# Introduction to Filters

## Colophon

An annotatable worksheet for this presentation is available as **Worksheet 15** (https://cpjobling.github.io/eg-247-textbook/fourier_transform/4/worksheet15.html).

- The source code for this page is fourier_transform/4/ft4.ipynb (https://github.com/cpjobling/eg-247-textbook/blob/master/fourier_transform/4/ft4.ipynb).
- You can view the notes for this presentation as a webpage (HTML (https://cpjobling.github.io/eg-247-textbook/fourier_transform/4/ft4.html)).
- This page is downloadable as a PDF (https://cpjobling.github.io/eg-247-textbook/fourier_transform/4/ft4.pdf) file.

## Scope and Background Reading

This section is Based on the section **Filtering** from Chapter 5 of Benoit Boulet, Fundamentals of Signals and Systems (https://ebookcentral.proquest.com/lib/swansea-ebooks/reader.action?ppg=221&docID=3135971&tm=1518715953782){cite} `boulet` from the **Recommended Reading List**.

This material is an introduction to analogue filters. You will find much more in-depth coverage on Pages 11-1—11-48 (https://ebookcentral.proquest.com/lib/swansea-ebooks/reader.action?ppg=429&docID=3384197&tm=1518716026573) of {cite} `karris` .

## Agenda

- Frequency Selective Filters
- Ideal low-pass filter
- Butterworth low-pass filter

- High-pass filter

- Bandpass filter

## Introduction

- Filter design is an important application of the Fourier transform
- Filtering is a rich topic often taught in graduate courses so we give only an introduction.
- Our introduction *will* illustrate the usefulness of the frequency domain viewpoint.
- We will explore how filters can shape the spectrum of a signal.

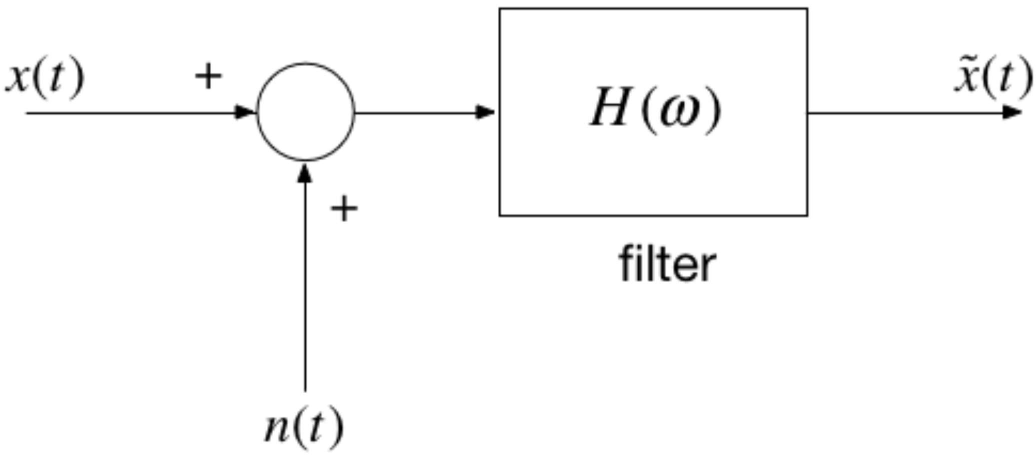
Other applications of the Fourier transform are sampling theory (introduced next week) and modulation.

## Frequency Selective Filters

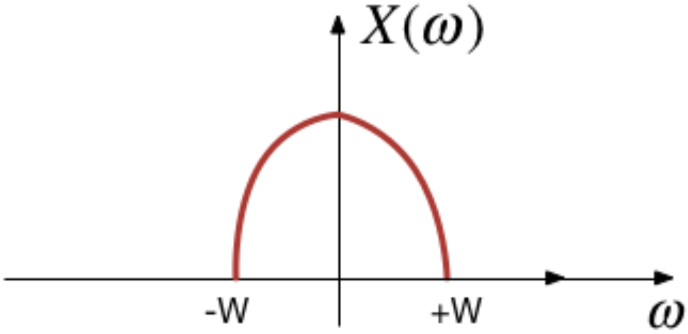
An ideal frequency-selective filter is a system that let's the frequency components of a signal through undistorted while frequency components at other components are completely cut off.

- The range of frequencies which are let through belong to the **pass Band**
- The range of frequencies which are cut-off by the filter are called the **stopband**
- A typical scenario where filtering is needed is when noise $n(t)$ is added to a signal $x(t)$ but that signal has most of its energy outside the bandwidth of a signal.

