

# Fourier Transforms for Circuit and LTI Systems Analysis

## Colophon

An annotatable worksheet for this presentation is available as [Worksheet 14](#).

- The source code for this page is [fourier\\_transform/3/fig3.ipynb](#).
- You can view the notes for this presentation as a webpage ([HTML](#)).
- This page is downloadable as a [PDF](#) file.

In this section we will apply what we have learned about Fourier transforms to some typical circuit problems. After a short introduction, the body of this chapter will form the basis of an examples class.

## Agenda

- The system function
- Examples

## The System Function

### System response from system impulse response

Recall that the convolution integral of a system with impulse response  $h(t)$  and input  $u(t)$  is

$$h(t) * u(t) = \int_{-\infty}^{\infty} h(t - \tau) u(\tau) d\tau.$$

We let

$$g(t) = h(t) * u(t)$$

Then by the time convolution property

$$h(t) * u(t) = g(t) \Leftrightarrow G(\omega) = H(\omega) \cdot U(\omega)$$

### The System Function

We call  $H(\omega)$  the *system function*.

We note that the system function  $H(\omega)$  and the impulse response  $h(t)$  form the Fourier transform pair

$$h(t) \Leftrightarrow H(\omega)$$

### Obtaining system response

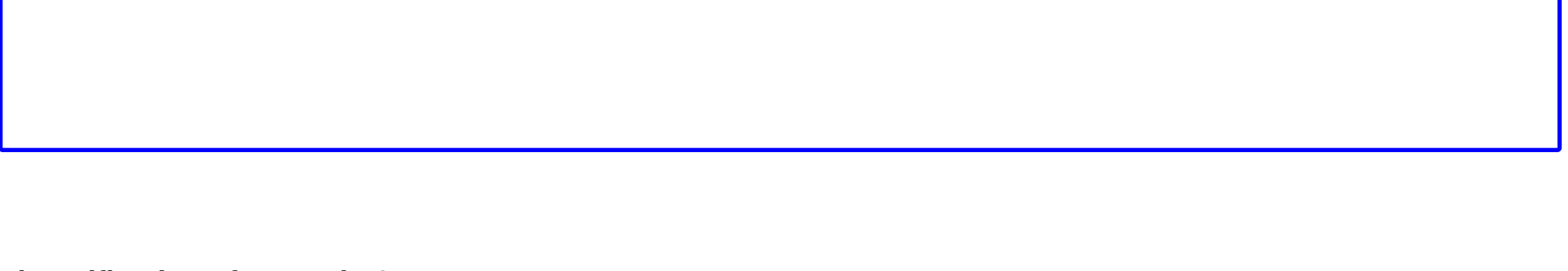
If we know the impulse response  $h(t)$ , we can compute the system response  $g(t)$  of any input  $u(t)$  by multiplying the Fourier transforms of  $H(\omega)$  and  $U(\omega)$  to obtain  $G(\omega)$ . Then we take the inverse Fourier transform of  $G(\omega)$  to obtain the response  $g(t)$ .

1. Transform  $h(t) \rightarrow H(\omega)$
2. Transform  $u(t) \rightarrow U(\omega)$
3. Compute  $G(\omega) = H(\omega) \cdot U(\omega)$
4. Find  $F^{-1}\{G(\omega)\} \rightarrow g(t)$

