Models of Discrete-Time Systems

Colophon

An annotatable worksheet for this presentation is available as **Worksheet 17**.

- The Jupytext source code for this page is dt_systems/4/dt_models.md.
- You can view the notes for this presentation as a webpage (HTML).
- This page is downloadable as a PDF file.

Scope and Background Reading

In this section we will explore digital systems and learn more about the z-transfer function model.

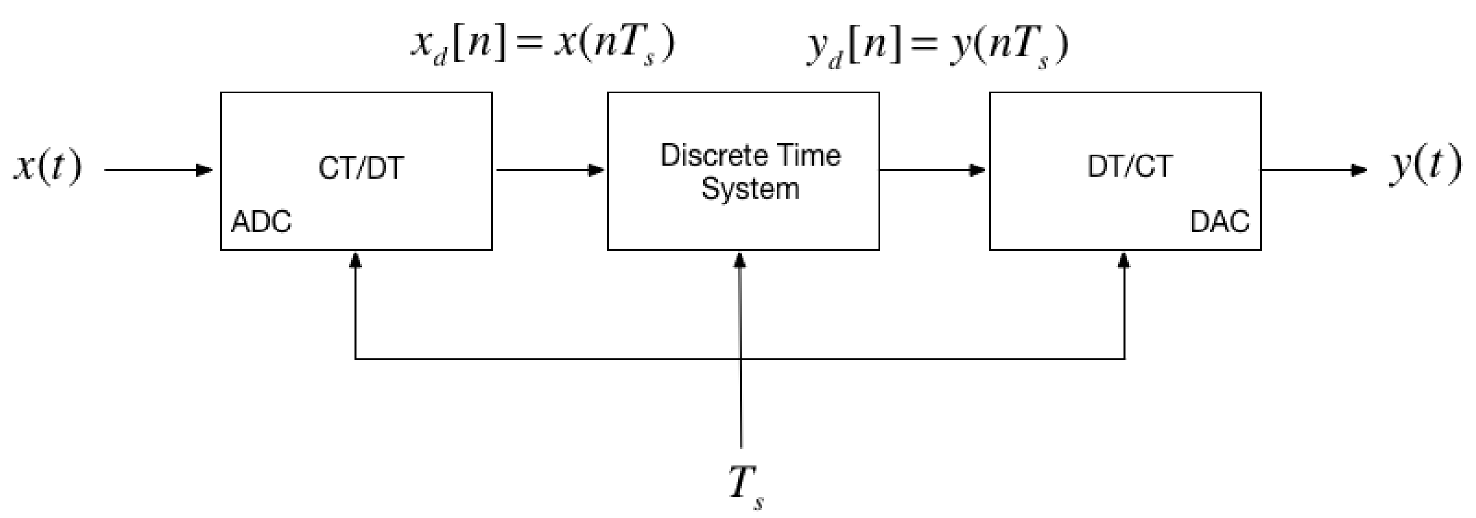
The material in this presentation and notes is based on Chapter 9 (Starting at Section 9.7) of [Kar12]. I have skipped the section on digital state-space models.

Agenda

- Discrete Time Systems
- Transfer Functions in the Z-Domain
- Modelling digital systems in Matlab/Simulink
- Converting Continuous Time Systems to Discrete Time Systems
- Example: Digital Butterworth Filter

Discrete Time Systems

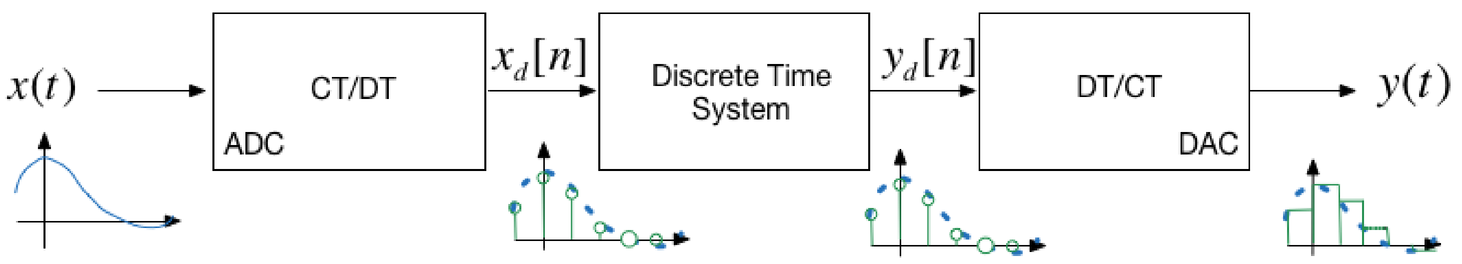
In the lecture that introduced the z-transform we talked about the representation of a discrete-time (DT) system by the model shown below:



