Introduction to Filters

Colophon

An annotatable worksheet for this presentation is available as **Worksheet 15**.

- The source code for this page is fourier_transform/4/ft4.ipynb.
- You can view the notes for this presentation as a webpage (HTML).
- This page is downloadable as a PDF file.

Scope and Background Reading

This section is Based on the section **Filtering** from Chapter 5 of Benoit Boulet, Fundamentals of Signals and Systems[Bou06] from the **Recommended Reading List**.

This material is an introduction to analogue filters. You will find much more in-depth coverage on Pages 11-1—11-48 of [Kar12].

Agenda

- Frequency Selective Filters
- Ideal low-pass filter
- Butterworth low-pass filter
- High-pass filter
- Bandpass filter

Introduction

- Filter design is an important application of the Fourier transform
- Filtering is a rich topic often taught in graduate courses so we give only an introduction.
- Our introduction *will* illustrate the usefulness of the frequency domain viewpoint.
- We will explore how filters can shape the spectrum of a signal.

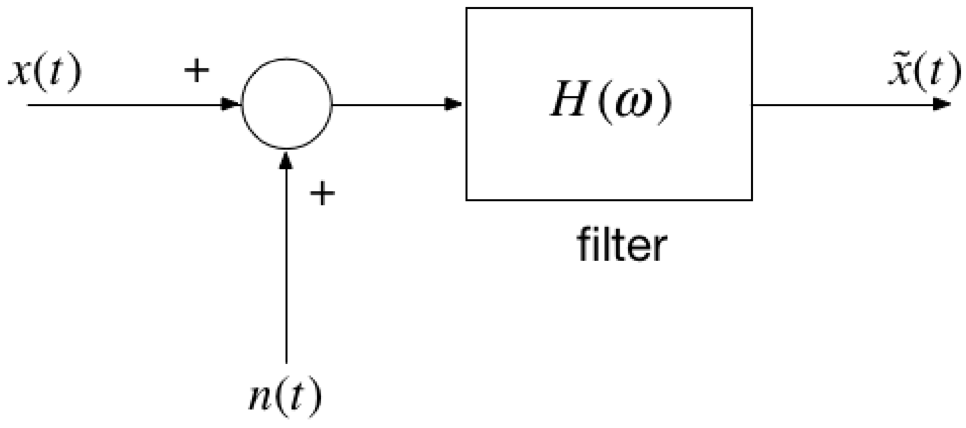
Other applications of the Fourier transform are sampling theory (introduced next week) and modulation.

Frequency Selective Filters

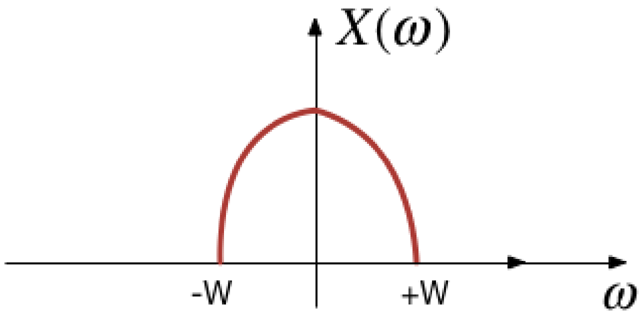
An ideal frequency-selective filter is a system that let's the frequency components of a signal through undistorted while frequency components at other frequency are completely cut off.

- The range of frequencies which are let through belong to the **pass Band**
- The range of frequencies which are cut-off by the filter are called the **stopband**
- A typical scenario where filtering is needed is when noise $n(t)$ is added to a signal $x(t)$ but that signal has most of its energy outside the bandwidth of a signal.

