# Answers to Frequently Asked Questions

A PDF version of this FAQ is FAQ.pdf.

## Acknowledgements

I thank the following students who helped me to recognise the holes in my course notes and were the inspiration for these answers.

- Charlotte Newman (Medical Engineering, 2020)

## Harmonic frequencies

*Fundamental frequency* -- A periodic signal $f(t) = f(t + nT),\ n \in \mathbb{Z}$ has period $T$ s and a fundamental frequency $f_0 = 1/T$ Hz. When used in Fourier series and Fourier transforms, frequencies are expressed as $\omega$ in radians/second. The **fundamental frequency** is $\omega = \Omega_0 = 2\pi f_0$ or, equivalently, $\Omega_0 = 2\pi/T$ rad/s.

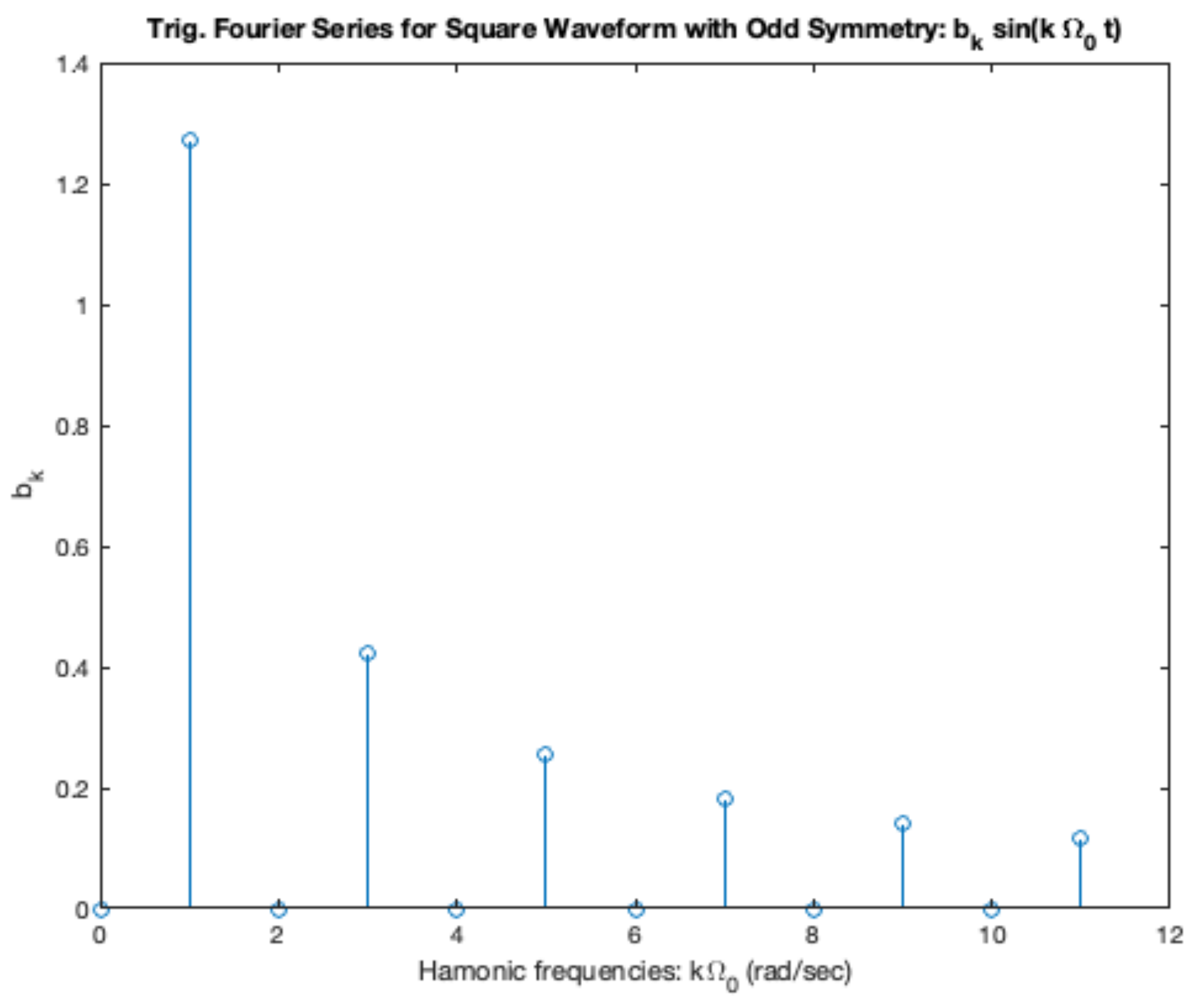
*Harmonic frquencies* (or *Harmonics*) are simply integer multiples of the fundamental frequency $\Omega_0$. So the zero-th harmonic is $\Omega_0 = 0$ rad/s or DC. The first harmonic is $1.\Omega_0 = \Omega_0$, the second is $2\Omega_0$, the third $3\Omega_0$ etc.