In this session, we want to explore the contents of the central block.

DT System as a Sequence Processor

- As noted in the previous slide, the discrete time system (DTS) `takes as an input the sequence $x_d[n]$[1] which in a physical signal would be obtained by sampling the continuous time signal $x(t)$ using an analogue to digital converter (ADC).
- It produces another sequence $y_d[n]$ by *processing* the input sequence in some way.
- The output sequence is converted into an analogue signal $y(t)$ by a digital to analogue converter (DAC).



What is the nature of the DTS?

- The discrete time system (DTS) is a block that converts a sequence $x_d[n]$ into another sequence $y_d[n]$
- The transformation will be a *difference equation* $h[n]$
- By analogy with CT systems, $h[n]$ is the impulse response of the DTS, and $y[n]$ can be obtained by *convolving* $h[n]$ with $x_d[n]$ so:

$$y_d[n] = h[n] * x_d[n]$$

- Taking the z-transform of $h[n]$ we get $H(z)$, and from the transform properties, convolution of the signal $x_d[n]$ by system $h[n]$ will be *multiplication* of the z-transforms:

$$Y_d(z) = H(z)X_d(z)$$

- So, what does $h[n]$ and therefore $H(z)$ look like?

Transfer Functions in the Z-Domain

Let us assume that the sequence transformation is a *difference equation* of the form[2]:

$$\begin{aligned} y[n] + a_1 y[n-1] + a_2 y[n-2] + \cdots + a_k y[n-k] \\ = b_0 x[n] + b_1 u[n-1] + b_2 u[n-2] + \cdots + b_k u[n-k] \end{aligned}$$

Take Z-Transform of both sides

From the z-transform properties

$$f[n-m] \Leftrightarrow z^{-m} F(z)$$

so....

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \cdots + a_k z^{-k} Y(z) = \ldots$$

$$b_0 U(z) + b_1 z^{-1} U(z) + b_2 z^{-2} U(z) + \cdots + b_k z^{-k} U(z)$$

Gather terms

$$\begin{aligned} \left(1 + a_1 z^{-1} + a_2 z^{-2} + \cdots a_k z^{-k}\right) Y(z) = \\ \left(b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_k z^{-k}\right) U(z) \end{aligned}$$

from which ...

$$Y(z) = \left(\frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_k z^{-k}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots a_k z^{-k}}\right) U(z)$$

Define transfer function

We define the *discrete time transfer function* $H(z) := Y(z)/U(z)$ so...

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_k z^{-k}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots a_k z^{-k}}$$

... or more conventionally[3]:

$$H(z) = \frac{b_0 z^k + b_1 z^{k-1} + b_2 z^{k-2} + \cdots b_{k-1} z + b_k}{z^k + a_1 z^{k-1} + a_2 z^{k-2} + \cdots a_{k-1} z + a_k}$$

DT impulse response

The *discrete-time impulse reponse* $h[n]$ is the response of the DT system to the input $x[n] = \delta[n]$

Last week we showed that \$$\mathcal{Z}\left\{\delta[n]\right\}$\$ was defined by the transform pair

$$\delta[n] \Leftrightarrow 1$$

so

$$h[n] = \mathcal{Z}^{-1}\left\{H(z).1\right\} = \mathcal{Z}^{-1}\left\{H(z)\right\}$$

We will work through an example in class.

