

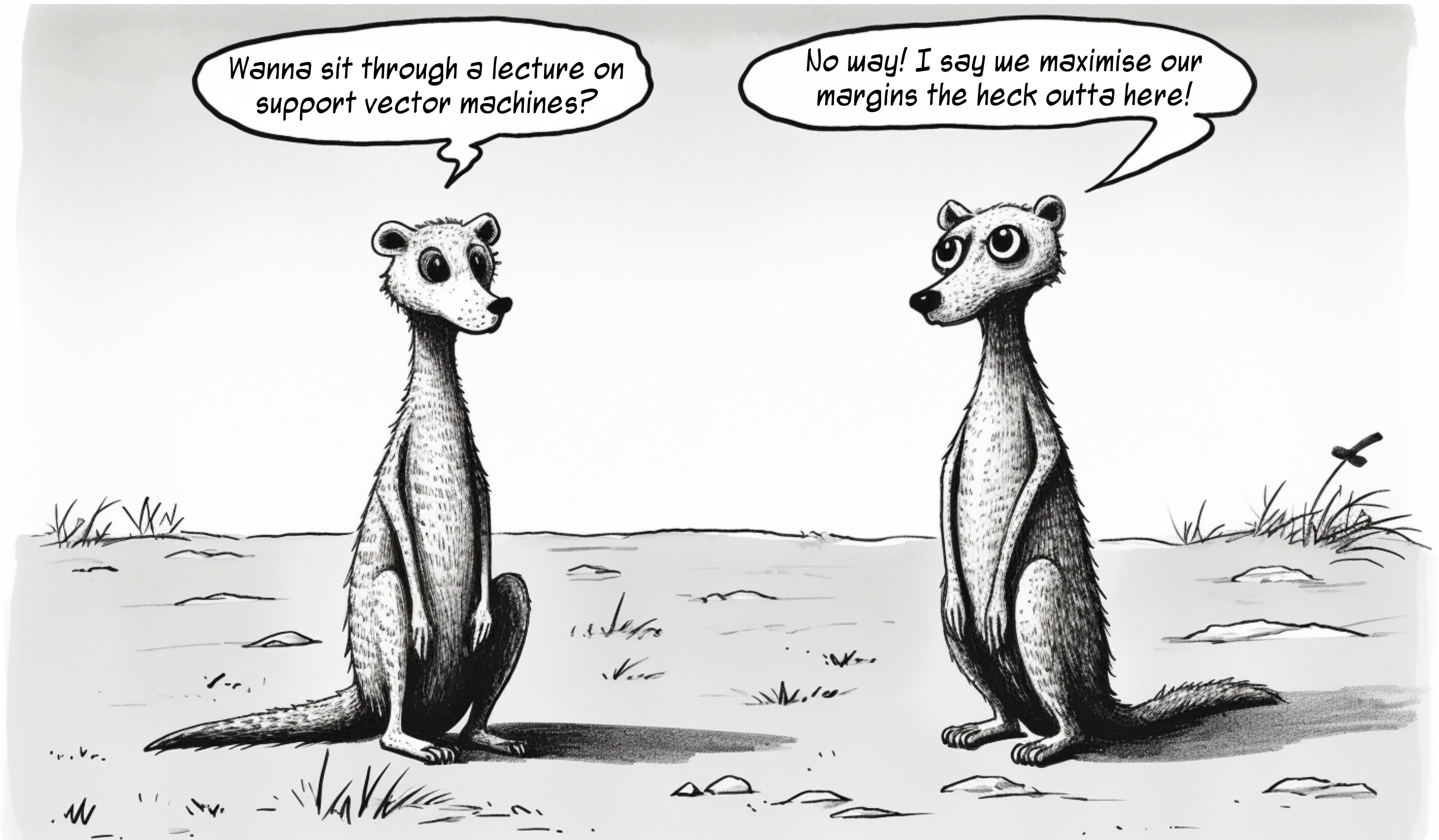
Week 4: Linear Models Revisited

Matthew Caldwell

COMP0088 Introduction to Machine Learning • UCL Computer Science • Autumn 2024

Week 4 Recap

Joining the Dots



Foreword by John Carmack of id Software

**Michael Abrash's
GRAPHICS
PROGRAMMING
Black Book
SPECIAL EDITION**

Michael Abrash

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^\top \mathbf{b} = \langle \mathbf{a}, \mathbf{b} \rangle = \sum_i a_i b_i$$

$$\mathbf{x} \cdot \mathbf{x} = \|\mathbf{x}\|^2$$

$$\mathbf{w} \cdot \mathbf{x} = w_1x_1 + w_2x_2 + \dots + w_dx_d$$

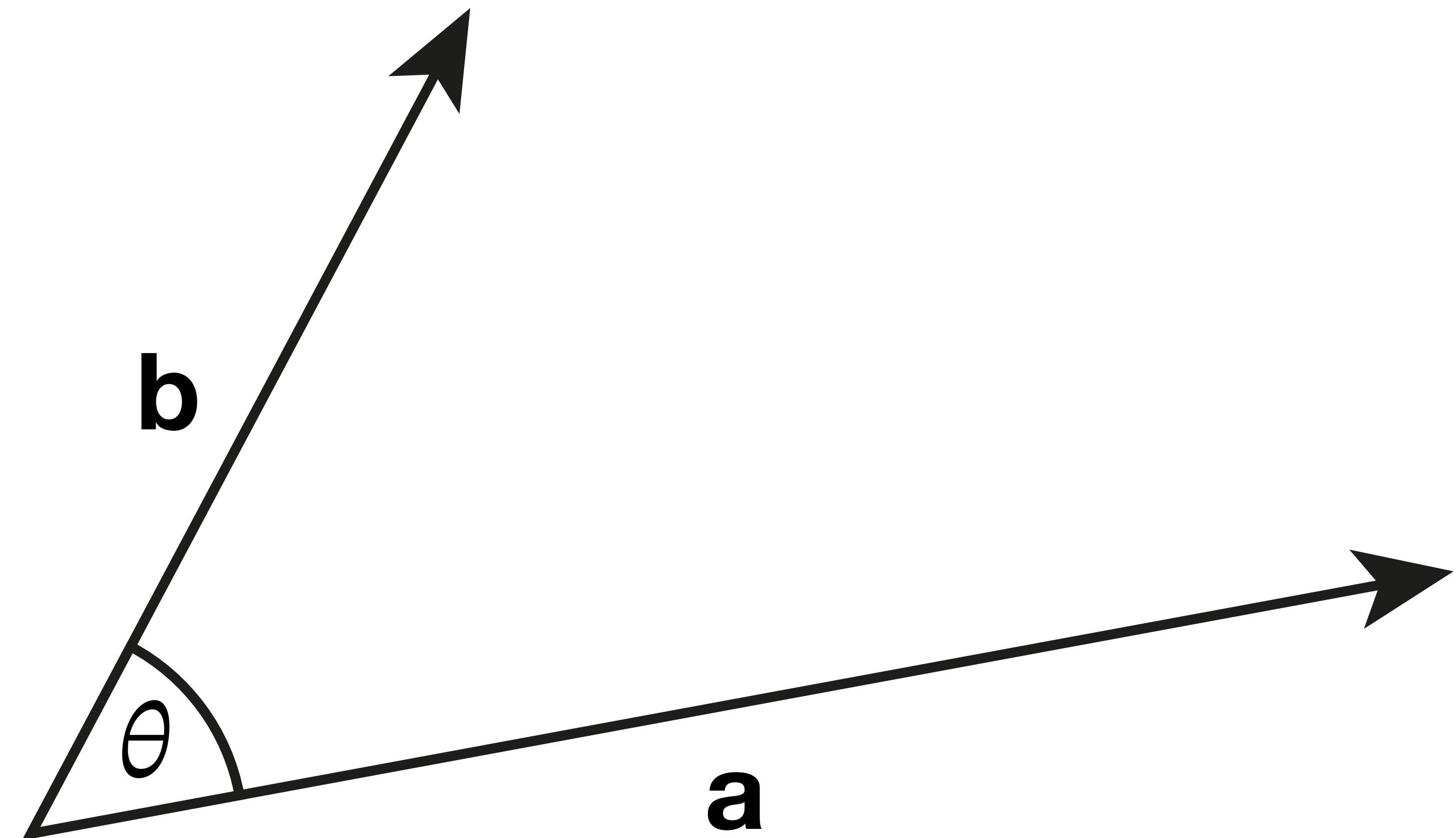
$$\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

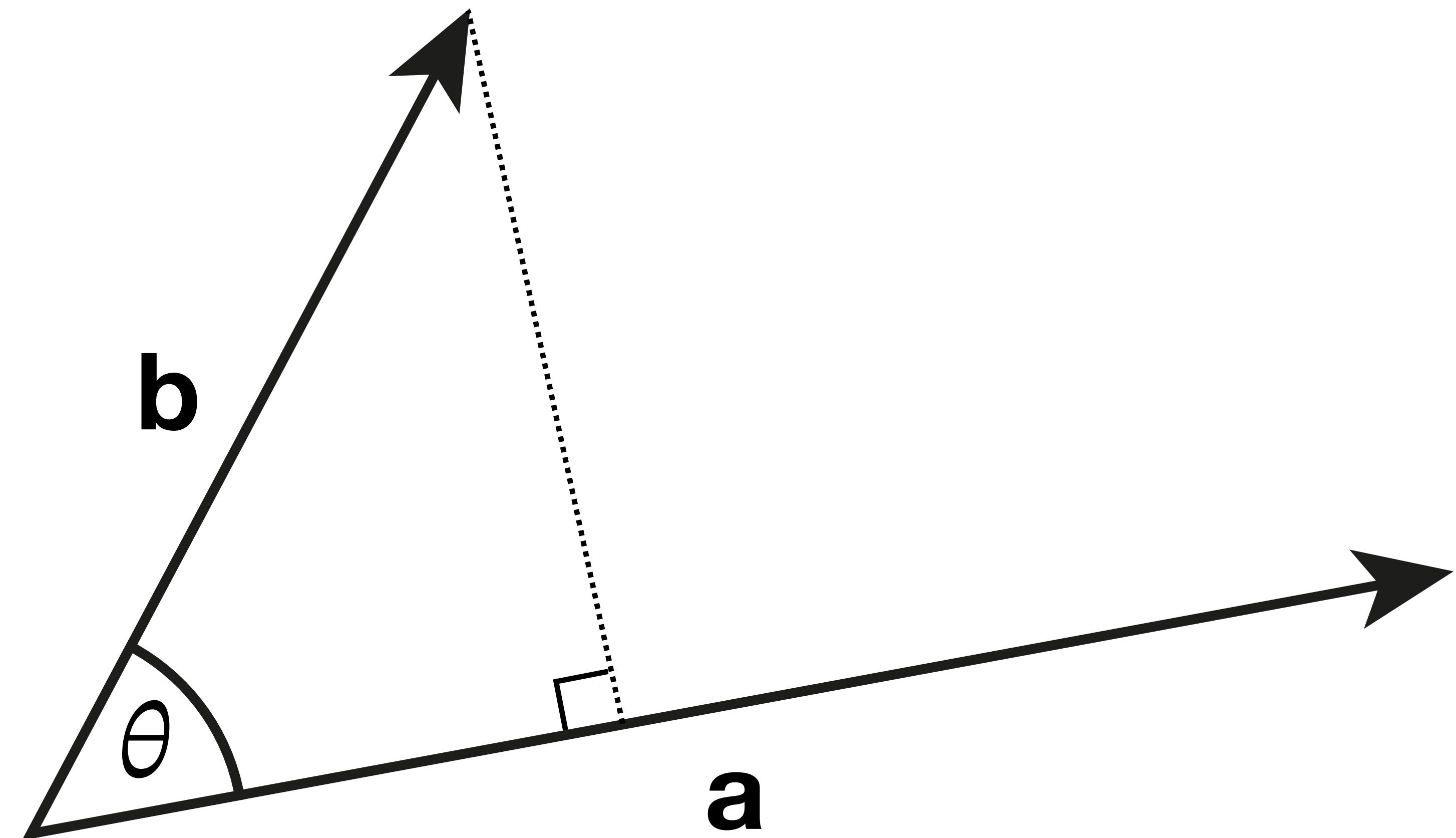
$$\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

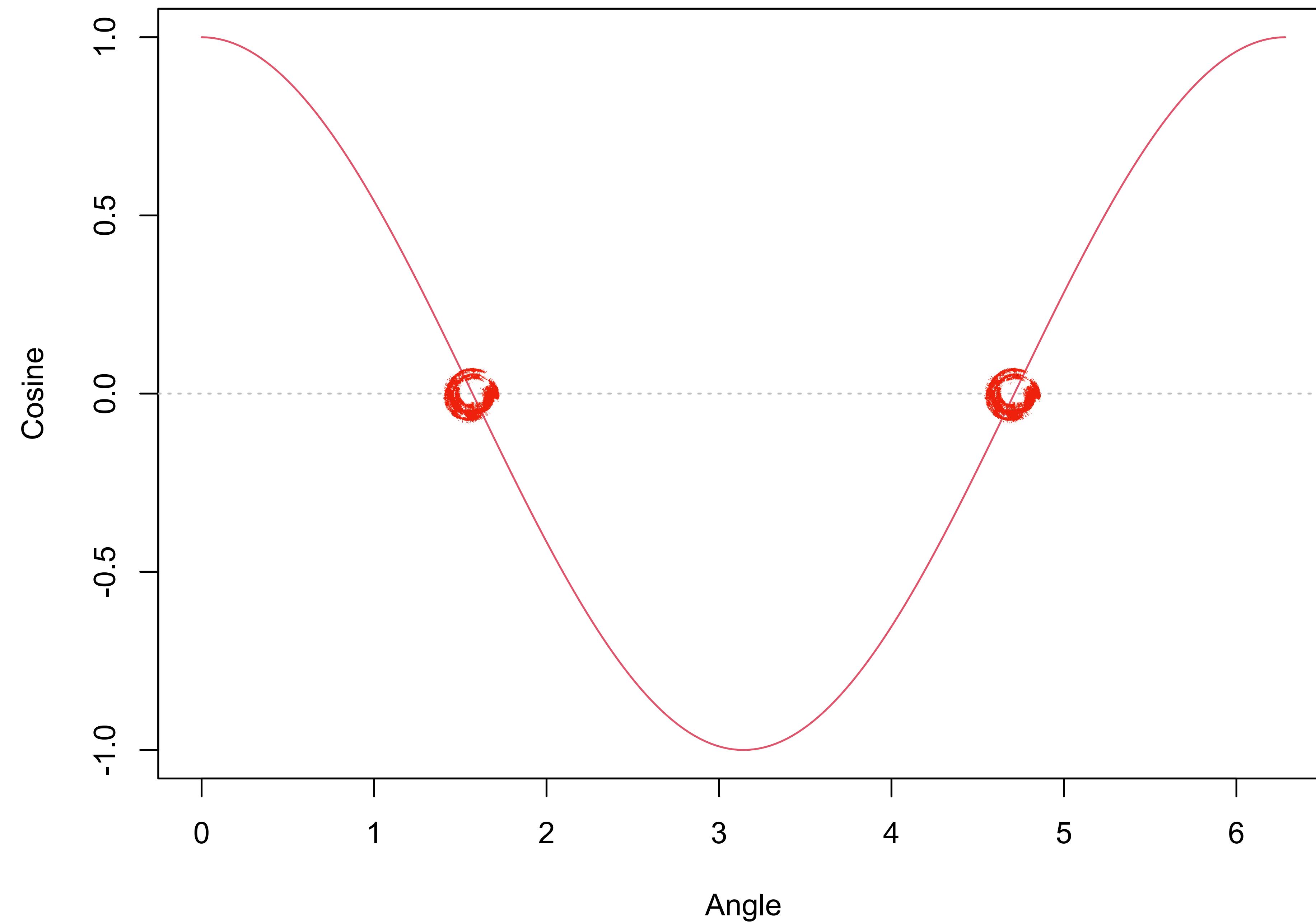
$$\mathbf{w} = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \\ \vdots \\ \mathbf{a}_n^\top \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_d \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 & \dots & \mathbf{a}_1 \cdot \mathbf{b}_d \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 & \dots & \mathbf{a}_2 \cdot \mathbf{b}_d \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_n \cdot \mathbf{b}_1 & \mathbf{a}_n \cdot \mathbf{b}_2 & \dots & \mathbf{a}_n \cdot \mathbf{b}_d \end{bmatrix}$$



$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

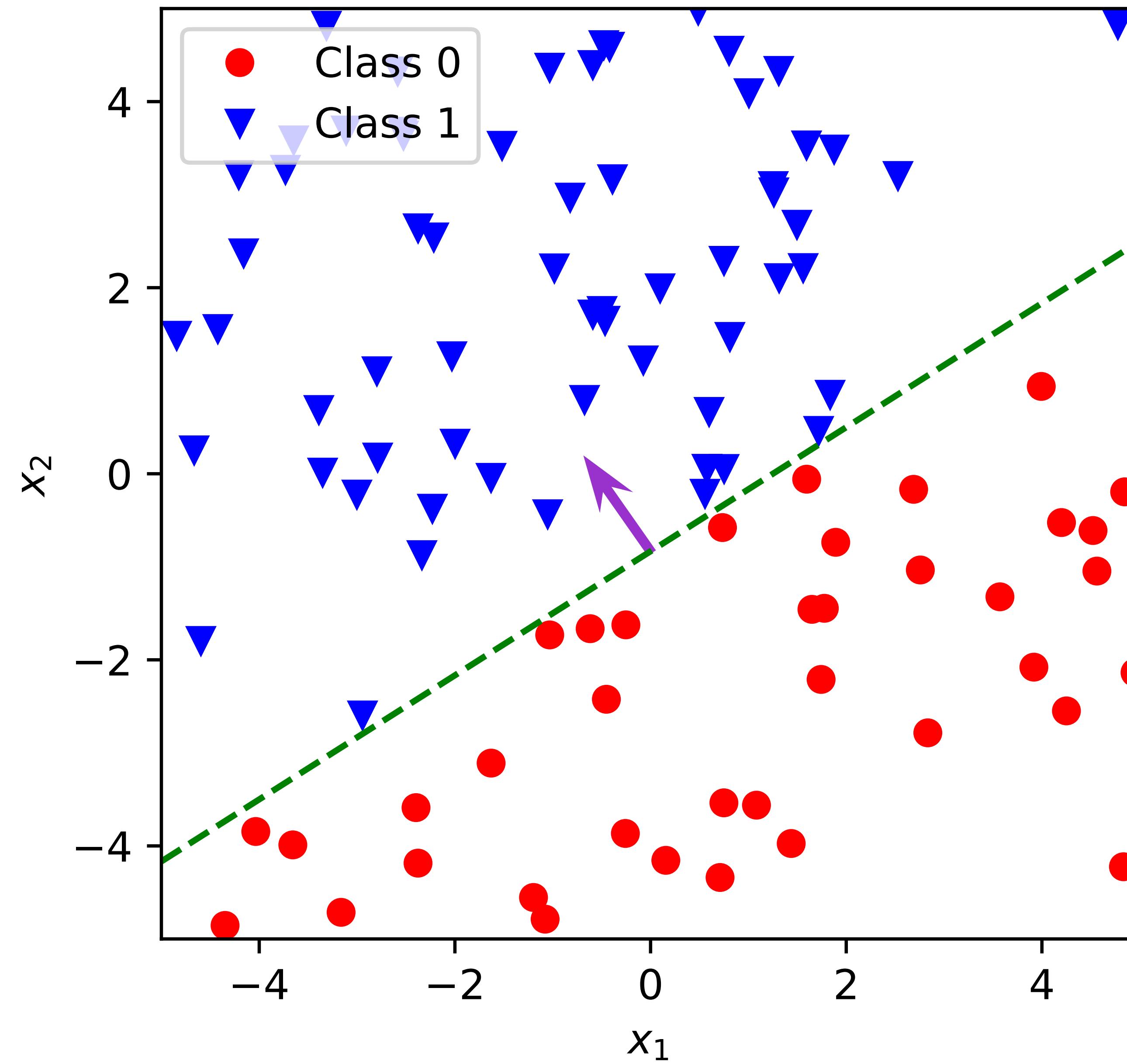


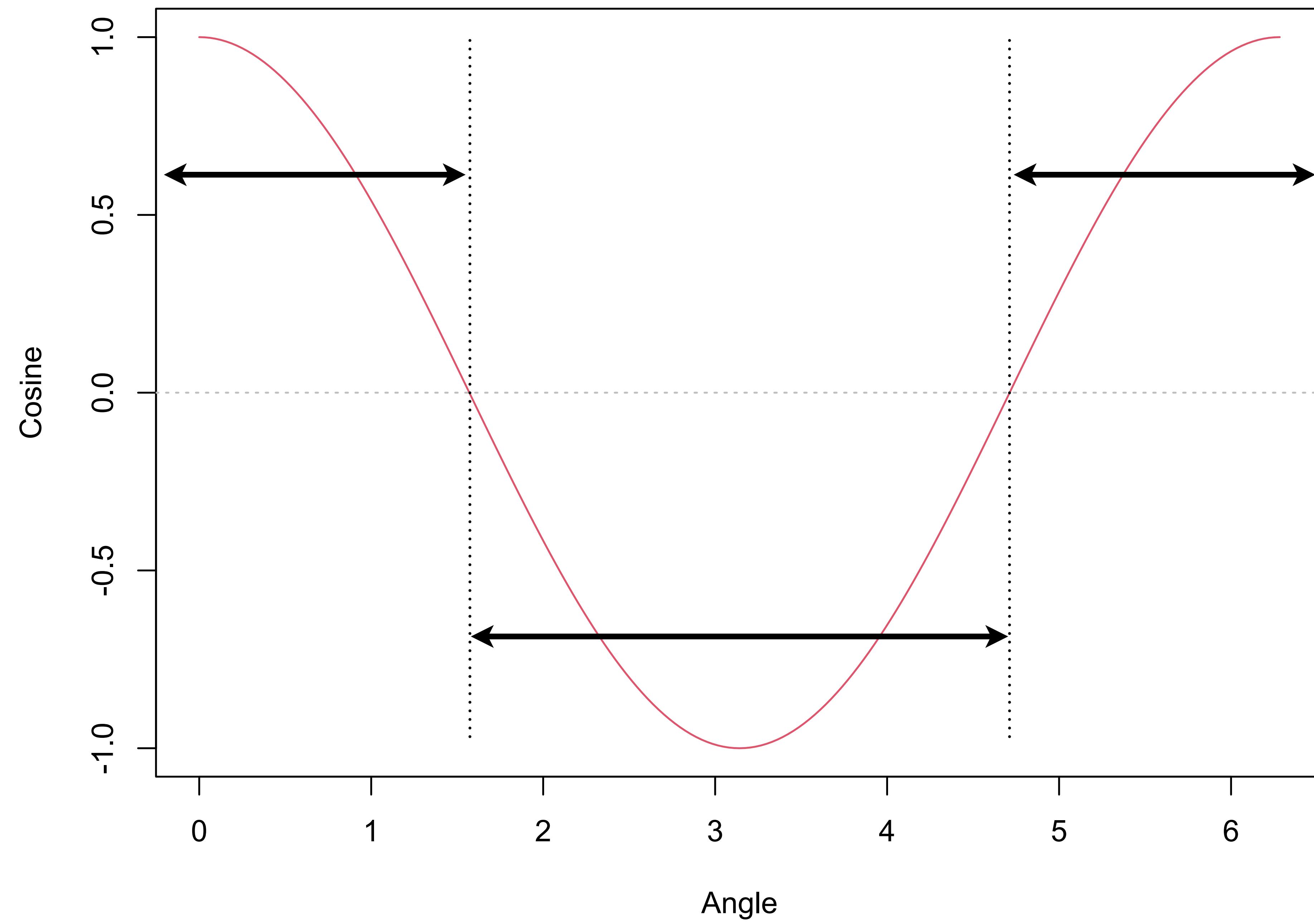


$$\mathbf{x} \cdot \mathbf{w} = 0$$

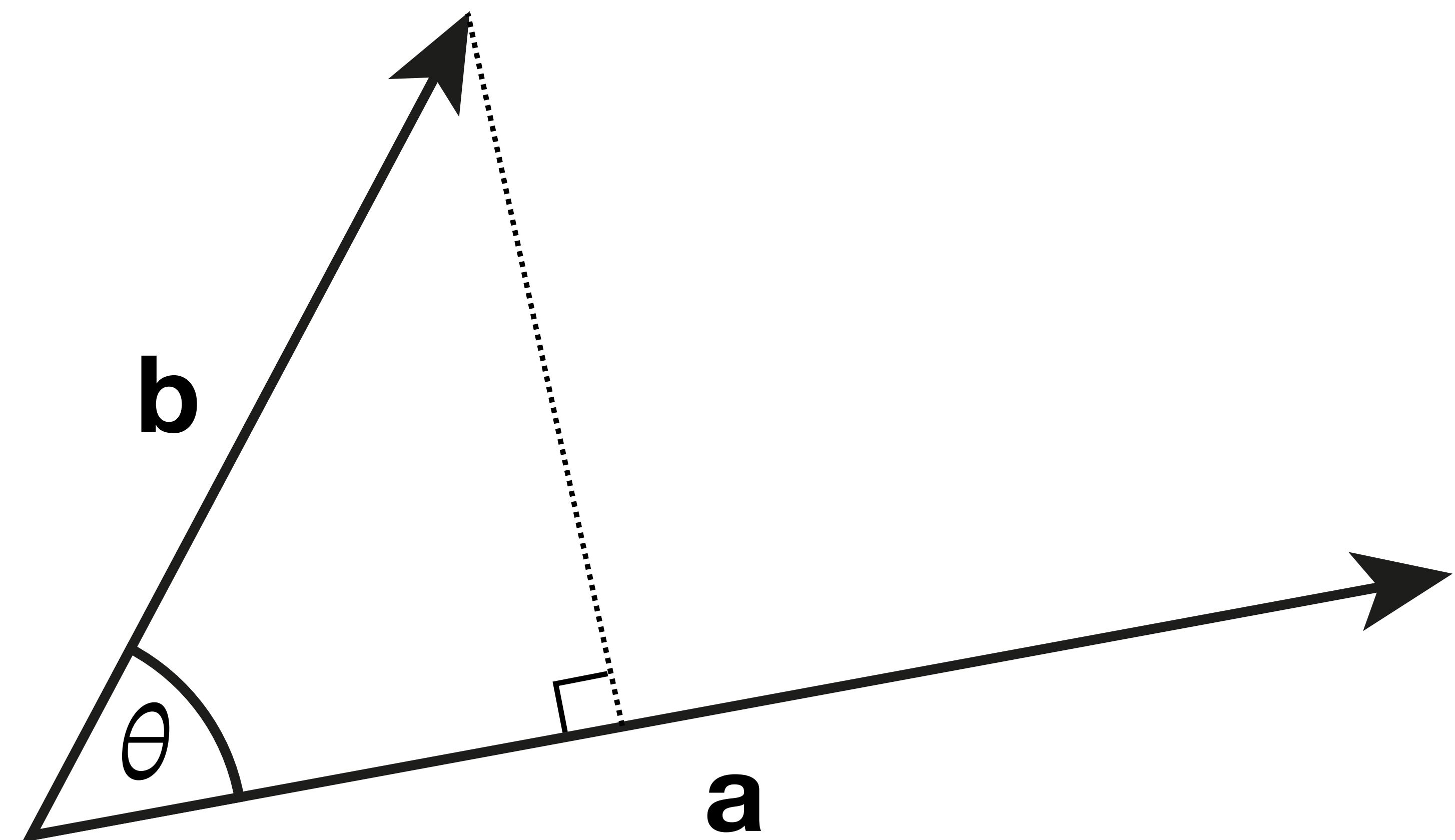
$$\implies \mathbf{x} \perp \mathbf{w}$$

