Worksheet 9

To accompany Chapter 4.1 Trigonometric Fourier Series

We will step through this worksheet in class.

You are expected to have at least watched the video presentation of Chapter 4.1 (https://cpjobling.github.io/eg-247-textbook/fourier_series/1/trig_fseries) of the notes (https://cpjobling.github.io/eg-247-textbook) before coming to class. If you haven't watch it afterwards!

Motivating Example

In the class I will demonstrate the Fourier Series demo (see Notes (trig_fs)).

The Trigonometric Fourier Series

Any periodic waveform $f(t)$ can be represented as

$$\begin{aligned} f(t) = & \frac{1}{2}a_0 + a_1 \cos \Omega_0 t + a_2 \cos 2\Omega_0 t + a_3 \cos 3\Omega_0 t + \cdots + a_n \cos n\Omega_0 t + \cdot \\ & + \; b_1 \sin \Omega_0 t + b_2 \sin 2\Omega_0 t + b_3 \sin 3\Omega_0 t + \cdots + b_n \sin n\Omega_0 t + \cdots \end{aligned}$$

or equivalently (if more confusingly)

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos n\Omega_0 t + b_n \sin n\Omega_0 t)$$

where Ω_0 rad/s is the *fundamental frequency*.

Evaluation of the Fourier series coefficients

The coefficients are obtained from the following expressions (valid for any periodic waveform with fundamental frequency Ω_0 so long as we integrate over one period $0 \rightarrow T_0$ where $T_0 = 2\pi/\Omega_0$), and $\theta = \Omega_0 t$:

$$\frac{1}{2}a_0 = \frac{1}{T_0}\int_0^{T_0} f(t)dt = \frac{1}{\pi}\int_0^{2\pi} f(\theta)d\theta$$

$$a_n = \frac{1}{T_0}\int_0^{T_0} f(t)\cos n\Omega_0 t\, dt = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta$$

$$b_n = \frac{1}{T_0}\int_0^{T_0} f(t)\sin n\Omega_0 t\, dt = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta$$

Odd, Even and Half-wave Symmetry

Odd- and even symmetry

- An *odd* function is one for which $f(t) = -f(-t)$. The function $\sin t$ is an *odd* function.
- An *even* function is one for which $f(t) = f(-t)$. The function $\cos t$ is an *even* function.

Half-wave symmetry

- A periodic function with period T is a function for which $f(t) = f(t + T)$
- A periodic function with period T, has *half-wave symmetry* if $f(t) = -f(t + T/2)$

Symmetry in Trigonometric Fourier Series

There are simplifications we can make if the original periodic properties has certain properties:

- If $f(t)$ is odd, $a_0 = 0$ and there will be no cosine terms so $a_n = 0 \; \forall n > 0$

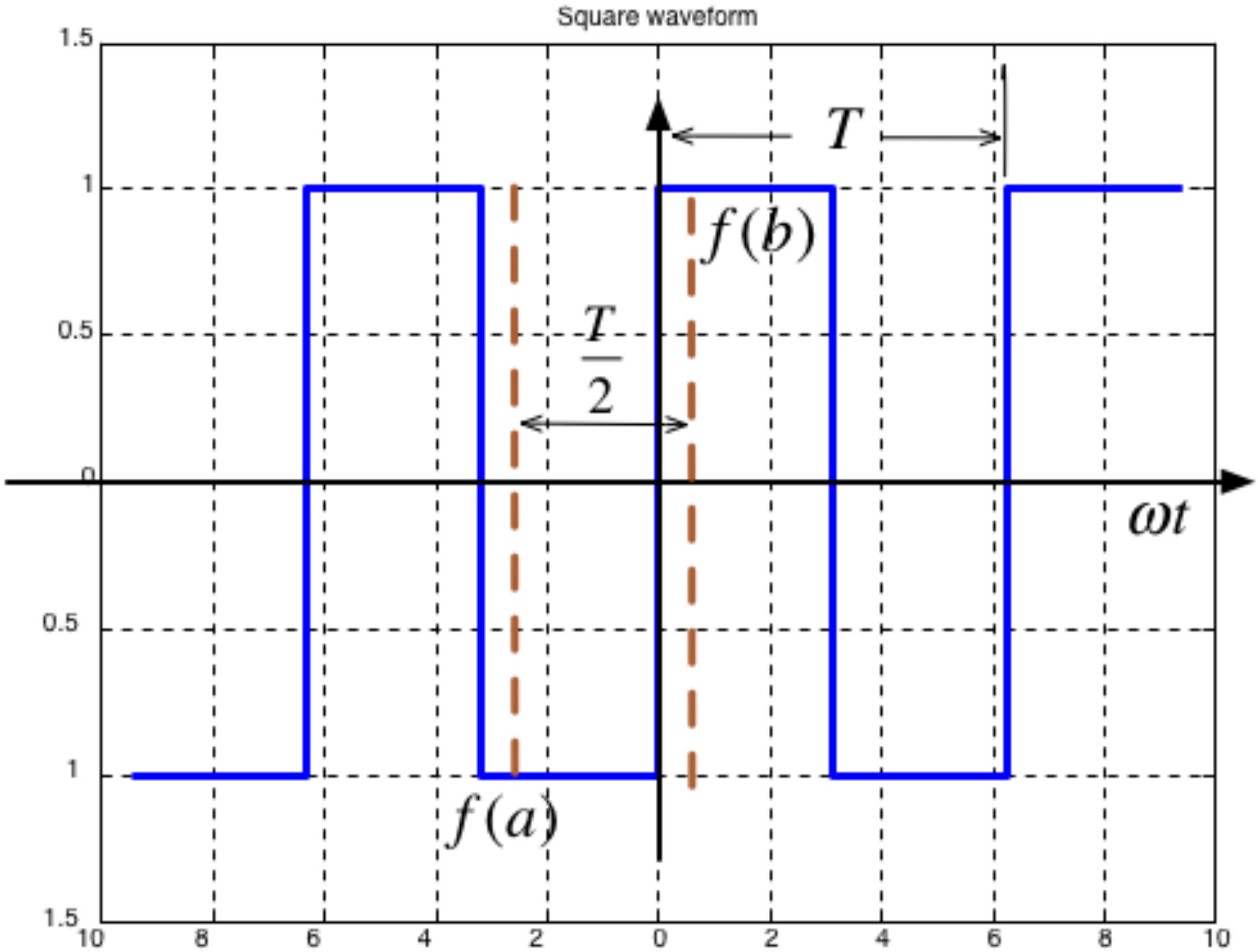
- If $f(t)$ is even, there will be no sine terms and $b_n = 0 \; \forall n > 0$. The DC may or may not be zero.

