Answers to Frequently Asked Questions

A PDF version of this FAQ is FAQ.pdf.

Acklowdgements

I thank the following students who helped me to recognise the holes in my course notes and were the inspiration for these answers.

- Charlotte Newman (Medical Engineering, 2020)

Harmonic frequencies

Fundamental frequency -- A periodic signal $f(t) = f(t + nT)$, $n \in \mathbb{Z}$ has period T s and a fundamental frequency $f_0 = 1/T$ Hz. When used in Fourier series and Fourier transforms, frequencies are expressed as ω in radians/second. The **fundamental frequency** is $\omega = \Omega_0 = 2\pi f_0$ or, equivalently, $\Omega_0 = 2\pi/T$ rad/s.

Harmonic frquencies (or *Harmonics*) are simply integer multiples of the fundamental frequency Ω_0. So the zero-th harmonic is $\Omega_0 = 0$ rad/s or DC. The first harmonic is $1.\Omega_0 = \Omega_0$, the second $2\Omega_0$, the third $3\Omega_0$ etc.

In general, we can express the k-th harmonic as $k\Omega_0$, $k \in \mathbb{Z}$.

Line spectra

In trig. Fourier series, the coefficients a_k and b_k are the amplitudes of the $\cos(k\Omega_0 t)$ and $\sin(k\Omega_0 t)$ terms respectively. We usually show these terms as lines (or *spectra*) with height a_k and/or b_k plotted against the harmonic frequency index k.

