Elementary Signals

The preparatory reading for this section is Chapter 1 of {cite} `karris` which

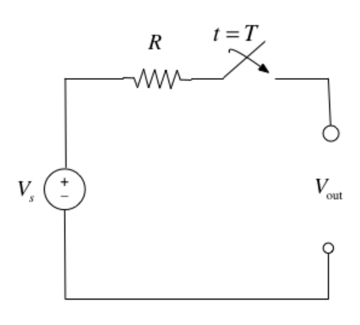
- begins with a discussion of the elementary signals that may be applied to electrical circuits
- introduces the unit step, unit ramp and dirac delta functions
- presents the sampling and sifting properties of the delta function and
- concludes with examples of how other useful signals can be synthesised from these elementary signals.

Colophon

An annotatable worksheet for this presentation is available as **Worksheet 3**.

- The source code for this page is elementary_signals/index.md.
- You can view the notes for this presentation as a webpage (HTML).
- This page is downloadable as a PDF file.

Consider the network shown in below where the switch is closed at time $t = T$ and all components are ideal.



Express the output voltage V_{out} as a function of the unit step function, and sketch the appropriate waveform.

Solution

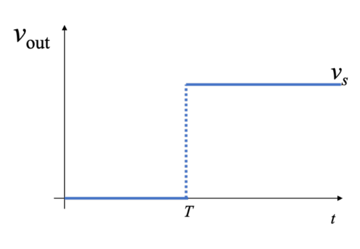
Before the switch is closed at $t < T$:

$$V_{\text{out}} = 0.$$

After the switch is closed for $t > T$:

$$V_{\text{out}} = V_s.$$

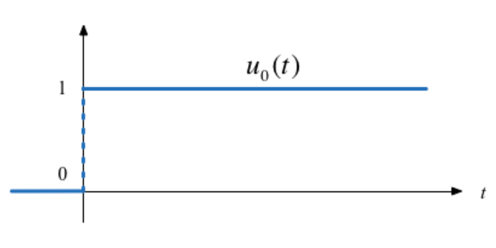
We imagine that the voltage jumps instantaneously from 0 to V_s volts at $t = T$ seconds as shown below.



We call this type of signal a step function.

The Unit Step Function

$$u_0(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$



In Matlab

In Matlab, we use the `heaviside` function (named after Oliver Heaviside).

File `plot_heaviside.m`

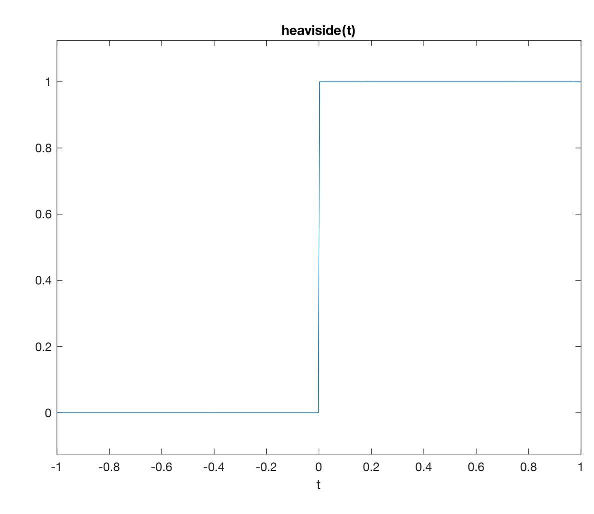
```
syms t
ezplot(heaviside(t),[-1,1])
heaviside(0)
```

In [11]:
```
imatlab_export_fig('print-svg')  % Static svg figures.
```

In [12]:
```
plot_heaviside
```

```
ans =

    0.5000
```



Note that, so that it can be plotted, Matlab defines the *heaviside function* slightly differently from the mathematically ideal unit step:

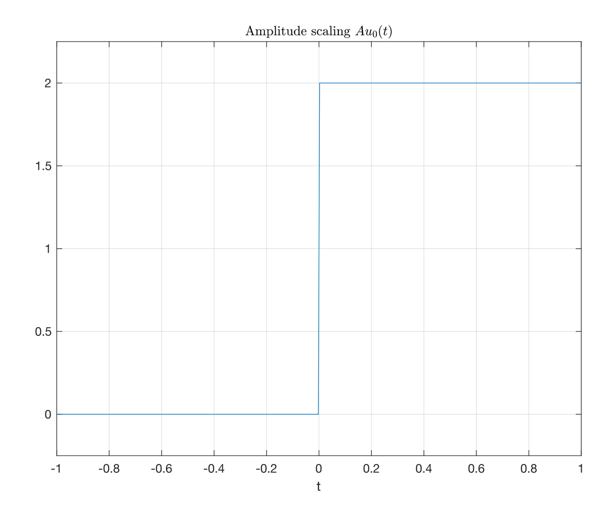
$$\text{heaviside}(t) = \begin{cases} 0 & t < 0 \\ 1/2 & t = 0 \\ 1 & t > 0 \end{cases}$$

Simple Signal Operations

Amplitude Scaling

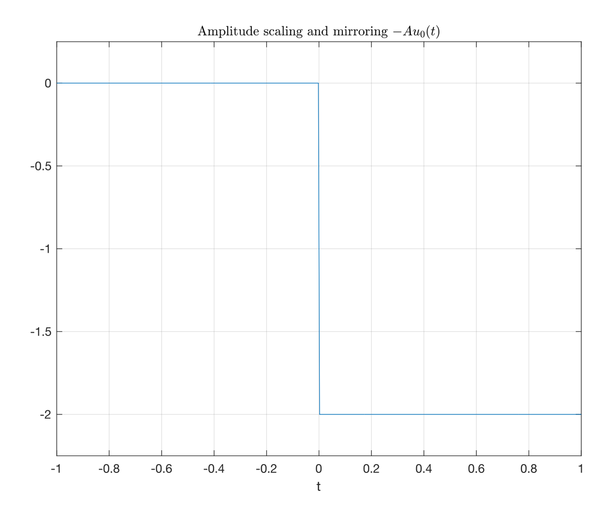
Sketch $Au_0(t)$ and $-Au_0(t)$

In [13]:
```
syms t;
u0(t) = heaviside(t); % rename heaviside function for ease of use
A = 2; % so signal can be plotted
ezplot(A*u0(t),[-1,1]),grid,title('Amplitude scaling $$Au_0(t)$$','interpreter','latex')
```



Note that the signal is scaled in the y direction.

In [14]:
```
ezplot(-A*u0(t),[-1,1]),grid,title('Amplitude scaling and mirroring $$-Au_0(t)$$','interpreter','latex')
```

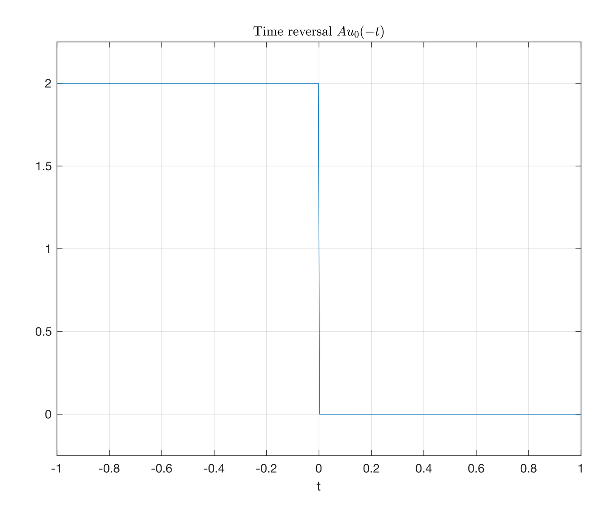


Note that, because of the sign, the signal is mirrored about the x axis as well as being scaled by 2.