Linearly Separable Binary Data

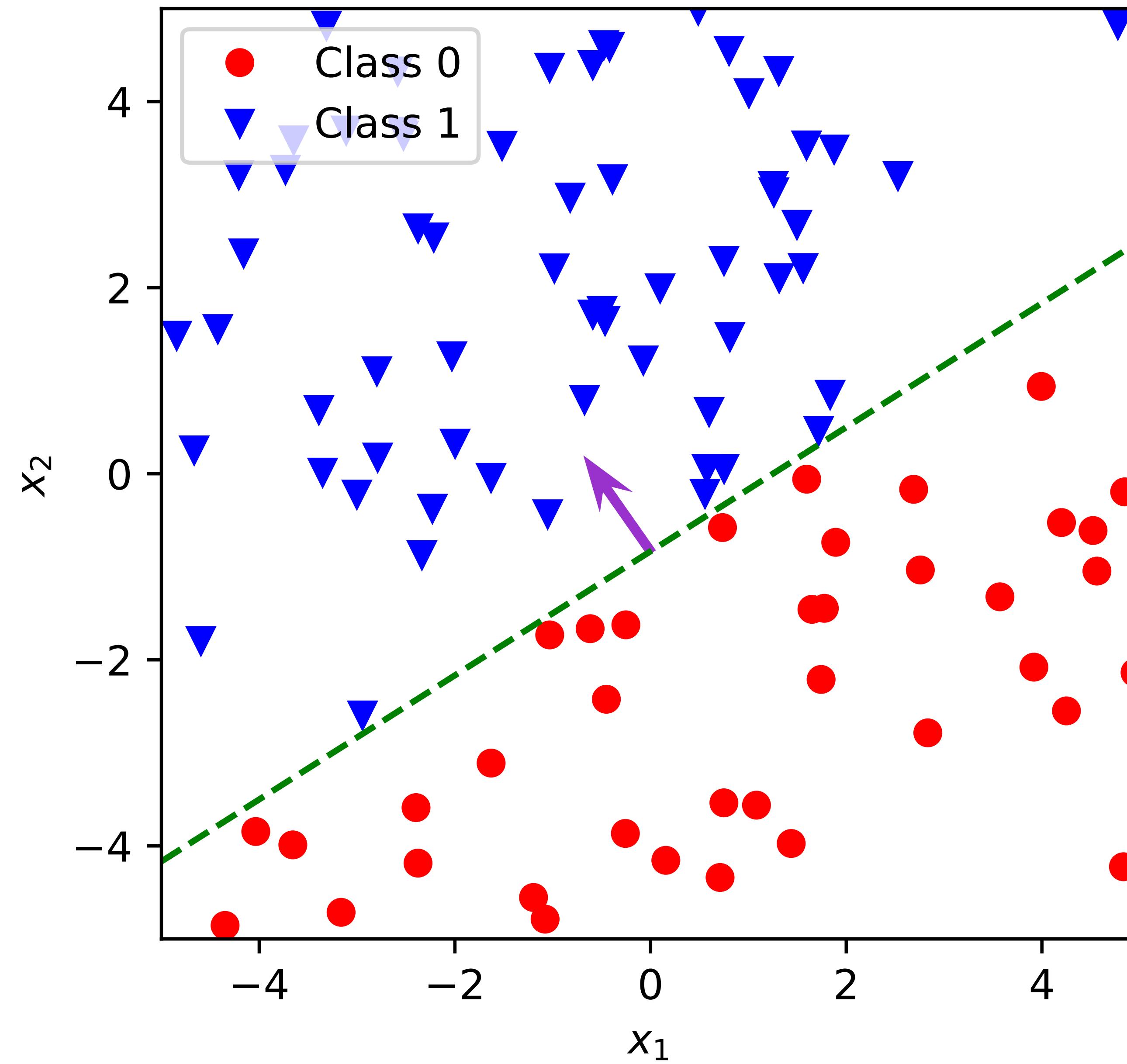




$$\hat{y} = \begin{cases} 1 & \text{if } \mathbf{x} \cdot \mathbf{w} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



Linearly Separable Binary Data



$x \cdot w$

$$\frac{x \cdot w}{|w|}$$

$$y \in \{0, 1\}$$

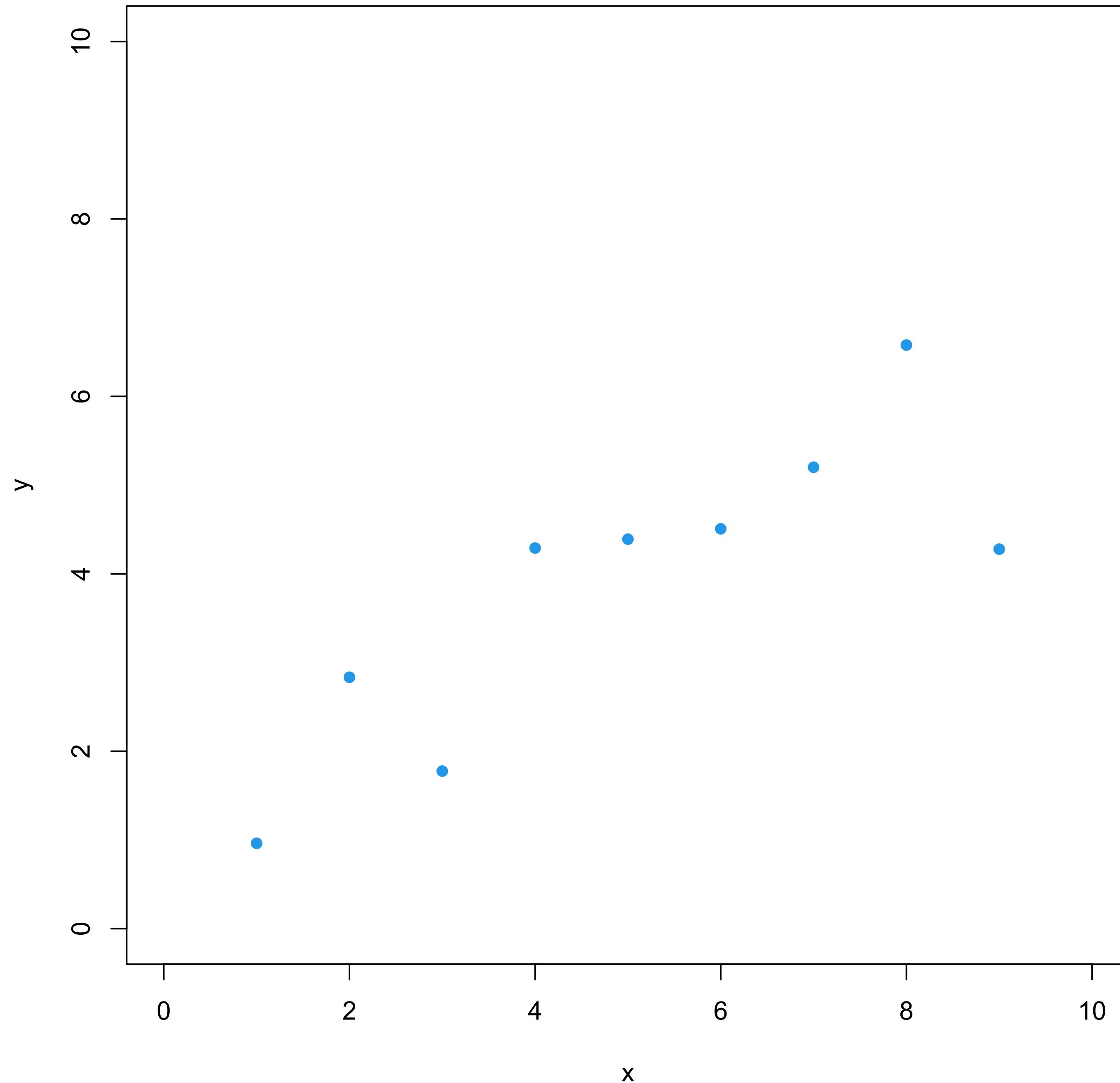
$$y \in \{-1, 1\}$$

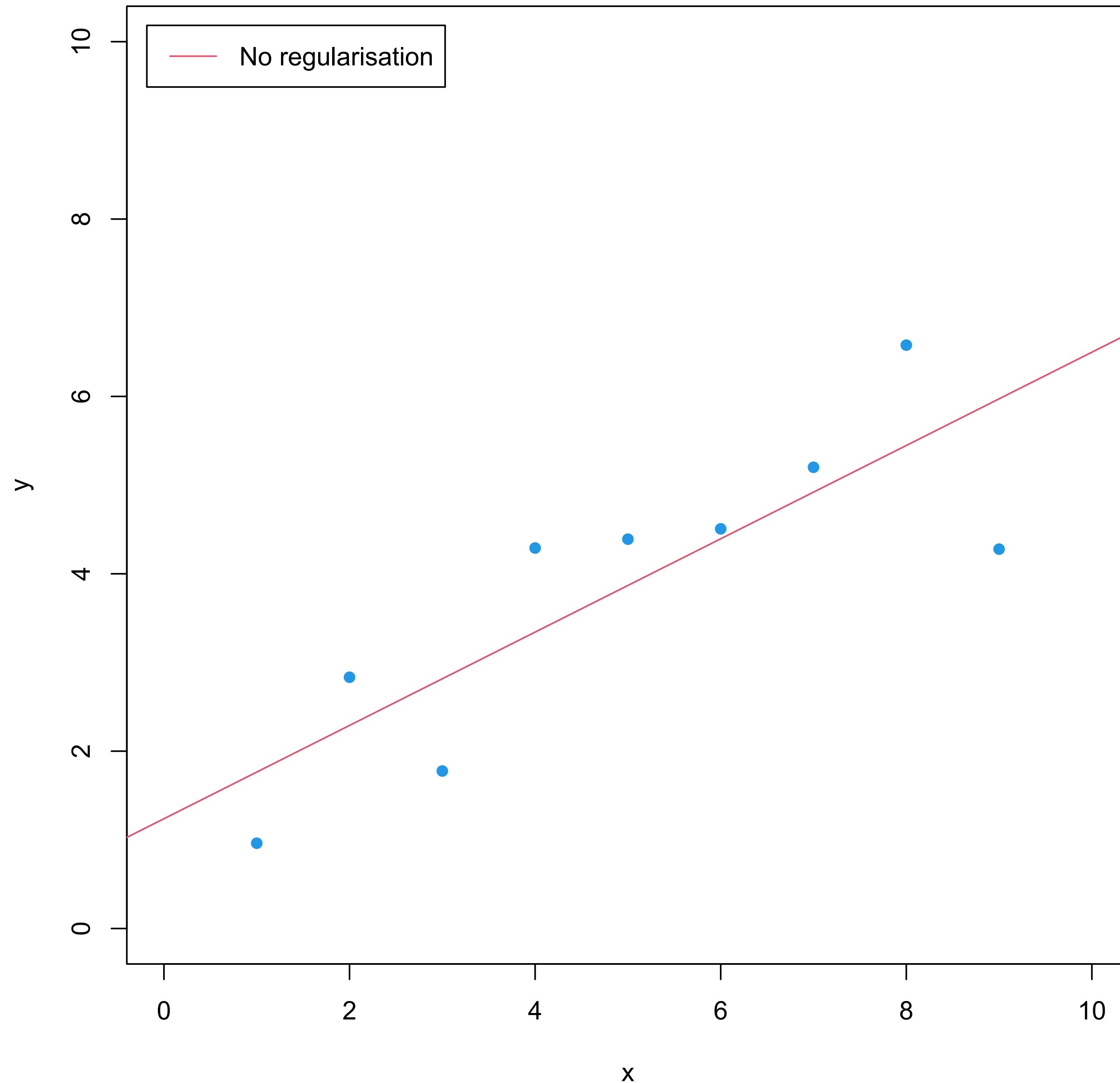
Functional Margin

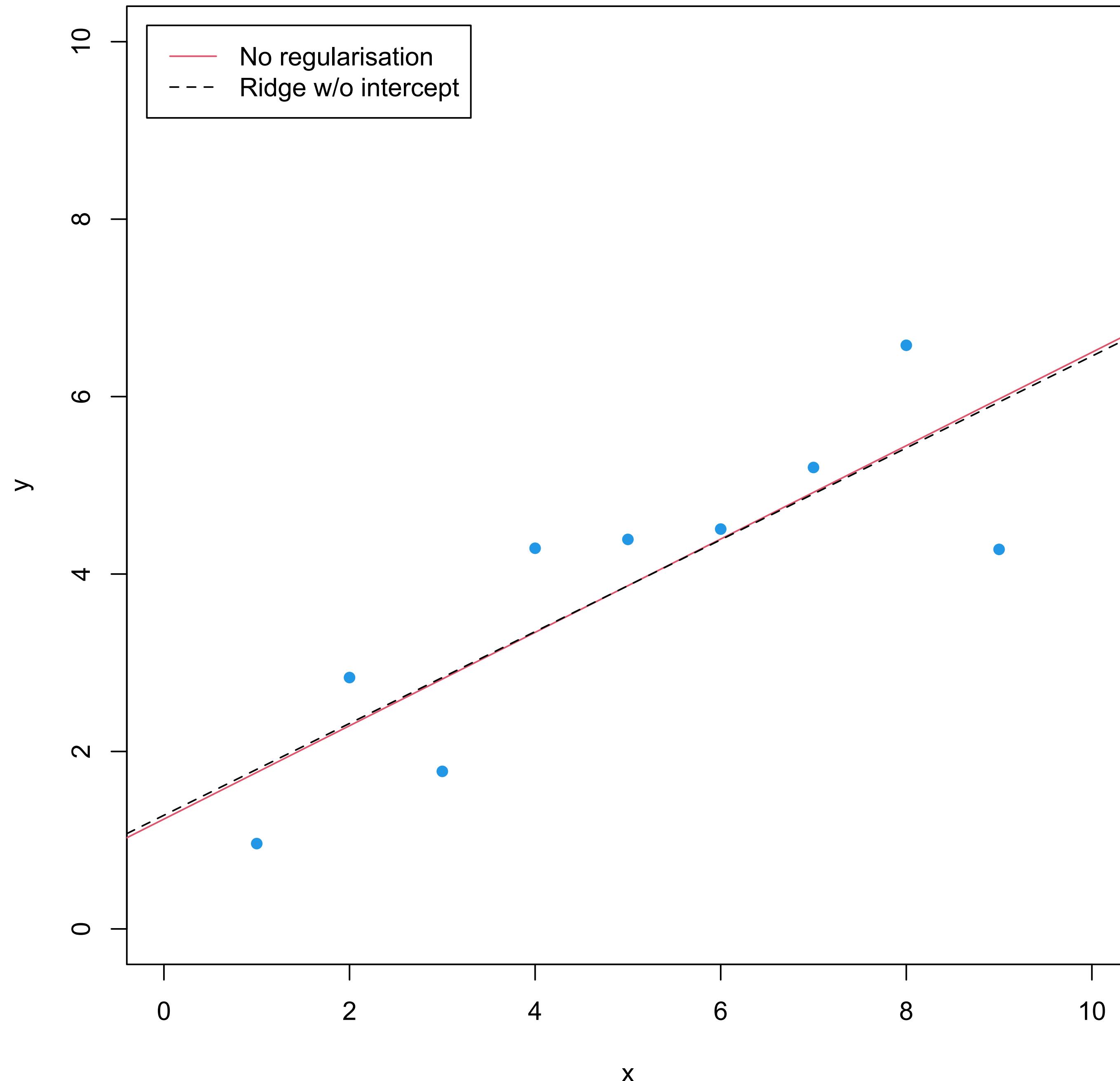
$$y\hat{y} = yf(\mathbf{x}) = y \mathbf{x} \cdot \mathbf{w}$$

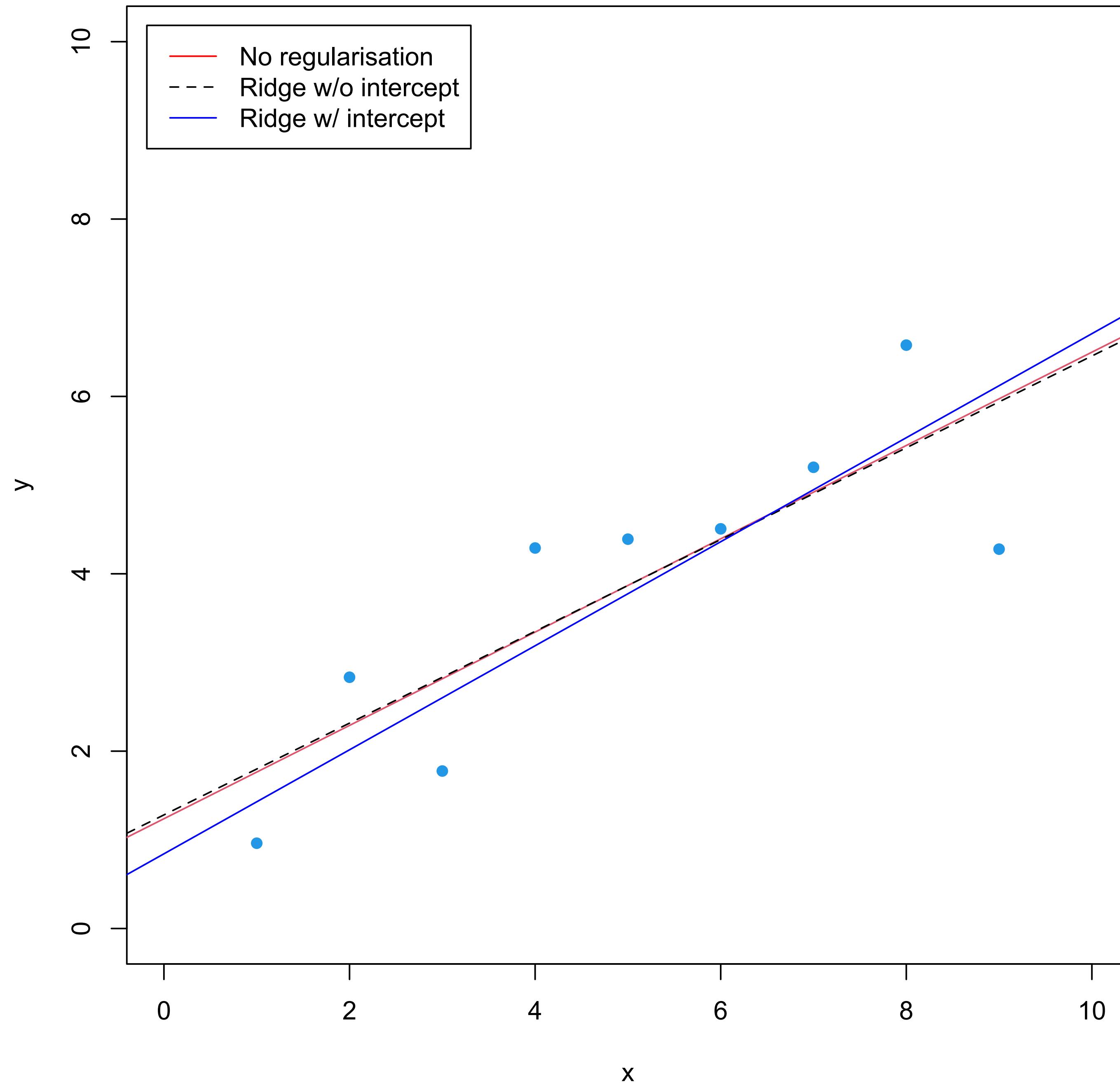
Geometric Margin

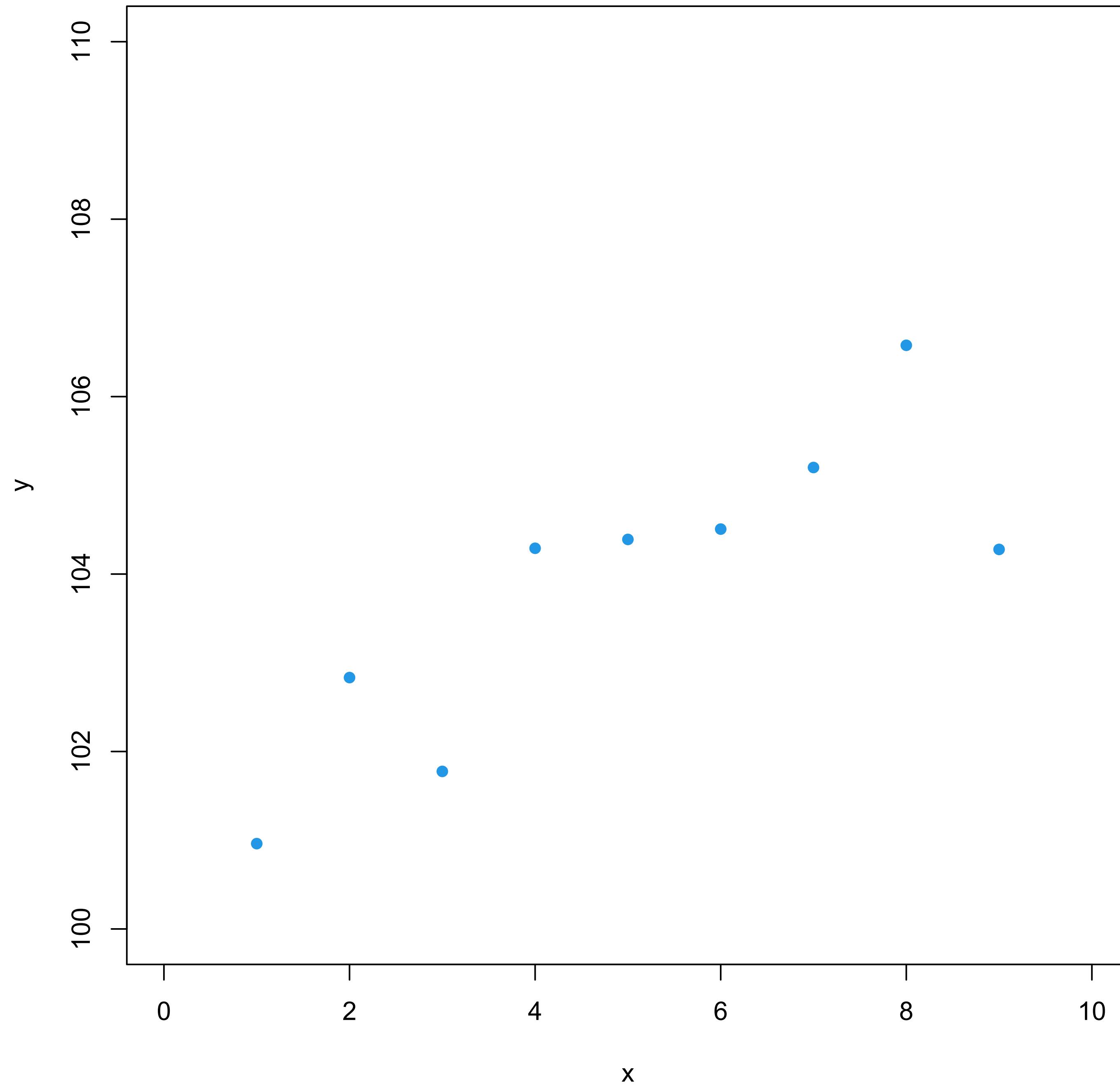
$$\frac{y \mathbf{x} \cdot \mathbf{w}}{\|\mathbf{w}\|}$$

