

# Worksheet 9

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## To accompany Unit 4.1 Trigonometric Fourier Series

### Colophon

This worksheet can be downloaded as a [PDF file](#). We will step through this worksheet in class.

An annotatable copy of the notes for this presentation will be distributed before the second class meeting as **Worksheet 9** in the **Week 4: Classroom Activities** section of the Canvas site. I will also distribute a copy to your personal **Worksheets** section of the **OneNote Class Notebook** so that you can add your own notes using OneNote.

You are expected to have at least watched the video presentation of [Chapter 4.1](#) of the [notes](#) before coming to class. If you haven't watch it afterwards!

After class, the lecture recording and the annotated version of the worksheets will be made available through Canvas.

### Motivating Examples

This [Fourier Series demo](#), developed by Members of the Center for Signal and Image Processing (CSIP) at the [School of Electrical and Computer Engineering](#) at the [Georgia Institute of Technology](#), shows how periodic signals can be synthesised by a sum of sinusoidal signals.

It is here used as a motivational example in our introduction to [Fourier Series](#). (See also [Fourier Series](#) from Wolfram MathWorld referenced in the **Quick Reference** on Blackboard.)

To install this example, download the [zip file](#) and unpack it somewhere on your MATLAB path.

## The Trigonometric Fourier Series

Any periodic waveform  $f(t)$  can be represented as

$$f(t) = \frac{1}{2}a_0 + a_1 \cos \Omega_0 t + a_2 \cos 2\Omega_0 t + a_3 \cos 3\Omega_0 t + \dots + a_n \cos n\Omega_0 t + b_1 \sin \Omega_0 t + b_2 \sin 2\Omega_0 t + b_3 \sin 3\Omega_0 t + \dots + b_n \sin n\Omega_0 t + \dots$$

or equivalently (if more confusingly)

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\Omega_0 t + b_n \sin n\Omega_0 t)$$

where  $\Omega_0$  rad/s is the *fundamental frequency*.

## Evaluation of the Fourier series coefficients

The coefficients are obtained from the following expressions (valid for any periodic waveform with fundamental frequency  $\Omega_0$  so long as we integrate over one period  $0 \rightarrow T_0$  where  $T_0 = 2\pi/\Omega_0$ ), and  $\theta = \Omega_0 t$ :

$$\begin{aligned} \frac{1}{2}a_0 &= \frac{1}{T_0} \int_0^{T_0} f(t) dt = \frac{1}{\pi} \int_0^{2\pi} f(\theta) d\theta \\ a_n &= \frac{1}{T_0} \int_0^{T_0} f(t) \cos n\Omega_0 t dt = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \cos n\theta d\theta \\ b_n &= \frac{1}{T_0} \int_0^{T_0} f(t) \sin n\Omega_0 t dt = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \sin n\theta d\theta \end{aligned}$$