[Skip next slide in Pre-Lecture]

Example 5

Karris Example 9.10:

The difference equation describing the input-output relationship of a DT system with zero initial conditions, is:

$$y[n] - 0.5y[n-1] + 0.125y[n-2] = x[n] + x[n-1]$$

Compute:

1. The transfer function $H(z)$
2. The DT impulse response $h[n]$
3. The response $y[n]$ when the input $x[n]$ is the DT unit step $u_0[n]$

5.1. The transfer function

$$H(z) = \frac{Y(z)}{U(z)} = \ldots ?$$

5.2. The DT impulse response

Start with:

$$\frac{H(z)}{z} = \frac{z-1}{z^2 + 0.5z + 0.125}$$

Matlab Solution

```
clear all
imatlab_export_fig('print-svg')  % Static svg figures.
cd matlab
pwd
format compact
```

See [dtm_ex1_2.mlx](dtm_ex1_2.mlx). (Also available as [dtm_ex1_2.m](dtm_ex1_2.m).)

The difference equation describing the input-output relationship of the DT system with zero initial conditions, is:

$$y[n] - 0.5y[n-1] + 0.125y[n-2] = x[n] + x[n-1]$$

Transfer function

Numerator $z^2 + z$

```
Nz = [1 1 0];
```

Denominator $z^2 - 0.5z + 0.125$

```
Dz = [1 -0.5 0.125];
```

Poles and residues

```
[r,p,k] = residue(Nz,Dz)
```

Impulse Response

```
Hz = tf(Nz,Dz,1)
hn = impulse(Hz, 15);
```

Plot the response

```
stem([0:15], hn)
grid
title('Example 5 - Part 2')
xlabel('n')
ylabel('Impulse response h[n]')
```

Response as stepwise continuous y(t)

```
impulse(Hz,15)
grid
title('Example 5 - Part 2 - As Analogue Signal')
xlabel('nTs [s]')
ylabel('Impulse response h(t)')
```

5.3. The DT step response

$$Y(z) = H(z)X(z)$$

$$u_0[n] \Leftrightarrow \frac{z}{z-1}$$

We will work through this example in class.

[Skip next slide in Pre-Lecture]

$$\begin{aligned} Y(z) = H(z)U_0(z) &= \tfrac{z^2+z}{z^2+0.5z+0.125} \cdot \tfrac{z}{z-1} \\ &= \tfrac{z(z^2+z)}{(z^2+0.5z+0.125)(z-1)} \end{aligned}$$

$$\frac{Y(z)}{z} = \frac{z^2+z}{(z^2+0.5z+0.125)(z-1)}$$

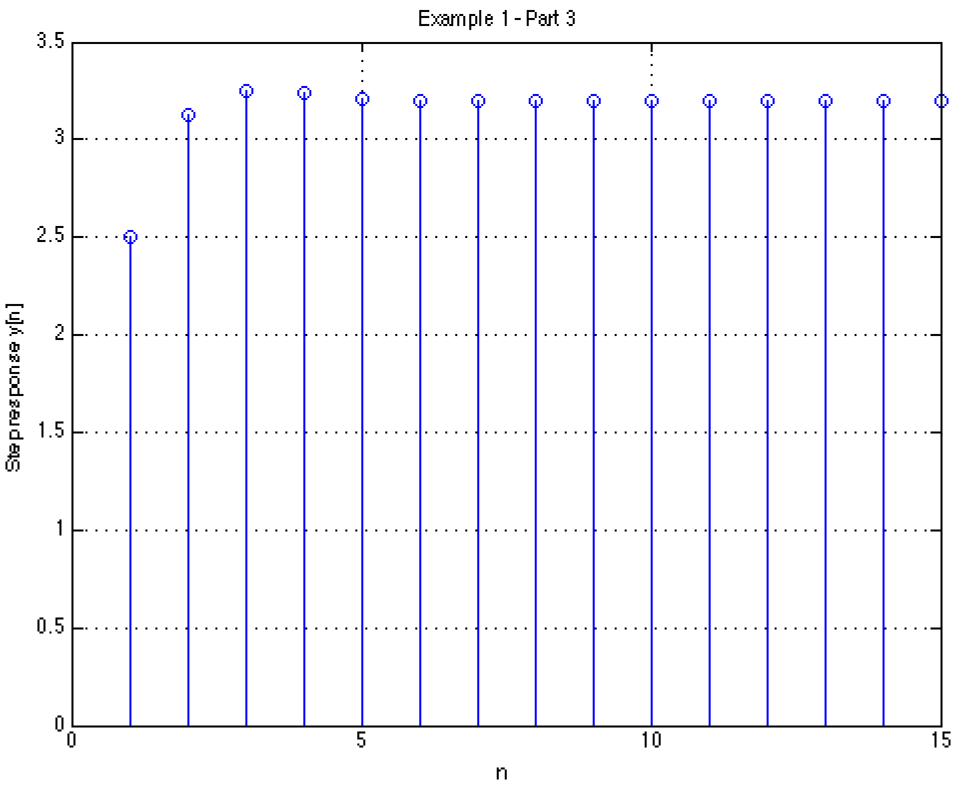
Solved by inverse Z-transform.

