Typical self-affine sets with non-empty interior

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Fractal Geometry

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Self-affine set

- Let $T_1, \ldots, T_m$ be $d \times d$ invertible real matrices with $\|T_j\| < 1/2$ for $1 \le j \le m$. Write $\mathbf{T} := (T_1, \ldots, T_m)$.

- For $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^{dm}$, we consider the affine iterated function system,

$$\{f_j^{\mathbf{a}}(x) = T_j x + a_j\}_{j=1}^m.$$

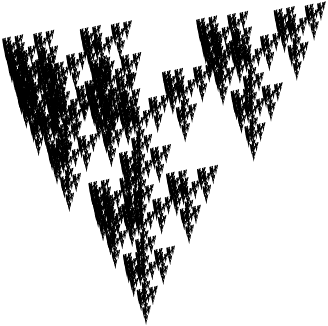
- It is well known that there is a unique non-empty compact set $K^{\mathbf{a}}$, called self-affine set, such that

$$K^{\mathbf{a}} = \bigcup_{j=1}^m f_j^{\mathbf{a}}(K^{\mathbf{a}}).$$

- In this talk, we fix the linear part $\mathbf{T}$ and study $K^{\mathbf{a}}$ with the translations $\mathbf{a}$ changing.

Research target

For example, some self-affine set $K^{\mathbf{a}}$ looks like



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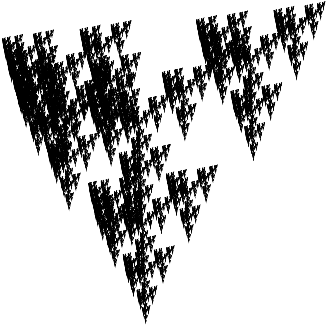

Goal

To provide some sufficient conditions on $\mathbf{T}$ such that $K^{\mathbf{a}}$ has non-empty interior for $\mathcal{L}^{dm}$-a.e. (typical) $\mathbf{a}$.

Motivation

In 1988, Falconer introduced a quantity $\dim_{\mathrm{AFF}} \mathbf{T}$ called affinity dimension which only depends on the linear part $\mathbf{T}$.

Classical results

- (Falconer, 1988; Solomyak, 1998) For $\mathcal{L}^{dm}$-a.e. $\mathbf{a}$,

$$\dim_{\mathrm{H}} K^{\mathbf{a}} = \dim_{\mathrm{B}} K^{\mathbf{a}} = \min\{d, \dim_{\mathrm{AFF}} \mathbf{T}\}.$$

- (Jordan, Pollicott, and Simon, 2007) If $\dim_{\mathrm{AFF}} \mathbf{T} > d$, then $\mathcal{L}^d(K^{\mathbf{a}}) > 0$ for $\mathcal{L}^{dm}$-a.e. $\mathbf{a}$.

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Question

How about the interior of typical $K^{\mathbf{a}}$?

Although this seems a rather fundamental question, it has hardly been studied.

Main results: general case

To state the result, we define

$$\gamma(\mathbf{T}) = \inf \left\{ \gamma \geq 0 \colon \sup_{n \geq 1} \sum_{|I|=n} \alpha_d(T_I)^{\gamma} |\det T_I| \leq 1 \right\}.$$

where $\alpha_d(T)$ denotes the smallest singular value of a matrix T and $T_I = T_{i_1} \cdots T_{i_n}$ for $I = i_1 \ldots i_n \in \{1, \ldots, m\}^n$.

In short, $\gamma(\mathbf{T})$ is a quantity only depending on $\mathbf{T}$.

Theorem A

If $\gamma(\mathbf{T}) > d$, then $K^{\mathbf{a}}$ has non-empty interior for $\mathcal{L}^{dm}$-a.e. $\mathbf{a}$.

