Unit 3.5: The Impulse Response and Convolution

Contents

Colophon

An annotatable worksheet for this presentation is available as **Worksheet 8**.

- The source code for this page is laplace_transform/5/convolution.ipynb.
- You can view the notes for this presentation as a webpage (HTML).
- This page is downloadable as a PDF file.

Scope and Background Reading

This section is an introduction to the impulse response of a system and time convolution. Together, these can be used to determine a Linear Time Invariant (LTI) system's time response to any signal.

As we shall see, in the determination of a system's response to a signal input, time convolution involves integration by parts and is a tricky operation. But time convolution becomes multiplication in the Laplace Transform domain, and is much easier to apply.

The material in this presentation and notes is based on Chapter 6 of Karris[Karris, 2012].

Agenda

The material to be presented is:

- Even and Odd Functions of Time
- Time Convolution

- Graphical Evaluation of the Convolution Integral
- System Response by Laplace

Even and Odd Functions of Time

(**This should be revision!**)

We need to be reminded of *even* and *odd* functions so that we can develop the idea of time convolution which is a means of determining the time response of any system for which we know its *impulse response* to any signal.

The development requires us to find out if the Dirac delta function ($\delta(t)$) is an *even* or an *odd* function of time.

Even Functions of Time

A function $f(t)$ is said to be an *even function* of time if the following relation holds

$$f(-t) = f(t)$$

that is, if we relace t with $-t$ the function $f(t)$ does not change.

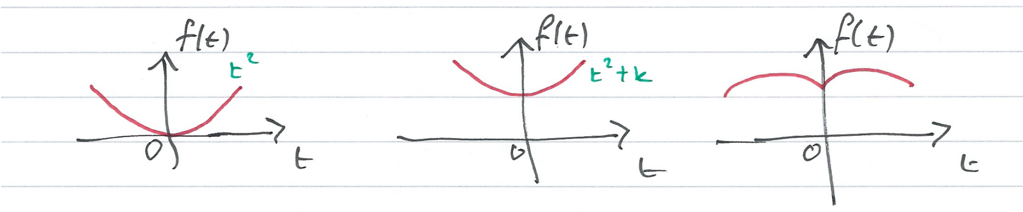
Polynomials with even exponents only, and with or without constants, are even functions.

For example:

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \ldots$$

is even.

Other Examples of Even Functions



Odd Functions of Time

A function $f(t)$ is said to be an *odd function* of time if the following relation holds

$$-f(-t) = f(t)$$

that is, if we relace t with $-t$, we obtain the negative of the function $f(t)$.

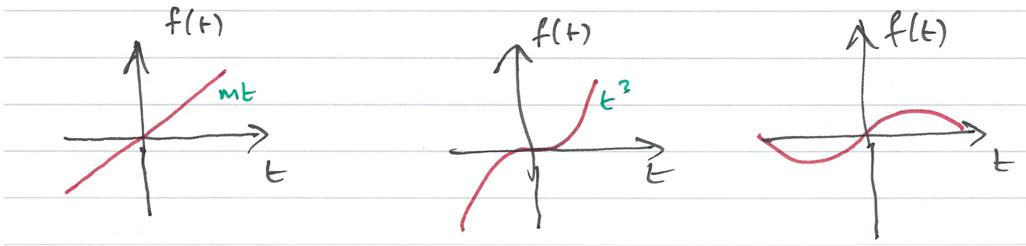
Polynomials with odd exponents only, and no constants, are odd functions.

For example:

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots$$

is odd.

Other Examples of Odd Functions



Observations

- For odd functions $f(0) = 0$.
- If $f(0) = 0$ we should not conclude that $f(t)$ is an odd function. *c.f.* $f(t) = t^2$ is even, not odd.
- The product of *two even* or *two odd* functions is an even function.
- The product of an even and an odd function, is an odd function.

In the following $f_e(t)$ will denote an even function and $f_o(t)$ an odd function.

