Machine Learning
Research task (task **1**)

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Description In this part of the project you are supposed to develop the algorithm that can use trained denoising models for the other image restoration problem: image deblurring. Non-blind image deblurring is the task of restoration clean image x given blurred observation y, degradation operator H that represents convolution with known kernel k and standard deviation of the noise level σ. So, we consider the following linear degradation process:

$$y = Hx + \sigma \cdot n\,,\; n \sim \mathcal{N}(0, I),$$

where $y \in \mathbb{R}^n$ and $x \in \mathbb{R}^m$ are vector representations of blurred and clean images respectively. $H \in \mathbb{R}^{nm}$ matrix represents valid convolution of clean image x with known kernel k. Read the paper https://www.ceremade.dauphine lier/FISTA about ISTA and FISTA optimization algorithms. Suppose we can solve denoising problem with the given level of noise σ:

$$T_{\sigma^2} \stackrel{\text{def}}{=} arg_u \min \left\{ g(u) + \frac{1}{2\sigma^2} \|u - u\|_2^2 \right\}.$$

Derive formulas based on ISTA and FISTA algorithms that solve the deblurring problem

$$T_{H,\sigma^2} \stackrel{\text{def}}{=} arg_u \min \left\{ g(u) + \frac{1}{2\sigma^2} \|Hu - y\|_2^2 \right\}$$

Solution:

Firstly, let's recognise that linear degradation process resembles us the basic linear inverse problem for the deblurring task:

$$Ax = b + w$$

, where $A \in \mathbb{R}^{mn}$ and $b \in \mathbb{R}^m$ are known and rpresents blur operator and the blurred picture respectively, w is an unknown noise (or perturbation) vector, and x is the "true" and unknown signal/image to be estimated.

and the T_{H,σ^2} formula resembles us the regularised least squares approached to solve this problem:

$$arg_x \min\{F(x) \equiv \|Ax - b\|^2 + \lambda \|x\|_1\},$$

but in our case $A = H \cdot \frac{1}{\sqrt{2}\sigma}$, $b = \frac{\sigma \cdot n}{\sqrt{2}\sigma}$ and the u is such a parameter, that $g(u) = \lambda \|x\|_1$

And the problem described can be derived by the Iterative shrinkage thresholding algorithm (ISTA), which is set by the formula:

$x_{k+1} = \mathcal{T}_{\lambda t}(x_k - 2tA^T(Ax_k - b))$, where t is an appropriate stepsize, which turns out to be $\frac{2}{\sigma_{\max} + \sigma_{\min}}$ and it can be approximated through the Gershgorin theorem to save some computational power, because A matrix is too big to find all the eigenvalues. $\mathcal{T}_\alpha : \mathbb{R} \to \mathbb{R}^n$ is the shrinkage operator defined by: $\mathcal{T}_\alpha(x)_i = (|x_i| - \alpha)_+ \operatorname{sgn}(x_i)$.

and initial step just at some $x_0 \in \mathbb{R}^n$

This method is proven to be converging.

Also the problem could be solve via Fast iterative shrinkage thresholding algorithm (FISTA), which in this way will be set by formula of k-th step:

$x_k = p_L(y_k)\,,\ t_{k+1} = \frac{1+\sqrt{1+4t_k^2}}{2}\,,\ y_{k+1} = x_k + \left(\frac{t_k-1}{t_{k+1}}\right)(x_k - x_{k-1})$

and starting at $y_1 = x_0 \in \mathbb{R}^n\,,\ t_1 = 1$

where $p_L(\cdot)$ is the iterative shrinkage operator

And this method is also proven to be converging even faster.