### Demo 1

Building up wave forms from sinusoids.

```
fseriesdemo
```

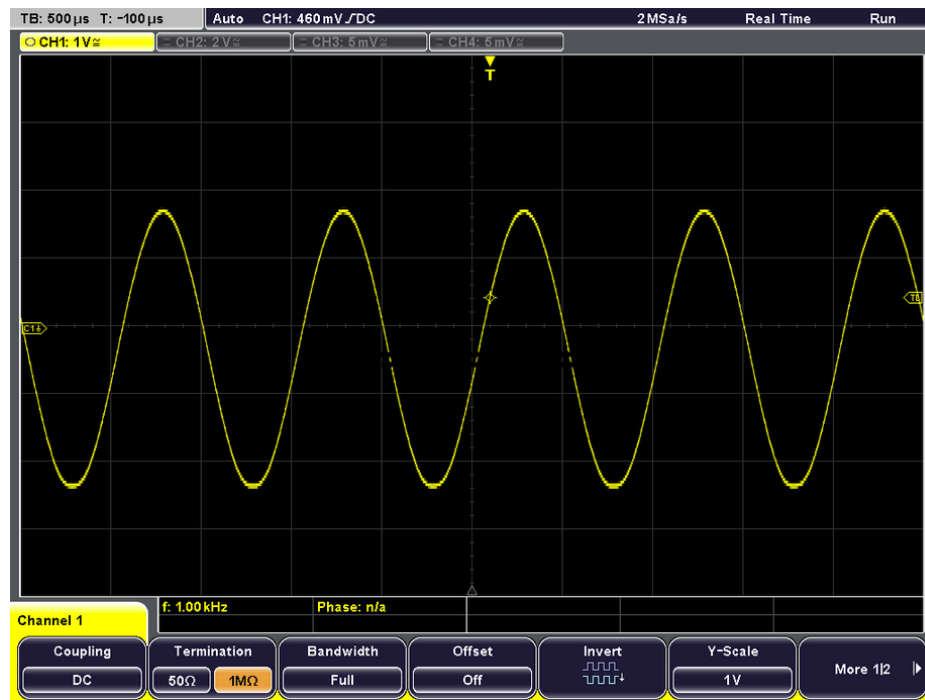
### Demo 2

Actual measurements

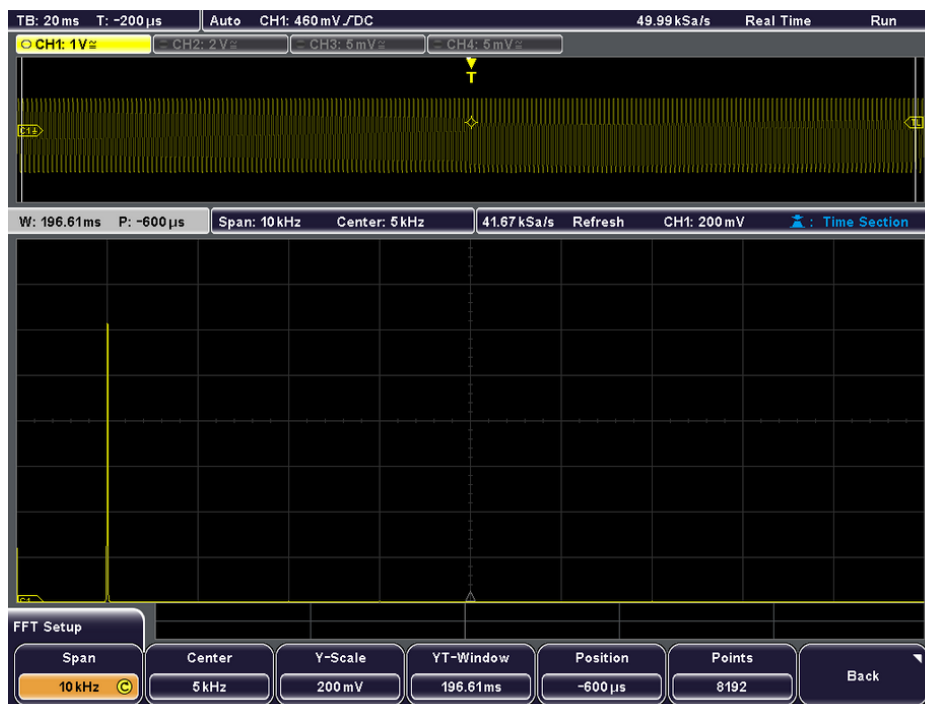
Taken by Dr Tim Davies with a Rhode&Schwarz Oscilloscope.

Note all spectra shown in these slides are generated numerically from the input signals by sampling and the application of the Fast Fourier Transform (FFT).

