# The Inverse Z-Transform

## Colophon

An annotatable worksheet for this presentation is available as **Worksheet 16 (https://cpjobling.github.io/eg-247-textbook/dt_systems/3/worksheet16.html)**.

- The source code for this page is content/dt_systems/3/i_z_transform.ipynb (https://github.com/cpjobling/eg-247-textbook/blob/master/content/dt_systems/3/i_z_transform.ipynb).
- You can view the notes for this presentation as a webpage (HTML (https://cpjobling.github.io/eg-247-textbook/dt_systems/3/i_z_transform.html)).
- This page is downloadable as a PDF (https://cpjobling.github.io/eg-247-textbook/dt_systems/3/i_z_transform.pdf) file.

## Scope and Background Reading

This session we will talk about the Inverse Z-Transform and illustrate its use through an examples class.

The material in this presentation and notes is based on Chapter 9 (Starting at Section 9.6 (https://ebookcentral.proquest.com/lib/swansea-ebooks/reader.action?docID=3384197&ppg=351)) of {% cite karris %}.

## Agenda

- Inverse Z-Transform

- Examples using PFE

- Examples using Long Division

- Analysis in MATLAB

# Performing the Inverse Z-Transform

The inverse Z-Transform enables us to extract a sequence $f[n]$ from $F(z)$. It can be found by any of the following methods:

- Partial fraction expansion
- The inversion integral
- Long division of polynomials

## Partial fraction expansion

We expand $F(z)$ into a summation of terms whose inverse is known. These terms have the form:

$$k,\ \frac{r_1 z}{z-p_1},\ \frac{r_1 z}{(z-p_1)^2},\ \frac{r_3 z}{z-p_2},\ldots$$

where $k$ is a constant, and $r_i$ and $p_i$ represent the residues and poles respectively, and can be real or complex[1].

#### Notes

1. If complex, the poles and residues will be in complex conjugate pairs

$$\frac{r_i z}{z-p_i}+\frac{r_i^* z}{z-p_i^*}$$

## Step 1: Make Fractions Proper

- Before we expand $F(z)$ into partial fraction expansions, we must first express it as a *proper* rational function.
- This is done by expanding $F(z)/z$ instead of $F(z)$
- That is we expand

$$\frac{F(z)}{z}=\frac{k}{z}+\frac{r_1}{z-p_1}+\frac{r_2}{z-p_2}+\cdots$$

## Step 2: Find residues

- Find residues from

$$r_k = \lim_{z \to p_k}(z - p_k)\frac{F(z)}{z} = (z - p_k)\frac{F(z)}{z}\bigg|_{z=p_k}$$

## Step 3: Map back to transform tables form

- Rewrite $F(z)/z$:

$$z\frac{F(z)}{z} = F(z) = k + \frac{r_1 z}{s - p_1} + \frac{r_2 z}{s - p_2} + \cdots$$

We will work through an example in class.

[Skip next slide in Pre-Lecture]

## Example 1

Karris Example 9.4: use the partial fraction expansion to compute the inverse z-transform of

$$F(z) = \frac{1}{(1 - 0.5z^{-1})(1 - 0.75z^{-1})(1 - z^{-1})}$$

## MATLAB solution for example 1

See example1.mlx (matlab/example1.mlx). (Also available as example1.m (matlab/example1.m).)

Uses MATLAB functions:

- `collect` – expands a polynomial
- `sym2poly` – converts a polynomial into a numeric polymial (vector of coefficients in descending order of exponents)
- `residue` – calculates poles and zeros of a polynomial
- `ztrans` – symbolic z-transform
- `iztrans` – symbolic inverse ze-transform
- `stem` – plots sequence as a "lollipop" diagram

In [1]:
```
clear all
cd matlab
format compact
```

In [2]:
```
syms z n
```

The denoninator of $F(z)$

In [3]:
```
Dz = (z - 0.5)*(z - 0.75)*(z - 1);
```

Multiply the three factors of Dz to obtain a polynomial

In [4]:
```
Dz_poly = collect(Dz)
```

```
Dz_poly =
z^3 - (9*z^2)/4 + (13*z)/8 - 3/8
```

## Make into a rational polynomial

$z^2$

In [5]:
```
num = [0, 1, 0, 0];
```

$z^3 - 9/4z^2 - 13/8z - 3/8$

In [6]:
```
den = sym2poly(Dz_poly)
```

```
den =
    1.0000   -2.2500    1.6250   -0.3750
```