Time Reversal

Sketch $u_0(-t)$

In [15]:
```
ezplot(A*u0(-t),[-1,1]),grid,title('Time reversal $$Au_0(-t)$$','interpreter','latex')
```

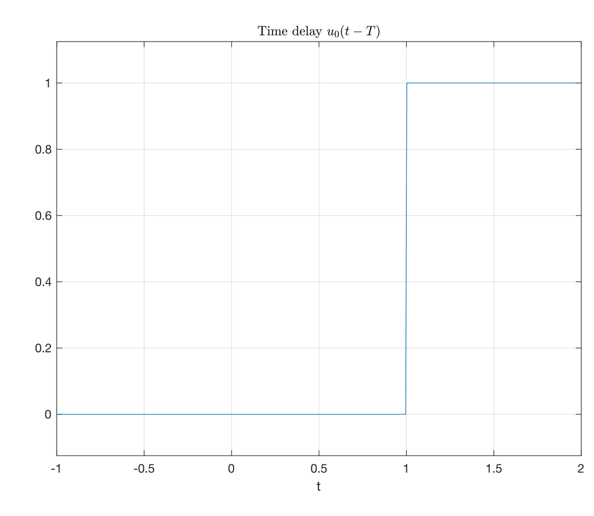


The sign on the function argument $-t$ causes the whole signal to be reversed in time. Note that another way of looking at this is that the signal is mirrored about the y axis.

Time Delay and Advance

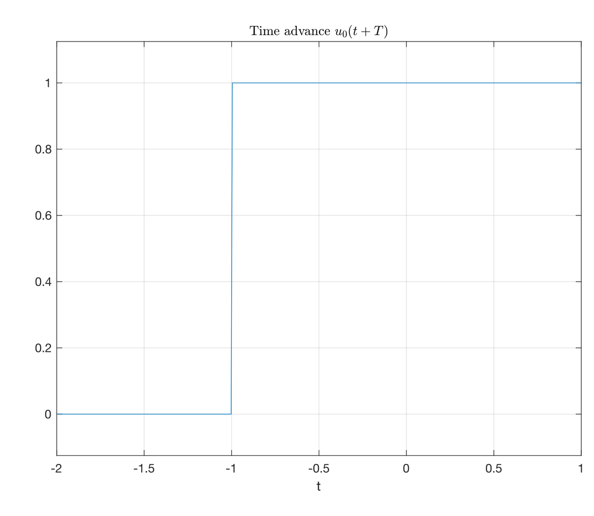
Sketch $u_0(t - T)$ and $u_0(t + T)$

In [16]:
```
T = 1; % again to make the signal plottable.
ezplot(u0(t - T),[-1,2]),grid,title('Time delay $$u_0(t - T)$$','interpreter','latex')
```



This is a *time delay* ... note for $u_0(t - T)$ the step change occurs T seconds **later** than it does for $u_o(t)$.

In [17]:
```
ezplot(u0(t + T),[-2,1]),grid,title('Time advance $$u_0(t + T)$$','interpreter','latex')
```



This is a *time advance* ... note for $u_0(t + T)$ the step change occurs T seconds **earlier** than it does for $u_o(t)$.

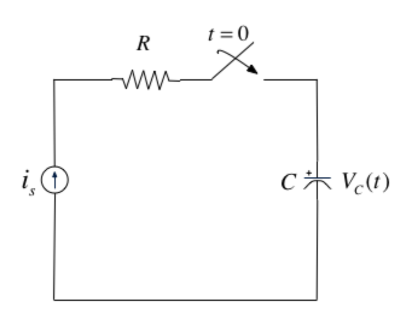
Examples

We will work through some examples in class. See Worksheet 3.

Synthesis of Signals from the Unit Step

Unit step functions can be used to represent other time-varying functions such as rectangular pulses, square waves and triangular pulses. See Worksheet 3 for the examples that we will look at in class.