- If $f(t)$ has *half-wave symmetry* only the odd harmonics will be present. That is a_n and b_n is zero for all even values of n (0, 2, 4, ...)

Symmetry in Common Waveforms

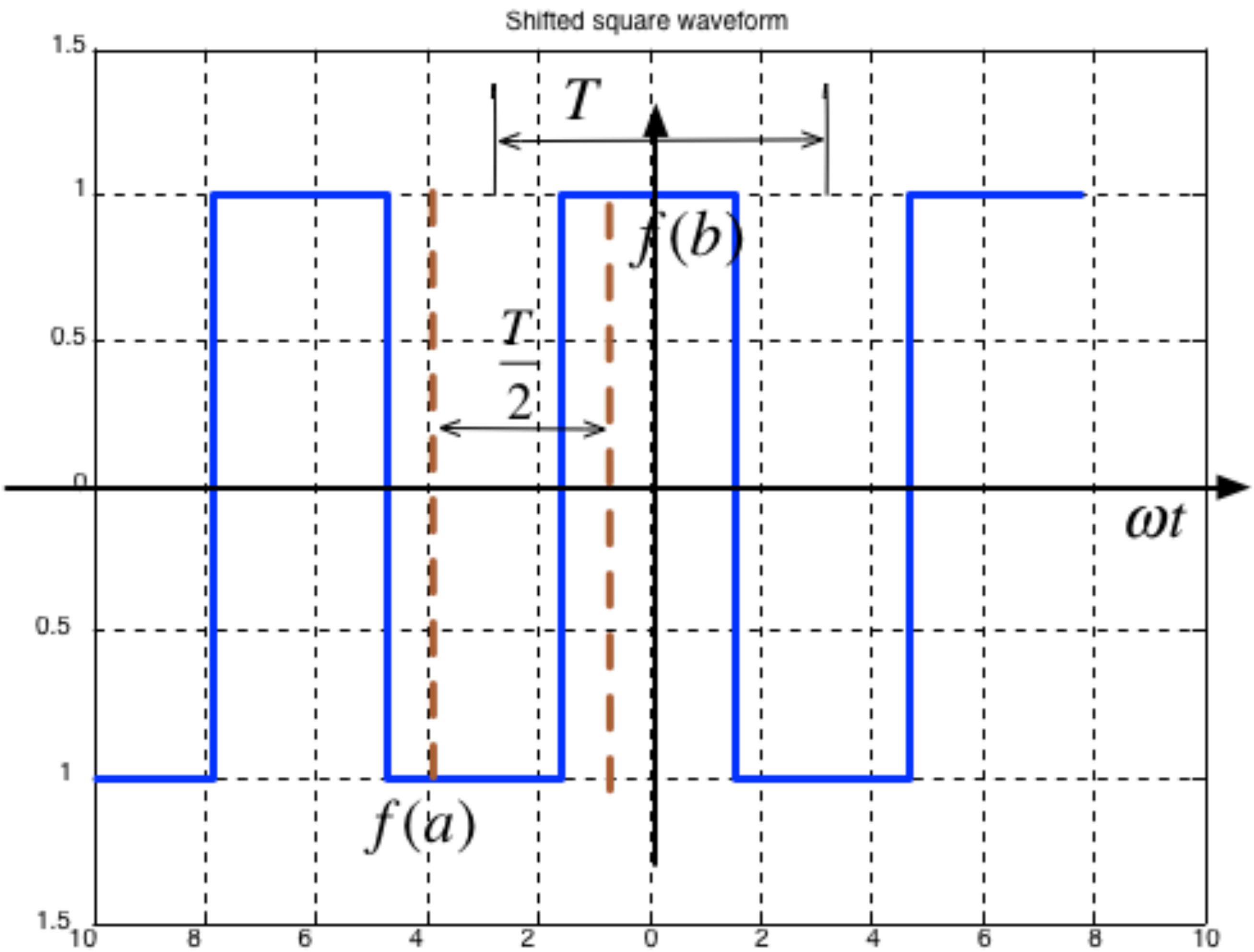
To reproduce the following waveforms (without annotation) publish the script waves.m (waves.m).

