

# The Impulse Response and Convolution

## Colophon

An annotatable worksheet for this presentation is available as [Worksheet 8](https://cpjobling.github.io/eg-247-textbook/laplace_transform/5/worksheet8.html) ([https://cpjobling.github.io/eg-247-textbook/laplace\\_transform/5/worksheet8.html](https://cpjobling.github.io/eg-247-textbook/laplace_transform/5/worksheet8.html)).

- The source code for this page is [content/laplace\\_transform/5/convolution.ipynb](https://github.com/cpjobling/eg-247-textbook/blob/master/content/laplace_transform/5/convolution.ipynb) ([https://github.com/cpjobling/eg-247-textbook/blob/master/content/laplace\\_transform/5/convolution.ipynb](https://github.com/cpjobling/eg-247-textbook/blob/master/content/laplace_transform/5/convolution.ipynb)).
- You can view the notes for this presentation as a webpage (HTML ([https://cpjobling.github.io/eg-247-textbook/laplace\\_transform/5/convolution.html](https://cpjobling.github.io/eg-247-textbook/laplace_transform/5/convolution.html))).
- This page is downloadable as a PDF ([https://cpjobling.github.io/eg-247-textbook/laplace\\_transform/5/convolution.pdf](https://cpjobling.github.io/eg-247-textbook/laplace_transform/5/convolution.pdf)) file.

## Scope and Background Reading

This section is an introduction to the impulse response of a system and time convolution. Together, these can be used to determine a Linear Time Invariant (LTI) system's time response to any signal.

As we shall see, in the determination of a system's response to a signal input, time convolution involves integration by parts and is a tricky operation. But time convolution becomes multiplication in the Laplace Transform domain, and is much easier to apply.

The material in this presentation and notes is based on [Chapter 6](https://ebookcentral.proquest.com/lib/swansea-ebooks/reader.action?ppg=185&docID=3384197&tm=1518698533541) (<https://ebookcentral.proquest.com/lib/swansea-ebooks/reader.action?ppg=185&docID=3384197&tm=1518698533541>) of {% cite karris %}.

## Agenda

The material to be presented is:

- Even and Odd Functions of Time
- Time Convolution

- Graphical Evaluation of the Convolution Integral
- System Response by Laplace

## Even and Odd Functions of Time

(This should be revision!)

We need to be reminded of *even* and *odd* functions so that we can develop the idea of time convolution which is a means of determining the time response of any system for which we know its *impulse response* to any signal.

The development requires us to find out if the Dirac delta function ( $\delta(t)$ ) is an *even* or an *odd* function of time.

### Even Functions of Time

A function  $f(t)$  is said to be an *even function* of time if the following relation holds

$$f(-t) = f(t)$$

that is, if we relace  $t$  with  $-t$  the function  $f(t)$  does not change.

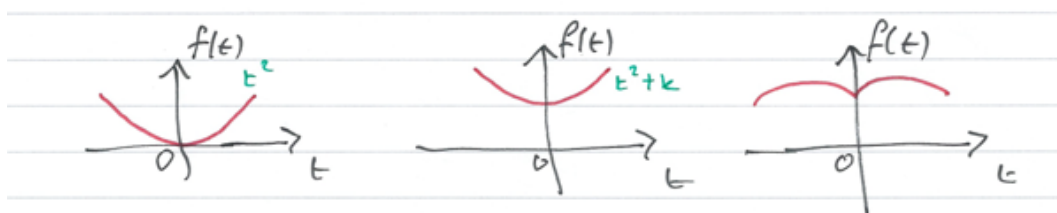
Polynomials with even exponents only, and with or without constants, are even functions.

For example:

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots$$

is even.

### Other Examples of Even Functions



## Odd Functions of Time

A function  $f(t)$  is said to be an *odd function* of time if the following relation holds

$$-f(-t) = f(t)$$

that is, if we relace  $t$  with  $-t$ , we obtain the negative of the function  $f(t)$ .