The idea for proving Theorem A

- By a classical result (see e.g. (Mattila, 2015)), it suffices to find measures $\mu^{\mathbf{a}}$ supported on $K^{\mathbf{a}}$ such that the Fourier transform $\widehat{\mu^{\mathbf{a}}}$ satisfies

$$\int_{B_\rho} \int_{\mathbb{R}^d} |\widehat{\mu^{\mathbf{a}}}(\xi)|^2 |\xi|^\gamma \, d\xi d\mathbf{a} < \infty \quad \text{for some } \gamma > d.$$

- By the transversality arguments of Falconer and Solomyak, and some key inequalities, the problem is reduced to finding some measure on $\{1, \ldots, m\}^{\mathbb{N}}$ with enough regularity.

- The condition $\gamma(\mathbf{T}) > d$ is discovered and provides such a regular measure.

Main results: commutative case

Theorem B

Suppose $T_i T_j = T_j T_i$ for $1 \le i, j \le m$. If $\sum_{j=1}^{m} |\det T_j|^2 > 1$, then $K^{\mathbf{a}}$ has non-empty interior for $\mathcal{L}^{dm}$-a.e. $\mathbf{a}$.

Theorem B

Suppose $T_iT_j = T_jT_i$ for $1 \le i,j \le m$. If $\sum_{j=1}^m |\det T_j|^2 > 1$, then $K^{\mathbf{a}}$ has non-empty interior for $\mathcal{L}^{dm}$-a.e. $\mathbf{a}$.

For simplicity, we show the idea of the proof for the homogeneous case where

$$\mathbf{T} = (T, \dots, T).$$

- By symbolic expression,

$$K^{\mathbf{a}} = E^{\mathbf{a}} + TE^{\mathbf{a}},$$

  where $E^{\mathbf{a}}$ is the self-affine set generated by $\{T^2x + a_j\}_{j=1}^m$.
- Then $m \cdot |\det T|^2 > 1$ implies typically $\mathcal{L}^d(E^{\mathbf{a}}) > 0$ (Jordan, Pollicott, and Simon, 2007).
- The proof is finished by Steinhaus theorem.

An open question

By definition,

$$(1) \qquad \gamma(\mathbf{T}) > d \iff \exists\, n \text{ such that } \sum_{|I|=n} \alpha_d(T_I)^d |\det T_I| > 1$$

$$(2) \qquad \dim_{\mathrm{AFF}} \mathbf{T} > 2d \iff \sum_{j=1}^{m} |\det T_j|^2 > 1$$

$$(3) \qquad \dim_{\mathrm{AFF}} \mathbf{T} > d \iff \sum_{j=1}^{m} |\det T_j| > 1.$$

Recall that (1) and (2) are respectively assumed in Theorems A and B. And (3) implies typically $\mathcal{L}^d(K^{\mathbf{a}}) > 0$.

Note (1) $\implies$ (2) $\implies$ (3).

An open question

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$$(3) \qquad \dim_{\mathrm{AFF}} \mathbf{T} > d \iff \sum_{j=1}^{m} |\det T_j| > 1.$$

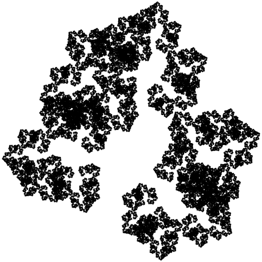
Recall that (1) and (2) are respectively assumed in Theorems A and B. And (3) implies typically $\mathcal{L}^d(K^{\mathbf{a}}) > 0$.

Note (1) $\implies$ (2) $\implies$ (3).

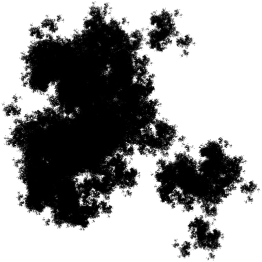
Open question

Does typical $K^{\mathbf{a}}$ have non-empty interior under the condition (3)?

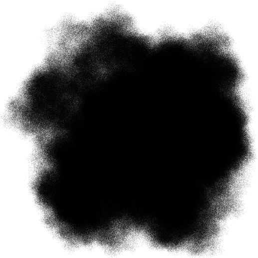
Some numerical experiments



$\dim_{\mathrm{AFF}} \mathbf{T} < 2$



(3) satisfied



(1) & (2) satisfied

Thank you for listening!

Cheers for fractal!