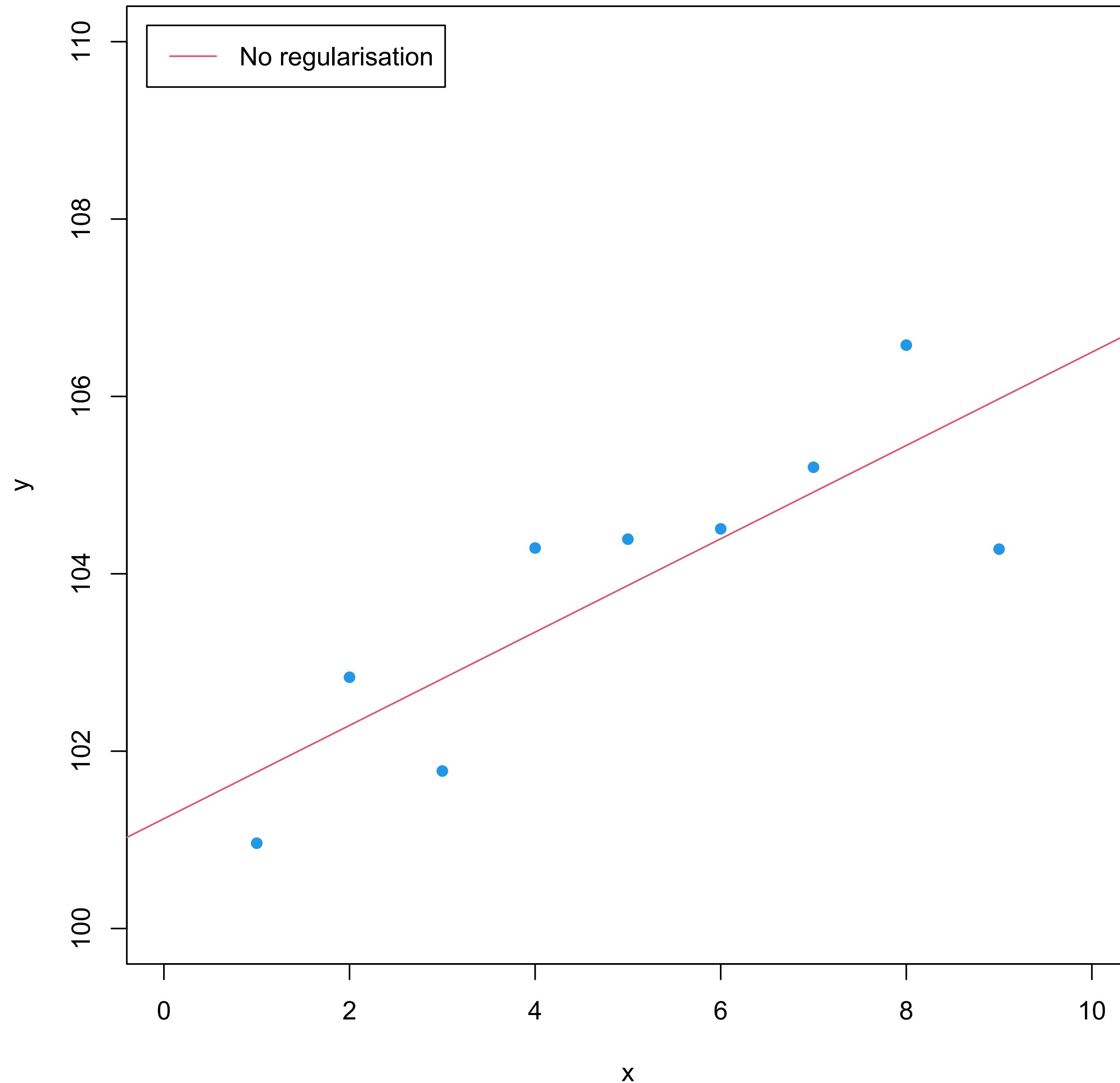


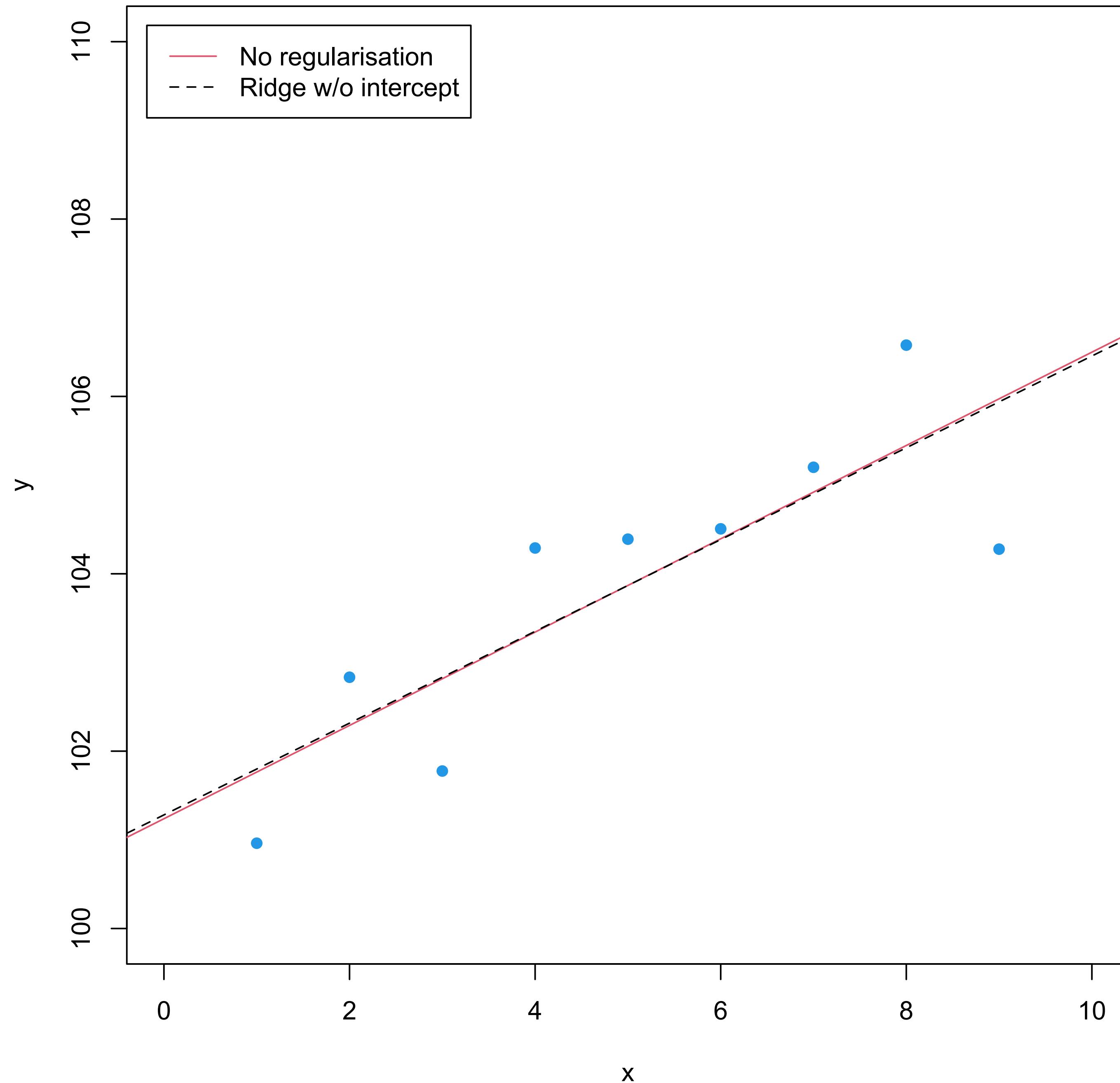


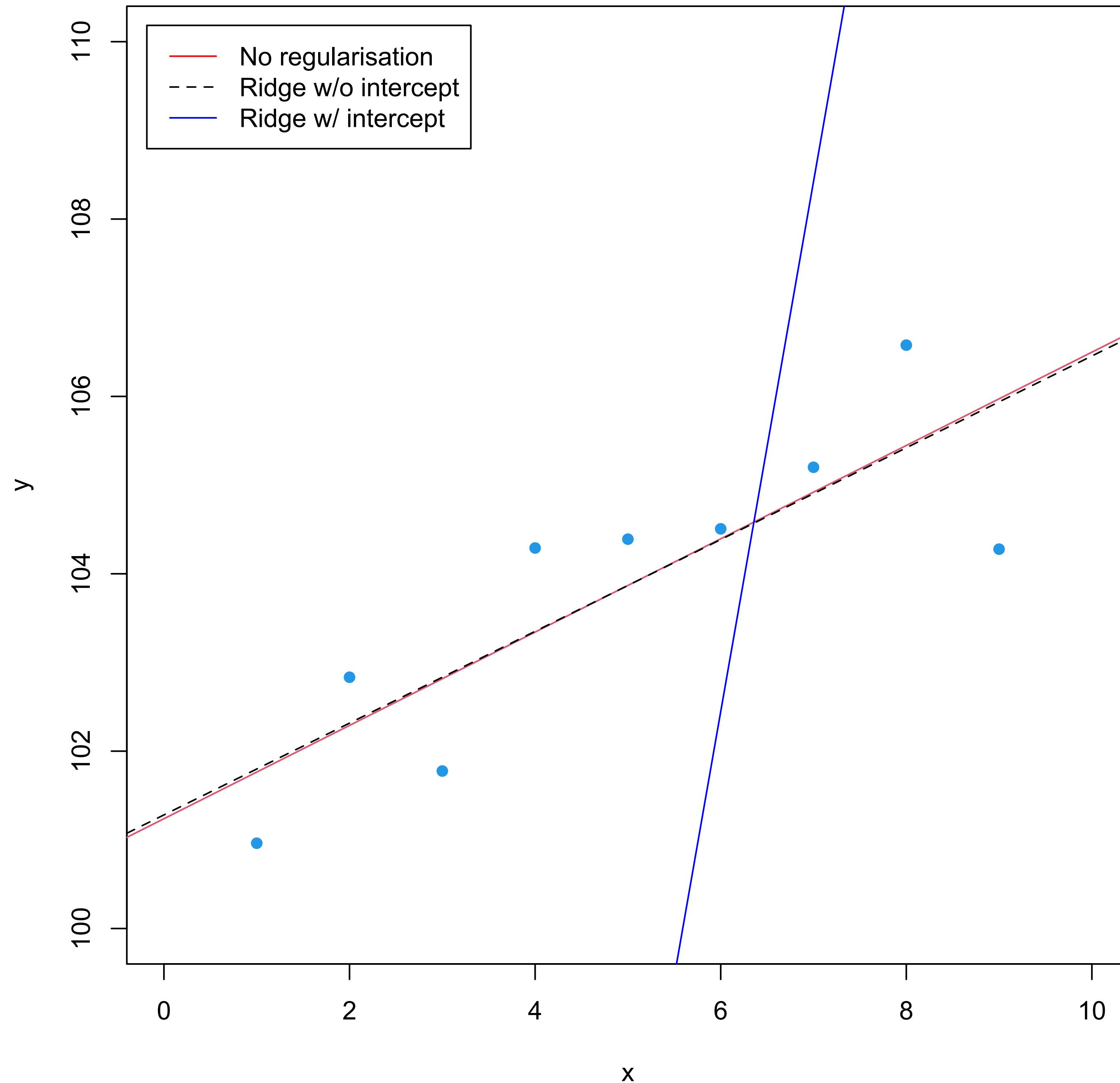


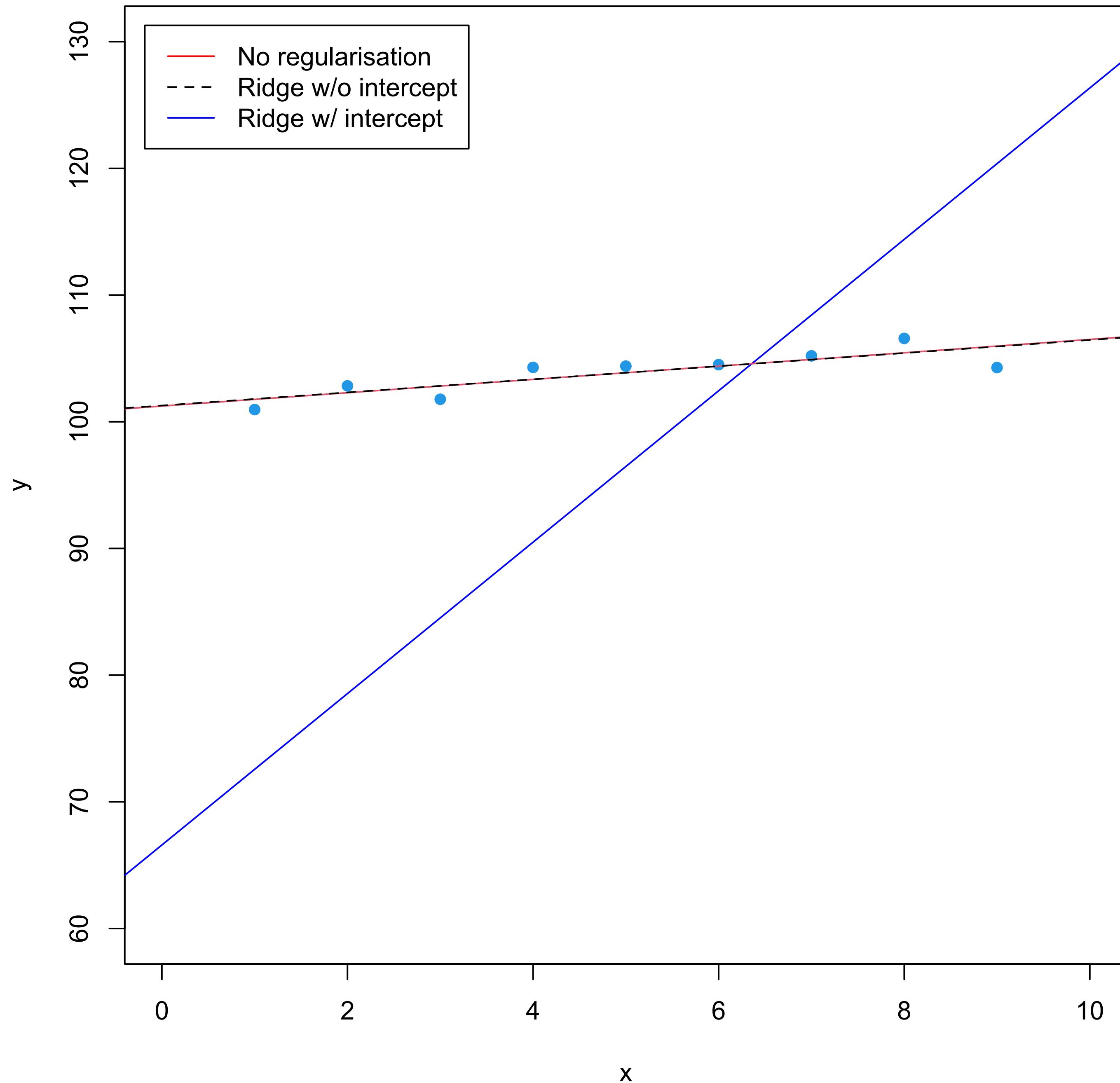




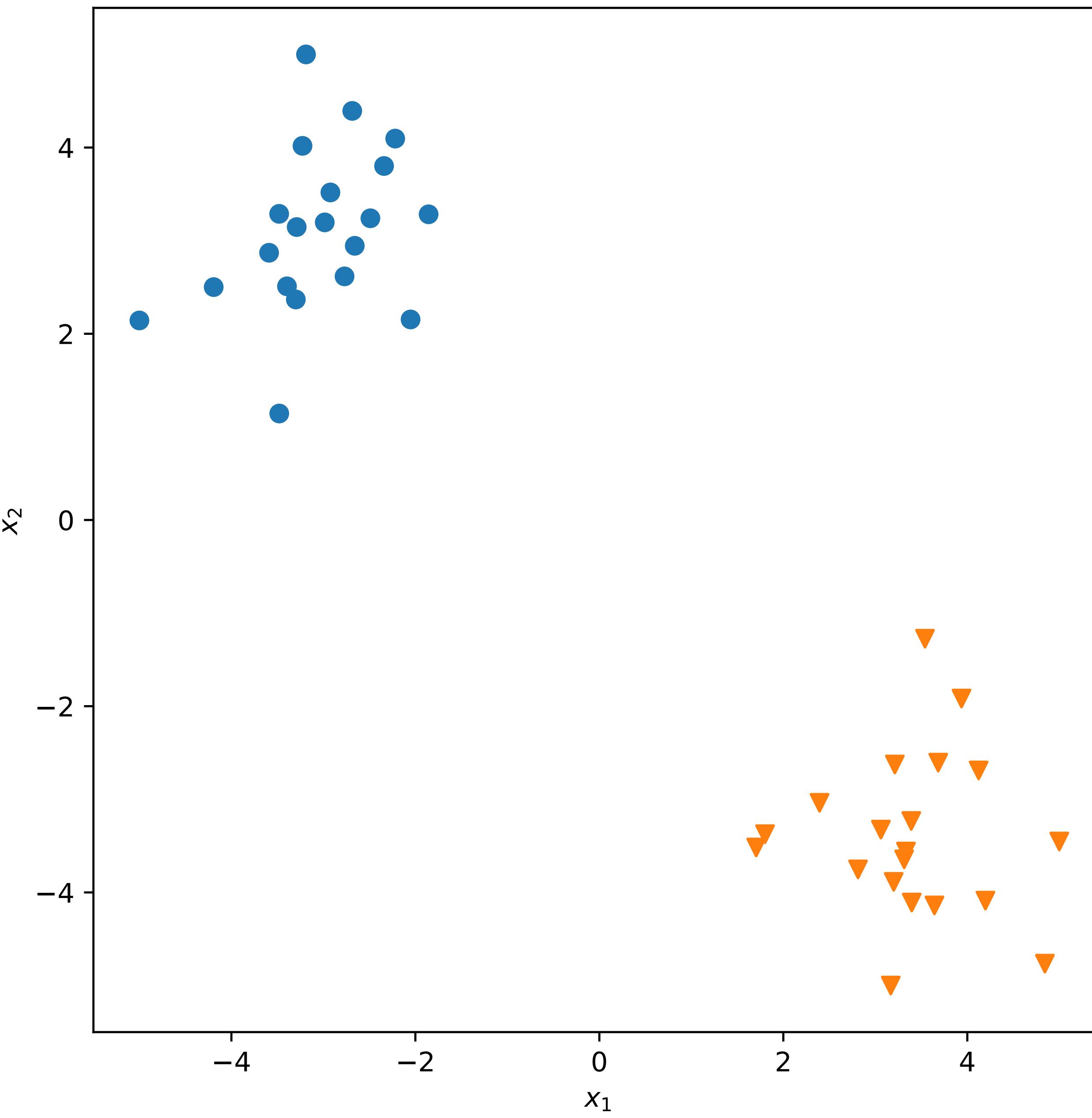


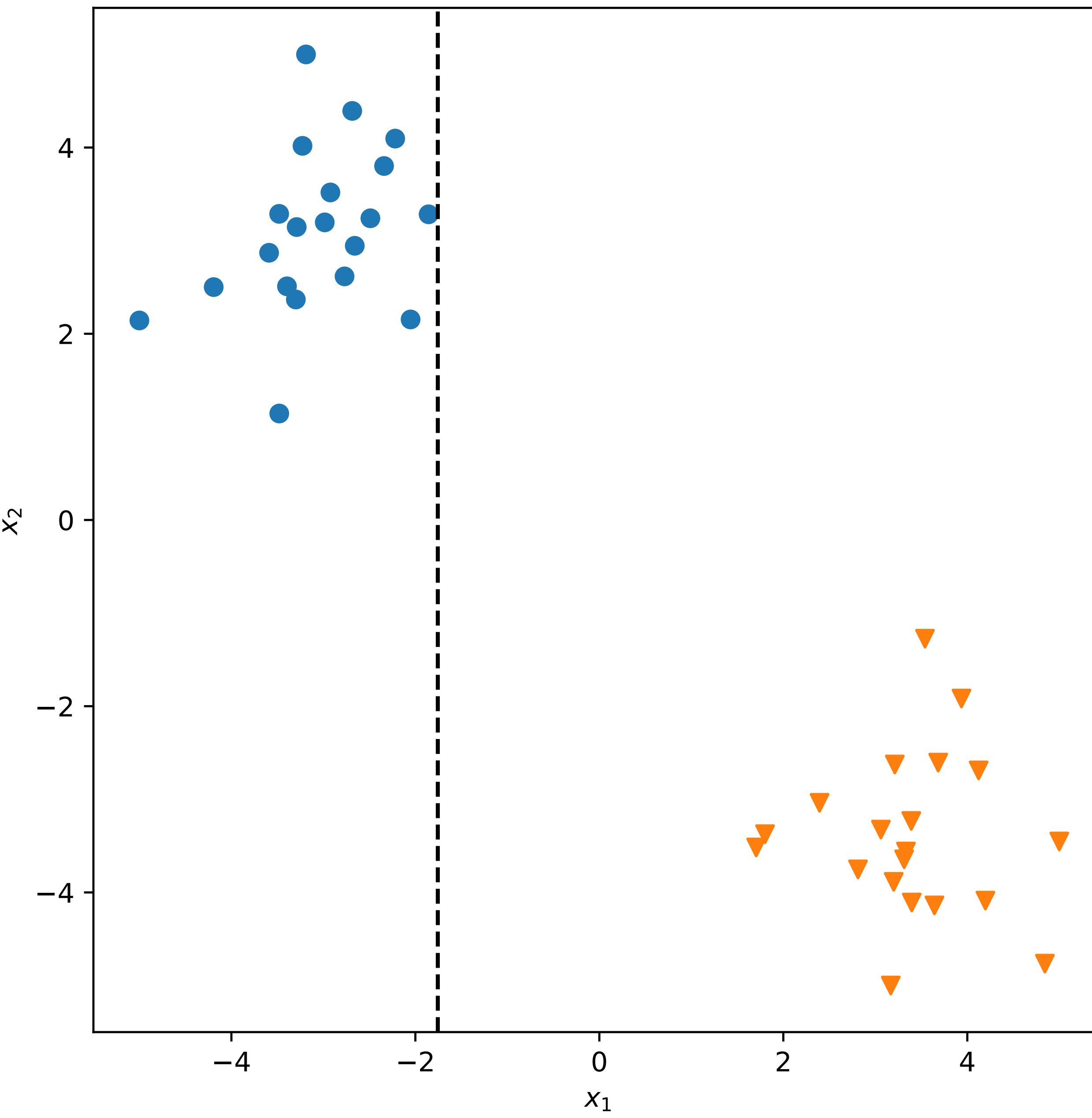


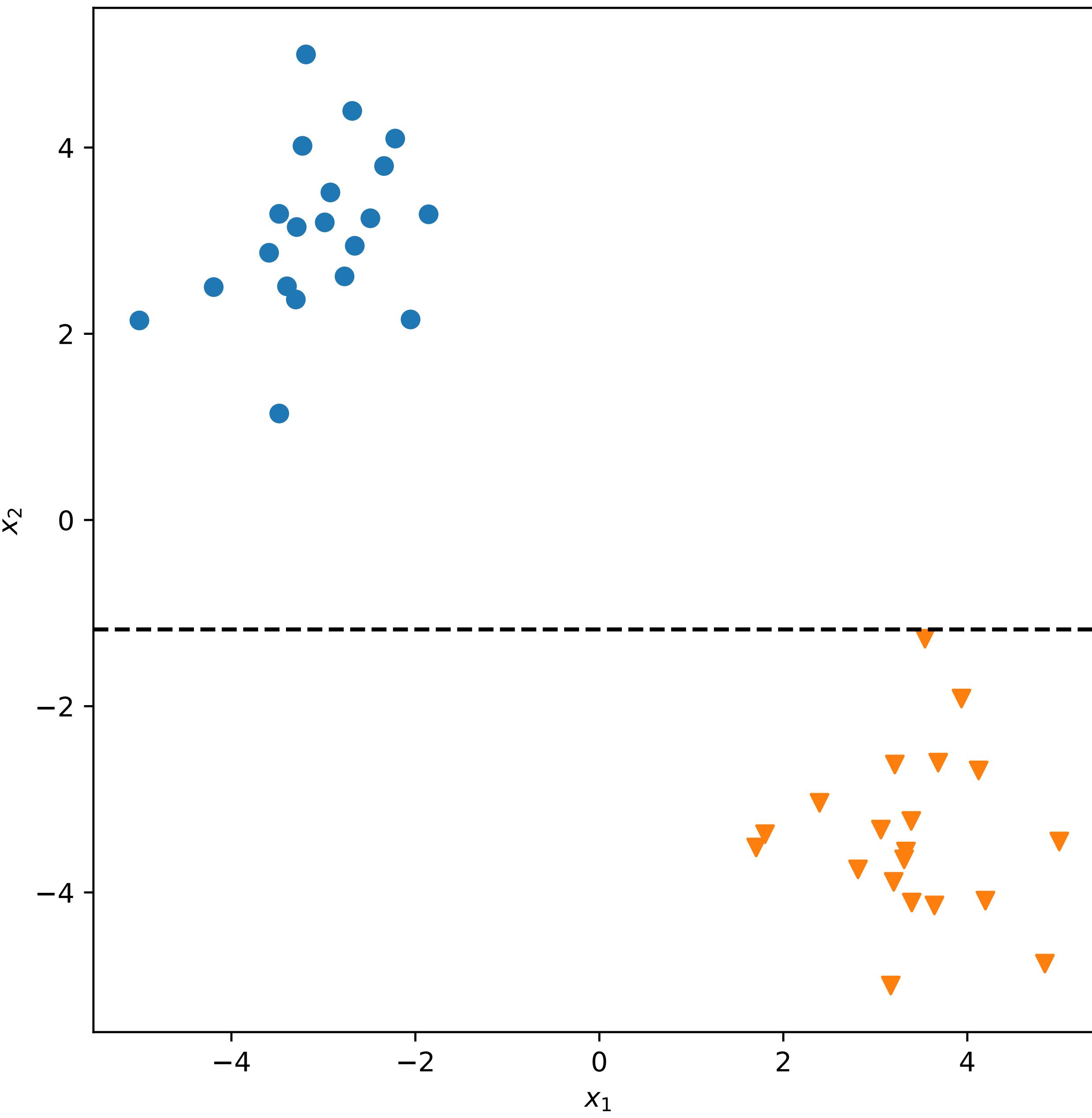


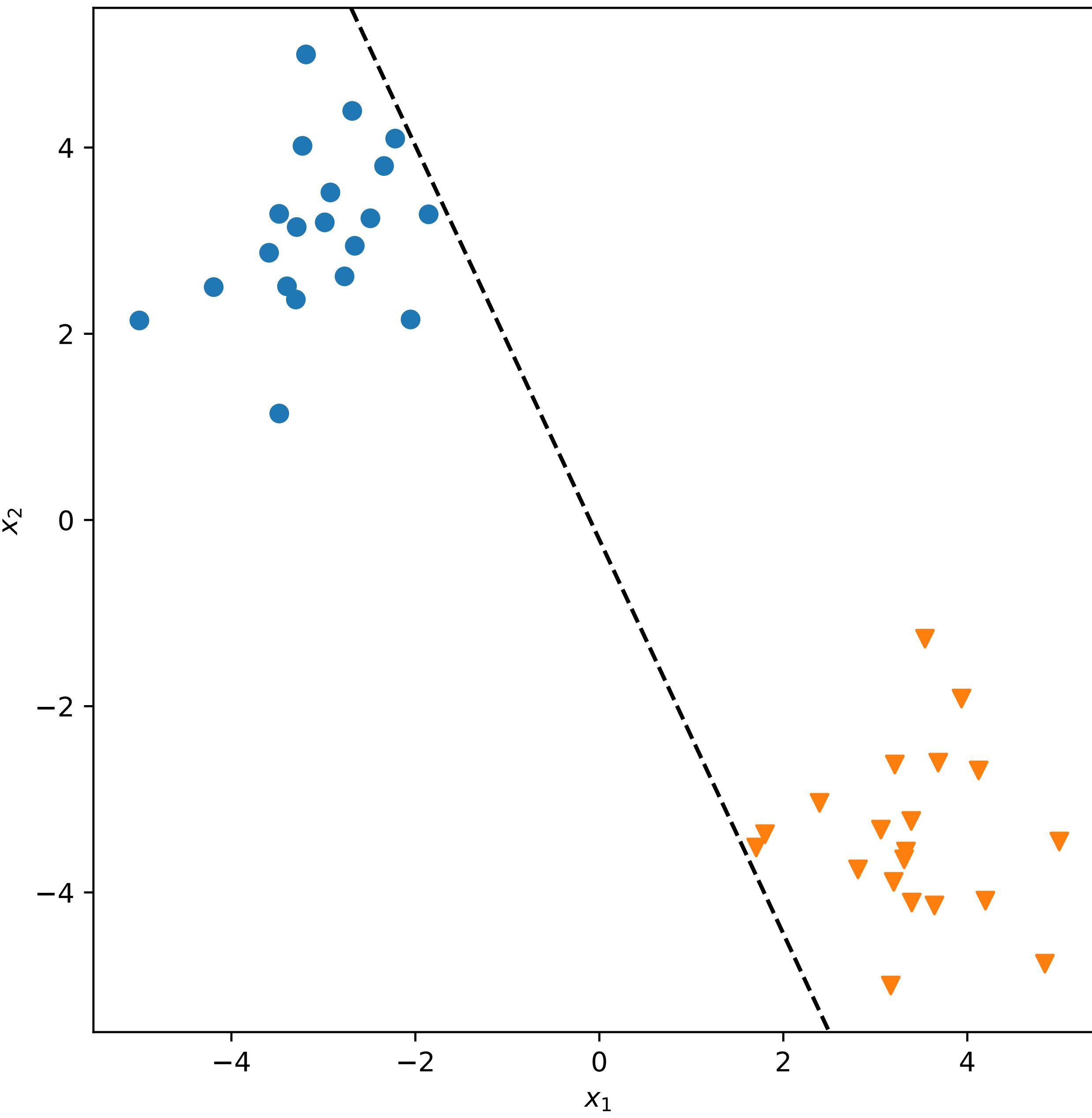


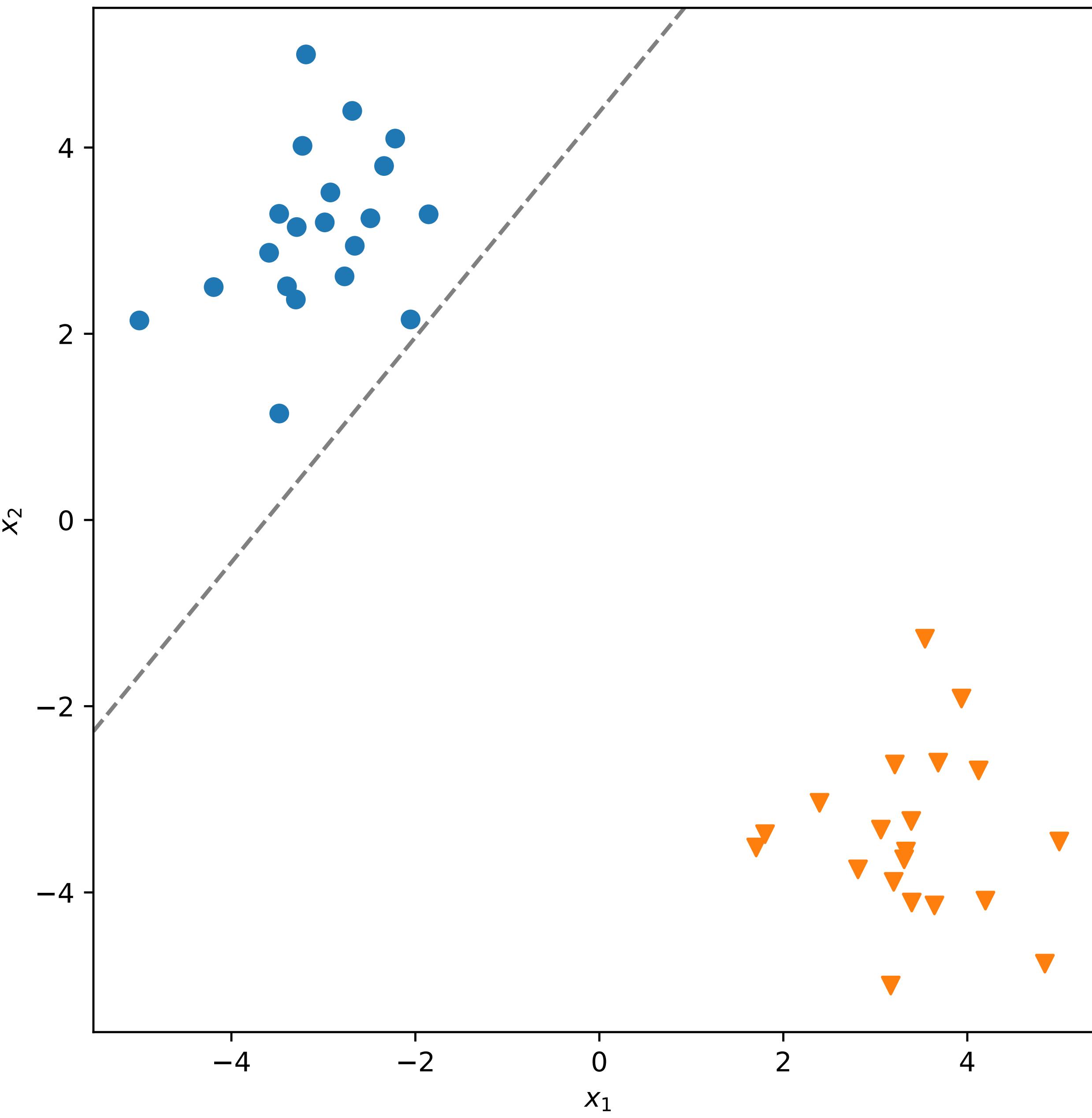
$$\hat{y} = \begin{cases} 1 & \text{if } \mathbf{x} \cdot \mathbf{w} + b \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