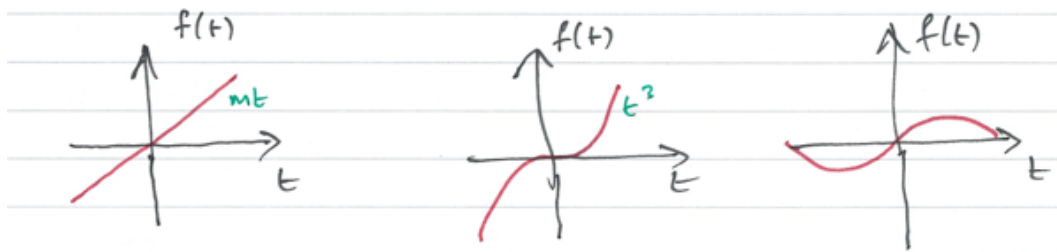
Polynomials with odd exponents only, and no constants, are odd functions.

For example:

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots$$

is odd.

## Other Examples of Odd Functions



## Observations

- For odd functions  $f(0) = 0$ .
- If  $f(0) = 0$  we should not conclude that  $f(t)$  is an odd function. *c.f.*  $f(t) = t^2$  is even, not odd.
- The product of *two even* or *two odd* functions is an even function.
- The product of an even and an odd function, is an odd function.

In the following  $f_e(t)$  will denote an even function and  $f_o(t)$  an odd function.

## Time integrals of even and odd functions

For an even function  $f_e(t)$

$$\int_{-T}^T f_e(t) dt = 2 \int_0^T f_e(t) dt$$

For an odd function  $f_o(t)$

$$\int_{-T}^T f_o(t) dt = 0$$

## Even/Odd Representation of an Arbitrary Function

A function  $f(t)$  that is neither even nor odd can be represented as an even function by use of:

$$f_e(t) = \frac{1}{2}[f(t) + f(-t)]$$

or as an odd function by use of:

$$f_o(t) = \frac{1}{2}[f(t) - f(-t)]$$

Adding these together, an arbitrary signal can be represented as

$$f(t) = f_e(t) + f_o(t)$$

That is, any function of time can be expressed as the sum of an even and an odd function.

### Example 1

Is the Dirac delta  $\delta(t)$  an *even* or an *odd* function of time?

We'll decide in class.

### Solution to example 1

Let  $f(t)$  be an arbitrary function of time that is continuous at  $t = t_0$ . Then by the sifting property of the delta function

$$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)$$

and for  $t_0 = 0$

$$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$$

Also for an even function  $f_e(t)$

$$\int_{-\infty}^{\infty} f_e(t)\delta(t)dt = f_e(0)$$

and for an odd function  $f_o(t)$

$$\int_{-\infty}^{\infty} f_o(t)\delta(t)dt = f_o(0)$$

### Even or odd?

An odd function  $f_o(t)$  evaluated at  $t = 0$  is zero, that is  $f_o(0) = 0$ .

Hence

$$\int_{-\infty}^{\infty} f_o(t)\delta(t)dt = f_o(0) = 0$$

Hence the product  $f_o(t)\delta(t)$  is odd function of  $t$ .

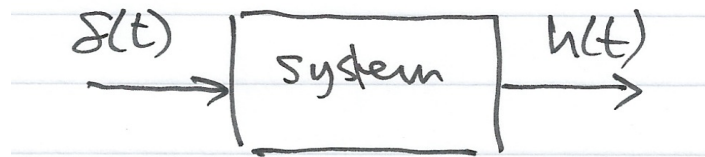
Since  $f_o(t)$  is odd,  $\delta(t)$  must be even because only an *even* function multiplied by an *odd* function can result in an *odd* function.

(Even times even or odd times odd produces an even function. See earlier slide)

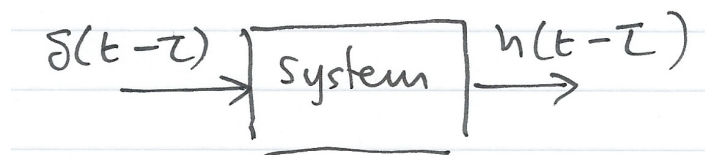
## Time Convolution

Consider a system whose input is the Dirac delta ( $\delta(t)$ ), and its output is the **impulse response**  $h(t)$ .

