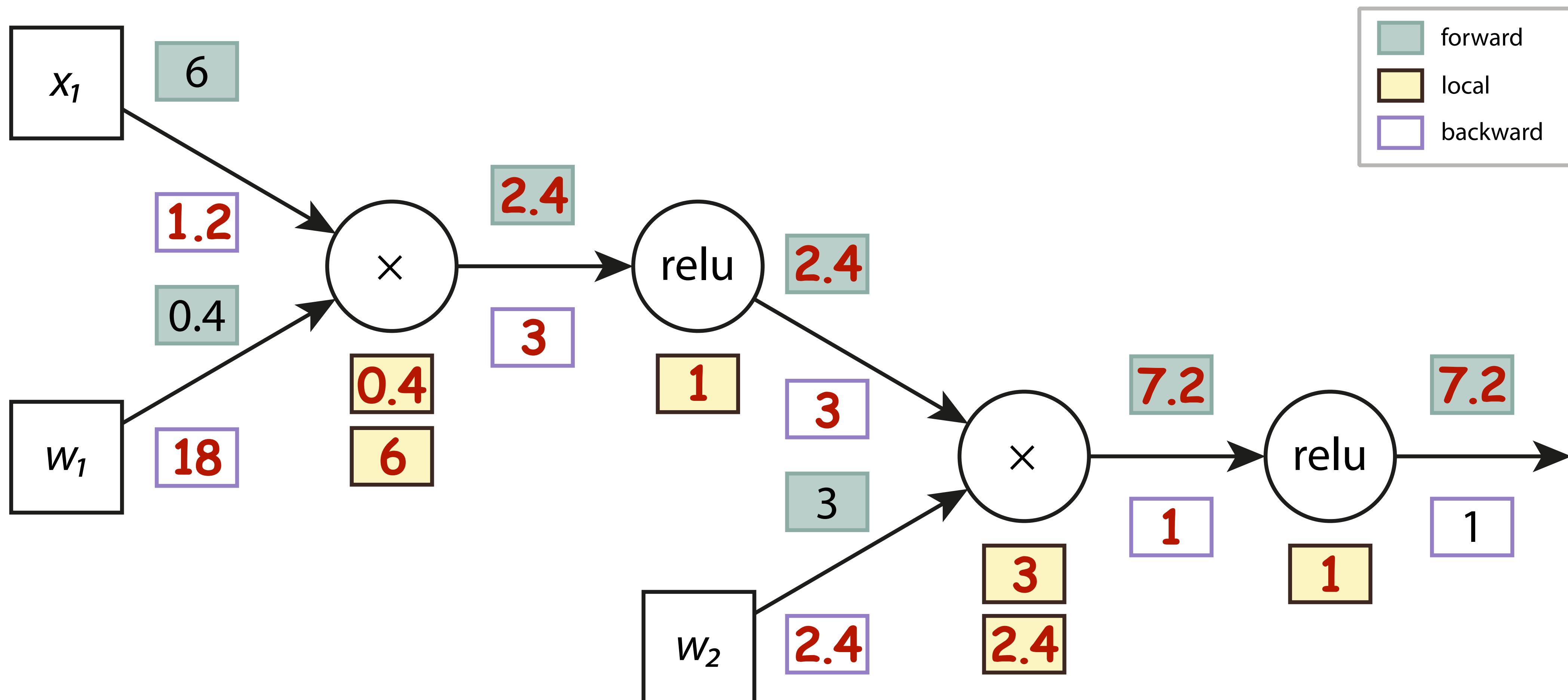
- evaluate model on data**
- calculate local gradients**
- save ins and outs**
- calculate final loss**

backward pass

**calculate final loss gradient
propagate gradients backward
multiply downstream by local**

optimise

update weights with gradients



Questions?

Next: More Neural Networks

