→ We look at games with 2 players but no. of strategies can vary
→ How to solve a problem with no saddle point?

- For any 2x2 two person zero sum game without a saddle point having the pay-off matrix for player A as -

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & a_{11} & a_{12} \\ A_2 & a_{21} & a_{22} \end{array}$$ → from A's POV

- The optimum mixed strategies given by

$$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix} \rightarrow \text{probability}, \quad p_1 + p_2 = 1$$

$$S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}, \quad q_1 + q_2 = 1$$

S_A and S_B are determined by the equations

$$\frac{p_1}{p_2} = \frac{a_{22} - a_{21}}{a_{11} - a_{12}}, \quad \frac{q_1}{q_2} = \frac{a_{22} - a_{12}}{a_{11} - a_{21}}$$

The value of the game for player A is given by:

$$v = \frac{a_{22}a_{11} - a_{12}a_{21}}{(a_{22} + a_{11}) - (a_{21} - a_{12})}$$

Q. For the game with the following payoff matrix, determine the optimum strategies and the value of the game.

$$\begin{array}{c|cc|c} & B_1 & B_2 & \\ \hline A_1 & 5 & 1 & -1 \\ A_2 & 3 & 4 & -3 \\ \hline & 5 & 4 & \end{array}$$

$\underline{v} = 3$

$\therefore \bar{v} \neq \underline{v}$

$\bar{v} = 4$

$$E(p,q) = p^T A q = (p_1 \; p_2)\begin{pmatrix} 5 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

$$= (p_1 \; 1-p_1)\begin{pmatrix} 5 & 1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} q_1 \\ 1-q_1 \end{pmatrix}$$

$$= \begin{pmatrix} 5p_1 + 3(1-p_1) & p_1 + 4(1-p_1) \end{pmatrix} \begin{pmatrix} q_1 \\ 1-q_1 \end{pmatrix}$$

$$= q_1\left(5p_1 + 3(1-p_1)\right) + (1-q_1)(p_1 + 4 - 4p_1)$$

$$= q_1(2p_1 + 3) + (1-q_1)(4 - 3p_1)$$

$$= 2q_1p_1 + 3q_1 + 4 - 3p_1 - 4q_1 + 3p_1q_1$$

$$E(p,q) = 5p_1q_1 - q_1 + 4 - 3p_1$$

$$5\left(p_1q_1 - \frac{1}{5}q_1 - \frac{3}{5}p_1\right) + 4$$

$$= 5\left(p_1 - \frac{1}{5}\right)\left(q_1 - \frac{3}{5}\right) - \frac{3}{5} + 4$$

$$E(p,q) = 5\left(p_1 - \frac{1}{5}\right)\left(q_1 - \frac{3}{5}\right) + \frac{17}{5}$$ → payoff equation when A plays his strategies

'A' wants to maximize the min. probability needed for his min loss.

'B' wants to minimize the max probability of getting the max loss.

for A's perspective $\therefore p_1 = \frac{1}{5}$ as then $E(p,q) =$ constant part $= 17/5$

$\therefore p_2 = 4/5$

$q_1 = 3/5$

$q_2 = 2/5$

$$\frac{p_1}{p_2} = \frac{a_{22} - a_{21}}{a_{11} - a_{12}} = \frac{4-3}{5-1} = \frac{1}{4}$$

$$\frac{p_1}{1-p_1} = \frac{1}{4} \Rightarrow p_1 = 1/5 \quad p_2 = 4/5$$

Therefore the optimum mixed strategy

$$S_A = \begin{pmatrix} A_1 & A_2 \\ 1/5 & 4/5 \end{pmatrix}$$

$$S_B = \begin{pmatrix} B_1 & B_2 \\ 3/5 & 2/5 \end{pmatrix}$$

The value of game $\vartheta = 17/5$

Graphical solⁿ of games with mixed strategy

Q. Solve the 2x2 game graphically.

$$\begin{array}{ccccc} & B_1 & B_2 & B_3 & B_4 \\ A_1 & 2 & 1 & 0 & -2 \\ A_2 & 1 & 0 & 3 & 2 \end{array}$$

Pl. A

→ First we identify best 2 strategies of B. And then solve 2x2 problem.

