Advection Diffusion

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1 Introduction

Advection-diffusion is often utilized as a mechanism for heat removal(and hence cooling) at a variety of length scales in a large variety of machines, from large power plant cooling systems to cooling chips in electronic systems.

Here, advection-diffusion for electronic cooling systems is considered. Typical examples are the use of coolant liquid circulations to cool server racks and individual high performance chips.

Strictly speaking, the most efficient design would be a two phas evaporative loop, typical products of this kind are made by companies such as Parker, who makes a two phase evaporative precision cooling system that is capable of heat loads from 3 to 300 kW. Flow boiling and microchannels both enhance the heat transfer. Here, however, we are ignoring both. The justification for ignoring these mechanisms is that we are limiting our analysis to lower powered desktop chips and not considering high power servers. Products of this sort, that rely on single phase heat transfer are produced by companies like Antec or Cooler Master, among others.

These products don't come with a microchannel based design and heat generated in the desktop is usually not sufficient to cause boiling of the flowing liquid. Usually water coolers are preferred owing to its high heat capacity.

2 Domain of the Problem

The problem is considered as a flow through a square channel The flowing liquid enters from the left side and exits out from the right. The velocity field is assumed to be known and is taken to be a constant for simplification. If the limit of known velocity is removed, then the problem becomes a pressure-

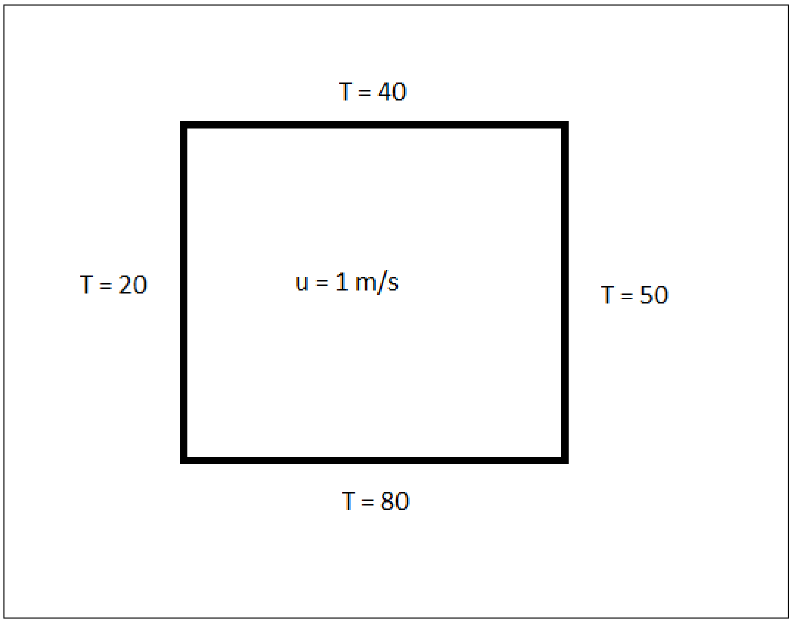

Figure 1: Schematic of the flow domain

velocity coupling and must be solved by a more complex scheme like SIMPLE, which is beyond the scope of the current investigation.

The flow domain considered is represented as shown in the figure below.

3 Solution of the Problem

The problem is solved using finite volume method implemented in Fortran 90. As there is virtually no native support for diagrams in Fortran, the plotting of graphs of the obtained data is done using the matplotlib library from python.

Discretization A uniform Cartesian grid is composed on the flow field. For this the flow domain is divided into 'n' subdivisions along both x and y directions. The convection chosen here for the numbering system is clarified below, as fortran numbers its arrays from 1(or generally arbitrarily) while the numpy module of python(that provides the array object) numbers them from 0.

