Worksheet 9

To accompany Chapter 4.1 Trigonometric Fourier Series

Colophon

This worksheet can be downloaded as a PDF file. We will step through this worksheet in class.

An annotatable copy of the notes for this presentation will be distributed before the second class meeting as **Worksheet 9** in the **Week 4: Classroom Activities** section of the Canvas site. I will also distribute a copy to your personal **Worksheets** section of the **OneNote Class Notebook** so that you can add your own notes using OneNote.

You are expected to have at least watched the video presentation of Chapter 4.1 of the notes before coming to class. If you haven't watch it afterwards!

After class, the lecture recording and the annotated version of the worksheets will be made available through Canvas.

Motivating Examples

This Fourier Series demo, developed by Members of the Center for Signal and Image Processing (CSIP) at the School of Electrical and Computer Engineering at the Georgia Institute of Technology, shows how periodic signals can be synthesised by a sum of sinusoidal signals.

It is here used as a motivational example in our introduction to Fourier Series. (See also Fourier Series from Wolfram MathWorld referenced in the **Quick Reference** on Blackboard.)

To install this example, download the zip file and unpack it somewhere on your MATLAB path.

The Trigonometric Fourier Series

Any periodic waveform $f(t)$ can be represented as

$$\begin{aligned} f(t) = & \frac{1}{2}a_0 + a_1 \cos \Omega_0 t + a_2 \cos 2\Omega_0 t + a_3 \cos 3\Omega_0 t + \cdots + a_n \cos n\Omega_0 t + \cdots \\ & + b_1 \sin \Omega_0 t + b_2 \sin 2\Omega_0 t + b_3 \sin 3\Omega_0 t + \cdots + b_n \sin n\Omega_0 t + \cdots \end{aligned}$$

or equivalently (if more confusingly)

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\Omega_0 t + b_n \sin n\Omega_0 t\right)$$

where Ω_0 rad/s is the *fundamental frequency*.

Evaluation of the Fourier series coefficients

The coefficients are obtained from the following expressions (valid for any periodic waveform with fundamental frequency Ω_0 so long as we integrate over one period $0 \to T_0$ where $T_0 = 2\pi/\Omega_0$), and $\theta = \Omega_0 t$:

$$\frac{1}{2}a_0 = \frac{1}{T_0}\int_0^{T_0} f(t)dt = \frac{1}{\pi}\int_0^{2\pi} f(\theta)d\theta$$

$$a_n = \frac{1}{T_0}\int_0^{T_0} f(t)\cos n\Omega_0 t\, dt = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta$$

$$b_n = \frac{1}{T_0}\int_0^{T_0} f(t)\sin n\Omega_0 t\,dt = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta$$

Demo 1

Building up wave forms from sinusoids.

```
FourierSeriesDemo
```

Demo 2

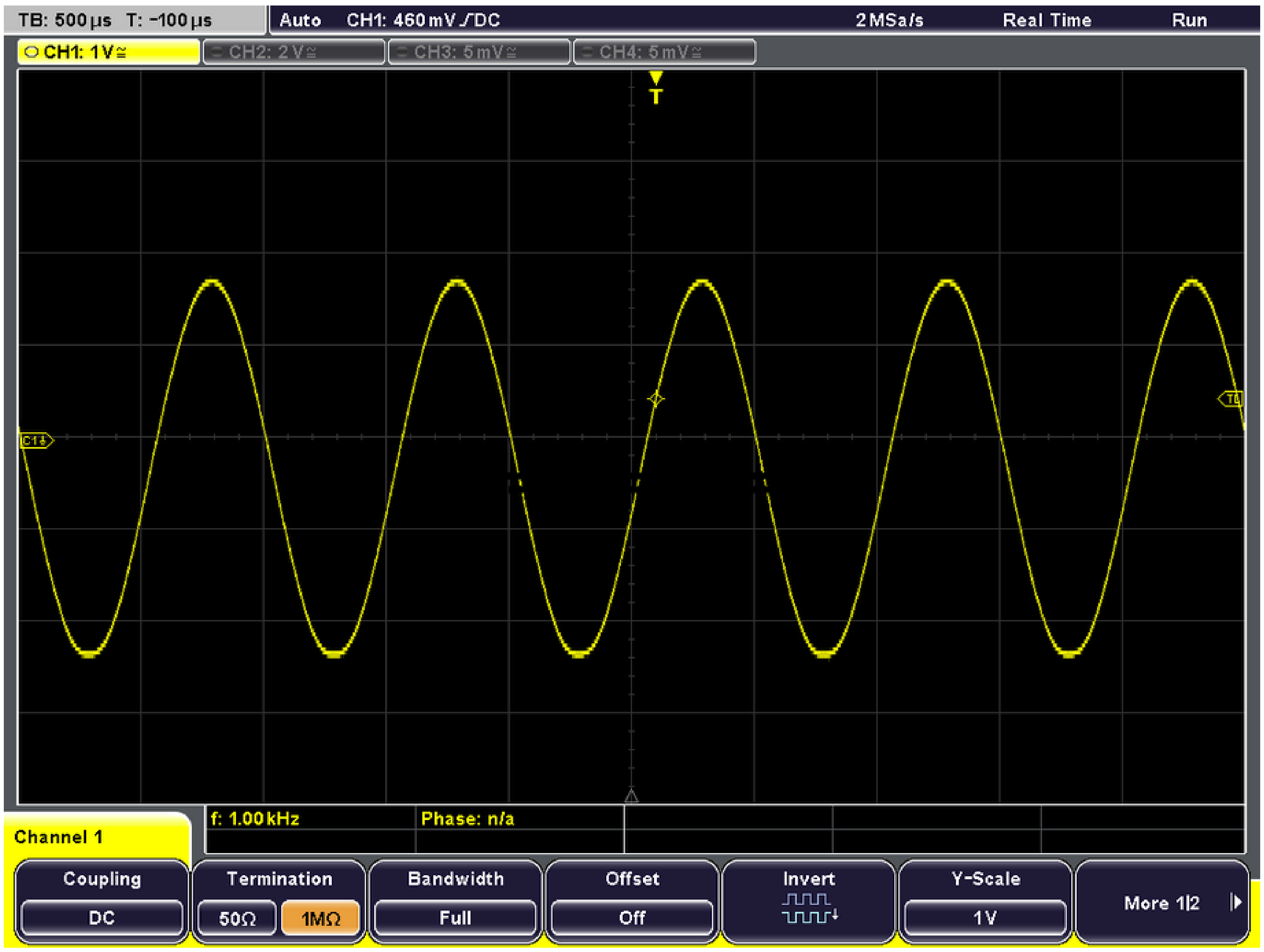
Actual measurements

Taken by Dr Tim Davies with a Rhode&Schwarz Oscilloscope.

