Absolute Continuity of Bernoulli Convolution

[Varjú 2019]

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§ Setup

Let $\lambda, p \in (0,1)$ and $\xi_0, \xi_1, \cdots \overset{\text{i.i.d.}}{\sim} p\delta_{+1} + (1-p)\delta_{-1}$.

Define the Bernoulli convolutions (BC)

$$\mu_{\lambda,p} :\sim \sum_{n=0}^{\infty} \xi_n \lambda^n$$

Q: Whether $\mu_{\lambda,p} \ll \mathcal{L}$ (Lebesgue measure) ?

(pure-type)

Thm (Main) $\forall \varepsilon > 0$, $p \in (0,1)$. $\exists\ c = c(\varepsilon, p) > 0$.

s.t. let $\lambda \in \overline{\mathbb{Q}} \cap (0,1)$

Suppose $1-\lambda < c \min\left\{ \log M_\lambda, (\log M_\lambda)^{-1-\varepsilon} \right\}$

Then the BC $\mu_{\lambda,p} \ll \mathcal{L}$ with density in $L\log L$.

Notation: "log" with base 2 ; "ln" with base e.

- Let $P = a_d x^d + \cdots + a_0 \in \mathbb{C}[x]$ with roots $z_1, \cdots, z_d$. The Mahler measure is

$$M(P) := a_d \prod_{k=1}^{d} \max\{1, |z_k|\}.$$

In particular, for $\lambda \in \bar{\mathbb{Q}}$ with minimal poly. $P_\lambda \in \mathbb{Z}[x]$,

define $\quad M_\lambda := M(P_\lambda)$

- Recall $f: \mathbb{R} \to \mathbb{R}$ in $L\log L$ if $\int |f| \log(|f|+2)\, dx < \infty$

In particular, if $\text{supp}(f)$ compact, then

$$f \text{ in } L\log L \Leftrightarrow \int |f| \log|f|\, dx < \infty$$

Pf:

$$\int_{|f|>0} |f| \log(|f|+2)\, dx = \int_{|f|>0} |f| \log|f|\, dx + \underbrace{\int_{|f|>0} |f| \log\left(1+\frac{2}{|f|}\right) dx}$$

$$0 \leqslant \quad \leqslant \int_{|f|>0} |f| \frac{2}{|f|} dx \leq 2\lambda(\text{supp} f) < \infty$$

$\square$

The const. $c = c(\varepsilon, p)$ is effective and continuous w.r.t. ε and p. However, we omit the explicit examples but focus on the proof of Main Thm. Maybe we will return to the explicit examples after we have a better understanding of the dependencies of the param's in Main Thm.

§ Progress

Take $p = \frac{1}{2}$.

- Basics:

$$\begin{cases} \lambda \in (0, \frac{1}{2}) & \mu_\lambda \perp \mathcal{L} \text{ as Cantor measure} \\ \lambda = \frac{1}{2} & \mu_{\frac{1}{2}} = \mathcal{L}_{[-1,1]} \text{ by binary expansion} \\ \Rightarrow \lambda = (\frac{1}{2})^{\frac{1}{k}} \,, k \in \mathbb{N} \,, & \mu_{(\frac{1}{2})^k} = \mu_{\frac{1}{2}} * \nu \ll \mathcal{L} \\ \lambda \in (\frac{1}{2}, 1) & \text{interesting.} \end{cases}$$

($(\frac{1}{2})^{\frac{1}{k}} \in (\frac{1}{2}, 1)$)

Consider $\lambda \in (\frac{1}{2}, 1)$

- (Erdös 1939) λ^{-1} Pisot $\Rightarrow$ $\mu_\lambda \perp \mathcal{L}$

  Open conj: " $\Leftarrow$ "

- (Erdös 1940) $\exists c < 1$, s.t. for $\mathcal{L}$-a.e. $\lambda \in (c, 1)$, $\mu_\lambda \ll \mathcal{L}$.

- (Garsia 1962) $\lambda^{-1} \in \overline{\mathbb{Z}} \subset \overline{\mathbb{Q}}$ & $M_\lambda = 2$ (explicit) $\Rightarrow$ $\mu_\lambda \ll \mathcal{L}$

• (Solomyak 1995) $\mathcal{L}$-a.e. $\lambda \in (\frac{1}{2}, 1)$. $\mu_\lambda \ll \mathcal{L}$.

Companying to AC problem is the dim. problem. To talk about a "unified" dim, need to establish exact-dim.

