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Unit - I
Matrix

Types of Matrix

* Arrangement of row & column is called Matrix.

• eg: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$

• for matrix
Mat Lab

i) Row Matrix

eg: $[1 \quad 2 \quad 3]_{1\times3}$ order

ii) Column Matrix

eg: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times1}$

iii) Square Matrix

$m=n \Rightarrow$ no. of rows = no. of column

* if $A = [a_{ij}]_{m\times n} = [a_{ij}]_{n\times n}$

iv) Identity Matrix (Unit Matrix) (I)

• 'I' is always a square matrix.

eg: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$

v) Singular Matrix

$$|A| = 0$$
$\downarrow$
Singular matrix

vi) Non-Singular Matrix

$$|A| \neq 0$$

vii) Upper Triangular Matrix

$$A = [a_{ij}]_{m\times n}$$
is upper triangular if –

$a_{ij} = 0$ when $i > j$

⇒ elements below the diagonal are zero.

eg: $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}_{3\times 3}$

viii) Lower Triangular Matrix

$A = [a_{ij}]_{m\times n}$
is lower triangular if –
i.e. $a_{ij} = 0$ & $i < j$
⇒ elements above diagonal are zero.

eg: $\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$

ix) Symmetric Matrix

⇒ On interchanging row & column matrix remains same.

if $A^T = A$ or $A' = A$

eg: $\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$

x) Skew Symmetric Matrix

eg: $\boxed{A^T = -A}$ ** Here diagonal elements are always zero.

$$\begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -9 \\ 4 & 9 & 0 \end{bmatrix}$$

*Imp
Complex Matrix

* Matrix made of complex nos.

eg: $A = \begin{bmatrix} 1 & 2+i & i \\ 3 & -3i & 4 \\ 2i & 4 & 5-i \end{bmatrix}$

* Conjugate of complex matrix –

$$\bar{A} = \begin{bmatrix} 1 & 2-i & -i \\ 3 & 3i & 4 \\ -2i & 4 & 5+i \end{bmatrix}$$

Hermitian Matrix

* if $\boxed{(\bar{A})' = A}$

Skew Hermitian Matrix

* if $\boxed{A = -(\bar{A})'}$

** V. Imp

Unitary Matrix

* if $\boxed{(\bar{A})^T \times A = I}$

Orthogonal Matrix

$$\boxed{A^T . A = I}$$

Q1: If $A = \frac{1}{2}\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i \end{bmatrix}$

Prove that A is unitary matrix.

Sol^n: To prove : $(\bar{A})^T . A = I$

$$\bar{A} = \frac{1}{2}\begin{bmatrix} -i & \sqrt{3} \\ \sqrt{3} & -i \end{bmatrix}$$

$$(\bar{A})^T = \frac{1}{2}\begin{bmatrix} -i & \sqrt{3} \\ \sqrt{3} & -i \end{bmatrix}$$

$$(\bar{A})' . A = \begin{bmatrix} -i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -i/2 \end{bmatrix}\begin{bmatrix} i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & i/2 \end{bmatrix}$$

$$= \begin{bmatrix} -i^2/4 + 3/4 & -\frac{\sqrt{3}}{2}i + \frac{\sqrt{3}i}{2} \\ \frac{\sqrt{3}}{2} \times \frac{i}{2} - \frac{\sqrt{3}i}{2 \times 2} & \frac{3}{4} - \frac{i^2}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} + \frac{3}{4} & 0 \\ 0 & \frac{3}{4} + \frac{1}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Q2: If $A = \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}_{2 \times 3}$

then prove that $(\bar{A})' . A$ is hermitian.

Sol^n:

$$\bar{A} = \begin{bmatrix} 2-i & 3 & -1-3i \\ -5 & -i & 4+2i \end{bmatrix}$$

$$(\bar{A})' = \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}$$

$$(\bar{A})' . A = \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}_{3 \times 2}\begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}_{2 \times 3}$$

$$(\bar{A})' \cdot A = \begin{bmatrix} 2^2 - i^2 + 25 & 6 - 3i - 5i & (2-i)(3i-1) + (-5)(4-2i) \\ 6 + 3i + 5i & 9 - i^2 & \\ (-1-3i)(2+i) + (-5)(4+2i) & -3-9i+4i+2i^2 & (-1)^2 - (3i)^2 + (4)^2 - (2i)^2 \end{bmatrix}_{3\times3}$$

$$B = \begin{bmatrix} 30 & 6-8i & 20i-22 \\ 6+8i & 9-i^2 & 5i-5 \\ -20i-22 & -5i-5 & 30 \end{bmatrix}_{3\times3}$$

$$\bar{B} = \begin{bmatrix} 30 & 6+8i & -20i-22 \\ 6-8i & 9-i^2 & -5i-5 \\ 20i-22 & 5i-5 & 30 \end{bmatrix}_{3\times3}$$

$$(\bar{B})^T = \begin{bmatrix} 30 & 6-8i & 20i-22 \\ 6+8i & 9+i^2 & 5i-5 \\ -20i-22 & -5i-5 & 30 \end{bmatrix}$$

Inverse of the matrix using elementary transform -ation.

Q: find the inverse using elementary transformation.

Solⁿ:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

$A = IA$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \rightarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

$R_1 \rightarrow R_1 - 2R_2$ $\quad$ $R_3 \rightarrow R_3 + 5R_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$$

$R_3 \rightarrow \frac{1}{2} R_3$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5/2 & -3/2 & 1/2 \end{bmatrix} A$$

$R_1 \rightarrow R_1 + R_3$

$R_2 \rightarrow R_2 - 2R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}}_{A^{-1}} A$$

because $\boxed{A \cdot A^{-1} = I}$

Q: find inverse using elementary transformation.

$$\therefore A = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$$

Soln:

$$\begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 1 & 3 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 - R_1$ $\quad$ $R_4 \to R_4 + R_1$

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 2 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - R_3$

$$\begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - R_2$ $\quad$ $R_4 \to R_4 - 2R_2$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & -2 & 2 & 0 \\ -2 & -1 & 2 & 1 \end{bmatrix} A$$

$R_1 \to R_1 + R_3$ $\quad$ $R_4 \to R_4 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -2 & 3 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & -2 & 2 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - R_4$ $\quad$ $R_3 \to R_3 - 3R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix} A$$

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H.W

Q: find inverse using elementary transformations –

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

Solⁿ: $$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

Rank of the matrix

- **Rank:** No. of linear independent solⁿ.

* There are three methods to find the rank of the matrix –

i) Determinant Method

If $A = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}_{3\times3}$

* If $|A|_{3\times3} \neq 0$ (Rank is 3)
* If $|A|_{3\times3} = 0$ (Rank is less than 3)
Here Rank will be 2.

Choosing minor of A → [Atleast one minor is non zero for rank 2]

Check two minors at least

- $|A|_{2\times2} \neq 0$ (Rank is 2)
- $|A|_{2\times2} = 0$ (Rank is 1)

* Doubt: What will we do if we have to find Rank of 4×4 matrix by determinant method.

