Exponential Fourier Series

Colophon

An annotatable worksheet for this presentation is available as **Worksheet 10**.

- The source code for this page is fourier_series/2/exp_fs1.ipynb.
- You can view the notes for this presentation as a webpage (HTML).
- This page is downloadable as a PDF file.

This section builds on our Revision of the to Trigonometrical Fourier Series.

Trigonometric Fourier series uses integration of a periodic signal multiplied by sines and cosines at the fundamental and harmonic frequencies. If performed by hand, this can a painstaking process. Even with the simplifications made possible by exploiting waveform symmetries, there is still a need to integrate cosine and sine terms, be aware of and able to exploit the trigonometrc identities, and the properties of *orthogonal functions* before we can arrive at the simplified solutions. This is why I concentrated on the properties and left the computation to a computer.

However, by exploiting the exponential function e^{at}, we can derive a method for calculating the coefficients of the harmonics that is much easier to calculate by hand and convert into an algorithm that can be executed by computer.

The result is called the *Exponential Fourier Series*.

Agenda

- Exponents and Euler's Equation
- The Exponential Fourier series
- Symmetry in Exponential Fourier Series
- Example

The Exponential Function e^{at}

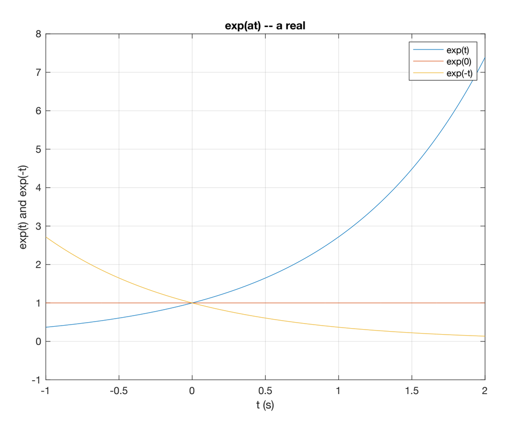
- You should already be familiar with e^{at} because it appears in the solution of differential equations.
- It is also a function that appears in the definition of the Laplace and Inverse Laplace Transform.
- It pops up again and again in tables and properies of the Laplace Transform.

Case when a is real.

When a is real the function e^{at} will take one of the two forms illustrated below:

In [1]:
```
clear all
imatlab_export_fig('print-svg')  % Static svg figures.
```

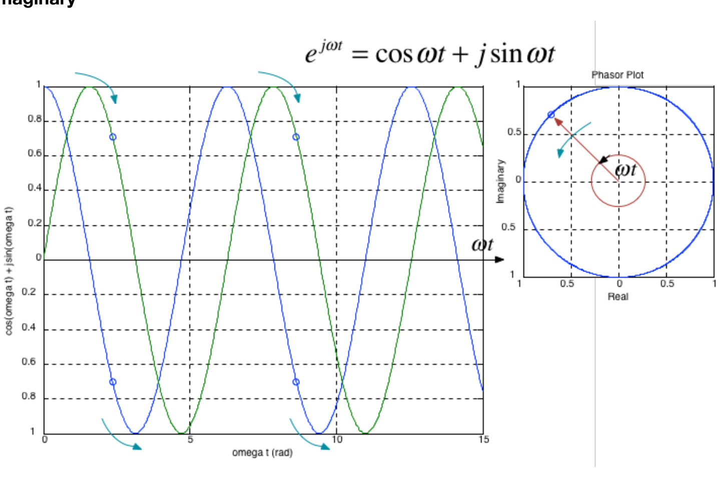
In [2]:
```
%% The decaying exponential
t=linspace(-1,2,1000);
figure
plot(t,exp(t),t,exp(0.*t),t,exp(-t))
axis([-1,2,-1,8])
title('exp(at) -- a real')
xlabel('t (s)')
ylabel('exp(t) and exp(-t)')
legend('exp(t)','exp(0)','exp(-t)')
grid
hold off
```



You can regenerate this image generated with this Matlab script: expon.m.