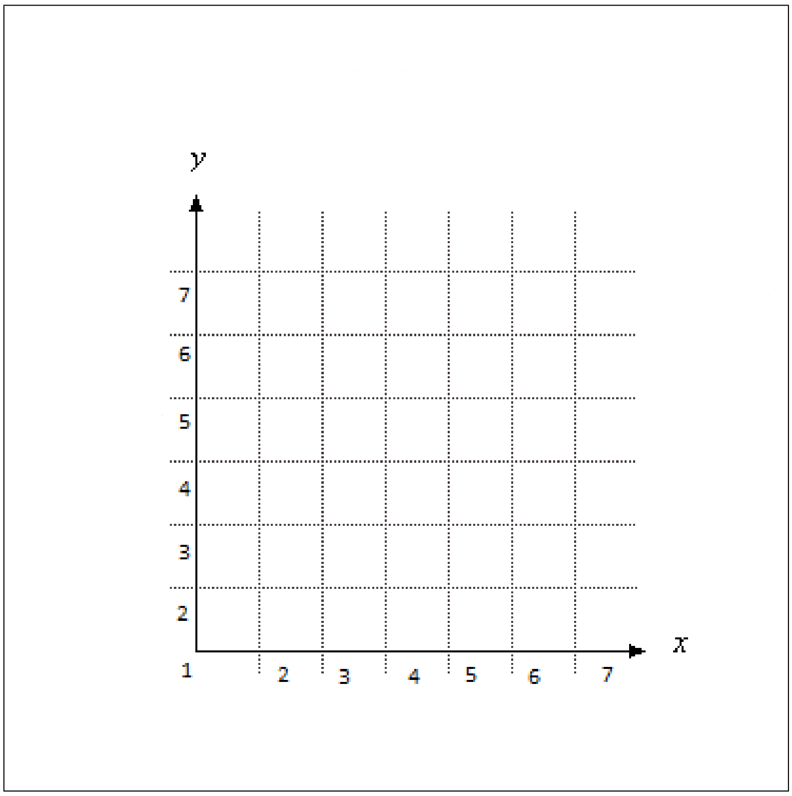

Figure 2: Numbering convention

The Equation We are considering the steady-state advection-diffusion problem.

$$\frac{\partial}{\partial x_j}(\rho u_j \Phi) = \frac{\partial}{\partial x_j}(\Gamma \frac{\partial \Phi}{\partial x_j}) \tag{1}$$

Assuming that there are no sources and Φ represents the Temperature field.

Since flow field is constant, we can rewrite the above equation as:

$$\rho u \frac{\partial \Phi}{\partial x} = \frac{\partial}{\partial x}(\Gamma \frac{\partial \Phi}{\partial x}) + \frac{\partial}{\partial y}(\Gamma \frac{\partial \Phi}{\partial y}) \tag{2}$$

Boundary conditions are shown in the accompanying diagram.

Solution Strategy The given domain is thus divided into a number of volumes. The above equation is integrated and we apply Green's theorem to convert the area integrals to line integrals and the gradient is converted into total flux.

$$\int_A \mathbf{n} \cdot (\rho \Phi \mathbf{u}) dA = \int_A \mathbf{n} \cdot (\Gamma grad \Phi) dA \tag{3}$$

Now consider a volume cell P inside the domain. Adopting the shown nomenclature in figure 3, we can write the following equation:

$$\Delta y(\rho u \Phi)_e - \Delta y(\rho u \Phi)_w = (\Gamma A \frac{\partial \Phi}{\partial x})_e - (\Gamma A \frac{\partial \Phi}{\partial x})_w + (\Gamma A \frac{\partial \Phi}{\partial y})_n - (\Gamma A \frac{\partial \Phi}{\partial y})_s \tag{4}$$

It is convenient to define two variable F and D to represent the convective mass flux per unit area and diffusion conductance at cell faces:

$$F = \rho u \tag{5}$$

$$D = \frac{\Gamma}{\Delta x} \tag{6}$$

Note that since the velocity field is taken as constant, the conservation of mass condition is satisfied implicitly.

4 Schemes of Solution

The finite volume method allows to measure the variation of a property, by allowing us to specify the property at cell centers(or equivalently, over the

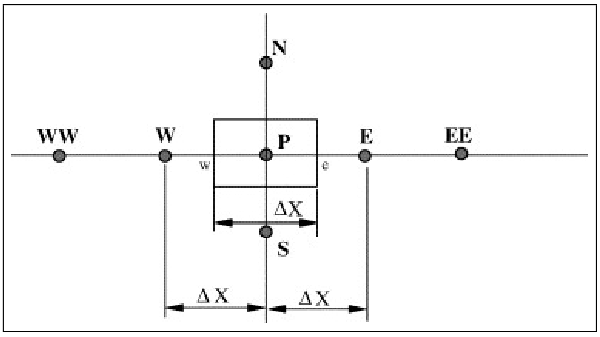

Figure 3: Stencil Nomenclature

cell volume). However, the equation above requires the determination of a property at the cell faces(or at the cell boundary in case of 1D).

