Mathematical Background

COMP4500/7500

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Finite Summations

See CLRS A.1; CLR §3.1.
Sequence: $\langle a_1, a_2, \ldots, a_n \rangle$

Definition (Finite sum)

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n \quad (1)$$

$$\sum_{k=1}^{0} a_k = 0 \quad \text{(by definition)} \quad (2)$$

Finite sums may be added in any order (commutativity).

For n nonintegral: could use floor, $\lfloor n \rfloor$, or ceiling, $\lceil n \rceil$.

Infinite sums

Infinite sum $a_1 + a_2 + \dots$

Definition (Infinite summation)

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k \tag{3}$$

This limit need not be well behaved:

- Diverges (limit does not exist): e.g. $\sum_{n=0}^{\infty}(-1)^n$
- Converges
- Absolutely convergent (any order):
  $$\sum_{k=1}^{\infty} |a_k| \text{ converges}$$

Linearity

$$\sum_{k=1}^{n}(ca_k + b_k) = c\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \tag{4}$$

(also holds for infinite sums: $\sum_{k=1}^{\infty}$)

Proof.

By induction: CLRS A.2; CLR 3.2; do Revision 1.

- Base case (eg, $n = 0$)
- Inductive step
  (true for $n = m$ implies true for $n = m + 1$)

$\square$

Arithmetic series

Theorem (Arithmetic series)

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{5}$$

Geometric series

Theorem (Geometric series)

Finite:

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \ldots + x^n \tag{6}$$

$$= \frac{x^{n+1} - 1}{x - 1} \quad \textit{for } x \neq 1 \tag{7}$$

Infinite:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x} \quad \textit{for } |x| < 1 \tag{8}$$

Harmonic series

Definition (Harmonic series)

$$H_n = 1 + 1/2 + 1/3 + \ldots + 1/n \quad (9)$$

$$= \sum_{k=1}^{n} 1/k \quad (10)$$

$H_n = \ln n + O(1)$ $(\to \gamma \sim 0.577\ldots)$

$$\sum_{i=1}^{n} i^k \sim \frac{n^{k+1}}{|k+1|} \text{ if } k \neq -1$$

Asymptotic notation

Definition (Little o)

$$o(g(n)) = \{f(n) : \forall c > 0, \exists n_0 > 0 \mid 0 \leq f(n) < cg(n), \forall n \geq n_0\}$$

Requires both f and g to be asymptotically nonnegative ($0 \leq$).

Little o and limits:

$$f(n) \in o(g(n)) \text{ if } \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

and f(n) is asymptotically nonnegative .

Using limits: L'Hôpital's rule

How do you evaluate $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ if it looks hard?

Consider $\lim_{n\to a} f(n)/g(n)$ when either

$$\lim_{n\to a} f(n) = 0 = \lim_{n\to a} g(n)$$

or

$$\lim_{n\to a} f(n) = \pm\infty = \lim_{n\to a} g(n)$$

L'Hôpital's rule gives:

$$\lim_{n\to a} \frac{f(n)}{g(n)} = \lim_{n\to a} \frac{f'(n)}{g'(n)}$$

Telescoping series

$$\sum_{k=1}^{n}(a_k - a_{k-1}) = (a_1 - a_0) + (a_2 - a_1) + \ldots + (a_n - a_{n-1})$$

$$= a_n - a_0 \quad (11)$$

$$\sum_{k=0}^{n-1}(a_k - a_{k+1}) = a_0 - a_n \quad (12)$$

Example:

$$\sum_{k=1}^{n-1} \frac{1}{k(k+1)} = \sum_{k=1}^{n-1} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n}$$

because $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$

Calculus: differentiation

Example: $\sum_{k=0}^{\infty} kx^k$ where $|x| < 1$. We know:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \qquad \text{if } |x| < 1$$

Differentiate both sides wrt x:

$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

Now,

$$\sum_{k=0}^{\infty} kx^k = x\left(\sum_{k=0}^{\infty} kx^{k-1}\right) = \frac{x}{(1-x)^2}$$

Algebraic manipulation

Example: $s = \sum_{i=1}^{\infty} i/2^i$

$$\begin{aligned}
s &= \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \dots \\
2s &= 1 + \frac{2}{2} + \frac{3}{2^2} + \dots \\
2s - s &= 1 + \frac{1}{2} + \frac{1}{2^2} + \dots = \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{1 - \frac{1}{2}} = 2
\end{aligned}$$

so $s = 2$.