Squarewave



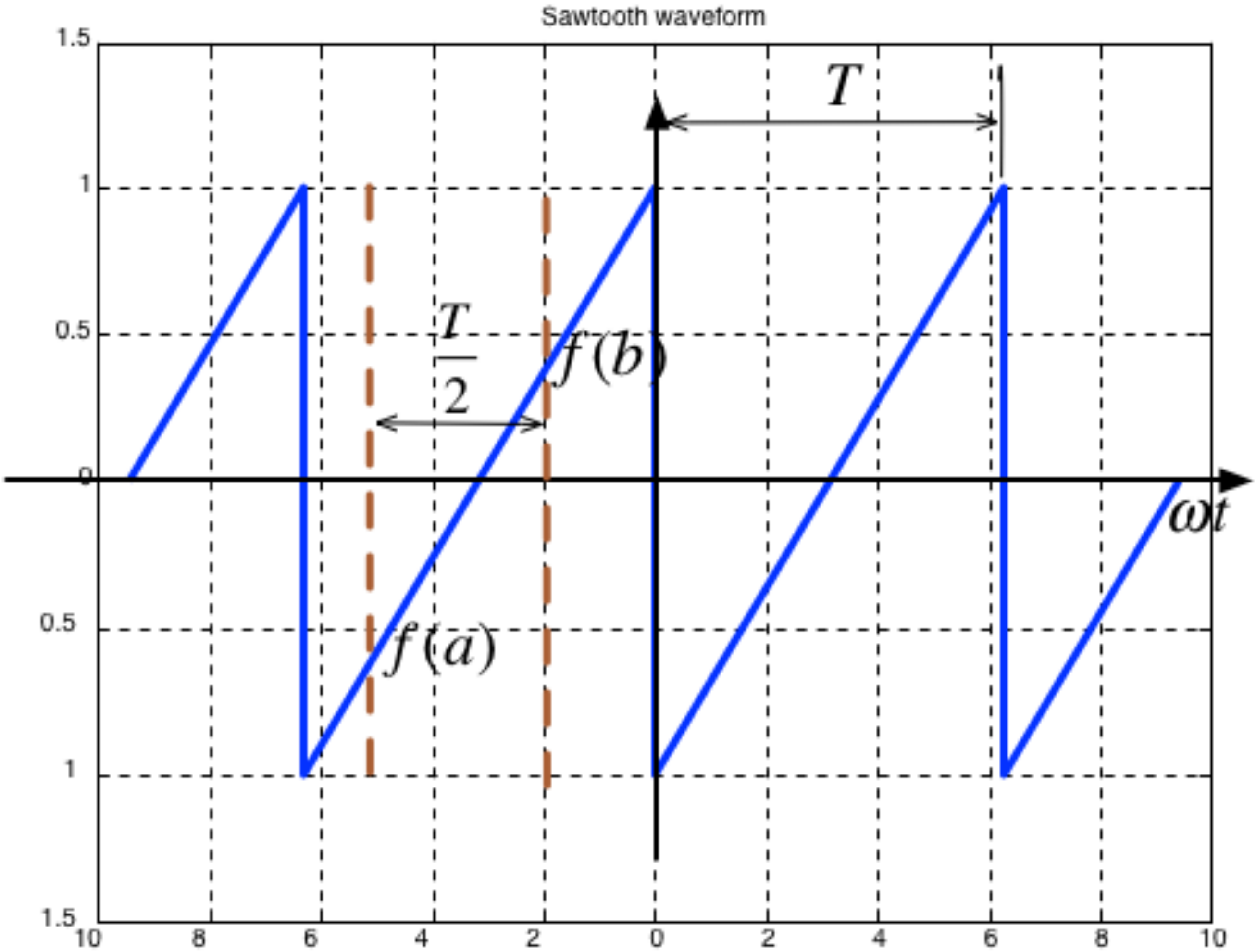
- Average value over period T is **...**?
- It is an **odd**/**even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Shifted Squarewave



