Worksheet 5

To accompany Chapter 3.2 Inverse Laplace Transform

This worksheet can be downloaded as a PDF file. We will step through this worksheet in class.

An annotatable copy of the notes for this presentation will be distributed before the second class meeting as **Worksheet 5** in the **Week 2: Classroom Activities** section of the Canvas site. I will also distribute a copy to your personal **Worksheets** section of the **OneNote Class Notebook** so that you can add your own notes using OneNote.

You are expected to have at least watched the video presentation of Chapter 3.2 of the notes before coming to class. If you haven't watch it afterwards!

After class, the lecture recording and the annotated version of this worksheet will be made available through Canvas.

In [ ]:
```
clear all
format compact
imatlab_export_fig('print-svg')  % Static svg figures.
```

Quiz

Question 3: Did you watch the pre lecture video (or read the notes) for the Inverse Laplace Transform?

A. Yes, watched video

B. Yes, watched the video and read the notes

C. Just read the notes

D. Didn't prepare anything

-> Launch Poll 1

Question 2: Inverse Laplace transforms

Inverse laplace transform the following

1.	1	A.	e^{-at}
2.	$\frac{1}{s}$	B.	$\delta'(t)$
3.	$\frac{1}{s+a}$	C.	$u_0(t)$
4.	$\frac{\omega}{(s+a)^2+\omega^2}$	D.	$e^{-at}\cos\omega t$
5.	$\frac{s+a}{(s+a)^2+\omega^2}$	E.	$e^{-at}\sin\omega t$
6.	s	F.	$\delta(t)$

-> Launch Poll 2

Question 3: Fill in the blanks

Complete this sentence: The [----] of a rational polynomial are the zeros of the numerator. The [----] of a rational polynomial are the zeros of the [-----].

-> Launch Poll 3

Question 4: Knowledge check

Is there anything in this quiz that you think we should go over in more detail in class?

Write your answers in the chat or add to the 💬 ❓ **Questions and Discussion on the Laplace Transformation and its Applications** board in Canvas after class.

The case of the distinct poles

Example 1

Use the PFE method to simplify $F_1(s)$ below and find the time domain function $f_1(t)$ corresponding to $F_1(s)$

$$F_1(s) = \frac{2s+5}{s^2+5s+6}$$

Quick solution: Wolfram Alpha%2F(s%5E2+%2B+5s+%2B+6)%7D)

Matlab Solution - Numerical

In [ ]:
```
format compact
clear all
```

In [ ]:
```
Ns = [2, 5]; Ds = [1, 5, 6];
[r,p,k] = residue(Ns, Ds)
```

Interpreted as:

$$F_1(s) = \frac{1}{s+3} + \frac{1}{s+2}$$

which because of the linearity property of the Laplace Transform and using tables results in the *Inverse Laplace Transform*

$$f_1(t) = e^{-3t} + e^{-2t}$$

Matlab solution - symbolic

In [ ]:
```
syms s t;
Fs = (2*s + 5)/(s^2 + 5*s + 6);
ft = ilaplace(Fs);
pretty(ft)
```

Example 2

Determine the Inverse Laplace Transform of

$$F_2(s) = \frac{3s^2+2s+5}{s^3+9s^2+23s+15}$$

Quick solution: Wolfram Alpha

Solution 2

Because the denominator of $F_2(s)$ is a cubic, it will be difficult to factorise without computer assistance so we use MATLAB to factorise $D(s)$

In [ ]:
```
syms s;
factor(s^3 + 9*s^2 + 23*s + 15)
```

In an exam you'd be given the factors

We can now use the previous technique to find the solution which according to Matlab should be

$$f_1(t) = \frac{3}{4}e^{-t} - \frac{13}{2}e^{-3t} + \frac{35}{4}e^{-5t}$$

The case of the complex poles

Quite often the poles of $F(s)$ are complex and because the complex poles occur as complex conjugate pairs, the number of complex poles is even. Thus if p_k is a complex root of $D(s)$ then its complex conjugate p_k^* is also a root of $D(s)$.

You can still use the PFE with complex poles, as demonstrated in Pages 3-5—3-7 in the textbook. However it is easier to use the fact that complex poles will appear as quadratic factors of the form s^2+as+b and then call on the two transforms in the PFE

$$\frac{\omega}{(s-a)^2+\omega^2} \Leftrightarrow e^{at}\sin\ \omega t$$
$$\frac{s+a}{(s-a)^2+\omega^2} \Leftrightarrow e^{at}\cos\ \omega t$$

Example 3

Rework Example 3-2 from the text book using quadratic factors.