Note all spectra shown in these slides are generated numerically from the input signals by sampling and the application of the Fast Fouriemr Transform (FFT).

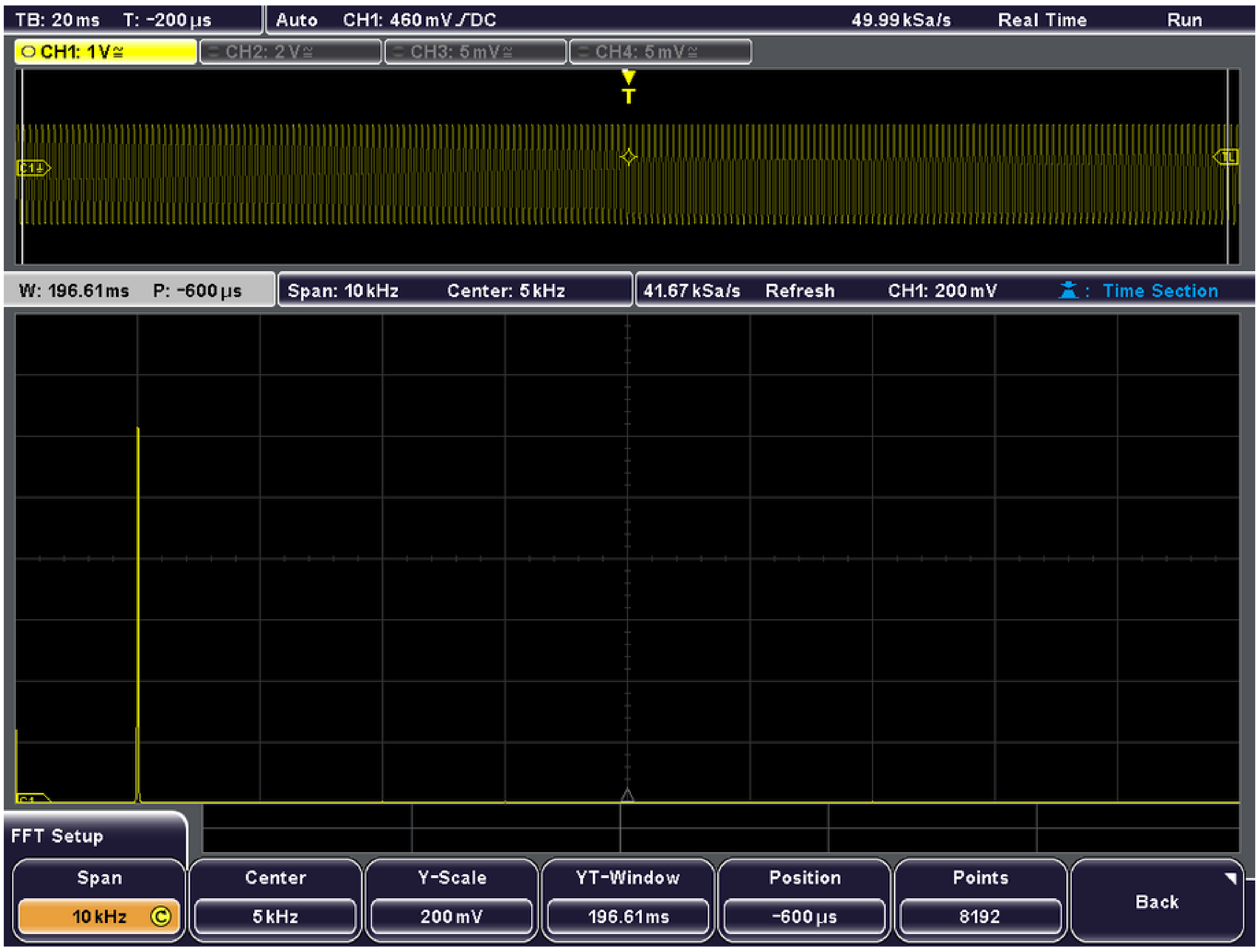
1 kHz Sinewave

m



Spectrum of 1kHz sinewave

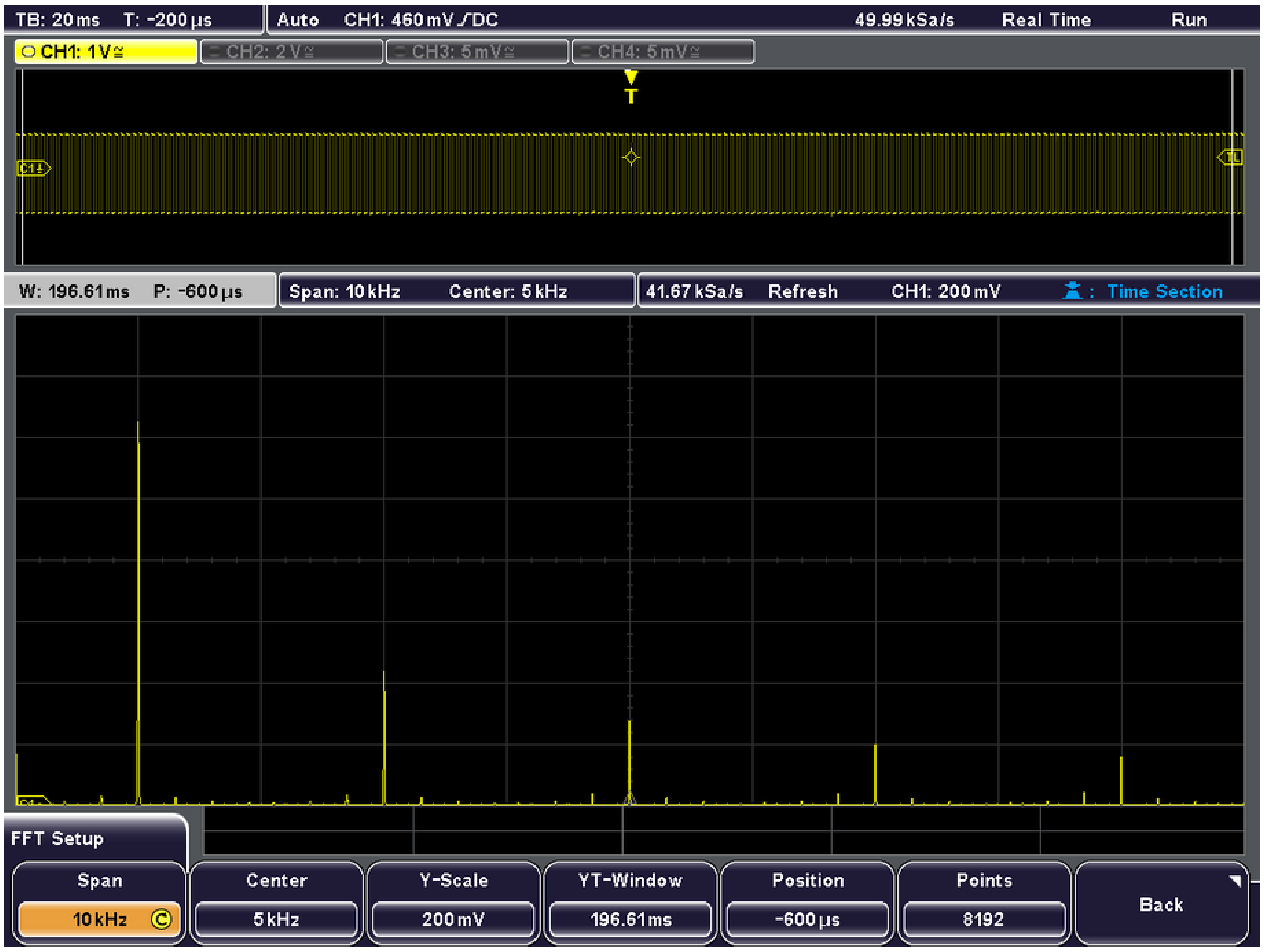