• (Feng-Hu 2009)

self-similar measures exact-dim.

$\downarrow$ (Feng 2019).

self-affine measure

• (Rapaport 2020)

Let $\nu \in \mathcal{M}(P(V))$ be the Furstenberg measure w.r.t. $\mu \in \mathcal{M}(GL(V))$.

$\left\{ \begin{array}{l} \mu \text{ finitely supp.} \\ \overline{\text{Semi-grp}\{\text{supp}\,\mu\}} \left\{ \begin{array}{l} \text{Strongly Irreducible} \\ \text{proximal.} \end{array} \right. \end{array} \right.$

non-degeracy conditions

Then ν exact-dim.

- (Ledrappier–Lessa 2021)

Let ν be the Furstenberg measure on the space of flags w.r.t. $\mu \in \mathcal{M}(SL_d(\mathbb{R}))$

with $\begin{cases} \mu \text{ discrete } \& \underbrace{\sum_g \log\|g\|\mu(g)<\infty}_{\text{"Shannon entropy"}} \\ \text{general } \underline{\text{non-degeneracy conditions}}. \end{cases}$

semi-grp (supp μ) $\begin{cases} \text{①. strongly irreducible} \\ \text{②: Zariski dense.} \end{cases}$

Then ν exact-dim.

- (Feng 2021). In progress.

- Partial results for special cases:

(Fraser-Jordan-Jurga) (Tanaka 2017) ....

- (Hochman 2014) $\dim_p \{\lambda \in (\frac{1}{2},1) : \dim \mu_\lambda < 1\} = 0$

- (Shmerkin 2014) $\dim_H \{\lambda \in (\frac{1}{2},1) : \mu_\lambda \perp \mathcal{L}\} = 0$

Solomyak-Shmerkin Arguments $\Rightarrow$ EOC for λ-Cantor set

(Hochman 2014) "algebraic"

(Varjú 2019) "transcendental"

exact-overlap conj hold for BC.

$$\lambda \in \bar{\mathbb{Q}}, \qquad \dim \mu_\lambda = \min\left\{1, \frac{h_\lambda}{\log \lambda^{-1}}\right\}$$

where $$h_\lambda := \lim_{n\to\infty} \frac{H(\mu_\lambda^{(n)})}{n} = \lim_{n\to\infty} \frac{H\left(\sum_{k=0}^{n-1} \xi_k \lambda^k\right)}{n}.$$

Garsia entropy / RW entropy

(B-V 2020, entropy-BC)

$$\boxed{0.44 \approx c_0 \min\{\log M_\lambda, 1\} \le h_\lambda \le \min\{\log M_\lambda, 1\}}$$

Connect Mahler measure to (Garsia) entropy !

(Rapaport 2020) EOC ✓ for $\begin{cases} \text{algebraic ratios} \\ \text{arbitrary translations.} \end{cases}$

From above, we have a criterion

$$\lambda \in \overline{\mathbb{Q}},\ \lambda > \min\{2, M_\lambda\}^{-\frac{1}{c_0}} \Rightarrow \dim \mu_\lambda = 1 .$$

are related.

Compare this to the Main Thm:

$$\lambda > 1 - c \min\left\{\log M_\lambda,\ \frac{1}{(\log M_\lambda)^{1+\varepsilon}}\right\}$$

(Shmerkin 2014) doesn't apply for specific param...
only for generic param's.

§ Proof Strategy

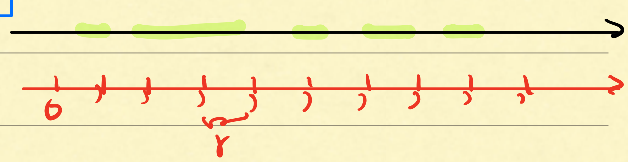
• Basic Notions

$\boxed{\text{"Entropy"}}$ big enough "$\Rightarrow$" smooth enough "$\Rightarrow$" A.C.

[Garsia 1963, "Garsia Link"]

Let X be a bded. r.v. $r>0$ "scale".

