Worksheet 17

To accompany Chapter 6.4 Models of Discrete-Time Systems

Colophon

This worksheet can be downloaded as a PDF file. We will step through this worksheet in class.

An annotatable copy of the notes for this presentation will be distributed before the second class meeting as **Worksheet 17** in the **Week 9: Classroom Activities** section of the Canvas site. I will also distribute a copy to your personal **Worksheets** section of the **OneNote Class Notebook** so that you can add your own notes using OneNote.

You are expected to have at least watched the video presentation of Chapter 6.4 of the notes before coming to class. If you haven't watch it afterwards!

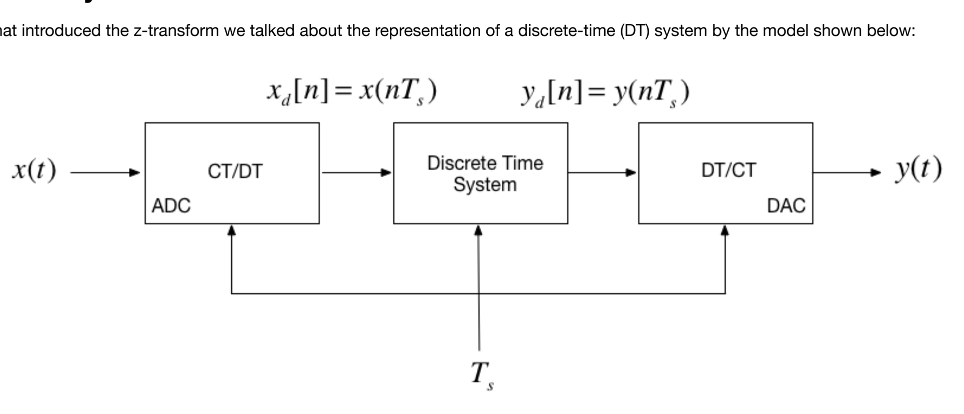
After class, the lecture recording and the annotated version of the worksheets will be made available through Canvas.

Agenda

- Discrete Time Systems (Notes)
- Transfer Functions in the Z-Domain (Notes)
- Modelling digital systems in MATLAB/Simulink
- Continuous System Equivalents
- In-class demonstration: Digital Butterworth Filter

Discrete Time Systems

In the lecture that introduced the z-transform we talked about the representation of a discrete-time (DT) system by the model shown below:



In this session, we want to explore the contents of the central block.

Example 5

Karris Example 9.10:

The difference equation describing the input-output relationship of a DT system with zero initial conditions, is:

$$y[n] - 0.5y[n-1] + 0.125y[n-2] = x[n] + x[n-1]$$

Compute:

1. The transfer function $H(z)$
2. The DT impulse response $h[n]$
3. The response $y[n]$ when the input $x[n]$ is the DT unit step $u_0[n]$

5.1. The transfer function

$$H(z) = \frac{Y(z)}{U(z)} = \ldots ?$$

5.2. The DT impulse response

Start with:

$$\frac{H(z)}{z} = \frac{z+1}{z^2 + 0.5z + 0.125}$$

MATLAB Solution

```
In [ ]: clear all
        imatlab_export_fig('print-svg')  % Static svg figures.
        cd matlab
        pwd
        format compact
```

See dtm_ex1_2.mlx. (Also available as dtm_ex1_2.m.)

The difference equation describing the input-output relationship of the DT system with zero initial conditions, is:

$$y[n] - 0.5y[n-1] + 0.125y[n-2] = x[n] + x[n-1]$$

Transfer function

Numerator $z^2 + z$

```
In [ ]: Nz = [1 1 0];
```

Denominator $z^2 - 0.5z + 0.125$

```
In [ ]: Dz = [1 -0.5 0.125];
```

Poles and residues

```
In [ ]: [r,p,k] = residue(Nz,Dz)
```

Impulse Response

```
In [ ]: Hz = tf(Nz,Dz,-1)
        hn = impulse(Hz, 15);
```

