Randomised algorithms
COMP4500/7500
Advanced Algorithms & Data Structures

October 19, 2020

Overview

- Admin/reminders
- Probabilistic Analysis and randomised algorithms
- Quicksort
    - Partition
    - Randomised quicksort

Admin

Lectures

- Week 11 (this week): probabilistic analysis and randomised algorithms
- Week 12: revision. Overview/recap followed by questions

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Tutorials:

- Week 11 (this week): revision and assignment 2 help
- Week 12: open revision of all material
- Note: No tutorial specific to this week's material

Average case analysis

$$
\begin{aligned}
T_{\text{Worst-case}}(n) &= \max_{|x|=n} T(x) \\
T_{\text{Best-case}}(n) &= \min_{|x|=n} T(x) \\
T_{\text{Average-case}}(n) &= \sum_{|x|=n} T(x) \cdot Pr\{x\}
\end{aligned}
$$

Hire assistant example

HIRE-ASSISTANT(n)

$best = 0$ // candidate 0 is a least-qualified dummy candidate
for $j = 1$ **to** n
    interview candidate j (cost c_i)
    if candidate j is better than the best candidate
        $best = j$
        hire candidate j (cost c_h)

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n the total number of candidates

m the number of candidates hired

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- Average case is?

Hire assistant example: average-case analysis

Probability of hiring the j^{th} candidate

- Assume candidates are in random order
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$$c_h \sum_{j=1}^{n} \frac{1}{j}$$

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Recall the Harmonic series $\sum_{j=1}^{n} \frac{1}{j} = \ln n + O(1)$

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For **deterministic (i.e. not randomised) algorithms** we can calculate the **average** running time, based on a probability distribution of inputs.

For **randomized algorithms** we calculate the **expected** running time – without having to make an assumption about the probability distribution of inputs.

Randomised hire assistant

RANDOMIZED-HIRE-ASSISTANT(n)

randomly permute the list of candidates
$best = 0$ // candidate 0 is a least-qualified dummy candidate
for $j = 1$ **to** n
    interview candidate j (cost c_i)
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Randomly permuting arrays: first method

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$1/n!$ chance of each permutation of the n elements of array A

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PERMUTE-BY-SORT(A)

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n = A.length
let P[1..n] be a new array
for i = 1 to n
        P[i] = RANDOM(1, n^3)
sort A, using P as the sort keys
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$$\frac{n^3}{n^3} \times \frac{n^3 - 1}{n^3} \times \frac{n^3 - 2}{n^3} \times \cdots \times \frac{n^3 - n}{n^3}$$

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Randomly permuting arrays: randomize in place

```
RANDOMIZE-IN-PLACE(A)
n = A.length
for i = 1 to n
        swap A[i] ↔ A[RANDOM(i, n)]
```

Randomly permuting arrays: randomize in place

RANDOMIZE-IN-PLACE(A)

$n = A.length$
for $i = 1$ **to** n
    swap $A[i] \leftrightarrow A[\textsc{Random}(i, n)]$

A k-permutation of set of n elements is defined to be a sequence containing k of the n elements.

Randomly permuting arrays: randomize in place

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A k-permutation of set of n elements is defined to be a sequence containing k of the n elements.

Invariant $Inv(i)$:
$A[1..i]$ contains any i-permutation of A with probability

$$\frac{(n-i)!}{n!}$$

Randomise-In-Place invariant initially

Before the first iteration $A[1..0]$ contains a 0-permutation with probability

$$1 = \frac{(n-0)!}{n!}$$

RANDOMISE-IN-PLACE maintains invariant

Assuming $Inv(i-1)$ holds before an iteration,
we must show $Inv(i)$ holds after.

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Assuming $Inv(i-1)$ holds before an iteration,
we must show $Inv(i)$ holds after.

- Assume $A[1..i-1]$ contains any $(i-1)$-permutation with probability $\frac{(n-i+1)!}{n!}$.
- Consider the i-permutation $\langle x_1, \ldots, x_i \rangle$.

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- Assume $A[1..i-1]$ contains any $(i-1)$-permutation with probability $\frac{(n-i+1)!}{n!}$.
- Consider the i-permutation $\langle x_1, \ldots, x_i \rangle$.
- Let E_1 be the event that $A[1..i-1]$ is $\langle x_1, \ldots, x_{i-1} \rangle$

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- Let E_2 be the event that the i^{th} iteration places x_i in $A[i]$.