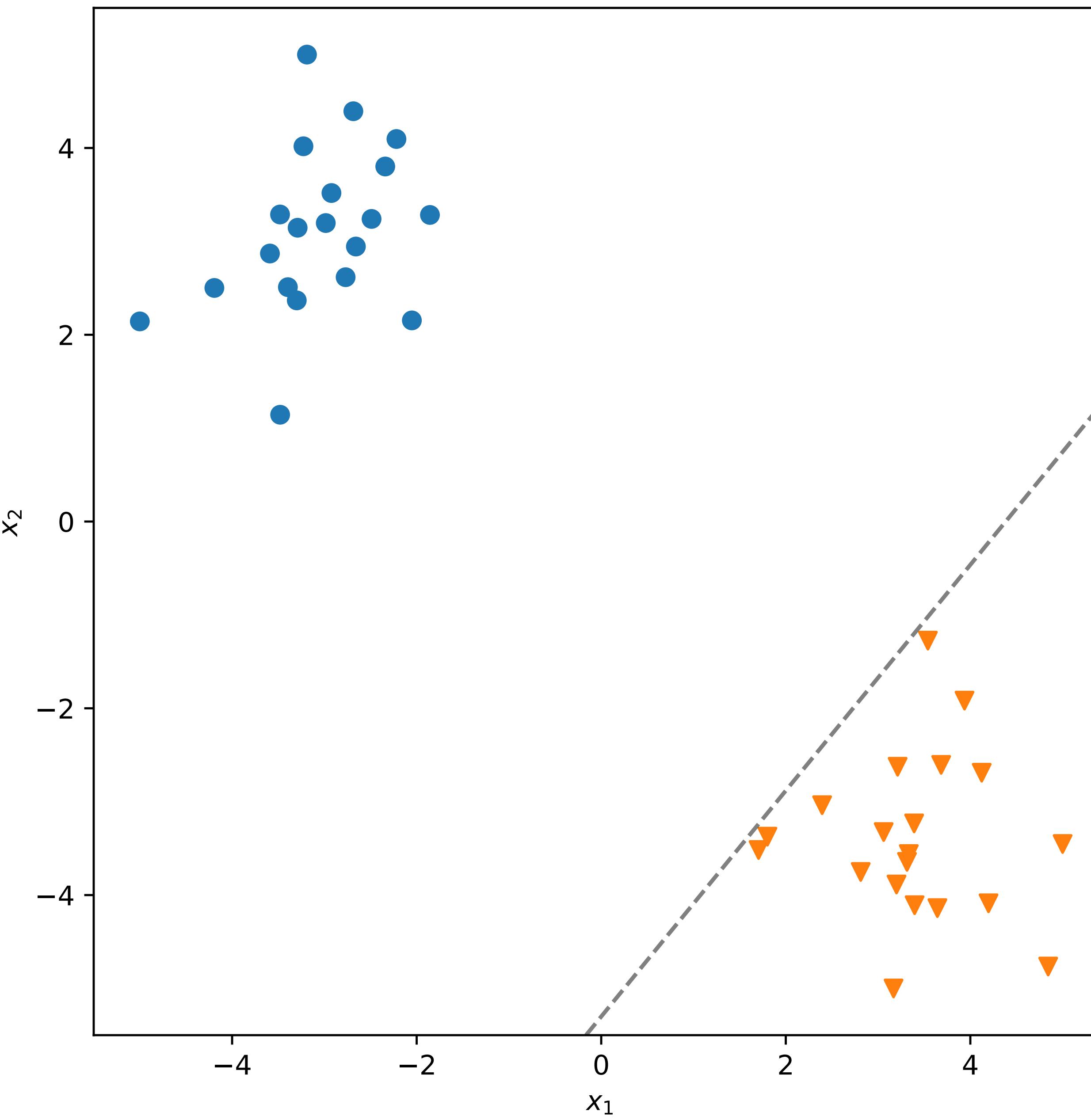


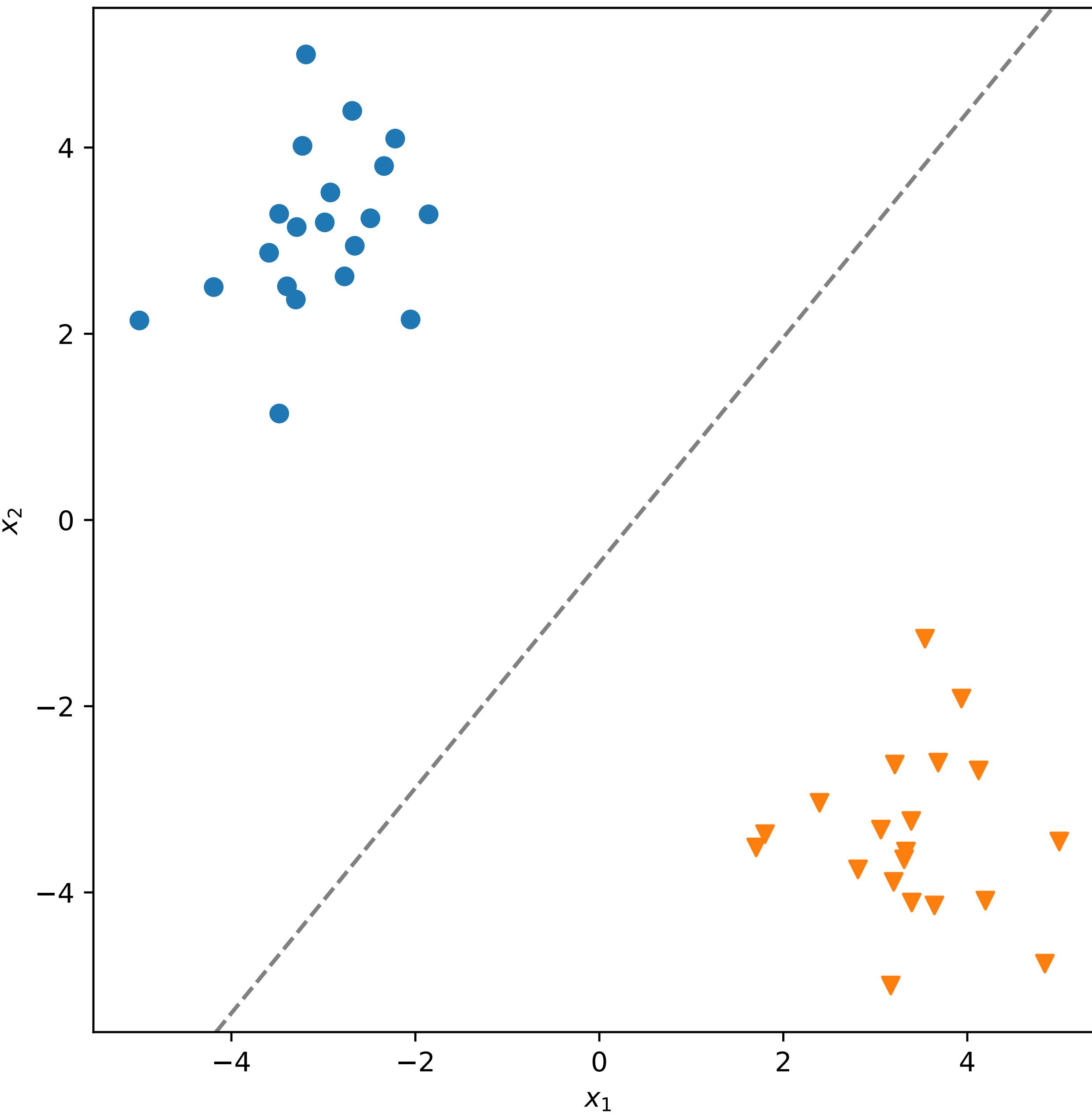


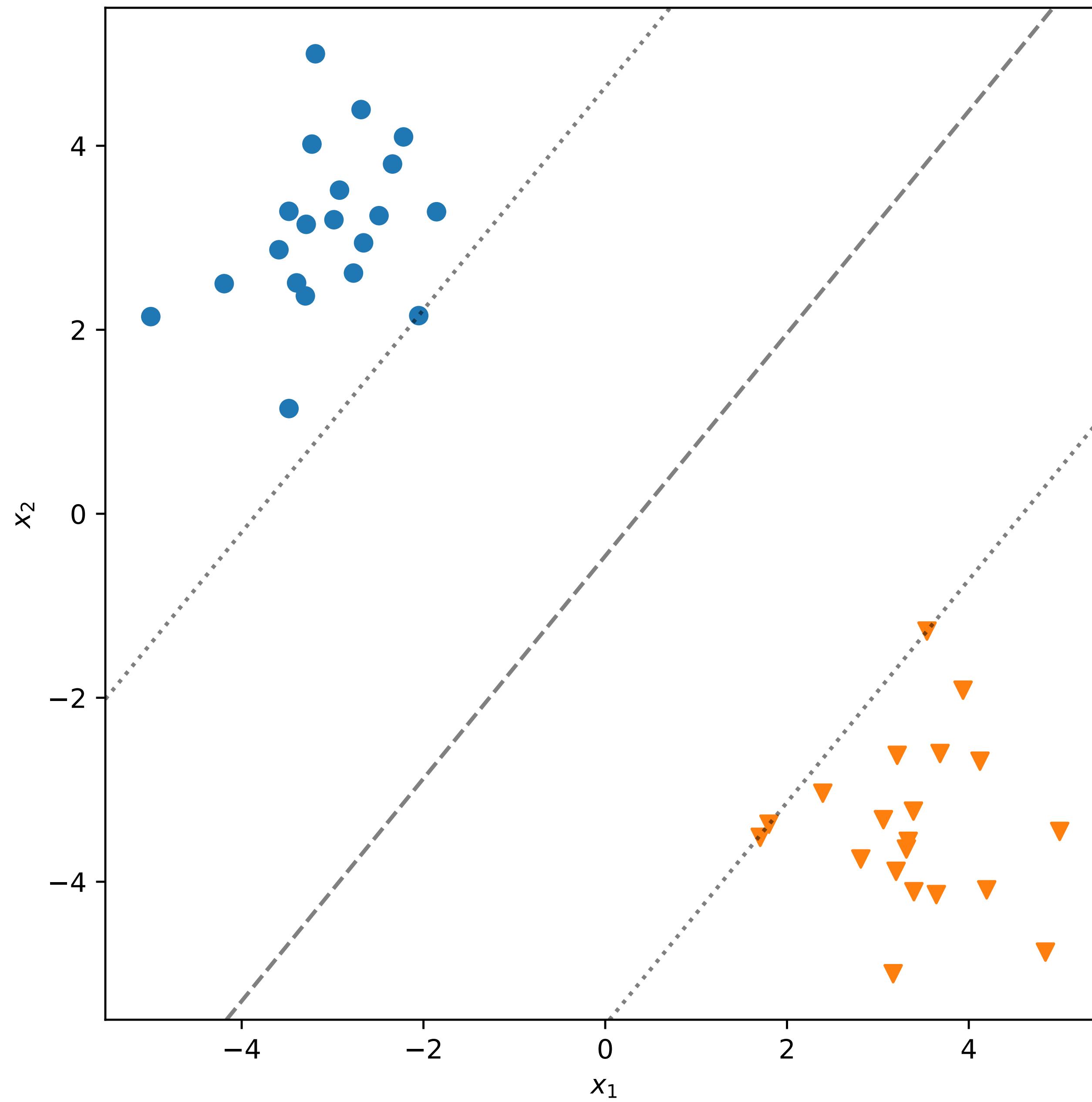


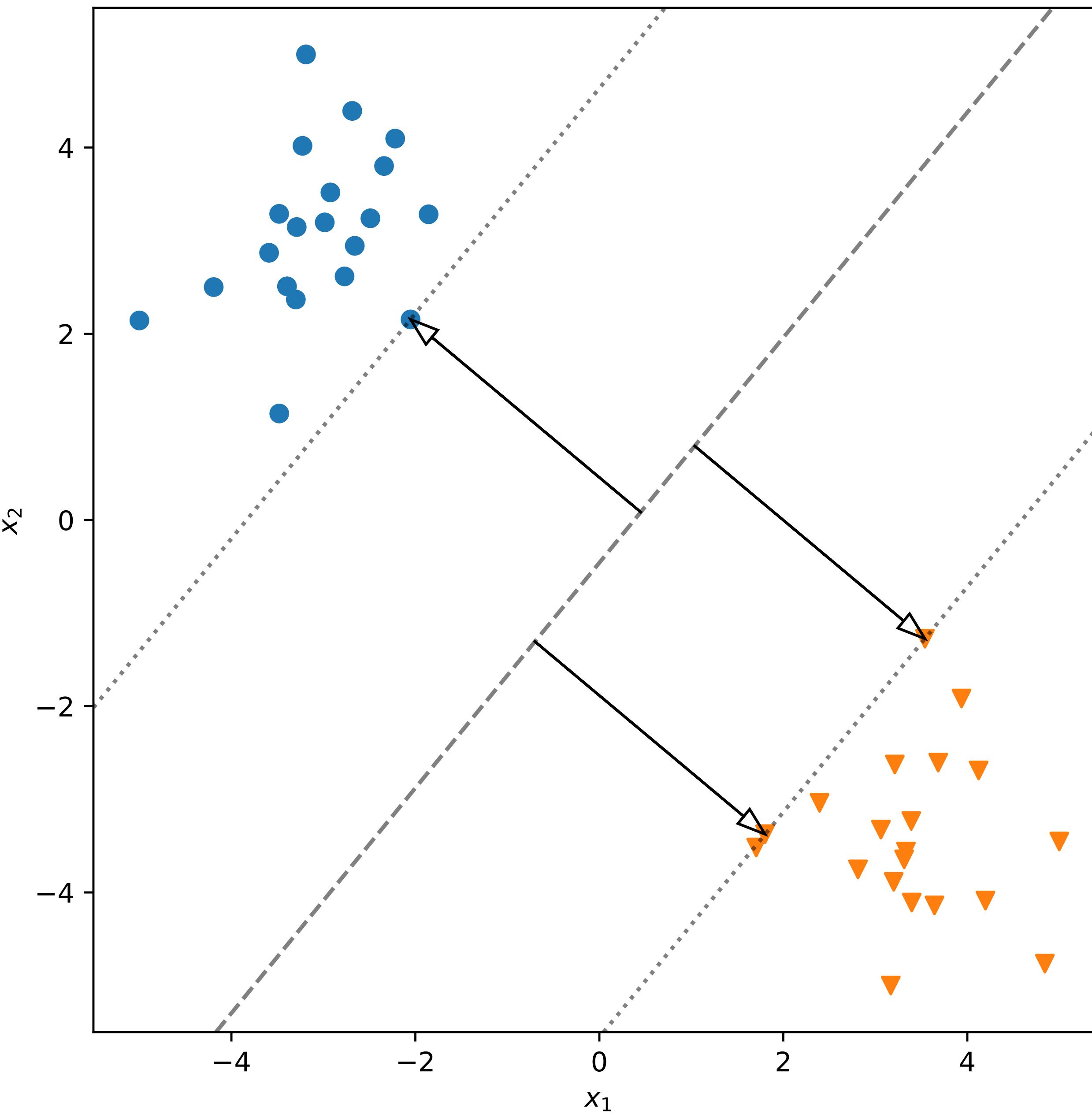


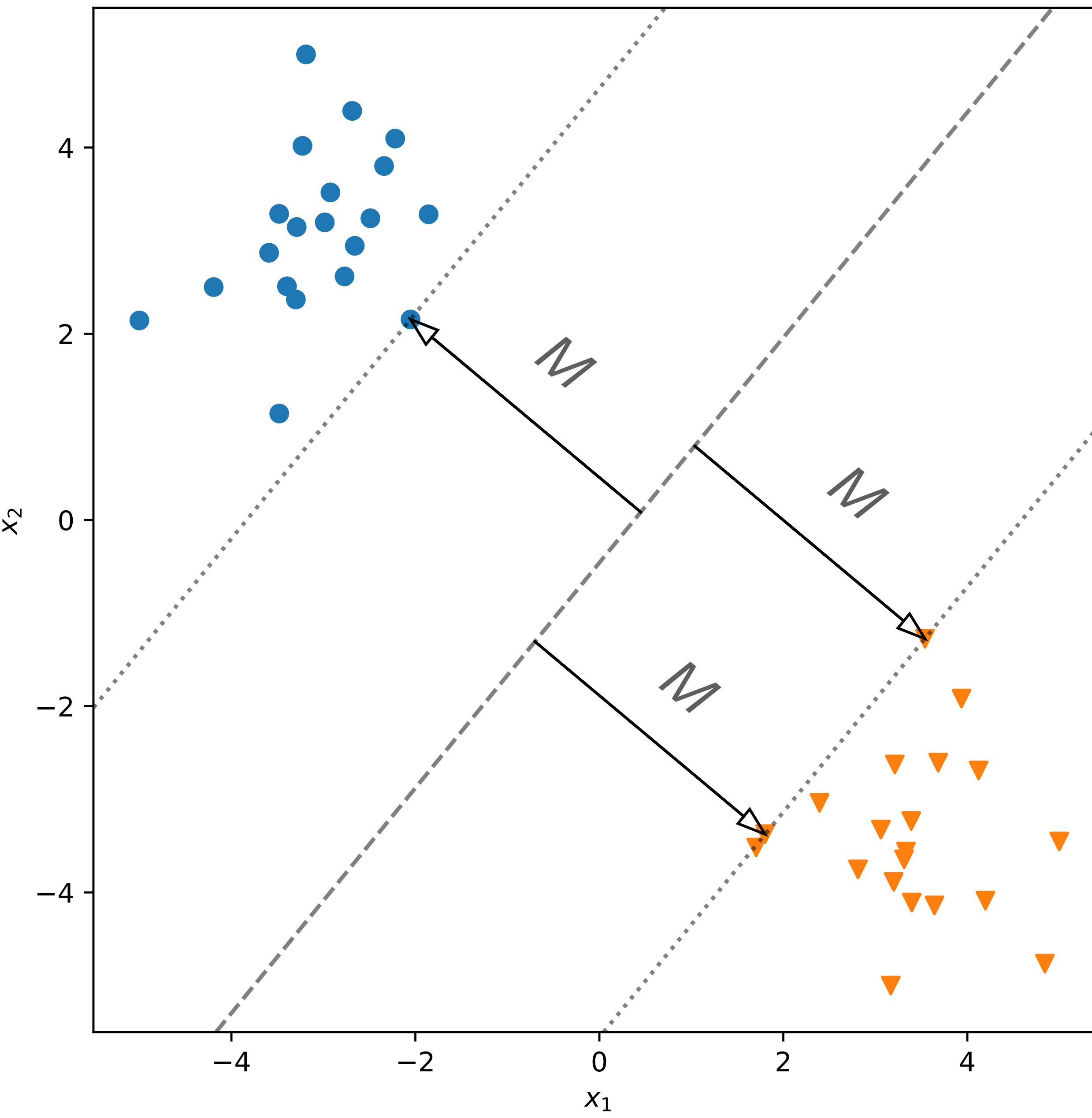












$$M = yf(\mathbf{x}) = y (\mathbf{x} \cdot \mathbf{w} + b)$$

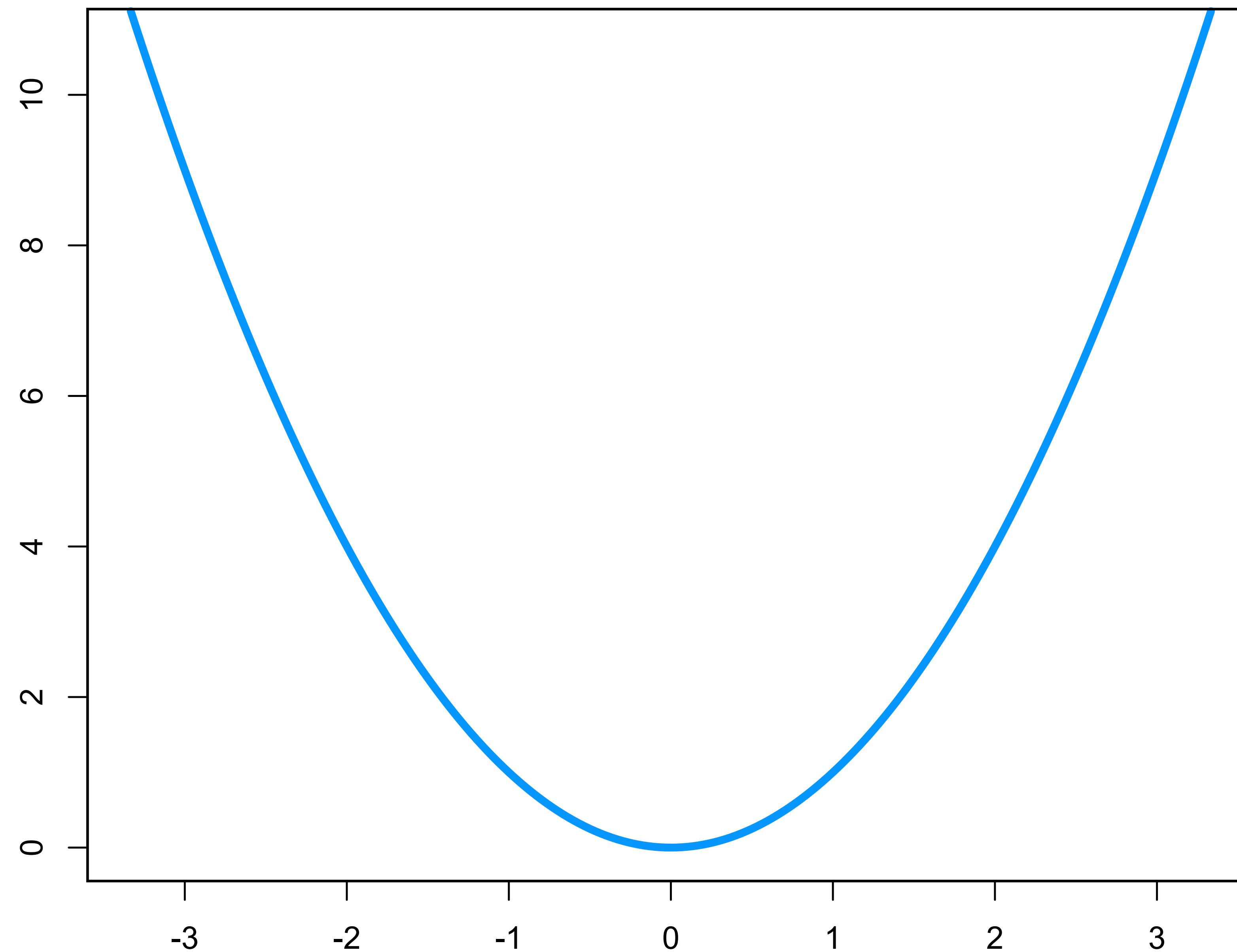
$$\|\mathbf{w}\| = 1$$

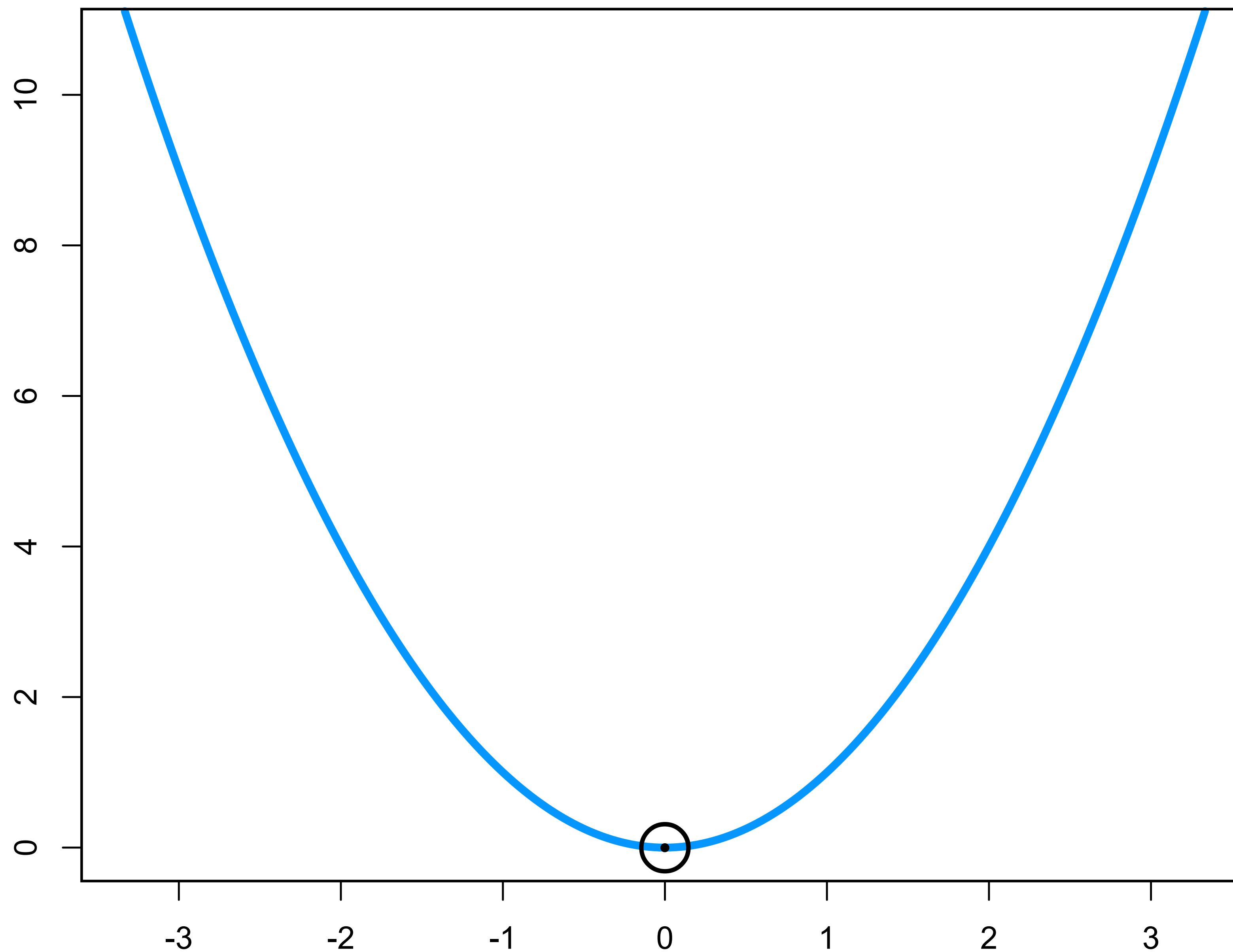
maximise M
 \mathbf{w}, b

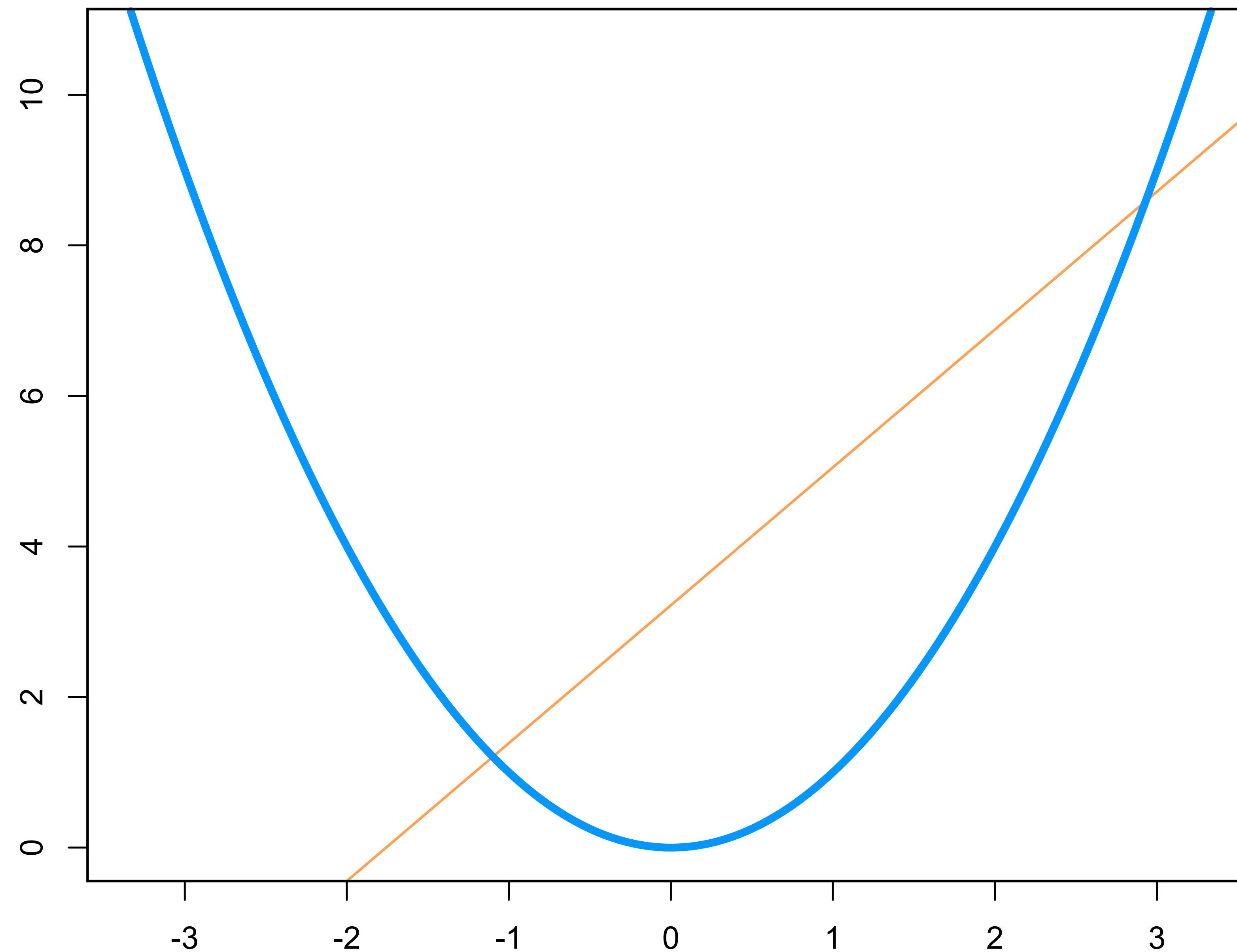
subject to:

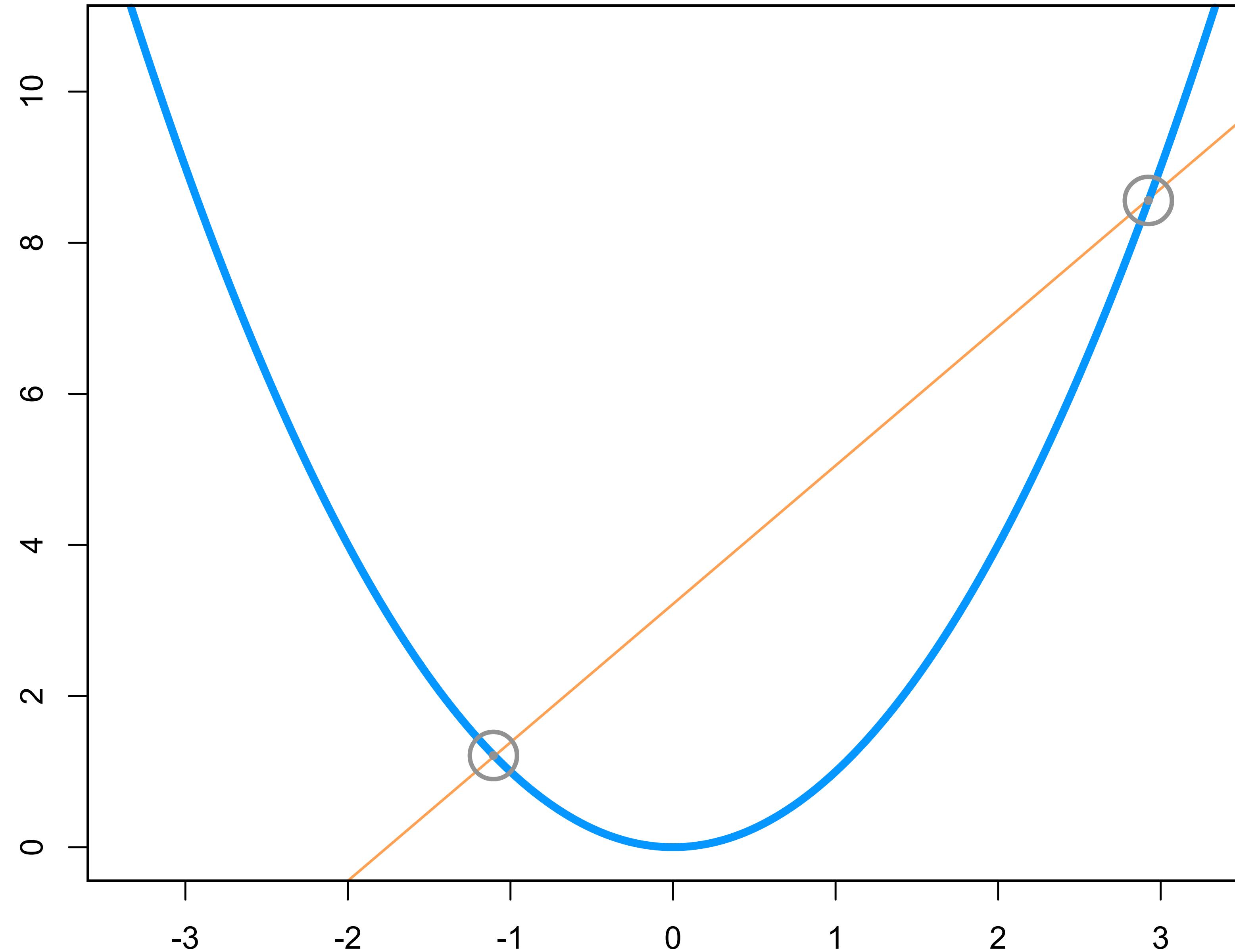
$$y_i (\mathbf{x}_i \cdot \mathbf{w} + b) \geq M$$

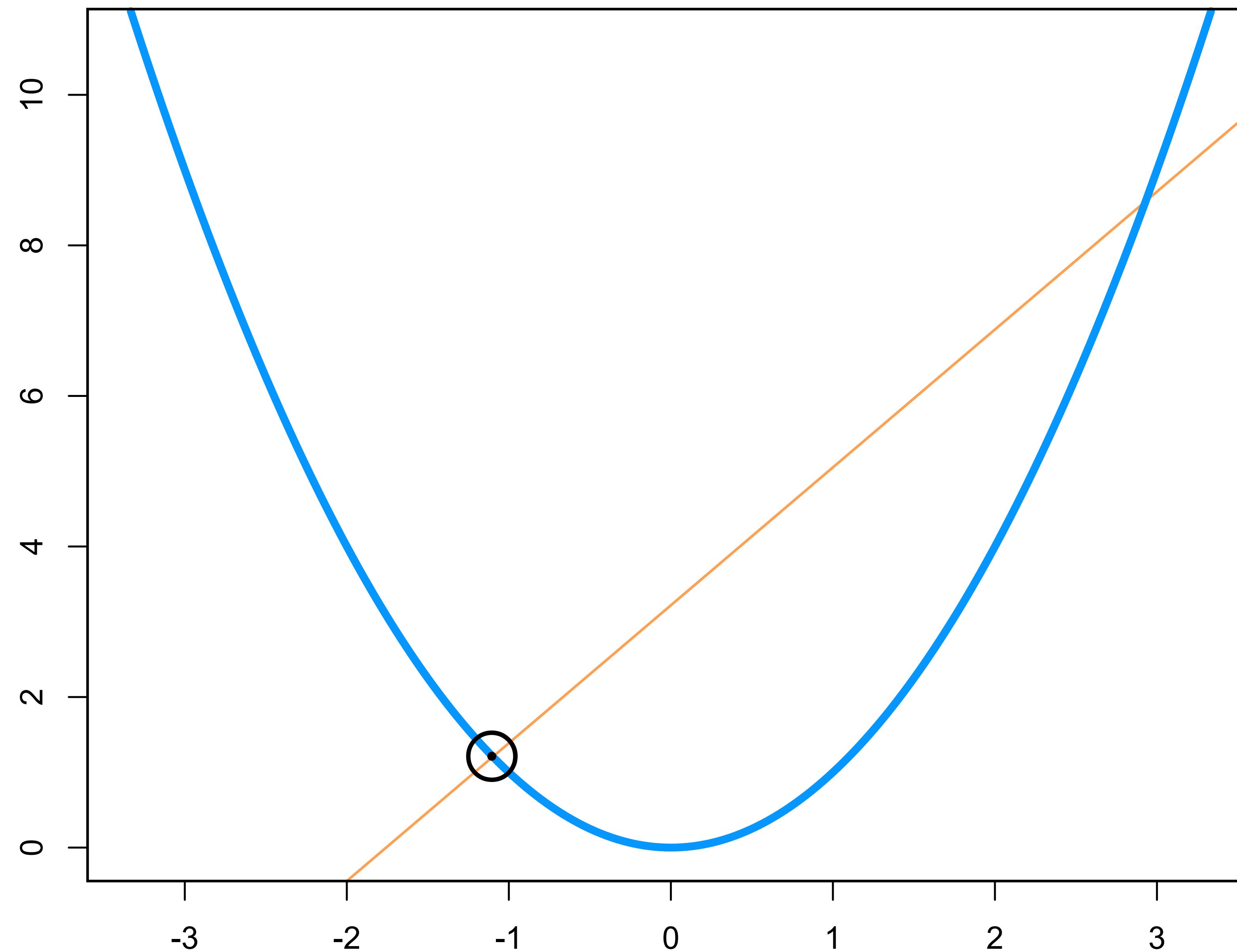
$$\|\mathbf{w}\| = 1$$

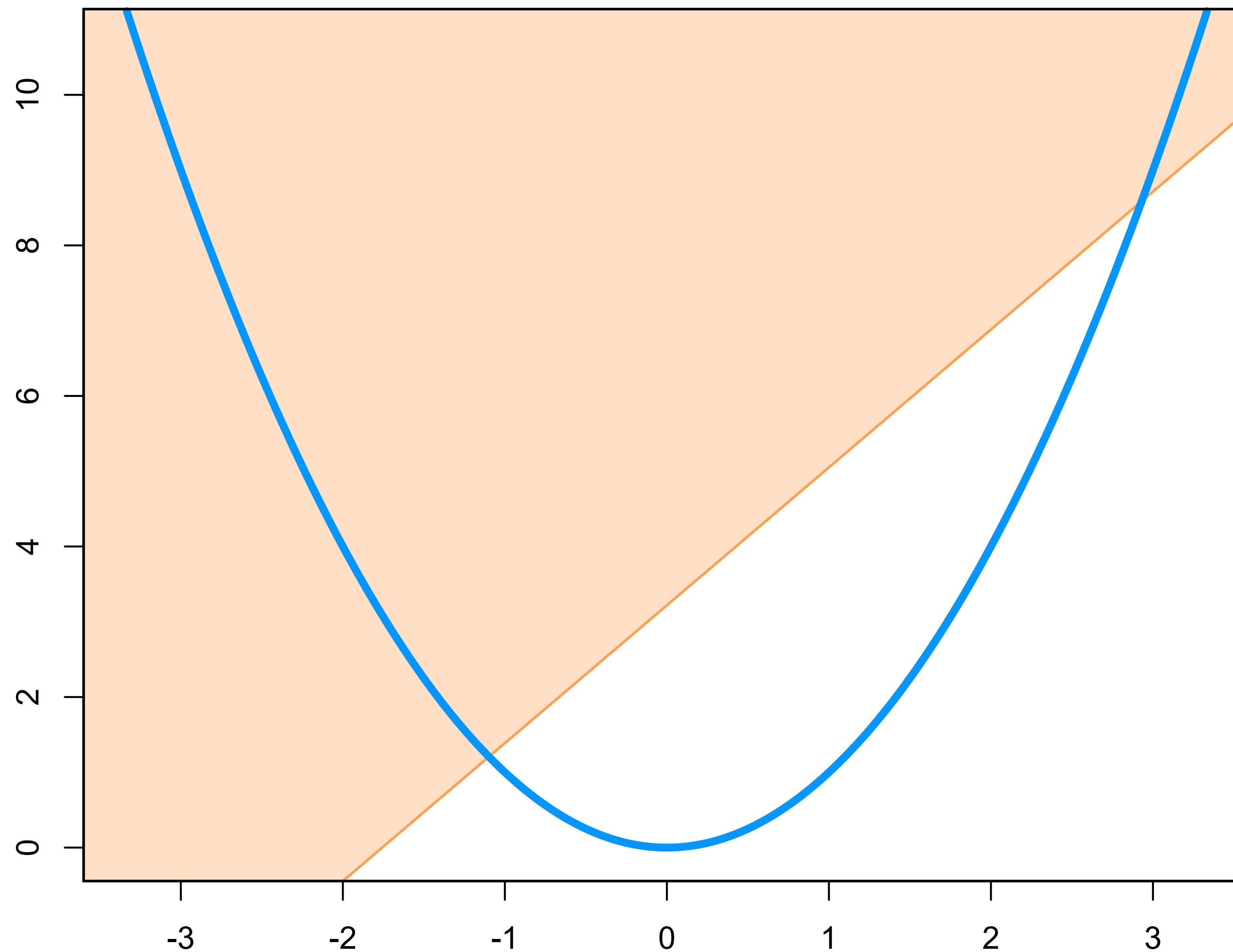


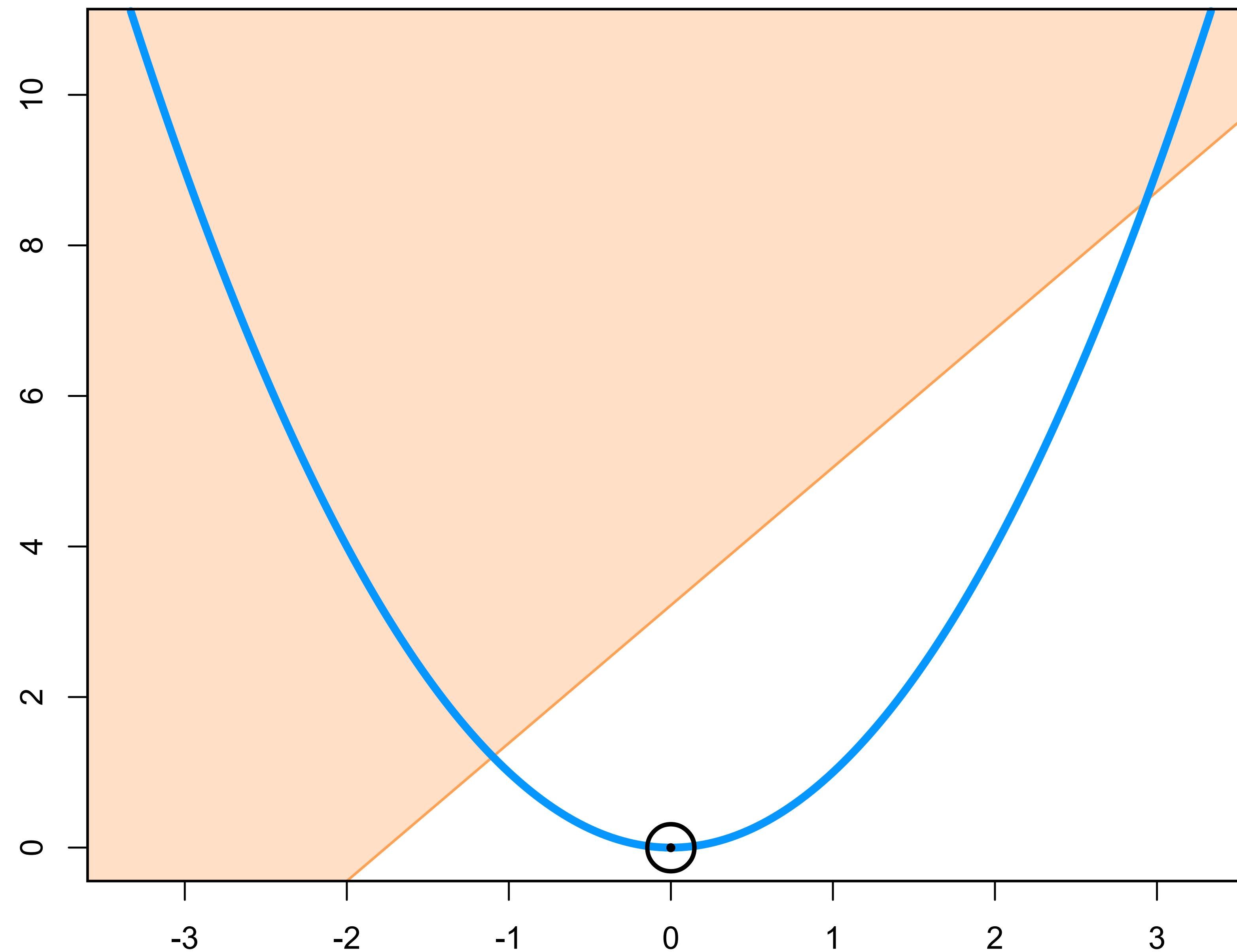


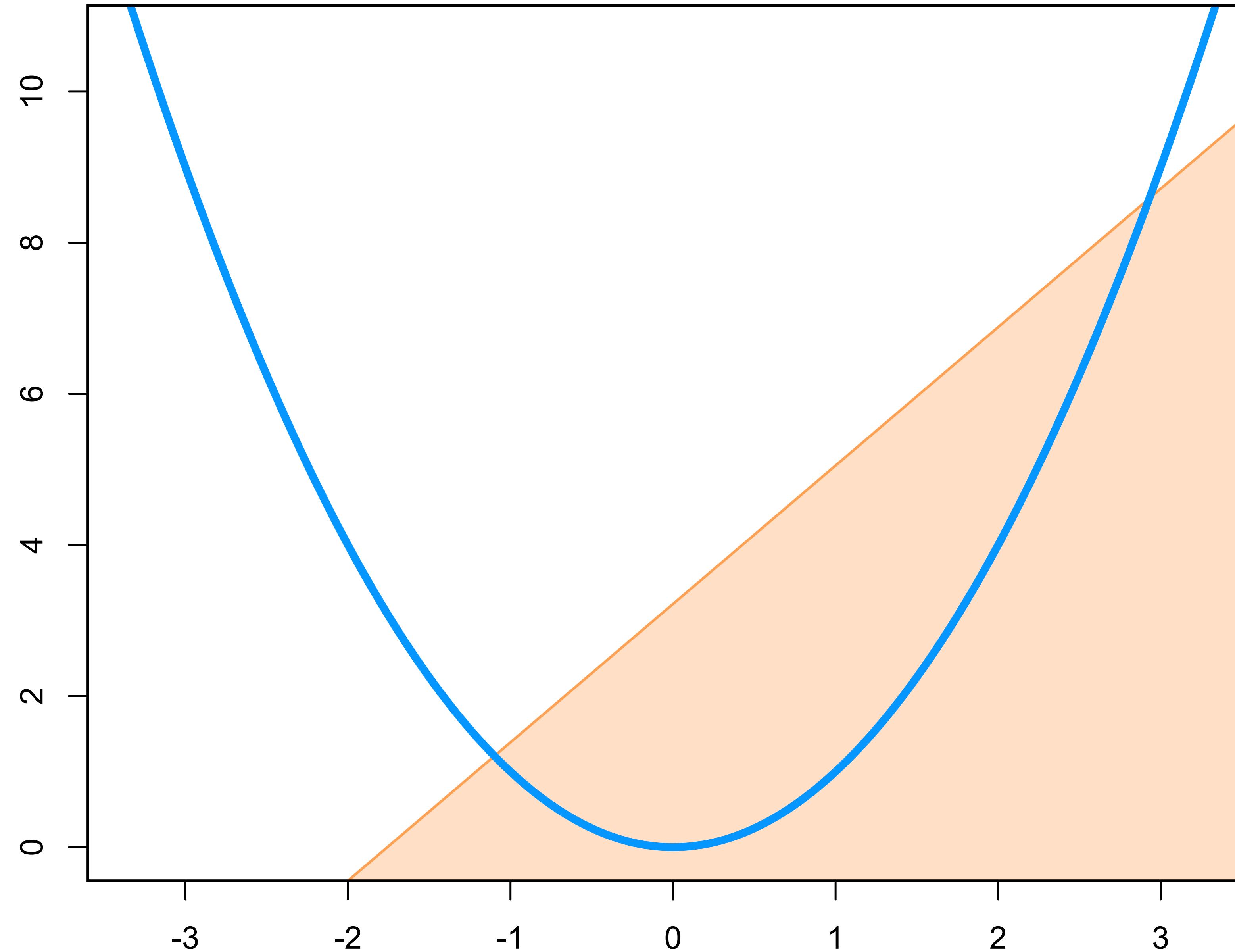