Matlab Solution

See dtm_ex1_3.mlx. (Also available as dtm_ex1_3.m.)

```
open dtm_ex1_3
```

Results



Modelling DT systems in MATLAB and Simulink

We will consider some examples in class

MATLAB

Code extracted from [dtm_ex1_3.m]:

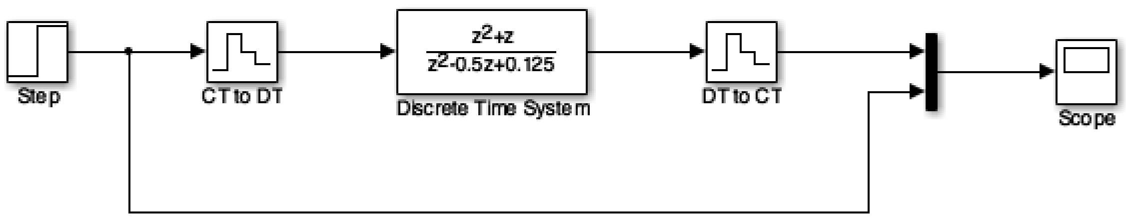
```
Ts = 1;
z = tf('z', Ts);
```

```
Hz = (z^2 + z)/(z^2 - 0.5 * z + 0.125)
```

```
step(Hz)
grid
title('Example 1 - Part 3 - As Analogue Signal')
xlabel('nTs [s]')
ylabel('Step response y(t)')
axis([0,15,0,3.5])
```