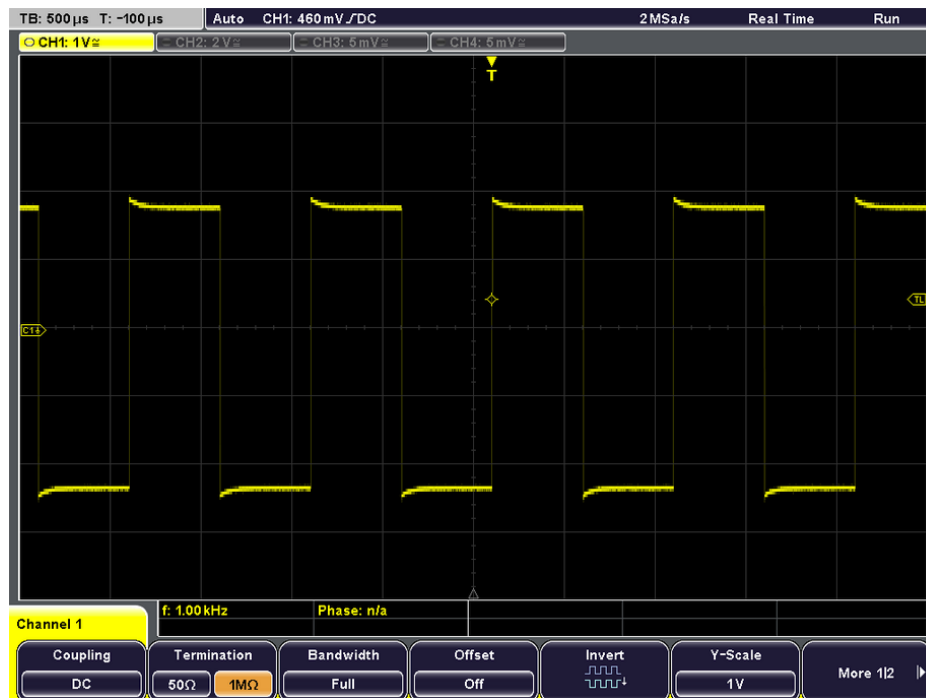
## 1 kHz Sinewave



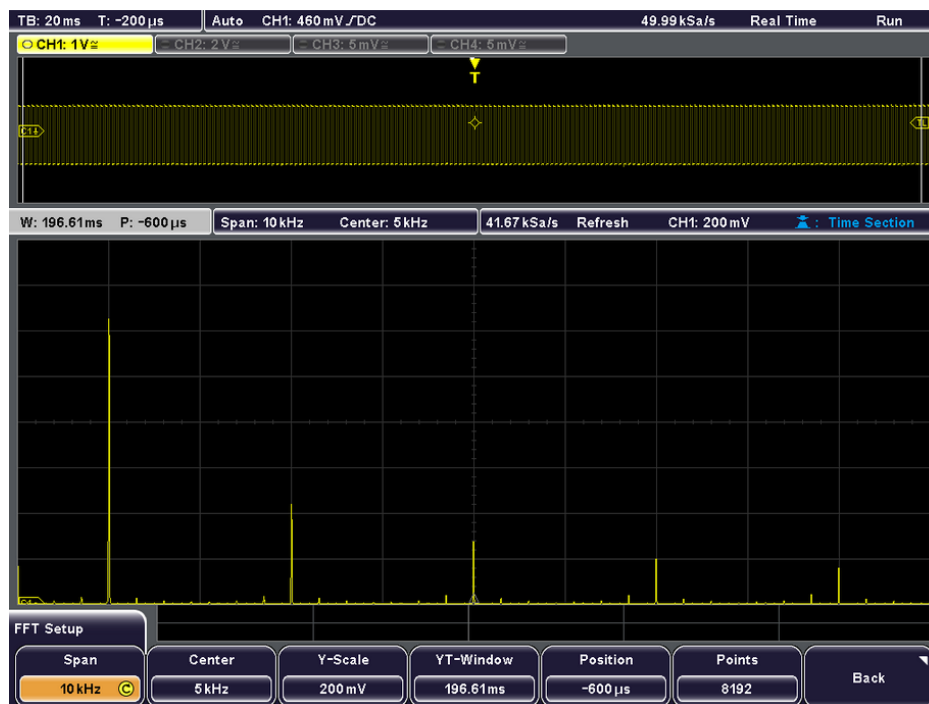
## Spectrum of 1kHz sinewave



## 1 kHz Squarewave

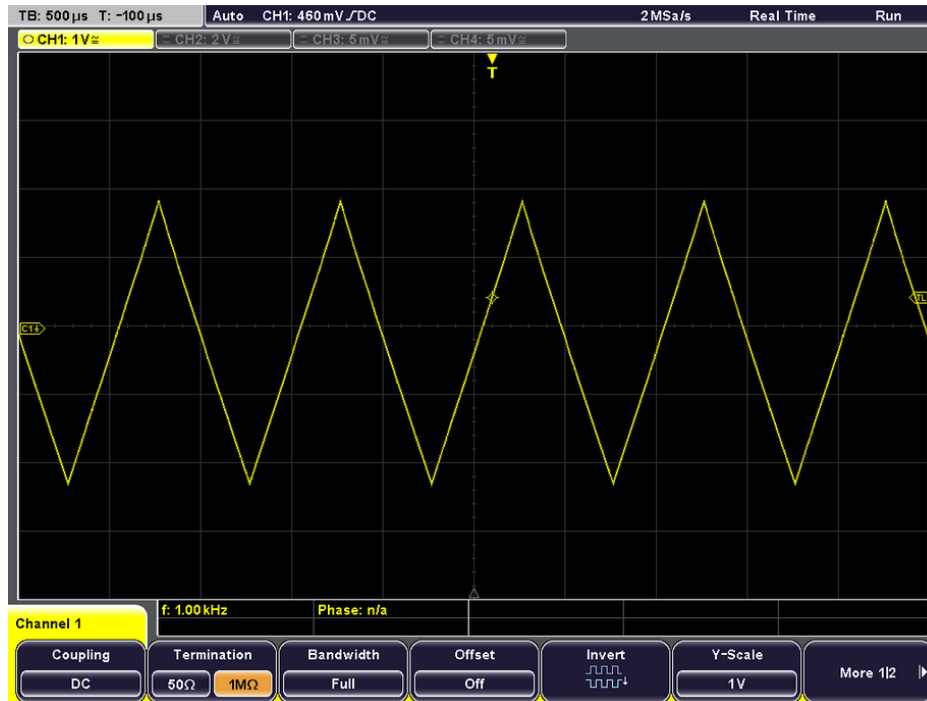


Spectrum of 1kHz square wave

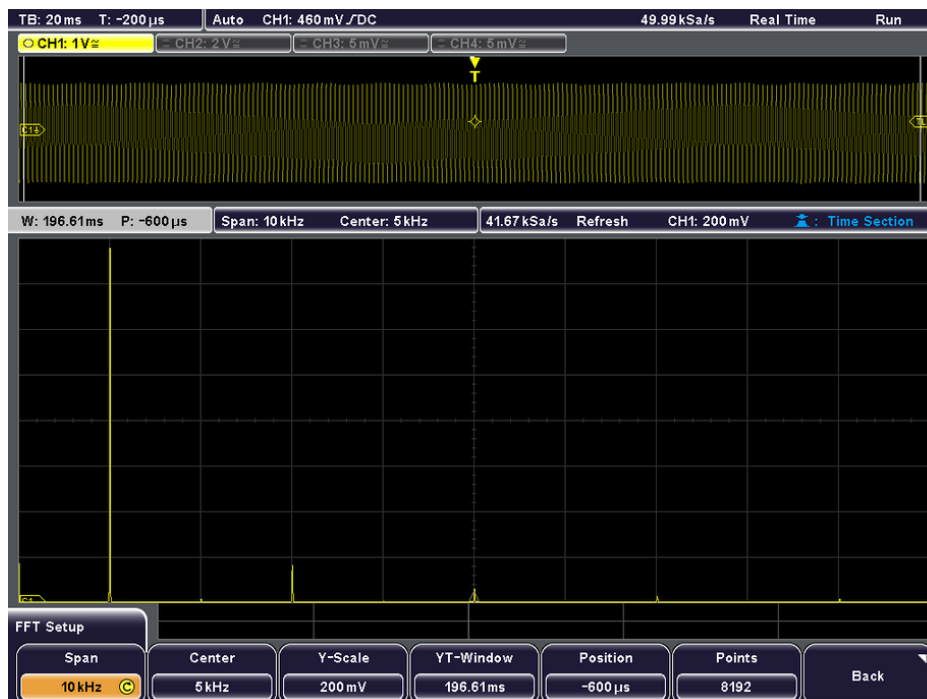


Clearly showing peaks at fundamental,  $1/3$ ,  $1/5$ ,  $1/7$  and  $1/9$  at 3rd, 5th and 7th harmonic frequencies. Note for sawtooth, harmonics decline in amplitude as the reciprocal of the of harmonic number  $n$ .

1 kHz triangle waveform



Spectrum of 1kHz triangle waveform



Clearly showing peaks at fundamental,  $1/9$ ,  $1/25$ ,  $1/7$  and  $1/49$  at 3rd, 5th and 7th harmonic frequencies. Note for triangle, harmonics decline in amplitude as the reciprocal of the square of  $n$ .

## Odd, Even and Half-wave Symmetry

### Odd- and even symmetry

- An *odd* function is one for which  $f(t) = -f(-t)$ . The function  $\sin t$  is an *odd* function.
- An *even* function is one for which  $f(t) = f(-t)$ . The function  $\cos t$  is an *even* function.