Showing one peak at harmonic frequency.



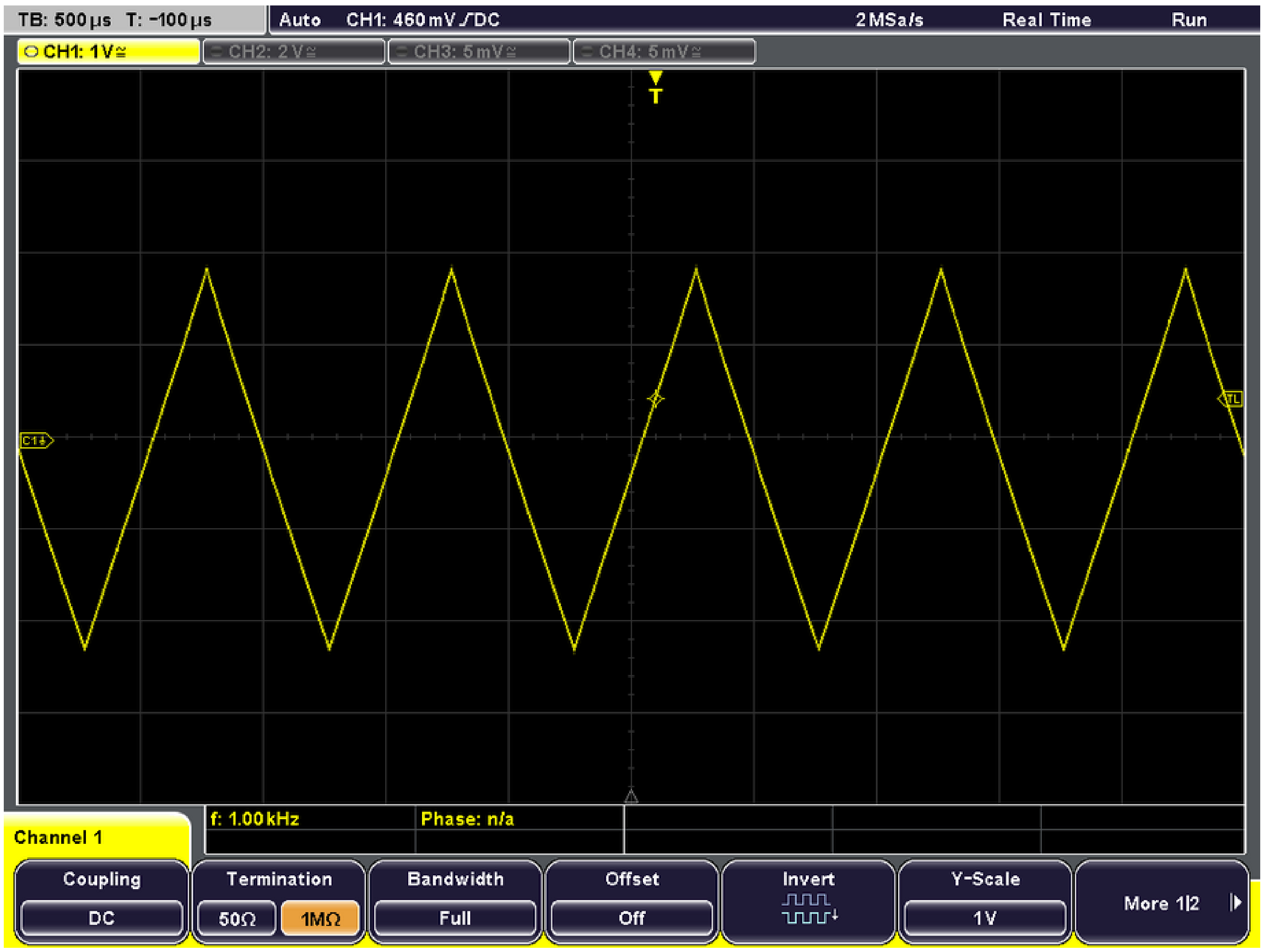
1 kHz Squarewave

Spectrum of 1kHz square wave

Clearly showing peaks at fundamental, 1/3, 1/5, 1/7 and 1/9 at 3rd, 5th and 7th harmonic frequencies. Note for sawtooth, harmonics decline in amplitude as the reciprocal of the of harmonic number n.