- When $a < 0$ the response is a decaying exponential (red line in plot)
- When $a = 0$ $e^{at} = 1$ -- essentially a model of DC
- When $a > 0$ the response is an *unbounded* increasing exponential (blue line in plot)

Case when a is imaginary



This is the case that helps us simplify the computation of sinusoidal Fourier series.

It was Leonhard Euler who discovered the formula visualized above.

Some important values of ωt

These are useful when simplifying expressions that result from integrating functions that involve the imaginary exponential

Give the following:

- $e^{j\omega t}$ when $\omega t = 0$
- $e^{j\omega t}$ when $\omega t = \pi/2$
- $e^{j\omega t}$ when $\omega t = \pi$
- $e^{j\omega t}$ when $\omega t = 3\pi/2$
- $e^{j\omega t}$ when $\omega t = 2\pi$

Case where a is complex

We shall not say much about this case except to note that the Laplace transform equation includes such a number. The variable s in the Laplace Transform

$$\int_0^\infty f(t)e^{-st}\,dt$$

is a *complex exponential*.

The consequences of a complex s have particular significance in the development of system stability theories and in control systems analysis and design. Look out for them in EG-243.

Two Other Important Properties

By use of trig. identities, it is relatively straight forward to show that:

$$\cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

and

$$\sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{j2}$$

We can use this result to convert the *Trigonometric Fourier Series* into an *Exponential Fourier Series* which has only one integral term to solve per harmonic.

The Exponential Fourier Series

As as stated in the notes on the Trigonometric Fourier Series any periodic waveform $f(t)$ can be represented as

$$\begin{aligned} f(t) = & \frac{1}{2}a_0 + a_1\cos\Omega_0 t + a_2\cos 2\Omega_0 t + \cdots \\ & + b_1\sin\Omega_0 t + b_2\sin 2\Omega_0 t + \cdots \end{aligned}$$

If we replace the $\cos$ and $\sin$ terms with their imaginary expontial equivalents:

$$\begin{aligned} f(t) = & \frac{1}{2}a_0 + a_1\left(\frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}\right) + a_2\left(\frac{e^{j2\Omega_0 t} + e^{-j2\Omega_0 t}}{2}\right) + \cdots \\ & + b_1\left(\frac{e^{j\Omega_0 t} - e^{-j\Omega_0 t}}{j2}\right) + b_2\left(\frac{e^{j2\Omega_0 t} - e^{-j2\Omega_0 t}}{j2}\right) + \cdots \end{aligned}$$

Grouping terms with same exponents

$$f(t) = \cdots + \left(\frac{a_2}{2} - \frac{b_2}{j2}\right)e^{-j2\Omega_0 t} + \left(\frac{a_1}{2} - \frac{b_1}{j2}\right)e^{-j\Omega_0 t} + \frac{1}{2}a_0 + \left(\frac{a_1}{2} + \frac{b_1}{j2}\right)e^{j\Omega_0 t} + \left(\frac{a_2}{2} + \frac{b_2}{j2}\right)e^{j2\Omega_0 t} + \cdots$$

New coefficents

The terms in parentheses are usually denoted as

$$\begin{aligned} C_{-k} &= \frac{1}{2}\left(a_k - \frac{b_k}{j}\right) = \frac{1}{2}(a_k + jb_k) \\ C_k &= \frac{1}{2}\left(a_k + \frac{b_k}{j}\right) = \frac{1}{2}(a_k - jb_k) \\ C_0 &= \frac{1}{2}a_0 \end{aligned}$$

The Exponential Fourier Series is

$$f(t) = \cdots + C_{-2}e^{-j2\Omega_0 t} + C_{-1}e^{-j\Omega_0 t} + C_0 + C_1 e^{j\Omega_0 t} + C_2 e^{j2\Omega_0 t} + \cdots$$

or more compactly

$$f(t) = \sum_{k=-n}^{n} C_k e^{jk\Omega_0 t}$$

Important

The C_k coefficents, except for C_0 are *complex* and appear in conjugate pairs so

$$C_{-k} = C_k^*$$

Evaluation of the complex coefficients

The coefficients are obtained from the following expressions*:

$$C_k = \frac{1}{2\pi}\int_0^{2\pi} f(\Omega_0 t)e^{-jk(\Omega_0 t)}\,d(\Omega_0 t)$$

or

$$C_k = \frac{1}{T}\int_0^T f(t)e^{-jk\Omega_0 t}\,dt$$

These are much easier to derive and compute than the equivalent Trigonemetric Fourier Series coefficients.