We can represent the input-output relationship as a block diagram

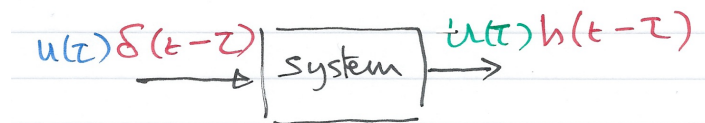


**In general**



**Add an arbitrary input**

Let  $u(t)$  be any input whose value at  $t = \tau$  is  $u(\tau)$ , Then because of the sampling property of the delta function



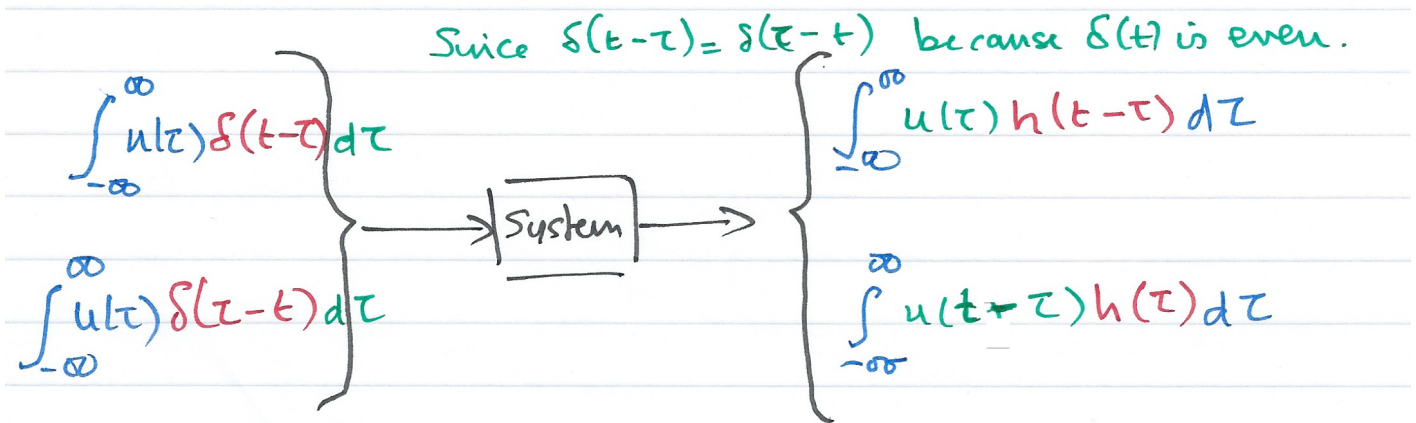
(output is  $u(\tau)h(t - \tau)$ )

**Integrate both sides**

Integrating both sides over all values of  $\tau$  ( $-\infty < \tau < \infty$ ) and making use of the fact that the delta function is even, i.e.

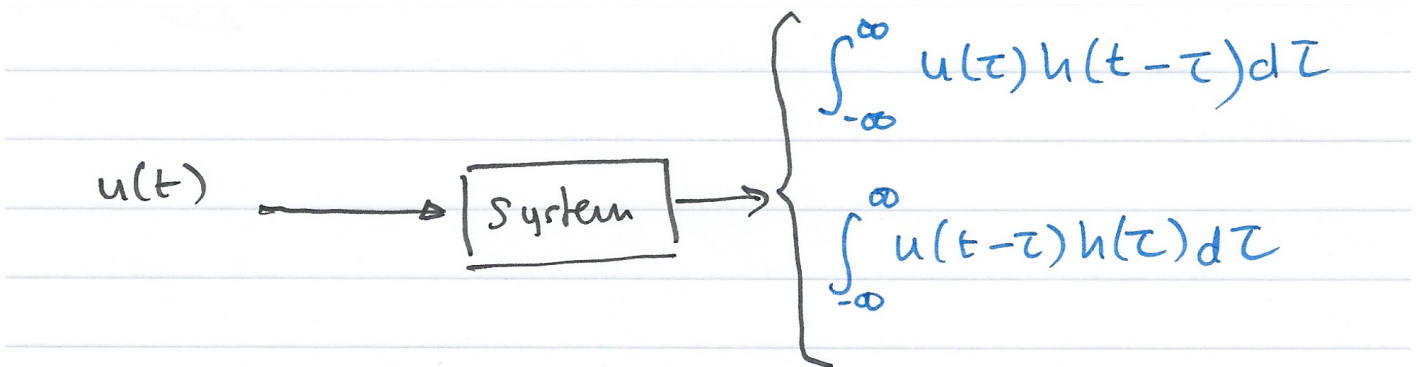
$$\delta(t - \tau) = \delta(\tau - t)$$

we have:



## Use the sifting property of delta

The second integral on the left side reduces to  $u(t)$



## The Convolution Integral

The integral

$$\int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau$$

or

$$\int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau$$

is known as the *convolution integral*; it states that if we know the impulse response of a system, we can compute its time response to any input by using either of the integrals.