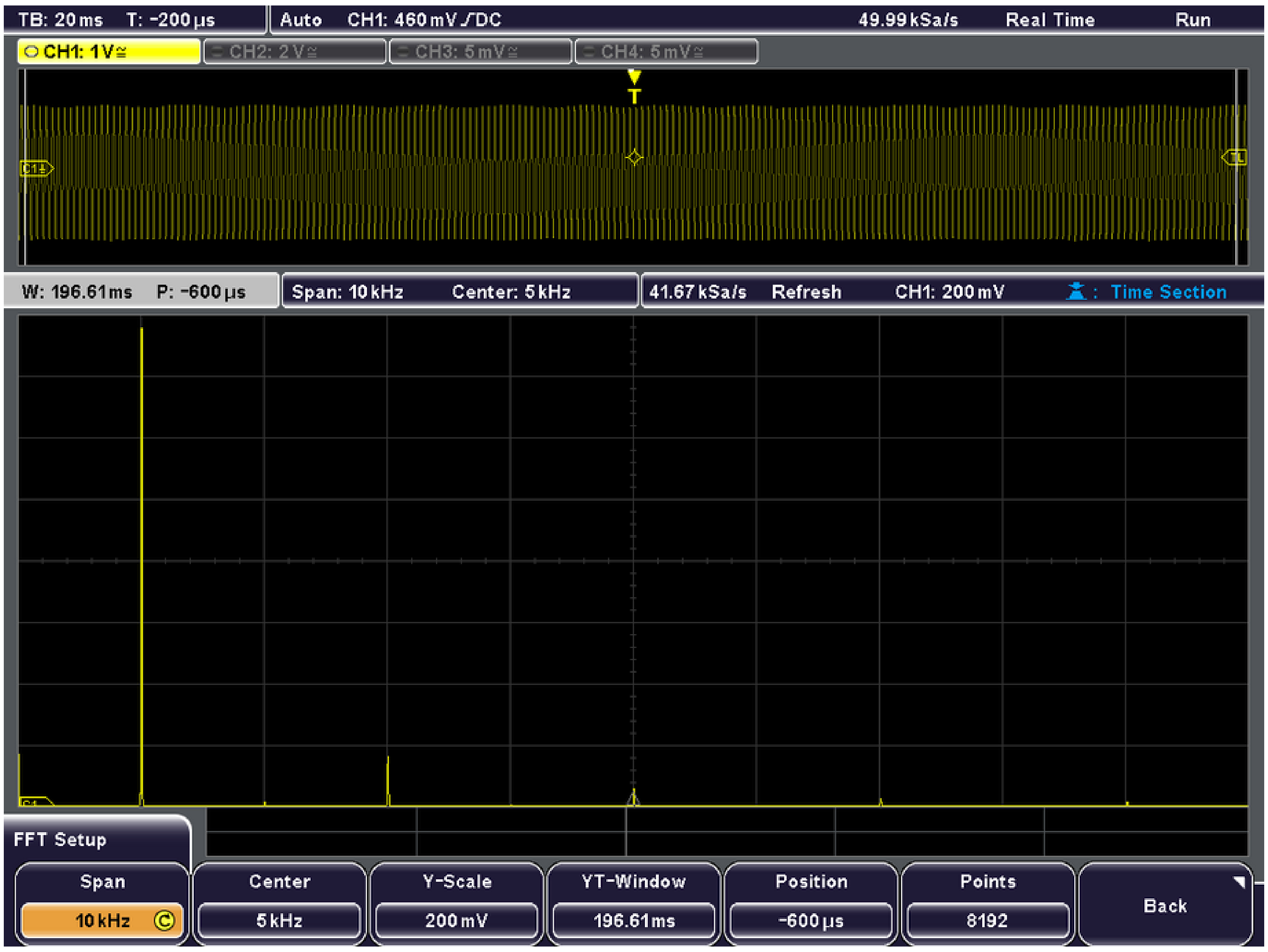


1 kHz triangle waveform



Spectrum of 1kHz triangle waveform

Clearly showing peaks at fundamental, 1/9, 1/25, 1/7 and 1/49 at 3rd, 5th and 7th harmonic frequencies. Note for triangle, harmonics decline in amplitude as the reciprocal of the square of n.



Odd, Even and Half-wave Symmetry

Odd- and even symmetry

- An *odd* function is one for which $f(t) = -f(-t)$. The function $\sin t$ is an *odd* function.
- An *even* function is one for which $f(t) = f(-t)$. The function $\cos t$ is an *even* function.

Half-wave symmetry

- A periodic function with period T is a function for which $f(t) = f(t + T)$
- A periodic function with period T, has *half-wave symmetry* if $f(t) = -f(t + T/2)$