## Compute residues and poles

In [7]:
```
[r,p,k] = residue(num,den);
```

## Print results

- `fprintf` works like the c-language function

In [8]:
```
fprintf('\n')
fprintf('r1 = %4.2f\t', r(1)); fprintf('p1 = %4.2f\n', p(1));...
fprintf('r2 = %4.2f\t', r(2)); fprintf('p2 = %4.2f\n', p(2));...
fprintf('r3 = %4.2f\t', r(3)); fprintf('p3 = %4.2f\n', p(3));
```

```
r1 = 8.00       p1 = 1.00
r2 = -9.00      p2 = 0.75
r3 = 2.00       p3 = 0.50
```

## Symbolic proof

$$f[n] = 2\left(\frac{1}{2}\right)^n - 9\left(\frac{3}{4}\right)^n + 8$$

In [9]:
```
% z-transform
fn = 2*(1/2)^n-9*(3/4)^n + 8;
Fz = ztrans(fn)
```

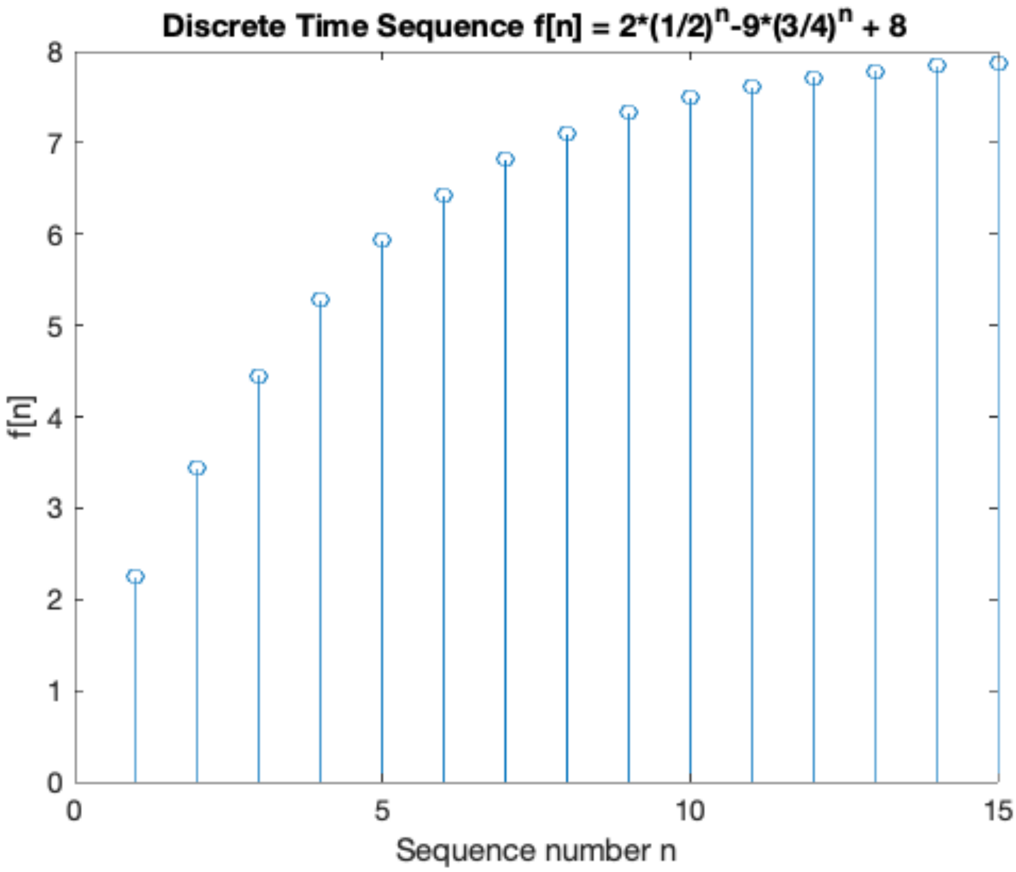
```
Fz =
(8*z)/(z - 1) + (2*z)/(z - 1/2) - (9*z)/(z - 3/4)
```

In [10]:
```
% inverse z-transform
iztrans(Fz)
```

```
ans =
2*(1/2)^n - 9*(3/4)^n + 8
```

## Sequence

In [11]:
```
n = 1:15;
sequence = subs(fn,n);
stem(n,sequence)
title('Discrete Time Sequence f[n] = 2*(1/2)^n-9*(3/4)^n + 8');
ylabel('f[n]')
xlabel('Sequence number n')
```




## Example 2

Karris example 9.5: use the partial fraction expansion method to to compute the inverse z-transform of

$$F(z) = \frac{12z}{(z+1)(z-1)^2}$$

## MATLAB solution for example 2

See example2.mlx (matlab/example2.mlx). (Also available as example2.m (matlab/example2.m).)