The convolution integral is usually written  $u(t) * h(t)$  or  $h(t) * u(t)$  where the asterisk (\*) denotes convolution.

## Graphical Evaluation of the Convolution Integral

The convolution integral is most conveniently evaluated by a graphical evaluation. The text book gives three examples (6.4-6.6) which we will demonstrate in class using a [graphical visualization tool](https://www.mathworks.co.uk/matlabcentral/fileexchange/25199-graphical-demonstration-of-convolution) (<https://www.mathworks.co.uk/matlabcentral/fileexchange/25199-graphical-demonstration-of-convolution>) developed by Teja Muppirala of the Mathworks.

The tool: [convolutiondemo.m](https://cpjobling.github.io/eg-247-textbook/laplace_transform/matlab/convolution_demo/convolutiondemo.m) ([https://cpjobling.github.io/eg-247-textbook/laplace\\_transform/matlab/convolution\\_demo/convolutiondemo.m](https://cpjobling.github.io/eg-247-textbook/laplace_transform/matlab/convolution_demo/convolutiondemo.m)) (see [license.txt](https://cpjobling.github.io/eg-247-textbook/laplace_transform/matlab/convolution_demo/license.txt) ([https://cpjobling.github.io/eg-247-textbook/laplace\\_transform/matlab/convolution\\_demo/license.txt](https://cpjobling.github.io/eg-247-textbook/laplace_transform/matlab/convolution_demo/license.txt))).

```
In [1]: clear all
        cd ../matlab/convolution_demo
        pwd
        format compact

ans =

      '/Users/eechris/dev/eg-247-textbook/content/laplace_transform/
matlab/convolution_demo'
```

```
In [2]: convolutiondemo % ignore warnings
```



```

Warning: The EraseMode property is no longer supported and will er
ror in a future release.
> In convolutiondemo>convolutiondemo_LayoutFcn (line 398)
    In convolutiondemo>gui_mainfcn (line 1188)
    In convolutiondemo (line 44)
Warning: The EraseMode property is no longer supported and will er
ror in a future release.
> In convolutiondemo>convolutiondemo_LayoutFcn (line 449)
    In convolutiondemo>gui_mainfcn (line 1188)
    In convolutiondemo (line 44)
Warning: The EraseMode property is no longer supported and will er
ror in a future release.
> In convolutiondemo>convolutiondemo_LayoutFcn (line 500)
    In convolutiondemo>gui_mainfcn (line 1188)
    In convolutiondemo (line 44)
Warning: The EraseMode property is no longer supported and will er
ror in a future release.
> In convolutiondemo>convolutiondemo_LayoutFcn (line 551)
    In convolutiondemo>gui_mainfcn (line 1188)
    In convolutiondemo (line 44)
Warning: The EraseMode property is no longer supported and will er
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> In convolutiondemo>convolutiondemo_LayoutFcn (line 621)
    In convolutiondemo>gui_mainfcn (line 1188)
    In convolutiondemo (line 44)
Warning: The EraseMode property is no longer supported and will er
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> In convolutiondemo>convolutiondemo_LayoutFcn (line 672)
    In convolutiondemo>gui_mainfcn (line 1188)
    In convolutiondemo (line 44)
Warning: The EraseMode property is no longer supported and will er
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> In convolutiondemo>convolutiondemo_LayoutFcn (line 723)
    In convolutiondemo>gui_mainfcn (line 1188)
    In convolutiondemo (line 44)
Warning: The EraseMode property is no longer supported and will er
ror in a future release.
> In convolutiondemo>convolutiondemo_LayoutFcn (line 774)
    In convolutiondemo>gui_mainfcn (line 1188)
    In convolutiondemo (line 44)

```

## Convolution by Graphical Method - Summary of Steps

For simplicity, we give the rules for  $u(t)$ , but the procedure is the same if we reflect and slide  $h(t)$

1. Substitute  $u(t)$  with  $u(\tau)$  – this is a simple change of variable. It doesn't change the definition of  $u(t)$ .