Symmetry in Trigonometric Fourier Series

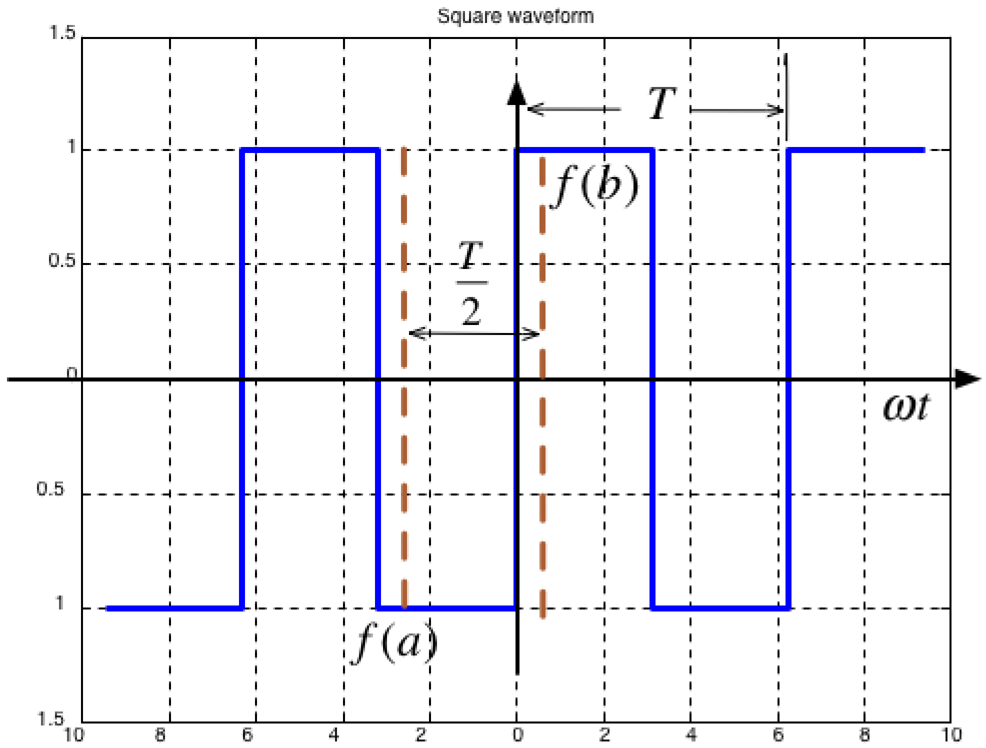
There are simplifications we can make if the original periodic properties has certain properties:

- If $f(t)$ is odd, $a_0 = 0$ and there will be no cosine terms so $a_n = 0 \ \forall n > 0$
- If $f(t)$ is even, there will be no sine terms and $b_n = 0 \ \forall n > 0$. The DC may or may not be zero.
- If $f(t)$ has *half-wave symmetry* only the odd harmonics will be present. That is a_n and b_n is zero for all even values of n (0, 2, 4, ...)

Symmetry in Common Waveforms

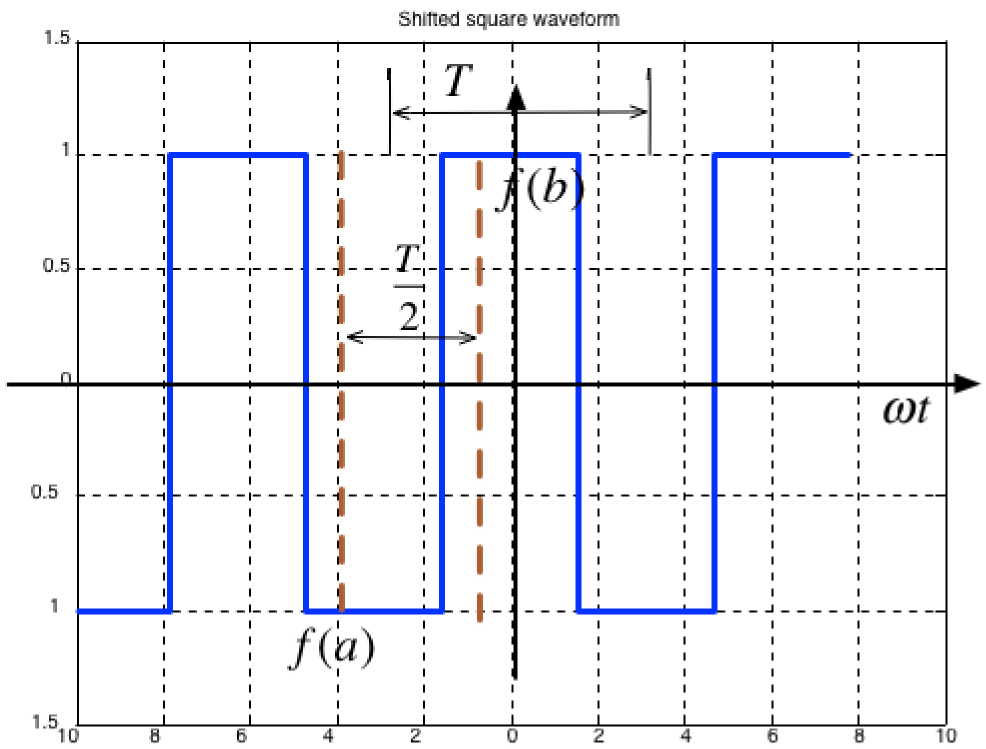
To reproduce the following waveforms (without annotation) publish the script waves.m.

Squarewave



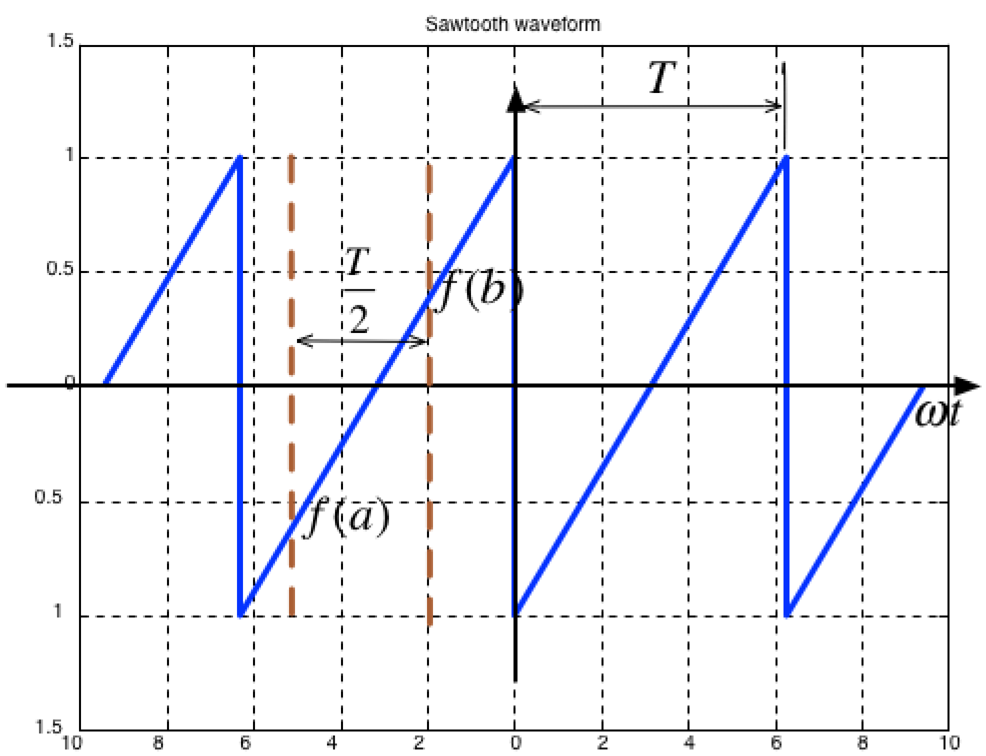
- Average value over period T is ...?
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Shifted Squarewave