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* Note: Only apply for finding the value of unknown in given matrix.

- eg: find the value of 'p' if rank is 1.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & p & 2 \end{bmatrix}_{3\times3}$$

Solⁿ: Minor must be zero for rank 1

$\therefore \begin{vmatrix} 5 & 6 \\ p & 2 \end{vmatrix} = 0$

or $10 - 6p = 0$
or $6p = 10$
$p = 10/6$
$\boxed{p = \frac{5}{3}}$

Q: Find the value of b if –

$$A = \begin{bmatrix} 3 & b & b \\ b & 3 & b \\ b & b & 3 \end{bmatrix}_{3\times3}$$ is of rank 1.

Solⁿ: Choosing minor of order 2×2

$\begin{vmatrix} 3 & b \\ b & 3 \end{vmatrix} = 0 \Rightarrow 9 - b^2 = 0$

$\boxed{b = \pm 3}$

Doubt

Q: find the value of μ if -

$$A = \begin{bmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & 11 & -6 & 1 \end{bmatrix}$$ is equal to rank 3

Sol^n:

$$|A| = \begin{vmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & 11 & -6 & 1 \end{vmatrix}$$

* Expanding along R_1 -

$$= \mu \begin{vmatrix} \mu & -1 & 0 \\ 0 & \mu & -1 \\ 11 & -6 & 1 \end{vmatrix} + 1 \begin{vmatrix} 0 & -1 & 0 \\ 0 & \mu & -1 \\ -6 & -6 & 1 \end{vmatrix} + 0 + 0$$

$$= \mu \left[\mu(\mu-6) + 1(11) \right] + 1 \left[0 + 1(-6) \right]$$

$$= \mu \left[\mu^2 - 6\mu + 11 \right] - 6$$

$$= \boxed{\mu^3 - 6\mu^2 + 11\mu - 6 = 0}$$

• put $\mu = 1$, we get -

$$(1)^3 - 6(1)^2 + 11(1) - 6$$
$$= 1 - 6 + 11 - 6 = 0$$

$\therefore$ $(\mu - 1)$ will be a factor of given eq^n:

vijeta

* if finding value of unknown use 1st method.

$$\mu - 1 \,)\, \mu^3 - 6\mu^2 + 11\mu - 6 \,(\, \mu^2 - 5\mu + 6$$
$$\mu^3 - \mu^2$$
$$(-)\ (+)$$
$$-5\mu^2 + 11\mu - 6$$
$$-5\mu^2 + 5\mu$$
$$(+)\ (-)$$
$$6\mu - 6$$
$$6\mu - 6$$
$$(-)\ (+)$$
$$\times$$

$$\therefore \mu^3 - 6\mu^2 + 11\mu - 6 = (\mu-1)(\mu^2 - 5\mu + 6)$$
$$0 = (\mu-1)[\mu^2 - 3\mu - 2\mu + 6]$$
$$0 = (\mu-1)[\mu(\mu-3) - 2(\mu-3)]$$
$$0 = (\mu-1)(\mu-2)(\mu-3)$$

$\therefore$ $\boxed{\mu = 1, 2, 3}$ $\Rightarrow$ $\boxed{\mu \notin R - \{1, 2, 3\}}$

• generally use this method (U) Imp

Echlone Form $\rightarrow$ (Upper Triangular Matrix) Diagonal elements should be non-zero

• Number of non-zero row in Echlone form is called 'Rank'.

$\Rightarrow$ No Requirement of square matrix for calculating rank.

Q: Find the rank of matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

vijeta

Solⁿ:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}_{3\times3}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Rank is } 2$$

Q: Find the rank of matrix

$$\begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$$

vijeta

Solⁿ:

$$\begin{bmatrix} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 5 & 2 & 0 & -1 \\ 2 & 3 & 4 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$$

$R_1 \to R_1 + R_3$

$$\begin{bmatrix} 1 & 7 & 12 & -2 \\ 2 & 3 & 4 & -1 \\ -4 & 5 & 12 & -1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 7 & 12 & -2 \\ 0 & -11 & -20 & 3 \\ -4 & 5 & 12 & -1 \end{bmatrix}$$

$R_3 \to R_3 + 4R_1$

$$\begin{bmatrix} 1 & 7 & 12 & -2 \\ 0 & -11 & -20 & 3 \\ 0 & 33 & 60 & -9 \end{bmatrix}$$

vijeta

$R_3 \to R_3 + 3R_2$

$$\begin{bmatrix} 1 & 7 & 12 & -2 \\ 0 & -11 & -20 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Rank is } 2$$

Done

Q: Find the rank of Matrix.

$$\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Solⁿ: $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - 6R_1$

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 1 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Solⁿ

$R_1 \leftrightarrow R_2$

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - 6R_1$

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_2 \to R_2 - R_3$

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_3 \to R_3 + 3$

$$\begin{bmatrix} 1 & 7 & 12 \\ 0 & -11 & -2c \\ 0 & 0 & 0 \end{bmatrix}$$

Done

Q: Find the rank

$$\begin{bmatrix} 2 \\ 1 \\ 3 \\ 6 \end{bmatrix}$$

Sol^n:

R_1

$R_4 \to R_4 - 9R_2$
$R_3 \to R_3 - 4R_2$

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$R_4 \to R_4 - 2R_3$

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Rank is } 3$$

vijeta

$R_2 \to R_2$
$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - 6R_1$

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 1 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

vijeta

$R_3 \to 5R_3$

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 1 & 7 \\ 0 & 20 & 45 & 50 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_4 \to 5R_4$

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 1 & 7 \\ 0 & 20 & 45 & 50 \\ 0 & 45 & 60 & 85 \end{bmatrix}$$

$R_3 \to R_3 - 4R_2$
$R_4 \to R_4 - 9R_2$

$$\begin{array}{r} 85 \\ 63 \\ \hline 22 \end{array}$$

$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 1 & 7 \\ 0 & 0 & 41 & 22 \\ 0 & 0 & 51 & 22 \end{bmatrix}$$

$\cancel{R_3 \to R_3 - 41R_2}$
$\cancel{R_4 \to R_4 - 51R_2}$

$$\Big[\quad \Big]$$

vijeta

We can use both column & row operations both.

iii) Normal Form (Canonical form)

To convert into Identity Matrix.

