Answers to Frequently Asked Questions

A PDF version of this FAQ is FAQ.pdf.

Harmonic frequencies

Fundamental frequency -- A periodic signal $f(t) = f(t + nT),\ n \in \mathbb{Z}$ has period T s and a fundamental frequency $f_0 = 1/T$ Hz. When used in Fourier series and Fourier transforms, frequencies are expressed as ω in radians/second. The **fundamental frequency** is $\omega = \Omega_0 = 2\pi f_0$ or, equivalently, $\Omega_0 = 2\pi/T$ rad/s.

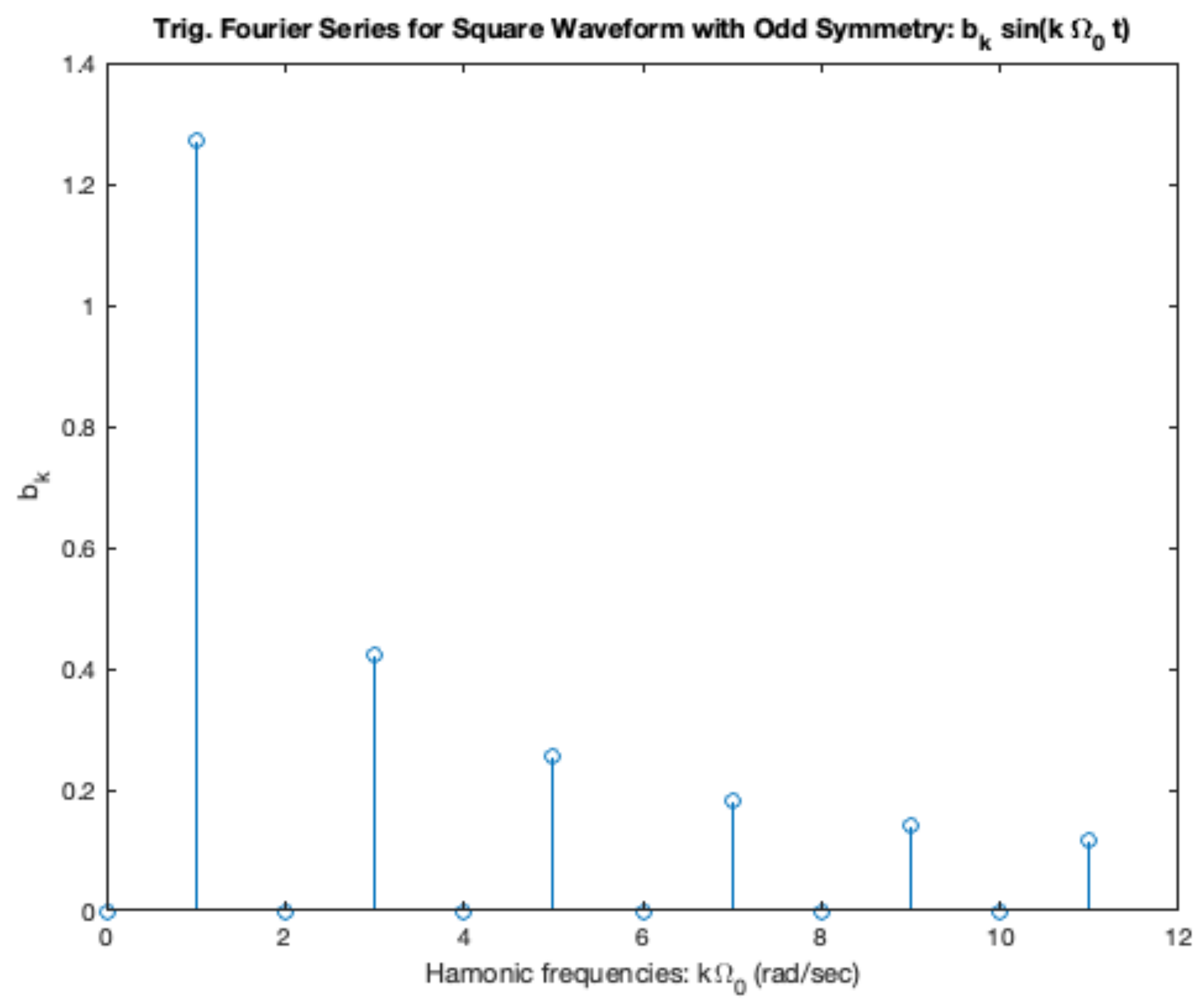
Harmonic frquencies (or *Harmonics*) are simply integer multiples of the fundamental frequency Ω_0 . So the zero-th harmonic is $\Omega_0 = 0$ rad/s or DC. The first harmonic is $1.\Omega_0 = \Omega_0$, the second is $2\Omega_0$, the third $3\Omega_0$ etc.