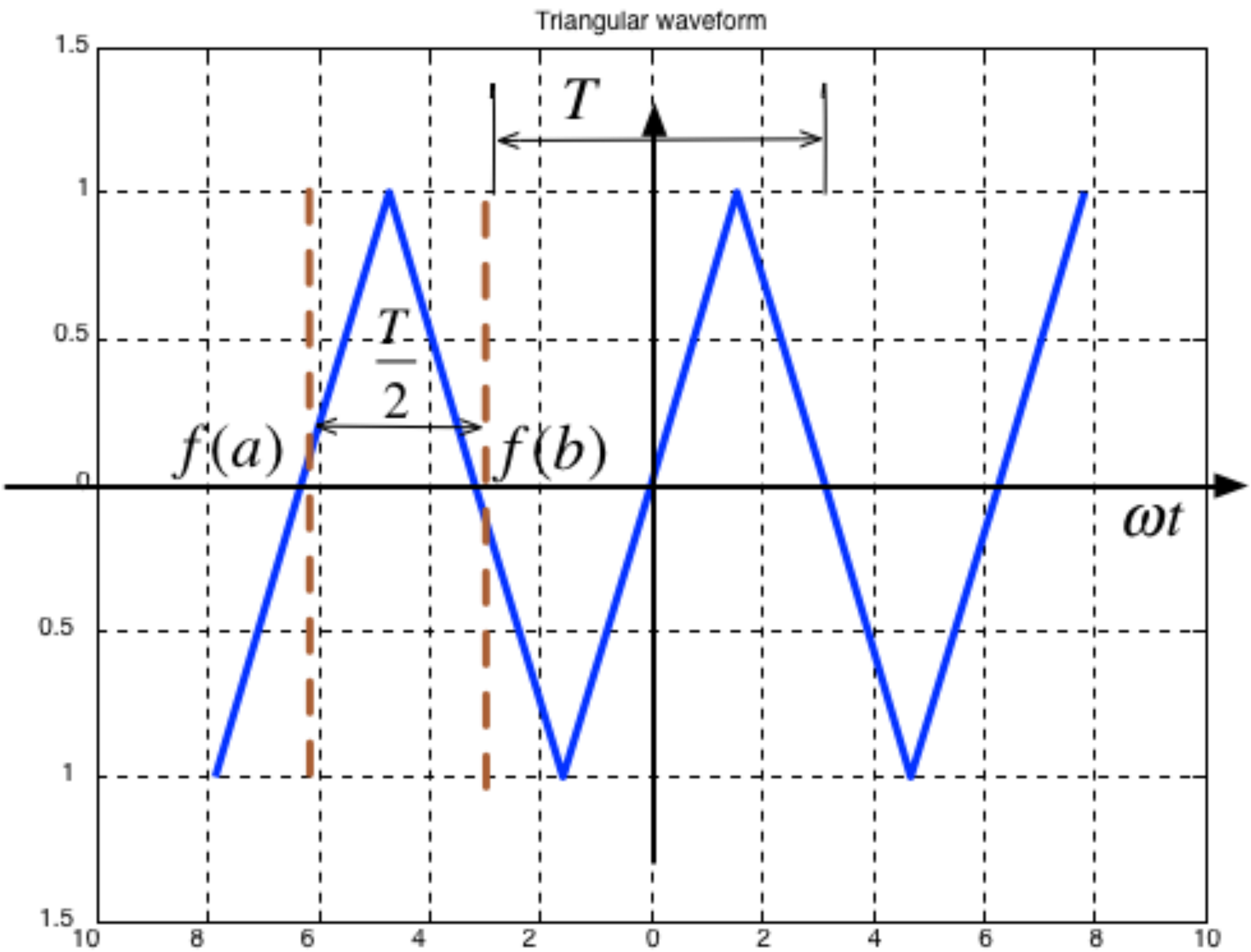
- Average value over period T is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Sawtooth



- Average value over period T is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Triangle

