

# Fourier Transforms for Circuit and LTI Systems Analysis

## Colophon

An annotatable worksheet for this presentation is available as [Worksheet 14](https://cpjobling.github.io/eg-247-textbook/fourier_transform/3/worksheet14.html) ([https://cpjobling.github.io/eg-247-textbook/fourier\\_transform/3/worksheet14.html](https://cpjobling.github.io/eg-247-textbook/fourier_transform/3/worksheet14.html)).

- The source code for this page is [content/fourier\\_transform/3/ft3.ipynb](https://github.com/cpjobling/eg-247-textbook/blob/master/content/fourier_transform/3/ft3.ipynb) ([https://github.com/cpjobling/eg-247-textbook/blob/master/content/fourier\\_transform/3/ft3.ipynb](https://github.com/cpjobling/eg-247-textbook/blob/master/content/fourier_transform/3/ft3.ipynb)).
- You can view the notes for this presentation as a webpage (HTML ([https://cpjobling.github.io/eg-247-textbook/fourier\\_transform/3/ft3.html](https://cpjobling.github.io/eg-247-textbook/fourier_transform/3/ft3.html))).
- This page is downloadable as a [PDF](https://cpjobling.github.io/eg-247-textbook/fourier_transform/3/ft3.pdf) ([https://cpjobling.github.io/eg-247-textbook/fourier\\_transform/3/ft3.pdf](https://cpjobling.github.io/eg-247-textbook/fourier_transform/3/ft3.pdf)) file.

In this section we will apply what we have learned about Fourier transforms to some typical circuit problems. After a short introduction, the body of this chapter will form the basis of an examples class.

## Agenda

- The system function
- Examples

## The System Function

### System response from system impulse response

Recall that the convolution integral of a system with impulse response  $h(t)$  and input  $u(t)$  is

$$h(t) * u(t) = \int_{-\infty}^{\infty} h(t - \tau)u(\tau) d\tau.$$

We let

$$g(t) = h(t) * u(t)$$

Then by the time convolution property

$$h(t) * u(t) = g(t) \Leftrightarrow G(\omega) = H(\omega) \cdot U(\omega)$$

## The System Function

We call  $H(\omega)$  the *system function*.

We note that the system function  $H(\omega)$  and the impulse response  $h(t)$  form the Fourier transform pair

$$h(t) \Leftrightarrow H(\omega)$$

## Obtaining system response

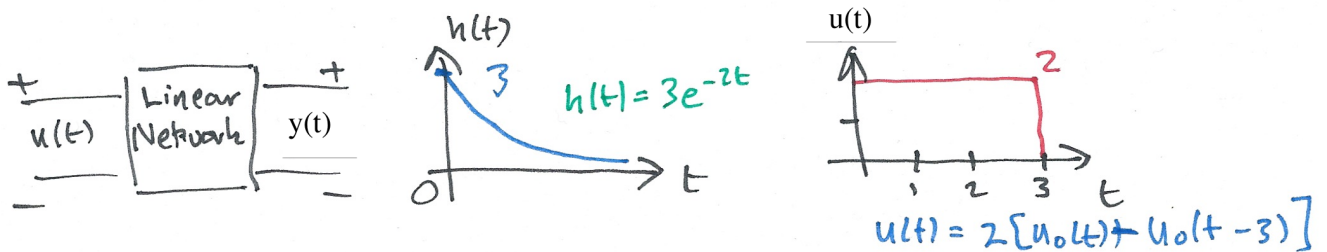
If we know the impulse response  $h(t)$ , we can compute the system response  $g(t)$  of any input  $u(t)$  by multiplying the Fourier transforms of  $H(\omega)$  and  $U(\omega)$  to obtain  $G(\omega)$ . Then we take the inverse Fourier transform of  $G(\omega)$  to obtain the response  $g(t)$ .