* The analysis that leads to this result is provided between pages 7-31 and 7-32 of the text book {cite}`karris`. It is not a difficult proof, but we are more interested in the result.

Trigonometric Fourier Series from Exponential Fourier Series

By substituting C_{-k} and C_k back into the original expansion

$$C_k + C_{-k} = \frac{1}{2}(a_k - jb_k + a_k + jb_k)$$

so

$$a_k = C_k + C_{-k}$$

Similarly

$$C_k - C_{-k} = \frac{1}{2}(a_k - jb_k - a_k - jb_k)$$

so

$$b_k = j\left(C_k - C_{-k}\right)$$

Thus we can easily go back to the Trigonetric Fourier series if we want to.

Symmetry in Exponential Fourier Series

Since the coefficients of the Exponential Fourier Series are complex numbers, we can use symmetry to determine the form of the coefficients and thereby simplify the computation of series for wave forms that have symmetry.

Even Functions

For even functions, all coefficients C_k are real.

Proof

Recall

$$C_{-k} = \frac{1}{2}\left(a_k - \frac{b_k}{j}\right) = \frac{1}{2}(a_k + jb_k)$$

and

$$C_k = \frac{1}{2}\left(a_k + \frac{b_k}{j}\right) = \frac{1}{2}(a_k - jb_k)$$

From knowledge of the trig. fourier series, even functions have no sine terms so the b_k coefficients are 0. Therefore both C_{-k} and C_k are real.

Odd Functions

For odd functions, all coefficients C_k are imaginary.

By a similar argument, all odd functions have no cosine terms so the a_k coefficients are 0. Therefore both C_{-k} and C_k are imaginary.

Half-wave symmetry

If there is *half-wave symmetry*, $C_k = 0$ for k even.

For proof see notes

Proof

From Trigonometric Fourier Series, if there is half-wave symmetry, all even harnonics are zero, thus both a_k and b_k are zero for k even. Hence C_{-k} and C_k are also zero when k is even.

No symmetry

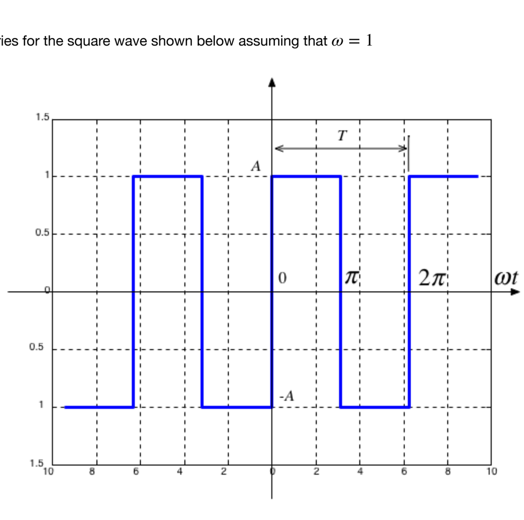
If there is no symmetry the Exponential Fourier Series of $f(t)$ is complex.

Relation of C_{-k} to C_k

$C_{-k} = C_k^*$ **always**

Example 1

Compute the Exponential Fourier Series for the square wave shown below assuming that $\omega = 1$



Solved in in Class

Some questions for you

- Square wave is an [**odd/even/neither**] function?
- DC component is [**zero/non-zero**]?
- Square wave [**has/does not have**] half-wave symmetry?

Hence

- C_0 =[**?**]
- Coefficients C_k are [**real/imaginary/complex**]?
- Subscripts k are [**odd only/even only/both odd and even**]?
- What is the integral that needs to be solved for C_k?