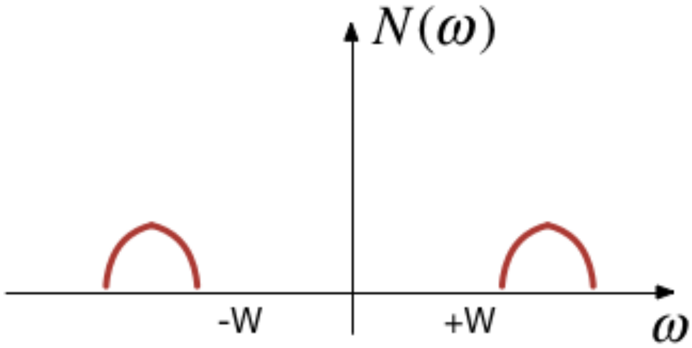
### Typical filtering problem



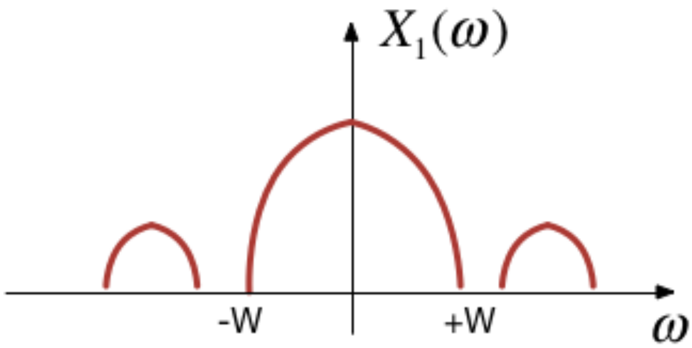
## Signal



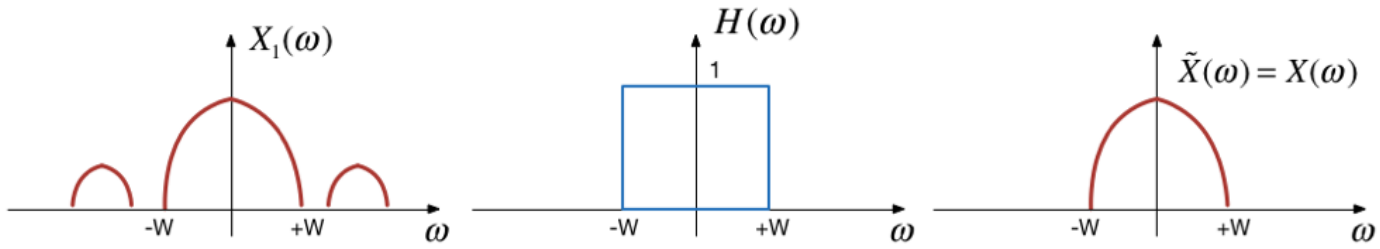

## Out-of Bandwidth Noise




## Signal plus Noise



### Results of filtering




### Motivating example

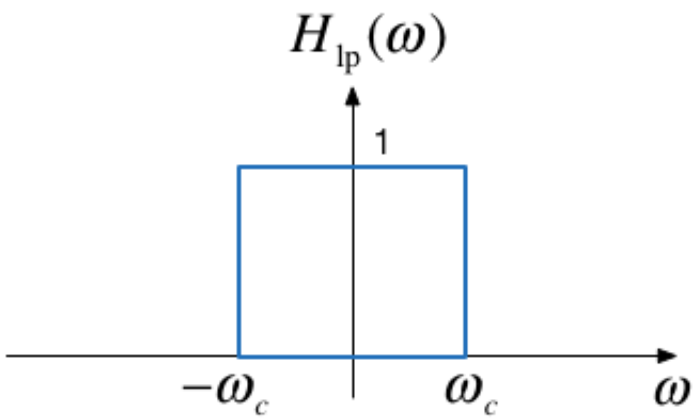
See the video and script on [*Canvas Week 7* (https://canvas.swansea.ac.uk/courses/646/pages/week-7-classroom-activities?module_item_id=398892)](https://canvas.swansea.ac.uk/courses/646/pages/week-7-classroom-activities?module_item_id=398892).

## Ideal Low-Pass Filter (LPF)

An ideal low pass filter cuts-off frequencies higher than its *cut-off frequency*, $\omega_c$ .

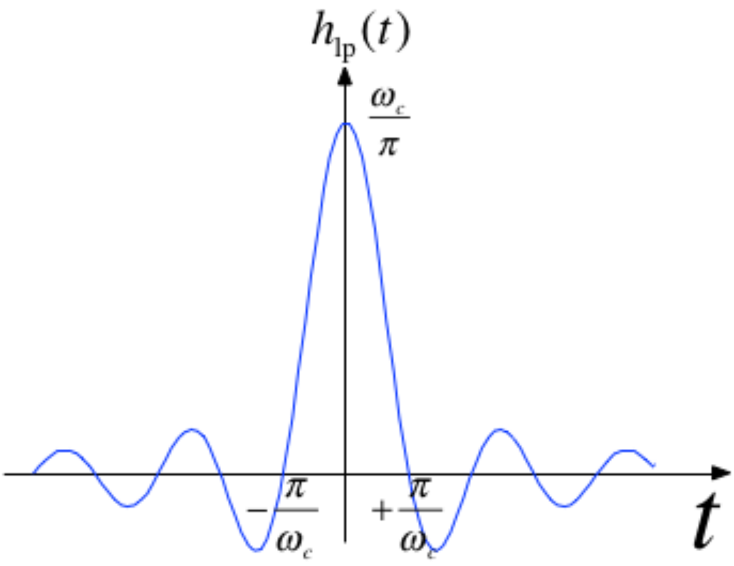
$$H_{\mathrm{lp}}(\omega) = \begin{cases} 1, & |\omega| \; < \omega_c \\ 0, & |\omega| \; \geq \omega_c \end{cases}$$