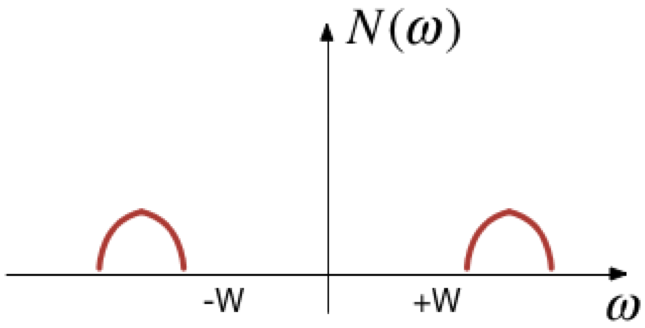
Typical filtering problem



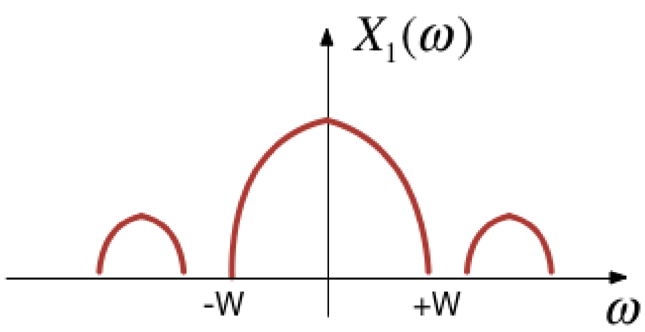
Signal



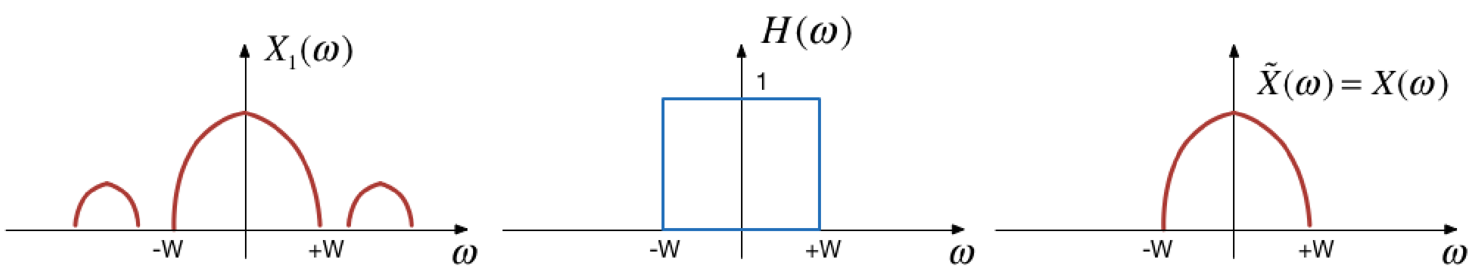
Out-of Bandwidth Noise



Signal plus Noise



Results of filtering



Motivating example

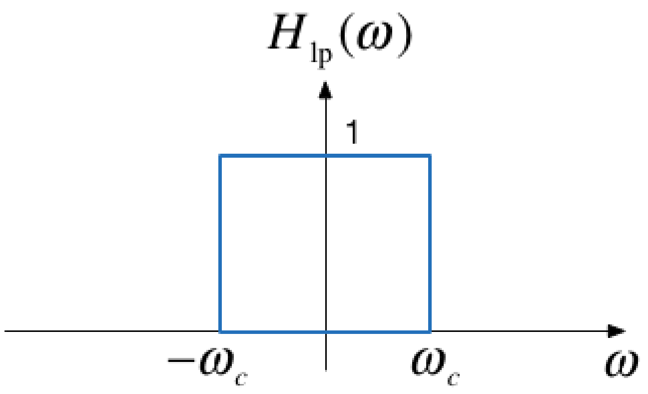
See the video and script on [Canvas Week 7](#).

Ideal Low-Pass Filter (LPF)

er cuts-off frequencies higher than its *cut-off frequency*, ω_c.