In general, we can express the $k$-th harmonic as $k\Omega_0,\ k \in \mathbb{Z}$.

## Line spectra

In trig. Fourier series, the coefficients $a_k$ and $b_k$ are the amplitudes of the $\cos(k\Omega_0 t)$ and $\sin(k\Omega_0 t)$ terms respectively. We usually show these terms as lines (or *spectra*) with height $a_k$ and/or $b_k$ plotted against the harmonic frequency index $k$.

An example of such a plot is reproduced from Line Spectra for Trig. FS.

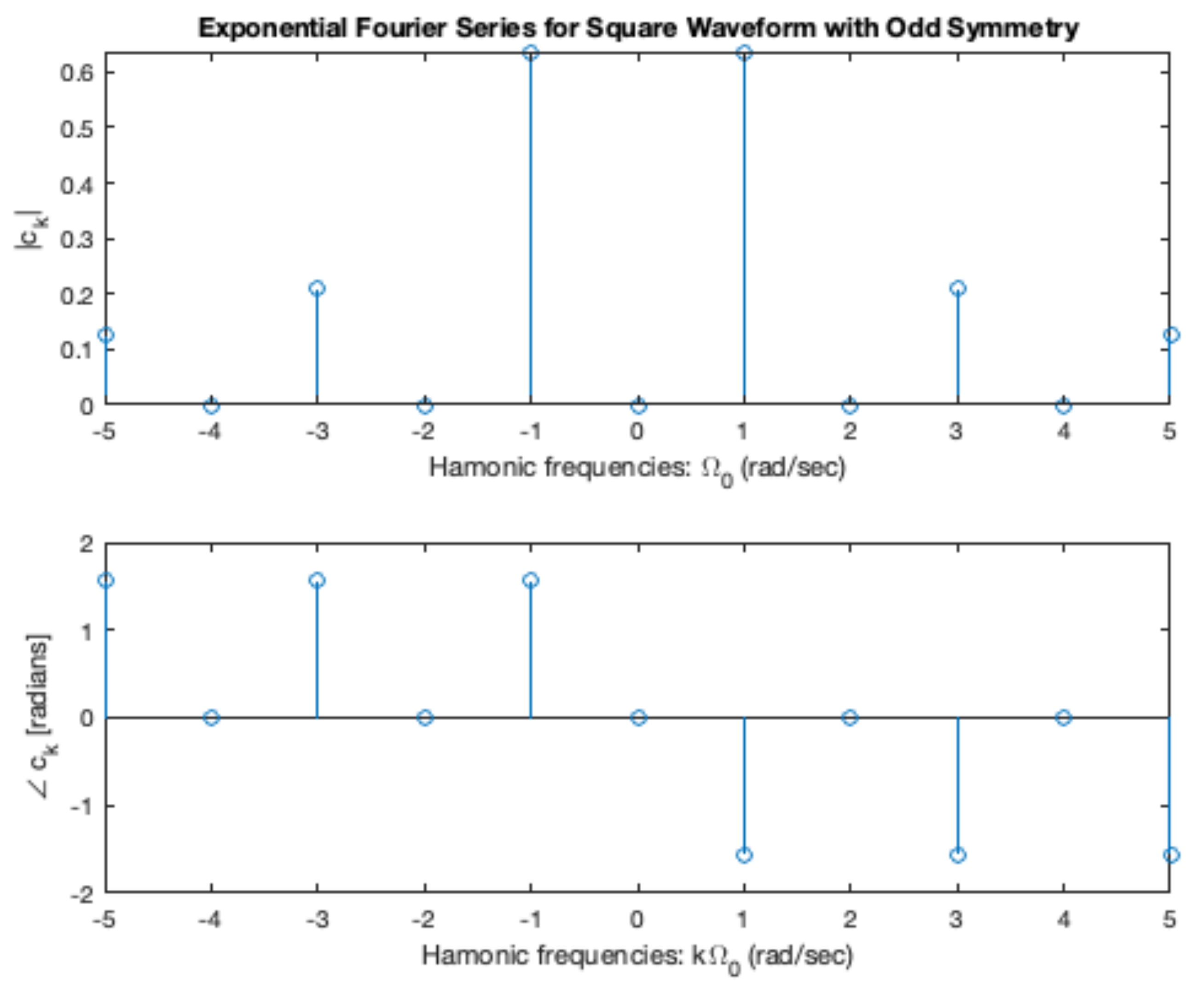



For exponential Fourier Series, the harmonic terms are defined as

$$C_k \exp(jk\Omega_0 t)\ k = -\infty, \ldots, -2, -1, 0, 1, 2, \ldots, \infty.$$

And $|C_k|$ and $\angle C_k$ are plotted as the line spectra against $k$. Note that $C_{-k}$ and $C_k$ are complex conjugates $\forall k > 0$. Hence, the spectrum will be symmetric around $k = 0$.

An example of such a plot is reproduced from Line Spectra for Exp. FS.




## Filter attenuation

In the notes Steady-State Response of an LTI System to a Periodic Signal we state (without proof) that the output of an LTI system to a periodic function with period $T$ represented by a Fourier series is given by:

$$y(t) = \sum_{k=-\infty}^{\infty} C_k H(jk\Omega_0) e^{jk\Omega_0 t}.$$

As a consequence

Thus $y(t)$ is a Fourier series itself with coefficients $D_k$:

$$D_k = C_k H(jk\Omega_0).$$

What is missing from this analysis is a discussion of what $H(jk\Omega_0)$ looks like.