In general, we can express the k-th harmonic as $k\Omega_0,\ k \in \mathbb{Z}$.

Line spectra

In trig. Fourier series, the coefficients a_k and b_k are the amplitudes of the $\cos(k\Omega_0 t)$ and $\sin(k\Omega_0 t)$ terms respectively. We usually show these terms as lines (or *spectra*) with height a_k and/or b_k plotted against the harmonic frequency index k.

An example of such a plot is reproduced from Line Spectra for Trig. FS.

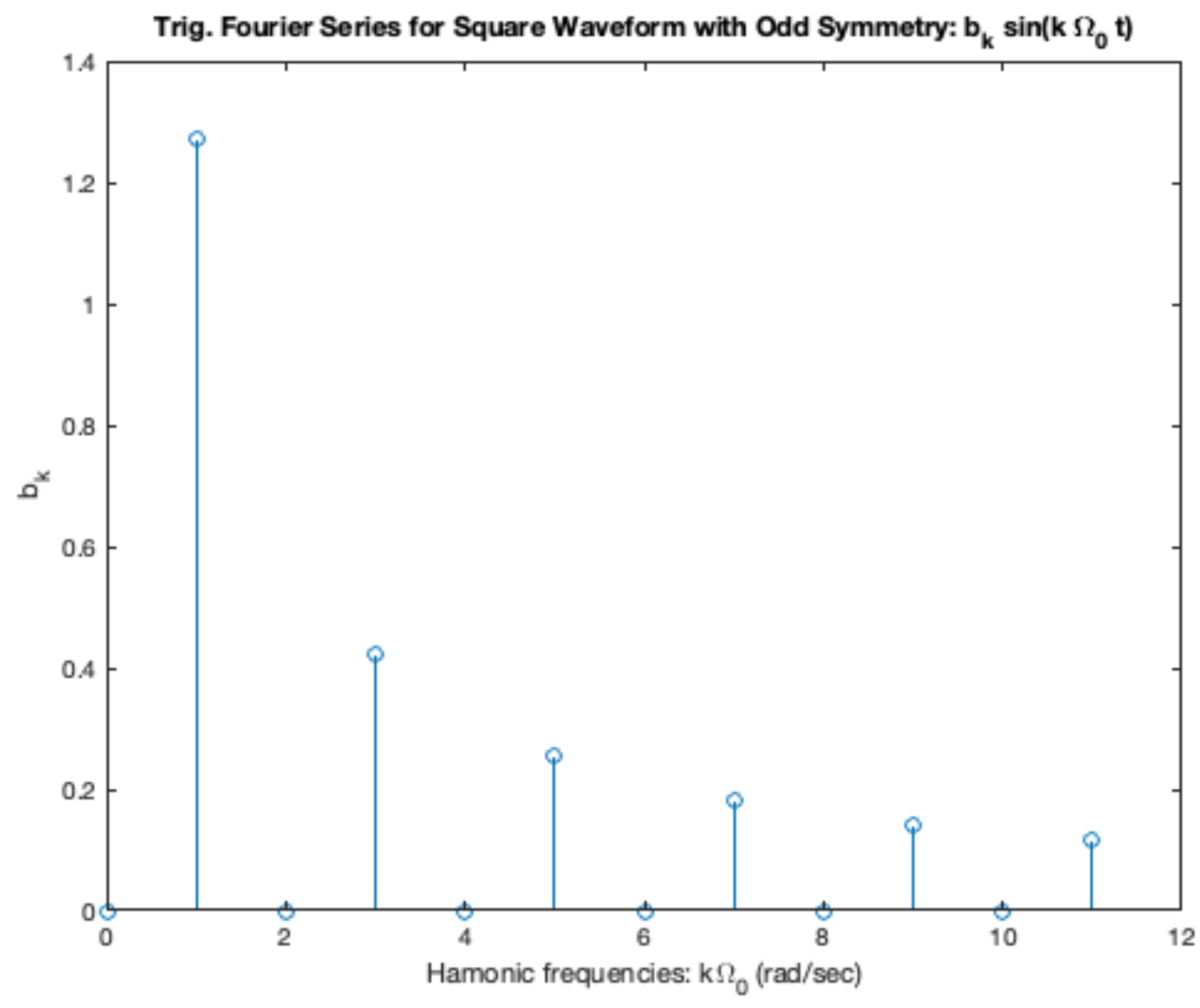


For exponential Fourier Series, the harmonic terms are defined as

$$C_k \exp(jk\Omega_0 t)\ k = -\infty, \ldots, -2, -1, 0, 1, 2, \ldots, \infty.$$

And $|C_k|$ and $\angle C_k$ are plotted as the line spectra against k. Note that C_{-k} and C_k are complex conjugates $\forall k > 0$. Hence, the spectrum will be symmetric around $k = 0$.

An example of such a plot is reproduced from Line Spectra for Exp. FS.



See the examples for exponential Fourier series and trig. Fourier Series in the notes Line Spectra.

Filter attenuation

In the notes Steady-State Response of an LTI System to a Periodic Signal we state (without proof) that the output of an LTI system to a periodic function with period T represented by a Fourier series is given by:

$$y(t) = \sum_{k=-\infty}^{\infty} C_k H(jk\Omega_0) e^{jk\Omega_0 t}.$$

As a consequence

Thus $y(t)$ is a Fourier series itself with coefficients D_k:

$$D_k = C_k H(jk\Omega_0).$$

What is missing from this anaalysis is a discussion of what $H(jk\Omega_0)$ looks like.