Time integrals of even and odd functions

For an even function $f_e(t)$

$$\int_{-T}^{T} f_e(t)dt = 2\int_{0}^{T} f_e(t)dt$$

For an odd function $f_o(t)$

$$\int_{-T}^{T} f_o(t)dt = 0$$

Even/Odd Representation of an Arbitrary Function

A function $f(t)$ that is neither even nor odd can be represented as an even function by use of:

$$f_e(t) = \frac{1}{2}[f(t) + f(-t)]$$

or as an odd function by use of:

$$f_o(t) = \frac{1}{2}[f(t) - f(-t)]$$

Adding these together, an abitrary signal can be represented as

$$f(t) = f_e(t) + f_o(t)$$

That is, any function of time can be expressed as the sum of an even and an odd function.

Example 1

Is the Dirac delta $\delta(t)$ an *even* or an *odd* function of time?

We'll decide in class.

Solution to example 1

Let $f(t)$ be an arbitrary function of time that is continous at $t = t_0$. Then by the sifting property of the delta function

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

and for $t_0 = 0$

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

Also for an even function $f_e(t)$

$$\int_{-\infty}^{\infty} f_e(t)\delta(t)dt = f_e(0)$$

and for an odd function $f_o(t)$

$$\int_{-\infty}^{\infty} f_o(t)\delta(t)dt = f_o(0)$$

Even or odd?

An odd function $f_o(t)$ evaluated at $t = 0$ is zero, that is $f_o(0) = 0$.

Hence

$$\int_{-\infty}^{\infty} f_o(t)\delta(t)dt = f_o(0) = 0$$

Hence the product $f_o(t)\delta(t)$ is odd function of t.

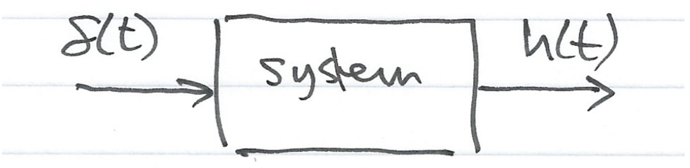
Since $f_o(t)$ is odd, $\delta(t)$ must be even because only an *even* function multiplied by an *odd* function can result in an *odd* function.

(Even times even or odd times odd produces an even function. See earlier slide)

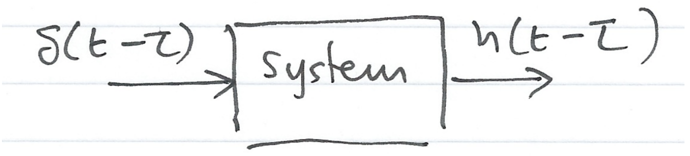
Time Convolution

Consider a system whose input is the Dirac delta ($\delta(t)$), and its output is the ***impulse response*** $h(t)$.

We can represent the input-output relationship as a block diagram

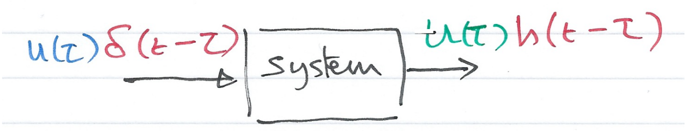


In general



Add an arbitrary input

Let $u(t)$ be any input whose value at $t = \tau$ is $u(\tau)$, Then because of the sampling property of the delta function



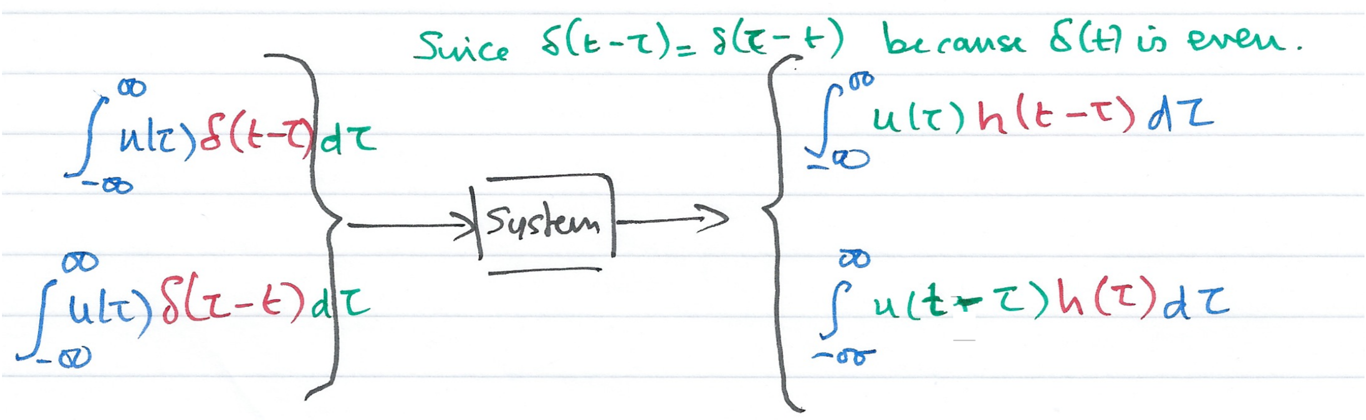
(output is $u(\tau)h(t-\tau)$)