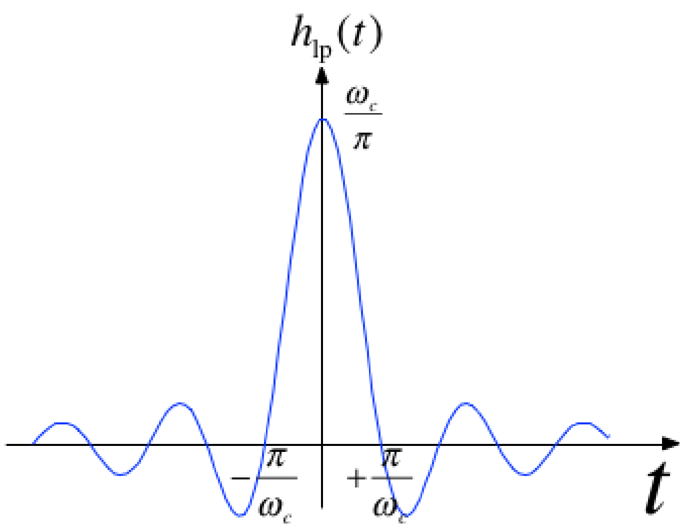
$$H_{\mathrm{lp}}(\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| \geq \omega_c \end{cases}$$

Frequency response of an ideal LPF



Impulse response of an ideal LPF

$$h_{\mathrm{lp}}(t) = \frac{\omega_c}{\pi}\mathrm{sinc}\left(\frac{\omega_c}{\pi}t\right)$$



Filtering is Convolution

The output of an LTI system with impulse response

$$h(t) \Leftrightarrow H(\omega)$$

subject to an input signal

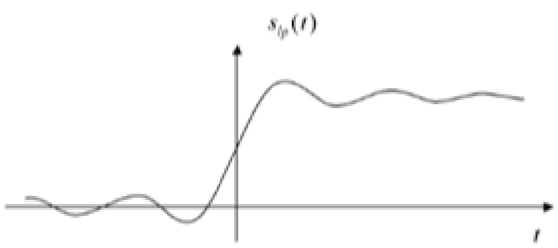
$$x(t) \Leftrightarrow X(\omega)$$

is given by

$$y(t) = h(t) * x(t) \Leftrightarrow Y(\omega) = H(\omega)X(\omega)$$

Issues with the “ideal” filter

This is the step response:



(reproduced from [Bou06] Fig. 5.23 p. 205)

Ripples in the impulse resonse would be undesireable, and because the impulse response is non-causal it cannot actually be implemented.

Butterworth low-pass filter

N-th Order Butterworth Filter

$$\left|H_B(\omega)\right| = \frac{1}{\left(1+\left(\frac{\omega}{\omega_c}\right)^{2N}\right)^{\frac{1}{2}}}$$

Remarks

- DC gain is

$$|H_B(j0)| = 1$$

- Attenuation at the cut-off frequency is

$$|H_B(j\omega_c)| = 1/\sqrt{2}$$

  for any N

More about the Butterworth filter: [Wikipedia Article]

Example 5: Second-order BW Filter

The second-order butterworth Filter is defined by is Characteristic Equation (CE):

$$p(s) = s^2 + \omega_c\sqrt{2}s + \omega_c^2 = 0^*$$

Calculate the roots of $p(s)$ (the poles of the filter transfer function) in both Cartesian and polar form.

Note: This has the same characteristic as a control system with damping ratio $\zeta = 1/\sqrt{2}$ and $\omega_n = \omega_c$!

Solution to example 5

Example 6

Derive the differential equation relating the input $x(t)$ to output $y(t)$ of the 2nd-Order Butterworth Low-Pass Filter with cutoff frequency ω_c.

Solution to example 6

Processing math: 100%

Example 7

Determine the frequency response $H_B(\omega) = Y(\omega)/X(\omega)$

Solution to example 7

Magnitude of frequency response of a 2nd-order Butterworth Filter

```
imatlab_export_fig('print-svg')  % Static svg figures.
```

```
wc = 100;
```

Transfer function

```
H = tf(wc^2,[1, wc*sqrt(2), wc^2])
```

```
H =


          10000
```