Since the value is not known, it needs to be interpolated from the values at the cell centers. Three common schemes are discussed here. The central difference, the upwind scheme and the hybrid scheme.

4.1 Central Difference Scheme

Here, the values at cell centers are approximated as the averages of the bounding cells. This can be stated mathematically as:

$$\Phi_e = \frac{\Phi_P + \Phi_E}{2} \tag{7}$$

Conservativeness Conservativeness is the property of conservation of the property being considered. To ensure conservation for the whole solution domain the flux entering a certain control volume must be equal to the flux leaving the same control volume. For this it is necessary to represent the flux through a common face in a consistent manner. It is found that when, as in the condition we are considering, Γ is constant, then CDS is conservative.

Boundedness Normally a finite volume approach reduces a problem to a set of algebraic equations that need to be solved. Then iterative numerical methods are used to solve the resulting large equation sets. These iterative methods start from some guess value of the distribution of Φ and continue variating it(according to some rules) until a converged solution is obtained. Scaraborough showed that the sufficient condition for boundedness is

$$\frac{|a_n b|}{|a'_p|} \leq 1 \quad \text{on all nodes} \tag{8}$$

$$\frac{|a_n b|}{|a'_p|} < 1 \quad \text{on one node at least} \tag{9}$$

where $a_n b$ is the coefficient of neighbouring points in the discretized equation and a'_p is the coefficient of the point P. Diagonal dominance is the necessary and sufficient criteria in this case, which is achieved, because we have no source terms and hence no off diagonal band terms. Another essential requirement is that all property values should be specified on the boundary and should be positive. This is ensured by using Dirichlet boundary conditions and temperatures in Kelvin so that they are always positive.

Transportiveness To understand transportiveness, a new non dimensional constant needs to be defined, the Peclét number:

$$Pe = \frac{F}{D} = \frac{\rho u}{\Gamma / \Delta x} \tag{10}$$

The Peclét number is a measure of the relative strength of the convection and the diffusion of the field. If $Pe \to \infty$ there is no diffusion and pure convection, while if $Pe \to 0$ then there is no convention and pure defection.

The transportiveness is a measure of how much the property values at neighbouring cell centers affect the value at the center. Here since there is no velocity in the y-direction, the flow is purely diffusive along the y axis.

4.2 The Upwind Differencing Scheme

While central differencing is a simple to use, robust scheme, it is not without its own share of inadequacies. One majore disadvantage of the central difference schme is that it cannot identify the flow dircetion of the flow. This is so because the value of Φ at any face is affected by its value on both the volumes on its either sides.

If the flow eg is highly convective, then this treatment is ineffective and unsuitable as there would be a lot more influence from the upstream cell than that of the downstream cell. Thus, this is taken into account by applying the upwind differencing scheme.

In this scheme, the value of Φ at a cell face is taken to be equal to the value at the upstream node. This can be stated mathematically as:

$$\Phi_w = \Phi_W \tag{11}$$

$$\Phi_e = \Phi_E \tag{12}$$

Conservativeness Since the formulation of Upwind differencing is consistent, it can be easily shown that the scheme is conservative. Upwind scheme satisfies conservation natively.

Boundedness Since the coefficients are always taken from the upstream terms, it immediately follows that all the coefficients of the final equation are positive. As such the coefficient matrix is diagonally dominant and solution is bounded.

Transportiveness Since the scheme accounts for the direction of the flow, transportiveness is built into the formulation.

Accuracy Since the scheme is based on backward differencing, the accuracy is only first order on the basis of Taylor series truncation.

A major drawback of the scheme is that, it leads to "false diffusion" when the flow is not aligned to the grid lines. This causes the transported property to get smeared out, thus even if the grid is made very fine, there is not enough comparative convergence with the true solution.

4.3 The Hybrid Differencing Scheme

This scheme is,as the name suggests a hybrid of the above two schemes, aimed at mitigating the flaws of both of them. It employs the central scheme to regions where the Peclet number is small($Pe < 2$) and the upwind scheme to the regions where it is large.