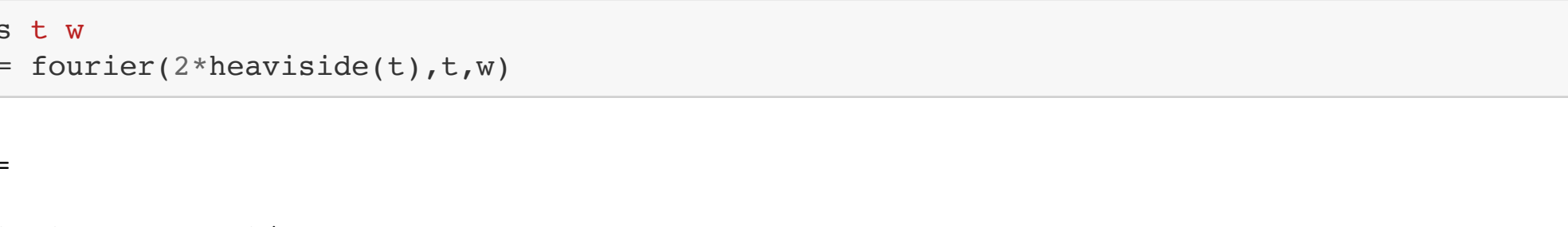
## Examples

### Example 1

Karris example 8.8: for the linear network shown below, the impulse response is  $h(t) = 3e^{-2t}$ . Use the Fourier transform to compute the response  $y(t)$  when the input  $u(t) = 2[u_0(t) - u_0(t - 3)]$ . Verify the result with MATLAB.



#### Solution to example 1



#### Matlab verification of example 1

```
In [1]: imatlab_export_fig('print-svg') % Static svg figures.

In [2]: syms t w
U1 = fourier(2*heaviside(t),t,w)

U1 =
2*pi*dirac(w) - 2i/w

In [3]: H = fourier(3*exp(-2*t)*heaviside(t),t,w)