Done

Q: $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$

Solⁿ:

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 + R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}$$

$C_2 \to C_2 + (C_3 + C_4)$

$$\begin{bmatrix} 1 & 5 & -1 & 4 \\ 0 & 1 & 5 & -4 \\ 0 & 4 & 4 & 0 \\ 0 & 2 & 5 & -3 \end{bmatrix}$$

$R_3 \to R_3 - 4R_2$
$R_4 \to R_4 - 2R_2$

$$\begin{bmatrix} 1 & 5 & -1 & 4 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & -16 & 16 \\ 0 & 0 & -3 & -3 \end{bmatrix}$$

$R_1 \to R_1 - 5R_2$

$$\begin{bmatrix} 1 & 0 & -6 & 24 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & -16 & 8 \\ 0 & 0 & -3 & -3 \end{bmatrix}$$

$R_3 \to \frac{-1}{16} R_3$

$$\begin{bmatrix} 1 & 0 & -6 & 24 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & -3 & -3 \end{bmatrix}$$

$R_1 \to R_1 + 6R_3$
$R_2 \to R_2 - 5R_3$
$R_4 \to R_4 + 3R_3$

$-4 + \frac{5}{2}$

$\frac{-8+5}{2}$

$-3 - \frac{3}{2}$

$\frac{-6-3}{2}$

$= \frac{-9}{2}$

$$\begin{bmatrix} 1 & 0 & 0 & 21 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & -9/2 \end{bmatrix}$$

$R_4 \to -\frac{2}{9} R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 21 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - 21R_4$

$R_2 \to R_2 + \frac{3}{2}R_4$

$R_3 \to R_3 + \frac{1}{2}R_4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Ans: Rank of $A = 3$

$$\left[\begin{array}{c|c} I_{3\times 3} & O \\ \hline O & O \end{array}\right]$$

Normal Form (Canonical Form).

Practice questions

Q1: $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 4 \\ -1 & 2 & 3 & 4 \\ -1 & -2 & 6 & -7 \end{bmatrix}$

Sol^n: $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$R_4 \to R_4 + R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -4 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 5 & -3 \end{bmatrix}$$

$C_3 \leftrightarrow C_2$

$$\begin{bmatrix} 1 & -1 & 2 & 4 \\ 0 & 5 & 0 & -4 \\ 0 & 4 & 0 & 0 \\ 0 & 5 & 0 & -3 \end{bmatrix}$$

$C_2 \to C_2 + C_4$

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & 0 & -4 \\ 0 & 4 & 0 & 0 \\ 0 & 2 & 0 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 2 & 4 \\ 0 & 1 & 0 & -4 \\ 0 & 4 & 0 & 0 \\ 0 & 2 & 0 & -3 \end{bmatrix}$$

$R_1 \to R_1 - 3R_2$
$R_3 \to R_3 - 4R_2$
$R_4 \to R_4 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 2 & 16 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 16 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$C_3 \leftrightarrow C_4$

$$\begin{bmatrix} 1 & 0 & 16 & 2 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 16 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 3R_4$

$$\begin{bmatrix} 1 & 0 & 16 & 2 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

R_1

$R_1 \to R_1 - 16R_3$
$R_2 \to R_2 + 4R_3$
$R_4 \to R_4 - 5R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

<u>OR</u>

$$\left[\begin{array}{c|c} I_{3\times 3} & O \\ \hline O & O \end{array}\right]$$

Rank of $A = 3$

Done

**

Q: Find the rank of the matrix reducing it into normal form

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 & 1 \\ 0 & 3 & 4 & 1 & 2 \end{bmatrix}$$

Solⁿ:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 & 1 \\ 0 & 3 & 4 & 1 & 2 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$A = \begin{bmatrix} 0 & 2 & 3 & 4 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 1 & 2 \end{bmatrix}$$

$C_4 \leftrightarrow C_1$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 4 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 4 & 1 & 0 \end{bmatrix}$$

** (Soln of this Question)

Q: Find the rank of Matrix by reducing it into normal form

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 & 1 \\ 0 & 3 & 4 & 1 & 2 \end{bmatrix}$$

Solⁿ: $R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 0 & 2 & 3 & 4 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 1 & 2 \end{bmatrix}$$

$C_5 \leftrightarrow C_1$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 4 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 4 & 1 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 4R_1$
$R_4 \rightarrow R_4 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & -7 & -10 & -13 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & -7 & 0 \end{bmatrix}$$

-86

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 4 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 4 & 1 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 4R_1$
$R_4 \rightarrow R_4 - 2R_1$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & -7 & -10 & -13 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & -7 & 0 \end{bmatrix}$$

$R_4 \leftrightarrow R_3$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & -7 & -10 & -13 & 0 \\ 0 & -1 & -2 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \rightarrow -R_3$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & -7 & -10 & -13 & 0 \\ 0 & 1 & 2 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 + 8R_3$

vijeta

$A = \begin{bmatrix} 1 & \cdots \\ 4 & \cdots \\ 0 & \cdots \\ 2 & \cdots \end{bmatrix}$

$R_2 \rightarrow R$ …
$R_4 \rightarrow R$ …

$A =$ …

$R_4 \leftrightarrow$ …

$A =$ …

$\frac{49}{13}$
36

$\therefore$

R …

$\therefore$

$R_2 \rightarrow R_2 + 8R_3$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & -7 & -10 & -13 & 0 \\ 0 & -1 & -2 & -7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \leftrightarrow -R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 2 & 7 & 0 \\ 0 & -7 & -10 & -13 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \rightarrow R_1 - 2R_2$
$R_3 \rightarrow R_3 + 7R_2$

$$\begin{bmatrix} 1 & 0 & -1 & -10 & 0 \\ 0 & 1 & 2 & 7 & 0 \\ 0 & 0 & 4 & 36 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \rightarrow \frac{R_3}{4}$

$$\begin{bmatrix} 1 & 0 & -1 & -10 & 0 \\ 0 & 1 & 2 & 7 & 0 \\ 0 & 0 & 1 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

vijeta

$R_1 \rightarrow R_1 + R_3$
$R_2 \rightarrow R_2 - 2R_3$

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -11 & 0 \\ 0 & 0 & 1 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \rightarrow C_4 + C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -11 & 0 \\ 0 & 0 & 1 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \rightarrow C_4 + 11C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \rightarrow C_4 - 9C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

vijeta

vijeta

$A = \begin{bmatrix} 1 & \\ 4 & \\ 1 & \\ & \end{bmatrix}$

$R_2 \rightarrow$

$R_4 \rightarrow$

$A =$

$R_2 \rightarrow R_2 + 8R_3$

$$\left[\begin{array}{ccc|cc} & I_{3\times3} & & 0 & 0 \\ & & & 0 & 0 \\ & & & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \boxed{\text{Rank of } A = 3}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 1 & 6 & 43 & 0 \\ 0 & 1 & 2 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$R_1 \rightarrow R_1 - 2R_2$

$$A = \begin{bmatrix} 1 & 0 & -9 & -82 & 0 \\ 0 & 1 & 6 & 43 & 0 \\ 0 & 0 & -4 & -36 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \rightarrow -\frac{R_3}{4}$

$$A = \begin{bmatrix} 1 & 0 & -9 & -82 & 0 \\ 0 & 1 & 6 & 43 & 0 \\ 0 & 0 & 1 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_4 \rightarrow R_4 + R_3$

$$A = \begin{bmatrix} 1 & 0 & -9 & -82 & 0 \\ 0 & 1 & 6 & 43 & 0 \\ 0 & 0 & 1 & 9 & 0 \\ 0 & 0 & 1 & 9 & 0 \end{bmatrix}$$

$A = \Big[$

Sol^n:

Q1:

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$
$R_4 \to R_4 - 4R_1$

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & -11 & 5 & -3 \\ 0 & -11 & 5 & -3 \end{bmatrix}$$

$C_2 \to C_2 + 2C_3$

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -5 & 3 \\ 0 & -1 & 5 & -3 \\ 0 & -1 & 5 & -3 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -5 & 3 \\ 0 & -1 & 5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

oom

al form

Solution of System of Linear Equation (Non-Homogeneous Equation)

Consider linear equations –

$$a_1x + a_2y + a_3z = d_1$$
$$b_1x + b_2y + b_3z = d_2$$
$$c_1x + c_2y + c_3z = d_3$$

or

earlier

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Now

- Augmented Matrix $(A|B)$

$$[A|B] = \left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & d_1 \\ b_1 & b_2 & b_3 & d_2 \\ c_1 & c_2 & c_3 & d_3 \end{array}\right]$$

$\rho \to$ Rank

* Find the $\underset{\text{rank}}{\rho}(A|B)$ by Echelon form

- Consistent – System of linear eq^n has a sol^n

$$\rho(A|B) = \rho(A)$$

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 4 & -1 \\ 0 & 1 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} C_3 &\to C_3 + 5C_2 \\ C_4 &\to C_4 - 3C_2 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 4 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} C_3 &\to C_3 - 4C_1 \\ C_4 &\to C_4 + C_1 \end{aligned}$$

$$\begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{bmatrix}$$

∴ Rank of above matrix is 2.

Soln

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & +2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 \leftrightarrow C_2$

$$\begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$R_1 \to R_1 - 3R_2$

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \qquad \begin{aligned} C_3 &\to C_3 - 5C_1 \\ C_4 &\to C_4 - 2C_1 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Solution of System of Linear Equation (Non-Homogeneous Equation)

Consider linear equations –

$a_1 x + a_2 y + a_3 z = d_1$
$b_1 x + b_2 y + b_3 z = d_2$
$c_1 x + c_2 y + c_3 z = d_3$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Matrix (A/B)

$$\left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & d_1 \\ b_1 & b_2 & b_3 & d_2 \\ c_1 & c_2 & c_3 & d_3 \end{array}\right]$$

* Find the $\underset{\text{rank}}{\rho}(A/B)$ by Echelon form

- Consistent – System of linear eqn has a soln

$\rho(A/B) = \rho(A)$

$C_1 \to C_1 + 2C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Find the rank reducing it into normal form

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix}$$

Find the rank reducing it into normal form

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -4 & 7 \\ -1 & -2 & -1 & -2 \end{bmatrix}$$

Solution of System of Linear Equation (Non-Homogeneous Equation)

Consider linear equations —

$$a_1x + a_2y + a_3z = d_1$$
$$b_1x + b_2y + b_3z = d_2$$
$$c_1x + c_2y + c_3z = d_3$$

or

earlier

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

Now • Augmented Matrix $(A|B)$

$$[A|B] = \left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & d_1 \\ b_1 & b_2 & b_3 & d_2 \\ c_1 & c_2 & c_3 & d_3 \end{array}\right]$$

$\rho \to$ Rank

* Find the $\underset{\text{rank}}{\rho}(A|B)$ by Echelon form

• Consistent — System of linear eq^n has a sol^n

$$\rho(A|B) = \rho(A)$$

- **Inconsistent** – No solution

$$\rho(A|B) \neq \rho(A)$$

System of linear Eq^n $(AX = B)$

- Consistent: $\rho(A|B) = \rho(A)$
  - Unique sol^n: $\rho(A|B) = \rho(A)$ = Number of Unknown
  - Infinite sol^n: $\rho(A|B) = \rho(A) <$ Number of Unknown
- Inconsistent: $\rho(A|B) \neq \rho(A)$

Q: Find the solution of system of linear equation –

$$3x + 3y + 2z = 1$$
$$x + 2y = 4$$
$$10y + 3z = -2$$
$$2x - 3y - z = 5$$

Sol^n:

$$[A|B] = \left[\begin{array}{ccc|c} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right]$$

Date

Q: Determine the value of λ, μ such that the system -

$$2x - 5y + 2z = 8$$
$$2x + 4y + 6z = 5$$
$$x + 2y + \lambda z = \mu$$

has i) no solution
ii) Unique solution
iii) Infinite solution

Soln:

$$[A|B] = \left[\begin{array}{ccc|c} 2 & -5 & 2 & 8 \\ 2 & 4 & 6 & 5 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$R_2 \to R_2 - R_1$ & $R_3 \to 2R_3 - R_1$

$$= \left[\begin{array}{ccc|c} 2 & -5 & 2 & 8 \\ 0 & 9 & 4 & -3 \\ 0 & 9 & 2\lambda-2 & 2\mu-8 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$= \left[\begin{array}{ccc|c} 2 & -5 & 2 & 8 \\ 0 & 9 & 4 & -3 \\ 0 & 0 & 2\lambda-6 & 2\mu-5 \end{array}\right]$$

Date

$$\left[\begin{array}{ccc|c} 2 & -5 & 2 & 8 \\ 0 & 9 & 4 & -3 \\ 0 & 0 & 2\lambda-6 & 2\mu-5 \end{array}\right]$$

i) No sol^n

$$2\lambda-6 = 0 \Rightarrow \boxed{\lambda=3}$$
$$2\mu-5 \neq 0 \Rightarrow \mu \neq \frac{5}{2}$$

ii) Unique sol^n

$$2\lambda-6 \neq 0 \Rightarrow \lambda \neq 3 ✓$$
$$\cancel{2\mu-5 \neq 0 \Rightarrow \mu \neq \frac{5}{2}} \times$$

nam ne yeh nhi likh -vayi

iii) Infinite

$$2\lambda-6 = 0 \Rightarrow \lambda = 3$$
$$2\mu-5 = 0 \Rightarrow \mu = \frac{5}{2}$$

Q8: Determine the value of 'K' the equation

$$x+y+z = 1$$
$$2x+y+4z = K$$
$$4x+y+10z = K^2$$

have a sol^n and solve them.