The Ramp Function



In the circuit shown above i_s is a constant current source and the switch is closed at time $t = 0$.

When the current through the capacitor $i_c(t) = i_s$ is a constant and the voltage across the capacitor is

$$v_c(t) = \frac{1}{C}\int_{-\infty}^{t} i_c(\tau)\, d\tau$$

where τ is a dummy variable.

Since the switch closes at $t = 0$, we can express the current $i_c(t)$ as

$$i_c(t) = i_s u_0(t)$$

and if $v_c(t) = 0$ for $t < 0$ we have

$$v_c(t) = \frac{i_s}{C}\int_{-\infty}^{t} u_0(\tau)\, d\tau = \underbrace{\frac{i_s}{C}\int_{-\infty}^{0} 0\, d\tau}_{0} + \frac{i_s}{C}\int_{0}^{t} 1\, d\tau$$

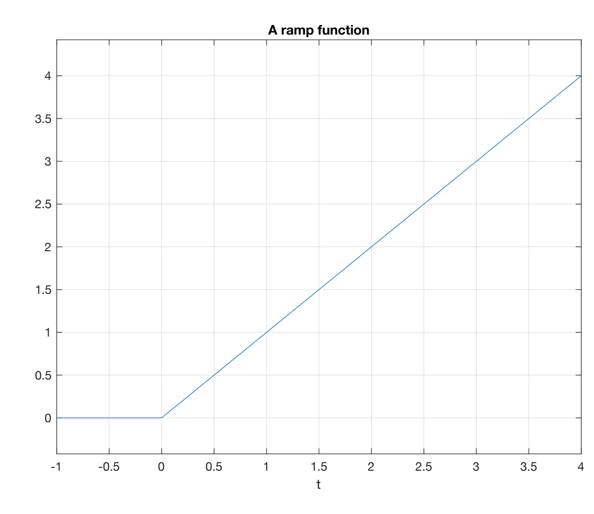
So, the voltage across the capacitor can be represented as

$$v_C(t) = \frac{i_s}{C} t u_0(t)$$

Note that in this as in other examples throughout these notes, and in published tables of transforms, the inclusion of $u_0(t)$ in $v_c(t)$ acts as a "gating function" that limits the definition of the signal to the causal range $0 \le t < \infty$.

To sketch the wave form, let's arbitrarily let C and i_s be one and then plot with MATLAB.

In [18]:
```
C = 1; is = 1;
vc(t)=(is/C)*t*u0(t);
ezplot(vc(t),[-1,4]),grid,title('A ramp function')
```



This type of signal is called a **ramp function**. Note that it is the *integral* of the step function (the resistor-capacitor circuit implements a simple integrator circuit).

The unit ramp function is defined as

$$u_1(t) = \int_{-\infty}^{t} u_0(\tau) d\tau$$

so

$$u_1(t) = \begin{cases} 0 & t < 0 \\ t & t \ge 0 \end{cases}$$

and

$$u_0(t) = \frac{d}{dt} u_1(t)$$

Note

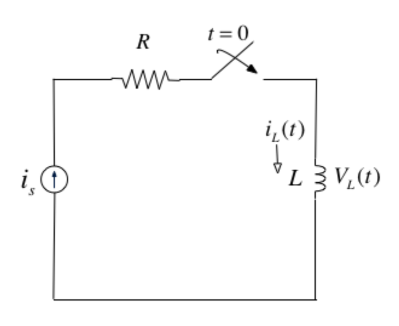
Higher order functions of t can be generated by the repeated integration of the unit step function.

For future reference, you should determine $u_2(t)$, $u_3(t)$ and $u_n(t)$ for yourself and make a note of the general rule:

$$u_{n-1} = \frac{1}{n}\frac{d}{dt} u_n(t)$$

Details are given in equations 1.26—1.29 in Karris.

The Dirac Delta Function



In the circuit shown above, the switch is closed at time $t = 0$ and $i_L(t) = 0$ for $t < 0$. Express the inductor current $i_L(t)$ in terms of the unit step function and hence derive an expression for $v_L(t)$.