Define $\boxed{\text{"average entropy"}}$ 平均熵



$$H(X;r) := \frac{1}{r}\int_0^r H(\lfloor X+s \rfloor_r)\, ds$$

Change of variable

$t = \frac{s}{r}$

$dt = \frac{1}{r} ds$

$$= \frac{1}{r}\int_0^1 H\big(\lfloor X+rt \rfloor_r\big)\, r\, dt$$

zoom-out the partition.

$$rn \le X+rt < r(n+1)$$
$$\Updownarrow$$
$$n \le \frac{X}{r}+t < n+1$$

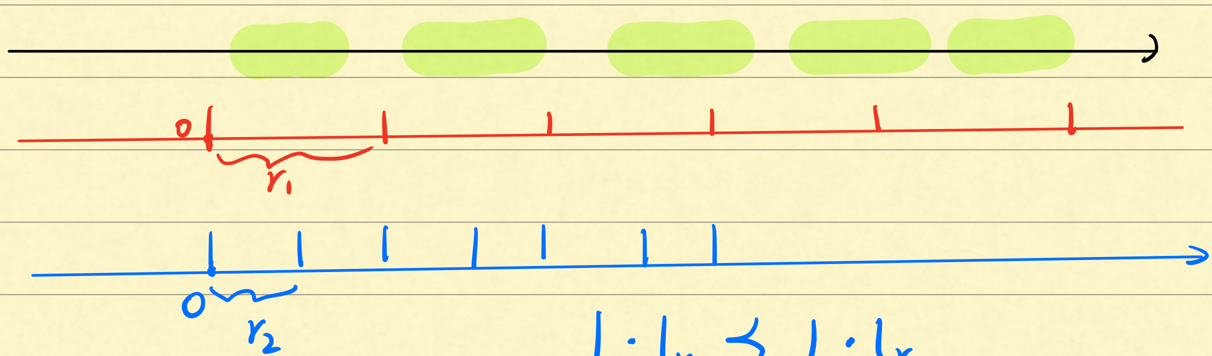
$$= \int_0^1 H\left(\left\lfloor \frac{X}{r}+t \right\rfloor\right) dt$$

To obtain LB of $H(X;r)$, we focus on the estimates on the increments between scales,
(in particular, the doubling scales) See below.

"difference/delta entropy" 差熵

$$H(X;r_1|r_2) := H(X;r_1) - H(X;r_2).$$

When $\frac{r_2}{r_1} \in \mathbb{N}$, we call it (average) conditional entropy.
平均条件熵.



$\lfloor \cdot \rfloor_{r_1} \prec \lfloor \cdot \rfloor_{r_2}$

$\Rightarrow$ $H(X;r_2|r_1) \leq \log N$ if $r_2/r_1 \in \mathbb{N}$.

<u>Notation</u>: For $I \subset [0,1]$, $\mu^I :\sim \sum_{n \in \mathbb{Z}_{\geq 0} : \lambda^n \in I} \xi_n \lambda^n$.

"scaling" is in push-forward sense.

$$\begin{cases} \text{scaling property: } H(\mu^{\lambda^k I}; \lambda^k r_2 | \lambda^k r_1) = H(\mu^I; r_2 | r_1) \\ \text{convolution structure: } \mu^I = \mu^{I_1} * \cdots * \mu^{I_k} \text{ if } I = I_1 \sqcup \cdots \sqcup I_k. \end{cases}$$

Starting point (due to Garsia, also a direct proof by height):

$$|P^{(\ell)}(I, \lambda) - P^{(\ell)}(J, \lambda)| \geq C_\lambda M_\lambda^{-\ell} \text{ when } \ell \text{ large enough.}$$

By "一个萝卜一个坑",

$$\underbrace{\forall \alpha < M_\lambda^{-1}, \quad \underset{\text{"average"}}{H(\mu^{(\lambda^\ell, 1]} | \alpha^\ell)} = \underset{\text{"Shannon"}}{H(\mu^{(\lambda^\ell, 1]})} \geq \frac{h_\lambda}{\ell}}$$

This establishes $H(\nu; r/2r) \geq 1-\alpha$ part in PHEP.

- **Logic diagram**

AC with $L\log L$ density

$\uparrow$ "Garsia Link"

$1 - H(\mu; 2^{-(n+1)} | 2^{-n}) \lesssim \frac{1}{n^2}$

$\uparrow$

(k-Level high entropy)