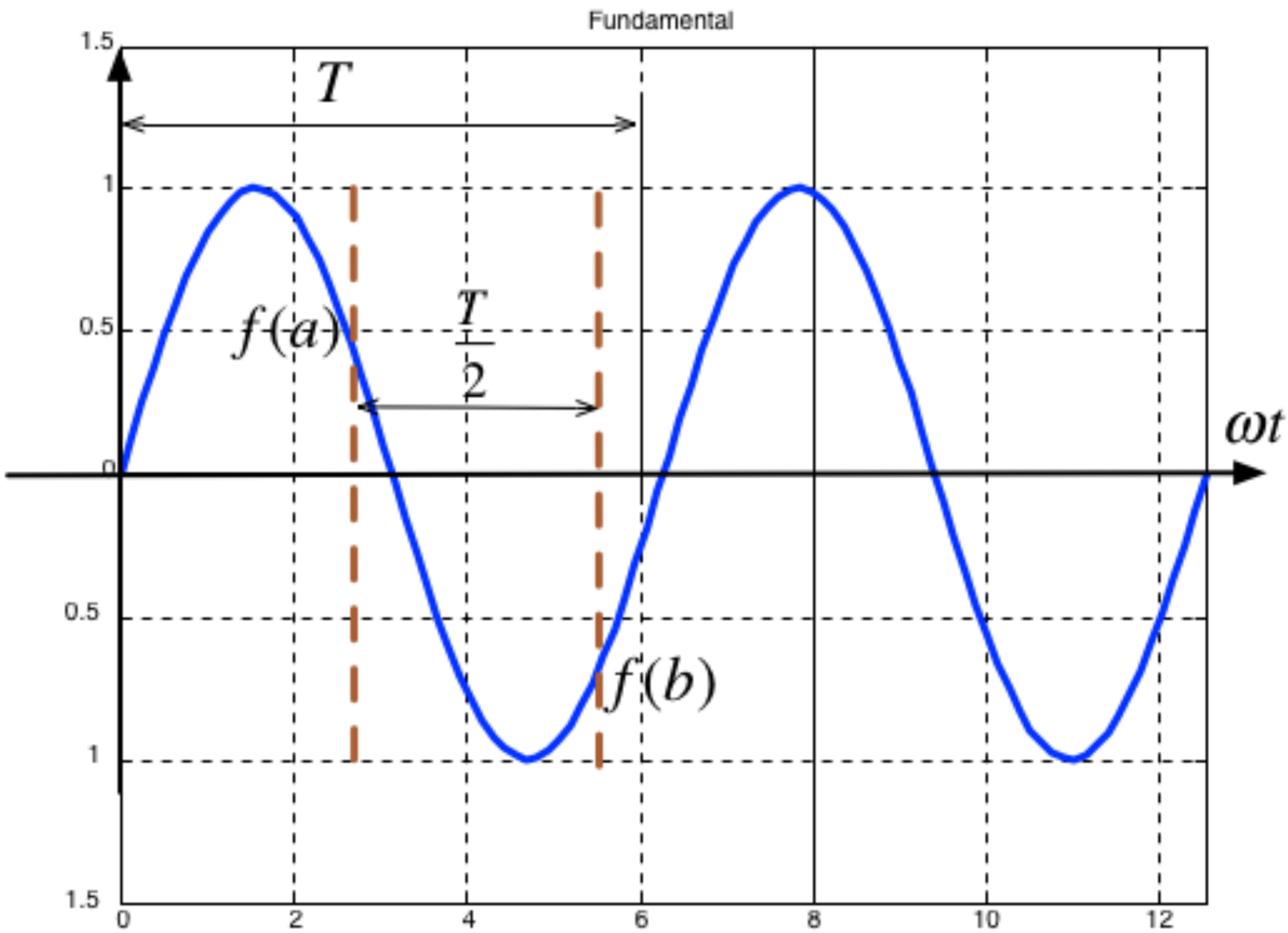


- Average value over period T is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Symmetry in fundamental, Second and Third Harmonics

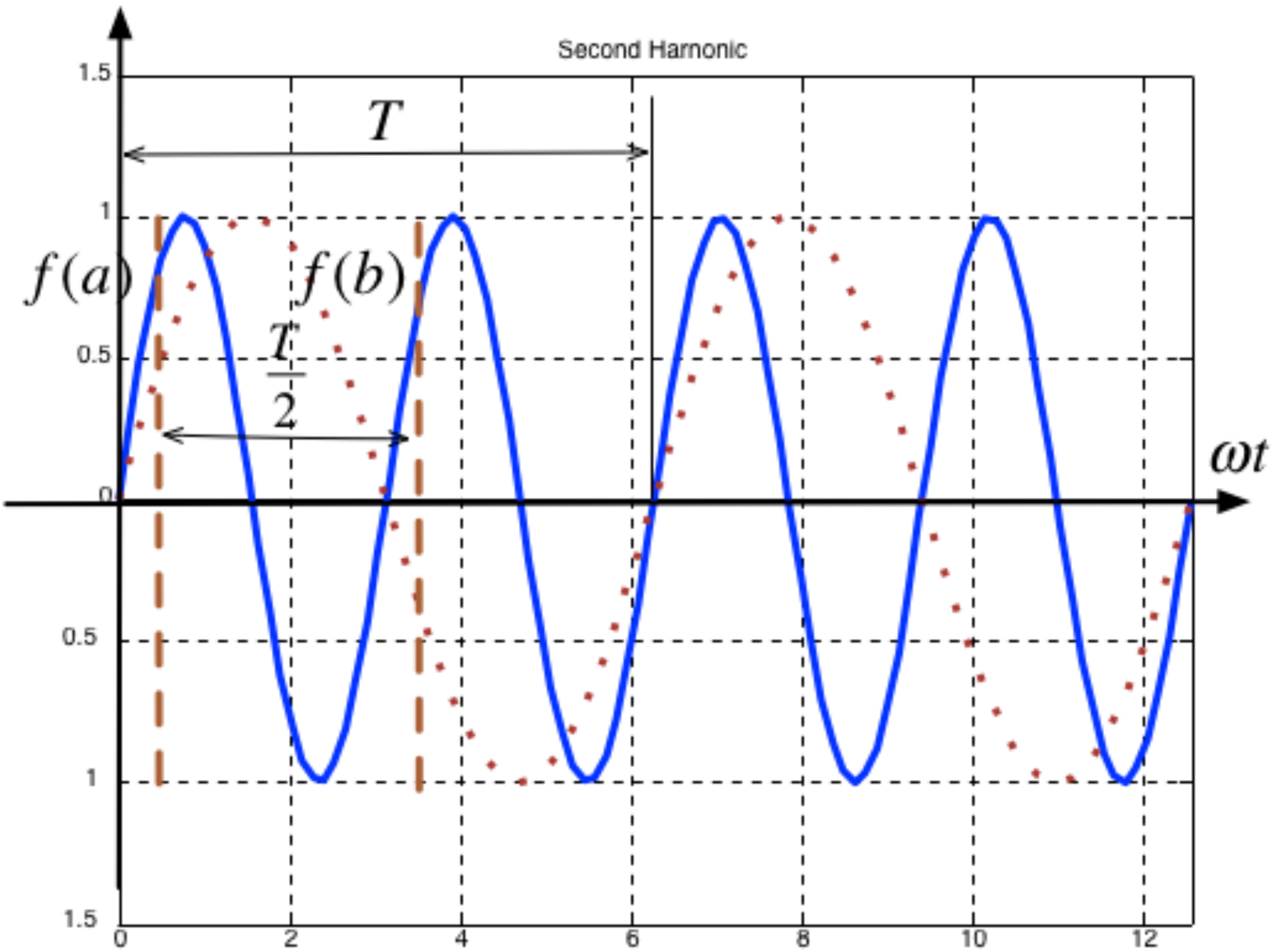
In the following, $T/2$ is taken to be the half-period of the fundamental sinewave.