### Frequency response of an ideal LPF




### Impulse response of an ideal LPF

$$h_{\mathrm{lp}}(t) = \frac{\omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c}{\pi} t\right)$$




## Filtering is Convolution

The output of an LTI system with impulse response

$$h(t) \Leftrightarrow H(\omega)$$

subject to an input signal

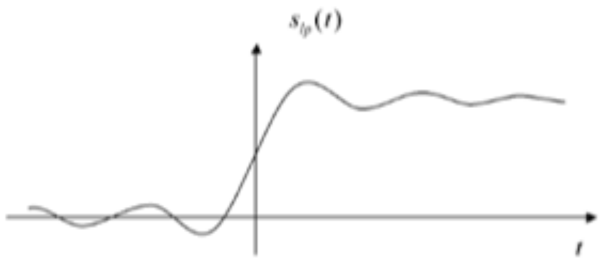
$$x(t) \Leftrightarrow X(\omega)$$

is given by

$$y(t) = h(t) * x(t) \Leftrightarrow Y(\omega) = H(\omega)X(\omega)$$

## Issues with the "ideal" filter

This is the step response:




(reproduced from {cite} `boulet` Fig. 5.23 p. 205)

Ripples in the impulse resonse would be undesireable, and because the impulse response is non-causal it cannot actually be implemented.

# Butterworth low-pass filter

N-th Order Butterworth Filter

$$|H_B(\omega)| = \frac{1}{\left(1+\left(\frac{\omega}{\omega_c}\right)^{2N}\right)^{\frac{1}{2}}}$$

#### Remarks

- DC gain is

$$|H_B(j0)| = 1$$

- Attenuation at the cut-off frequency is

$$|H_B(j\omega_c)| = 1/\sqrt{2}$$

  for any $N$

More about the Butterworth filter: [Wikipedia Article (http://en.wikipedia.org/wiki/Butterworth_filter)](http://en.wikipedia.org/wiki/Butterworth_filter)

## Example 5: Second-order BW Filter

The second-order butterworth Filter is defined by is Characteristic Equation (CE):

$$p(s) = s^2 + \omega_c\sqrt{2}s + \omega_c^2 = 0^*$$

Calculate the roots of $p(s)$ (the poles of the filter transfer function) in both Cartesian and polar form.

**Note**: This has the same characteristic as a control system with damping ratio $\zeta = 1/\sqrt{2}$ and $\omega_n = \omega_c$!

**Solution to example 5**

## Example 6

Derive the differential equation relating the input $x(t)$ to output $y(t)$ of the 2nd-Order Butterworth Low-Pass Filter with cutoff frequency $\omega_c$ .