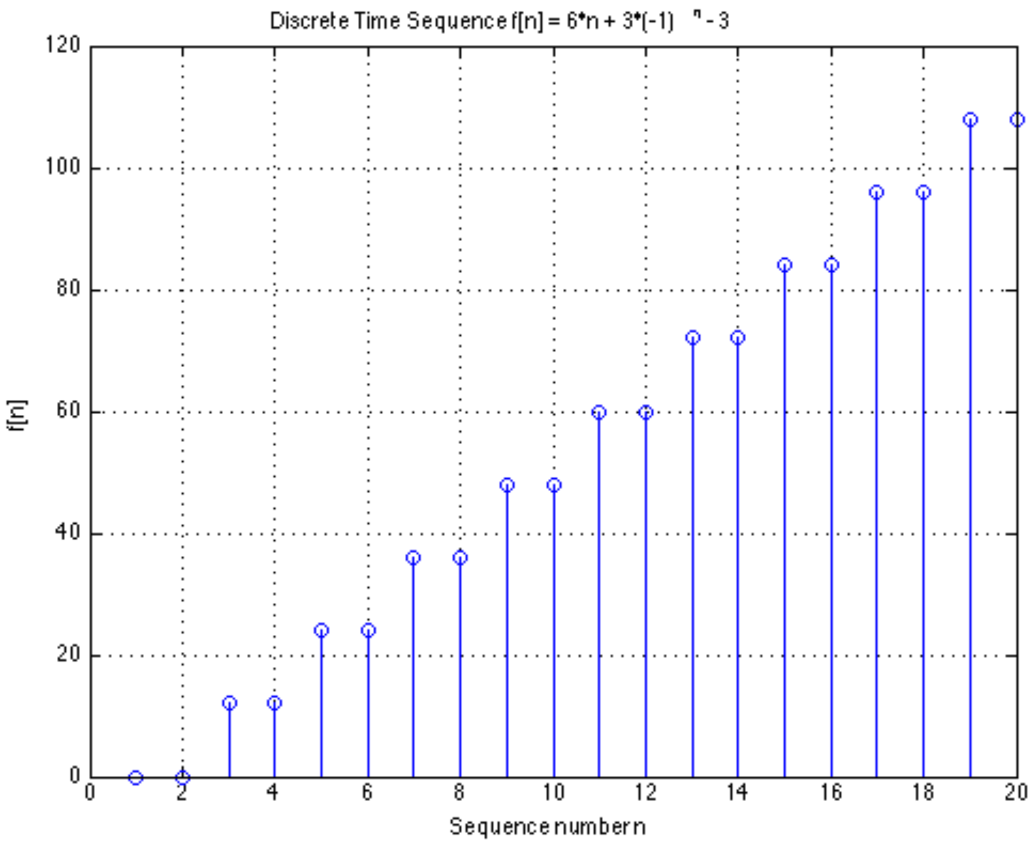
Uses additional MATLAB functions:

- `dimpulse` – computes and plots a sequence $f[n]$ for any range of values of $n$

