Lab 5 Task 1

May 4, 2023

Part 1

1)

The analytical solution is: $y(x) = 2\sin x + \cos x$

2)

We have the constraints:

$$
\begin{align}
y'' + y &= 0 \tag{1}\\
y(0) &= 1 \tag{2}\\
y(\tfrac{\pi}{2}) &= 2 \tag{3}\\
h = \frac{\frac{pi}{2}}{2} &= \frac{\pi}{8}. \notag
\end{align}
$$

The stepsize h and boundary-points 0 and $\frac{\pi}{2}$gives 5 discrete points with the boundary conditions:

$$
\begin{aligned}
(2) &\implies y_0 = 1\\
(3) &\implies y_4 = 2.
\end{aligned}
$$

We use central difference to approximate y'':

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = h^{-2}(y_{i-1} - 2y_i + y_{i+1}). \tag{4}$$

We substitute the y'' in equation (1) for (4):

$$
\begin{align}
(1) \iff & h^{-2}(y_{i-1} - 2y_i + y_{i+1}) + y_i \notag\\
&= h^{-2}y_{i-1} + (1 - 2h^{-2})y_i + h^{-2}y_{y+1} = 0. \tag{5}
\end{align}
$$

Now the boundary conditions (2) and (3) together with equation (5) gives us a sysetm of linear equations:

$$\begin{bmatrix} 1 & & & & \\ h^{-2} & 1-2h^{-2} & h^{-2} & & \\ & h^{-2} & 1-2h^{-2} & h^{-2} & \\ & & h^{-2} & 1-2h^{-2} & h^{-2} \\ & & & & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}. \tag{6}$$

3)

The LTE for this method is the same as the LTE for the finite difference method used, central difference in this case.

The central difference $\tilde{y}(x)$ approximates $y'(x)$ and is given by:

$$\tilde{y}(x) = \frac{y(x+h) - y(x-h)}{2h}. \tag{7}$$

We use tayor-expansion shenanigans to find the LTE. We begin by taylor expanding $y(x+h)$ around x:

$$\begin{aligned} y(x+h) &= \sum_{n=0}^{\infty} \frac{y^{(n)}(x)}{n!}(x+h-x)^n = \sum_{n=0}^{\infty} h^n \frac{y^{(n)}(x)}{n!} \\ &= y(x) + hy'(x) + \tfrac{h^2}{2}y''(x) + \tfrac{h^3}{6}y'''(x) + \cdots. \end{aligned} \tag{8}$$

We perform the same procedure for $y(x-h)$ around x:

$$\begin{aligned} y(x-h) &= \sum_{n=0}^{\infty} \cdots \\ &= y(x) - hy'(x) + \tfrac{h^2}{2}y''(x) - \tfrac{h^3}{6}y'''(x) + \cdots. \end{aligned} \tag{9}$$

No we substitute the $y(x+h)$ and $y(x-h)$ in (7) for (8) and (9):

$$\begin{aligned} (7) \iff \tilde{y}(x) &= \tfrac{1}{2h}[(y(x) + hy'(x) + \tfrac{h^2}{2}y''(x) + \tfrac{h^3}{6}y'''(x) + \cdots) \\ &\quad - (y(x) - hy'(x) + \tfrac{h^2}{2}y''(x) - \tfrac{h^3}{6}y'''(x) + \cdots)] \\ \iff \tilde{y}(x) &= \frac{2hy'(x) + \tfrac{h^3}{3}y'''(x) + \cdots}{2h} \\ \iff \tilde{y}(x) &= y'(x) + \tfrac{h^2}{6}y'''(x) + \cdots \end{aligned}$$

Now we can se how $\tilde{y}(x)$ differs from $y'(x)$ and that $\frac{h^2}{6}y'''(x) + \cdots$ is our LTE, which is $\mathcal{O}(h^2)$.

4)

Yes the method is consistent. The determinant of the leftmost matrix in (6) has a nonzero determinant and thus the equation has a solution.

Part 2

When LU-decomposing A with partial pivoting we want to find an upper triangular matrix U, a lower triangular matrix L and a permutation matrix P such that $PA = LU$. When doing the calculation by hand there are nice bookkeeping-tricks one can perform to save on time, but let's do it in a slightly more inneficent way that makes it more clear what's actually going on and why it works.

We begin by finding the upper triangular matrix U by applying elementary matrix operations on A. Partial pivoting is necesary when there's a 0 somewhere in the diagonal, and computers do it as it reduces rounding errors.

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & -2 & 1 \\ 2 & 1 & -4 \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & 1 \\ 0 & -\frac{3}{2} & \frac{1}{2} \\ 0 & 2 & -5 \end{pmatrix} \sim$$

$$\begin{pmatrix} 2 & -1 & 1 \\ 0 & 2 & -5 \\ 0 & -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \sim \begin{pmatrix} 2 & -1 & 1 \\ 0 & 2 & -5 \\ 0 & 0 & -\frac{13}{4} \end{pmatrix}$$

All this gauss-elimination notation really means is that we have some elementary matrix operations E_1, E_2, P_1 and E_3 coresponding to the four row-operations performed such that: $U = E_3 P_1 E_2 E_1 A$.

Now the equation $PA = LU$ can be rewritten as:

$$\begin{aligned} & PA = LE_3P_1E_2E_1A \\ \iff & P = LE_3P_1E_2E_1 \\ \iff & L = P(E_3P_1E_2E_1)^{-1} = PE_1^{-1}E_2^{-1}P_1^{-1}E_3^{-1} \\ & \quad \vdots \\ \iff & L = P \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{3}{4} & 1 \\ 1 & 1 & 0 \end{pmatrix}. \end{aligned}$$

Now we need L to be a lower triangular matrix, and so P is determined to be the matrix permutation switching row 2 and 3 (which is the same permutation as P_1 and this is *not* a coincidence).

In closing, we have now found that:

$$U = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 2 & -5 \\ 0 & 0 & -\frac{13}{4} \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{2} & -\frac{3}{4} & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$