H =
3/(2 + w*1i)

In [4]: Y1=simplify(H*U1)

Y1 =
3*pi*dirac(w) - 6i/(w*(2 + w*1i))

In [5]: y1 = simplify(ifourier(Y1,w,t))

y1 =
(3*exp(-2*t)*(sign(t) + 1)*(exp(2*t) - 1))/2

Get y2
Substitute t - 3 into f.

In [6]: y2 = subs(y1,t,t-3)

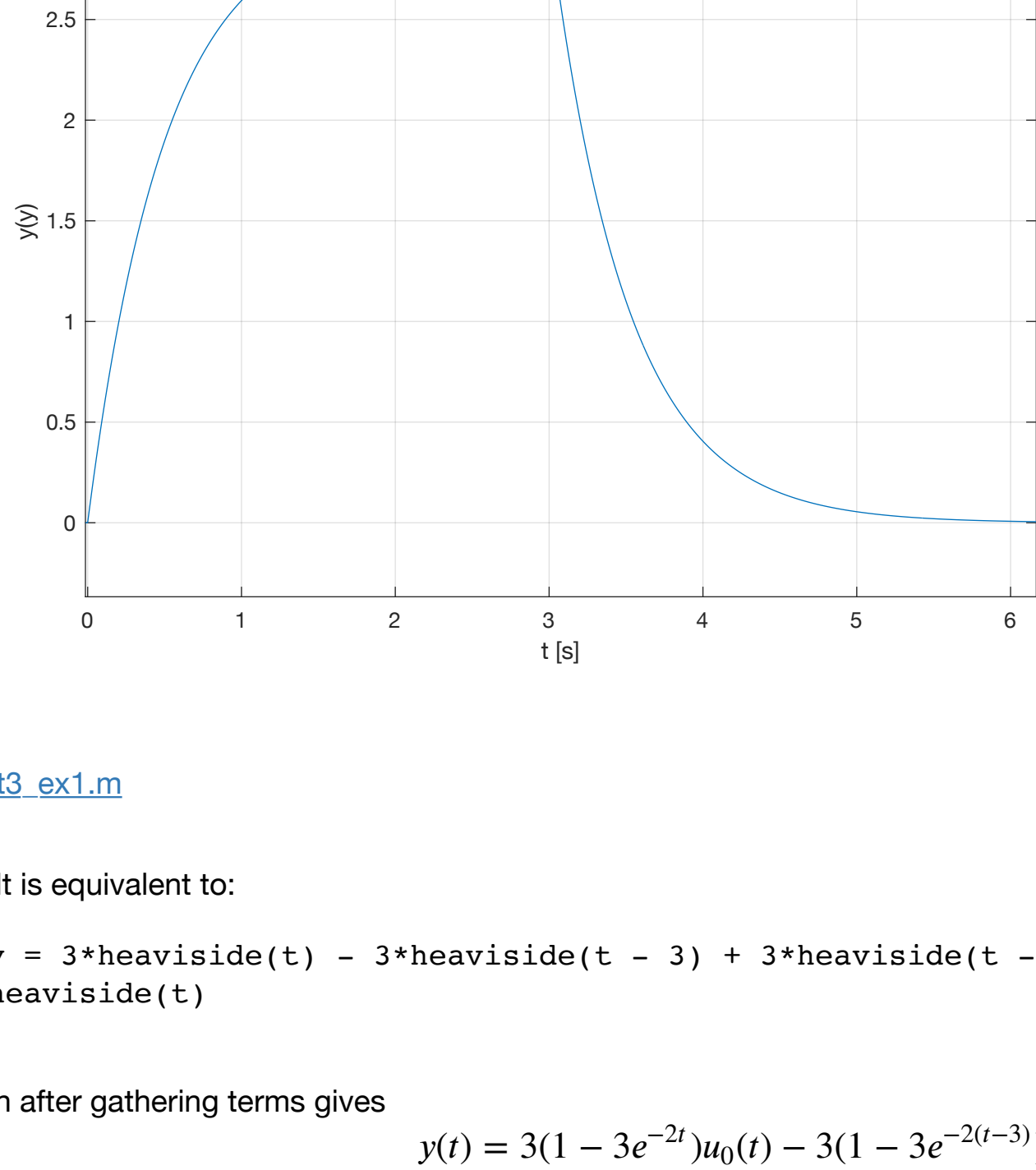
y2 =
(3*exp(6 - 2*t)*(sign(t - 3) + 1)*(exp(2*t - 6) - 1))/2

In [7]: y = y1 - y2

y =
(3*exp(-2*t)*(sign(t) + 1)*(exp(2*t) - 1))/2 - (3*exp(6 - 2*t)*(sign(t - 3) + 1)*(exp(2*t - 6) - 1))/2

Plot result

In [8]: ezplot(y)
title('Solution to Example 1')
ylabel('y(y)')
xlabel('t [s]')
grid
```