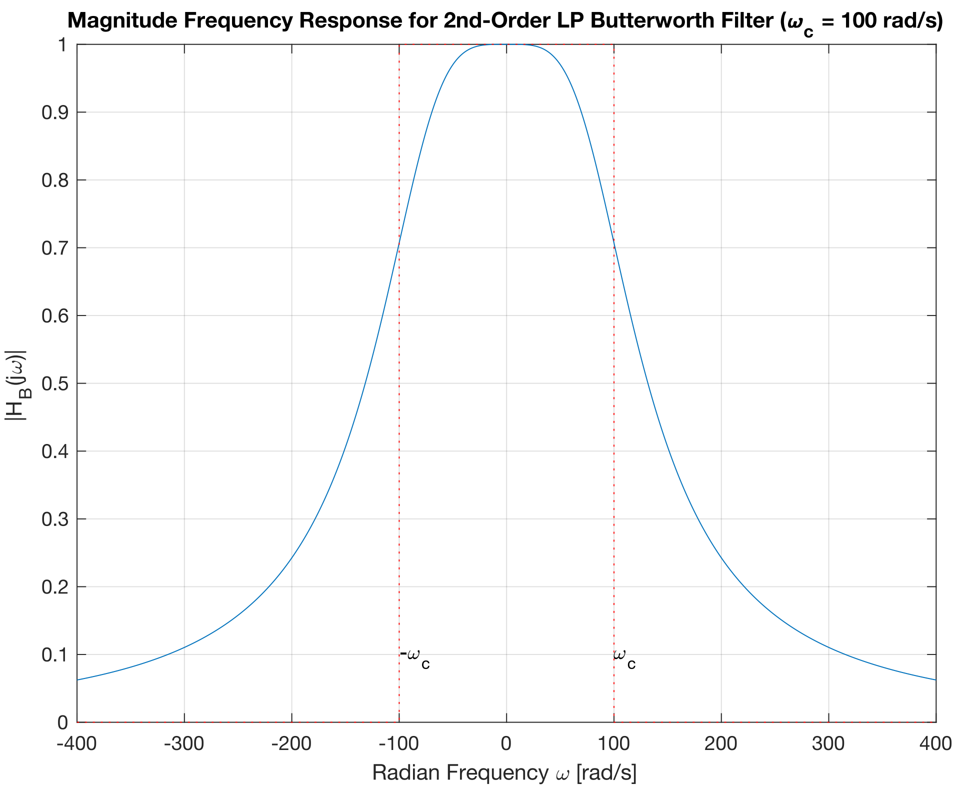
```
---------------------
```

```
s^2 + 141.4 s + 10000
```

```
Continuous-time transfer function.
```

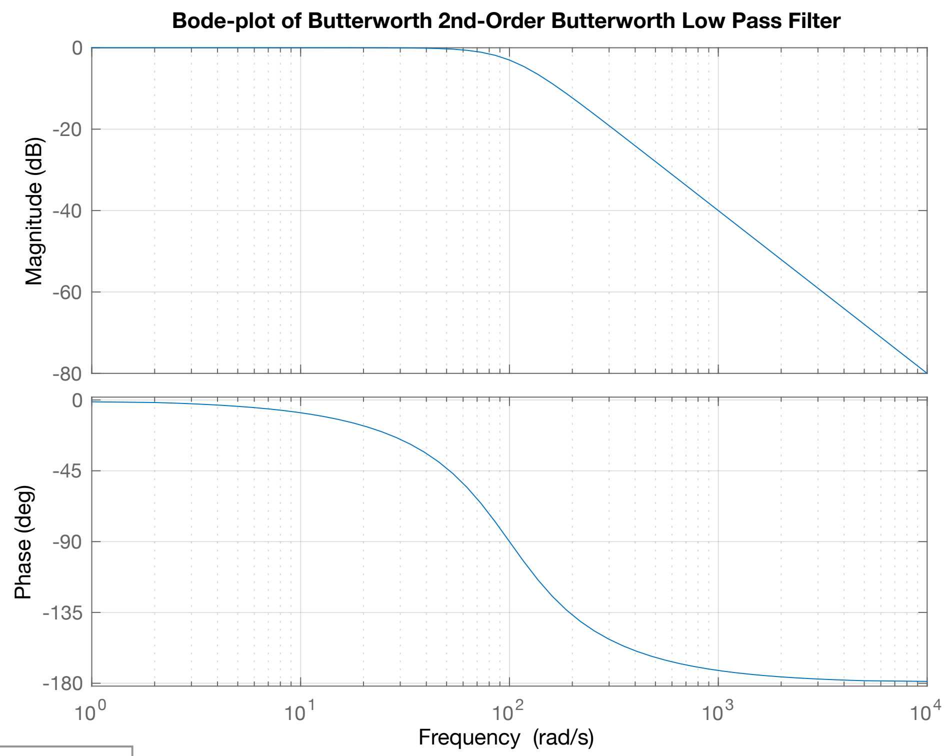
Magnitude frequency response

```
w = -400:400;
mHlp = 1./(sqrt(1 + (w./wc).^4));
plot(w,mHlp)
grid
ylabel('|H_B(j\omega)|')
title('Magnitude Frequency Response for 2nd-Order LP Butterworth Filter (\omega_c = 100
rad/s)')
xlabel('Radian Frequency \omega [rad/s]')
text(100,0.1,'\omega_c')
text(-100,0.1,'-\omega_c')
hold on
plot([-400,-100,-100,100,100,400],[0,0,1,1,0,0],'r:')
hold off
```



