Lecture 3
AH

classmate
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Page

2D transformations

Homogeneous coordinates

$(x, y) \Rightarrow (x_h, y_h, h)$ — should be non-zero

$P' = M_1 P + M_2$ — Harder to solve
$\Downarrow$
$P' = M_1 M_2 P$ — Multiplicative form is easier

$x = \dfrac{x_h}{h}$

$y = \dfrac{y_h}{h}$

usually done: column major $\rightarrow$ right to left
row major $\rightarrow$ left to right

writing is right to left and calculations are left to right!

For translation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$$

$$P' = P + T \Rightarrow P' = T \cdot P$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}}_{T} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

For rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

For scaling

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

~~For composite transformation~~

Successive translations

Translations:

t_{x_1} and t_{x_2}

t_{y_1} and t_{y_2}

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_{x_2} \\ 0 & 1 & t_{y_2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & t_{x_1} \\ 0 & 1 & t_{y_1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad - (I)$$

Both are same

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_{x_1} + t_{x_2} \\ 0 & 1 & t_{y_1} + t_{y_2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \quad - (II)$$

(I) =

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_{x_1} \\ 0 & 1 & t_{y_2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x + t_{x_1} \\ y + t_{x_1} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_{x_1} + t_{x_2} \\ y + t_{y_1} + t_{y_2} \\ 1 \end{bmatrix}$$

So successive translations are additive

(2) =

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_{x_1} + t_{x_2} \\ y + t_{y_1} + t_{y_2} \\ 1 \end{bmatrix}$$

Successive rotations

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x\cos\theta_1 - y\sin\theta_1 \\ x\sin\theta_1 + y\cos\theta_1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_2(x\cos\theta_1 - y\sin\theta_1) - \sin\theta_2(x\sin\theta_1 + y\cos\theta_1) \\ \sin\theta_2(x\cos\theta_1 - y\sin\theta_1) - \cos\theta_2(x\sin\theta_1 + y\cos\theta_1) \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\Rightarrow \boxed{P' = R(\theta_2) \cdot R(\theta_1) \cdot P}$$

$$\boxed{P' = R(\theta_1 + \theta_2) \cdot P}$$

Successive scaling

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_{x_2} & 0 & 0 \\ 0 & S_{y_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_{x_1} & 0 & 0 \\ 0 & S_{y_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} S_{x_2} & 0 & 0 \\ 0 & S_{y_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x S_{x_1} + 0 + 0 \\ 0 + y S_{y_2} + 0 \\ 1 \end{bmatrix}$$

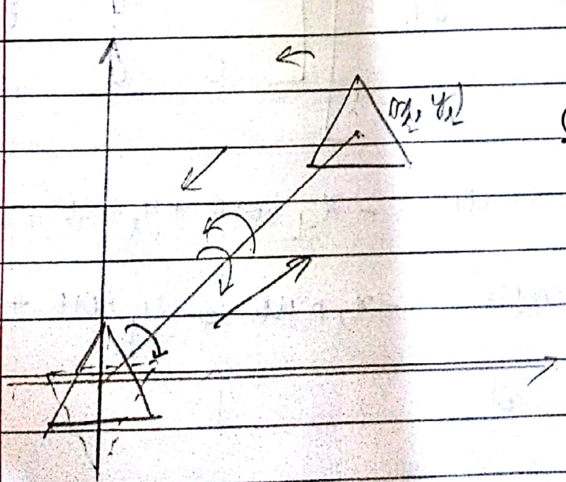
$$= \begin{bmatrix} x S_{x_1} S_{x_2} \\ y S_{x_1} S_{x_2} \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_{x_1} S_{x_2} & 0 & 0 \\ 0 & S_{y_1} S_{y_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{P' = S_1 \cdot S_2 \cdot P}$$

Composite Scaling is multiplicative.
(successive)

Composite transformation



Step 1: Translate to the origin
Step 2: Rotate
Step 3: Reverse translate to original position

Step 1:

$$T = \begin{bmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{bmatrix}$$

Step2:

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step3:

$$T^{-1} = \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ finally, $P' = T^{-1}.R(\theta).T.P$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & +x_1 \\ 0 & 1 & +y_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$T^{-1}.R(\theta).T$

$$= \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & x_1 \\ \sin\theta & \cos\theta & y_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & -x_1\cos\theta + y_1\sin\theta + x_1 \\ \sin\theta & \cos\theta & -x_1\sin\theta - y_1\cos\theta + y_1 \\ 0 & 0 & 1 \end{bmatrix}$$

Then,

$T^{-1} \cdot R(\theta) \cdot T \cdot P$

$$= \begin{bmatrix} \cos\theta & \sin\theta & y_2 \sin\theta + x_2(1-\cos\theta) \\ \sin\theta & \cos\theta & y_2(1-\cos\theta) + x_2 \sin\theta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} x\cos\theta \mp y\sin\theta + y_2 \sin\theta + x_2(1-\cos\theta) \\ x\sin\theta + y\cos\theta + y_2(1-\cos\theta) + x_2 \sin\theta \\ 1 \end{bmatrix}$$

$x' = (\cos\theta)(x - x_2) \mp (\sin\theta)(y - y_2) + x_2$

$y' = (y - y_r)\cos\theta + (x - x_2)\sin\theta + y_2$