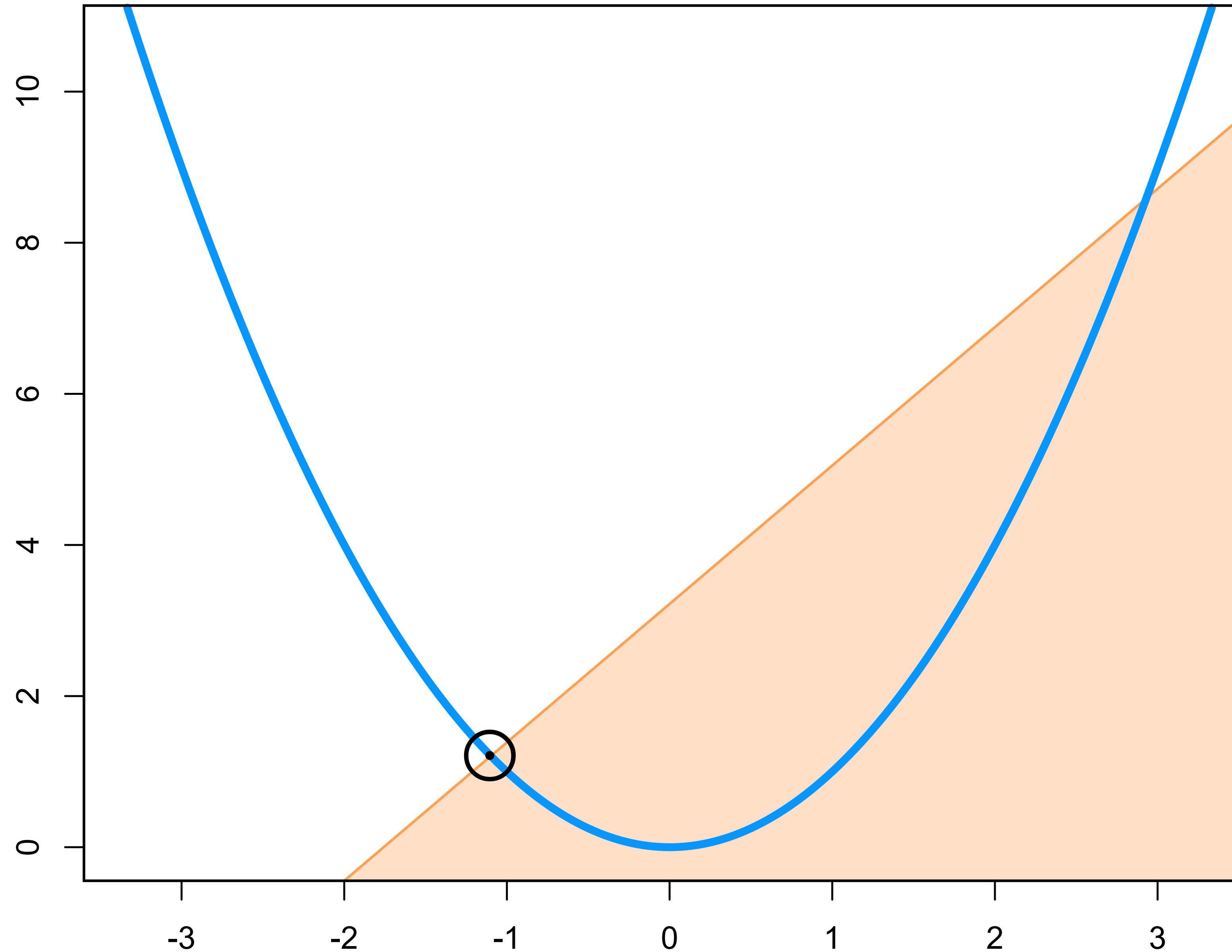












maximise $f(x)$
 x

subject to: $g(x) = 0$

$$\mathcal{L}(x, \alpha) = f(x) - \alpha g(x)$$

maximise $f(x)$
 x

subject to: $g(x) = 0$

maximise $f(x)$
 x

subject to: $g(x) \leq 0$

$$\alpha_i g_i(x) = 0, \quad \forall i$$

Complementary Slackness

Inequality constraints are only active at equality

maximise M
 \mathbf{w}, b

subject to:

$$y_i (\mathbf{x}_i \cdot \mathbf{w} + b) \geq M$$

$$\|\mathbf{w}\| = 1$$

$$\underset{\mathbf{w}, b}{\text{minimise}} \quad \frac{1}{2} \|\mathbf{w}\|^2$$

subject to: $y_i (\mathbf{x}_i \cdot \mathbf{w} + b) \geq 1$

$$L_p = \frac{1}{2} \|\mathbf{w}\|^2 - \sum_i^n \alpha_i \left(y_i (\mathbf{x}_i \cdot \mathbf{w} + b) - 1 \right)$$