In [12]:
```
open example2
```

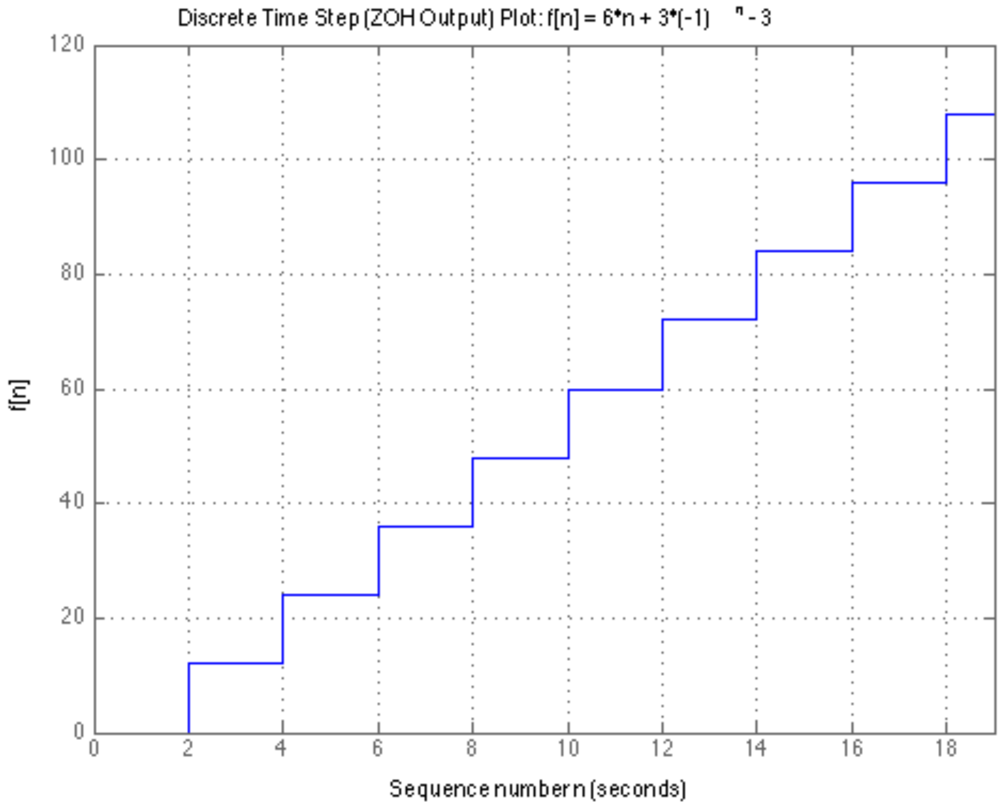
## Results for example 2

#### 'Lollipop' Plot




### 'Staircase' Plot

Simulates output of Zero-Order-Hold (ZOH) or Digital Analogue Converter (DAC)




## Example 3

Karris example 9.6: use the partial fraction expansion method to to compute the inverse z-transform of

$$F(z) = \frac{z+1}{(z-1)(z^2+2z+2)}$$

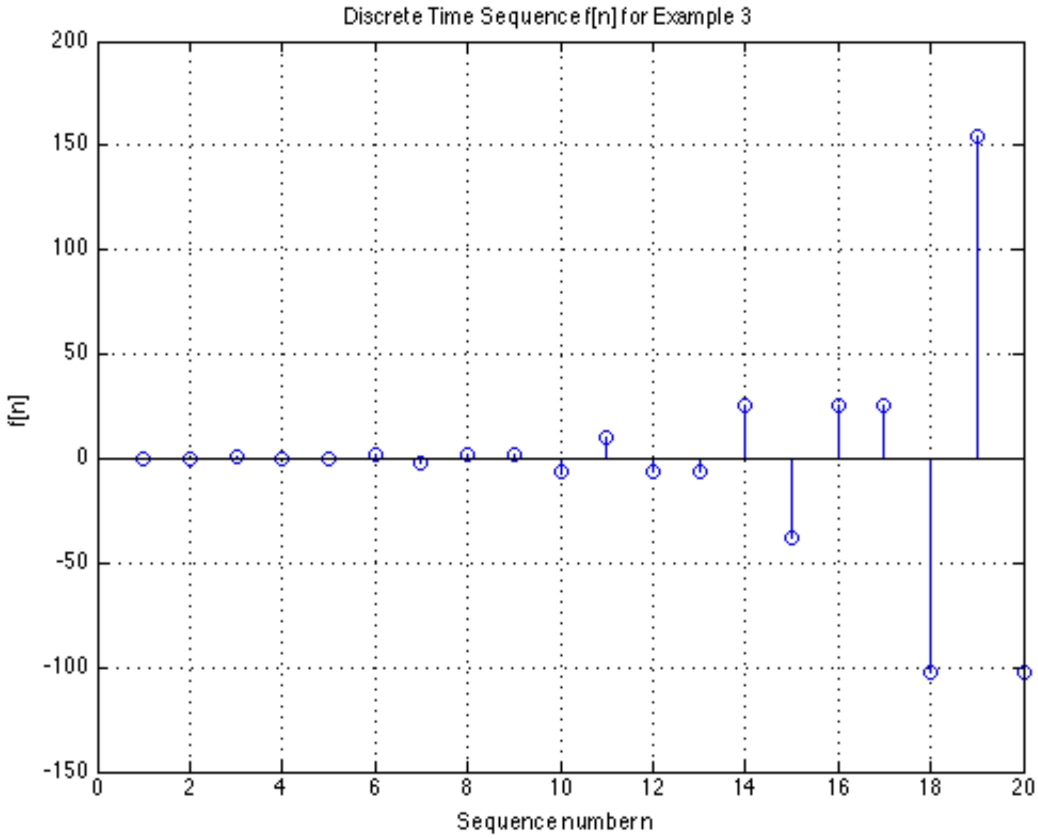
## MATLAB solution for example 3

See [example3.mlx (matlab/example3.mlx)](matlab/example3.mlx). (Also available as [example3.m (matlab/example3.m)](matlab/example3.m).)