### Solution to example 6

## Example 7

Determine the frequency response $H_B(\omega) = Y(\omega)/X(\omega)$

### Solution to example 7

## Magnitude of frequency response of a 2nd-order Butterworth Filter

In [2]:
```
wc = 100;
```

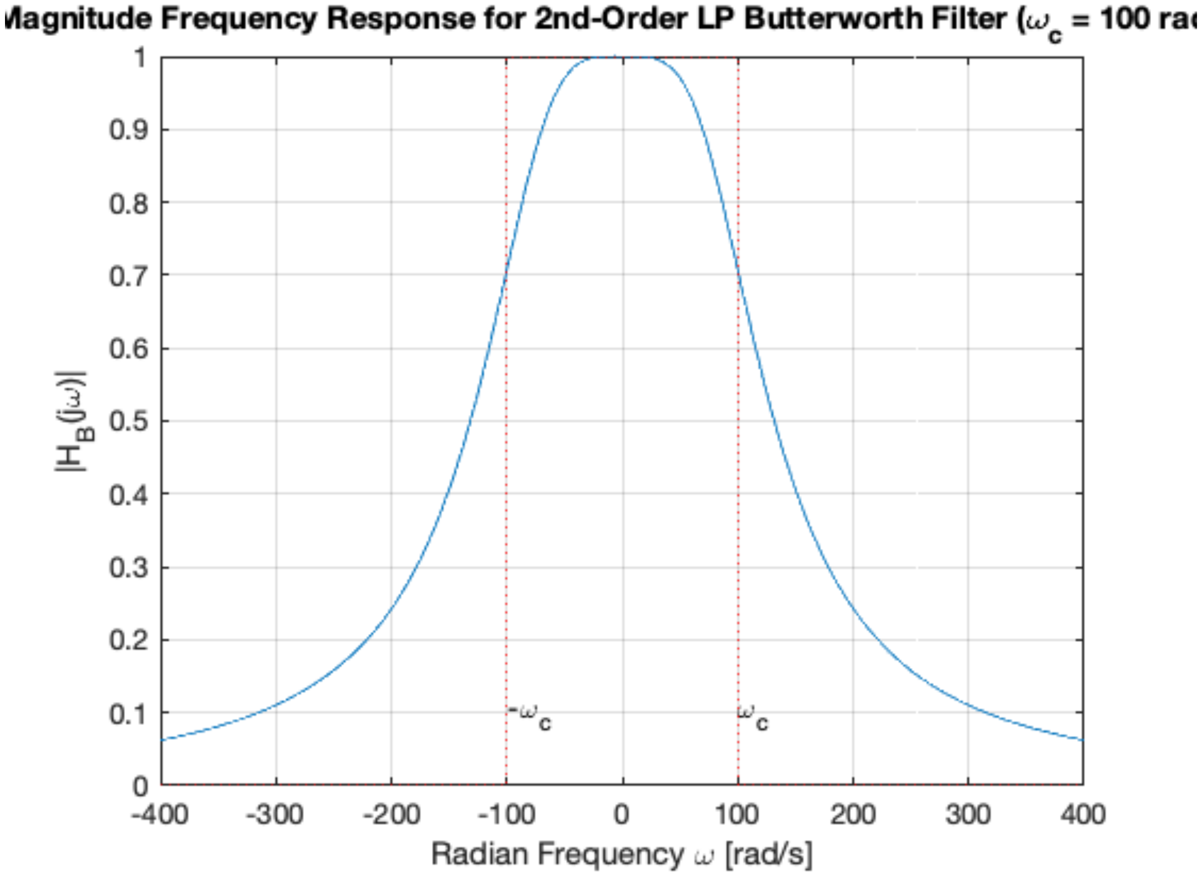
Transfer function

In [3]:
```
H = tf(wc^2,[1, wc*sqrt(2), wc^2])
```

```
H =
 
          10000
  ---------------------
  s^2 + 141.4 s + 10000
 
Continuous-time transfer function.
```