1. Reflect  $u(\tau)$  about the vertical axis to form  $u(-\tau)$
1. Slide  $u(-\tau)$  to the right a distance  $t$  to obtain  $u(t - \tau)$
1. Multiply the two signals to obtain the product  $u(t - \tau)h(\tau)$
1. Integrate the product over all  $\tau$  from  $-\infty$  to  $\infty$ .

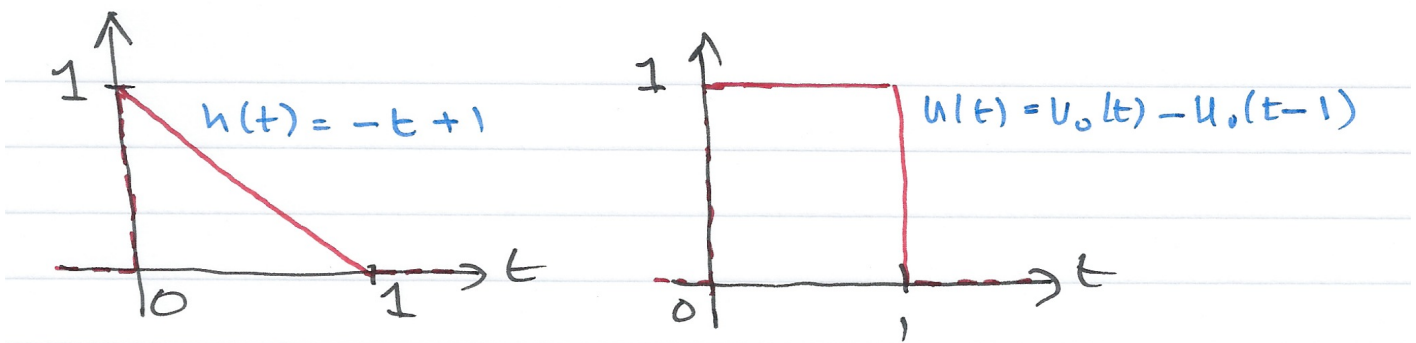
## Examples

We will do these live in class.

### Example 2

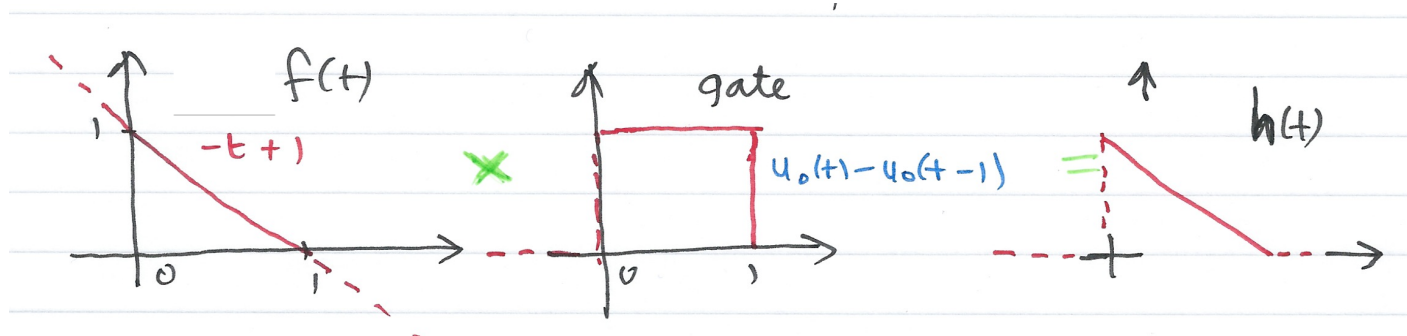
(This is example 6.4 in the textbook)

The signals  $h(t)$  and  $u(t)$  are shown below. Compute  $h(t) * u(t)$  using the graphical technique.