- Average value over period T is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

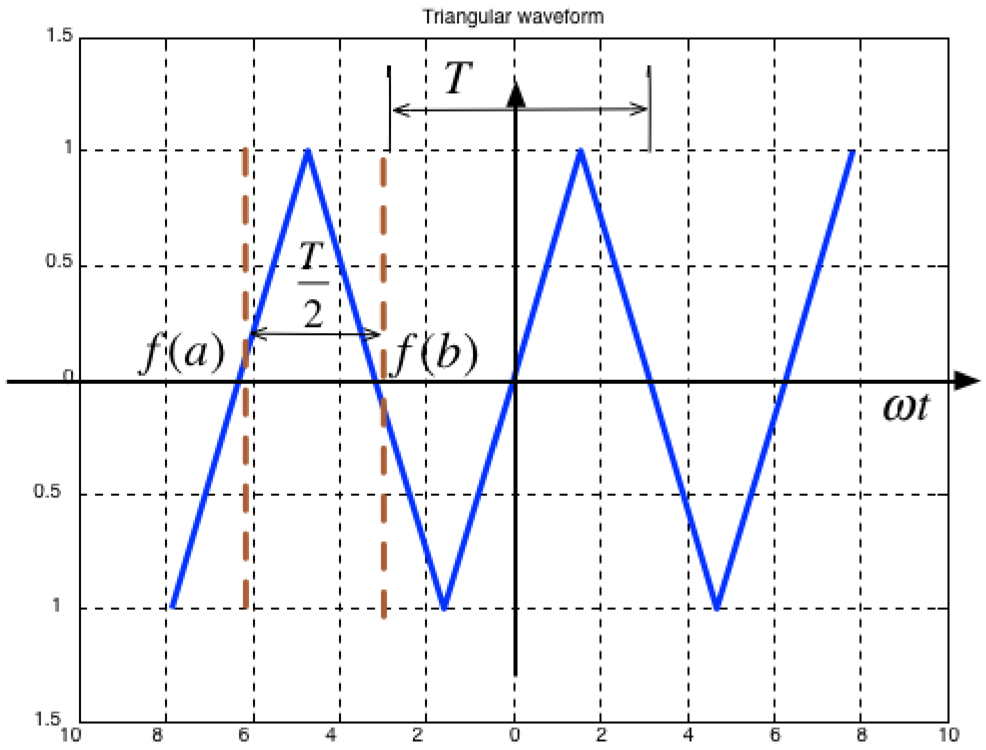
Sawtooth



- Average value over period T is

- Average value over period T is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Triangle

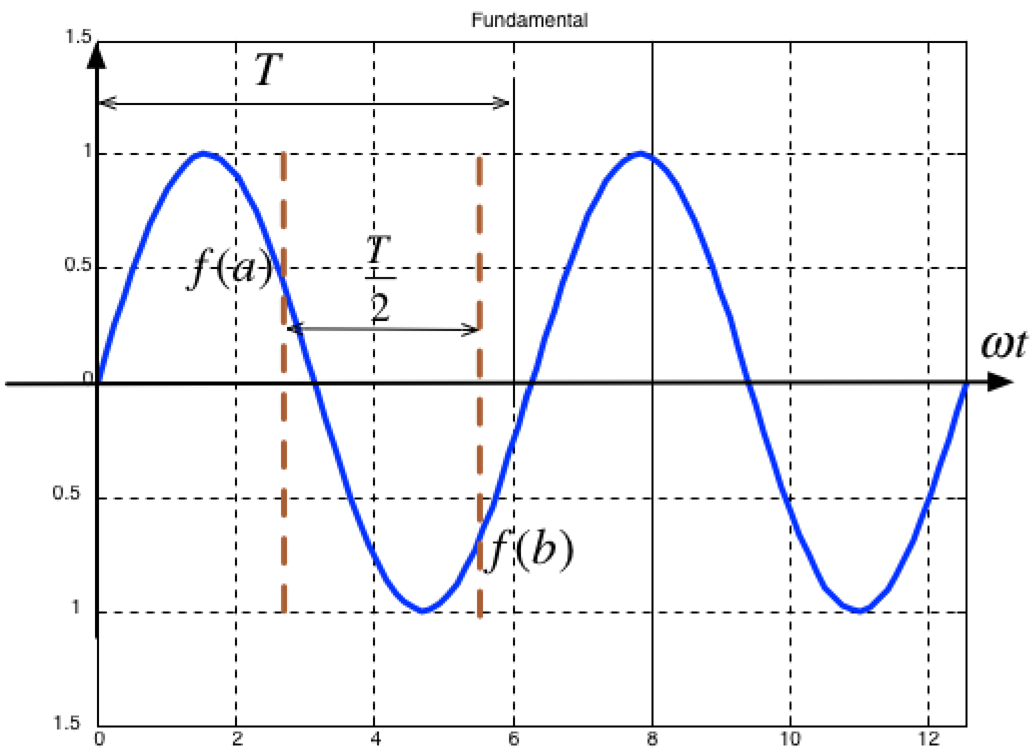


- Average value over period T is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Symmetry in fundamental, Second and Third Harmonics