See [fig\\_ex1.m](#)

Result is equivalent to:

$$y = 3 * \text{heaviside}(t) - 3 * \text{heaviside}(t - 3) + 3 * \text{heaviside}(t - 3) * \exp(6 - 2 * t) - 3 * \exp(-2 * t) * \text{heaviside}(t)$$

Which after gathering terms gives

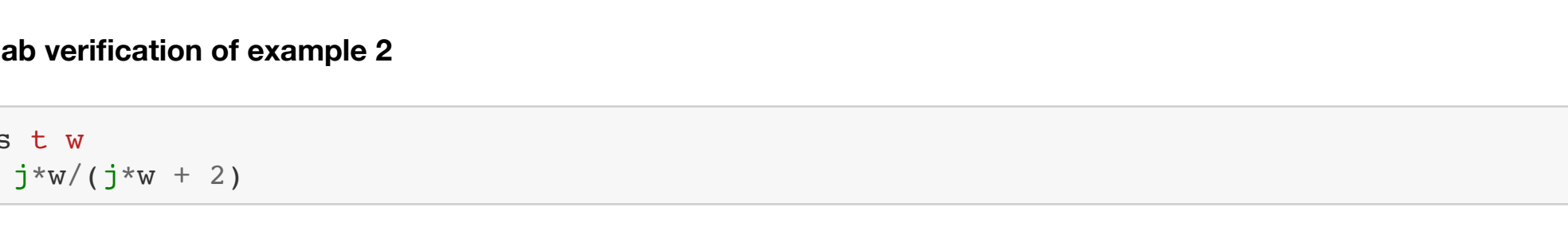
$$y(t) = 3(1 - 3e^{-2t})u_0(t) - 3(1 - 3e^{-2(t-3)})u_0(t - 3)$$

### Example 2

Karris example 8.9: for the circuit shown below, use the Fourier transform method, and the system function  $H(\omega)$  to compute  $V_L(t)$ . Assume  $i_L(0^-) = 0$ . Verify the result with Matlab.