1. Transform  $h(t) \rightarrow H(\omega)$
2. Transform  $u(t) \rightarrow U(\omega)$
3. Compute  $G(\omega) = H(\omega) \cdot U(\omega)$
4. Find  $\mathcal{F}^{-1} \{G(\omega)\} \rightarrow g(t)$

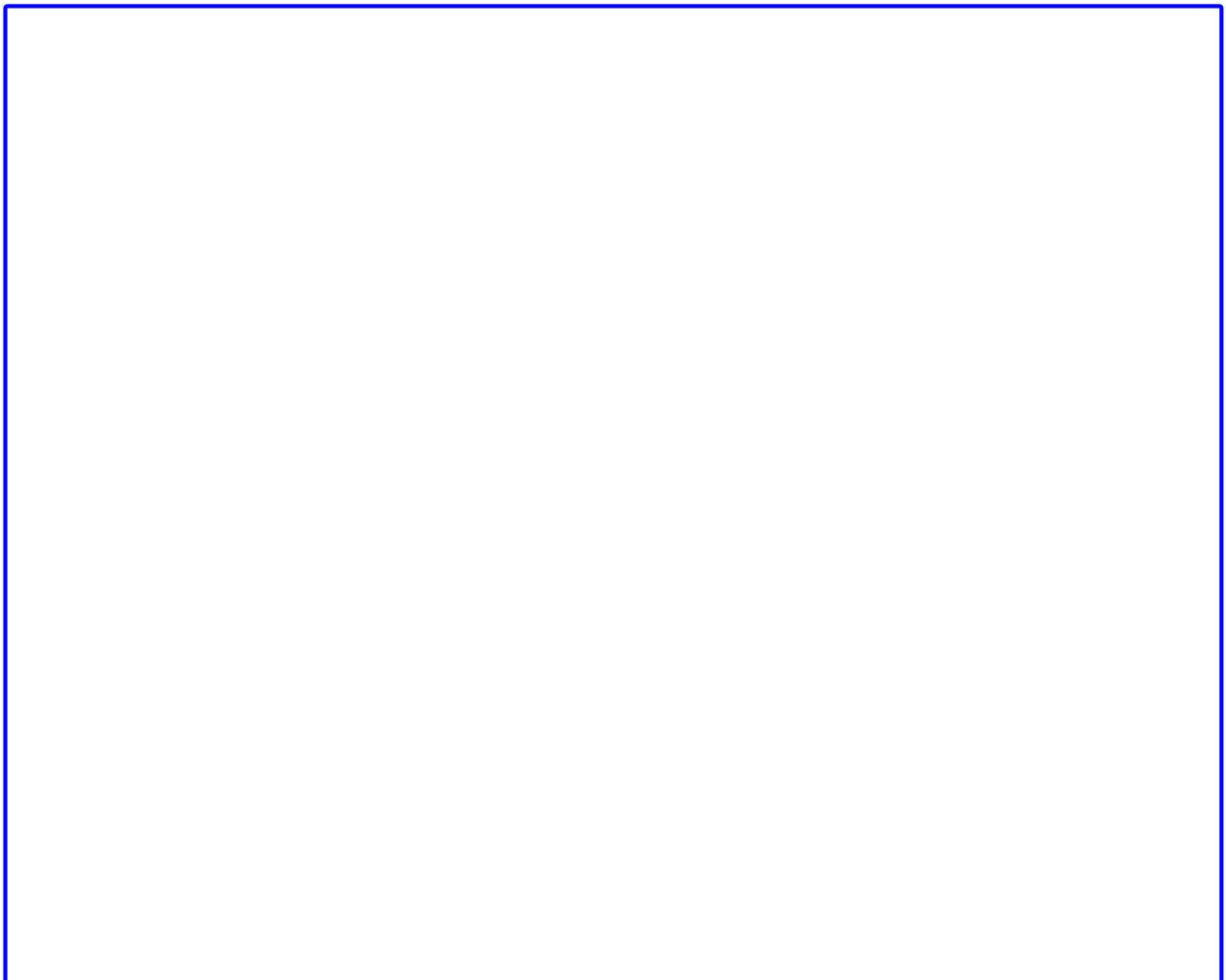
## Examples

## Example 1

Karris example 8.8: for the linear network shown below, the impulse response is  $h(t) = 3e^{-2t}$ . Use the Fourier transform to compute the response  $y(t)$  when the input  $u(t) = 2[u_0(t) - u_0(t - 3)]$ . Verify the result with MATLAB.