Sol^n:

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & K \\ 4 & 1 & 10 & K^2 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 4R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & K-2 \\ 0 & -3 & 6 & K^2-4 \end{array}\right]$$

$R_3 \to R_3 - 3R_2$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & K-2 \\ 0 & 0 & 0 & (K^2-4)-3(K-2) \end{array}\right]$$

$$= \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & K-2 \\ 0 & 0 & 0 & K^2-3K+2 \end{array}\right]$$

$\rho(A) = 2$

$\therefore \rho(B) = 2$

$\therefore K^2 - 3K + 2 = 0$

$K^2 - 2K - K + 2 = 0$

$K(K-2) - 1(K-2) = 0$

$\boxed{K = 1, 2}$

Case I: $K = 1$

$$x+y+z=1$$
$$2x+y+4z=1$$
$$4x+y+10z=1$$

Case II: $K=2$

$$x+y+z=1$$
$$2x+y+4z=2$$
$$4x+y+10z=4$$

OR

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & K-2 \\ 0 & 0 & 0 & K^2-3K+2 \end{array}\right]$$

Case I $K=1$

$$x+y+z=1 \quad \text{— (a)}$$
$$-y+2z=-1 \quad \text{— (b)}$$

Let $\boxed{z = K_1}$

$$-y+2K_1=-1$$
$$y-1=2K_1$$
$$\boxed{y=2K_1+1}$$

from (a)

$$x+2K_1+1+K_1=1$$
$$\boxed{x=-3K_1}$$

Case II $(K=2)$

$$x+y+z=1 \quad \text{— (a)}$$
$$-y+2z=0 \quad \text{— (b)}$$

Let $\boxed{z = K_2}$

$$-y+2K_2=0$$
$$\boxed{y=2K_2}$$

$$x+2K_2+K_2=1$$
$$\boxed{x=1-3K_2}$$

Q: Show that the equations –

$$-2x+y+z=a$$
$$x-2y+z=-b$$
$$x+y-2z=c$$

have no solution unless $a+b+c=0$: In which case they have infinitely many solⁿ when $a=1, b=1, c=2$

Solⁿ:

$$[A|B] = \left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 1 & -2 & 1 & -b \\ 1 & 1 & -2 & c \end{array}\right]$$

$\Rightarrow R_2 \to 2R_2 + R_1 \qquad R_3 \to 2R_3 + R_1$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 0 & -3 & 3 & -2b+a \\ 0 & 3 & -3 & 2c+a \end{array}\right]$$

vijeta

$R_3 \to R_3 + R_2$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & a \\ 0 & -3 & 3 & -2b+a \\ 0 & 0 & 0 & 2a-2b+2c \end{array}\right]$$

$$\begin{bmatrix} -2 & 1 & 1 & a \\ 0 & -3 & 3 & a-2b \\ 0 & 0 & 0 & 2(a-b+c) \end{bmatrix}$$

$$2(a-b+c) = 0$$
or $a - b + c = 0$

$\left.\begin{array}{l} c=3 \\ a=-1 \\ b=2 \end{array}\right\}$

Homogeneous Equation ($AX = 0$)

System of linear homogeneous equation $AX = 0$ Always has a solution

If $\rho(A) =$ Number of unknown $\Rightarrow$ Trivial solⁿ

If $\rho(A) <$ Number of unknown $\Rightarrow$ Non-Trivial solⁿ

eg: $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 3 & a \end{bmatrix}_{3\times3}$ Trivial solⁿ $|\ |_{3\times3} = 0$

for non Trivial $|\ |_{3\times3} \neq 0$

Q: Determine the value of b such that system of homogeneous equations –

$$2x + y + 2z = 0$$
$$x + y + 3z = 0$$
$$4x + 3y + bz = 0$$

has i) trivial solⁿ
ii) Non Trivial solⁿ

Solⁿ:

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & b \end{bmatrix}$$

or u can also use echlone form

i) Trivial Solⁿ

$$|A| = 2(b-9) - 1(b-12) + 2(3-4)$$
$$0 = 2b - 18 - b + 12 + (-2)$$
$$\boxed{b = 8}$$

ii) Non Trivial Solⁿ

$$|A|_{3\times3} \neq 0$$
so $b \neq 8$

Q: For what value of λ such that system of homogeneous equations –

$$2x - 2y + z = \lambda x$$
$$2x - 3y + 2z = \lambda y$$
$$-x + 2y + 0z = \lambda z$$

has i) Trivial solⁿ
ii) Non-Trivial

Solⁿ:

$$A = \begin{bmatrix} 2-\lambda & -2 & 1 \\ 2 & (-3-\lambda) & 2 \\ -1 & 2 & \lambda \end{bmatrix}$$

i) Trivial

$|A| = 0$

$(2-\lambda)(-3\lambda-4)$
$-6\lambda - 8 + 3\lambda^2 + 4\lambda$

$\therefore$ ~~$(2-\lambda)[-3\lambda-4] + 2(2\lambda+2) + 1(4-3) = 0$~~

~~$\Rightarrow (3\lambda^2 - 2\lambda - 8) + 4\lambda + 4 + 1 = 0$~~
~~$3\lambda^2 + 2\lambda - 3 = 0$~~

~~$\lambda = -2 \pm$~~

~~$\therefore (3\lambda^2 - 2\lambda - 8)$~~

$$\Rightarrow (2-\lambda)[\lambda(-3-\lambda) - 4] + 4\lambda + 4 + 1(4 + (-3-\lambda)) = 0$$

$$\Rightarrow (2-\lambda)[-3\lambda - \lambda^2 - 4] + 4\lambda + 4 + 4 - 3 - \lambda = 0$$

$(2-\lambda)(-3\lambda - \lambda^2 - 4)$

$\lambda^3 - \lambda^2 + 13\lambda - 3 = 0$ ✗

$(1) = 0$
$(\lambda - 1) = 0$
$\lambda = 2$

$\lambda^3 + \lambda^2 - 5\lambda + 3 = 0$

$$\begin{array}{r|l} (\lambda - 1) & \lambda^3 + \lambda^2 - 5\lambda + 3 \;(\; \lambda^2 + 2\lambda - 3 \\ & \lambda^3 - \lambda^2 \\ & (-)\;(+) \\ \hline & 2\lambda^2 - 5\lambda \\ & 2\lambda^2 - 2\lambda \\ & (-)\;(+) \\ \hline & -3\lambda + 3 \\ & -3\lambda + 3 \\ \hline & \times \end{array}$$

$$(\lambda^3 + \lambda^2 - 5\lambda + 3) = (\lambda - 1)(\lambda^2 + 2\lambda - 3)$$
$$= (\lambda-1)(\lambda^2 + 3\lambda - \lambda - 3)$$
$$= (\lambda - 1)(\lambda(\lambda+3) - 1(\lambda+3))$$
$$= (\lambda-1)(\lambda-1)(\lambda+3)$$

Rank

Revision

i) Determinant Method
ii) Echlone Form → only apply row operations
iii) Normal Form → row & column use both operations

Practice

Q1. For what value of λ & μ do the system of eq^n –

$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+\lambda z=\mu$$

have i) No solution
ii) Unique sol^n
iii) Infinite sol^n

Sol^n:

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$R_1 \to R_1 - R_2$
$R_3 \to R_3 - R_2$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & -\lambda & 12-\mu \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

i) No sol^n

$\lambda-3=0 \Rightarrow \lambda=3$
$\mu-10 \neq 0 \Rightarrow \mu \neq 10$

ii) Unique sol^n

$\lambda-3 \neq 0$
$\lambda \neq 3$

iii) Infinite sol^n

$\lambda-3=0 \Rightarrow \lambda=3$
$\mu-10=0 \Rightarrow \mu=10$

* Use of Eigen values & Eigen vectors

Characteristic Equation & [Eigen values & Eigen vectors]