Integrate both sides

Integrating both sides over all values of τ ($-\infty < \tau < \infty$) and making use of the fact that the delta function is even, i.e.

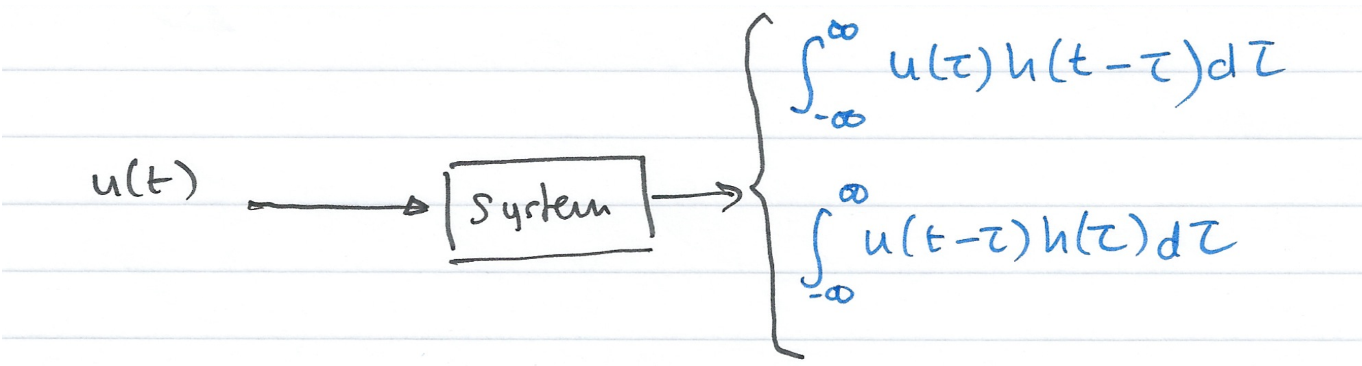
$$\delta(t-\tau) = \delta(\tau - t)$$

we have:



Use the sifting property of delta

The second integral on the left side reduces to $u(t)$



The Convolution Integral

The integral

$$\int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau$$

or

$$\int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau$$

is known as the *convolution integral*; it states that if we know the impulse response of a system, we can compute its time response to any input by using either of the integrals.

The convolution integral is usually written $u(t) * h(t)$ or $h(t) * u(t)$ where the asterisk $(*)$ denotes convolution.

Graphical Evaluation of the Convolution Integral

The convolution integral is most conveniently evaluated by a graphical evaluation. The text book gives three examples (6.4-6.6) which we will demonstrate in class using a graphical visualization tool developed by Teja Muppirala of the Mathworks and updated by Rory Adams for this module.