## Solution to example 1



## Matlab verification of example 1

```
In [1]: syms t w
        U1 = fourier(2*heaviside(t),t,w)
```

```
U1 =

2*pi*dirac(w) - 2i/w
```

```
In [2]: H = fourier(3*exp(-2*t)*heaviside(t),t,w)
```

```
H =

3/(2 + w*1i)
```

```
In [3]: Y1=simplify(H*U1)
```

```
Y1 =

3*pi*dirac(w) - 6i/(w*(2 + w*1i))
```

```
In [4]: y1 = simplify(ifourier(Y1,w,t))
```

```
y1 =

(3*exp(-2*t)*(sign(t) + 1)*(exp(2*t) - 1))/2
```

Get y2

Substitute  $t - 3$  into  $t$ .

```
In [5]: y2 = subs(y1,t,t-3)
```

```
y2 =

(3*exp(6 - 2*t)*(sign(t - 3) + 1)*(exp(2*t - 6) - 1))/2
```

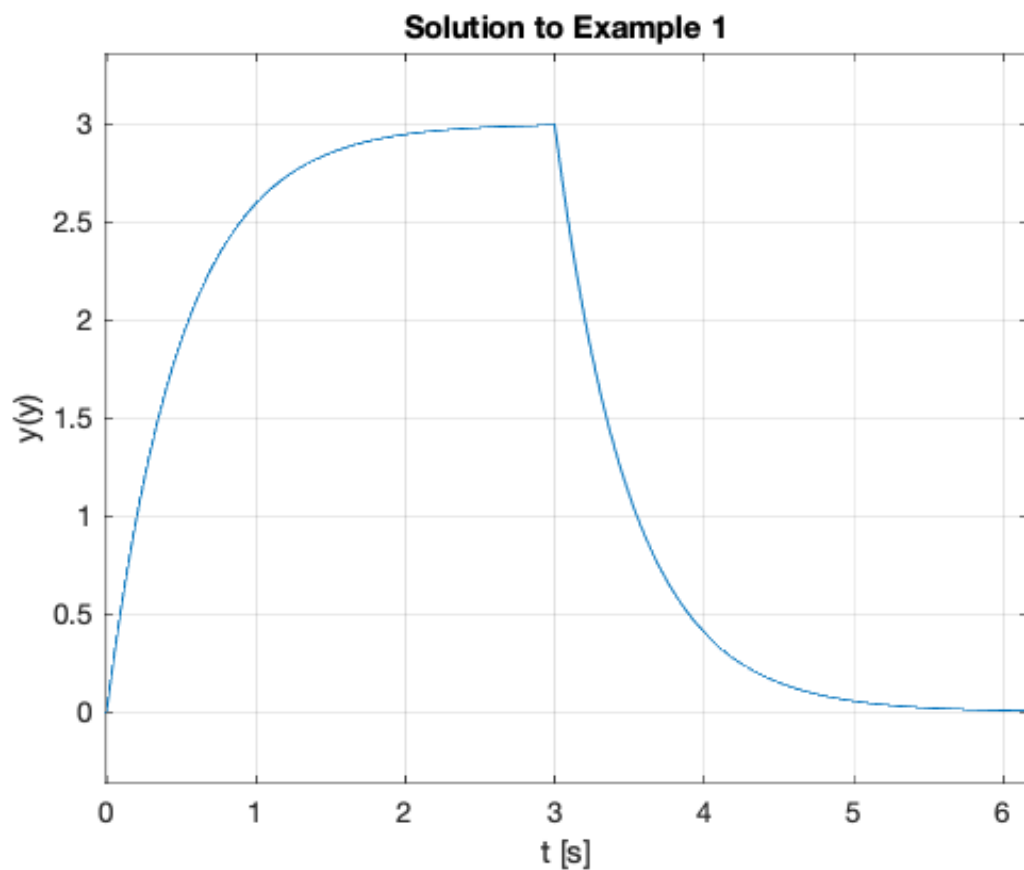
```
In [6]: y = y1 - y2
```

y =

$$(3 \exp(-2t) (\operatorname{sign}(t) + 1) (\exp(2t) - 1)) / 2 - (3 \exp(6 - 2t) (\operatorname{sign}(t - 3) + 1) (\exp(2t - 6) - 1)) / 2$$