#### Solution of example 2



#### Matlab verification of example 2

```
In [9]: syms t w
H = j*w/(j*w + 2)

H =
(w*1i)/(2 + w*1i)

In [10]: Vin = fourier(5*exp(-3*t)*heaviside(t),t,w)

Vin =
5/(3 + w*1i)

In [11]: Vout=simplify(H*Vin)

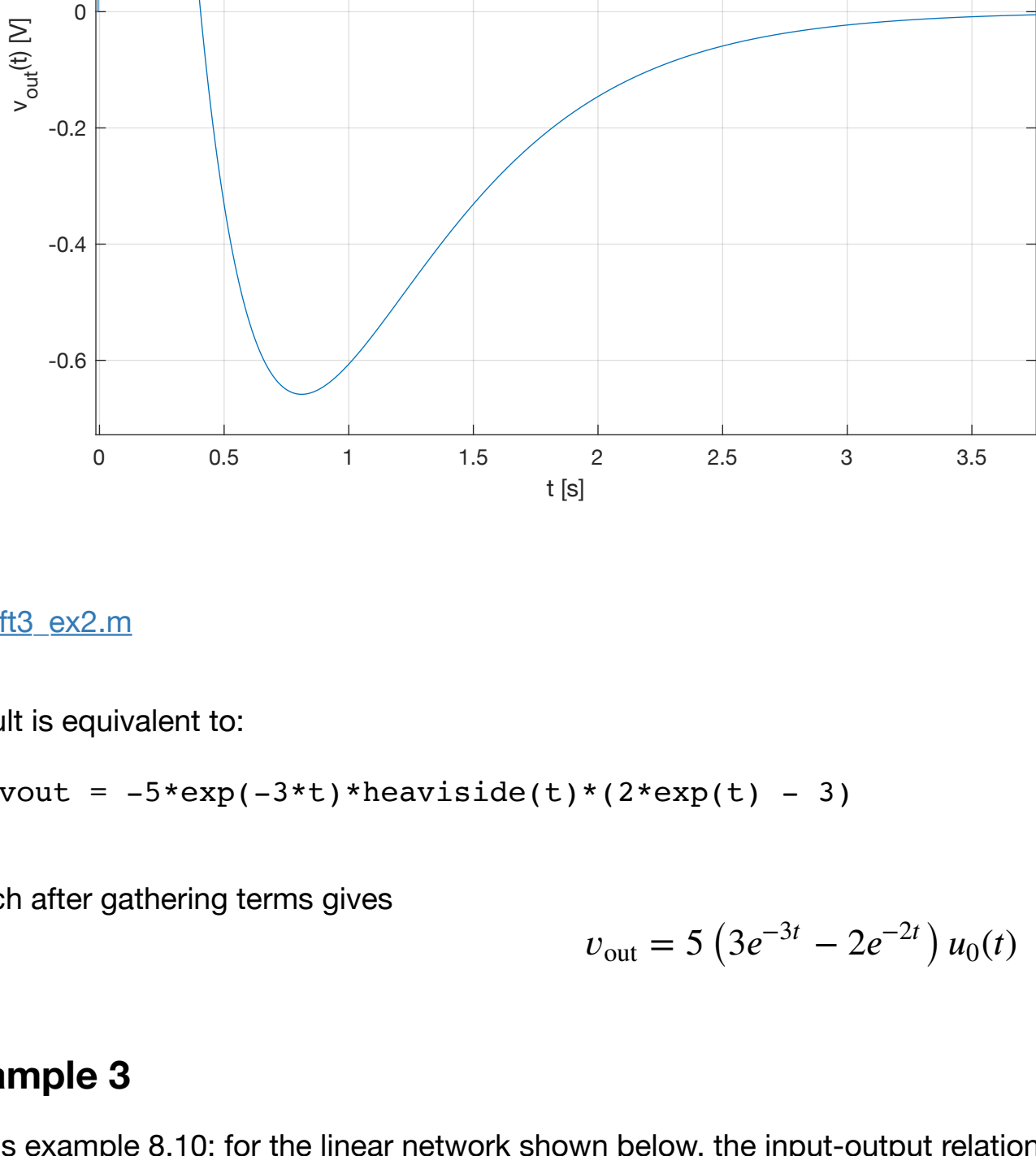
Vout =
(w*5i)/((2 + w*1i)*(3 + w*1i))

In [12]: vout = simplify(ifourier(Vout,w,t))

vout =
-(5*exp(-3*t)*(sign(t) + 1)*(2*exp(t) - 3))/2

Plot result

In [13]: ezplot(vout)
title('Solution to Example 2')
ylabel('v_{out}(t) [V]')
xlabel('t [s]')
grid
```