Bode plot

```
bode(H)
grid
title('Bode-plot of Butterworth 2nd-Order Butterworth Low Pass Filter')
```



Note that the attentuation of the filter is flat at 0 dB in the pass-band at frequencies below the cut-off frequency $\omega < \omega_c$; has a value of -3 dB at the cut-off frquency $\omega = \omega_c$; and has a "roll-off" (rate of decrease) of $N \times 20$ dB/decade in the stop-band.

In this case, $N = 2$, and $\omega_c = 100$ rad/s so the attenuation is -40 dB at $\omega = 10\omega_c = 1,000$ rad/s and $\omega = -80$ dB at $\omega = 100\omega_c = 10,000$ rad/s.

The phase is 0° at $\omega = 0$; $N \times 90^\circ$ at $\omega = \infty$; and $N \times 45^\circ$ and $\omega = \omega_c$.

Example 8

Determine the impulse and step response of a butterworth low-pass filter.

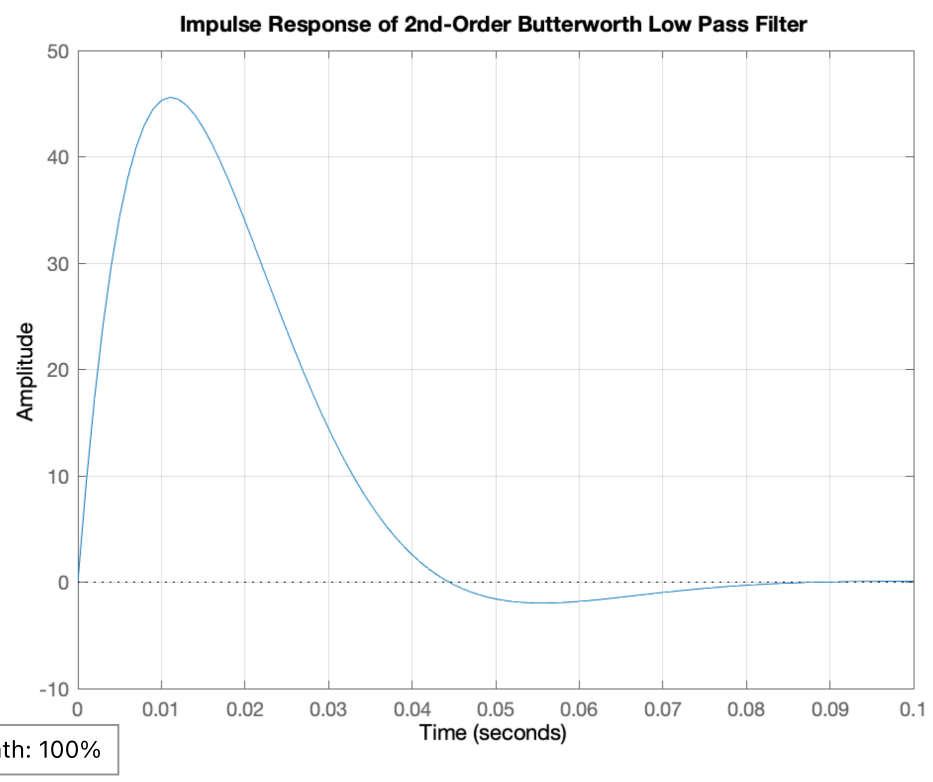
You will find this Fourier transform pair useful:

$$e^{-at}\sin\omega_0 t\; u_0(t) \Leftrightarrow \frac{\omega_0}{(j\omega + a)^2 + \omega_0^2}$$