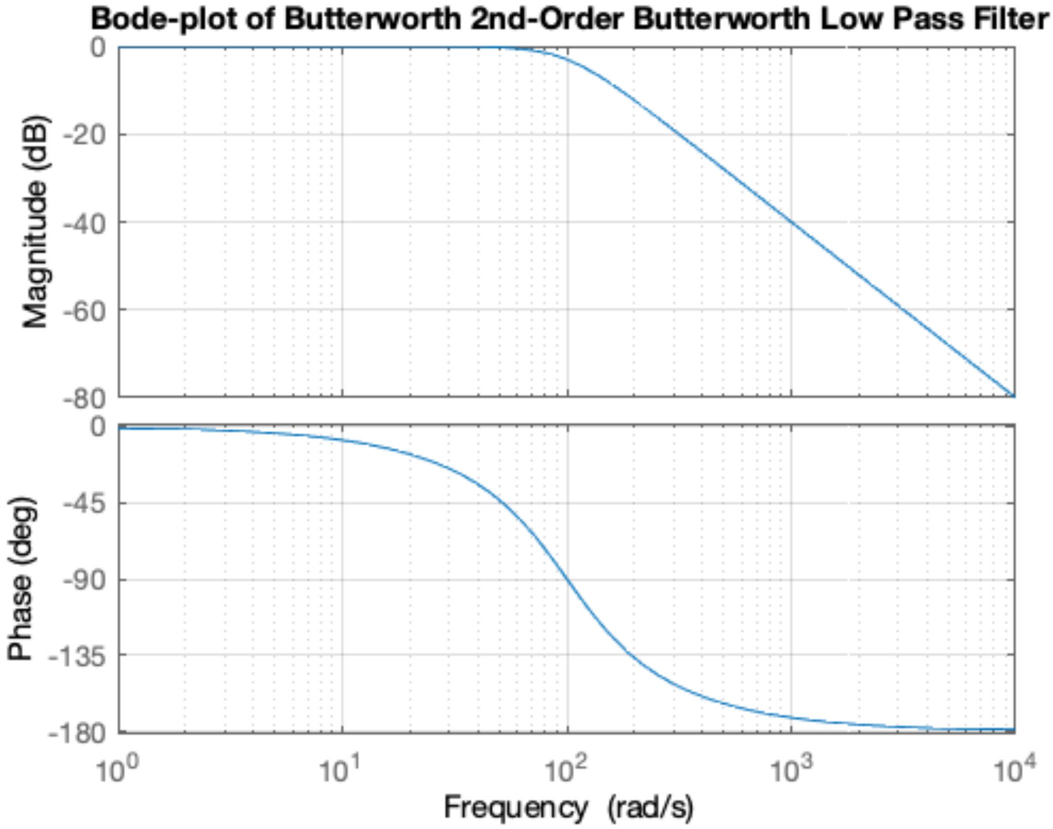
Magnitude frequency response

In [4]:
```
w = -400:400;
mHlp = 1./(sqrt(1 + (w./wc).^4));
plot(w,mHlp)
grid
ylabel('|H_B(j\omega)|')
title('Magnitude Frequency Response for 2nd-Order LP Butterworth Filter (\omega_c = 100 rad/s)')
xlabel('Radian Frequency \omega [rad/s]')
text(100,0.1,'\omega_c')
text(-100,0.1,'-\omega_c')
hold on
plot([-400,-100,-100,100,100,400],[0,0,1,1,0,0],'r:')
hold off
```


Magnitude Frequency Response for 2nd-Order LP Butterworth Filter (ω_c = 100 ra
|H_B(jω)|
Radian Frequency ω [rad/s]
-ω_c
ω_c
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-400
-300
-200
-100
0
100
200
300
400

Bode plot

In [5]:
```
bode(H)
grid
title('Bode-plot of Butterworth 2nd-Order Butterworth Low Pass Filt
er')
```




## Example 8

Determine the impulse and step response of a butterworth low-pass filter.

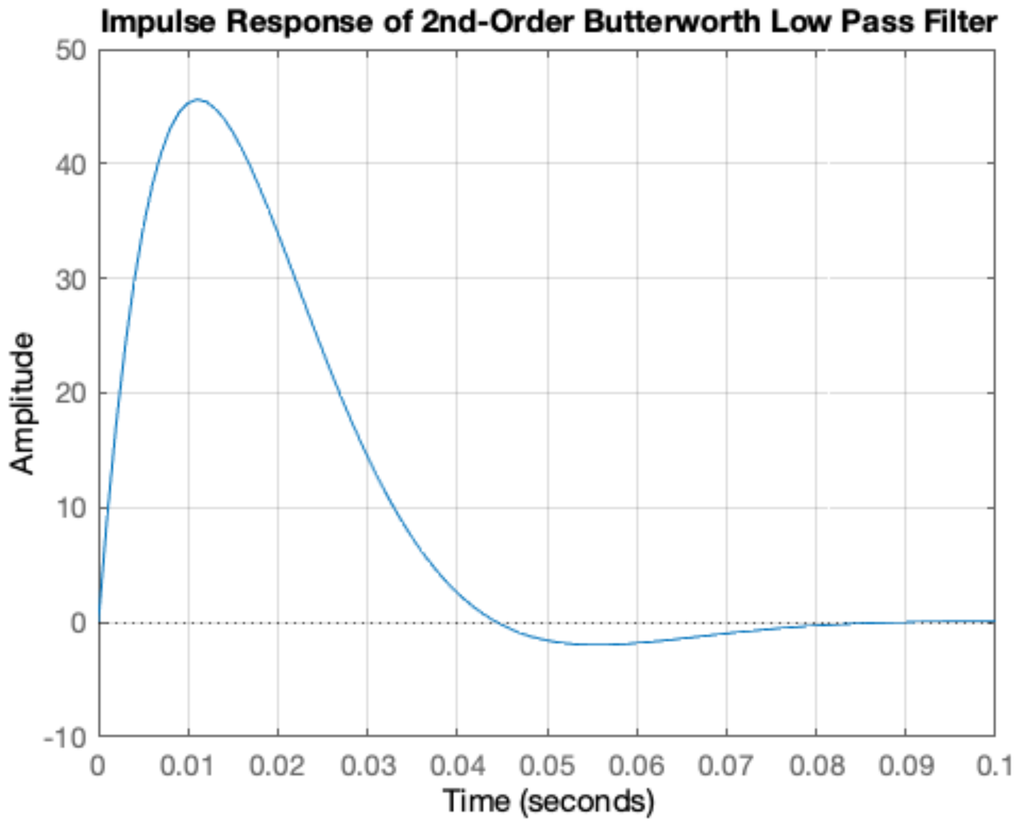
You will find this Fourier transform pair useful:

$$e^{-at}\sin\omega_0 t\;u_0(t) \Leftrightarrow \frac{\omega_0}{(j\omega + a)^2 + \omega_0^2}$$