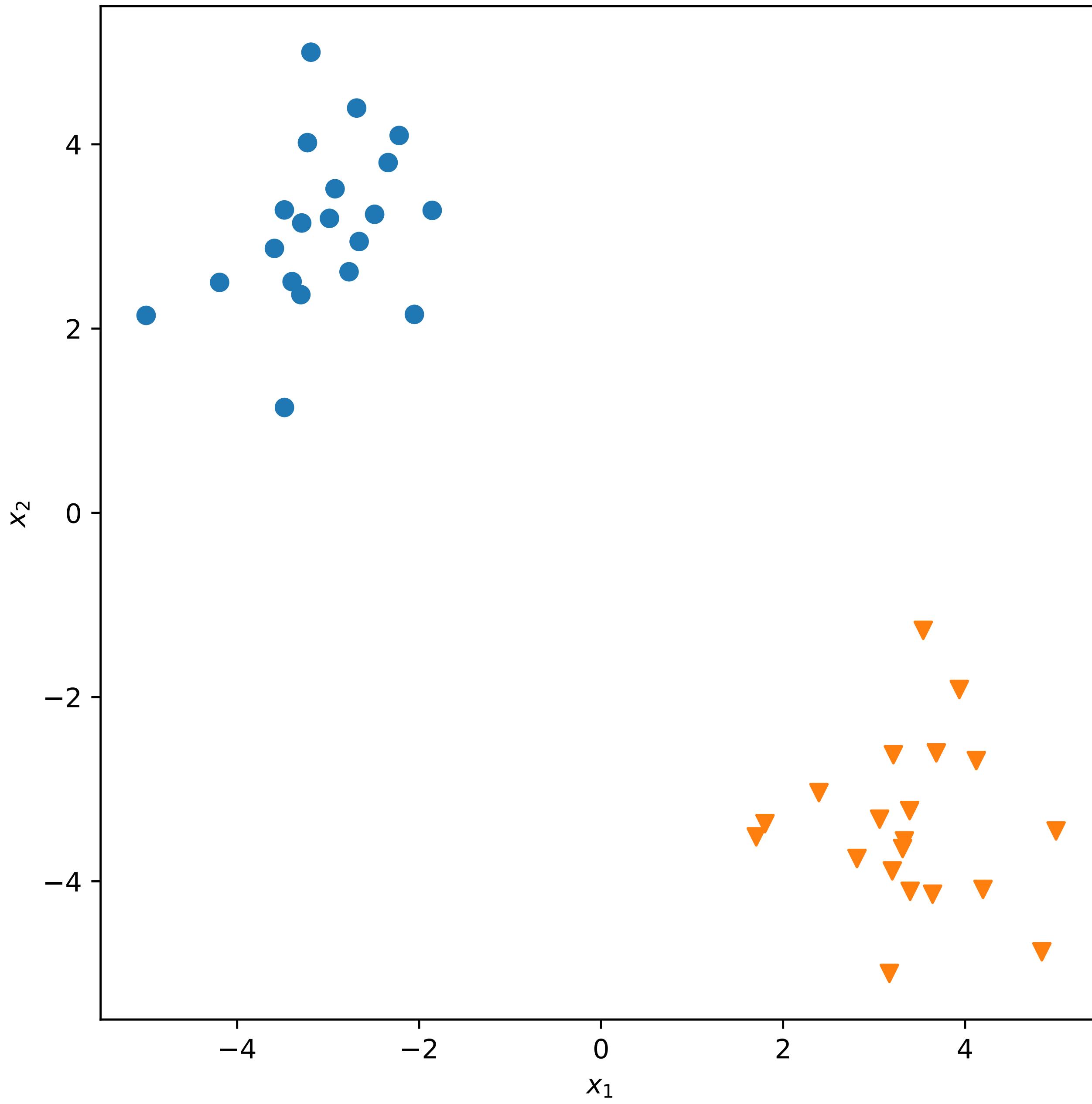
$$L_d = \sum_i^n \alpha_i - \frac{1}{2} \sum_i^n \sum_j^n \alpha_i \alpha_j y_i y_j (\mathbf{x}_i \cdot \mathbf{x}_j)$$

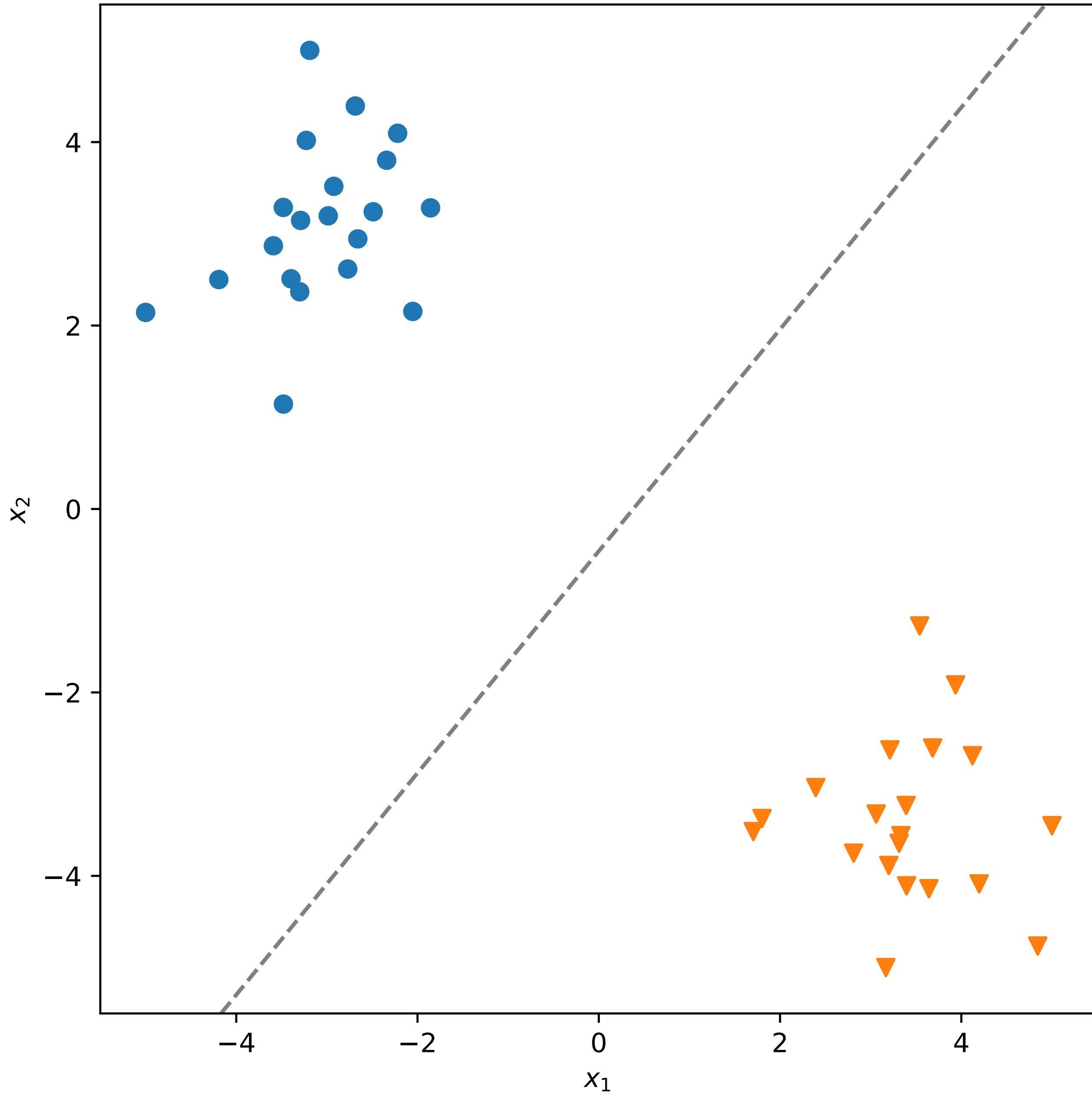
$$\mathbf{w} = \sum_i^n \alpha_i y_i \mathbf{x}_i$$

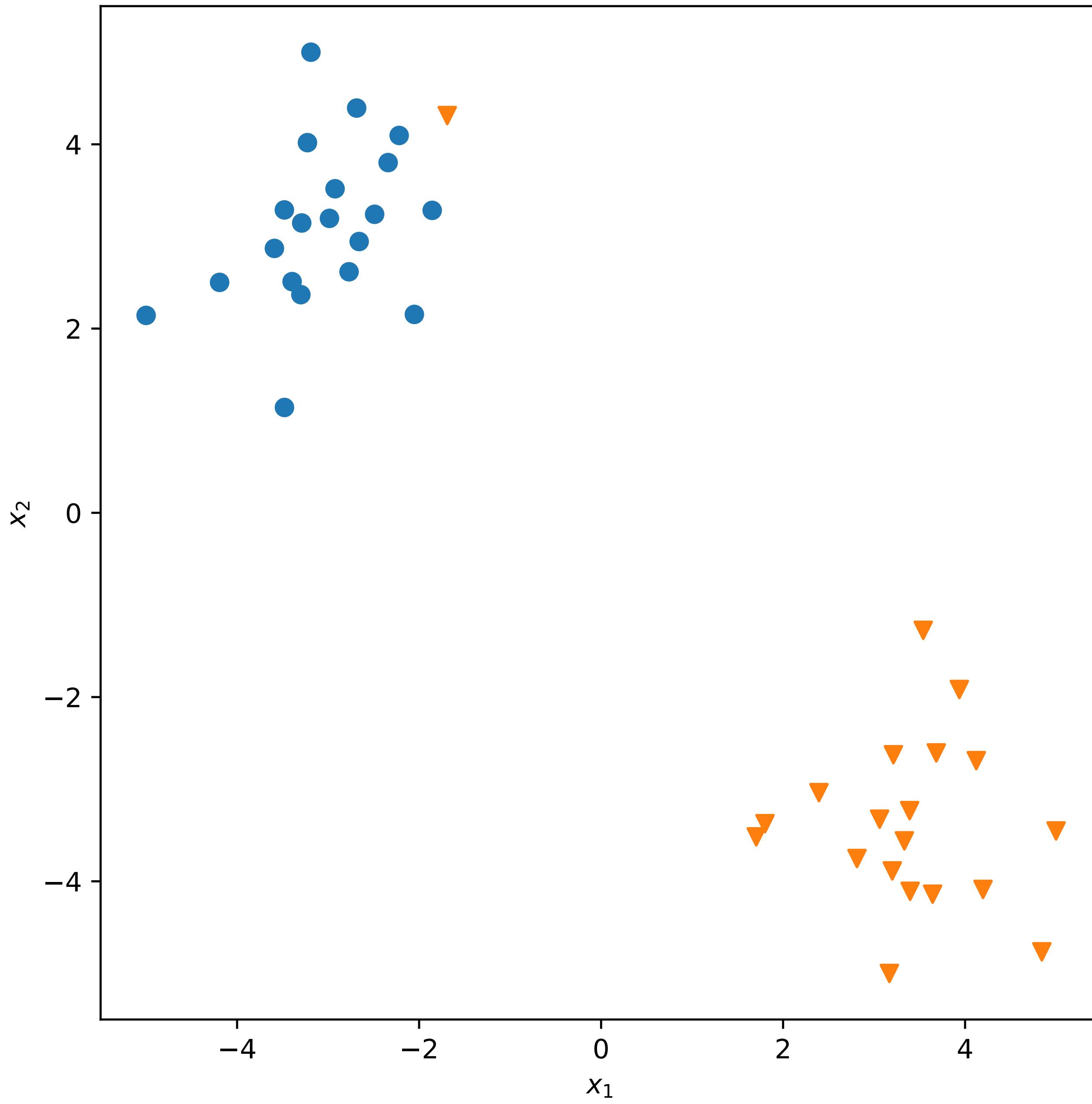
$$\alpha_i [y_i(\mathbf{x}_i \cdot \mathbf{w} + b) - 1] = 0 \quad \forall i$$

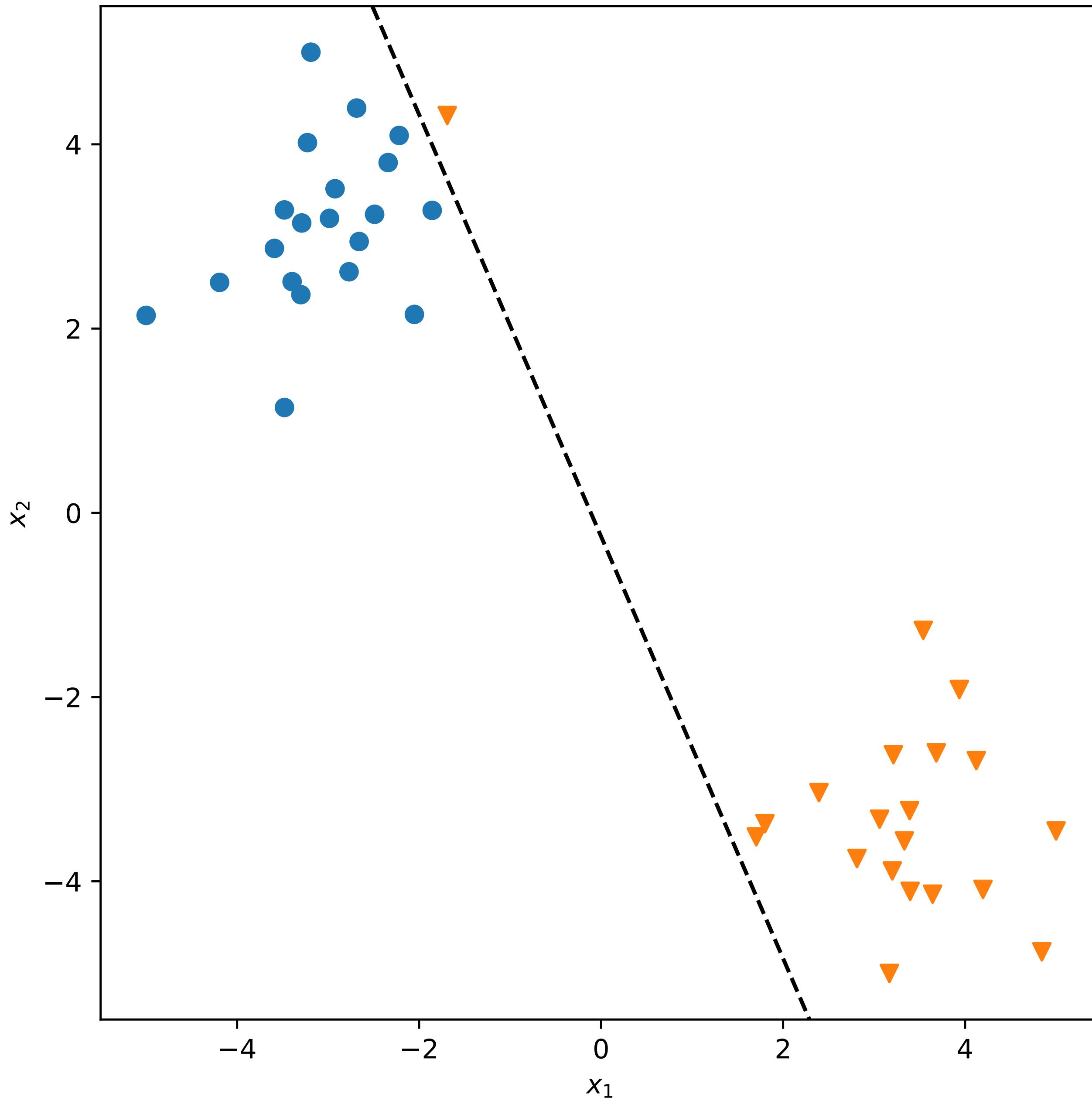
$$\underset{\alpha}{\text{maximise}} \quad \sum_i^n \alpha_i - \frac{1}{2} \sum_i^n \sum_j^n \alpha_i \alpha_j y_i y_j (\mathbf{x}_i \cdot \mathbf{x}_j)$$

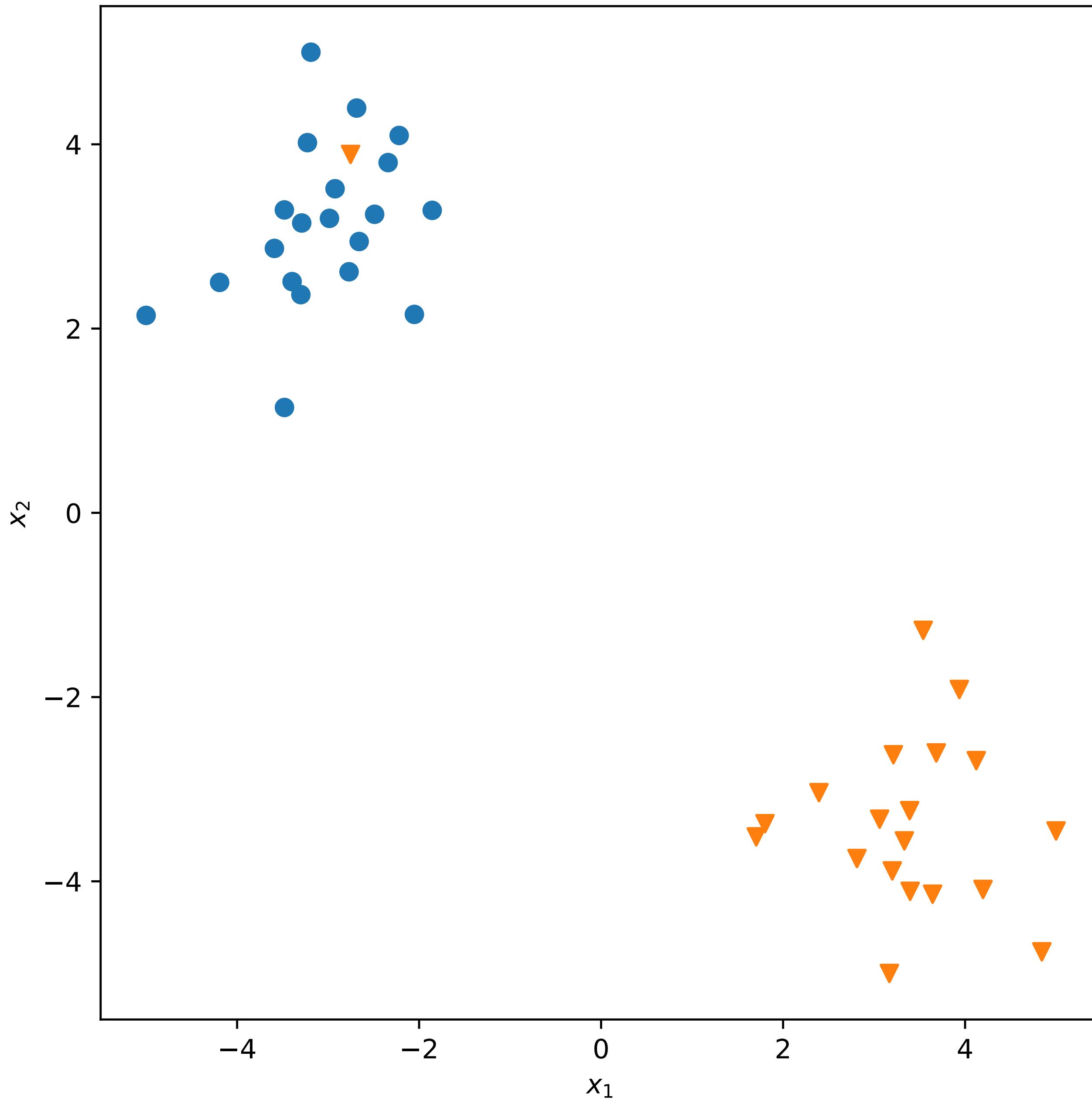
$$\hat{y} = \begin{cases} 1 & \text{if } \sum_i^n \alpha_i y_i (\mathbf{x}_i \cdot \mathbf{x}) + b \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

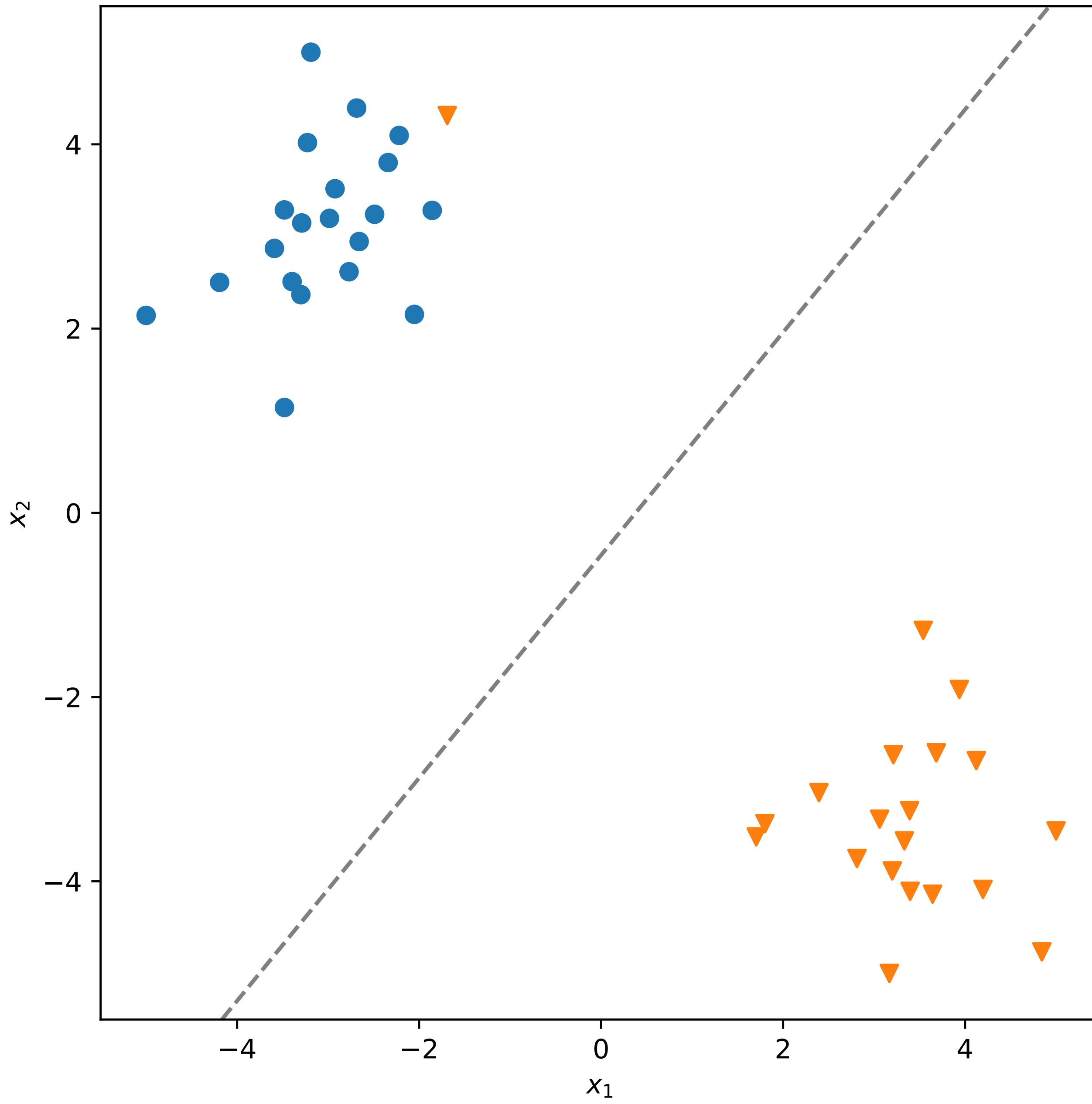


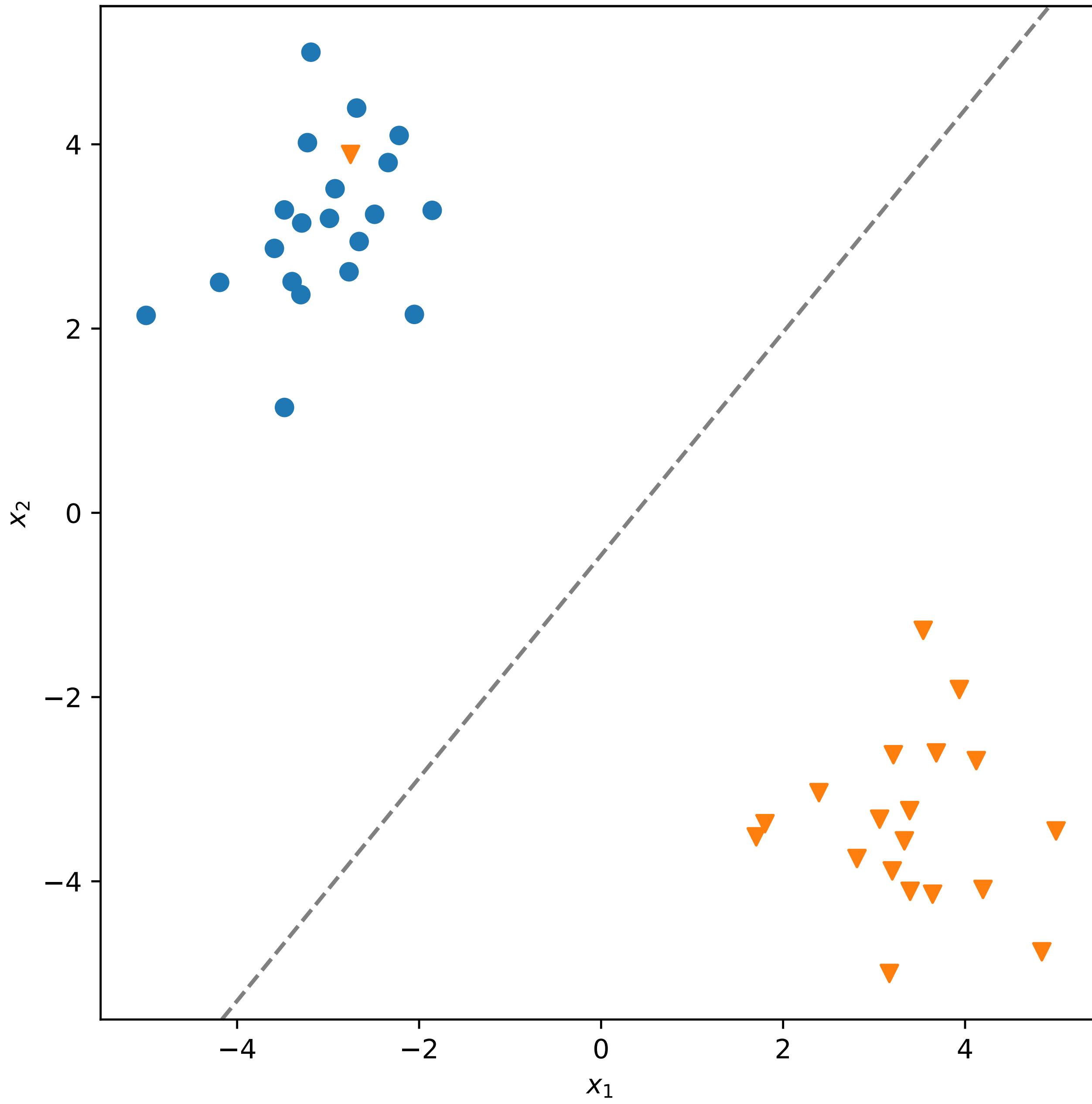












maximise M
 \mathbf{w}, b, ξ

subject to:

$$y_i (\mathbf{x}_i \cdot \mathbf{w} + b) \geq M(1 - \xi_i)$$

$$\|\mathbf{w}\| = 1$$

$$\xi_i \geq 0 \quad \forall i$$

$$\sum_i \xi_i \leq \text{constant}$$

$$\underset{\mathbf{w}, b, \xi}{\text{minimise}} \quad \frac{1}{2} \|\mathbf{w}\|^2 + C \sum_i^n \xi_i$$

subject to:

$$y_i (\mathbf{x}_i \cdot \mathbf{w} + b) \geq 1 - \xi_i$$

$$\xi_i \geq 0$$

$$L_p = \frac{1}{2} \|\mathbf{w}\|^2 + C \sum_i^n \xi_i - \sum_i^n \alpha_i [y_i(\mathbf{x}_i \cdot \mathbf{w} + b) - (1 - \xi_i)] - \sum_i^n \mu_i \xi_i$$