## Half-wave symmetry

- A periodic function with period  $T$  is a function for which  $f(t) = f(t + T)$
- A periodic function with period  $T$ , has *half-wave symmetry* if  $f(t) = -f(t + T/2)$

## Symmetry in Trigonometric Fourier Series

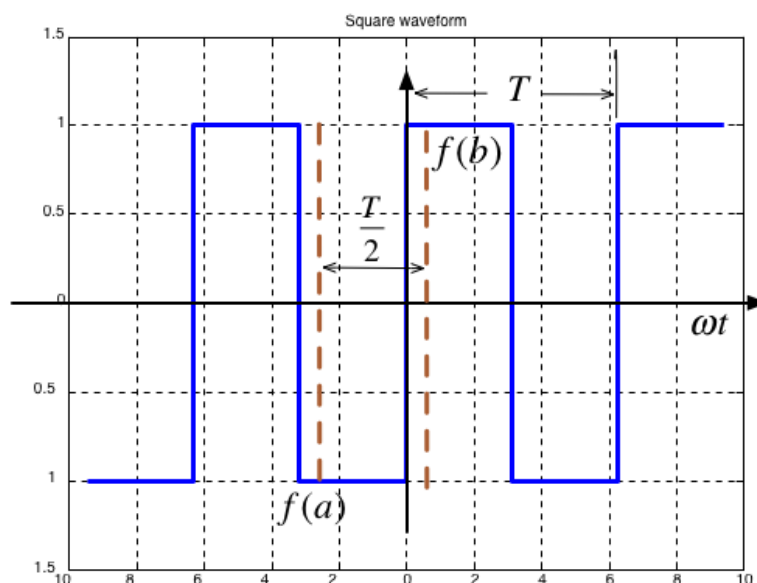
There are simplifications we can make if the original periodic properties has certain properties:

- If  $f(t)$  is odd,  $a_0 = 0$  and there will be no cosine terms so  $a_n = 0 \forall n > 0$
- If  $f(t)$  is even, there will be no sine terms and  $b_n = 0 \forall n > 0$ . The DC may or may not be zero.
- If  $f(t)$  has *half-wave symmetry* only the odd harmonics will be present. That is  $a_n$  and  $b_n$  is zero for all even values of  $n$  (0, 2, 4, ...)

## Symmetry in Common Waveforms

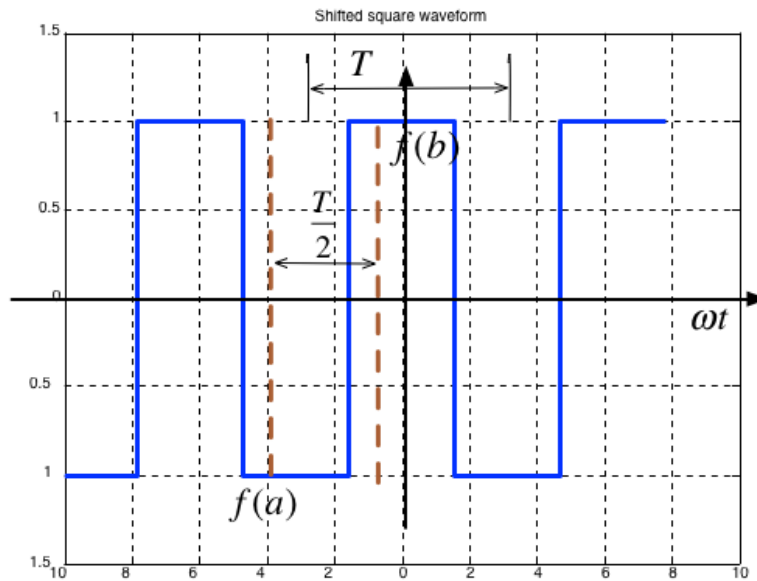
To reproduce the following waveforms (without annotation) publish the script [waves.m](https://cpjabling.github.io/eg-247-textbook/fourier_series/1/worksheet9.html).

### Squarewave



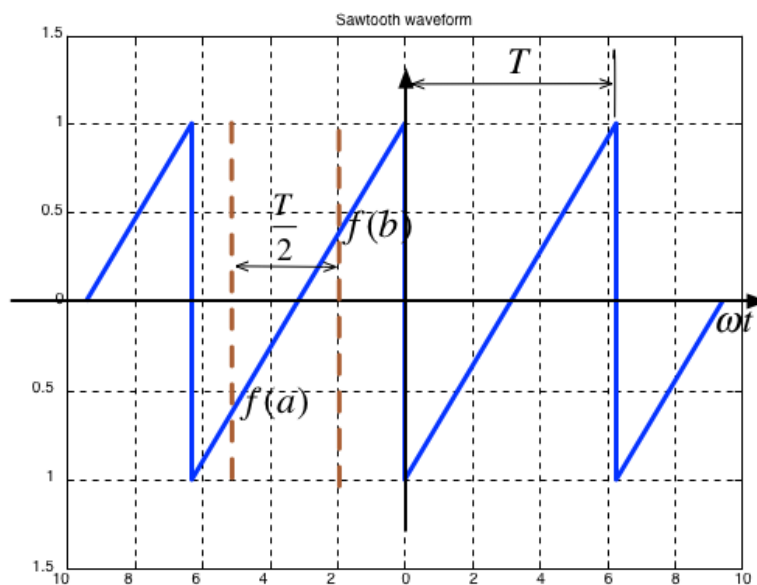
- Average value over period  $T$  is ...?
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry  $f(t) = -f(t + T/2)$ ?