Plot the response

```
In [ ]: stem([0:15], hn)
        grid
        title('Example 5 - Part 2')
        xlabel('n')
        ylabel('Impulse response h[n]')
```

Response as stepwise continuous y(t)

```
In [ ]: impulse(Hz,15)
        grid
        title('Example 5 - Part 2 - As Analogue Signal')
        xlabel('nTs [s]')
        ylabel('Impulse response h(t)')
```

5.3. The DT step response

$$Y(z) = H(z)X(z)$$

$$u_0[n] \Leftrightarrow \frac{z}{z-1}$$

$$\begin{aligned} Y(z) = H(z)U_0(z) &= \frac{z^2+z}{z^2+0.5z+0.125} \cdot \frac{z}{z-1} \\ &= \frac{z(z^2+z)}{(z^2+0.5z+0.125)(z-1)} \end{aligned}$$

$$\frac{Y(z)}{z} = \frac{z^2+z}{(z^2+0.5z+0.125)(z-1)}$$

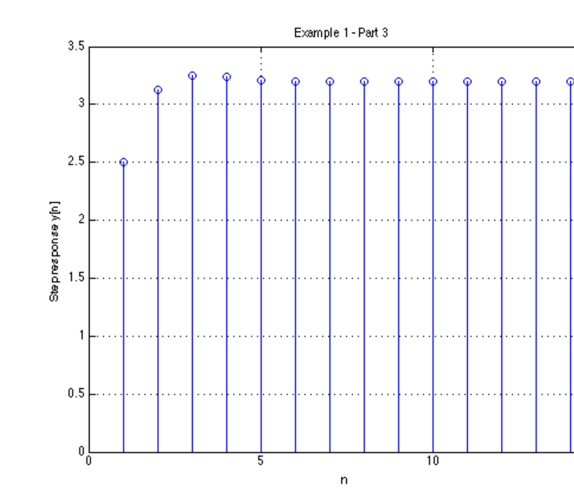
Solved by inverse Z-transform.

MATLAB Solution

See dtm_ex1_3.mlx. (Also available as dtm_ex1_3.m.)

```
In [ ]: open dtm_ex1_3
```

Results



Modelling DT systems in MATLAB and Simulink

We will consider some examples in class

MATLAB

Code extracted from dtm_ex1_3.m:

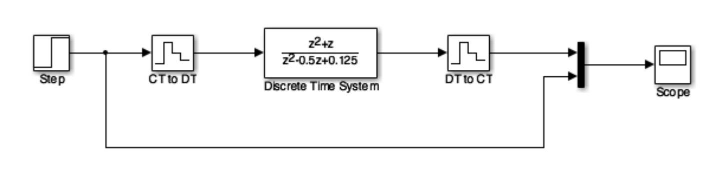
```
In [ ]: Ts = 1;
        z = tf('z', Ts);
```

```
In [ ]: Hz = (z^2 + z)/(z^2 - 0.5 * z + 0.125)
```

```
In [ ]: step(Hz)
        grid
        title('Example 1 - Part 3 - As Analogue Signal')
        xlabel('nTs [s]')
        ylabel('Step response y(t)')
        axis([0,15,0,3.5])
```