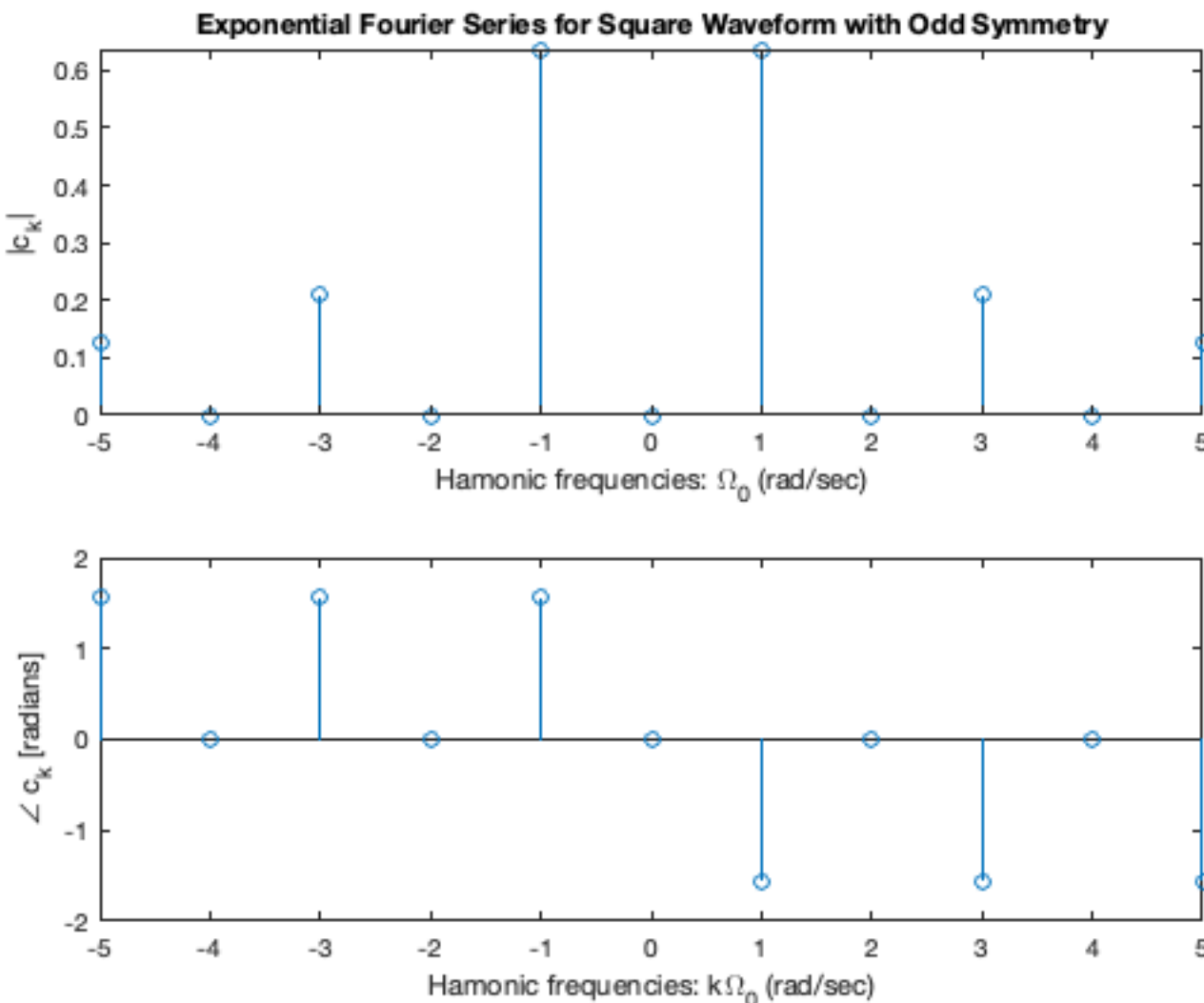An example of such a plot is reproduced from Line Spectra for Trig. FS.
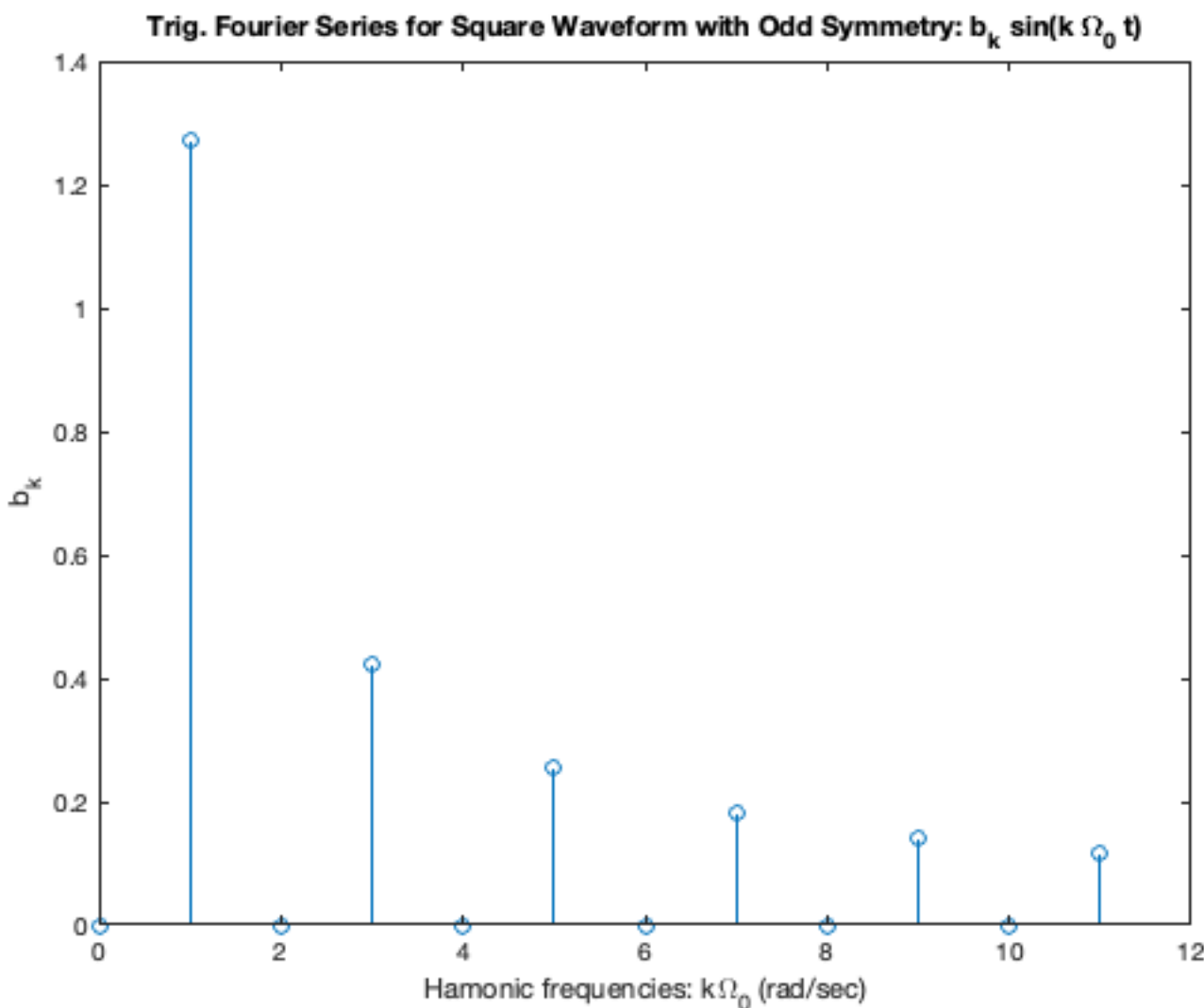
For exponential Fourier Series, the harmonic terms are defined as
$$C_k \exp(jk\Omega_0 t) \, k = -\infty, \dots, -2, -1, 0, 1, 2, \dots, \infty.$$
And $|C_k|$ and $\angle C_k$ are plotted as the line spectra against k. Note that C_{-k} and C_k are complex conjugates $\forall k > 0$. Hence, the spectrum will be symmetric around $k = 0$.