## Shifted Squarewave



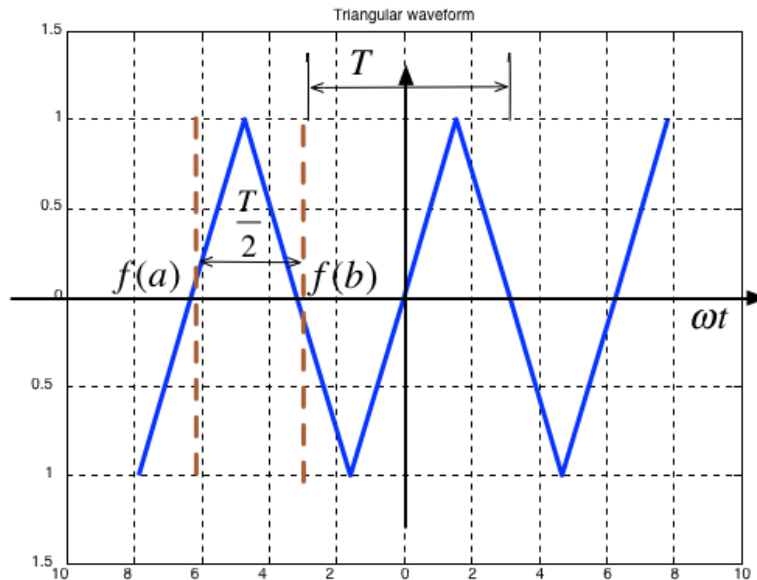
- Average value over period  $T$  is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry  $f(t) = -f(t + T/2)$ ?

## Sawtooth



- Average value over period  $T$  is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry  $f(t) = -f(t + T/2)$ ?

## Triangle

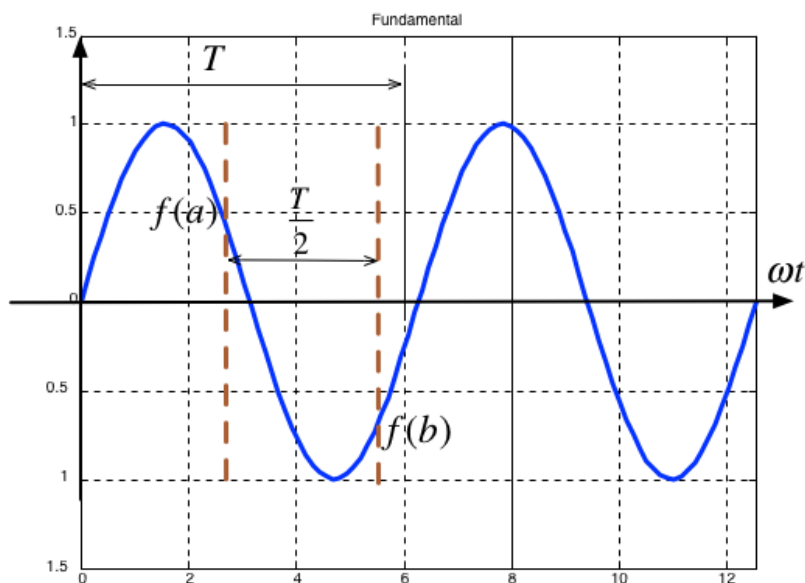


- Average value over period  $T$  is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry  $f(t) = -f(t + T/2)$ ?

## Symmetry in fundamental, Second and Third Harmonics

In the following,  $T/2$  is taken to be the half-period of the fundamental sinewave.