Solution to example 8

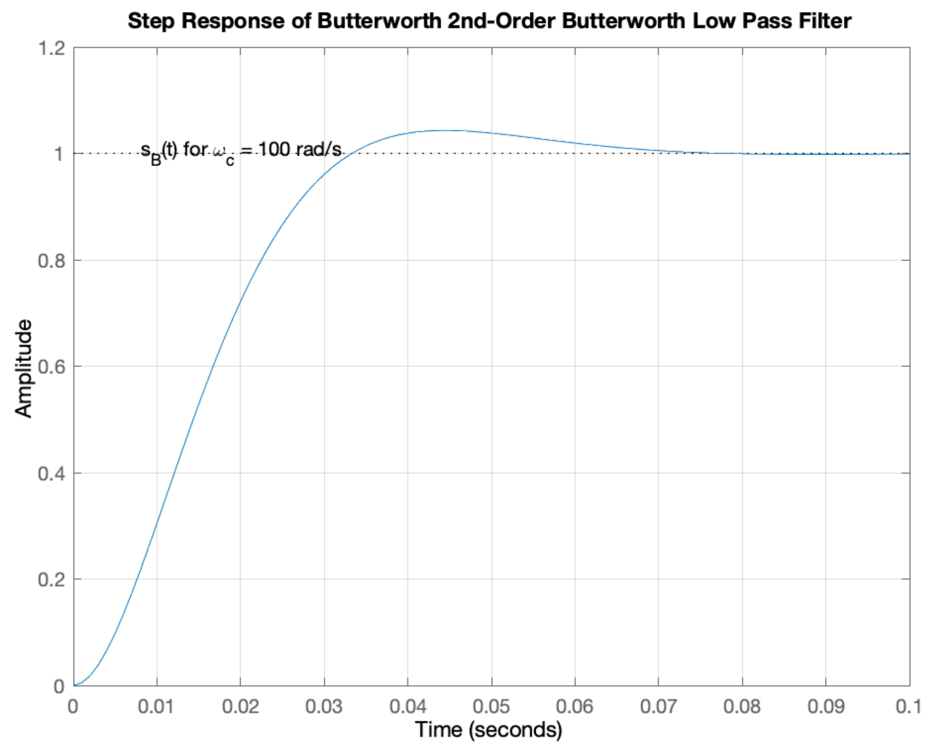
Impulse response

```
impulse(H,0.1)
grid
title('Impulse Response of 2nd-Order Butterworth Low Pass Filter')
```



Step response

```
step(H,0.1)
title('Step Response of Butterworth 2nd-Order Butterworth Low Pass Filter')
grid
text(0.008,1,'s_B(t) for \omega_c = 100 rad/s')
```

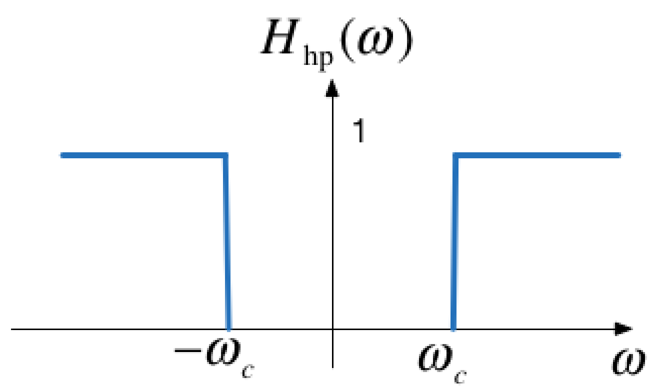


High-pass filter (HPF)

An ideal highpass filter cuts-off frequencies lower than its *cutoff frequency*, ω_c.

$$H_{\text{hp}}(\omega) = \begin{cases} 0, & |\omega| \leq \omega_c \\ 1, & |\omega| > \omega_c \end{cases}$$