Plot result

```
In [7]: ezplot(y)
title('Solution to Example 1')
ylabel('y(y)')
xlabel('t [s]')
grid
```



See [ft3\\_ex1.m](https://cpjobling.github.io/eg-247-textbook/fourier_transform/3/ft3_ex1.m) ([https://cpjobling.github.io/eg-247-textbook/fourier\\_transform/3/ft3\\_ex1.m](https://cpjobling.github.io/eg-247-textbook/fourier_transform/3/ft3_ex1.m))

Result is equivalent to:

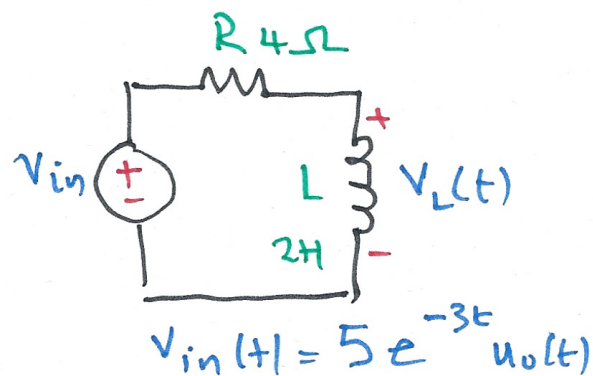
$$y = 3 \cdot \text{heaviside}(t) - 3 \cdot \text{heaviside}(t - 3) + 3 \cdot \text{heaviside}(t - 3) \cdot \exp(6 - 2 \cdot t) - 3 \cdot \exp(-2 \cdot t) \cdot \text{heaviside}(t)$$

Which after gathering terms gives

$$y(t) = 3(1 - 3e^{-2t})u_0(t) - 3(1 - 3e^{-2(t-3)})u_0(t - 3)$$

## Example 2

Karris example 8.9: for the circuit shown below, use the Fourier transform method, and the system function  $H(\omega)$  to compute  $V_L(t)$ . Assume  $i_L(0^-) = 0$ . Verify the result with Matlab.



## Solution of example 2



## Matlab verification of example 2

```
In [8]: syms t w  
H = j*w/(j*w + 2)
```

H =

$(w*1i)/(2 + w*1i)$

```
In [9]: Vin = fourier(5*exp(-3*t)*heaviside(t),t,w)
```

Vin =

$$5/(3 + w*1i)$$

```
In [10]: Vout=simplify(H*Vin)
```

Vout =

$$(w*5i)/((2 + w*1i)*(3 + w*1i))$$

```
In [11]: vout = simplify(iffourier(Vout,w,t))
```