**$h(t)$** 

The signal  $h(t)$  is the straight line  $f(t) = -t + 1$  but this is defined only between  $t = 0$  and  $t = 1$ . We thus need to gate the function by multiplying it by  $u_0(t) - u_0(t - 1)$  as illustrated below:



Thus

$$h(t) \Leftrightarrow H(s)$$

$$h(t) = (-t + 1)(u_0(t) - u_0(t - 1)) = (-t + 1)u_0(t) - (-(t - 1)u_0(t - 1)) = -tu_0(t) + u_0(t) + (t - 1)u_0(t - 1)$$

$$-tu_0(t) + u_0(t) + (t - 1)u_0(t - 1) \Leftrightarrow -\frac{1}{s^2} + \frac{1}{s} + \frac{e^{-s}}{s^2}$$

$$H(s) = \frac{s + e^{-s} - 1}{s^2}$$

 **$u(t)$** 

The input  $u(t)$  is the gating function:

$$u(t) = u_0(t) - u_0(t - 1)$$

so

$$U(s) = \frac{1}{s} - \frac{e^{-s}}{s} = \frac{1 - e^{-s}}{s}$$

**Prepare for convolutiondemo**

To prepare this problem for evaluation in the `convolutiondemo` tool, we need to determine the Laplace Transforms of  $h(t)$  and  $u(t)$ .

**convolutiondemo settings**

- Let  $g = (1 - \exp(-s))/s$
- Let  $h = (s + \exp(-s) - 1)/s^2$
- Set range  $-2 < \tau < 2$

**Summary of result**

1. For  $t < 0$ :

$$u(t - \tau)h(\tau) = 0$$

2. For  $t = 0$ :  $u(t - \tau) = u(-\tau)$  and  $u(-\tau)h(\tau) = 0$

3. For  $0 < t \leq 1$ :

$$h * u = \int_0^t (1)(-\tau + 1)d\tau = \tau - \tau^2/2 \Big|_0^t = t - t^2/2$$

4. For  $1 < t \leq 2$ :

$$h * u = \int_{t-1}^1 (-\tau + 1)d\tau = \tau - \tau^2/2 \Big|_{t-1}^1 = t^2/2 - 2t + 2$$

5. For  $2 \leq t$ :

$$u(t - \tau)h(\tau) = 0$$

**Example 3**

This is example 6.5 from the text book.

$$h(t) = e^{-t}$$

$$u(t) = u_0(t) - u_0(t - 1)$$

**Answer 3**

$$y(t) = \begin{cases} 0 : t \leq 0 \\ 1 - e^{-t} : 0 < t \leq 1 \\ e^{-t} (e - 1) : 1 < t < \infty \end{cases}$$

**Check with MATLAB**

```
In [11]: syms t tau
x1=int(exp(-tau),tau,0,t)

x1 =
1 - exp(-t)
```

```
In [12]: x2=int(exp(-tau),tau,t-1,t)

x2 =
exp(-t)*(exp(1) - 1)
```

**Example 4**

This is example 6.6 from the text book.

$$\begin{aligned}h(t) &= 2(u_0(t) - u_0(t - 1)) \\ u(t) &= u_0(t) - u_0(t - 2)\end{aligned}$$

**Answer 4**

$$y(t) = \begin{cases} 0 & : t \leq 0 \\ 2t & : 0 < t \leq 1 \\ 2 & : 1 < t \leq 2 \\ -2t + 6 & : 2 < t \leq 3 \\ 0 & : 3 \leq t \end{cases}$$

**System Response by Laplace**

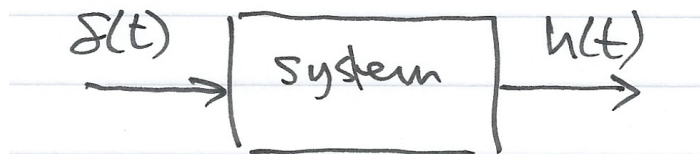
In the discussion of Laplace, we stated that

$$\mathcal{L}\{f(t) * g(t)\} = F(s)G(s)$$

We can use this property to make the solution of convolution problems even simpler.

## Impulse Response and Transfer Functions

Returning to the example we started with



Then the impulse response of the system  $h(t)$  will be given by:

$$\mathcal{L}\{h(t) * \delta(t)\} = H(s)\Delta(s)$$

Where  $H(s)$  be the laplace transform of the impulse response of the system  $h(t)$ . From properties of the Laplace transform we know that

$$\delta(t) \Leftrightarrow 1$$

so that  $\Delta(s) = 1$  and

$$h(t) * \delta(t) \Leftrightarrow H(s).1 = H(s)$$

A consequence of this is that the transform of the impulse response  $h(t)$  of a system with transfer function  $H(s)$  is completely defined by the transfer function itself.