### Solution to example 8

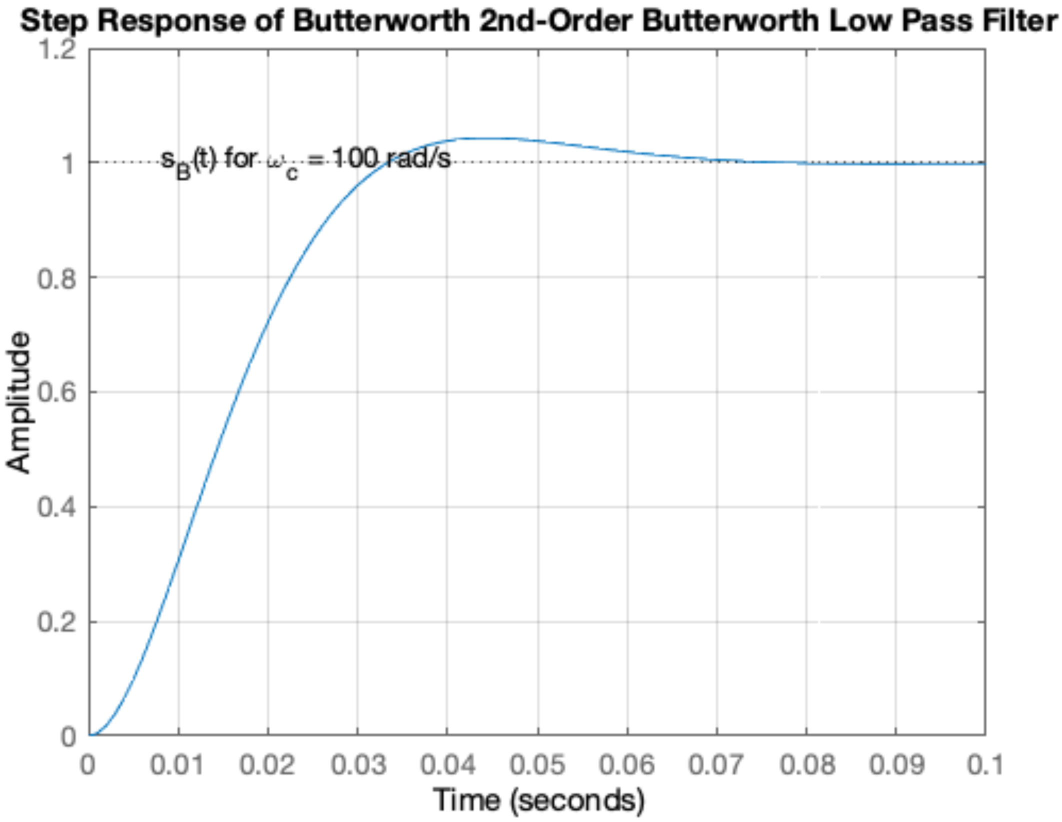
Impulse response

In [6]:
```
impulse(H,0.1)
grid
title('Impulse Response of 2nd-Order Butterworth Low Pass Filter')
```




Step response

In [7]:
```
step(H,0.1)
title('Step Response of Butterworth 2nd-Order Butterworth Low Pass Filter')
grid
text(0.008,1,'s_B(t) for \omega_c = 100 rad/s')
```

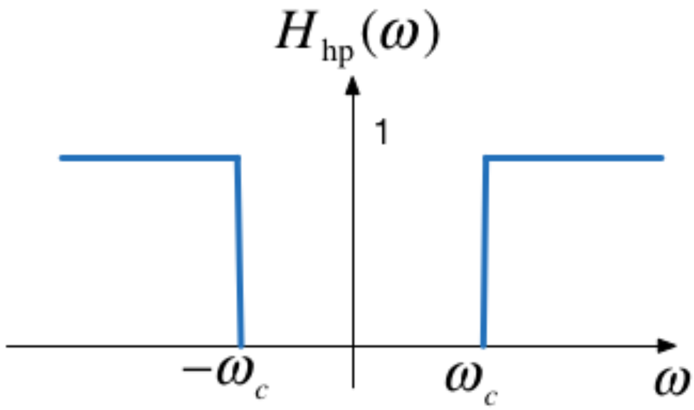



## High-pass filter (HPF)

An ideal highpass filter cuts-off frequencies lower than its *cutoff frequency*, $\omega_c$ .

$$H_{\text{hp}}(\omega) = \begin{cases} 0, & |\omega| \ \leq \omega_c \\ 1, & |\omega| \ > \omega_c \end{cases}$$