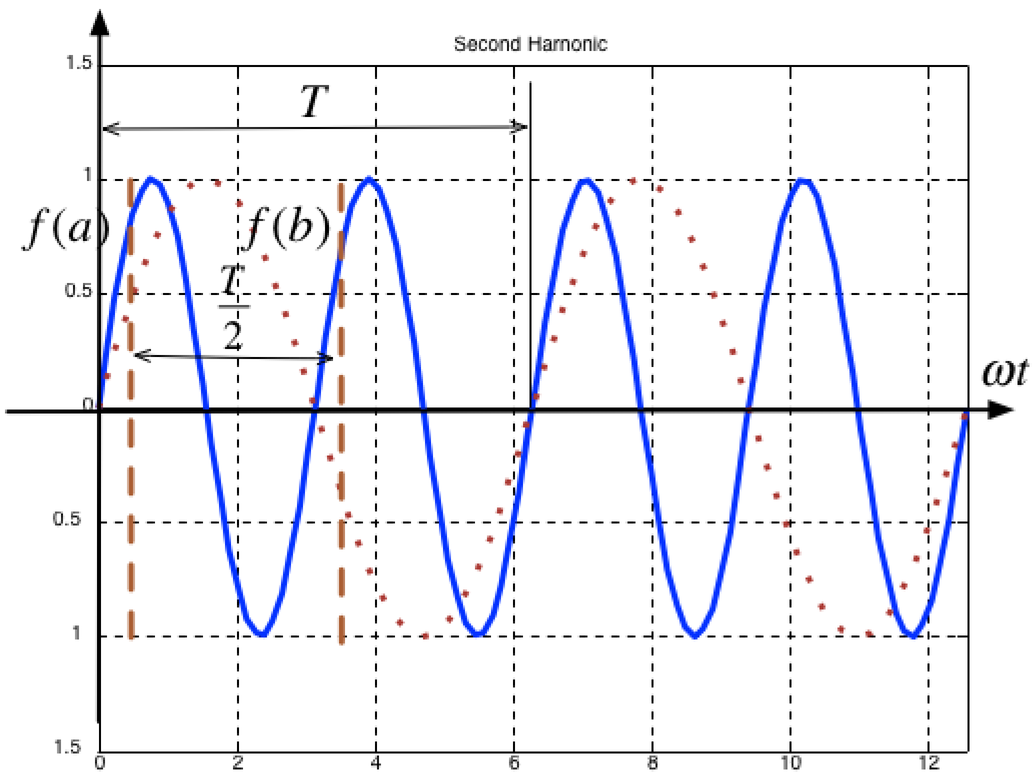
In the following, $T/2$ is taken to be the half-period of the fundamental sinewave.

Fundamental



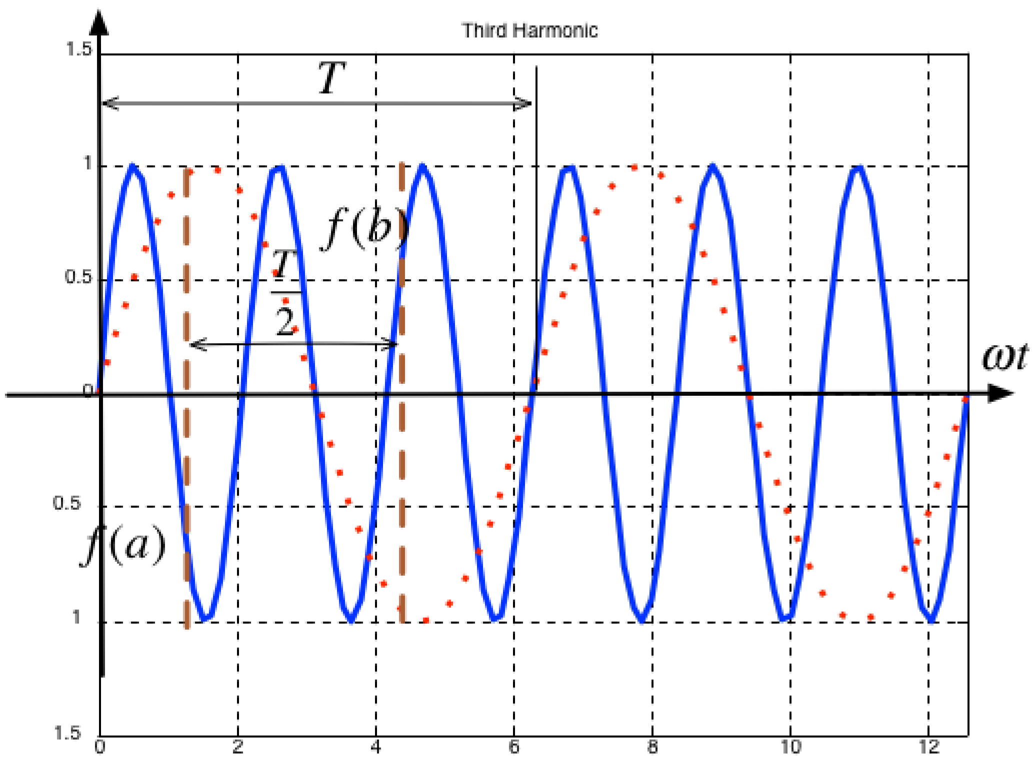
- Average value over period T is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Second Harmonic



- Average value over period T is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Third Harmonic



- Average value over period T is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

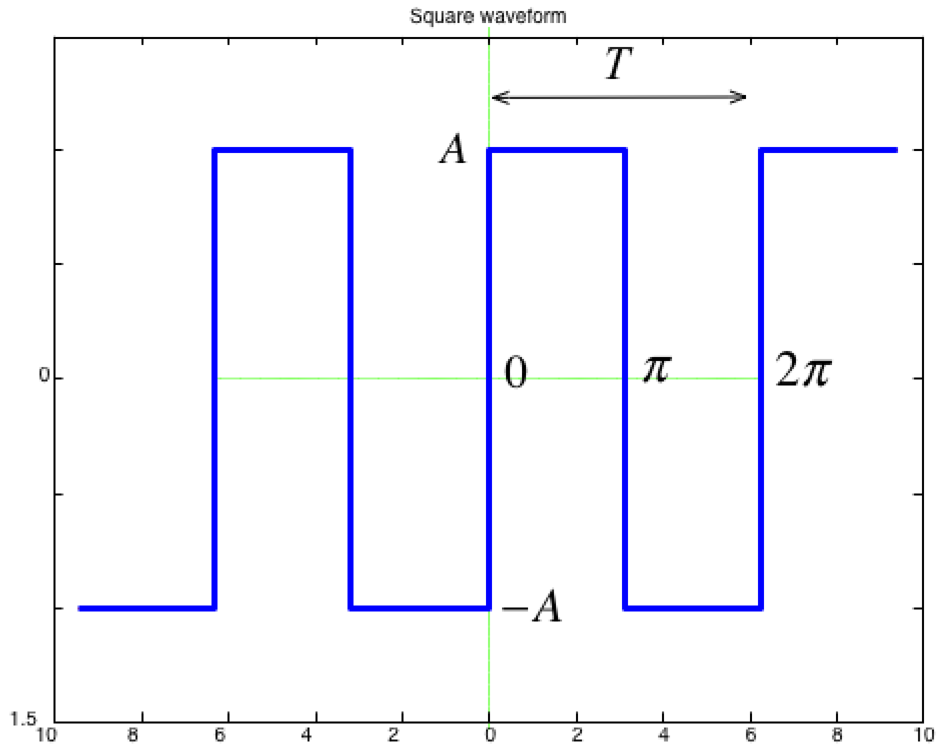
Some simplifications that result from symmetry