Frequency response of an ideal HPF



Responses

Frequency response

$$H_{\text{hp}}(\omega) = 1 - H_{\text{lp}}(\omega)$$

Impulse response

$$h_{\text{hp}}(t) = \delta(t) - h_{\text{lp}}(t)$$

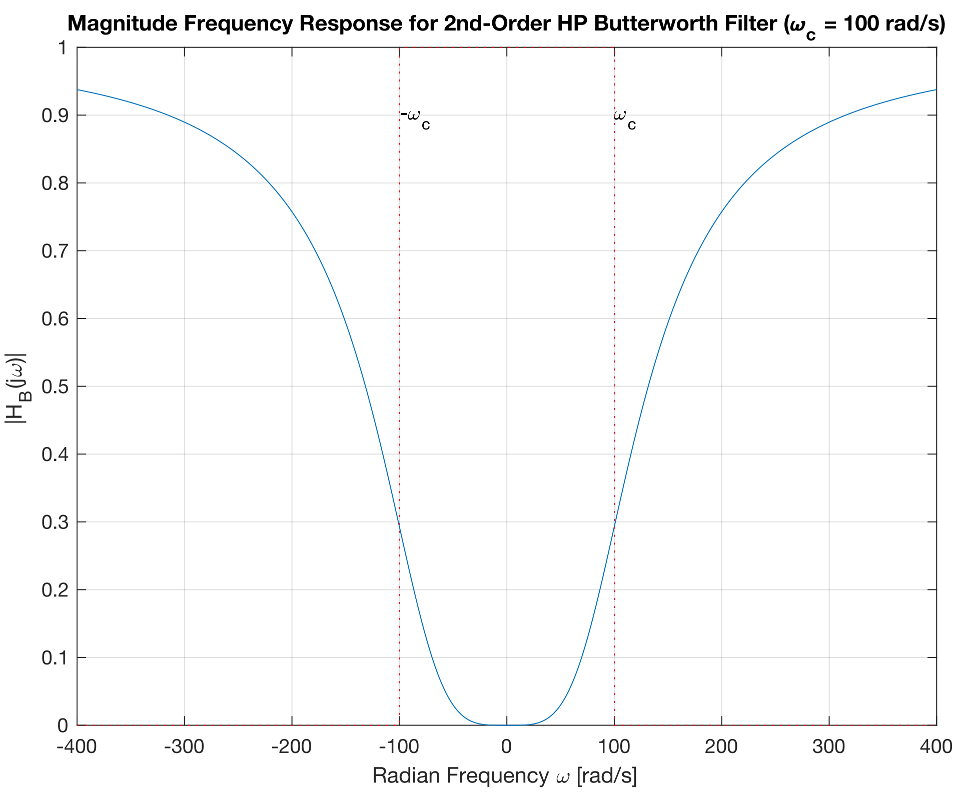
Example 9

Determine the frequency response of a 2nd-order butterworth highpass filter

Processing math: 100% mple 9

Magnitude frequency response

```
w = -400:400;
plot(w,1-mHlp)
grid
ylabel('|H_B(j\omega)|')
title('Magnitude Frequency Response for 2nd-Order HP Butterworth Filter (\omega_c = 100 rad/s)')
xlabel('Radian Frequency \omega [rad/s]')
text(100,0.9,'\omega_c')
text(-100,0.9,'-\omega_c')
hold on
plot([-400,-100,-100,100,100,400],[0,0,1,1,0,0],'r:')
hold off
```



High-pass filter

```
Hhp = 1 - H
bode(Hhp)
grid
title('Bode-plot of Butterworth 2nd-Order Butterworth High Pass Filter')
```