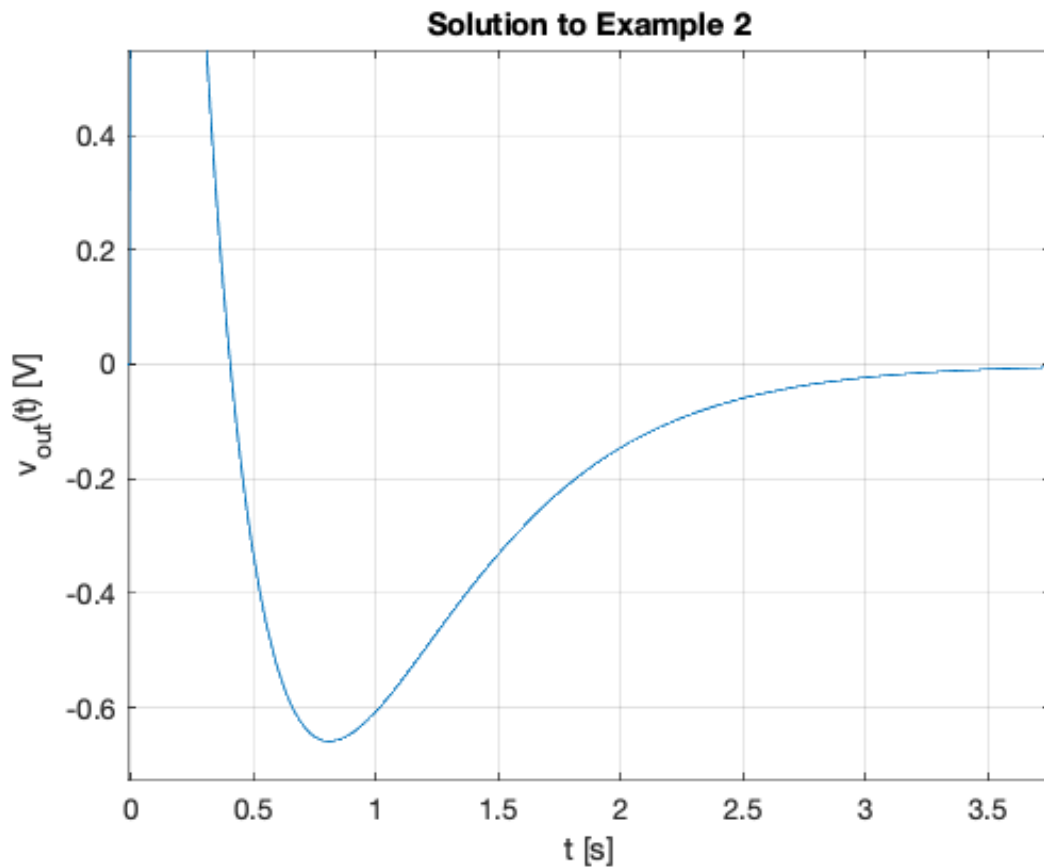
vout =

$$-(5*\exp(-3*t)*(sign(t) + 1)*(2*\exp(t) - 3))/2$$

Plot result



```
In [12]: ezplot(vout)
         title('Solution to Example 2')
         ylabel('v_{out}(t) [V]')
         xlabel('t [s]')
         grid
```



See [ft3\\_ex2.m \(https://cpjobling.github.io/eg-247-textbook/fourier\\_transform/3/ft3\\_ex2.m\)](https://cpjobling.github.io/eg-247-textbook/fourier_transform/3/ft3_ex2.m)

Result is equivalent to:

$$v_{\text{out}} = -5 \cdot \exp(-3 \cdot t) \cdot \text{heaviside}(t) \cdot (2 \cdot \exp(t) - 3)$$

Which after gathering terms gives

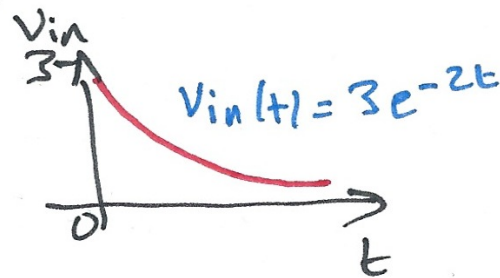
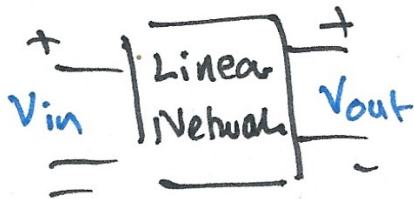
$$v_{\text{out}} = 5 \left( 3e^{-3t} - 2e^{-2t} \right) u_0(t)$$