k-HE $\qquad \mu_1^{(k)}$

(k-1)-HE $\qquad \mu_1^{(k-1)} \quad \mu_2^{(k-1)}$

$\cdots \qquad \cdots \quad \cdots$

1-HE $\qquad \mu_1^{(1)} \quad \cdots\cdots \quad \mu_{2^{k-1}}^{(1)}$

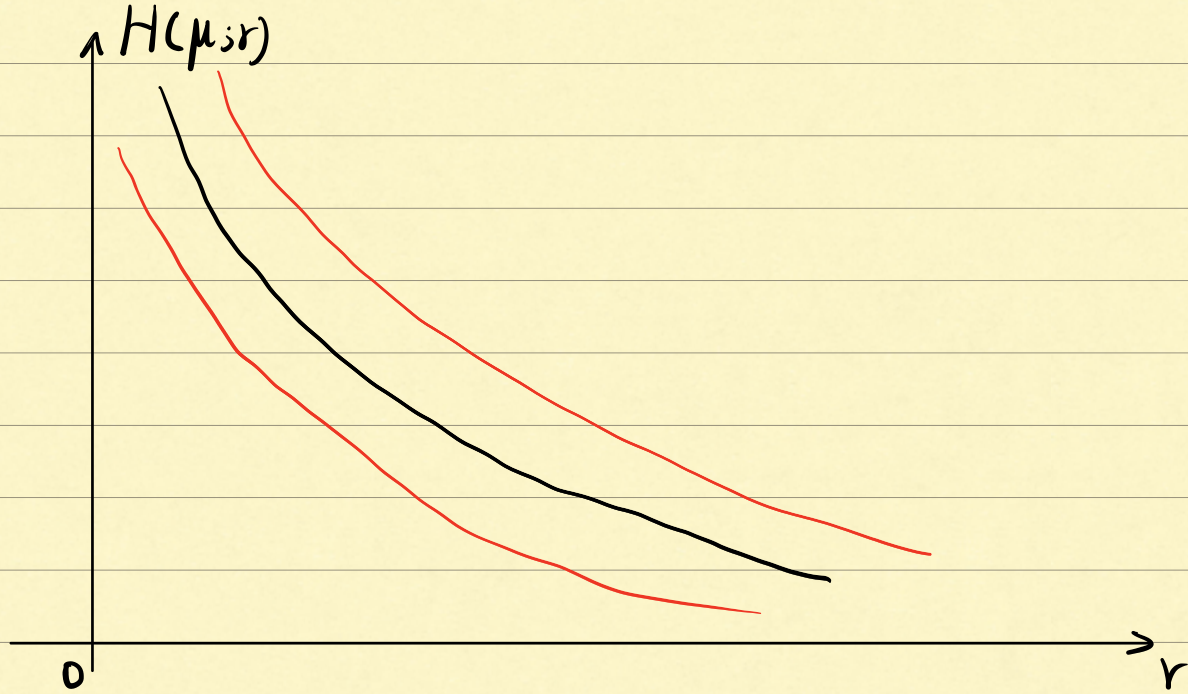
0-HE $\qquad \mu = \mu_1 * \mu_2 * \cdots * \mu_{2^k-1} * \mu_{2^k} * \nu$