$\underline{v} = 0$

$\bar{v} = 1$

Let the optimum mixed strategy for player A is

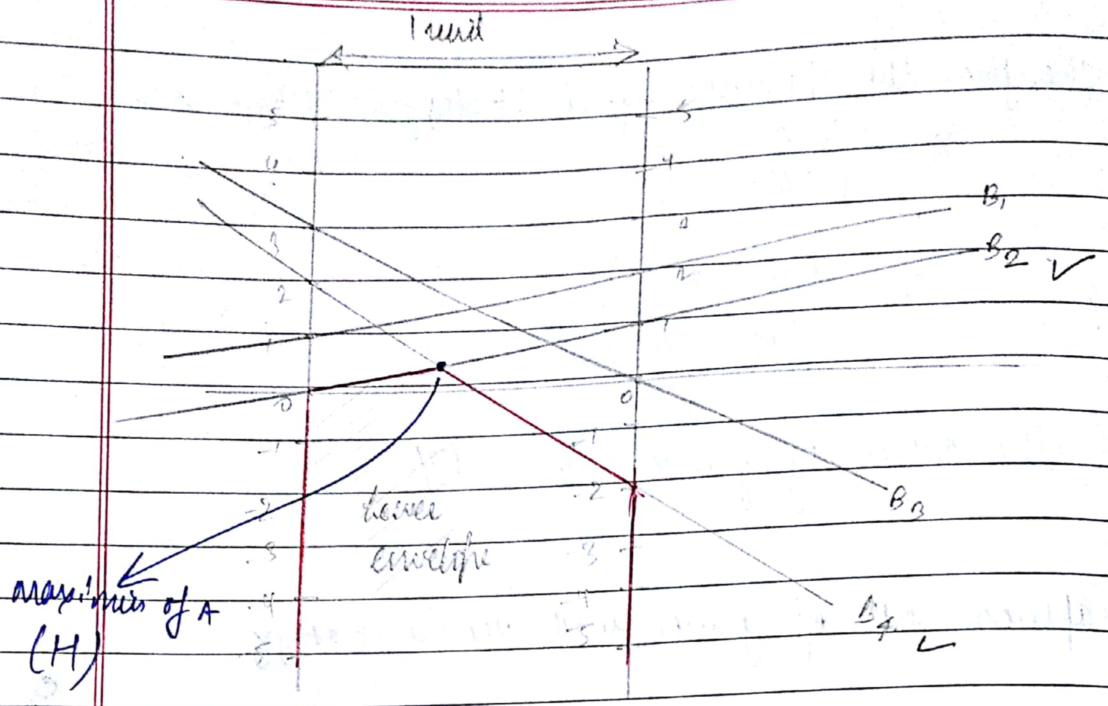
$$S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & 1-p_1 \end{pmatrix}$$

Now we need to find S_B

B's pure moves

A's expected pay-off's against B's pure moves are given as follows:

B's pure moves	net expected gain of A
B_1	$p^TAq = (p_1 \quad 1-p_1)\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 1+p_1 = E_1(p_1)$
B_2	$p^TAq = (p_1 \quad 1-p_1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = p_1 = E_2(p_1)$
B_3	$p^TAq = (p_1 \quad 1-p_1)\begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3-3p_1 = E_3(p_1)$
B_4	$p^TAq = (p_1 \quad 1-p_1)\begin{pmatrix} -2 \\ 2 \end{pmatrix} = 2-4p_1 = E_4(p_1)$



Lines that bind the lower envelope $= \underbrace{B_2, B_4}$

~~Best strategies of B~~

$$\therefore S_B = \begin{pmatrix} B_2 & B_4 \\ q_1 & 1-q_1 \end{pmatrix}$$

$\rightarrow$ a 2×2 problem

Now, $A = \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}$

$$\frac{p_1}{p_2} = \cancel{\frac{2-0}{-2-1}} = \cancel{\frac{-2}{3}} \quad \frac{2-0}{1-(-2)} = \frac{2}{3}$$

$$\frac{p_1}{1-p_1} = +\frac{2}{3} \Rightarrow 2 - 2p_1 = 3p_1$$

$$\Rightarrow p_1 = \frac{2}{5}$$

$$p_2 = 3/5$$

$$\frac{q_1}{q_2} = \frac{2-(-2)}{1-0} = 4$$

$$q_1 = 4 - 4q_1$$

$$\Rightarrow q_1 = 4/5 \qquad q_2 = 1/5$$

classmate

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$$S_A = \begin{pmatrix} A_1 & A_2 \\ 2/5 & 3/5 \end{pmatrix} \qquad S_B = \begin{pmatrix} B_2 & B_4 \\ 4/5 & 1/5 \end{pmatrix}$$