

# Worksheet 8

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## To accompany Unit 3.5 Convolution and the Impulse Response

## Colophon

This worksheet can be downloaded as a [PDF file](#). We will step through this worksheet in class.

An annotatable copy of the notes for this presentation will be distributed before the second class meeting as **Worksheet 7** in the **Week 3: Classroom Activities** section of the Canvas site. I will also distribute a copy to your personal **Worksheets** section of the **OneNote Class Notebook** so that you can add your own notes using OneNote.

You are expected to have at least watched the video presentation of [Chapter 3.5](#) of the [notes](#) before coming to class. If you haven't watch it afterwards!

After class, the lecture recording and the annotated version of the worksheets will be made available through Canvas.

## Agenda

The material to be presented is:

### First Hour

- Even and Odd Functions of Time
- Time Convolution

## Even and Odd Functions of Time

Fill in the Blanks Quiz.

### Even Functions of Time

A function  $f(t)$  is said to be an *even function* of time if the following relation holds

Polynomials with even exponents only, and with or without constants, are even functions. For example:

Write down the Taylor-series polynomial expansion of  $\cos t$ . Is  $\cos t$  even or odd?

Odd/Even?

## Odd Functions of Time

A function  $f(t)$  is said to be an *odd function* of time if the following relation holds

Polynomials with even exponents only, and with or without constants, are even functions.

Write down the Taylor-series polynomial expansion of  $\sin t$ . Is  $\sin t$  even or odd?

Odd/Even?

## Observations

For odd functions  $f(0) =$

The product of *two even* or *two odd* functions is an [Even/Odd] function.

The product of an even and an odd function, is an [Even/Odd] function.

## More about Even and Odd Functions

In the following  $f_e(t)$  will denote an even function and  $f_o(t)$  an odd function.

### Time integrals of even and odd functions

For an even function  $f_e(t)$

$$\int_{-T}^T f_e(t)dt = 2 \int_0^T f_e(t)dt$$

For an odd function  $f_o(t)$

$$\int_{-T}^T f_o(t)dt = 0$$

## Even/Odd Representation of an Arbitrary Function

A function  $f(t)$  that is neither even nor odd can be represented as an even function by use of:

$$f_e(t) = \frac{1}{2}[f(t) + f(-t)]$$

or as an odd function by use of:

$$f_o(t) = \frac{1}{2}[f(t) - f(-t)]$$

Adding these together, an abitrary signal can be represented as

$$f(t) = f_e(t) + f_o(t)$$

That is, any function of time can be expressed as the sum of an even and an odd function.

### Example 1

Is the Dirac delta  $\delta(t)$  an even or an odd function of time?

### Solution

Let  $f(t)$  be an arbitrary function of time that is continous at  $t = t_0$ . Then by the sifting property of the delta function

and for  $t_0 = 0$

Also for an even function  $f_e(t)$

and for an odd function  $f_o(t)$

### Even or odd?

An odd function  $f_o(t)$  evaluated at  $t = 0$  is zero, that is  $f_o(0) = 0$ .

Hence

$$\int_{-\infty}^{\infty} f_o(t)\delta(t)dt = f_o(0) = 0$$

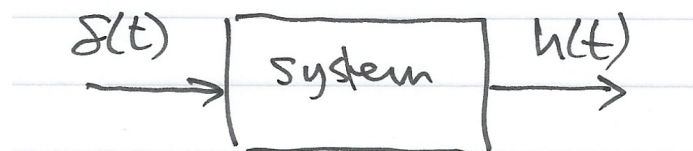
Hence the product  $f_o(t)\delta(t)$  is odd function of  $t$ .

Since  $f_o(t)$  is odd,  $\delta(t)$  must be even because only an *even* function multiplied by an *odd* function can result in an *odd* function.