Solution

$$v_L(t) = L\frac{di_L}{dt}$$

Because the switch closes instantaneously at $t = 0$

$$i_L(t) = i_s u_0(t)$$

Thus

$$v_L(t) = i_s L \frac{d}{dt} u_0(t).$$

To solve this problem we need to invent a function that represents the derivative of the unit step function. This function is called $\delta(t)$ or the *dirac delta* function (named after Paul Dirac).

The delta function

The unit impulse or the delta function, denoted as $\delta(t)$, is the derivative of the unit step.

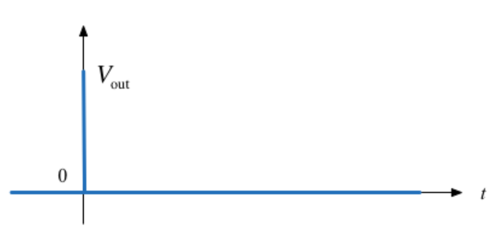
This function is tricky because $u_0(t)$ is discontinuous at $t = 0$ but it must have the properties

$$\int_{-\infty}^{t} \delta(\tau) d\tau = u_0(t)$$

and

$$\delta(t) = 0 \;\forall\; t \ne 0.$$

Sketch of the delta function



MATLAB Confirmation

In [19]:
```
syms is L;
vL(t) = is * L * diff(u0(t))
```

```
vL(t) =

L*is*dirac(t)
```

Note that we can't plot dirac(t) in MATLAB with `ezplot`.

Important properties of the delta function

Sampling Property

The *sampling property* of the delta function states that

$$f(t)\delta(t - a) = f(a)\delta(t - a)$$

or, when $a = 0$,

$$f(t)\delta(t) = f(0)\delta(t)$$

Multiplication of any function $f(t)$ by the delta function $\delta(t)$ results in sampling the function at the time instants for which the delta function is not zero.

The study of descrete-time (sampled) systems is based on this property.

You should work through the proof for youself.

Sifting Property

The *sifting property* of the delta function states that

$$\int_{-\infty}^{\infty} f(t)\delta(t - \alpha) dt = f(\alpha)$$

That is, if multiply any function $f(t)$ by $\delta(t - \alpha)$, and integrate from $-\infty$ to $+\infty$, we will get the value of $f(t)$ evaluated at $t = \alpha$.

You should also work through the proof for yourself.

Higher Order Delta Fuctions

the n*th*-order *delta function* is defined as the n*th* derivative of $u_0(t)$, that is

$$\delta^n(t) = \frac{d^n}{dt^n}[u_0(t)]$$

The function $\delta'(t)$ is called the *doublet*, $\delta''(t)$ is called the *triplet* and so on.

By a procedure similar to the derivation of the sampling property we can show that

$$f(t)\delta'(t - a) = f(a)\delta'(t - a) - f'(t)\delta(t - a)$$

Also, derivation of the sifting property can be extended to show that

$$\int_{-\infty}^{\infty} f(t)\delta^n(t - \alpha)dt = (-1)^n \frac{d^n}{dt^n}[f(t)]\Big|_{t=\alpha}$$

Summary

In this chapter we have looked at some elementary signals and the theoretical circuits that can be used to generate them.

Takeaways

- You should note that the unit step is the *heaviside function* $u_0(t)$.
- Many useful signals can be synthesized by use of the unit step as a "gating function" in combination with other signals
- That unit ramp function $u_1(t)$ is the integral of the step function.
- The *Dirac delta* function $\delta(t)$ is the derivative of the unit step function. We sometimes refer to it as the *unit impulse function*.
- The delta function has sampling and sifting properties that will be useful in the development of *time convolution* and *sampling theory*.

Examples

We will do some of these in class. See Worksheet 3.

Homework

These are for you to do later for further practice. See Homework 1.

References

See Bibliography.