Fundamental



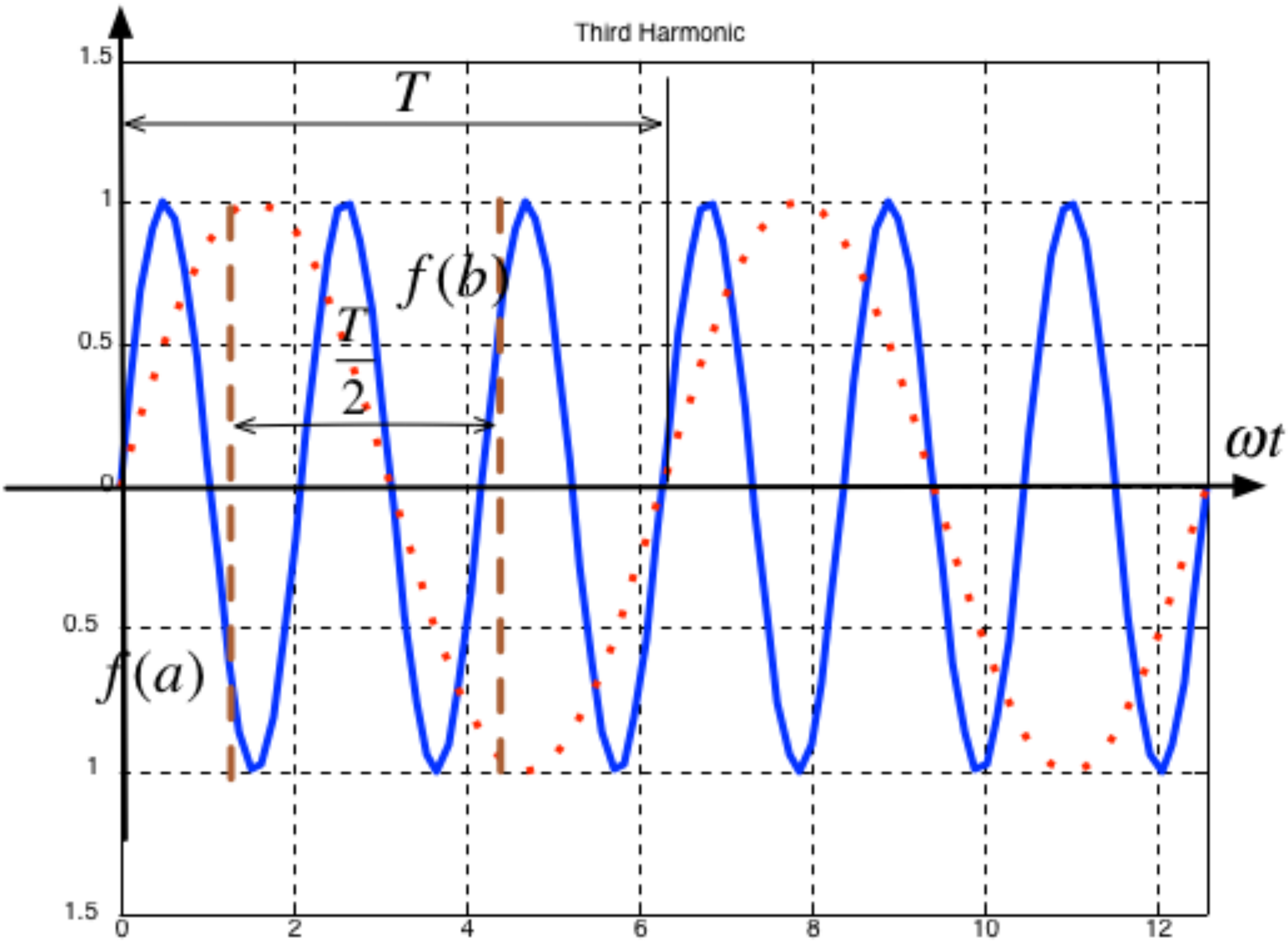
- Average value over period T is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Second Harmonic



- Average value over period T is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

Third Harmonic



- Average value over period T is
- It is an **odd/even** function?
- It **has/has not** half-wave symmetry $f(t) = -f(t + T/2)$?