The tool: convolutiondemo.m (see license.txt).

```
clear all
cd ../matlab/convolution_demo
pwd
```

```
ans =

    '/Users/eechris/code/src/github.com/cpjobling/eg-247-
textbook/laplace_transform/matlab/convolution_demo'
```

```
convolutiondemo % ignore warnings
```

Convolution by Graphical Method - Summary of Steps

For simplicity, we give the rules for $u(t)$, but the procedure is the same if we reflect and slide $h(t)$

1. Substitute $u(t)$ with $u(\tau)$ – this is a simple change of variable. It doesn't change the definition of $u(t)$.

1. Reflect $u(\tau)$ about the vertical axis to form $u(-\tau)$

1. Slide $u(-\tau)$ to the right a distance t to obtain $u(t-\tau)$

1. Multiply the two signals to obtain the product $u(t-\tau)h(\tau)$

1. Integrate the product over all τ from $-\infty$ to ∞.

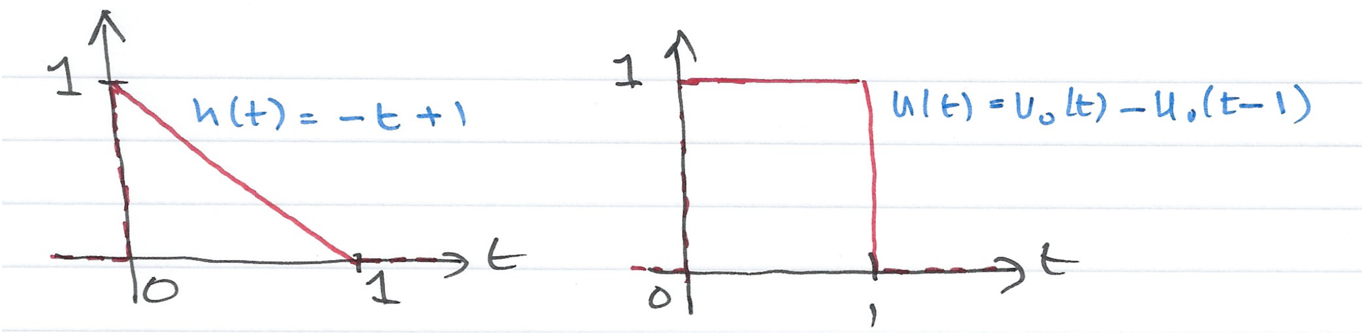
Examples

We will do these live in class.

Example 2

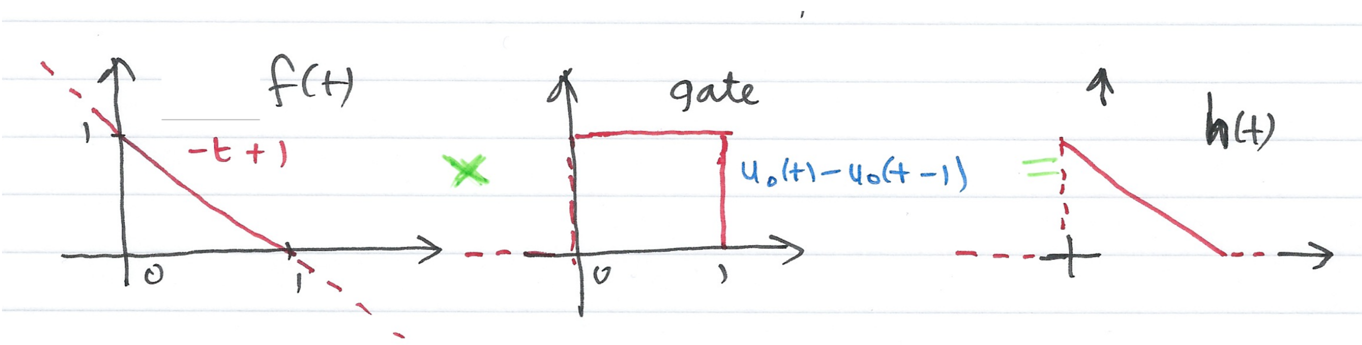
(This is example 6.4 in the textbook)

The signals $h(t)$ and $u(t)$ are shown below. Compute $h(t) * u(t)$ using the graphical technique.



h(t)

The signal $h(t)$ is the straight line $f(t) = -t + 1$ but this is defined only between $t = 0$ and $t = 1$. We thus need to gate the function by multiplying it by $u_0(t) - u_0(t-1)$ as illustrated below:



convolutiondemo settings

- Let `f = heaviside(t)-heaviside(t-1) % u(t)`
- Let `g = (t-1)*(heaviside(t)-heaviside(t-1)) % h(t)`
- Set range $-2 < \tau < -2$

Alternative solution

The original version of `convolutiondemo` didn't support the `heaviside` function so to prepare this problem for evaluation in the `convolutiondemo` tool, we needed to determine the Laplace Transforms of $h(t)$ and $u(t)$.