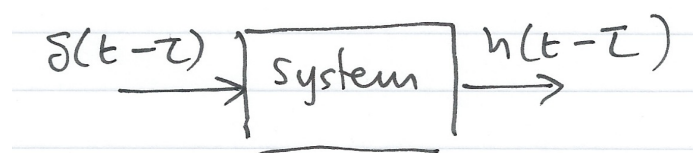
(Even times even or odd times odd produces an even function. See earlier slide)

## Time Convolution

Consider a system whose input is the Dirac delta ( $\delta(t)$ ), and its output is the **impulse response**  $h(t)$ . We can represent the input-output relationship as a block diagram

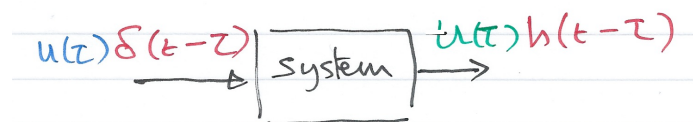


In general



### Add an arbitrary input

Let  $u(t)$  be any input whose value at  $t = \tau$  is  $u(\tau)$ , Then because of the sampling property of the delta function



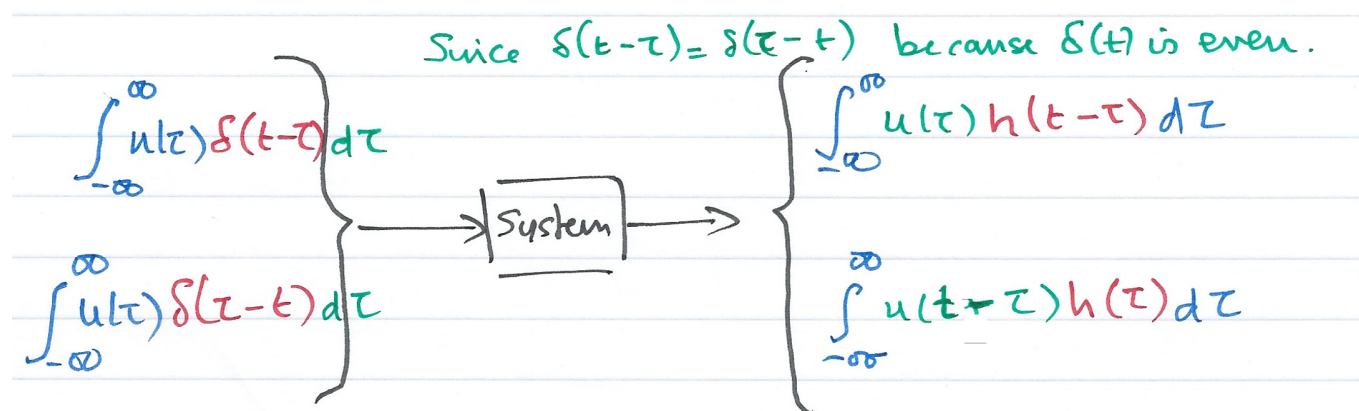
(output is  $u(\tau)h(t-\tau)$ )

### Integrate both sides

Integrating both sides over all values of  $\tau$  ( $-\infty < \tau < \infty$ ) and making use of the fact that the delta function is even, i.e.

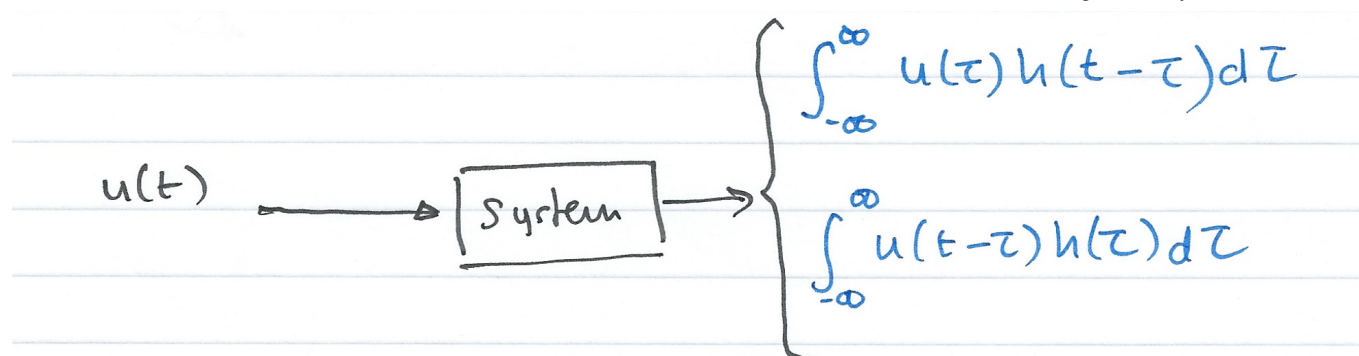
$$\delta(t-\tau) = \delta(\tau-t)$$

we have:



### Use the sifting property of delta

The second integral on the left side reduces to  $u(t)$



## The Convolution Integral

The integral

$$\int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau$$

or

$$\int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau$$

is known as the *convolution integral*; it states that if we know the impulse response of a system, we can compute its time response to any input by using either of the integrals.

The convolution integral is usually written  $u(t) * h(t)$  or  $h(t) * u(t)$  where the asterisk (\*) denotes convolution.

## Second Hour

- Graphical Evaluation of the Convolution Integral
- System Response by Laplace

## Graphical Evaluation of the Convolution Integral

The convolution integral is most conveniently evaluated by a graphical evaluation. The text book gives three examples (6.4-6.6) which we will demonstrate using a [graphical visualization tool](#) developed by Teja Muppirala of the Mathworks.

The tool: [convolutiondemo.m](#) (see [license.txt](#)).

```
clear all
cd ../matlab/convolution_demo
pwd
format compact
```

```
% cd ../matlab/convolution_demo
convolutiondemo % ignore warnings
```

## Convolution by Graphical Method - Summary of Steps

For simplicity, we give the rules for  $u(t)$ , but the procedure is the same if we reflect and slide  $h(t)$

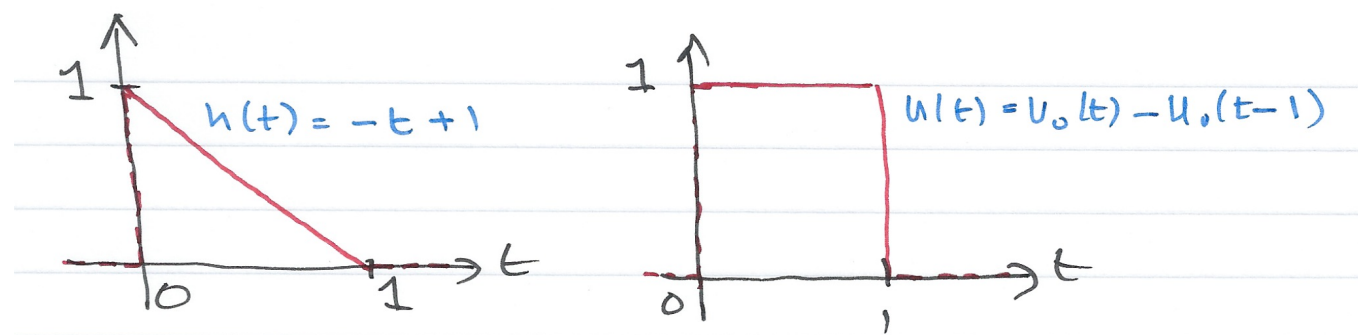
1. Substitute  $u(t)$  with  $u(\tau)$  – this is a simple change of variable. It doesn't change the definition of  $u(t)$ .
1. Reflect  $u(\tau)$  about the vertical axis to form  $u(-\tau)$
1. Slide  $u(-\tau)$  to the right a distance  $t$  to obtain  $u(t-\tau)$
1. Multiply the two signals to obtain the product  $u(t-\tau)h(\tau)$