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$$
\begin{aligned}
Pr\{E_2 \cap E_1\} &= Pr\{E_2 | E_1\} \cdot Pr\{E_1\} \\
&= \frac{1}{n-i+1} \cdot \frac{(n-i+1)!}{n!}
\end{aligned}
$$

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&= \frac{1}{n-i+1} \cdot \frac{(n-i+1)!}{n!} \\
&= \frac{(n-i)!}{n!}
\end{aligned}$$

RANDOMISE-IN-PLACE on termination

On termination $Inv(n)$ holds, i.e. $A[1..n]$ contains any n-permutation of the original array with probability

$$\frac{(n-n)!}{n!} = \frac{1}{n!}$$

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Therefore A contains any permutation of the original array with probability

$$\frac{1}{n!}$$

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Therefore A contains any permutation of the original array with probability

$$\frac{1}{n!}$$

Therefore each of the $n!$ possible permutations of the original array is equally likely.

Quicksort

1. Merge sort: divide & conquer
2. Heapsort: build and manipulate a heap
3. Quicksort: pre-process array by partitioning into elements greater-than and less-than some element (the “pivot”).

Quicksort

- Quicksort: pre-process data into “low” and “high” elements

  QUICKSORT(A, p, r)

  ```
  if p < r
        q = PARTITION(A, p, r)
        QUICKSORT(A, p, q − 1)
        QUICKSORT(A, q + 1, r)
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Quicksort

- Quicksort: pre-process data into “low” and “high” elements

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  if $p < r$
  $\quad q = $ PARTITION(A, p, r)
  $\quad$ QUICKSORT($A, p, q - 1$)
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- Mergesort: post-process sorted data into a single, sorted array

  MERGE-SORT(A, p, r)

  if $p < r$
  $\quad q = \lfloor (p + r)/2 \rfloor$
  $\quad$ MERGE-SORT(A, p, q)
  $\quad$ MERGE-SORT($A, q + 1, r$)
  $\quad$ MERGE(A, p, q, r)

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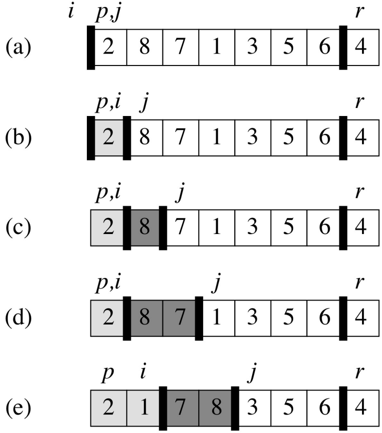

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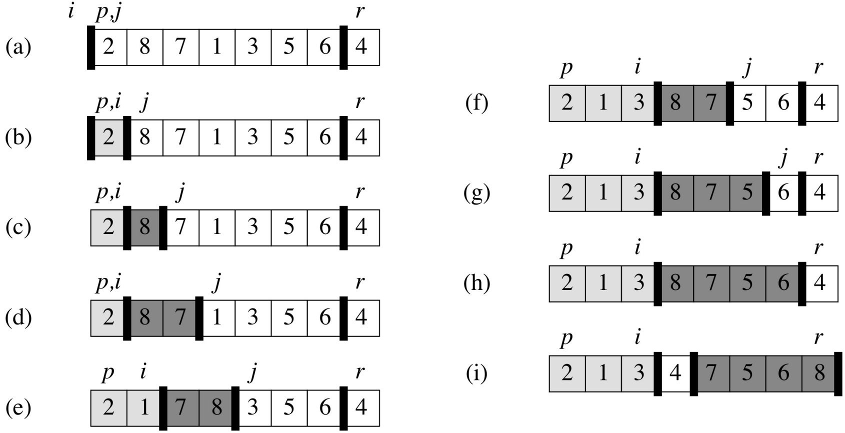

Quicksort: partition

PARTITION(A, p, r)