An example of such a plot is reproduced from Line Spectra for Exp. FS.

Filter attenuation

In the notes Steady-State Response of an LTI System to a Periodic Signal we state (without proof) that the output of an LTI system to a periodic function with period T represented by a Fourier series is given by:
$$y(t) = \sum_{k=-\infty}^{\infty} C_k H(jk\Omega_0) e^{jk\Omega_0 t}.$$

As a consequence

Thus $y(t)$ is a Fourier series itself with coefficients D_k:
$$D_k = C_k H(jk\Omega_0).$$

What is missing from this anaalysis is a discussion of what $H(jk\Omega_0)$ looks like.

As an example, consider the simple first-order Butterworth low-pass (LP) filter with cut-off frequency ω_c:
$$H(s) = \frac{\omega_c}{s + \omega_c}$$

For this filter
$$H(j\omega) = \frac{\omega_c}{j\omega + \omega_c}.$$

Let us say that we wish to compute the attenuation and phase of this filter at $\omega = \Omega_0$.

To compute the *magnitude*:
$$|H(j\Omega_0)| = \left| \frac{\omega_c}{j\Omega_0 + \omega_c} \right|$$
$$= \frac{\omega_c}{\sqrt{\Omega_0^2 + \omega_c^2}}$$

We note that is $|H(j\Omega_0)| < 1$ so the filter will *attenuate* the incoming harmonic frequency. This will be true for all harmonics, so in general, for a LP filter:
$$D_k = C_k |H(jk\Omega_0)| < C_k.$$

The phase will be given by
$$\phi = \angle H(j\omega) = \tan^{-1}\left(\frac{\Im(H(j\omega))}{\Re(H(j(\omega))} \right)$$

where
$$H(jk\Omega_0) = \frac{\omega_c^2}{(k\Omega_0)^2 + \omega_c^2} - j\frac{k\Omega_0 \omega_c}{(k\Omega_0)^2 + \omega_c^2}$$
$$\phi_k = \tan^{-1}\left(-\frac{K\Omega_0 \omega_c}{\omega_c^2} \right)$$
$$= \tan^{-1}\left(-\frac{k\Omega_0}{\omega_c} \right)$$

Phases are additive so
$$\angle D_k = \angle C_k + \phi_k.$$

By doing such analysis, we can examine the effect of a filter on a periodic signal, just by considering how the coefficients of the harmonic terms are changed (attenuated in magnitude and shifted in phase) by the filter.