Thus

$$h(t) \Leftrightarrow H(s)$$

$$h(t) = (-t+1)(u_0(t) - u_0(t-1)) = (-t+1)u_0(t) - (-(t-1)u_0(t-1)) = -tu_0(t)$$

$$-tu_0(t) + u_0(t) + (t-1)u_0(t-1) \Leftrightarrow -\frac{1}{s^2} + \frac{1}{s} + \frac{e^{-s}}{s^2}$$

$$H(s) = \frac{s + e^{-s} - 1}{s^2}$$

u(t)

The input $u(t)$ is the gating function:

$$u(t) = u_0(t) - u_0(t-1)$$

so

$$U(s) = \frac{1}{s} - \frac{e^{-s}}{s} = \frac{1-e^{-s}}{s}$$

convolutiondemo settings

- Let `f = (1 - exp(-s))/s % U(s)`
- Let `g= (s + exp(-s) - 1)/s^2 % H(s)`
- Set range $-2 < \tau < -2$

Summary of result

1. For $t < 0$: \$$u(t-\tau)h(\tau) = 0$\$
2. For $t = 0$: $u(t-\tau) = u(-\tau)$ and $u(-\tau)h(\tau) = 0$
3. For $0 < t \le 1$: \$$h * u = \int_0^t (1)(-\tau+1)d\tau = \tau - \tau^2/2\big|_0^t = t - t^2/2$\$
4. For $1 < t \le 2$: \$$h * u = \int_{t-1}^1 (-\tau+1)d\tau = \tau - \tau^2/2\big|_{t-1}^1 = t^2/2 - 2t + 2$\$
5. For $2 \le t$: \$$u(t-\tau)h(\tau) = 0$\$

Example 3

This is example 6.5 from the text book.

$$h(t) = e^{-t}$$

$$u(t) = u_0(t) - u_0(t-1)$$

Answer 3

$$y(t) = \begin{cases} 0 : t \le 0 \\ 1 - e^{-t} :\ 0 < t \le 1 \\ e^{-t}\,(e-1) :\ 1 < t < \infty \end{cases}$$

Check with MATLAB

```
syms t tau
x1=int(exp(-tau),tau,0,t)
```

```
x1 =
```

```
1 - exp(-t)
```

```
x2=int(exp(-tau),tau,t-1,t)
```

```
x2 =
```

```
exp(-t)*(exp(1) - 1)
```

Example 4

This is example 6.6 from the text book.

$$h(t) = 2(u_0(t) - u_0(t-1))$$

$$u(t) = u_0(t) - u_0(t-2)$$

Answer 4

$$y(t) = \begin{cases} 0 : t \le 0 \\ 2t : \ 0 < t \le 1 \\ 2 : \ 1 < t \le 2 \\ -2t + 6 : \ 2 < t \le 3 \\ 0 : \ 3 \le t \end{cases}$$

System Response by Laplace

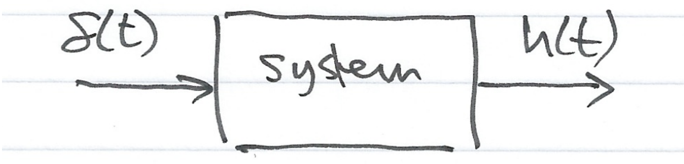
In the discussion of Laplace, we stated that

$$\mathcal{L}\left\{f(t) * g(t)\right\} = F(s)G(s)$$

We can use this property to make the solution of convolution problems even simpler.