### Example 3

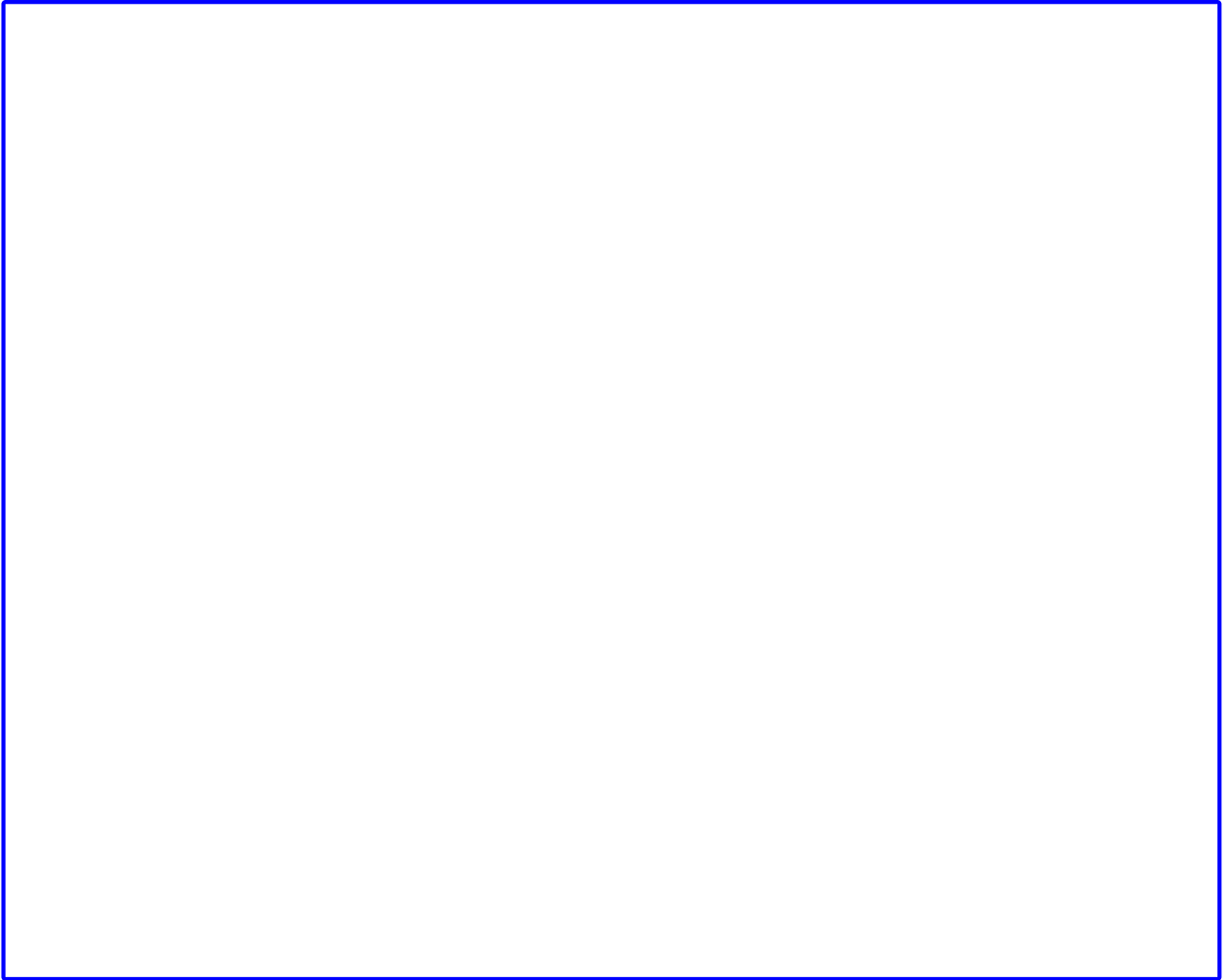
Karris example 8.10: for the linear network shown below, the input-output relationship is:

$$\frac{d}{dt}v_{\text{out}} + 4v_{\text{out}} = 10v_{\text{in}}$$

where  $v_{\text{in}} = 3e^{-2t}$ . Use the Fourier transform method, and the system function  $H(\omega)$  to compute the output  $v_{\text{out}}$ . Verify the result with Matlab.