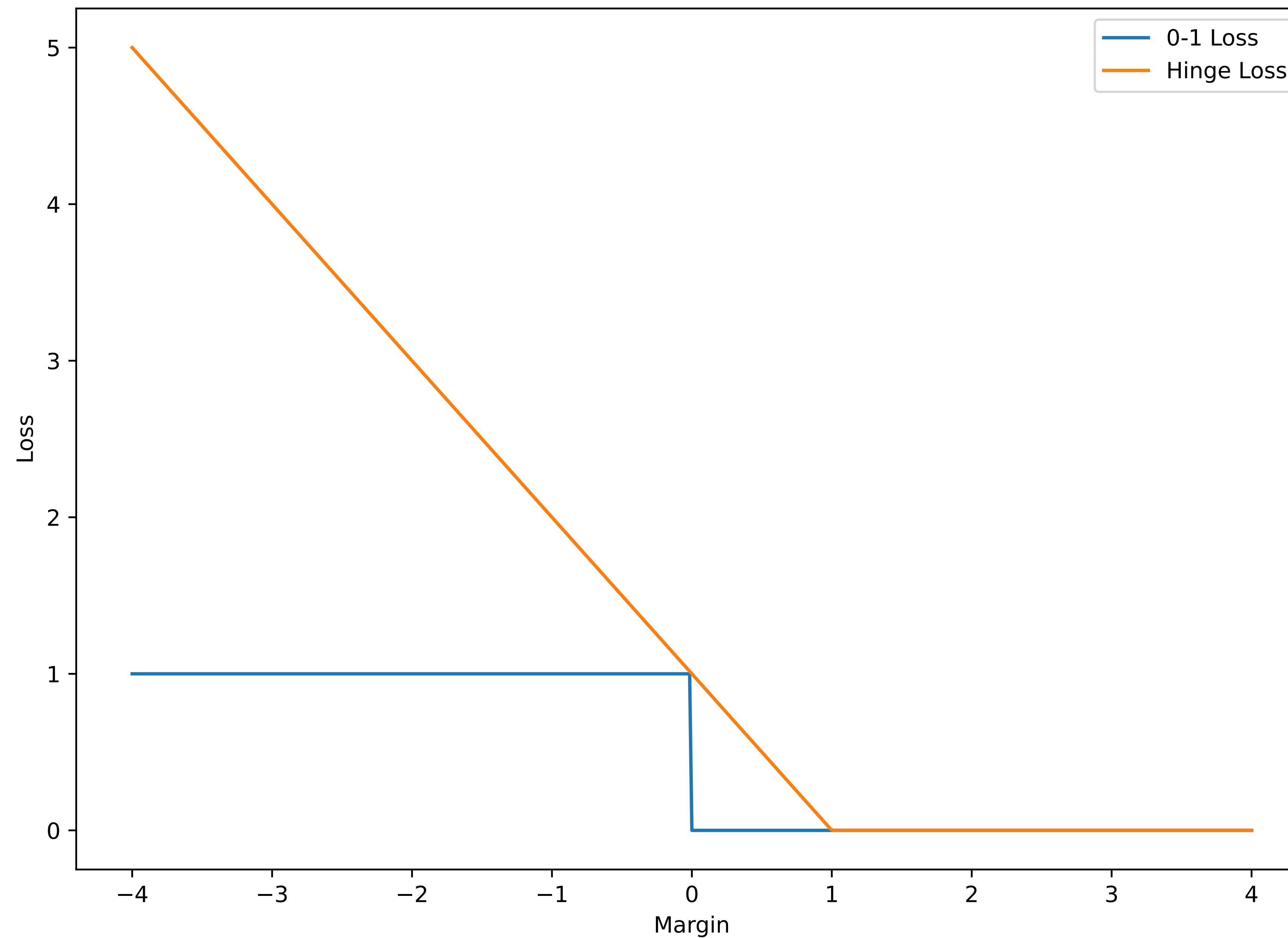
$$L_d = \sum_i^n \alpha_i - \frac{1}{2} \sum_i^n \sum_j^n \alpha_i \alpha_j y_i y_j (\mathbf{x}_i \cdot \mathbf{x}_j)$$

$$0 \leq \alpha_i \leq C$$

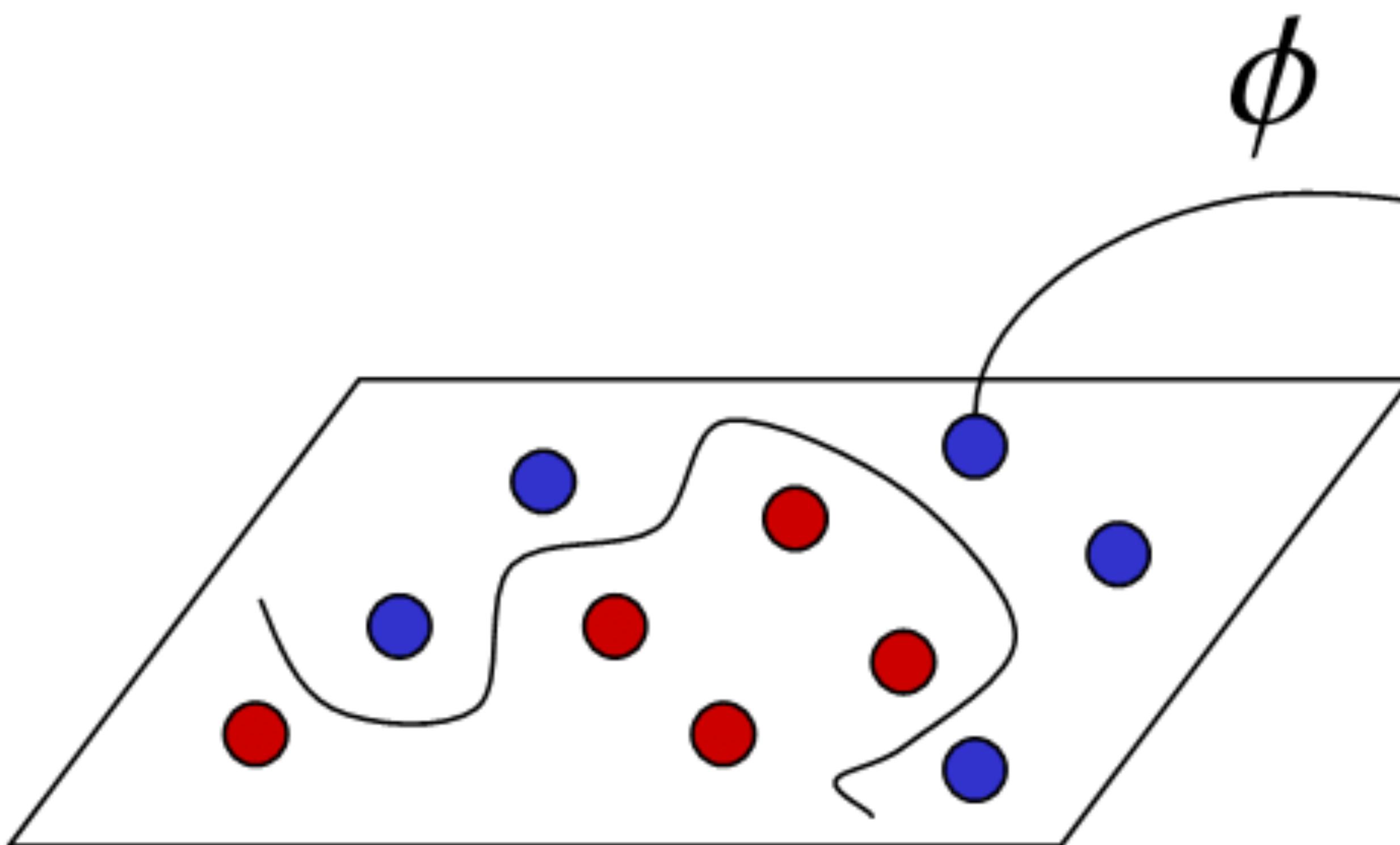
$$\alpha_i \left[y_i(\mathbf{x}_i \cdot \mathbf{w}) - (1 - \xi_i) \right] = 0$$

$$y_i(\mathbf{x}_i \cdot \mathbf{w}) - (1 - \xi_i) \geq 0$$

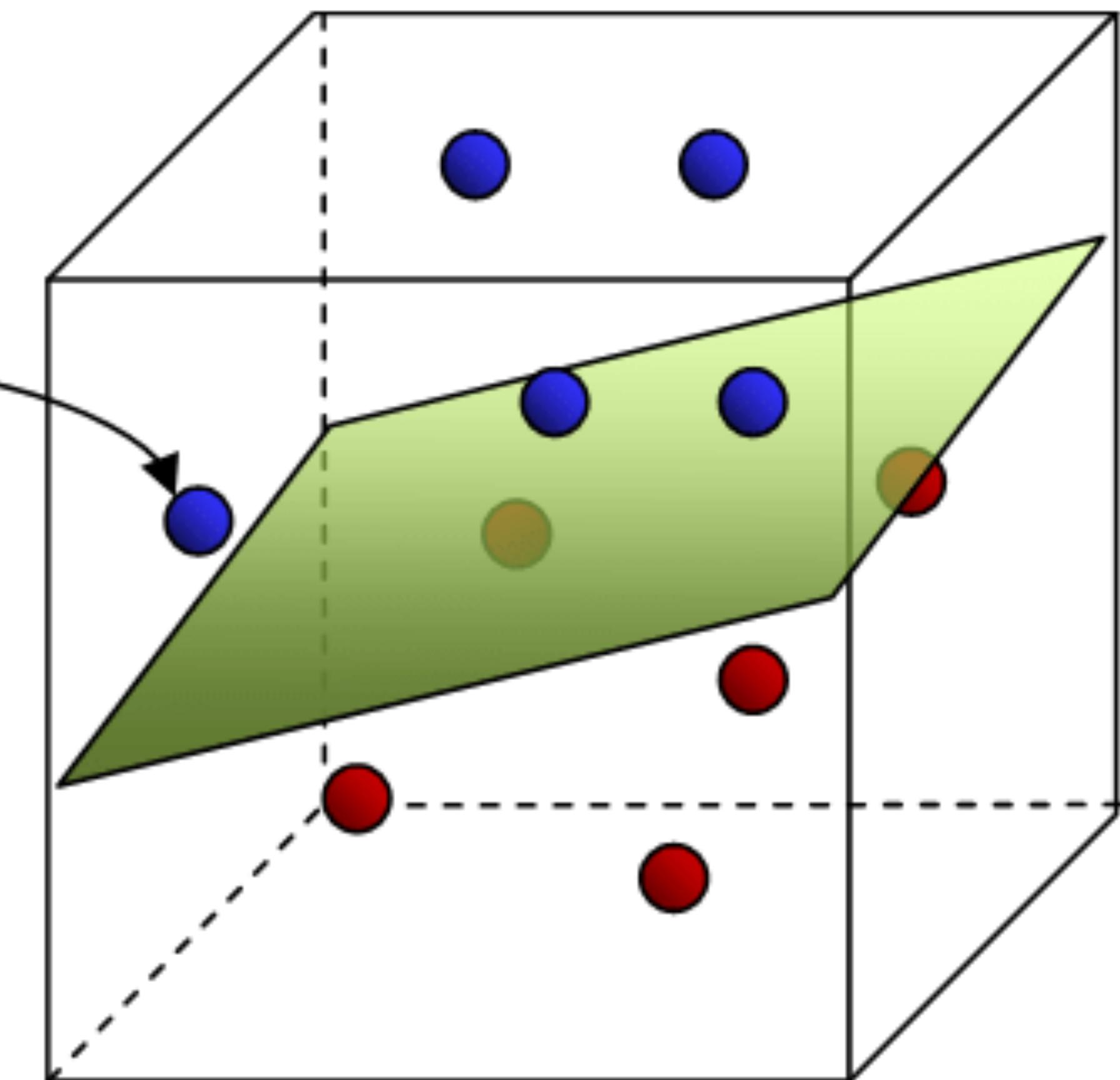
$$\begin{aligned} L(y, \mathbf{x}) &= \max(0, 1 - yf(\mathbf{x})) \\ &= \max(0, 1 - y(\mathbf{x} \cdot \mathbf{w} + b)) \end{aligned}$$



This One Weird Trick



a) Input Space



b) Feature Space

$$\mathbf{x}' = \phi(\mathbf{x})$$

$$\underset{\alpha}{\text{maximise}} \quad \sum_i^n \alpha_i - \frac{1}{2} \sum_i^n \sum_j^n \alpha_i \alpha_j y_i y_j (\mathbf{x}_i \cdot \mathbf{x}_j)$$

$$\hat{y} = \begin{cases} 1 & \text{if } \sum_i^n \alpha_i y_i (\mathbf{x}_i \cdot \mathbf{x}) + b \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

$$\underset{\alpha}{\text{maximise}} \sum_i^n \alpha_i - \frac{1}{2} \sum_i^n \sum_j^n \alpha_i \alpha_j y_i y_j (\phi(\mathbf{x}_i) \cdot \phi(\mathbf{x}_j))$$

$$\hat{y} = \begin{cases} 1 & \text{if } \sum_i^n \alpha_i y_i (\phi(\mathbf{x}_i) \cdot \phi(\mathbf{x})) + b \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

$$\phi(\mathbf{x}) \mapsto \mathbb{R}^{10,000,000}$$

$$K(\mathbf{u}, \mathbf{v}) = \phi(\mathbf{u}) \cdot \phi(\mathbf{v})$$

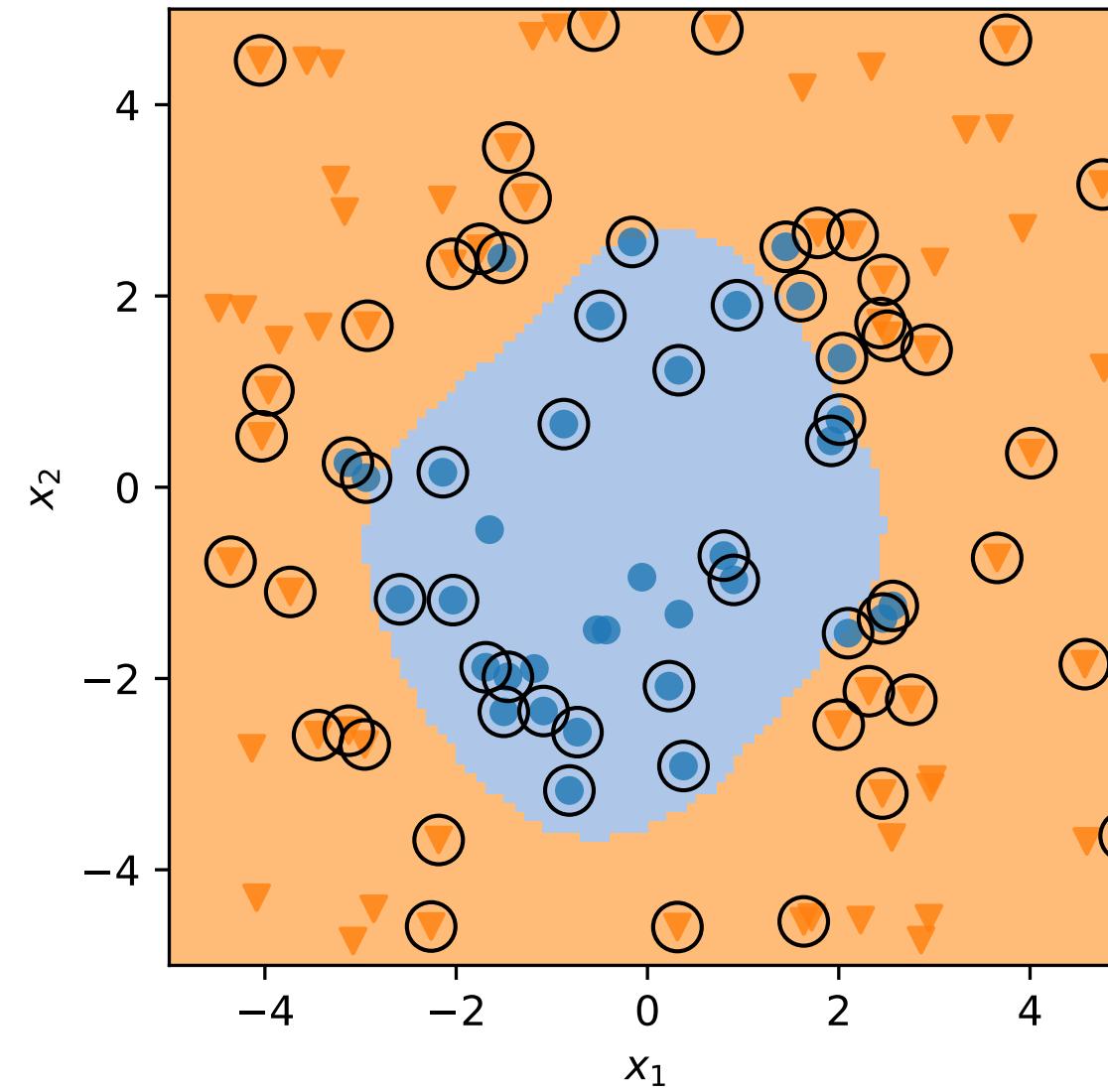
$$\underset{\alpha}{\text{maximise}} \sum_i^n \alpha_i - \frac{1}{2} \sum_i^n \sum_j^n \alpha_i \alpha_j y_i y_j K(\mathbf{x}_i, \mathbf{x}_j)$$

$$\hat{y} = \begin{cases} 1 & \text{if } \sum_i^n \alpha_i y_i K(\mathbf{x}_i, \mathbf{x}) + b \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

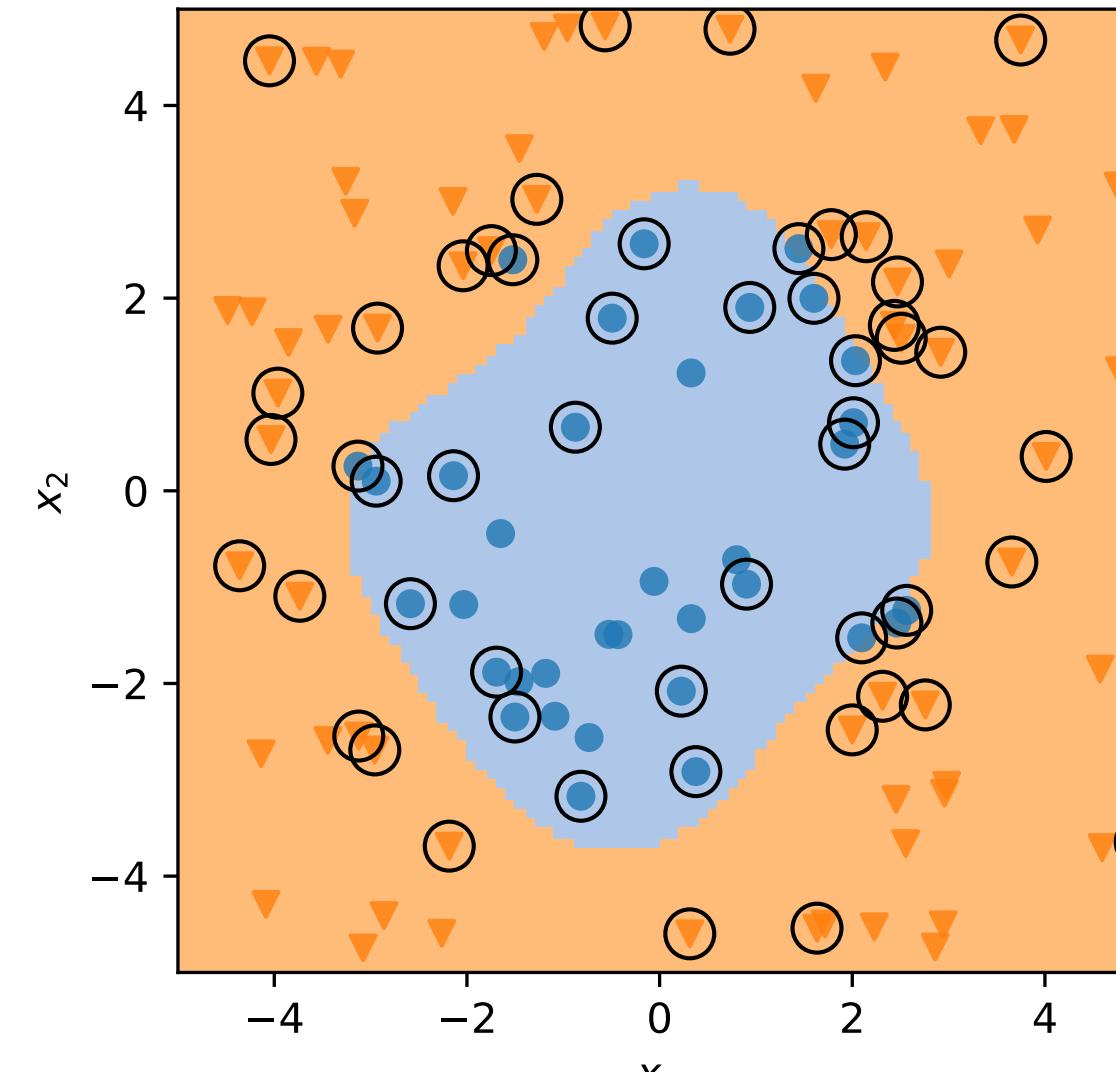
$$K(\mathbf{u}, \mathbf{v}) = e^{-\gamma \|\mathbf{u} - \mathbf{v}\|^2}$$