Impulse Response and Transfer Functions

Returning to the example we started with



Then the impulse response of the system $h(t)$ will be given by:

$$\mathcal{L}\left\{h(t) * \delta(t)\right\} = H(s)\Delta(s)$$

Where $H(s)$ be the laplace transform of the impulse response of the system $h(t)$. From properties of the Laplace transform we know that

$$\delta(t) \Leftrightarrow 1$$

so that $\Delta(s) = 1$ and

$$h(t) * \delta(t) \Leftrightarrow H(s).1 = H(s)$$

A consequence of this is that the transform of the impulse response $h(t)$ of a system with transfer function $H(s)$ is completely defined by the transfer function itself.

Previously we argued that the response of system with impulse response $h(t)$ was given by the convolution integrals:

$$h(t) * u(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau$$

Thus the Laplace transform of any system subject to an input $u(t)$ is simply

$$Y(s) = H(s)U(s)$$

and

$$y(t) = \mathcal{L}^{-1}\left\{H(s)U(s)\right\}$$

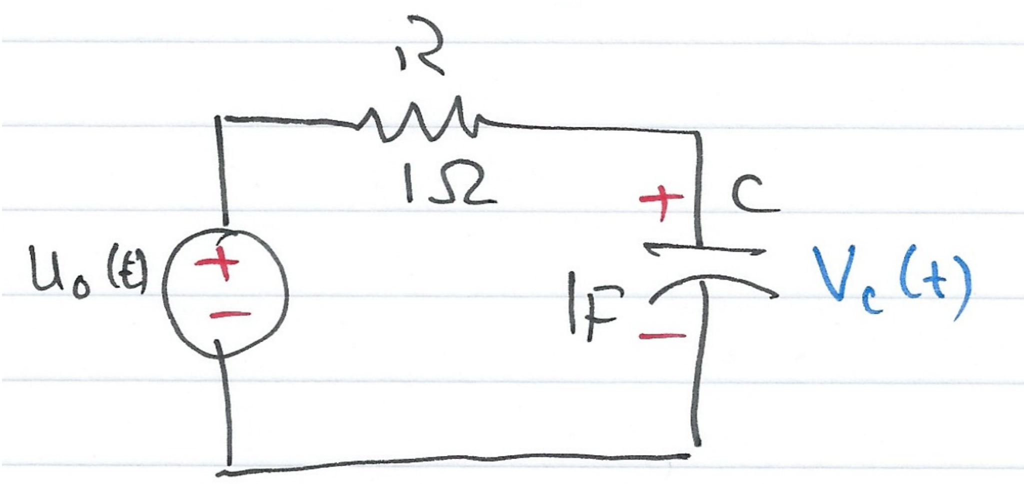
Using tables, solution of a convolution problem by Laplace is usually simpler than using convolution directly.

More Examples

We will work through these in class

Example 5

This is example 6.7 from the textbook.



For the circuit shown above, show that the transfer function of the circuit is:

$$H(s) = \frac{V_c(s)}{V_s(s)} = \frac{1/RC}{s + 1/RC}$$

Hence determine the impulse respone $h(t)$ of the circuit and the response of the capacitor voltage when the input is the unit step function $u_0(t)$ and $v_c(0^-) = 0$.

Assume $C = 1\ \mathrm{F}$ and $R = 1\ \Omega$.

Solution 5a - Impulse response

$$h(t) = \frac{1}{RC}e^{-t/RC}u_0(t)$$

which when $C = 1\ \mathrm{F}$ and $R = 1\ \Omega$ reduces to

You can't use 'macro parameter character #' in math mode

$$u(t) = u_0(t) \Leftrightarrow U(s) = \frac{1}{s}$$

$$y(t) = h(t) * u(t) \Leftrightarrow Y(s) = H(s)U(s) = \left(\frac{1}{s+1}\right) \times \left(\frac{1}{s}\right)$$

By PFE

$$Y(s) = \frac{r_1}{s+1} + \frac{r_2}{s}$$

The residues are $r_1 = -1$, $r_2 = 1$, so

$$Y(s) = -\frac{1}{s+1} + \frac{1}{s} \Leftrightarrow y(t) = \left(1 - e^{-t}\right)u_0(t)$$

Homework

Verify this result using the convolution integral

$$h(t) * u(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau$$

Reference

See Bibliography.

By Dr Chris P. Jobling

This page was created by Dr Chris P. Jobling for Swansea University .