## Frequency response of an ideal HPF




## Responses

### Frequency response

$$H_{\rm hp}(\omega) = 1 - H_{\rm lp}(\omega)$$

### Impulse response

$$h_{\rm hp}(t) = \delta(t) - h_{\rm lp}(t)$$

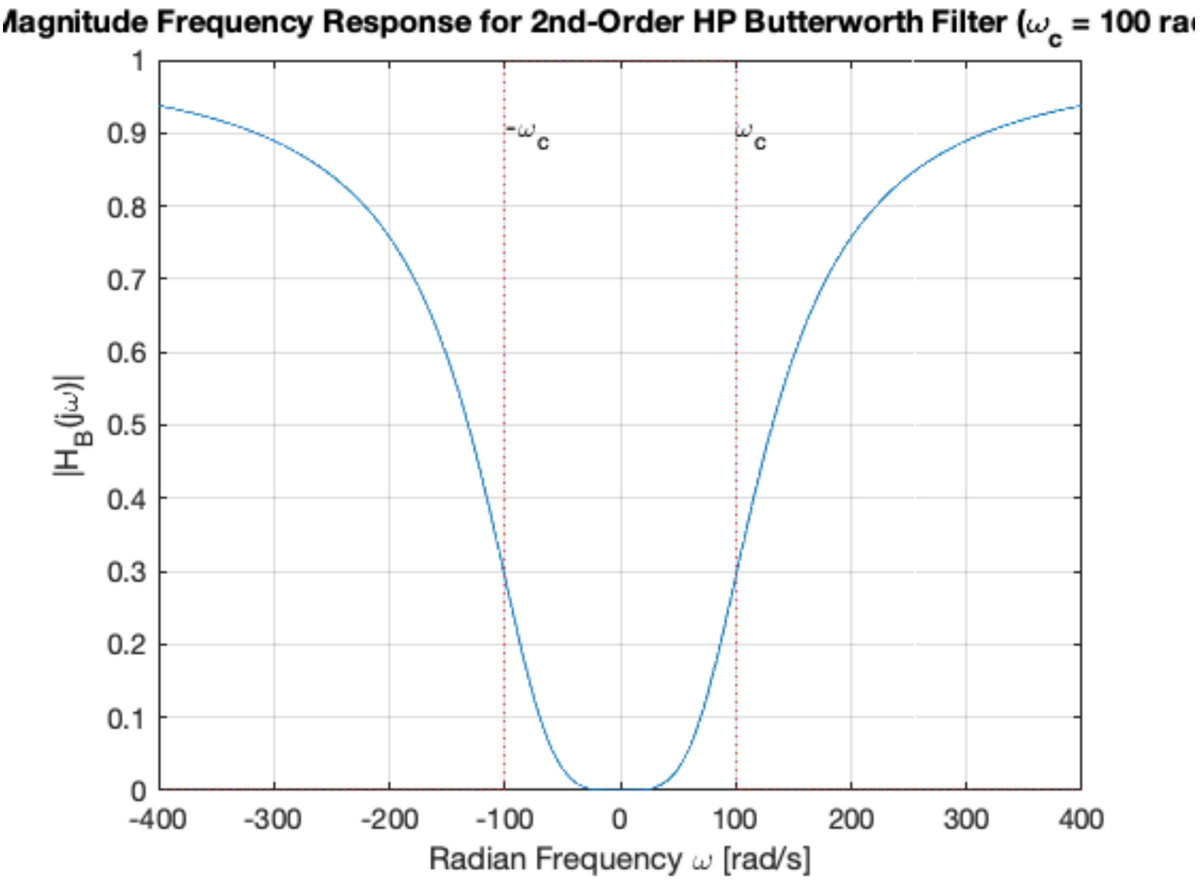
## Example 9

Determine the frequency response of a 2nd-order butterworth highpass filter

### Solution to example 9

Magnitude frequency response

In [8]:
```
w = -400:400;
plot(w,1-mHlp)
grid
ylabel('|H_B(j\omega)|')
title('Magnitude Frequency Response for 2nd-Order HP Butterworth Fi
lter (\omega_c = 100 rad/s)')
xlabel('Radian Frequency \omega [rad/s]')
text(100,0.9,'\omega_c')
text(-100,0.9,'-\omega_c')
hold on
plot([-400,-100,-100,100,100,400],[0,0,1,1,0,0],'r:')
hold off
```