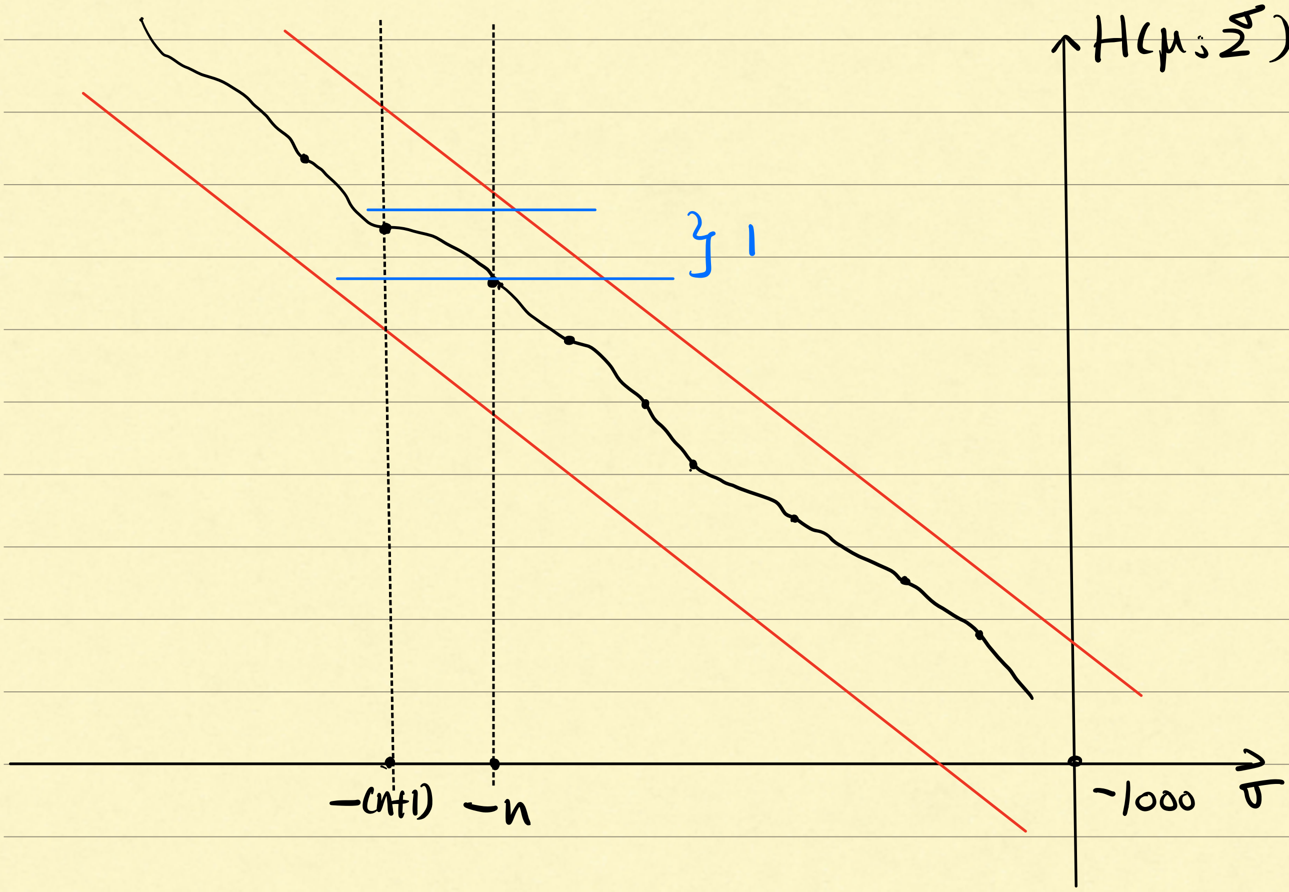
"conduction entropy bootstrap" (CEB) $\Leftarrow$ "discretization, decomposition, restriction, moving window."

"Plenty of high entry pieces" (PHEP) $\Leftarrow$ "additive combinatorics"

"Notions" $\Longrightarrow$ Main Thm

§ Basic notions in mind.



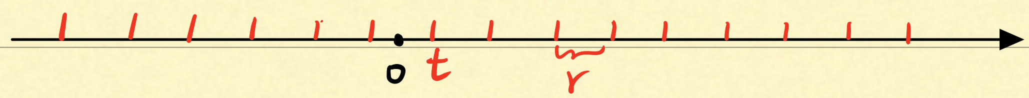


§ Notation & Conventions.

- c small constant ; C large constant.
- r, r_i, s, s_i ~ scales
- σ. n. ~ "log scales"
- α ~ missing entropy
- β ~ log-slope
- $\chi_N := \frac{1}{N} \sum_{i=1}^{N} \delta_i$ discrete uniform measure.
- I_r cts. uniform. r.v. on $[0, r]$. Denote $I_r \sim \chi_r$
- $\mathcal{M}(X)$ ~ (Borel) probability measures on X

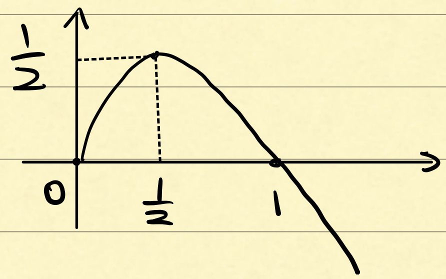
  $\mathcal{M}_c(X) := \{ \mu \in \mathcal{M}(X) : \operatorname{supp} \mu \text{ compact} \}$

- $\lfloor \cdot \rfloor_r^t$:= the lattice Γ with equal r length and $t \in \Gamma$. (translated to t).



- $\log x$ ~ base 2 ; $\ln x$ ~ natural base.

- $\varphi(x) := -x \log x$ ~ the entropy function



$$\begin{cases} \text{concavity} \Rightarrow \text{subadditivity} \\ \varphi(xy) = x\,\varphi(y) + y\,\varphi(x). \end{cases}$$

§ Outline of the series of seminars

Chp 0 : Intro , General Picture , Notions

Chp 1 : Entropy facts.

Chp 2 : CEB

Chp 3 : PHEP

Chp 4 : AC