$C.E = |A-\lambda I| = 0$

eg: for $A_{3\times3}$ $\Rightarrow$ $a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$
$\lambda = (\ ), (\ ), (\ ) \to$ Eigen values

Q1: Find Eigen value of Matrix A

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$$

Solⁿ: $|A - \lambda I| = 0$

$$\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & -3 & 1 \\ 3 & 1-\lambda & 3 \\ -5 & 2 & -4-\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -3 & 1 \\ 3 & 1-\lambda & 3 \\ -5 & 2 & -4-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left[(1-\lambda)(-4-\lambda) - 6\right] + 3\left[3(-4-\lambda) + 15\right] + 1\left[6 + 5(1-\lambda)\right]$$

$$= (2-\lambda)\left[-4 - \lambda + 4\lambda + \lambda^2 - 6\right] + 3\left[-12 + 15 - 3\lambda\right] + 1\left[6 + 5 - 5\lambda\right]$$

$$= (2-\lambda)\left[\lambda^2 + 3\lambda - 10\right] + 3\left[3 - 3\lambda\right] + 1\left[11 - 5\lambda\right]$$

$$= \left(2\lambda^2 - \lambda^3 + 6\lambda - 3\lambda^2 - 20 + 10\lambda\right) + 9 - 9\lambda + 11 - 5\lambda$$

$$\Rightarrow -\lambda^3 - \lambda^2 + 2\lambda = 0$$

or $\lambda^3 + \lambda^2 - 2\lambda = 0$

$\lambda(\lambda^2 + \lambda - 2) = 0$

$\lambda = 0$

or $\lambda^2 + \lambda - 2 = 0 \Rightarrow D = b^2 - 4ac$

$= (1)^2 - 4(1)(-2)$

$D < 0 = 1 + 8$

$= 9$

$\lambda = 0$

or $\lambda = \dfrac{-1 \pm \sqrt{9}}{2}$

$\lambda = 0, \dfrac{-1+\sqrt{9}}{2}, \dfrac{-1-\sqrt{9}}{2}$

$\lambda = 0, \dfrac{-1+3}{2}, \dfrac{-1-3}{2}$

$$\boxed{\lambda = 0, 1, -2}$$

OR

Trick

$\lambda^3 - (\text{sum of diagonal})\lambda^2 + (\text{minor } a_{11} + a_{22} + a_{33})\lambda - |A| = 0$

for forming eqⁿ in λ.

$$\Rightarrow \lambda^3 - (-1)\lambda^2 + \lambda\left[\begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix}\right] - |A| = 0$$

~~How to solve~~

10 marks

Q: Find the eigen value & Eigen vector of

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Sol^n:

$$A-\lambda I = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}$$

so $|A-\lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)\left[\lambda(\lambda-1)-12\right]-2\left[-2\lambda-6\right]-3\left[-4+(1-\lambda)\right]=0$$

$$(-2-\lambda)\left[\lambda^2-\lambda-12\right]+4\lambda+12+12-3+3\lambda=0$$

$$-2\lambda^2-\lambda^3+2\lambda+\lambda^2+24+12\lambda+4\lambda+24-3+3\lambda=0$$

$$-\lambda^3-\lambda^2+21\lambda+45=0$$

or $\lambda^3+\lambda^2-21\lambda-45=0$

$\therefore (\lambda+3)$ is a factor

$-8+4+42-45=0$ $(\lambda=2)$

$\cancel{-1+1+21}$

(-3) $-27+9-21\times-3-45$

$-27-45+9+63=0$

* We can also use Synthetic Division Rule Also

$$\lambda+3\ \overline{)\ \lambda^3+\lambda^2-21\lambda-45\ }\ \lambda^2-2\lambda-15$$

$\lambda^3+3\lambda^2$

$(-)\ (-)$

$-2\lambda^2-21\lambda$

$-2\lambda^2-6\lambda$

$(+)\ (+)$

$-15\lambda-45$

$$\lambda^3+\lambda^2-21\lambda-45=(\lambda^2-2\lambda-15)(\lambda+3)$$

$\Rightarrow (\lambda+3)(\lambda^2-2\lambda-15)=0$

$\Rightarrow \lambda=-3$

$\lambda^2-2\lambda-15=0$

$3 \mid 15$, $5 \mid 5$

$\lambda^2-5\lambda+3\lambda-15=0$

$\lambda(\lambda-5)+3(\lambda-5)=0$

$(\lambda+3)(\lambda-5)=0$

$\boxed{\lambda = -3, -3, 5}$ (Eigen Value)

Synthetic Division

	1	1	-21	-45
-3	0	-3	6	45
	1	-2	-15	$\boxed{0}$

Another trick: while using hit & trial we use value of x for finding first root of cubic eq^n is one among the factors of constant term.

3	45
3	15
5	5
	1

$\therefore$ Ist factor will be getted by putting $\lambda = 3, 5$ (-terms)

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$$AX = \lambda X$$
$$AX - \lambda X = 0$$
$$(A - \lambda I)X = 0$$

for

Eigen Vector

$$(A - \lambda I)X = 0$$

- for $\lambda = -3$

$$A - \lambda I = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix}$$

$$A + 3I = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}$$

$$(A+3I)X = 0$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 - 3x_3 = 0 \text{ — (i)}$$
$$2x_1 + 4x_2 - 6x_3 = 0 \text{ — (ii)}$$
$$-x_1 - 2x_2 + 3x_3 = 0 \text{ — (iii)}$$

All same → Infinite solⁿ

only with eqⁿ (i)

$\therefore \quad x_1 + 2x_2 - 3x_3 = 0$

Let $x_1 = K_1$
$x_2 = K_2$

$$K_1 + 2K_2 - 3x_3 = 0$$
$$3x_3 = K_1 + 2K_2$$
$$x_3 = \frac{K_1 + 2K_2}{3}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} K_1 \\ K_2 \\ \frac{K_1+2K_2}{3} \end{bmatrix} = \begin{bmatrix} 3K_1 \\ 3K_2 \\ K_1+2K_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = K_1 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + K_2 \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

Same with $\lambda = 5$

$$A - 5I = \begin{bmatrix} -2-5 & 2 & -3 \\ 2 & 1-5 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

$$(A-5I)X = 0 \Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-7x_1 + 2x_2 - 3x_3 = 0 \text{ — (i)}$$
$$2x_1 - 4x_2 - 6x_3 = 0 \text{ — (ii)}$$
$$-x_1 - 2x_2 - 5x_3 = 0 \text{ — (iii)}$$

Let $x_1 = K_1$
& $x_2 = K_2$

With eq^n (1)