```
x = A[r]
i = p - 1
for j = p to r - 1
    if A[j] ≤ x
        i = i + 1
        exchange A[i] with A[j]
exchange A[i + 1] with A[r]
return i+1
```

Partition is $\Theta(n)$.

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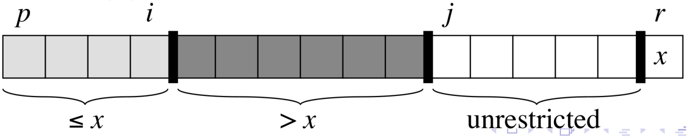

Analysis of Quicksort

Performance depends on the element chosen as the pivot

- Best and average case: $\Theta(n \lg n)$.
- Worst case: $\Theta(n^2)$

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- Best and average case: $\Theta(n \lg n)$.
- Worst case: $\Theta(n^2) =$ insertion sort.
  This case occurs if the array is already sorted. In fact in this special case, insertion sort is $\Theta(n)$.

Analysis of Quicksort (cont.)

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QUICKSORT(A, p, r)
if p < r
    q = PARTITION(A, p, r)
    QUICKSORT(A, p, q − 1)
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1. Best case: $q = \lfloor (p + r)/2 \rfloor$
   $$T(n) = 2T(n/2) + \Theta(n)$$

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2. Constant ratio split: $q = p + (r - p)/c$
   $$T(n) = T(n/c) + T((c-1)n/c) + \Theta(n)$$

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3. Worst case, no partitioning: $q = p$.
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- In the partitioning:
  - the pivot 4 is compared with every other element
  - but no element front he first set is or ever will be compared with an element of the second set

Probabiity z_i is compared with z_j

- For any elements z_i and z_j once a pivot x is chosen such that

$$z_i < x < z_j$$

  z_i and z_j can never be compared in the future.

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- If z_i is chosen as a pivot before any other element in Z_{ij} then z_i will be compared with every other element in Z_{ij}. Similarly for z_j.
- Thus z_i and z_j are compared if and only if the first element chosen as a pivot in Z_{ij} is either z_i or z_j

Probabiity z_i is compared with z_j

- For any elements z_i and z_j once a pivot x is chosen such that

$$z_i < x < z_j$$

  z_i and z_j can never be compared in the future.
- If z_i is chosen as a pivot before any other element in Z_{ij} then z_i will be compared with every other element in Z_{ij}. Similarly for z_j.
- Thus z_i and z_j are compared if and only if the first element chosen as a pivot in Z_{ij} is either z_i or z_j
- Any element in Z_{ij} is equally likely and Z_{ij} has $j - i + 1$ elements

Analysis of QUICKSORT

Each pair of elements is compared at most once because in PARTITION elements are compared with the pivot only at most once, and an element is only used as a pivot in at most one call to PARTITION

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= & \; O(n \lg n)
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Randomised Quicksort

- Despite the bad worst-case bound, Quicksort is regarded by many to be a good sorting algorithm.

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- Good expected-case performance can be achieved by randomly permuting the input before sorting. (Permutation can be done in $\Theta(n)$)
- In practice, an even simpler approach specific to Quicksort is to choose a pivot from a random location. This just requires a constant-time swap of some random element $A[i]$ with $A[r]$.
- Having done this, the worst-case is unlikely, and the **expected time complexity** of the algorithm becomes $\Theta(n \lg n)$.

Recap

1. Randomised algorithms
2. Quicksort
    - Low overheads, sorts in-place, generally regarded as the fastest/most practical general sorting algorithm
    - Worst case is avoided by adding some randomness

Quick question

Given procedure BIASED-RANDOM() that returns 0 with probability p and 1 with probability $1-p$, where $0 < p < 1$,

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Given procedure BIASED-RANDOM() that returns 0 with probability p and 1 with probability $1-p$, where $0 < p < 1$, how can you implement procedure RANDOM that returns 0 or 1 with equal probability?

Quick question

RANDOM()

1 a = BIASED-RANDOM()
2 $\cdots$

- probability $a = 0$ is p
- probability $a = 1$ is $1-p$

Quick question

RANDOM()

1 $a = $ BIASED-RANDOM()
2 $b = $ BIASED-RANDOM()
3 $\cdots$

- probability $a = 0 \wedge b = 0$ is p^2
- probability $a = 0 \wedge b = 1$ is $p \times (1-p)$
- probability $a = 1 \wedge b = 0$ is $p \times (1-p)$
- probability $a = 1 \wedge b = 1$ is $(1-p)^2$

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- probability $a = 1 \wedge b = 1$ is $(1-p)^2$

So, the probability that $a = 0$ given that $a \neq b$ is:

$$\frac{p \times (1-p)}{p \times (1-p) + p \times (1-p)} = \frac{1}{2}$$

.

Quick question

RANDOM()

$a =$ BIASED-RANDOM()
$b =$ BIASED-RANDOM()
if $a \neq b$
    return a
else $\cdots$

Quick question

```
RANDOM()
a = BIASED-RANDOM()
b = BIASED-RANDOM()
while a = b
    a = BIASED-RANDOM()
    b = BIASED-RANDOM()
return a
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a = BIASED-RANDOM()
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What is the expected running time?

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Let $\alpha = 2(p \times (1-p))$ be the probability that $a \neq b$.

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Expected number of loop iterations:

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= \; & \alpha \times (\textstyle\sum_{i=0}^{\infty} i \times ((1-\alpha)^i)) \\
= \; & \alpha \times \frac{1-\alpha}{(1-(1-\alpha))^2} \\
= \; & \alpha \times \frac{1-\alpha}{\alpha^2} \\
= \; & \frac{1-\alpha}{\alpha} \\
= \; & \frac{1}{\alpha} - 1
\end{aligned}
$$