Solution to example 1

$$\begin{aligned} & \frac{1}{2\pi}\left[\int_0^\pi Ae^{-jk(\Omega_0 t)}\,d(\Omega_0 t) + \int_\pi^{2\pi}(-A)e^{-jk(\Omega_0 t)}\,d(\Omega_0 t)\right] = \frac{1}{2\pi}\left[\frac{A}{-jk}e^{-jk(\Omega_0 t)}\Big|_0^\pi + \frac{-A}{-jk}e^{-jk(\Omega_0 t)}\Big|_\pi^{2\pi}\right] \\ & = \frac{1}{2\pi}\left[\frac{A}{-jk}\left(e^{-jk\pi} - 1\right) + \frac{A}{jk}\left(e^{-j2k\pi} - e^{-jk\pi}\right)\right] = \frac{A}{2j\pi k}\left(1 - e^{-jk\pi} + e^{-j2k\pi} - e^{-jk\pi}\right) \\ & \frac{A}{2j\pi k}\left(e^{-j2k\pi} - 2e^{-jk\pi} - 1\right) = \frac{A}{2j\pi k}\left(e^{-jk\pi} - 1\right)^2 \end{aligned}$$

For n odd*, $e^{-jk\pi} = -1$. Therefore

$$\underset{k=\text{odd}}{C_k} = \frac{A}{2j\pi k}\left(e^{-jk\pi} - 1\right)^2 = \frac{A}{2j\pi k}(-1-1)^2 = \frac{A}{2j\pi k}(-2)^2 = \frac{2A}{j\pi k}$$

* You may want to verify that $C_0 = 0$ and

$$\underset{k=\text{even}}{C_k} = 0.$$

Exponential Fourier series for the square wave with odd symmetry

From the definition of the exponential Fourier series

$$f(t) = \cdots + C_{-2}e^{-j2\Omega_0 t} + C_{-1}e^{-j\Omega_0 t} + C_0 + C_1 e^{j\Omega_0 t} + C_2 e^{j2\Omega_0 t} + \cdots$$

the *exponential Fourier series for the square wave with odd symmetry* is

$$f(t) = \frac{2A}{j\pi}\left(\cdots - \frac{1}{3}e^{-j3\Omega_0 t} - e^{-j\Omega_0 t} + e^{j\Omega_0 t} + \frac{1}{3}e^{j3\Omega_0 t} + \cdots\right) = \frac{2A}{j\pi}\sum_{k=\text{odd}}\frac{1}{k}e^{jk\Omega_0 t}$$

Note sign change in first two terms. This is due to the fact that $C_{-k} = C_k^*$.

E.g. since $C_3 = 2A/j3\pi$, $C_{-3} = C_3^* = -2A/j3\pi$

Trig. Fourier Series from Exponential Fourier Series

Since

$$f(t) = \frac{2A}{j\pi}\left(\cdots - \frac{1}{3}e^{-j3\Omega_0 t} - e^{-j\Omega_0 t} + e^{j\Omega_0 t} + \frac{1}{3}e^{j3\Omega_0 t} + \cdots\right)$$

gathering terms at each harmonic frequency gives

$$\begin{aligned} f(t) &= \frac{4A}{\pi}\left(\cdots + \left(\frac{e^{j\Omega_0 t} - e^{-j\Omega_0 t}}{2j}\right) + \frac{1}{3}\left(\frac{e^{j3\Omega_0 t} - e^{-j3\Omega_0 t}}{2j}\right) + \cdots\right) \\ &= \frac{4A}{\pi}\left(\sin\Omega_0 t + \frac{1}{3}\sin 3\Omega_0 t + \cdots\right) \\ &= \frac{4A}{\pi}\sum_{k=\text{odd}}\frac{1}{k}\sin k\Omega_0 t. \end{aligned}$$

Computing coefficients of Exponential Fourier Series in MATLAB

Example 2

Verify the result of Example 1 using MATLAB.

Solution to example 2

Solution: See efs_sqw.m.