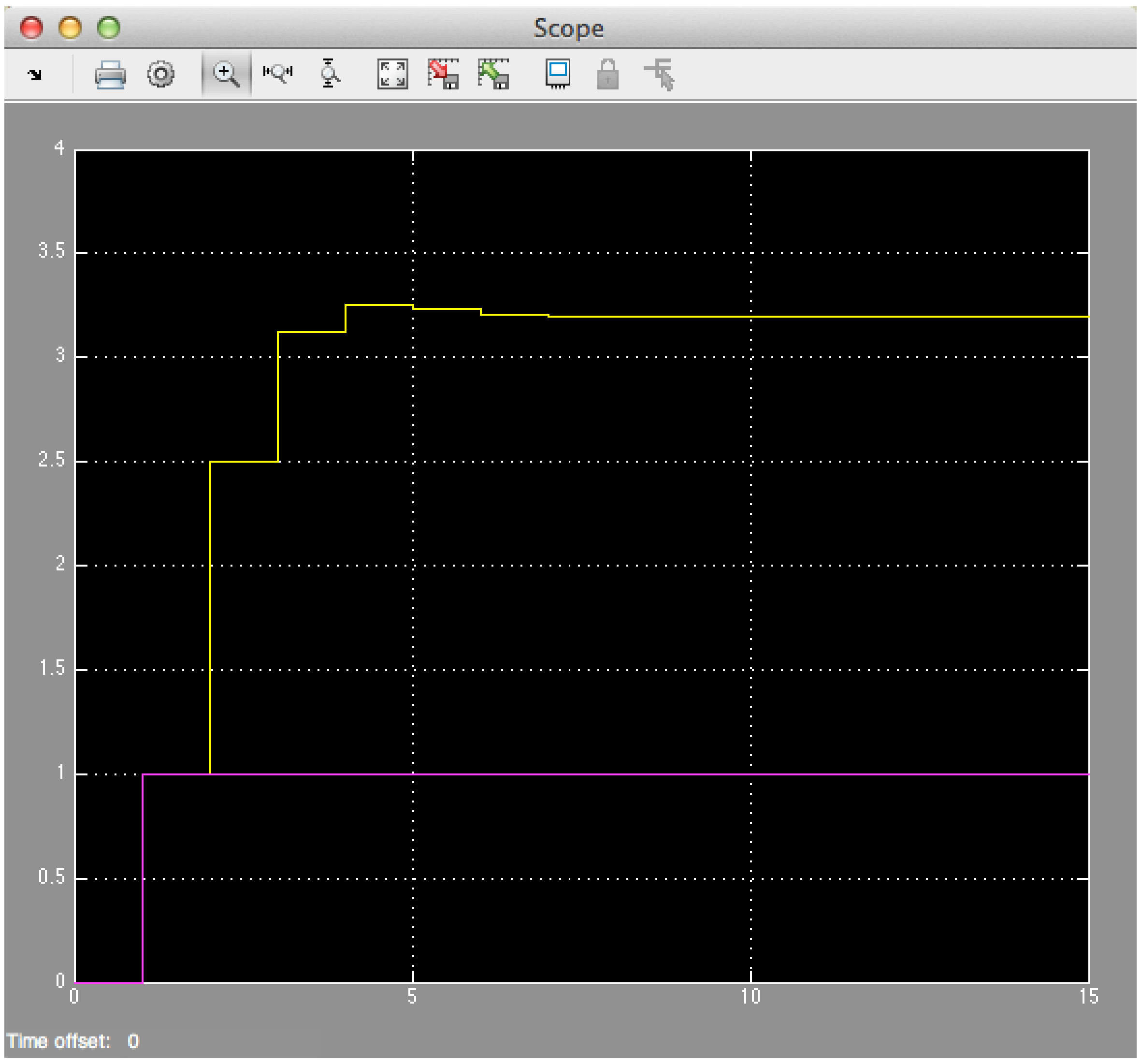
Simulink Model

See [dtm.slx]:




```
dtm
```

Results



Converting Continuous Time Systems to Discrete Time Systems

In analogue electronics, to implement a filter we would need to resort to op-amp circuits with resistors, capacitors and inductors acting as energy dissipation, storage and release devices.

- In modern digital electronics, it is often more convenient to take the original transfer function $H(s)$ and produce an equivalent $H(z)$.
- We can then determine a *difference equation* that will respresent $h[n]$ and implement this as *computer algorithm*.
- Simple storage of past values in memory becomes the repository of past state rather than the integrators and derivative circuits that are needed in the analogue world.

To achieve this, all we need is to be able to do is to *sample* and *process* the signals quickly enough to avoid violating Nyquist-Shannon's sampling theorem.

Continuous System Equivalents

- There is no digital system that uniquely represents a continuous system
- This is because as we are sampling, we only have knowledge of signals being processed at the sampling instants, and need to *reconstruct* the inter-sample behaviour.
- In practice, only a small number of transformations are used.
- The derivation of these is beyond the scope of this module, but **in class** we'll demonstrate the ones that MATLAB provides in a function called `c2d`

MATLAB c2d function

Let's see what the help function says:

```
help c2d
```

```
doc c2d
```

Example: Digital Butterworth Filter

- Design a 2nd-order butterworth low-pass anti-aliasing filter with transfer function $H(s)$ for use in sampling music.
- The cut-off frequency $\omega_c = 20$ kHz and the filter should have an attenuation of at least -80 dB in the stop band.
- Choose a suitable sampling frequency for the audio signal and give the transfer function $H(z)$ and an algorithm to implement $h[n]$

Solution

See digi_butter.mlx.