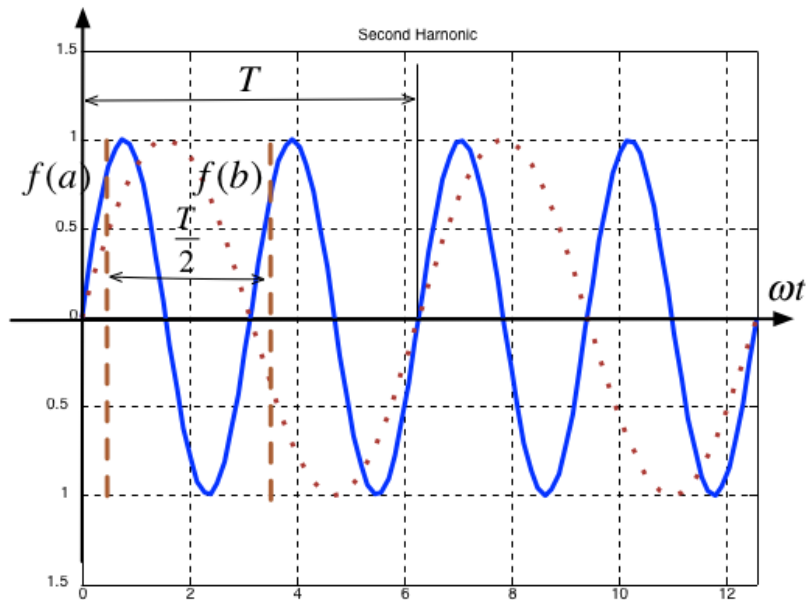
### Fundamental



- Average value over period  $T$  is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry  $f(t) = -f(t + T/2)$ ?

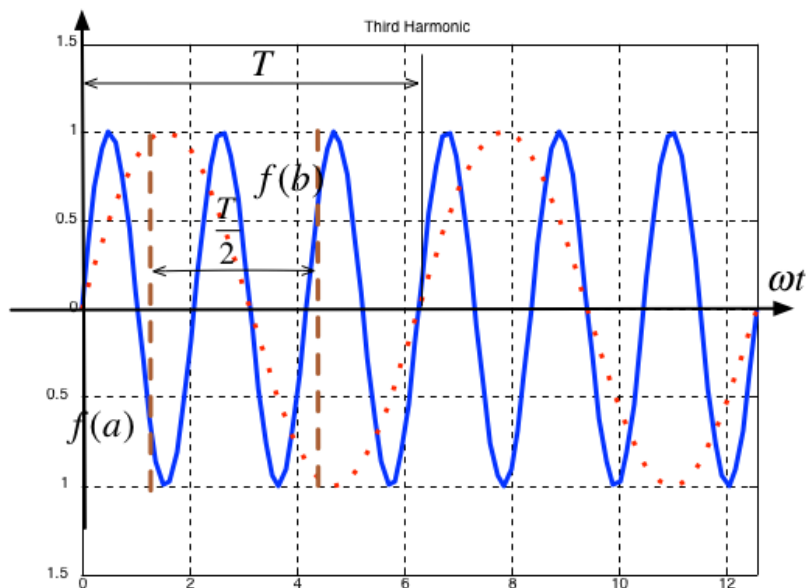
### Second Harmonic





- Average value over period  $T$  is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry  $f(t) = -f(t + T/2)$ ?

### Third Harmonic



- Average value over period  $T$  is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry  $f(t) = -f(t + T/2)$ ?

### Some simplifications that result from symmetry

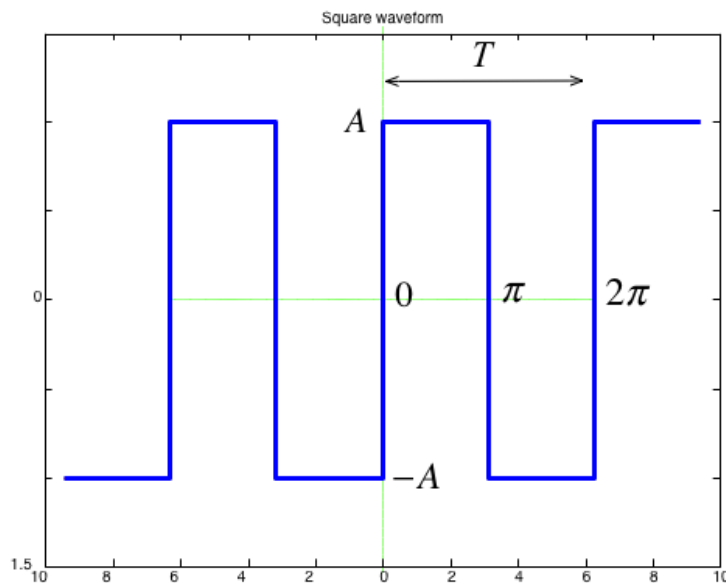
- The limits of the integrals used to compute the coefficients  $a_n$  and  $b_n$  of the Fourier series are given as  $0 \rightarrow 2\pi$  which is one period  $T$
- We could also choose to integrate from  $-\pi \rightarrow \pi$
- If the function is *odd*, or *even* or has *half-wave symmetry* we can compute  $a_n$  and  $b_n$  by integrating from  $0 \rightarrow \pi$  and multiplying by 2.