Quick problem: using algebraic manipulation

Calculate the solution to $s = \sum_{k=0}^{n} x^k$.

$$s = 1 + x + x^2 + \ldots + x^n$$

Products

Definition (Product)

$$\prod_{k=1}^{n} a_k = a_1 a_2 \ldots a_n \tag{13}$$

$$\prod_{k=1}^{0} a_k = 1 \tag{14}$$

Products can be converted to sums using logarithms

$$\lg \prod_{k=1}^{n} a_k = \sum_{k=1}^{n} \lg a_k$$

Logarithms

$$
\begin{aligned}
x^a = b &\iff \log_x b = a && (15)\\
b^{\log_b y} &= y && (16)\\
a^{\log_b n} &= n^{log_b a} && (17)\\
\log_a b &= \frac{\log_c b}{\log_c a} \quad \text{for } c > 0 && (18)\\
\log(ab) &= \log a + \log b && (19)\\
\log(a^b) &= b * \log a && (20)\\
\log(\frac{1}{a}) &= -\log a && (21)\\
\log x &< x \quad \text{for all } x > 0 && (22)
\end{aligned}
$$

Notation $\lg x = \log_2 x$ and $\ln x = \log_e x$

Bounding sums: upper bounds

$$\sum_{k=1}^{n} a_k \leq n a_{\max}$$

Example:

$$\sum_{k=1}^{n} k \leq \sum_{k=1}^{n} n = n^2$$

Improving the bound: Say $\frac{a_{k+1}}{a_k} \leq r$, $\forall k \geq 0$ with $r < 1$ constant. Then:

$$\sum_{k=0}^{n} a_k \leq \sum_{k=0}^{n} a_0 r^k = a_0 \sum_{k=0}^{n} r^k \leq \frac{a_0}{1-r}$$

Bounding sums: lower bounds

$$\sum_{k=1}^{n} a_k \geq n a_{\text{min}}$$

Example:

$$\sum_{k=1}^{n} k \geq \sum_{k=1}^{n} 1 = n$$

This is a linear bound, which is poor. Split the sum:

$$\sum_{k=1}^{n} k = \sum_{k=1}^{\frac{n}{2}} k + \sum_{k=\frac{n}{2}+1}^{n} k \geq \sum_{k=1}^{\frac{n}{2}} 0 + \sum_{k=\frac{n}{2}+1}^{n} \frac{n}{2} \geq \left(\frac{n}{2}\right)^2 = \frac{n^2}{4}$$

This is a quadratic bound, which is OK (Θ)

Splitting sums is a powerful technique

Consider $H_n = \sum_{k=1}^{n} \frac{1}{k}$

Split the range into $\lfloor \lg n \rfloor$ pieces, with each piece summing ≤ 1

$$(1) + (1/2 + 1/3) + (1/4 + 1/5 + 1/6 + 1/7) + \dots$$

$$\sum_{k=1}^{n} \frac{1}{k} \leq \sum_{i=0}^{\lfloor \lg n \rfloor} \left(\sum_{j=0}^{2^i - 1} \frac{1}{2^i + j} \right) \leq \sum_{i=0}^{\lfloor \lg n \rfloor} \left(\sum_{j=0}^{2^i - 1} \frac{1}{2^i} \right) \leq \sum_{i=0}^{\lfloor \lg n \rfloor} 1 \leq 1 + \lg n$$

Approximation by integrals

Consider $\sum_{k=m}^{n} f(k)$ where $f(k)$ is monotonically increasing:

$$\int_{m-1}^{n} f(x)\,dx \le \sum_{k=m}^{n} f(k) \le \int_{m}^{n+1} f(x)\,dx$$

See pictorial “proof” CLRS (p1155 (3rd))
There is a similar result for monotonically decreasing:

$$\int_{m}^{n+1} f(x)\,dx \le \sum_{k=m}^{n} f(k) \le \int_{m-1}^{n} f(x)\,dx$$

Thus $H_n \ge \int_{1}^{n+1} \frac{1}{x} dx = \ln(n+1)$
and $\sum_{k=2}^{n} \frac{1}{k} \le \int_{1}^{n} \frac{1}{x} dx = \ln n$ so $H_n \le \ln n + 1$