Assessment of the Hybrid Differencing Scheme The hybrid scheme aims to merge the good points of both the central and the upwind differencing scheme. It applies the central scheme in regions where the upwind scheme is impractical($Pe < 2$) and an upwind scheme is applied in the regions where the accuracy of the central scheme is questionable.

In this particular problem, it can be easily demonstrated that $Pe > 2$ everywhere in the flow domain and hence the hybrid scheme, when applied, reduces to a upwind difference scheme globally.

5 Solution

The given problem is solved by central and upwind difference schemes. As the hybrid scheme reduces to central scheme, it is not considered here. For each of the two schemes, we begin with a small value of the number of cells and gradually go on increasing the number of cells. The difference in values of the grid is computed and this deviation is considered. The process is continued till the difference of values between the coarser and finer grids becomes negligible, at which point, grid independence could be claimed to be achieved.

6 Results

6.1 From Central Difference

The resulting equation is solved by central difference method. For this, each side is divided into 10,20,40 and 80 cells respectively. Then the difference between two consecutive levels is measured. The standard deviation of these is calculated and plotted against the grid size. Also plotted are the contours for 80×80 and finally the trend for the difference between two consecutive levels.

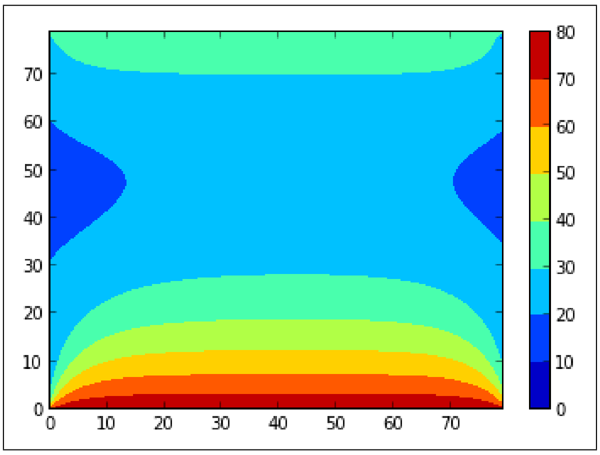

Figure 4: Central Difference:Contour plot for 80×80

6.2 From Upwind Difference

The same procedure is repeated for upwind difference, the only extra step being a comparison of the two methods for 80×80 grid by computing the difference of the two explicitly.

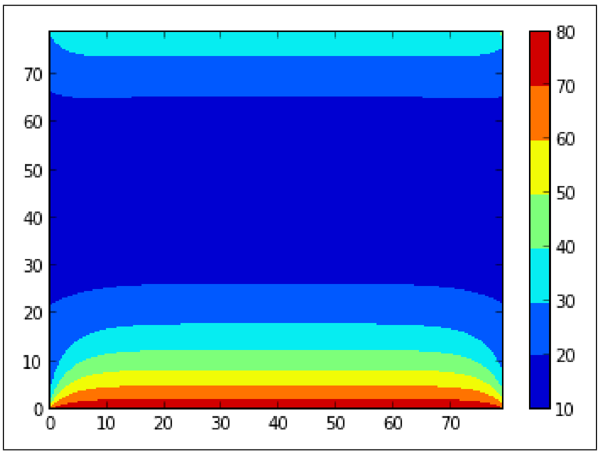

Figure 5: Upwind Difference: Contour plot for 80×80

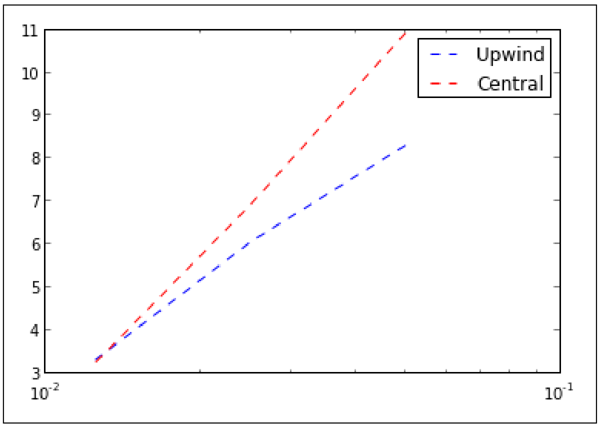

Figure 6: Comparison of the trends of rms vs log of grid size