Find the Inverse Laplace Transform of

$$F_3(s) = \frac{s+3}{(s+1)(s^2+4s+8)}$$

Quick solution: Wolfram Alpha – Shows that the computer is not always best!

1. We complete the square in the denominator
2. Then compare with the desired form $(s-a)^2+\omega^2$
3. Solve this by finding the PFE for the assumed solution:
$$F_3(s) = \frac{r_1}{s+1} + \frac{r_2(s-a)}{(s-a)^2+\omega^2} + \frac{ar_3}{(s-a)^2+\omega^2}.$$
   expecting the solution
$$f_3(t) = r_1e^{-t} - r_2e^{-at}\cos\omega t + r_3e^{-at}\sin\omega t.$$

You can use trig. identities to simplify this further if you wish.

Solution 3

The case of the repeated poles

When a rational polynomial has repeated poles

$$F(s) = \frac{N(s)}{(s-p_1)^m(s-p_2)\cdots(s-p_{n-1})(s-p_0)}$$

and the PFE will have the form:

$$F(s) = \frac{r_{11}}{(s-p_1)^m} + \frac{r_{12}}{(s-p_1)^{m-1}} + \frac{r_{13}}{(s-p_1)^{m-2}} + \cdots + \frac{r_1}{(s-p_1)} + \frac{r_2}{(s-p_2)} + \frac{r_3}{(s-p_3)} + \cdots + \frac{r_n}{(s-p_n)}$$

The ordinary residues r_k can be found using the rule used for distinct roots.

To find the residuals for the repeated term r_{1k} we need to multiply both sides of the expression by $(s+p_1)^m$ and take repeated derivatives as described in detail in Pages 3-7—3-9 of the text book. This yields the general formula

$$r_{1k} = \lim_{s\to p_1}\frac{1}{(k-1)!}\frac{d^{k-1}}{ds^{k-1}}\left[(s-p_1)^m F(s)\right]$$

which in the age of computers is rarely needed.

Example 4

Find the inverse Laplace Transform of

$$F_4(s) = \frac{s+3}{(s+2)(s+1)^2}$$

Note that the transform

$$te^{at} \Leftrightarrow \frac{1}{(s-a)^2}$$

will be useful.

Quick solution: Wolfram Alpha

Solution 4

We will leave the solution that makes use of the residude of repeated poles formula for you to study from the text book. In class we will illustrate the slightly simpler approach also presented in the text.

For exam preparation, I would recommend that you use whatever method you find most comfortable.

The case of the improper rational polynomial

If $F(s)$ is an improper rational polynomial, that is $m \geq n$, we must first divide the numerator $N(s)$ by the denomonator $D(s)$ to derive an expression of the form

$$F(s) = k_0 + k_1s + k_2s^2 + \cdots + k_{m-n}s^{m-n} + \frac{N(s)}{D(s)}$$

and then $N(s)/D(s)$ will be a proper rational polynomial.

Example 5

$$F_6(s) = \frac{s^2+2s+2}{s+1}$$

Quick solution: Wolfram Alpha

Dividing s^2+2s+2 by $s+1$ gives

$$F_6(s) = s+1+\frac{1}{s+1}$$

$$f_6(t) = e^{-t} + \delta(t) + \delta'(t)$$

See notes for proof.

Matlab verification for solition 5

In [ ]:
```
Ns = [1, 2, 2]; Ds = [1 1];
[r, p, k] = residue(Ns, Ds)
```

In [ ]:
```
syms s;
F6 = (s^2 + 2*s + 2)/(s + 1);
f6 = ilaplace(F6)
```

Matlab Solutions

For convenience, single script MATLAB solutions to the examples are provided and can be downloaded from the accompanying MATLAB folder.

- Example 1 - Real poles [ex3_1.m]
- Example 2 - Real poles cubic denominator [ex3_2.m]
- Example 3 - Complex poles [ex3_3.m]
- Example 4 - Repeated real poles [ex3_4.m]
- Example 5 - Non proper rational polynomial [ex3_5.m]

In [ ]:
```
cd ../matlab
ls
open ex3_1
```