1. Integrate the product over all  $\tau$  from  $-\infty$  to  $\infty$ .

## Example 2

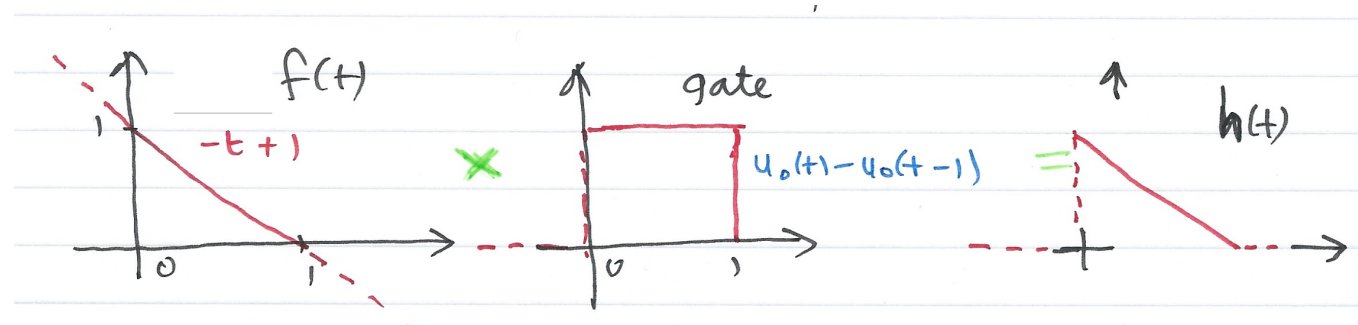
(This is example 6.4 in the Karris)

The signals  $h(t)$  and  $u(t)$  are shown below. Compute  $h(t) * u(t)$  using the graphical technique.



$h(t)$

The signal  $h(t)$  is the straight line  $f(t) = -t + 1$  but this is defined only between  $t = 0$  and  $t = 1$ . We thus need to gate the function by multiplying it by  $u_0(t) - u_0(t - 1)$  as illustrated below:



## convolutiondemo settings

- Let `f = heaviside(t)-heaviside(t-1) % u(t)`
- Let `g = (t-1)*(heaviside(t)-heaviside(t-1)) % h(t)`
- Set range  $-2 < \tau < 2$

## Summary of result

1. For  $t < 0$ :  $u(t - \tau)h(\tau) = 0$
2. For  $t = 0$ :  $u(t - \tau) = u(-\tau)$  and  $u(-\tau)h(\tau) = 0$
3. For  $0 < t \leq 1$ :  $h * u = \int_0^t (1)(-\tau + 1)d\tau = \tau - \tau^2/2 \Big|_0^t = t - t^2/2$
4. For  $1 < t \leq 2$ :  $h * u = \int_{t-1}^1 (-\tau + 1)d\tau = \tau - \tau^2/2 \Big|_{t-1}^1 = t^2/2 - 2t + 2$
5. For  $2 \leq t$ :  $u(t - \tau)h(\tau) = 0$

## MATLAB Solution

```
syms t tau
u0(t) = heaviside(t)
f(t) = u0(t)-u0(t-1)
g(t) = f(t)*(-t + 1)

int(f(tau)*g(t-tau),tau,-inf,inf)

fplot(ans)
```

## Example 3

This is example 6.5 from Karris.

$$h(t) = e^{-t}$$

$$u(t) = u_0(t) - u_0(t - 1)$$

Answer 3

$$y(t) = \begin{cases} 0 & : t \leq 0 \\ 1 - e^{-t} & : 0 < t \leq 1 \\ e^{-t}(e - 1) & : 1 < t < \infty \end{cases}$$

Check with MATLAB

```
syms t tau
x1=int(exp(-tau),tau,0,t)
```

```
x2=int(exp(-tau),tau,t-1,t)
```

Example 4

This is example 6.6 from the text book.

$$h(t) = 2(u_0(t) - u_0(t - 1))$$
$$u(t) = u_0(t) - u_0(t - 2)$$

Answer 4

$$y(t) = \begin{cases} 0 & : t \leq 0 \\ 2t & : 0 < t \leq 1 \\ 2 & : 1 < t \leq 2 \\ -2t + 6 & : 2 < t \leq 3 \\ 0 & : 3 \leq t \end{cases}$$

System Response by Laplace

In the discussion of Laplace, we stated that

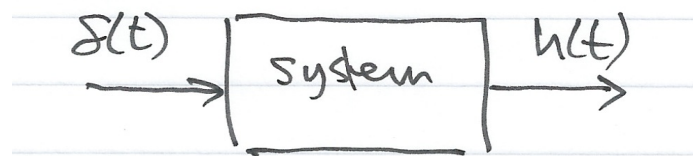


$$\mathcal{L}\{f(t) * g(t)\} = F(s)G(s)$$

We can use this property to make the solution of convolution problems even simpler.