- If we have *half-wave symmetry* we can compute  $a_n$  and  $b_n$  by integrating from  $0 \rightarrow \pi/2$  and multiplying by 4.

(For more details see page 7-10 of the textbook)

## Computing coefficients of Trig. Fourier Series in Matlab

As an example let's take a square wave with amplitude  $\pm A$  and period  $T$ .



### Solution

Solution: See [square\\_ftrig.mlx](#). Script confirms that:

- $a_0 = 0$
- $a_i = 0$ : function is odd
- $b_i = 0$ : for  $i$  even - half-wave symmetry

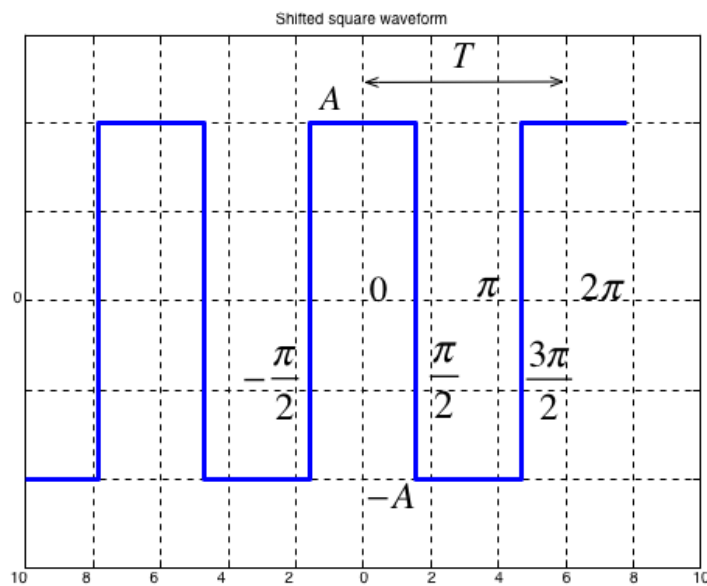
```
ft =
(4*A*sin(t))/pi + (4*A*sin(3*t))/(3*pi) + (4*A*sin(5*t))/(5*pi) +
(4*A*sin(7*t))/(7*pi) + (4*A*sin(9*t))/(9*pi) +
(4*A*sin(11*t))/(11*pi)
```

```
clear all
cd ../matlab
format compact
open square_ftrig
```

Note that the coefficients match those given in the textbook (Section 7.4.1).

$$f(t) = \frac{4A}{\pi} \left( \sin \Omega_0 t + \frac{1}{3} \sin 3\Omega_0 t + \frac{1}{5} \sin 5\Omega_0 t + \dots \right) = \frac{4A}{\pi} \sum_{n=\text{odd}} \frac{1}{n} \sin n\Omega$$

## Using symmetry - computing the Fourier series coefficients of the shifted square wave



- As before  $a_0 = 0$
- We observe that this function is even, so all  $b_k$  coefficients will be zero
- The waveform has half-wave symmetry, so only odd indexed coefficients will be present.
- Furthermore, because it has half-wave symmetry we can just integrate from  $0 \rightarrow \pi/2$  and multiply the result by 4.

See [shifted\\_sq\\_ftrig.mlx](#).

```
ft =
(4*A*cos(t))/pi - (4*A*cos(3*t))/(3*pi) + (4*A*cos(5*t))/(5*pi) -
(4*A*cos(7*t))/(7*pi) + (4*A*cos(9*t))/(9*pi) -
(4*A*cos(11*t))/(11*pi)
```

```
[ open shifted_sq_ftrig
```

Note that the coefficients match those given in the textbook (Section 7.4.2).

$$f(t) = \frac{4A}{\pi} \left( \cos \Omega_0 t - \frac{1}{3} \cos 3\Omega_0 t + \frac{1}{5} \cos 5\Omega_0 t - \dots \right) = \frac{4A}{\pi} \sum_{n=\text{odd}} (-1)^{\frac{n-1}{2}}$$

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