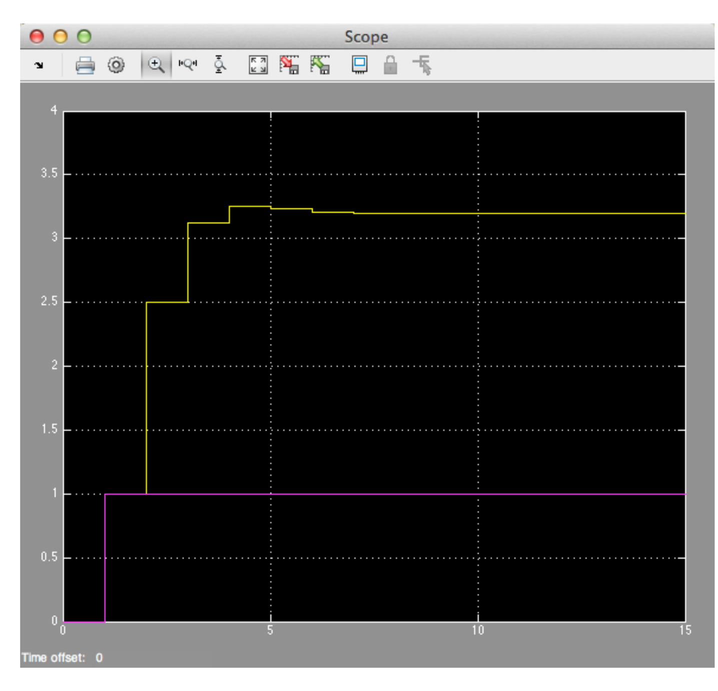
Simulink Model

See dtm.slx:




```
In [ ]: dtm
```

Results



Converting Continuous Time Systems to Discrete Time Systems

Continuous System Equivalents

- There is no digital system that uniquely represents a continuous system
- This is because as we are sampling, we only have knowledge of signals being processed at the sampling instants, and need to *reconstruct* the inter-sample behaviour.
- In practice, only a small number of transformations are used.
- The derivation of these is beyond the scope of this module, but here we'll demonstrate the ones that MATLAB provides in a function called `c2d`

MATLAB c2d function

Let's see what the help function says:

```
In [ ]: help c2d
```

```
In [ ]: help c2d
```

Example 6

- Design a 2nd-order butterworth low-pass anti-aliasing filter with transfer function $H(s)$ for use in sampling music.
- The cut-off frequency $\omega_c = 20$ kHz and the filter should have an attenuation of at least -80 dB in the stop band.
- Choose a suitable sampling frequency for the audio signal and give the transfer function $H(z)$ and an algorithm to implement $h[n]$

Solution

See digi_butter.mlx.

First determine the cut-off frequency ω_c

$$\omega_c = 2\pi f_c = 2 \times \pi \times 20 \times 10^3 \text{ rad/s}$$

```
In [ ]: wc = 2*pi*20e3
```

$$\omega_c = 125.66 \times 10^3 \text{ rad/s}$$

From the lecture on filters, we know the 2nd-order butterworth filter has transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{\omega_c^2}{s^2 + \omega_c\sqrt{2}\,s + \omega_c^2}$$

Substituting for $\omega_c = 125.6637 \times 10^3$ this is ...?

```
In [ ]: Hs = tf(wc^2,[1 wc*sqrt(2), wc^2])
```

$$H(s) = \frac{15.79 \times 10^9}{s^2 + 177.7 \times 10^3 s + 15.79 \times 10^9}$$

Bode plot

MATLAB:

```
In [ ]: bode(Hs,{1e4,1e8})
        grid
```

Sampling Frequency

From the bode diagram, the frequency at which $|H(j\omega)|$ is -80 dB is approx 12.6×10^6 rad/s.

To avoid aliasing, we should choose a sampling frequency twice this = ?

$\omega_s = 2 \times 12.6 \times 10^6$ rad/s.

```
In [ ]: ws = 2* 12.6e6
```

So

$\omega_s = 25.2 \times 10^6$ rad/s.

Sampling frequency (f_s) in Hz = ?

$$f_s = \omega_s/(2\pi) \text{ Mhz}$$

```
In [ ]: fs = ws/(2*pi)
```

$$f_s = 40.11 \text{ Mhz}$$

Sampling time T_s =?