First determine the cut-off frequency ω_c

$$\omega_c = 2\pi f_c = 2 \times \pi \times 20 \times 10^3 \text{ rad/s}$$

```
wc = 2*pi*20e3
```

$$\omega_c = 125.66 \times 10^3 \text{ rad/s}$$

From the lecture on filters, we know the 2nd-order butterworth filter has transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{\omega_c^2}{s^2 + \omega_c \sqrt{2}\, s + \omega_c^2}$$

Substituting for $\omega_c = 125.6637 \times 10^3$ this is ...?

```
Hs = tf(wc^2,[1 wc*sqrt(2), wc^2])
```

$$H(s) = \frac{15.79 \times 10^9}{s^2 + 177.7 \times 10^3 s + 15.79 \times 10^9}$$

Bode plot

MATLAB:

```
bode(Hs,{1e4,1e8})
grid
```

Sampling Frequency

From the bode diagram, the frequency at which $|H(j\omega)|$ is -80 dB is approx 12.6×10^6 rad/s.

To avoid aliasing, we should choose a sampling frequency twice this = ?

$\omega_s = 2 \times 12.6 \times 10^6$ rad/s.

```
ws = 2* 12.6e6
```

So

$\omega_s = 25.2 \times 10^6$ rad/s.

Sampling frequency (f_s) in Hz = ?

$$f_s = \omega_s/(2\pi) \text{ Mhz}$$

```
fs = ws/(2*pi)
```

$$f_s = 40.11 \text{ Mhz}$$

Sampling time $T_s = ?$

$T_s = 1/fs$ s

```
Ts = 1/fs
```

$$T_s = 1/f_s \approx 0.25\ \mu\text{s}$$

Digital Butterworth

zero-order-hold equivalent

```
Hz = c2d(Hs, Ts)
```

Step response

```
step(Hz)
```

Algorithm

From previous result:

$$H(z) = \frac{Y(z)}{U(z)} = \frac{486.6 \times 10^{-6} z + 476.5 \times 10^{-6}}{z^2 - 1.956z + 0.9567}$$

Dividing top and bottom by z^2 ...

$$H(z) = \frac{Y(z)}{U(z)} = \frac{486.6 \times 10^{-6} z^{-1} + 476.5 \times 10^{-6} z^{-2}}{1 - 1.956z^{-1} + 0.9567z^{-2}}$$

expanding out ...

$$\begin{aligned} &Y(z) - 1.956z^{-1}Y(z) + 0.9567z^{-2}Y(z) = \\ &\quad 486.6 \times 10^{-6} z^{-1} U(z) + 476.5 \times 10^{-6} z^{-2} U(z) \end{aligned}$$

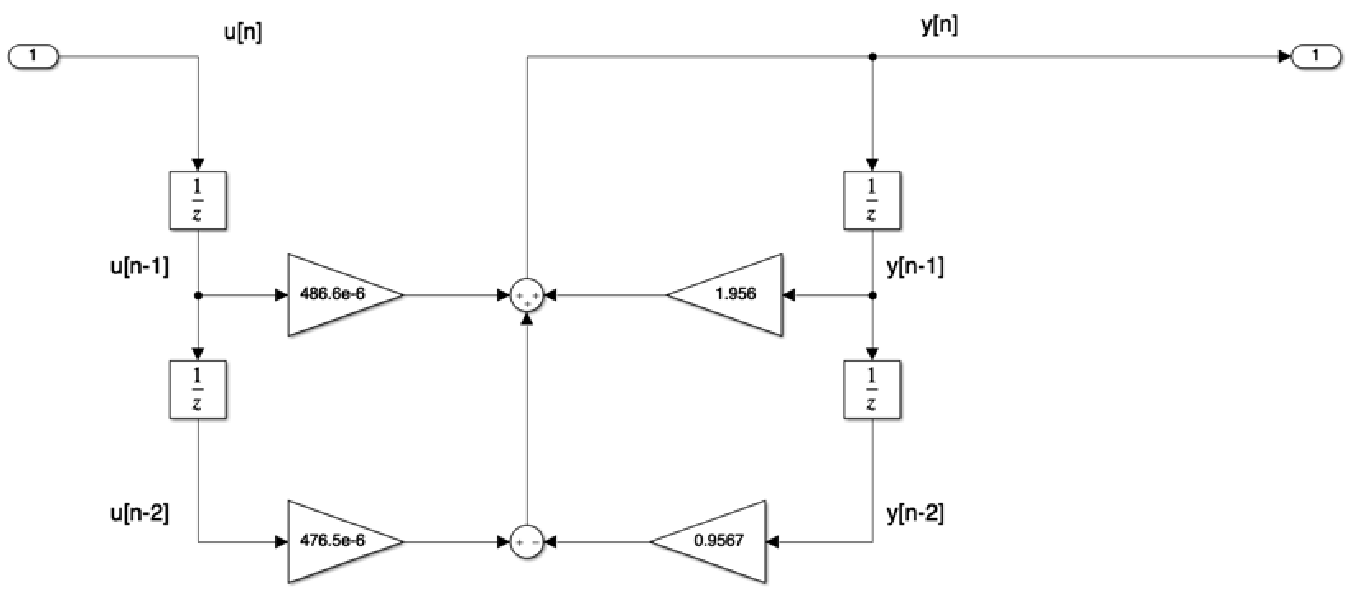
Inverse z-transform gives ...