$\therefore \quad -7k_1 + 2k_2 - 3x_3 = 0$

$$3x_3 = -7k_1 + 2k_2$$

$$\left(x_3 = \frac{-7k_1 + 2k_2}{3} \right)$$

No need to do

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \frac{-7k_1+2k_2}{3} \end{bmatrix} = \begin{bmatrix} 3k_1 \\ 3k_2 \\ -7k_1+2k_2 \end{bmatrix}$$

or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k_1 \begin{bmatrix} 3 \\ 0 \\ -7 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

$$-7x_1 + 2x_2 - 3x_3 = 0 \quad \text{---(i)}$$

$$\checkmark\ 2x_1 - 4x_2 - 6x_3 = 0 \quad \text{---(ii)}$$

$$\checkmark\ -x_1 - 2x_2 - 5x_3 = 0 \quad \text{---(iii)}$$

Taking last two

$$\frac{x_1}{-4 \times -5 - (-2)(-6)} = \frac{-x_2}{-10-6} = \frac{x_3}{-4-4} = k_1$$

$$x_1 = 8k_1$$

$$x_2 = 16k_1$$

$$x_3 = -8k_1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 8k_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Q: Find Eigen value & Eigen vector

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Sol^n:

$$A - \lambda I = \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}$$

So, $|A - \lambda I| = 0$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)\left[(3-\lambda)^2 - 1\right] + 2\left[2(\lambda-3)+2\right] + 2\left[2 - 2(3-\lambda)\right] = 0$$

$$(6-\lambda)\left[9 + \lambda^2 - 6\lambda - 1\right] + 2\left[2\lambda - 4\right] + 2\left[2\lambda - 4\right] = 0$$

$$\Rightarrow 54 - 9\lambda + 6\lambda^2 - \lambda^3 - 36\lambda + 6\lambda^2 - 6 + \lambda + 4\lambda - 8 + 4\lambda - 8 = 0$$

$$\Rightarrow -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$$

$$\text{or} \quad \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$8 - 48 - 72 + 32$

$8 - 48 + 72$

54

$8 - 48 + 72 - 32$

2	32
2	16
2	8
2	4
2	2
	1

$(\lambda - 2)\ \overline{)\ \lambda^3 - 12\lambda^2 + 36\lambda - 32\ }\ (\lambda^2 - 10\lambda + 16$

$\lambda^3 - 2\lambda^2$

$(-)\ (+)$

$-10\lambda^2 + 36\lambda$

$-10\lambda^2 + 20\lambda$

$(+)\ (-)$

$16\lambda - 32$

$16\lambda - 32$

$\times$

$$= (\lambda-2)(\lambda^2-10\lambda+16)$$

$$= (\lambda-2)(\lambda^2-8\lambda-2\lambda+16)$$

$$= (\lambda-2)[\lambda(\lambda-8)-2(\lambda-8)]$$

$$0 = (\lambda-2)(\lambda-2)(\lambda-8)$$

$\therefore$ $\boxed{\lambda = 2, 2, 8}$

for $\lambda = 2$

$$A - 2I = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}$$

$$(A-2I)X = 0$$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x_1 - 2x_2 + 2x_3 = 0 \quad \text{— (I)}$$
$$-2x_1 + x_2 - x_3 = 0 \quad \text{— (II)}$$
$$2x_1 - x_2 + x_3 = 0 \quad \text{— (III)}$$

{Identical}

$\therefore$ Let $4x_1 - 2x_2 + 2x_3 = 0$

let $x_1 = k_1$

& $x_2 = k_2$

$$4k_1 - 2k_2 + 2x_3 = 0$$

$$2x_3 = -4k_1 + 2k_2$$

$$x_3 = \frac{-4k_1 + 2k_2}{2}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \frac{-4k_1+2k_2}{2} \end{bmatrix} = \begin{bmatrix} 2k_1 \\ 2k_2 \\ -4k_1+2k_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k_1 \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

$\therefore$ with $\lambda = 8$

$$A - 8I = \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}$$

By $[A-8I]X = 0$

$$\therefore -2x_1 - 2x_2 + 2x_3 = 0 \quad \text{— (I)}$$
$$-2x_1 - 5x_2 - x_3 = 0 \quad \text{— (II)}$$
$$2x_1 - x_2 - 5x_3 = 0 \quad \text{— (III)}$$

$$\frac{x_1}{\begin{matrix} -2 & 2 \\ -5 & -1 \end{matrix}} = \frac{x_2}{\begin{matrix} 2 & -2 \\ -1 & -2 \end{matrix}} = \frac{x_3}{\begin{matrix} -2 & -2 \\ -2 & -5 \end{matrix}} = k_1$$

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4} = k_1$$

$$x_1 = 12k_1$$
$$x_2 = -6k_1$$
$$x_3 = 6k_1$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_3 \\ x_3 \end{bmatrix} = 6k_1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Q: Find Eigen values & Eigen vector if

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$$

Solⁿ:

$$|A - \lambda I| = 0 \qquad = \begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix}$$

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(2-\lambda)(5-\lambda) - 0 + 4(0)\right] = 0$$

$$\therefore \boxed{\lambda = 3, 2, 5}$$

for $\lambda = 3$

$$A - 3I = \begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

$$(A - 3I)X = 0$$

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_2 + 4x_3 = 0 \quad \text{—— (i)}$$
$$-x_2 + 6x_3 = 0 \quad \text{—— (ii)}$$
$$2x_3 = 0 \quad \text{—— (iii)}$$

$x_3 = 0$, $x_2 = 0$ & $x_1 = K_1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} K_1 \\ 0 \\ 0 \end{bmatrix} = K_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

for $\lambda = 2$

$$(A - 2I)X = 0$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_2 + 4x_3 = 0 \quad \text{—— (i)}$$
$$6x_3 = 0 \quad \text{—— (ii)}$$
$$3x_3 = 0 \quad \text{—— (iii)}$$

$x_3 = 0$

& $x_1 + x_2 = 0$

$x_1 = -x_2$

let $x_1 = K_1$

$\therefore$ $x_2 = -K_1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} K_1 \\ -K_1 \\ 0 \end{bmatrix} = K_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

for $\lambda = 5$

$(A - 5I)X = 0$

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_1 + x_2 + 4x_3 = 0 \quad \text{--- (I)}$

$-3x_2 + 6x_3 = 0 \quad \text{--- (II)}$

~~put $x_1 = k_1$~~

~~$x_2 = k_2$~~

~~$\therefore \quad -2k_1 + k_2 + 4x_3 = 0$~~

~~$4x_3 = \frac{-k_2 + 2k_1}{1}$~~

~~$x_3 = \frac{2k_1 - k_2}{4}$~~

~~$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ \frac{2k_1 - k_2}{4} \end{bmatrix} = \begin{bmatrix} 4k_1 \\ 4k_2 \\ 2k_1 - k_2 \end{bmatrix}$$~~

~~$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k_1 \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}$$~~

put $x_3 = k_3$

$+3x_2 = +6k_3$

$x_2 = 2k_3$

$\therefore \quad -2x_1 + 2k_3 + 4k_3 = 0$

$-2x_1 = -6k_3$

$\boxed{x_1 = 3k_3}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3k_3 \\ 2k_3 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Q: Find Eigen value & Eigen vector if

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Sol^n:

$|A - \lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (8-\lambda)\left[(7-\lambda)(3-\lambda) - 16\right] + 6\left[6(\lambda-3) + 8\right] + 2\left[24 - 2(7-\lambda)\right] = 0$