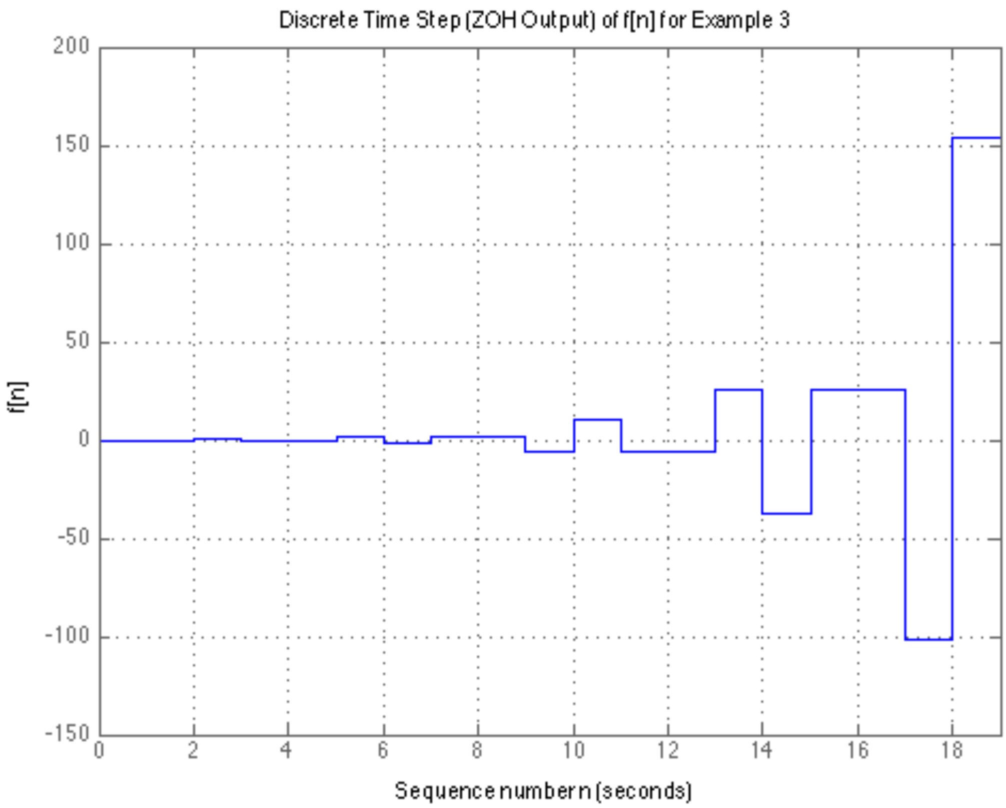
In [13]:
```
open example3
```

## Results for example 3

### Lollipop Plot




## Staircase Plot



## Inverse Z-Transform by the Inversion Integral

The inversion integral states that:

$$f[n] = \frac{1}{j2\pi}\oint_C F(z)z^{n-1}\,dz$$

where $C$ is a closed curve that encloses all poles of the integrant.

This can (*apparently*) be solved by Cauchy's residue theorem!!

Fortunately (:-), this is beyond the scope of this module!

See Karris Section 9.6.2 (pp 9-29—9-33) if you want to find out more.

## Inverse Z-Transform by the Long Division

To apply this method, $F(z)$ must be a rational polynomial function, and the numerator and denominator must be polynomials arranged in descending powers of $z$.

We will work through an example in class.

[Skip next slide in Pre-Lecture]

## Example 4

Karris example 9.9: use the long division method to determine $f[n]$ for $n = 0,\ 1,\ \text{and}\ 2$, given that

$$F(z) = \frac{1 + z^{-1} + 2z^{-2} + 3z^{-3}}{(1 - 0.25z^{-1})(1 - 0.5z^{-1})(1 - 0.75z^{-1})}$$

## MATLAB solution for example 4

See [example4.mlx (matlab/example4.mlx)](matlab/example4.mlx). (also available as [example4.m (matlab/example4.m)](matlab/example4.m).)