- The limits of the integrals used to compute the coefficents a_n and b_n of the Fourier series are given as $0 \rightarrow 2\pi$ which is one period T
- We could also choose to integrate from $-\pi \rightarrow \pi$
- If the function is *odd*, or *even* or has *half-wave symmetry* we can compute a_n and b_n by integrating from $0 \rightarrow \pi$ and multiplying by 2.
- If we have *half-wave symmetry* we can compute a_n and b_n by integrating from $0 \rightarrow \pi/2$ and multiplying by 4.

(For more details see page 7-10 of the textbook)

Computing coefficients of Trig. Fourier Series in Matlab

As an example let's take a square wave with amplitude $\pm A$ and period T.



Solution

Solution: See square_ftrig.mlx. Script confirms that:

- $a_0 = 0$
- $a_i = 0$: function is odd
- $b_i = 0$: for i even - half-wave symmetry

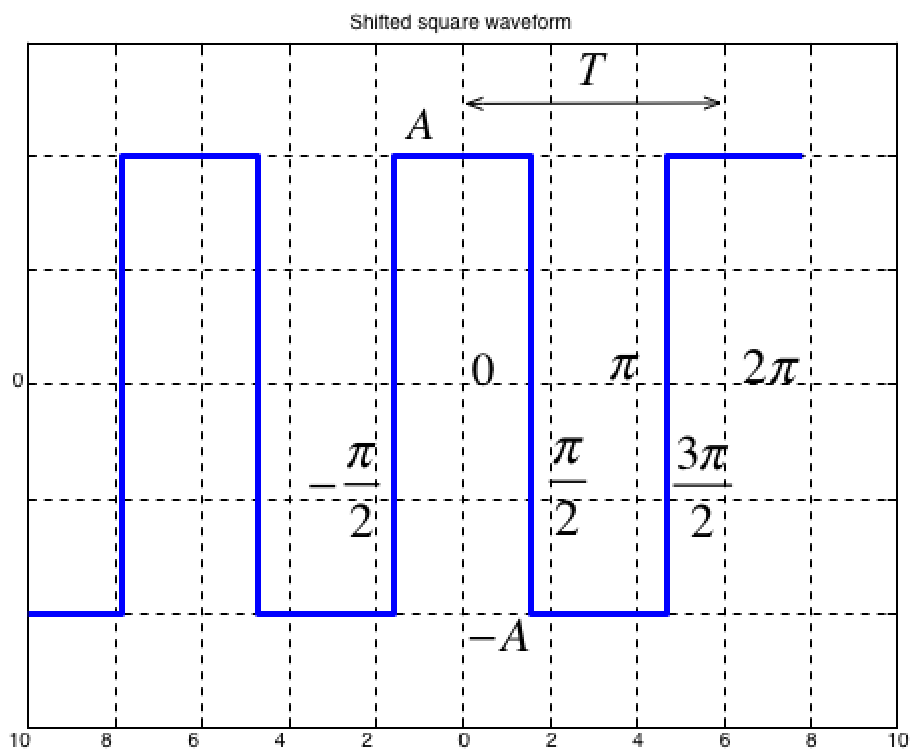
```
ft =

(4*A*sin(t))/pi + (4*A*sin(3*t))/(3*pi) + (4*A*sin(5*t))/(5*pi) + (4*A*sin(7*t))/(7*pi) +
(4*A*sin(9*t))/(9*pi) + (4*A*sin(11*t))/(11*pi)
```

```
clear all
cd ../matlab
format compact
imatlab_export_fig('print-svg')  % Static svg figures.cd ../matlab
open square_ftrig
```

Note that the coefficients match those given in the textbook (Section 7.4.1).

$$f(t) = \frac{4A}{\pi}\left(\sin\Omega_0 t + \frac{1}{3}\sin 3\Omega_0 t + \frac{1}{5}\sin 5\Omega_0 t + \cdots\right) = \frac{4A}{\pi}\sum_{n=\text{odd}}\frac{1}{n}\sin n\Omega_0 t$$

Using symmetry - computing the Fourier series coefficients of the shifted square wave



- As before $a_0 = 0$

- We observe that this function is even, so all b_k coefficents will be zero
- The waveform has half-wave symmetry, so only odd indexed coeeficents will be present.
- Further more, because it has half-wave symmetry we can just integrate from $0 \to \pi/2$ and multiply the result by 4.

See shifted_sq_ftrig.mlx.

```
ft =

(4*A*cos(t))/pi - (4*A*cos(3*t))/(3*pi) + (4*A*cos(5*t))/(5*pi) - (4*A*cos(7*t))/(7*pi) +
(4*A*cos(9*t))/(9*pi) - (4*A*cos(11*t))/(11*pi)
```

```
open shifted_sq_ftrig
```

Note that the coefficients match those given in the textbook (Section 7.4.2).

$$f(t) = \frac{4A}{\pi}\left(\cos\Omega_0 t - \frac{1}{3}\cos 3\Omega_0 t + \frac{1}{5}\cos 5\Omega_0 t - \cdots\right) = \frac{4A}{\pi}\sum_{n=\mathrm{odd}}(-1)^{\frac{n-1}{2}}\frac{1}{n}\cos n\Omega_0 t$$

By Dr Chris P. Jobling

This page was created by Dr Chris P. Jobling for Swansea University .