As an example, consider the simple first-order Butterworth low-pass (LP) filter with cut-off frequency $\omega_c$:

$$H(s) = \frac{\omega_c}{s + \omega_c}$$

For this filter

$$H(j\omega) = \frac{\omega_c}{j\omega + \omega_c}.$$

Let us say that we wish to compute the attenuation and phase of this filter at $\omega = \Omega_0$.

To compute the *magnitude*:

$$\begin{aligned} |H(j\Omega_0)| &= \left|\frac{\omega_c}{j\Omega_0 + \omega_c}\right| \\ &= \frac{\omega_c}{\sqrt{\Omega_0^2 + \omega_c^2}} \end{aligned}$$

We note that is $|H(j\Omega_0| < 1$ so the filter will *attenuate* the incoming harmonic frequency. This will be true for all harmonics, so in general, for a LP filter:

$$D_k = C_k\, |H(jk\Omega_0)| < C_k.$$

The phase will be given by

$$\phi = \angle H(j\omega) = \tan^{-1}\left(\frac{\Im\,(H(j\omega))}{\Re\,(H(j(\omega))}\right)$$

where

$$\begin{aligned} H(jk\Omega_0) &= \frac{\omega_c^2}{(k\Omega_0)^2 + \omega_c^2} - j\frac{k\Omega_0\omega_c}{(k\Omega_0)^2 + \omega_c^2} \\ \phi_k &= \tan^{-1}\left(-\frac{K\Omega_0\omega_c}{\omega_c^2}\right) \\ &= \tan^{-1}\left(-\frac{k\Omega_0}{\omega_c}\right) \end{aligned}$$

Phases are additive so

$$\angle D_k = \angle C_k + \phi_k.$$

By doing such analysis, we can examine the effect of a filter on a periodic signal, just by considering how the coefficients of the harmonic terms are changed (attenuated in magnitude and shifted in phase) by the filter.