Formal Model-based Validation for Tally Systems

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Abstract. Existing commercial and open source e-voting systems have horrifically poor testing frameworks. Most tally systems, for example, are tested by re-running all past elections and seeing if the new system gives the same answer as an older, perhaps erroneous, system did. This amounts to a few dozen system tests and, typically, few-to-no unit tests. These systems are used today in a dozen countries to determine the outcome of national elections. This state-of-affairs cannot continue because it calls into question the legitimacy of elections in major European and North American democracies.
In this work, the ballot counting process for one of the most complex electoral schemes used in the world, Proportional Representation by Single Transferable Vote (PR-STV), is mechanically formally modeled. The purpose of such a formalization is to generate, using an algorithm of our design, a complete set of non-isomorphic test cases *per electoral scheme*, once and for all. Using such a system test suite, any digital election technology (proprietary or open source) can be rigorously evaluated for correctness. Doing so will vastly improve the confidence experts have—and can only improve the level of trust citizens have—in these digital elections systems.

1 Introduction

The electoral process consists of various different stages, from voter registration, through vote casting and tallying, to the final declaration of results. Some, but not all, aspects of the election process are apparently suitable for automation. For example, voter registration records can be stored in computer databases, and ballot counting can be done by machine. However, many attempts to introduce electronic counting of ballots have failed, or at least received much criticism, due to software and hardware errors, *including potential counting errors*, many of which are avoided through the appropriate use of formal methods and careful testing. The security aspects of elections, including voter privacy and election integrity, are an important concern, but are beyond the scope of this paper.

One of the potential advantages from automation is the *accuracy of vote counting*, so it is important to be able to prove that software can actually count ballots more accurately than the manual, labor-intensive process of counting paper ballots by hand, especially for complex voting schemes. Measured error rates for manual tallying of even simple electoral schemes range from around 0.5% to 1.5%. Mechanical tallying of (well-formed, unadulterated) digital ballots must have an error rate of 0%.

In this paper we focus mainly on the Irish voting scheme, as a case study, as it is one of the most complex electoral schemes in the world. By virtue of the design of this scheme and the manner in which we formalize it, we also mechanize two other popular voting schemes. We use the Alloy model finder [9] to describe the elections in terms of scenarios, consisting of equivalence classes of possible outcomes for each candidate in the election, where each outcome represents one branch through the algorithm. We show how test data is generated from a first-order logic representation of the counting algorithm using the Alloy model finder. This algorithm guarantees that we find the smallest number of ballots needed to test each scenario.

1.1 The Irish Voting Scheme

The Republic of Ireland uses Proportional Representation by Single Transferable Vote (PR-STV) for its national, local and European elections. Ireland uses Instant Runoff Voting (IRV) for its presidential elections and for by-elections to fill casual vacancies in Dáil Éireann. PR-STV is a multi-seat ranked choice voting system in which each voter ranks the candidates from first to last preference. IRV is PR-STV with just one remaining seat.

Manual recounts are often called for closely contested seats, as the results often vary slightly, indicating small errors in the manual process of counting votes. Paper-based voting with counting by hand is popular in Ireland, and recent attempts at automation were frustrated by subtle logic errors in the ballot counting software [2]. The potential for logic errors exist, in part, due to the complexities and idiosyncrasies with regard to tie breaking, especially involving the rounding up or down of vote transfers.

There has been some desire in Ireland to simplify matters. Referenda to introduce plurality (first past the post, where the candidate with the most votes is the winner, as is used in the U.S.A. and the U.K.) voting were rejected twice by the Irish electorate, once in 1959 and again in 1968 [20]. Since then, there have been no further legislative proposals to change the voting scheme used in Ireland.

The following are selected quotes from the Irish Commission on Electronic Voting (CEV) report on the previous electronic voting system used in Ireland (emphasis added) [3]:

- Design weaknesses, including *an error in the implementation of the count rules that could compromise the accuracy of an election*, have been identified and these have reduced the Commission's confidence in this software.
- The achievement of the full potential of the chosen system in terms of secrecy and accuracy depends upon a number of software and hardware modifications, both major and minor, and more significantly, is dependent on the *reliability of its software being adequately proven.*