### Solution to example 3



### Matlab verification of example 3

```
In [13]: syms t w  
H = 10/(j*w + 4)
```

H =

$10/(4 + w*1i)$

```
In [14]: Vin = fourier(3*exp(-2*t)*heaviside(t),t,w)
```

Vin =

$$3/(2 + w*1i)$$

```
In [15]: Vout=simplify(H*Vin)
```

Vout =

$$30/((2 + w*1i)*(4 + w*1i))$$

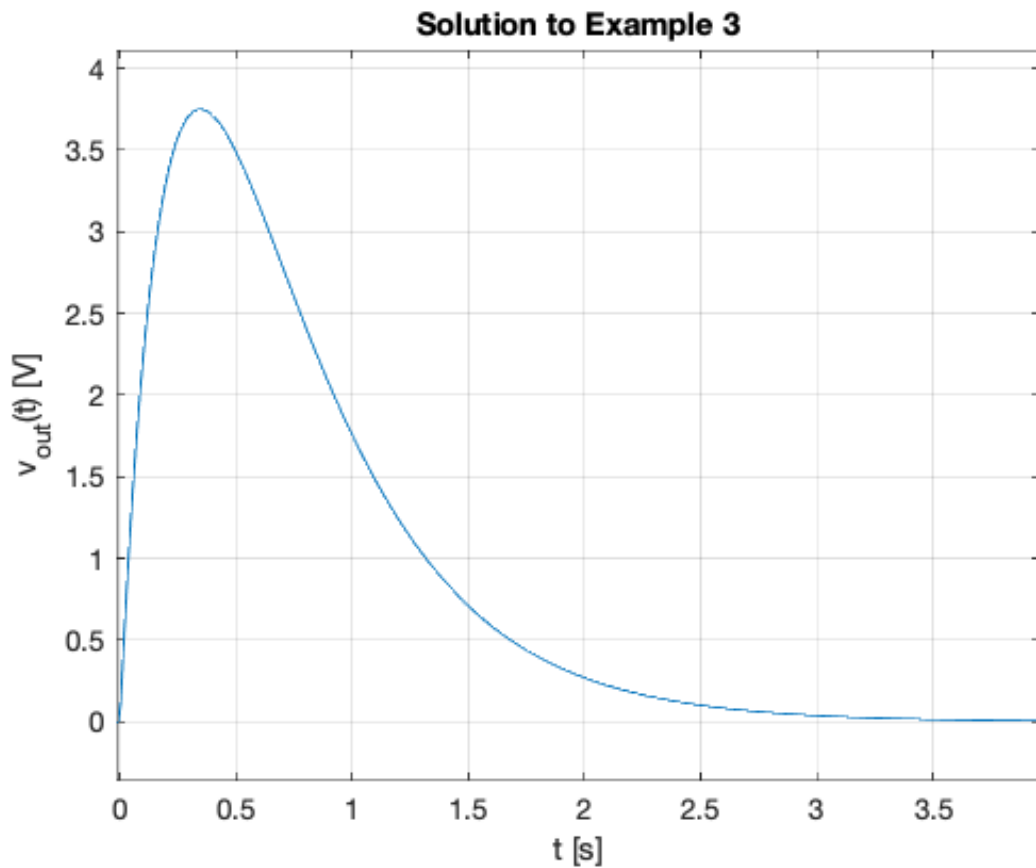
```
In [16]: vout = simplify(ifourier(Vout,w,t))
```

vout =

$$(15*\exp(-4*t)*(sign(t) + 1)*(exp(2*t) - 1))/2$$

Plot result

```
In [17]: ezplot(vout)
         title('Solution to Example 3')
         ylabel('v_{out}(t) [V]')
         xlabel('t [s]')
         grid
```



See [ft3\\_ex3.m](https://cpjobling.github.io/eg-247-textbook/fourier_transform/3/ft3_ex3.m) ([https://cpjobling.github.io/eg-247-textbook/fourier\\_transform/3/ft3\\_ex3.m](https://cpjobling.github.io/eg-247-textbook/fourier_transform/3/ft3_ex3.m))