Previously we argued that the response of system with impulse response  $h(t)$  was given by the convolution integrals:

$$h(t) * u(t) = \int_{-\infty}^{\infty} u(\tau)h(t - \tau)d\tau = \int_{-\infty}^{\infty} u(t - \tau)h(\tau)d\tau$$

Thus the Laplace transform of any system subject to an input  $u(t)$  is simply

$$Y(s) = H(s)U(s)$$

and

$$y(t) = \mathcal{L}^{-1}\{G(s)U(s)\}$$

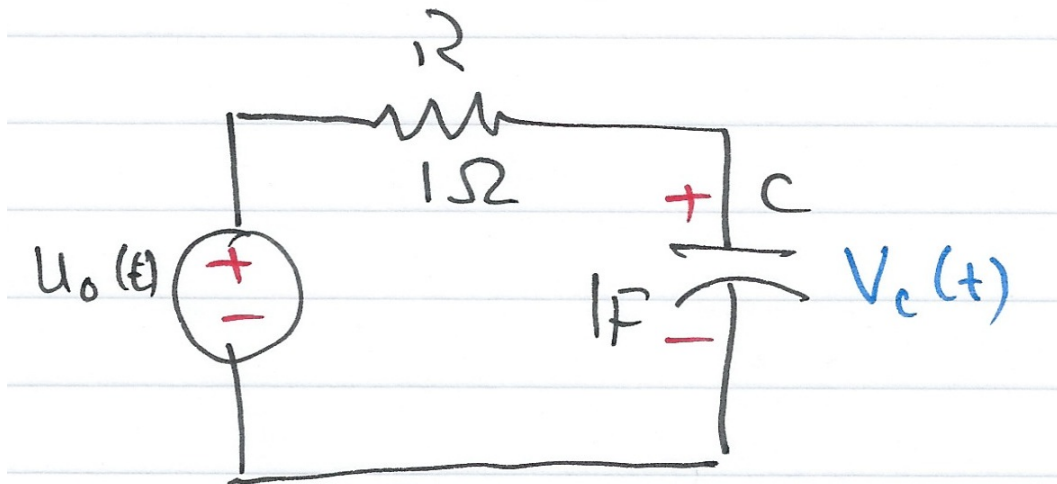
Using tables, solution of a convolution problem by Laplace is usually simpler than using convolution directly.

## More Examples

We will work through these in class

**Example 5**

This is example 6.7 from the textbook.



For the circuit shown above, show that the transfer function of the circuit is:

$$H(s) = \frac{V_c(s)}{V_s(s)} = \frac{1/RC}{s + 1/RC}$$

Hence determine the impulse response  $h(t)$  of the circuit and the response of the capacitor voltage when the input is the unit step function  $u_0(t)$  and  $v_c(0^-) = 0$ .

Assume  $C = 1 \text{ F}$  and  $R = 1 \Omega$ .

**Solution 5a - Impulse response**

$$h(t) = \frac{1}{RC} e^{-t/RC} u_0(t)$$

which when  $C = 1 \text{ F}$  and  $R = 1 \Omega$  reduces to

$$h(t) = e^{-t} u_0(t)$$

**Solution 5b - Step response**

$$\begin{aligned}
 h(t) &= e^{-t} u_0(t) \Leftrightarrow H(s) = \frac{1}{s+1} \\
 u(t) &= u_0(t) \Leftrightarrow U(s) = \frac{1}{s} \\
 y(t) &= h(t) * u(t) \Leftrightarrow Y(s) = H(s)U(s) = \left( \frac{1}{s+1} \right) \times \left( \frac{1}{s} \right)
 \end{aligned}$$

By PFE

$$Y(s) = \frac{r_1}{s+1} + \frac{r_2}{s}$$

The residues are  $r_1 = -1$ ,  $r_2 = 1$ , so

$$Y(s) = -\frac{1}{s+1} + \frac{1}{s} \Leftrightarrow y(t) = (1 - e^{-t}) u_0(t)$$

**Homework**

Verify this result using the convolution integral

$$h(t) * u(t) = \int_{-\infty}^{\infty} u(\tau) h(t - \tau) d\tau$$

**Reference**

{% bibliography --cited %}