In [14]:
```
open example4
```

## Results for example 4

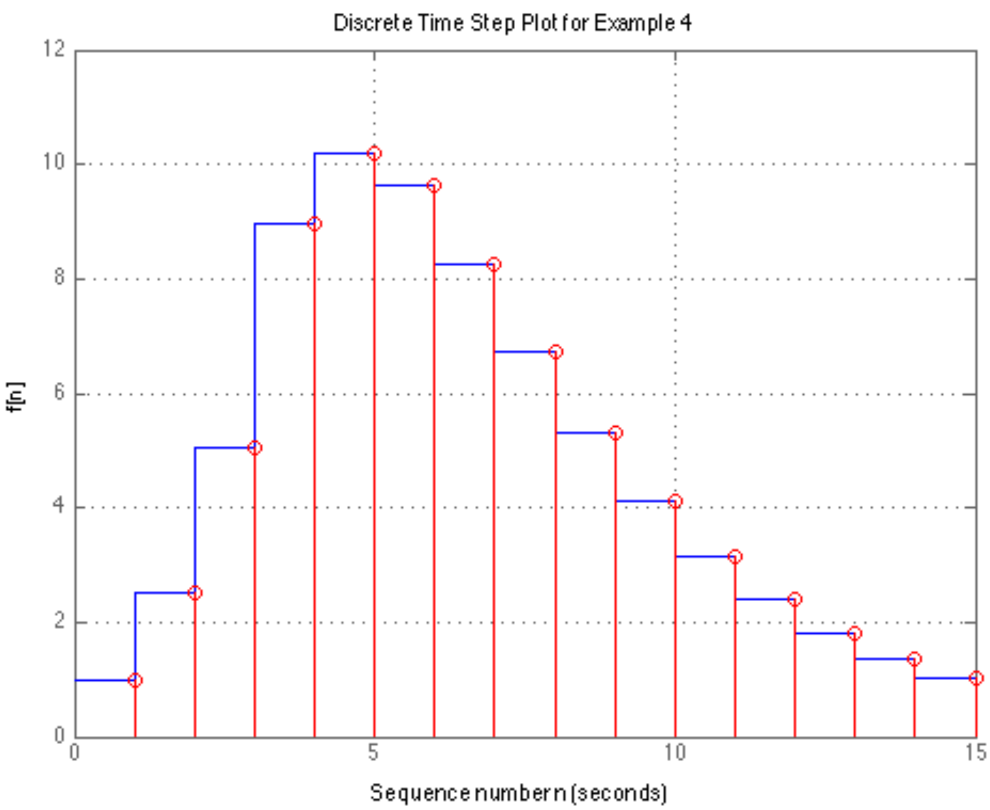
```
sym_den =

z^3 - (3*z^2)/2 + (11*z)/16 - 3/32


fn =

    1.0000
    2.5000
    5.0625
    ....
```

### Combined Staircase/Lollipop Plot




# Methods of Evaluation of the Inverse Z-Transform

## Partial Fraction Expansion

*Advantages*

- Most familiar.
- Can use MATLAB `residue` function.

*Disadvantages*

- Requires that $F(z)$ is a proper rational function.

## Invsersion Integral

*Advantage*

- Can be used whether $F(z)$ is rational or not

*Disadvantages*

- Requires familiarity with the *Residues theorem* of complex variable analaysis.</li></ul>

## Long Division

*Advantages*

- Practical when only a small sequence of numbers is desired.
- Useful when z-transform has no closed-form solution.

*Disadvantages*

- Can use MATLAB `dimpulse` function to compute a large sequence of numbers.
- Requires that $F(z)$ is a proper rational function.
- Division may be endless.

## Summary

- Inverse Z-Transform
- Examples using PFE
- Examples using Long Division
- Analysis in MATLAB

*Coming Next*

- DT transfer functions, continuous system equivalents, and modelling DT systems in Matlab and Simulink.

## Reference

{% bibliography --cited %}

## Answers to Examples

### Answer to Example 1

$$f[n] = 2\left(\frac{1}{2}\right)^n - 9\left(\frac{3}{4}\right)^n + 8$$

### Answer to Example 2

$$f[n] = 3(-1)^n + 6n - 3$$

### Answer to Example 3

$$f[n] = -0.5\delta[n] + 0.4 + \frac{(\sqrt{2})^n}{10}\cos\frac{3n\pi}{4} - \frac{3(\sqrt{2})^n}{10}\sin\frac{3n\pi}{4}$$

### Answer to Example 4

$f[0] = 1$, $f[1] = 5/2$, $f[2] = 81/16$, ....