## Impulse Response and Transfer Functions

Returning to the example we started with



Then the impulse response of the system  $h(t)$  will be given by:

$$\mathcal{L}\{h(t) * \delta(t)\} = H(s)\Delta(s)$$

Where  $H(s)$  be the laplace transform of the impulse response of the system  $h(t)$ . From properties of the Laplace transform we know that

$$\delta(t) \Leftrightarrow 1$$

so that  $\Delta(s) = 1$  and

$$h(t) * \delta(t) \Leftrightarrow H(s).1 = H(s)$$

A consequence of this is that the transform of the impulse response  $h(t)$  of a system with transfer function  $H(s)$  is completely defined by the transfer function itself.

Previously we argued that the response of system with impulse response  $h(t)$  was given by the convolution integrals:

$$h(t) * u(t) = \int_{-\infty}^{\infty} u(\tau)h(t - \tau)d\tau = \int_{-\infty}^{\infty} u(t - \tau)h(\tau)d\tau$$

Thus the Laplace transform of any system subject to an input  $u(t)$  is simply

$$Y(s) = H(s)U(s)$$

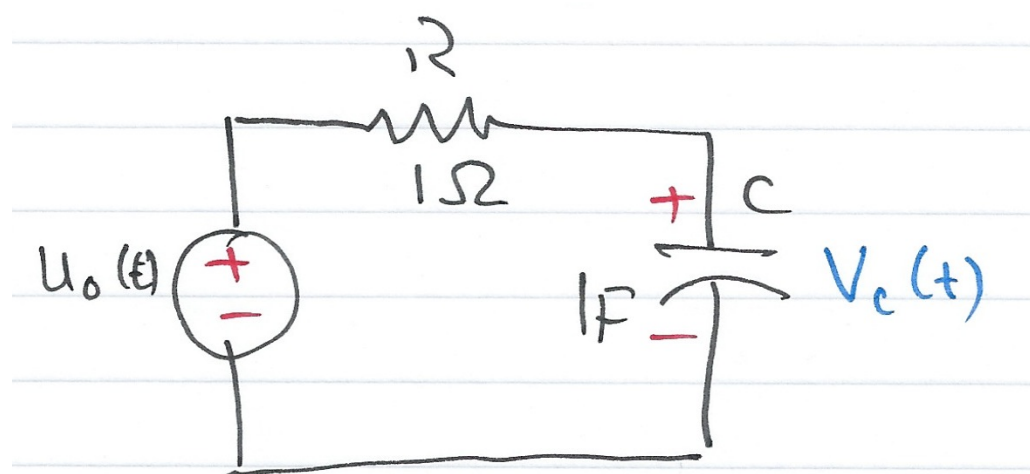
and

$$y(t) = \mathcal{L}^{-1}\{H(s)U(s)\}$$

Using tables, solution of a convolution problem by Laplace is usually simpler than using convolution directly.

## Example 5

This is example 6.7 from Karris.



For the circuit shown above, show that the transfer function of the circuit is:

$$H(s) = \frac{V_c(s)}{V_s(s)} = \frac{1/RC}{s + 1/RC}$$



Determine the impulse response  $h(t)$  of the circuit and the response of the capacitor voltage when the input is the unit step function  $u_0(t)$  and  $v_c(0^-) = 0$ .

Assume  $C = 1 \text{ F}$  and  $R = 1 \text{ } \Omega$ .

## Solution 5a - Impulse response

## Solution 5b - Step response

## Homework

Verify this result using the convolution integral

$$h(t) * u(t) = \int_{-\infty}^{\infty} u(\tau)h(t - \tau)d\tau$$

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