Result is equivalent to:

$$15 \cdot \exp(-4 \cdot t) \cdot \text{heaviside}(t) \cdot (\exp(2 \cdot t) - 1)$$

Which after gathering terms gives

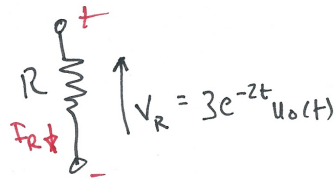
$$v_{\text{out}}(t) = 15 \left( e^{-2t} - e^{-4t} \right) u_0(t)$$

## Example 4

Karris example 8.11: the voltage across a  $1\ \Omega$  resistor is known to be  $V_R(t) = 3e^{-2t}u_0(t)$ . Compute the energy dissipated in the resistor for  $0 < t < \infty$ , and verify the result using Parseval's theorem. Verify the result with Matlab.

Note from [tables of integrals \(https://en.wikipedia.org/wiki/Lists\\_of\\_integrals\)](https://en.wikipedia.org/wiki/Lists_of_integrals)

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C.$$



## Solution to example 4

## Matlab verification of example 4

```
In [18]: syms t w
```

Calculate energy from time function

```
In [19]: Vr = 3*exp(-2*t)*heaviside(t);
R = 1;
Pr = Vr^2/R
Wr = int(Pr,t,0,inf)
```

Pr =

$9 \cdot \exp(-4 \cdot t) \cdot \text{heaviside}(t)^2$

Wr =

$9/4$

Calculate using Parseval's theorem

```
In [20]: Fw = fourier(Vr,t,w)
```

Fw =

$3/(2 + w \cdot 1i)$

```
In [21]: Fw2 = simplify(abs(Fw)^2)
```

Fw2 =

$9/\text{abs}(2 + w \cdot 1i)^2$

```
In [22]: Wr=2/(2*pi)*int(Fw2,w,0,inf)

Wr =

(51607450253003931*pi)/72057594037927936
```

See [ft3\\_ex4.m](https://cpjobling.github.io/eg-247-textbook/fourier_transform/3/ft3_ex4.m) ([https://cpjobling.github.io/eg-247-textbook/fourier\\_transform/3/ft3\\_ex4.m](https://cpjobling.github.io/eg-247-textbook/fourier_transform/3/ft3_ex4.m))

## Solutions

- Example 1: [ft3-ex1.pdf](https://cpjobling.github.io/eg-247-textbook/fourier_transform/solutions/ft3-ex1.pdf) ([https://cpjobling.github.io/eg-247-textbook/fourier\\_transform/solutions/ft3-ex1.pdf](https://cpjobling.github.io/eg-247-textbook/fourier_transform/solutions/ft3-ex1.pdf))
- Example 2: [ft3-ex2.pdf](https://cpjobling.github.io/eg-247-textbook/fourier_transform/solutions/ft3-ex2.pdf) ([https://cpjobling.github.io/eg-247-textbook/fourier\\_transform/solutions/ft3-ex2.pdf](https://cpjobling.github.io/eg-247-textbook/fourier_transform/solutions/ft3-ex2.pdf))
- Example 3: [ft3-ex3.pdf](https://cpjobling.github.io/eg-247-textbook/fourier_transform/solutions/ft3-ex4.pdf) ([https://cpjobling.github.io/eg-247-textbook/fourier\\_transform/solutions/ft3-ex4.pdf](https://cpjobling.github.io/eg-247-textbook/fourier_transform/solutions/ft3-ex4.pdf))
- Example 3: [ft3-ex4.pdf](https://cpjobling.github.io/eg-247-textbook/fourier_transform/solutions/ft3-ex4.pdf) ([https://cpjobling.github.io/eg-247-textbook/fourier\\_transform/solutions/ft3-ex4.pdf](https://cpjobling.github.io/eg-247-textbook/fourier_transform/solutions/ft3-ex4.pdf))