See [fig\\_ex2.m](#)

Result is equivalent to:

$$v_{out} = -5 * \exp(-3 * t) * \text{heaviside}(t) * (2 * \exp(t) - 3)$$

Which after gathering terms gives

$$v_{out} = 5(3e^{-3t} - 2e^{-2t})u_0(t)$$

### Example 3

Karris example 8.10: for the linear network shown below, the input-output relationship is:

$$\frac{d}{dt} v_{out} + 4v_{out} = 10v_{in}$$

where  $v_{in} = 3e^{-2t}$ . Use the Fourier transform method, and the system function  $H(\omega)$  to compute the output  $v_{out}$ . Verify the result with Matlab.



#### Solution to example 3



#### Matlab verification of example 3

```
In [14]: syms t w
H = 10/(j*w + 4)

H =
10/(4 + w*1i)

In [15]: Vin = fourier(3*exp(-2*t)*heaviside(t),t,w)

Vin =
3/(2 + w*1i)

In [16]: Vout=simplify(H*Vin)

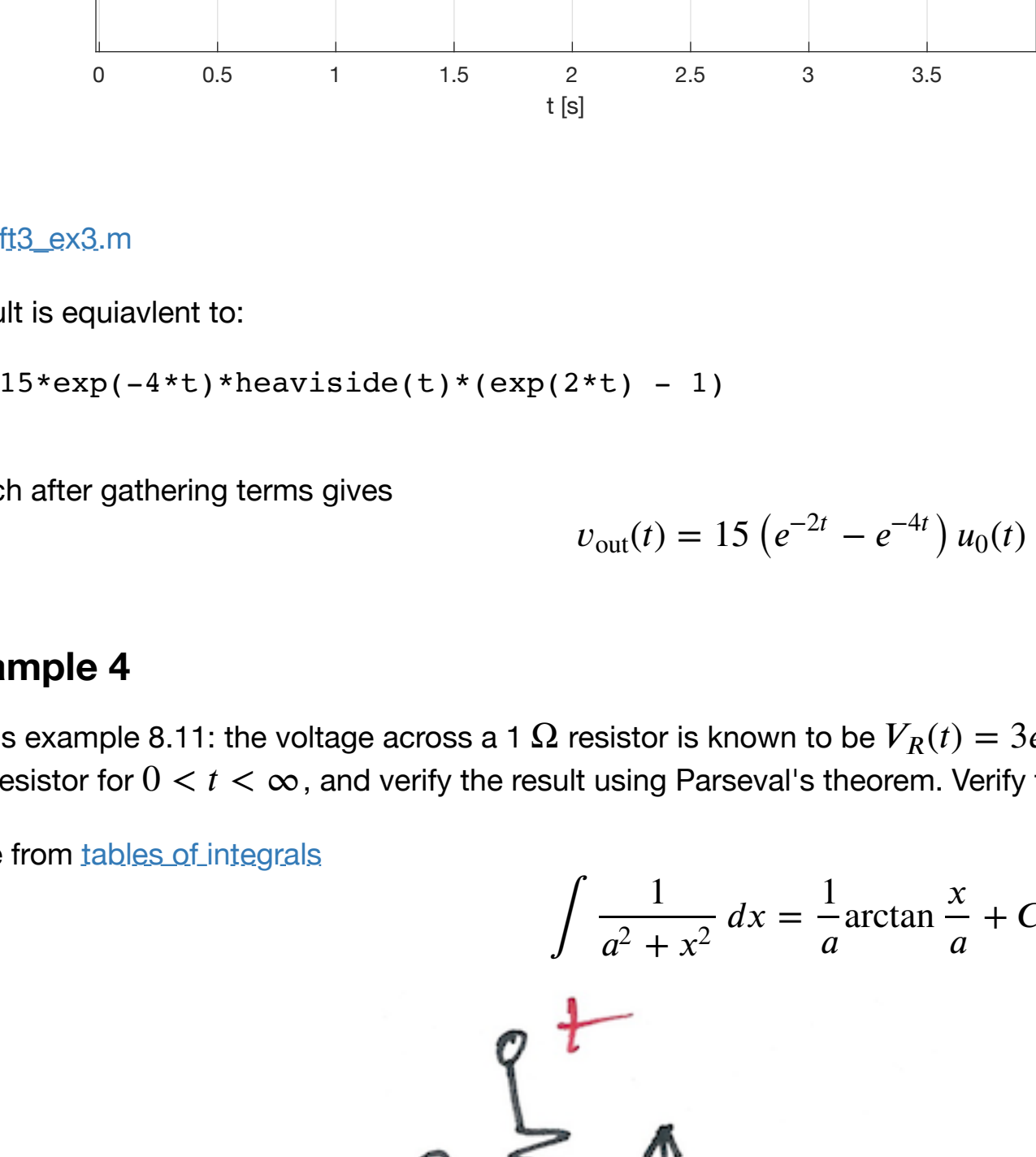
Vout =
30/((2 + w*1i)*(4 + w*1i))

In [17]: vout = simplify(ifourier(Vout,w,t))

vout =
(15*exp(-4*t)*(sign(t) + 1)*(exp(2*t) - 1))/2

Plot result

In [18]: ezplot(vout)
title('Solution to Example 3')
ylabel('v_{out}(t) [V]')
xlabel('t [s]')
grid
```



See [fig\\_ex3.m](#)

Result is equivalent to:

$$15 * \exp(-4 * t) * \text{heaviside}(t) * (\exp(2 * t) - 1)$$

Which after gathering terms gives

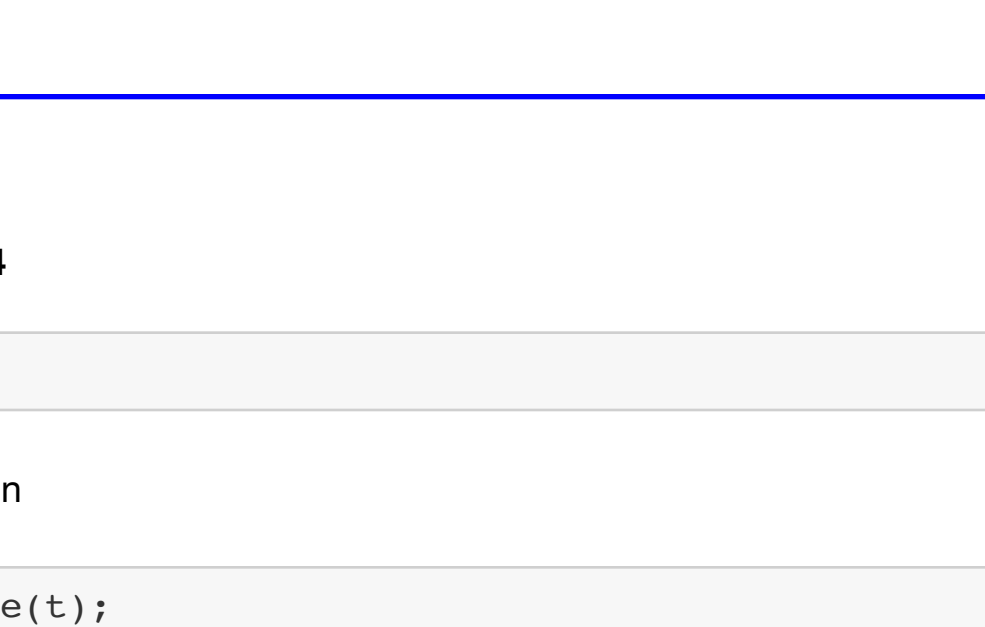
$$v_{out}(t) = 15(e^{-2t} - e^{-4t})u_0(t)$$

### Example 4

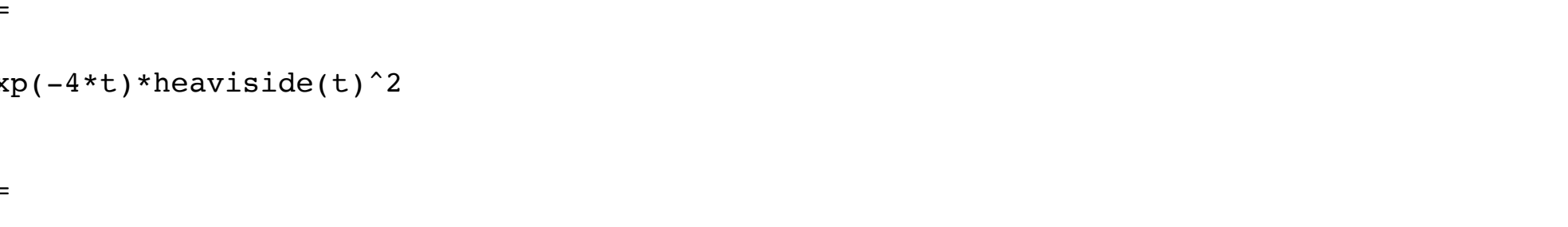
Karris example 8.11: the voltage across a  $1 \Omega$  resistor is known to be  $V_R(t) = 3e^{-2t}u_0(t)$ . Compute the energy dissipated in the resistor for  $0 < t < \infty$ , and verify the result using Parseval's theorem. Verify the result with Matlab.

Note from [tables\\_of\\_integrals](#)

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C.$$



#### Solution to example 4



#### Matlab verification of example 4

```
In [19]: syms t w

Calculate energy from time function

In [20]: Vr = 3*exp(-2*t)*heaviside(t);
R = 1;
Pr = Vr^2/R
Wr = int(Pr,t,0,inf)

Pr =
9*exp(-4*t)*heaviside(t)^2

Wr =
9/4

Calculate using Parseval's theorem

In [21]: Fw = fourier(Vr,t,w)

Fw =
3/(2 + w*1i)

In [22]: Fw2 = simplify(abs(Fw)^2)

Fw2 =
9/abs(2 + w*1i)^2

In [23]: Wc=2/(2*pi)*int(Fw2,w,0,inf)

Wc =
(51607450253003931*pi)/72057594037927936

See fig\_ex4.m
```

## Solutions

- Example 1: [fig2-ex1.pdf](#)
- Example 2: [fig2-ex2.pdf](#)
- Example 3: [fig2-ex3.pdf](#)
- Example 3: [fig2-ex4.pdf](#)