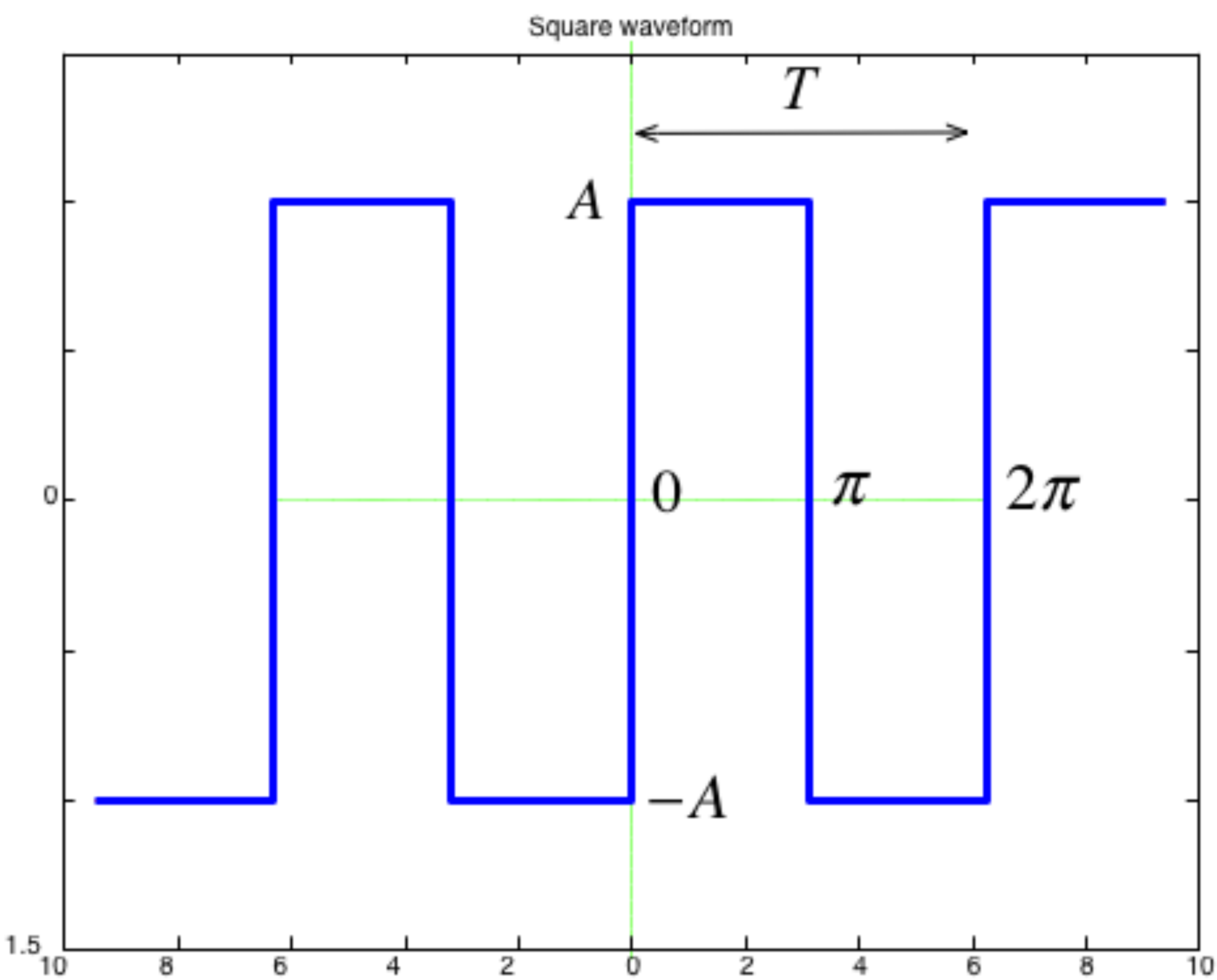
Some simplifications that result from symmetry

- The limits of the integrals used to compute the coefficents a_n and b_n of the Fourier series are given as $0 \rightarrow 2\pi$ which is one period T
- We could also choose to integrate from $-\pi \rightarrow \pi$
- If the function is *odd*, or *even* or has *half-wave symmetry* we can compute a_n and b_n by integrating from $0 \rightarrow \pi$ and multiplying by 2.
- If we have *half-wave symmetry* we can compute a_n and b_n by integrating from $0 \rightarrow \pi/2$ and multiplying by 4.

(For more details see page 7-10 of the textbook)

Computing coefficients of Trig. Fourier Series in Matlab

As an example let's take a square wave with amplitude $\pm A$ and period T.



Solution

Solution: See square_ftrig.mlx (square_ftrig.mlx). Script confirms that:

- $a_0 = 0$
- $a_i = 0$: function is odd
- $b_i = 0$: for i even - half-wave symmetry