$T_s = 1/fs$ s

```
In [ ]: Ts = 1/fs
```

$$T_s = 1/f_s \approx 0.25\ \mu\text{s}$$

Digital Butterworth

zero-order-hold equivalent

```
In [ ]: Hz = c2d(Hs, Ts)
```

Step response

```
In [ ]: step(Hz)
```

Algorithm

From previous result:

$$H(z) = \frac{Y(z)}{U(z)} = \frac{486.6 \times 10^{-6} z + 476.5 \times 10^{-6}}{z^2 - 1.956z + 0.9567}$$

Dividing top and bottom by z^2 ...

$$H(z) = \frac{Y(z)}{U(z)} = \frac{486.6 \times 10^{-6} z^{-1} + 476.5 \times 10^{-6} z^{-2}}{1 - 1.956z^{-1} + 0.9567z^{-2}}$$

expanding out ...

$$\begin{gathered} Y(z) - 1.956z^{-1}Y(z) + 0.9567z^{-2}Y(z) = \\ 486.6 \times 10^{-6} z^{-1} U(z) + 476.5 \times 10^{-6} z^{-2} U(z) \end{gathered}$$

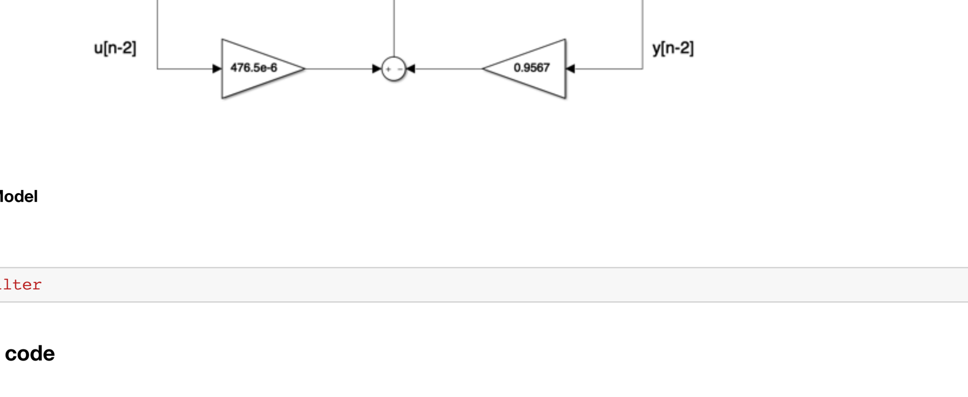
Inverse z-transform gives ...

$$\begin{gathered} y[n] - 1.956y[n-1] + 0.9567y[n-2] = \\ 486.6 \times 10^{-6} u[n-1] + 476.5 \times 10^{-6} u[n-2] \end{gathered}$$

in algorithmic form (compute $y[n]$ from past values of u and y) ...

$$\begin{gathered} y[n] = 1.956y[n-1] - 0.9567y[n-2] + 486.6 \times 10^{-6} u[n-1] + \ldots \\ 476.5 \times 10^{-6} u[n-2] \end{gathered}$$

Block Diagram of the digital BW filter



As Simulink Model

digifilter.slx

```
In [ ]: open digifilter
```

Convert to code

To implement:

$$y[n] = 1.956y[n-1] - 0.9567y[n-2] + 486.6 \times 10^{-6} u[n-1] + 476.5 \times 10^{-6} u[n-2]$$

```
/* Initialize */
Ts = 2.4933e-07; /* more probably some fraction of clock speed */
ynm1 = 0; ynm2 = 0; unm1 = 0; unm2 = 0;
while (true) {
    un = read_adc;
    yn = 1.956*ynm1 - 0.9567*ynm2 + 486.6e-6*unm1 + 476.5e-6*unm2;
    write_dac(yn);
    /* store past values */
    ynm2 = ynm1; ynm1 = yn;
    unm2 = unm1; unm1 = un;
    wait(Ts);
}
```

Comments

PC soundcards can sample audio at 44.1 kHz so this implies that the anti-aliasing filter is much sharper than this one as $f_s/2 = 22.05$ kHz.

You might wish to find out what order butterworth filter would be needed to have $f_c = 20$ kHz and f_{stop} of 22.05 kHz.