As an example, consider the simple first-order Butterworth low-pass (LP) filter with cut-off frequency ω_c:

$$H(s) = \frac{\omega_c}{s + \omega_c}$$

For this filter

$$H(j\omega) = \frac{\omega_c}{j\omega + \omega_c}.$$

Let us say that we wish to compute the attenuation and phase of this filter at $\omega = \Omega_0$.

To compute the *magnitude*:

$$\begin{aligned} |H(j\Omega_0)| &= \left|\frac{\omega_c}{j\Omega_0 + \omega_c}\right| \\ &= \frac{\omega_c}{\sqrt{\Omega_0^2 + \omega_c^2}} \end{aligned}$$

We note that is $|H(j\Omega_0| < 1$ so the filter will *attenuate* the incoming harmonic frequency. This will be true for all harmonics, so in general, for a LP filter:

$$D_k = C_k\, |H(jk\Omega_0)| < C_k.$$

The phase will be given by

$$\phi = \angle H(j\omega) = \tan^{-1}\left(\frac{\Im\,(H(j\omega))}{\Re\,(H(j(\omega))}\right)$$

where

$$\begin{aligned} H(jk\Omega_0) &= \frac{\omega_c^2}{(k\Omega_0)^2 + \omega_c^2} - j\frac{k\Omega_0\omega_c}{(k\Omega_0)^2 + \omega_c^2} \\ \phi_k &= \tan^{-1}\left(-\frac{K\Omega_0\omega_c}{\omega_c^2}\right) \\ &= \tan^{-1}\left(-\frac{k\Omega_0}{\omega_c}\right) \end{aligned}$$

Phases are additive so

$$\angle D_k = \angle C_k + \phi_k.$$

By doing such analysis, we can examine the effect of a filter on a periodic signal, just by considering how the coefficients of the harmonic terms are changed (attenuated in magnitude and shifted in phase) by the filter.