$$= e^{-\frac{\|\mathbf{u} - \mathbf{v}\|^2}{2\sigma^2}}$$

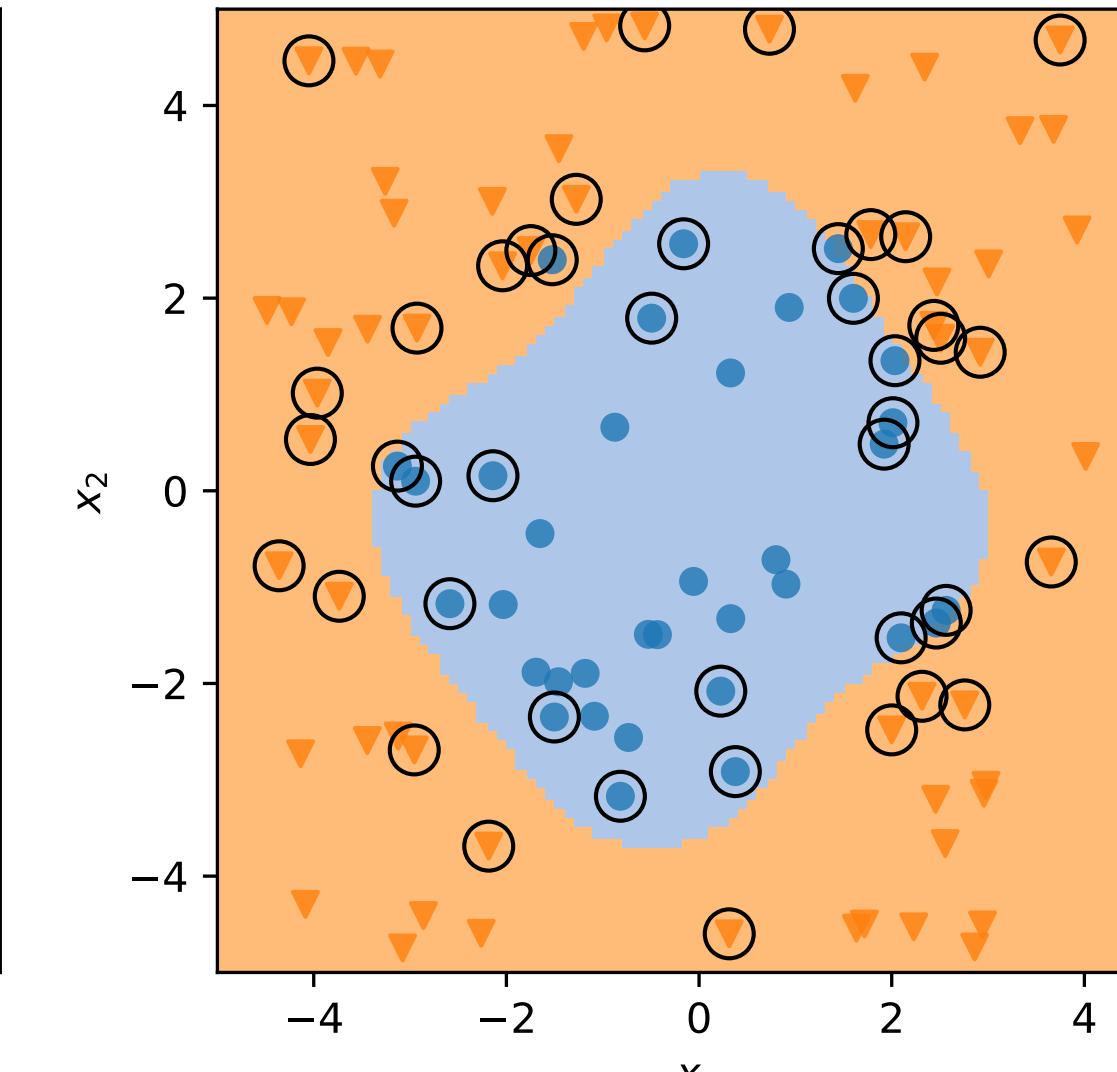
RBF SVM, C=0.3, gamma=0.25



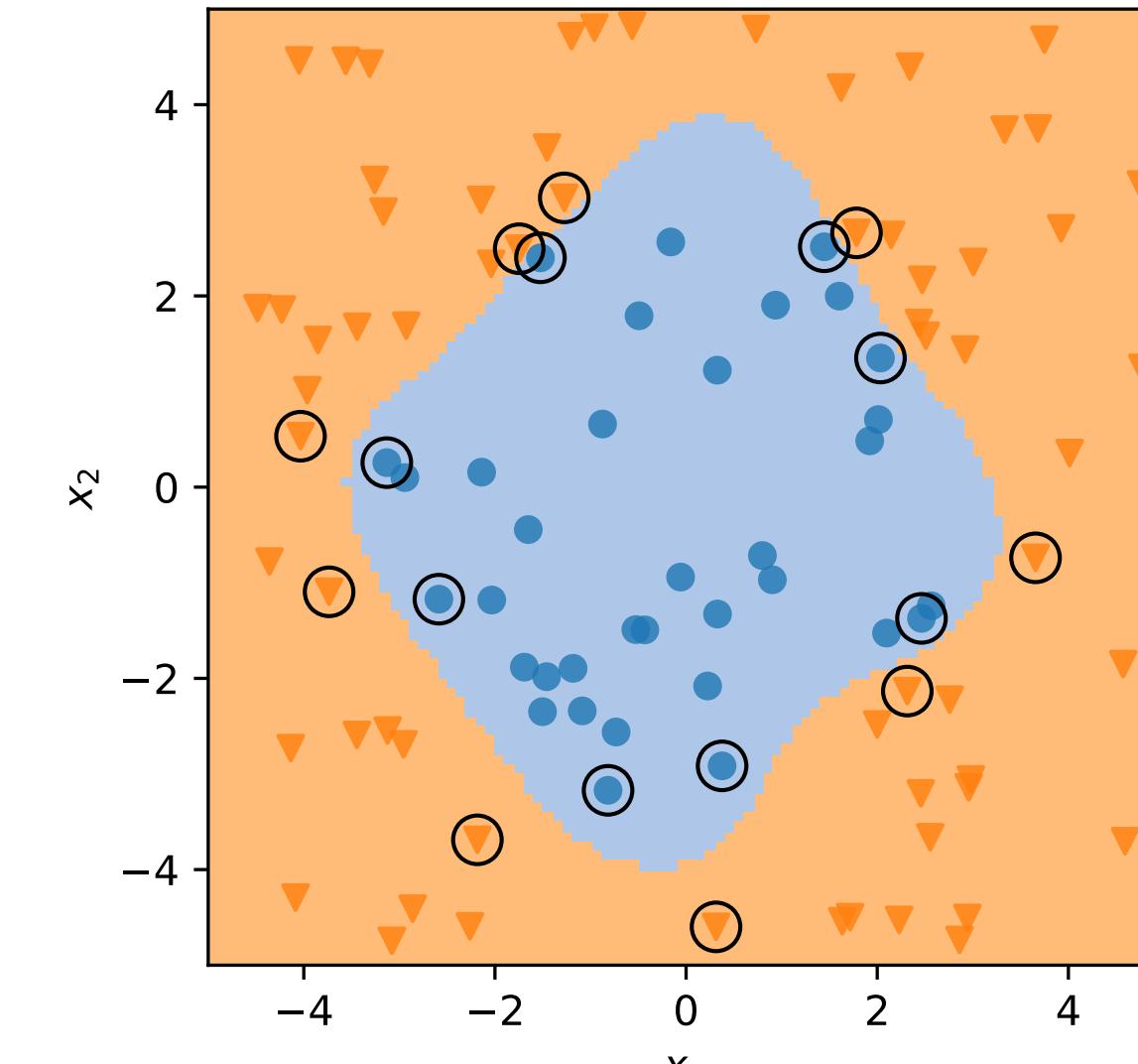
RBF SVM, C=0.6, gamma=0.25



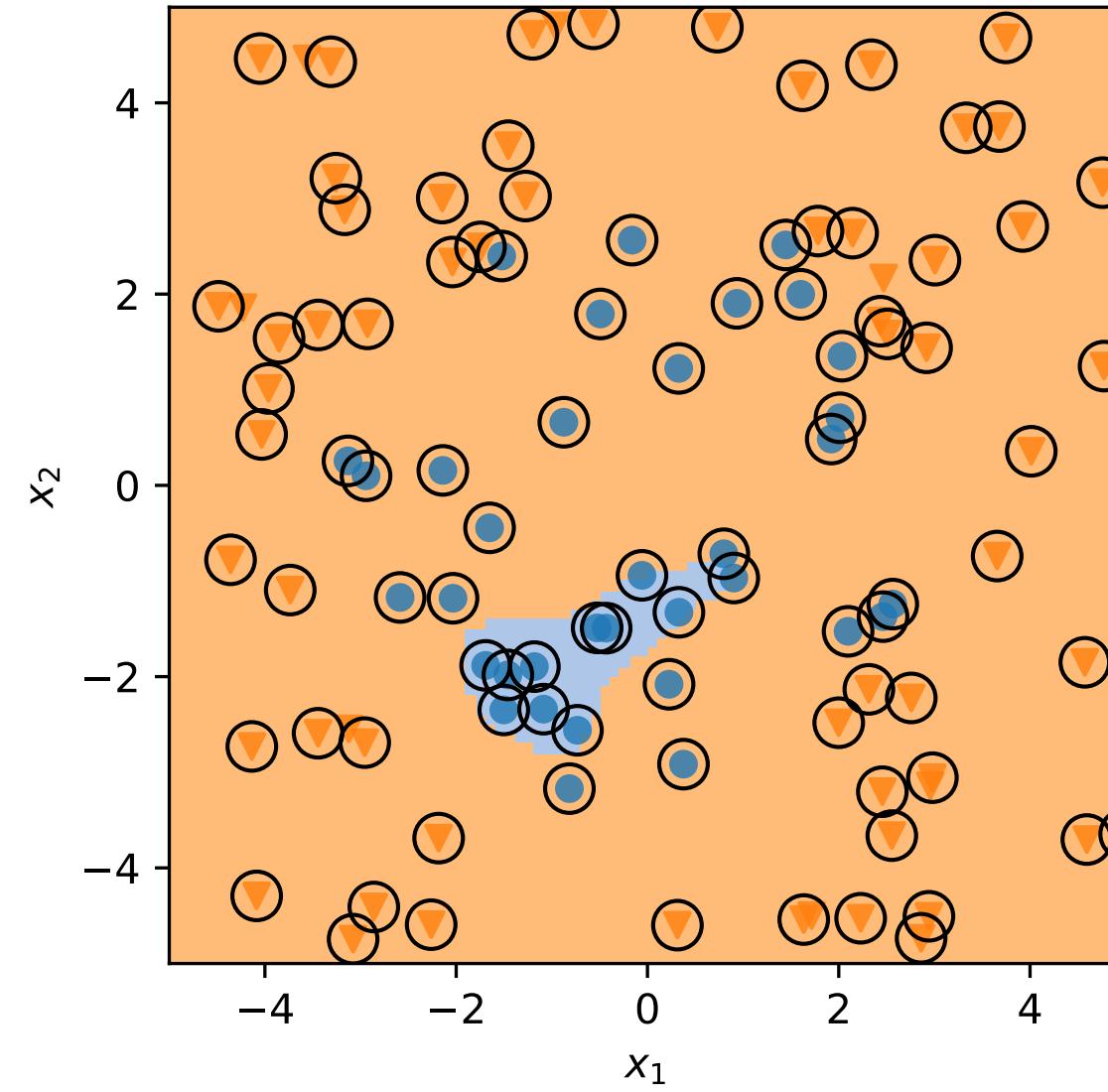
RBF SVM, C=1, gamma=0.25



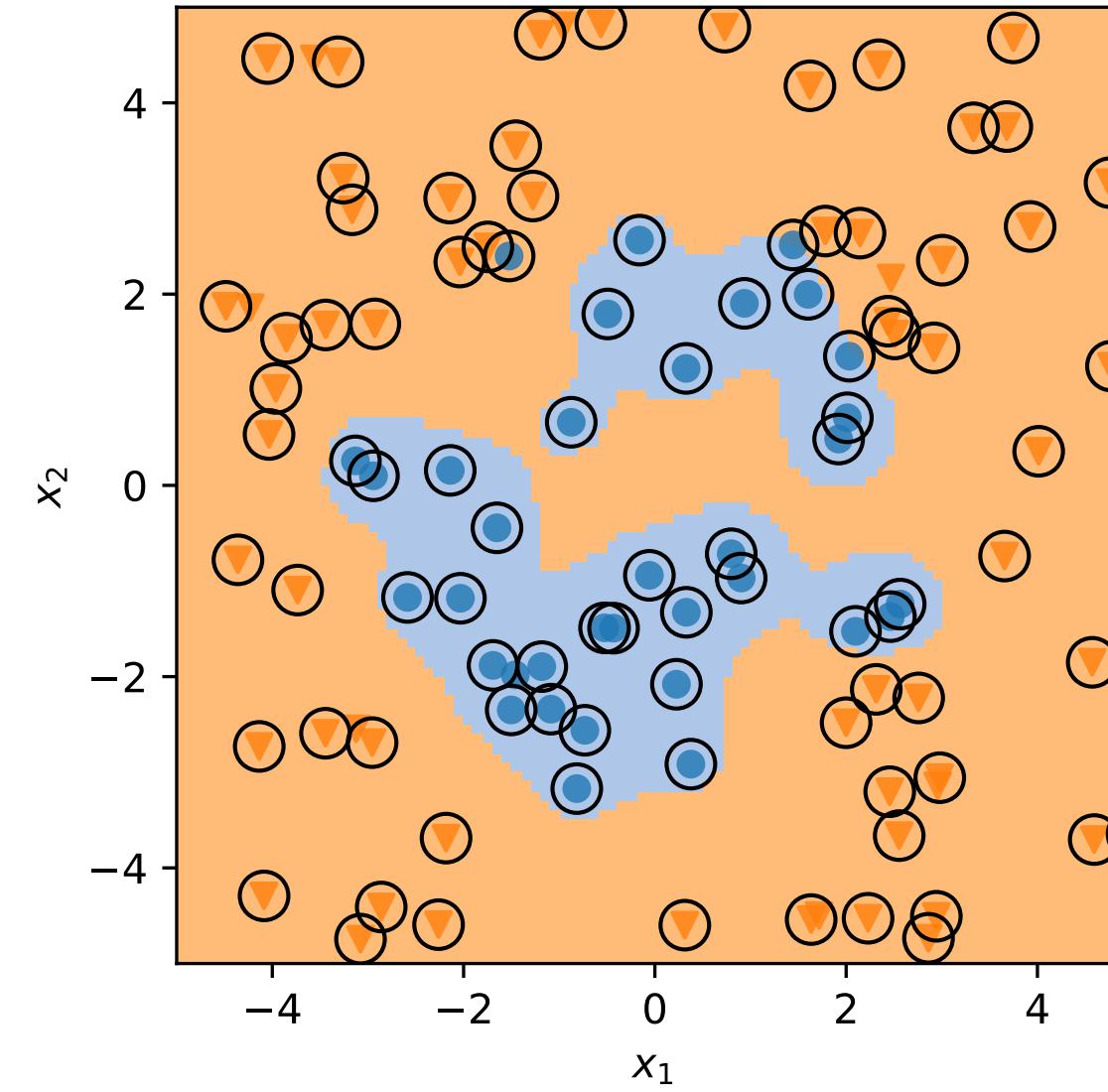
RBF SVM, C=1000, gamma=0.25



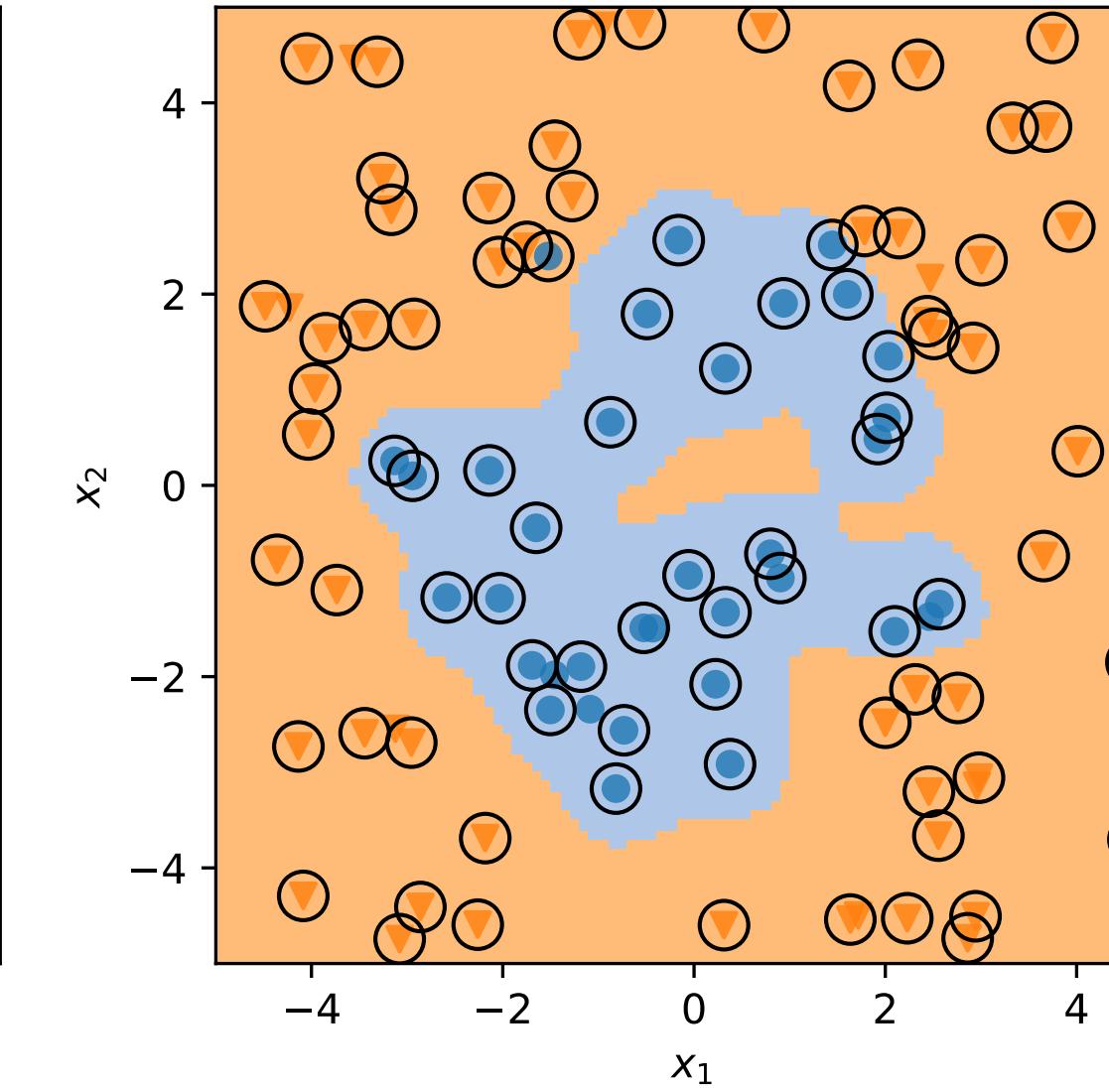
RBF SVM, C=0.3, gamma=2.5



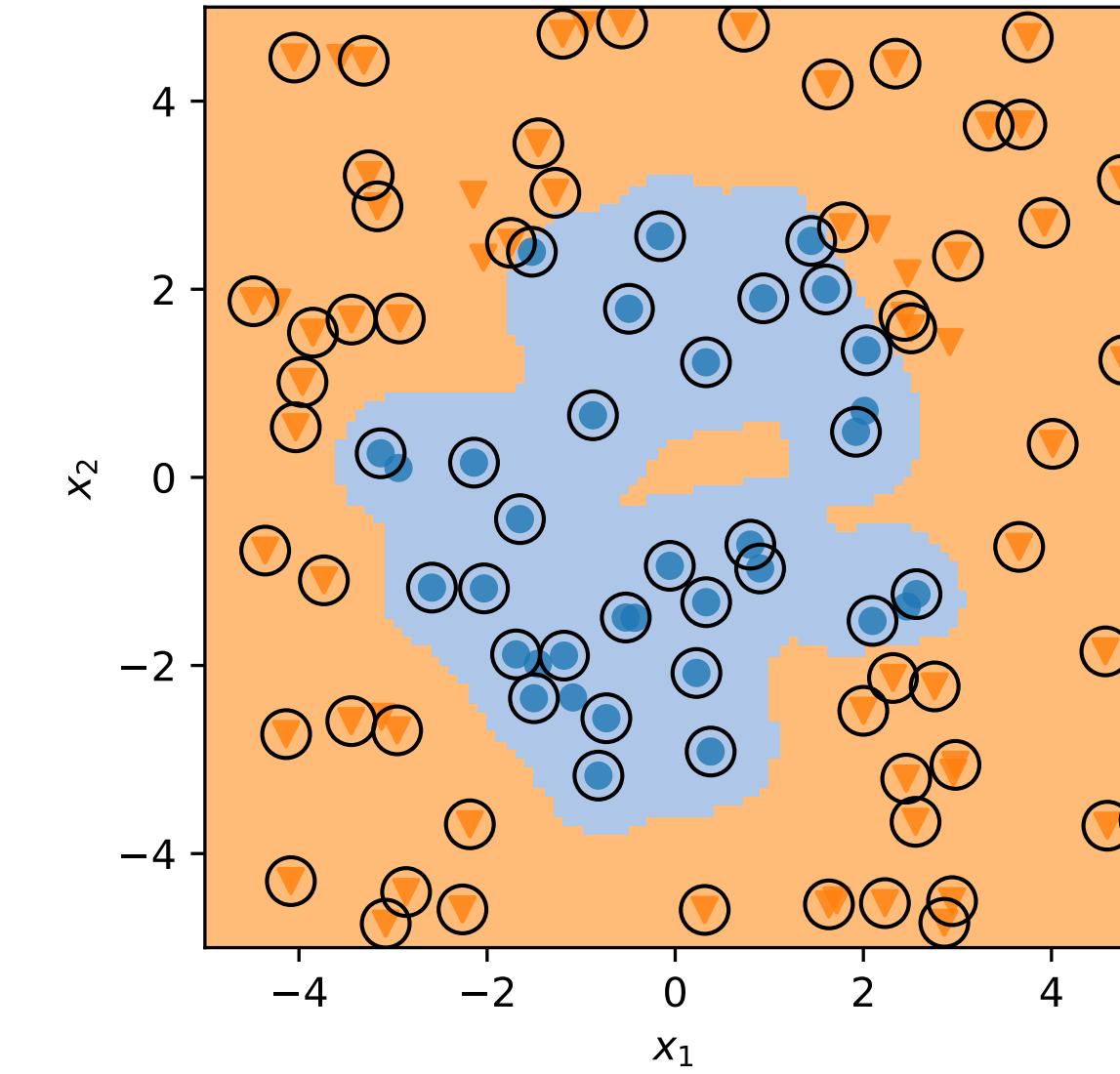
RBF SVM, C=0.6, gamma=2.5

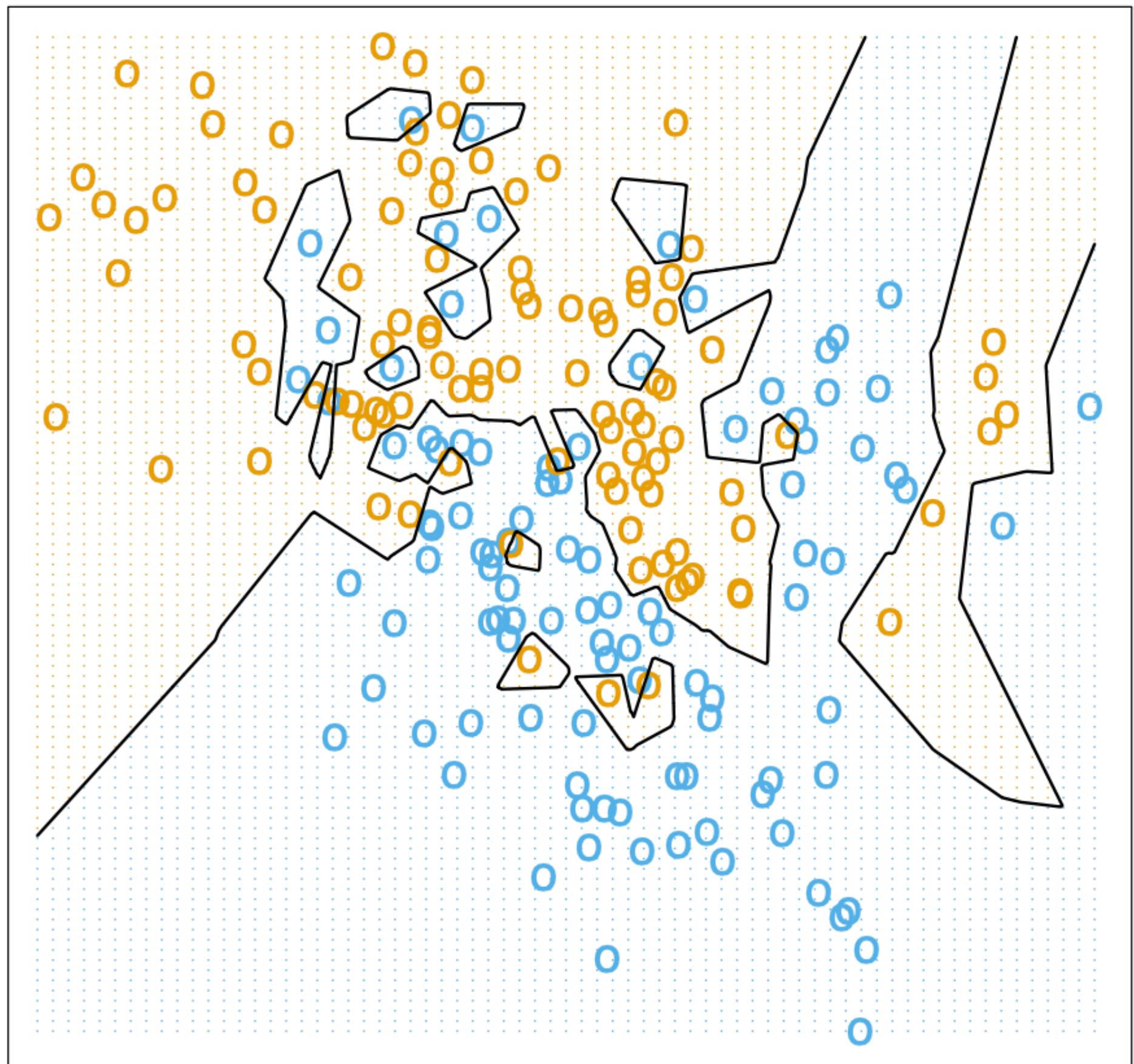


RBF SVM, C=1, gamma=2.5

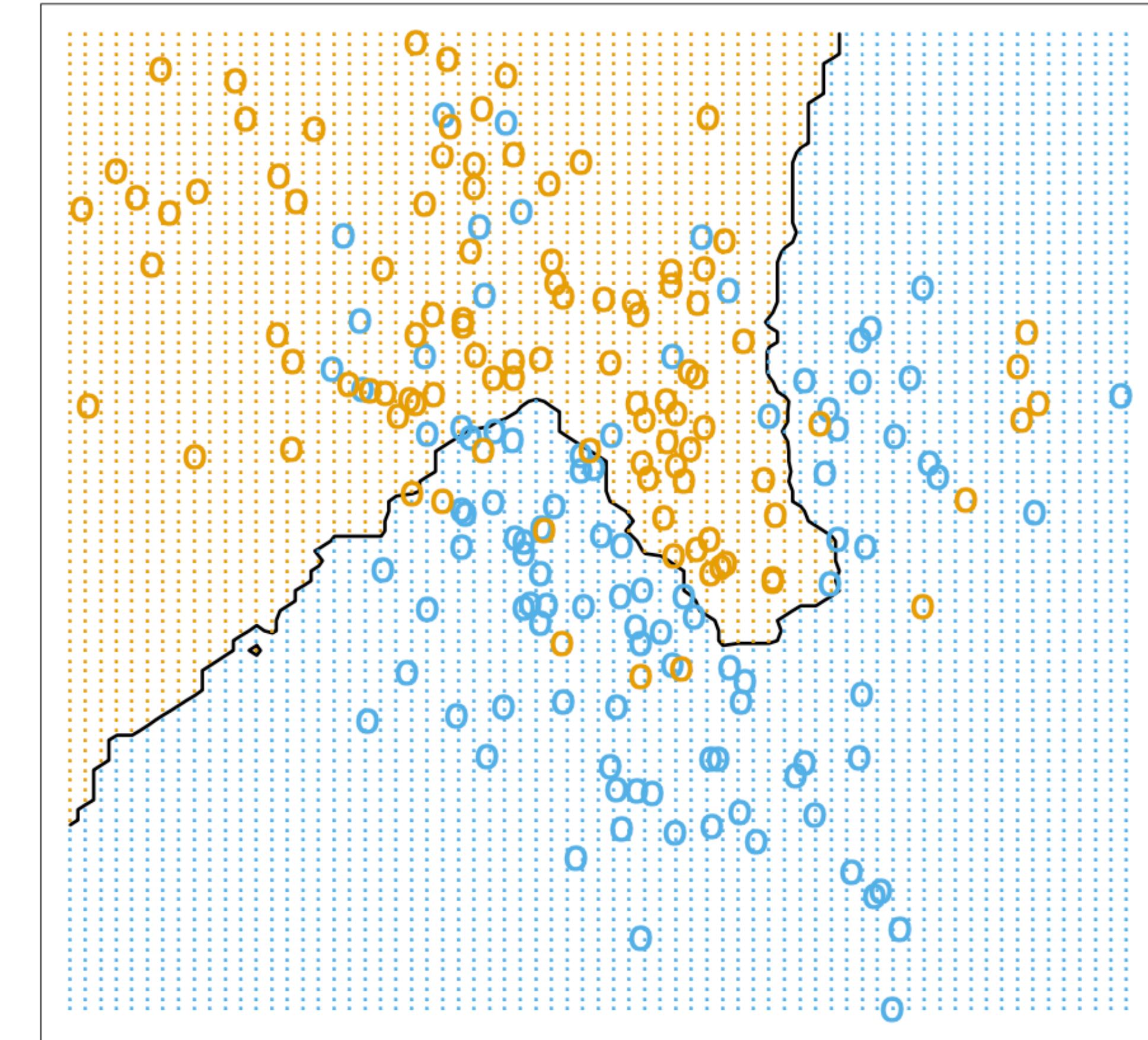


RBF SVM, C=1000, gamma=2.5





$k=1$



$k=15$

$$K(\mathbf{u}, \mathbf{v}) = \phi(\mathbf{u}) \cdot \phi(\mathbf{v})$$

$$\mathbf{K} = \begin{bmatrix} K(\mathbf{x}_1, \mathbf{x}_1) & K(\mathbf{x}_1, \mathbf{x}_2) & \cdots & K(\mathbf{x}_1, \mathbf{x}_n) \\ K(\mathbf{x}_2, \mathbf{x}_1) & K(\mathbf{x}_2, \mathbf{x}_2) & \cdots & K(\mathbf{x}_2, \mathbf{x}_n) \\ \vdots & \vdots & \ddots & \vdots \\ K(\mathbf{x}_n, \mathbf{x}_1) & K(\mathbf{x}_n, \mathbf{x}_2) & \cdots & K(\mathbf{x}_n, \mathbf{x}_n) \end{bmatrix}$$

Mercer's Condition

$$K(\mathbf{u}, \mathbf{v}) = \phi(\mathbf{u}) \cdot \phi(\mathbf{v}) \iff K \text{ is PSD}$$

Questions?

Next week: Neural Networks