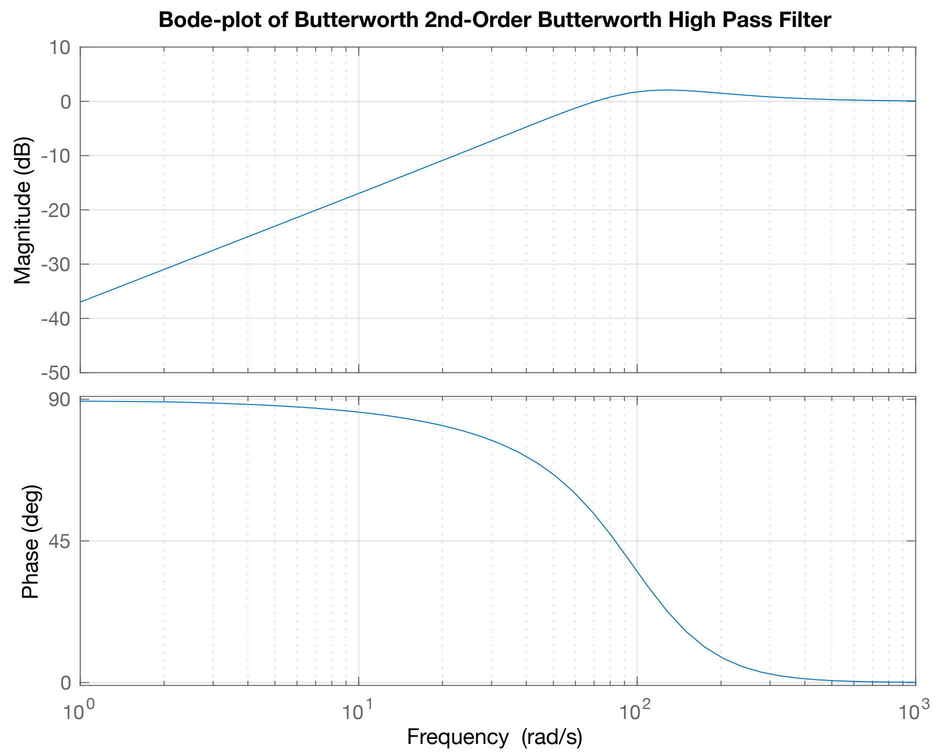
```
Hhp =


      s^2 + 141.4 s

  -----------
```

```
s^2 + 141.4 s + 10000
```

```
Continuous-time transfer function.
```

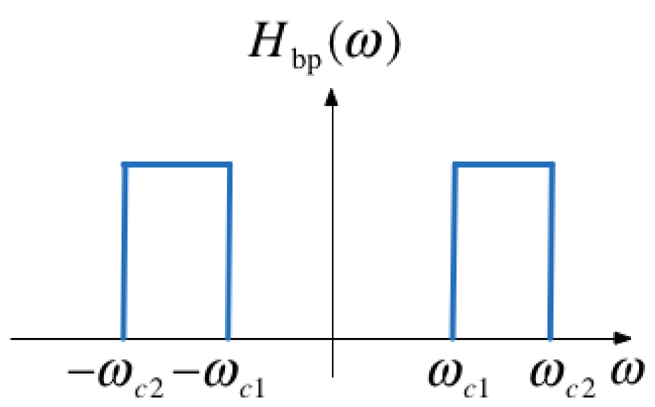


Band-pass filter (BPF)

An ideal bandpass filter cuts-off frequencies lower than its first *cutoff frequency* ω_{c1}, and higher than its second *cutoff frequency* ω_{c2}.

$$H_{\text{bp}}(\omega) = \begin{cases} 1, & \omega_{c1} < \ |\omega|\ < \omega_{c2} \\ 0, & \text{otherwise} \end{cases}$$

Frequency response of an ideal BPF



Bandpass filter design

A bandpass filter can be obtained by multiplying the frequency responses of a lowpass filter by a highpass filter.

$$H_{\text{bp}}(\omega) = H_{\text{hp}}(\omega) H_{\text{lp}}(\omega)$$

- The highpass filter should have cut-off frequency of ω_{c1}
- The lowpass filter should have cut-off frequency of ω_{c2}

To generate all the plots shown in this presentation, you can use [butter2_ex.mlx]

Summary

- Frequency-Selective Filters

- Ideal low-pass filter
- Butterworth low-pass filter
- High-pass filter
- Bandpass filter

Solutions

Solutions to Examples 5-9 are captured as a PenCast in filters.pdf.

By Dr Chris P. Jobling