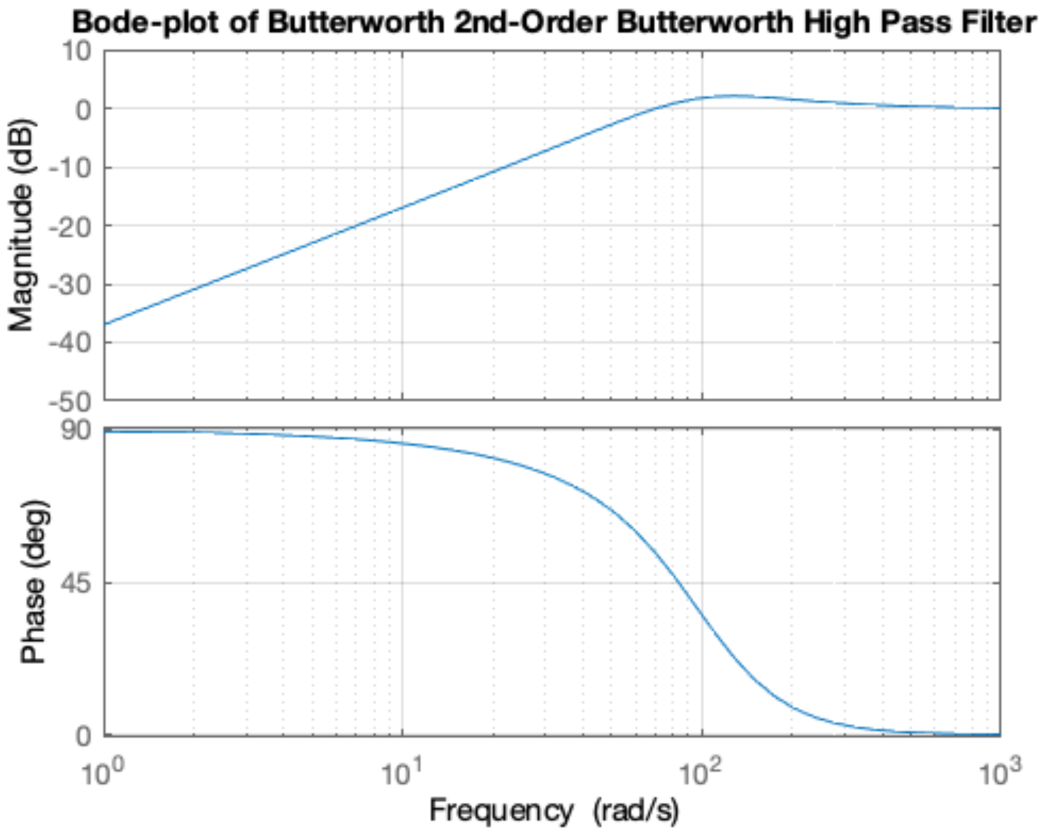

High-pass filter

In [9]:
```
Hhp = 1 - H
bode(Hhp)
grid
title('Bode-plot of Butterworth 2nd-Order Butterworth High Pass Filter')
```

```
Hhp =

      s^2 + 141.4 s
  ---------------------
  s^2 + 141.4 s + 10000

Continuous-time transfer function.
```

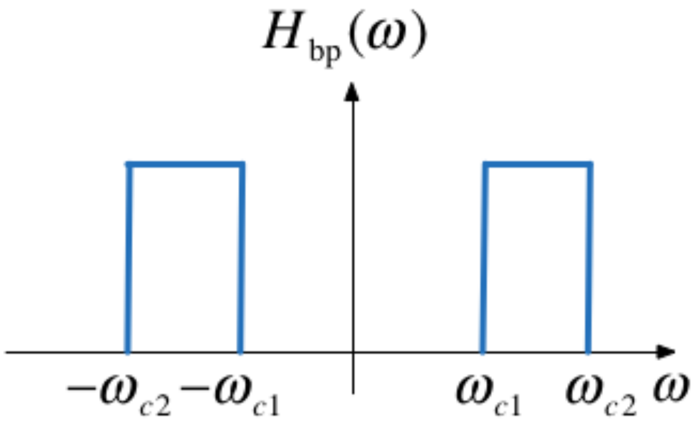



## Band-pass filter (BPF)

An ideal bandpass filter cuts-off frequencies lower than its first *cutoff frequency* $\omega_{c1}$ , and higher than its second *cutoff frequency* $\omega_{c2}$ .

$$H_{\mathrm{bp}}(\omega) = \begin{cases} 1, & \omega_{c1} < |\omega| < \omega_{c2} \\ 0, & \text{otherwise} \end{cases}$$

### Frequency response of an ideal BPF




### Bandpass filter design

A bandpass filter can be obtained by multiplying the frequency responses of a lowpass filter by a highpass filter.

$$H_{\text{bp}}(\omega) = H_{\text{hp}}(\omega)H_{\text{lp}}(\omega)$$

- The highpass filter should have cut-off frequency of $\omega_{c1}$

- The lowpass filter should have cut-off frequency of $\omega_{c2}$

To generate all the plots shown in this presentation, you can use [butter2_ex.mlx (https://cpjobling.github.io/eg-247-textbook/fourier_transform/matlab/butter2_ex.mlx)](https://cpjobling.github.io/eg-247-textbook/fourier_transform/matlab/butter2_ex.mlx)

## Summary

- Frequency-Selective Filters
- Ideal low-pass filter
- Butterworth low-pass filter
- High-pass filter
- Bandpass filter

# Solutions

Solutions to Examples 5-9 are captured as a PenCast in filters.pdf (https://cpjobling.github.io/eg-247-textbook/fourier_transform/solutions/filters2.pdf).