EFS_SQW

Calculates the Exponential Fourier for a Square Wave with Odd Symmetry.

In [3]:
```
cd ../matlab
format compact
```

Set up parameters

In [4]:
```
syms t A;

tau = 1;
T0 = 2*pi; % w = 2*pi*f -> t = 2*pi/omega
k_vec = [-5:5];
```

Define f(t)

IMPORTANT: the signal definition must cover [0 to T0]

In [5]:
```
xt = A*(heaviside(t)-heaviside(t-T0/2)) - A*(heaviside(t-T0/2)-heaviside(t-T0));
```

Compute EFS

In [6]:
```
[X, w] = FourierSeries(xt, T0, k_vec)
```

```
X =
[(A*2i)/(5*pi), 0, (A*2i)/(3*pi), 0, (A*2i)/pi, 0, -(A*2i)/pi, 0, -(A*2i)/(3*pi), 0, -(A*2i)/(5*pi)]
w =
    -5    -4    -3    -2    -1     0     1     2     3     4     5
```

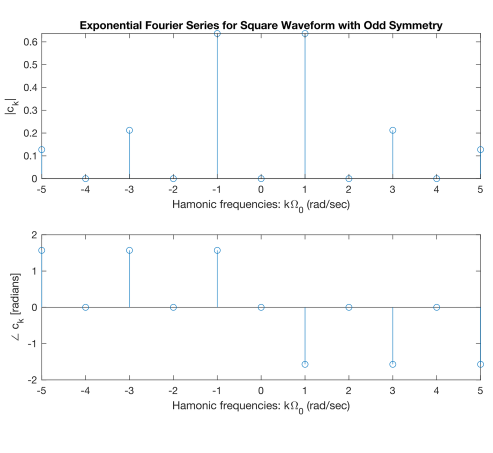
Plot the numerical results from MATLAB calculation.

Convert symbolic to numeric result

In [7]:
```
Xw = subs(X,A,1);
```

Plot

In [8]:
```
subplot(211)
stem(w,abs(Xw), 'o-');
title('Exponential Fourier Series for Square Waveform with Odd Symmetry')
xlabel('Hamonic frequencies: k\Omega_0 (rad/sec)');
ylabel('|c_k|');
subplot(212)
stem(w,angle(Xw), 'o-');
xlabel('Hamonic frequencies: k\Omega_0 (rad/sec)');
ylabel('\angle c_k [radians]');
```



Summary

- Exponents and Euler's Equation
- The exponential Fourier series
- Symmetry in Exponential Fourier Series
- Example

Answers to in-class problems

Some important values of ωt - Solution

- When $\omega t = 0$: $e^{j\omega t} = e^{j0} = 1$
- When $\omega t = \pi/2$: $e^{j\omega t} = e^{j\pi/2} = j$
- When $\omega t = \pi$: $e^{j\omega t} = e^{j\pi} = -1$
- When $\omega t = 3\pi/2$: $e^{j\omega t} = e^{j3\pi/2} = -j$
- When $\omega t = 2\pi$: $e^{j\omega t} = e^{j2\pi} = e^{j0} = 1$

It is also worth being aware that $n\omega t$, when n is an integer, produces rotations that often map back to the simpler cases given above. For example see $e^{j2\pi}$ above.

Some answers for you

- Square wave is an **odd** function!
- DC component is **zero**!
- Square wave **has** half-wave symmetry!

Hence

- $C_0 = 0$
- Coefficients C_k are **imaginary**!
- Subscripts k are **odd only**!
- What is the integral that needs to be solved for C_k?

$$C_k = \frac{1}{2\pi}\int_0^{2\pi} f(\Omega_0 t)e^{-jk(\Omega_0 t)}\,d(\Omega_0 t) = \frac{1}{2\pi}\left[\int_0^\pi Ae^{-jk(\Omega_0 t)}\,d(\Omega_0 t) + \int_\pi^{2\pi}(-A)e^{-jk(\Omega_0 t)}\,d(\Omega_0 t)\right]$$