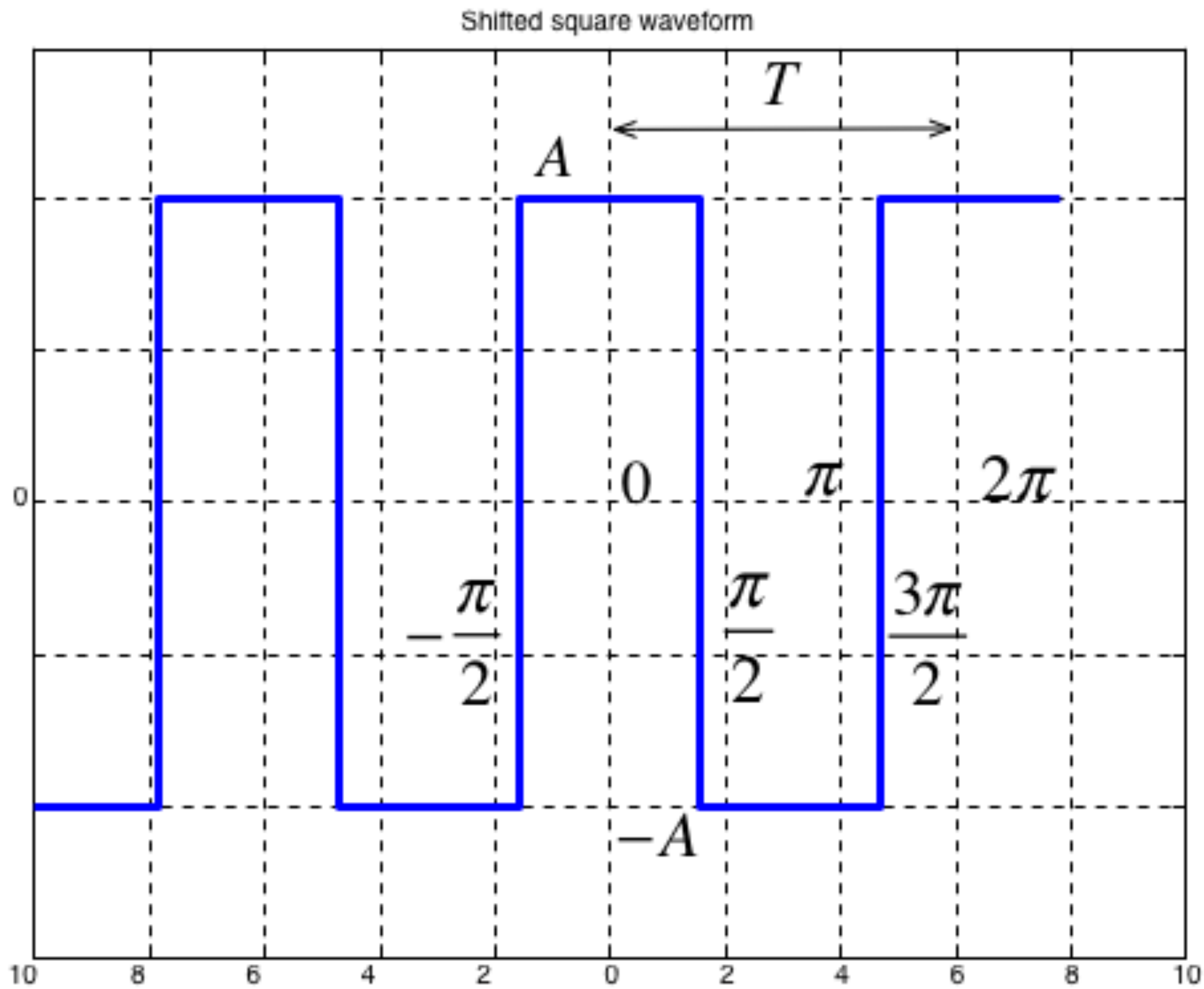
```
ft =

(4*A*sin(t))/pi + (4*A*sin(3*t))/(3*pi) + (4*A*sin(5*t)
)/(5*pi) + (4*A*sin(7*t))/(7*pi) + (4*A*sin(9*t))/(9*pi
) + (4*A*sin(11*t))/(11*pi)
```

Note that the coefficients match those given in the textbook (Section 7.4.1).

$$f(t) = \frac{4A}{\pi}\left(\sin \Omega_0 t + \frac{1}{3}\sin 3\Omega_0 t + \frac{1}{5}\sin 5\Omega_0 t + \cdots\right) = \frac{4A}{\pi}\sum_{n=\text{odd}} \frac{1}{n}\sin n\Omega_0 t$$

Using symmetry - computing the Fourier series coefficients of the shifted square wave



- As before $a_0 = 0$
- We observe that this function is even, so all b_k coefficents will be zero
- The waveform has half-wave symmetry, so only odd indexed coeeficents will be present.
- Further more, because it has half-wave symmetry we can just integrate from $0 \to \pi/2$ and multiply the result by 4.

See [shifted_sq_ftrig.mlx (shifted_sq_ftrig.mlx)](shifted_sq_ftrig.mlx).

```
ft =

(4*A*cos(t))/pi - (4*A*cos(3*t))/(3*pi) + (4*A*cos(5*t)
)/(5*pi) - (4*A*cos(7*t))/(7*pi) + (4*A*cos(9*t))/(9*pi
) - (4*A*cos(11*t))/(11*pi)
```

Note that the coefficients match those given in the textbook (Section 7.4.2).

$$f(t) = \frac{4A}{\pi}\left(\cos \Omega_0 t - \frac{1}{3}\cos 3\Omega_0 t + \frac{1}{5}\cos 5\Omega_0 t - \cdots\right) = \frac{4A}{\pi}\sum (-1)^{\frac{n-1}{2}}\frac{1}{n}\mathrm{c}$$