$\Rightarrow (8-\lambda)\left[21 - 10\lambda + \lambda^2 - 16\right] + 6\left[6\lambda - 10\right] + 2\left[24 - 14 + 2\lambda\right] = 0$

$\Rightarrow (8-\lambda)\left[\lambda^2 - 10\lambda + 5\right] + 36\lambda - 60 + 48 - 28 + 4\lambda = 0$

$\Rightarrow 8\lambda^2 - \lambda^3 - 80\lambda + 10\lambda^2 + 40 - 5\lambda + 36\lambda - 60 + 48 - 28 + 4\lambda = 0$

$\Rightarrow -\lambda^3 + 18\lambda^2 - 45\lambda = 0$

$\Rightarrow \quad \lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\lambda(\lambda^2 - 18\lambda + 45) = 0$

$\therefore \quad \lambda = 0$

or $\quad \lambda^2 - 18\lambda + 45 = 0$

$$\begin{array}{r|l} 3 & 45 \\ 3 & 15 \\ 5 & 5 \\ & 1 \end{array}$$

$\lambda^2 - 15\lambda - 3\lambda + 45 = 0$

$\lambda(\lambda - 15) - 3(\lambda - 15) = 0$

$(\lambda - 3)(\lambda - 15) = 0$

$$\boxed{\lambda = 0, 3, 15}$$

for $\lambda = 3$

$(A - 3I)X = 0$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5x_1 - 6x_2 + 2x_3 = 0$

$-6x_1 + 4x_2 - 4x_3 = 0$

$2x_1 - 4x_2 = 0$

$$\frac{x_1}{\begin{smallmatrix} -6 & 2 \\ 4 & -4 \end{smallmatrix}} = \frac{x_2}{\begin{smallmatrix} 2 & 5 \\ -4 & -6 \end{smallmatrix}} = \frac{x_3}{\begin{smallmatrix} 5 & -6 \\ -6 & 4 \end{smallmatrix}} = k_1$$

$$\frac{x_1}{24 - 8} = \frac{x_2}{-12 + 20} = \frac{x_3}{20 - 36} = k_1$$

$x_1 = 16k_1$

$x_2 = 8k_1$

$x_3 = -16k_1$

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$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16k_1 \\ 8k_1 \\ -16k_1 \end{bmatrix} = k_1 \begin{bmatrix} 16 \\ 8 \\ -16 \end{bmatrix} = 8k_1 \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

for $\lambda = 15$

$(A - 15I)X = 0$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-7x_1 - 6x_2 + 2x_3 = 0$ — (I)

$-6x_1 - 8x_2 - 4x_3 = 0$ — (II)

$2x_1 - 4x_2 - 12x_3 = 0$ — (III)

$$\frac{x_1}{\begin{smallmatrix} -6 & 2 \\ -8 & -4 \end{smallmatrix}} = \frac{x_2}{\begin{smallmatrix} 2 & -7 \\ -4 & -6 \end{smallmatrix}} = \frac{x_3}{\begin{smallmatrix} -7 & -6 \\ -6 & -8 \end{smallmatrix}} = k_1$$

$$\frac{x_1}{24 + 16} = \frac{x_2}{-12 - 28} = \frac{x_3}{56 - 36} = k_1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 40k_1 \\ -40k_1 \\ 20k_1 \end{bmatrix} = k_1 \begin{bmatrix} 40 \\ -40 \\ 20 \end{bmatrix} = 20k_1 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$\frac{28}{12}$

for $\lambda = 0$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$8x_1 - 6x_2 + 2x_3 = 0$ — (I)

$-6x_1 + 7x_2 - 4x_3 = 0$ — (II)

$2x_1 - 4x_2 + 3x_3 = 0$ — (III)

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$$\frac{x_1}{\begin{matrix}-6 & 2\\ 7 & -4\end{matrix}} = \frac{x_2}{\begin{matrix}2 & 8\\ -4 & -6\end{matrix}} = \frac{x_3}{\begin{matrix}8 & -6\\ -6 & 7\end{matrix}} = k_1$$

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36} = k_1$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k_1 \begin{bmatrix} 10 \\ 20 \\ 20 \end{bmatrix} = 10k_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Diagonal Matrix

eg

$\lambda = 0, 3, 15$

Eigen vector

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

* Modal Matrix, $M = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

$$D = M^{-1}AM = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

Properties of Eigen Value

i) Sum of Eigen values is equal to the sum of principle diagonal.

eg:
$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 2 & 5 \end{bmatrix}$$

find sum of Eigen value

$= 1+1+5 = 7$ $\boxed{\text{(trace)}}$

ii) The product of Eigen values is equal to the determinant of the matrix.

eg: if product of eigen values is 16. Find the third eigen value & determinant is 32

Soln:
$$\lambda_1 \lambda_2 \lambda_3 = |A| = 32$$
$$\downarrow$$
$$16\,\lambda_3 = 32$$
$$\boxed{\lambda_3 = 2}$$

iii) If λ be the eigen value of matrix A then $\frac{1}{\lambda}$ be the eigen value of A^{-1}.

eg: eigen values of A = 2, 3, 4 then eigen values of A^{-1}?

Soln: eigen values of $A^{-1} = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$

iv) If $\lambda_1, \lambda_2, \ldots \lambda_m$ are the eigen values of matrix A then A^n has eigen values – $\lambda_1^n, \lambda_2^n, \lambda_3^n, \ldots, \lambda_m^n$.

eg: A = 2, 3, 4 (eigen values)

$A^3 = 8, 27, 64$ (eigen values)

v) Matrix A & A^T has same eigen values (Eigen values)

vi) The characteristic root (Latent Root) of Hermitian Matrix are all real.

vii) Characteristic roots of Unitary Matrix are unit modulus.

$$|\lambda| = 1$$

eg:
$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

& Eigen values of A are equal & $\frac{1}{5}$ times of the third. Find the Eigen values.

Soln:

$\cancel{\lambda_1 + \lambda_2 = \frac{\lambda_3}{5}}$ $\quad$ $\cancel{\lambda_1 + \lambda_2 = 7 - \lambda_3}$

$\cancel{7 - \lambda_3 = \frac{\lambda_3}{5}} \Rightarrow \cancel{35 - 5\lambda = \lambda}$

$\cancel{6\lambda = 35}$

$\cancel{\lambda = \frac{35}{6}}$

$$\frac{1}{5}\lambda_3 + \frac{1}{5}\lambda_3 + \lambda_3 = 2+3+2$$

$$\frac{7\lambda_3}{5} = 7$$

$$\boxed{\lambda_3 = 5}$$

$$\Rightarrow \lambda_1 = \lambda_2 = 1$$

Q: For singular matrix of order 3, 2 & 3 are the eigen values find the third eigen value.

Solⁿ: $|A| = 0$

$$\lambda_1 \lambda_2 \lambda_3 = 0$$
$$2 \times 3 \times \lambda_3 = 0$$
$$\lambda_3 = 0$$