$$\begin{aligned} &y[n] - 1.956y[n-1] + 0.9567y[n-2] = \\ &\quad 486.6 \times 10^{-6} u[n-1] + 476.5 \times 10^{-6} u[n-2] \end{aligned}$$

in algorithmic form (compute $y[n]$ from past values of u and y) ...

$$\begin{aligned} &y[n] = 1.956y[n-1] - 0.9567y[n-2] + 486.6 \times 10^{-6} u[n-1] + \ldots \\ &\quad 476.5 \times 10^{-6} u[n-2] \end{aligned}$$

Block Diagram of the digital BW filter



As Simulink Model

digifilter.slx

```
open digifilter
```

Convert to code

To implement:

$$y[n] = 1.956y[n-1] - 0.9567y[n-2] + 486.6\times 10^{-6}u[n-1] + 476.5\times 10^{-6}u[n-2]$$

```
/* Initialize */
Ts = 2.4933e-07; /* more probably some fraction of clock speed */
ynm1 = 0; ynm2 = 0; unm1 = 0; unm2 = 0;
while (true) {
    un = read_adc;
    yn = 1.956*ynm1 - 0.9567*ynm2 + 486.6e-6*unm1 + 476.5e-6*unm2;
    write_dac(yn);
    /* store past values */
    ynm2 = ynm1; ynm1 = yn;
    unm2 = unm1; unm1 = un;
    wait(Ts);
}
```

Comments

PC soundcards can sample audio at 44.1 kHz so this implies that the anti-aliasing filter is much sharper than this one as $f_s/2 = 22.05$ kHz.

You might wish to find out what order butterworth filter would be needed to have $f_c = 20$ kHz and f_{stop} of 22.05 kHz.

Summary

- Discrete Time Systems
- Transfer Functions in the Z-Domain
- Modelling digital systems in MATLAB/Simulink
- Continuous System Equivalents
- In-class demonstration: Digital Butterworth Filter

Reference

[Kar12]
Steven T. Karris. *Signals and systems with MATLAB computing and Simulink modeling*. Orchard Publishing, Fremont, CA., 2012. ISBN 9781934404232. Library call number: TK5102.9 K37 2012 LOCATE. URL: https://ebookcentral.proquest.com/lib/swansea-ebooks/reader.action?docID=3384197.

Solutions to Example 5

Solution to 5.1.

The transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2 + z}{z^2 - 0.5z + 0.125}$$

Solution to 5.2.

The DT impulse response:

$$h[n] = \left(\frac{\sqrt{2}}{4}\right)^n \left(\cos\left(\frac{n\pi}{4}\right) + 5\sin\left(\frac{n\pi}{4}\right)\right)$$

Solution to 5.3.

Step response:

$$y[n] = \left(3.2 - \left(\frac{\sqrt{2}}{4}\right)^n \left(2.2\cos\left(\frac{n\pi}{4}\right) + 0.6\sin\left(\frac{n\pi}{4}\right)\right)\right) u_0[n]$$