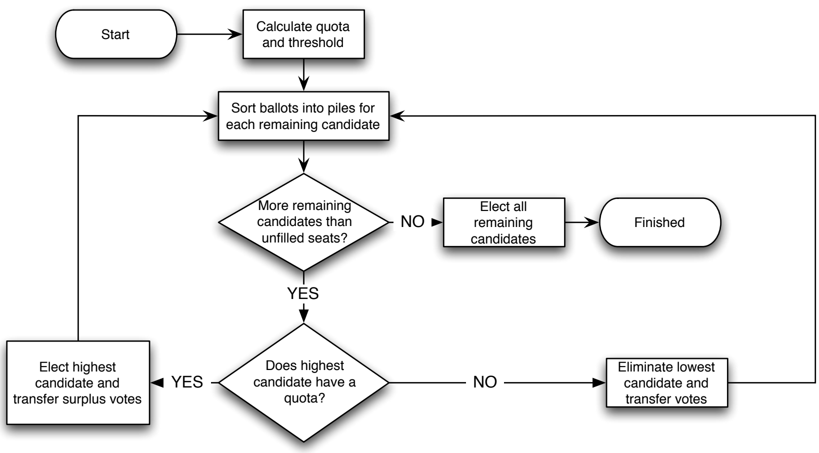
– Taking account of the ease and relative cost of making some of these modifications, the potential advantages of the chosen system, once modified in accordance with the Commission's recommendations, can make it a *viable alternative to the existing paper system in terms of secrecy and accuracy.*

Thus, Ireland wishes to keep its current complicated voting scheme, is critical of the existing attempts to implement that scheme in e-voting, but keeps the door slightly ajar for the introduction of e-voting in the future.

1.2 Proportional Representation by Single Transferable Vote

Proportional Representation by Single Transferable Vote (PR-STV) achieves proportional representation in multi-winner elections, and reduces to IRV for single-winner elections.

The following flowchart outlines the algorithm used for counting preferences ballots by PR-STV. A quota of preferences is chosen so that at most $N - 1$ candidates can reach the quota, where N is the number of seats to be filled. A *threshold* number or percentage of votes is introduced to discourage unserious candidates—to be on a ballot a candidate must put down a non-trivial deposit (say, €1,000) and, if the number of vote for them does not reach the threshold, they lose their deposit. The threshold is always less than the quota. The surplus for a candidate is the number of votes in excess of the quota.



1.3 Vótáil

Vótáil is an open source Java implementation of Irish Proportional Representation by Single Transferable Vote (PR-STV). Its functional requirements, derived from Irish electoral law, were formally specified using the Business Object Notation (BON) and refined to a model-based Java Modeling Language (JML) specification. While Normal and Extended Static Checking (ESC) were used to help verify and validate the correctness of the software, the system was not rigorously validated using system testing when it was first developed. Consequently, this new research enables us to rigorously validate this (lightly, formally) verified tally system, further increasing ones confidence in the system's correctness.

1.4 Related Work

Related work in this area is thin in peer-reviewed publication venues, as few groups in the world work on applying software engineering to electionic election systems. Some industrial and governmental work exists, but much that we are aware of is either under NDA or is word-of-mouth summaries of development practices at commercial firms.

Academic Work Meagher wrote a Z and B specification (both are traditional formal methods with tool support [23]) for election to the board of Waterford Institute of Technology, which uses a variant of the Irish PR-STV system [17]. Kjölbro used a similar methodology for specification and implementation of the Danish Voting System [12]. Neither system has been rigorously validated through unit or system testing.

We are also aware of some unpublished work relating to a formalization of PR-STV in λ-Prolog by Lee Naish. It is our understanding that this system was verified but not validated.

Researchers at the Radboud University Nijmegen attempted to test two closed-source binaries implementing Scotland's tally system to ensure its compliance with the WIG-rule [18]. To do so they did a clean-room implementation of the Scottish STV system in the purely functional programming language CLEAN and then compared nearly 6,000 hand-written and automatically generated test runs between all three implementations [13,14]. It is perhaps surprising that they found a number of errors in the commercial implementations, given the ad hoc nature of their testing.

Researchers at the University College Dublin performed a similar exercise on behalf of the aforemented CEV to test the closed source "PowerVote" tally system. Their clean-room implementation was run in parallel to the closed source binary on a network of workstations for over one month on millions of randomly generated elections. Using this completely ad hoc technique they too found correctness errors in the closed source tally system.

Also of interest is a protocol for the tallying of encrypted STV ballots [22] and other work verifying properties of voting protocols (e.g., several papers by Delaune and colleagues), but none of this work focuses on rigorously developed or validated tally systems.

All of these systems, even those that are semi-rigorously validated (like those from Nijmegen), and *especially* all of those that are formally verified benefit from this new work. The latter is true since formally verified system often have errors due to un- or under-stated simplifications in the reasoning framework or verification tools that introduce soundness and completeness problems. Likewise, commercial systems that we and others have examined (research, open source, closed source, or leaked) tally systems all benefit from our work as well.

1.5 Outline of Paper

The next section of the paper describes voting schemes in more detail. The third section describes the system-under-test using a mathematical theory of ballots and ballot boxes. The fourth section outlines the process of deriving test data needed for each election configuration. The final section contains our conclusions and plans for future work.

2 Formalisation

We must represent the input data space in a precise mathematical way to formally reason about its properties with respect to the algorithm. In the following, all components of our model are described verbally, but of course the entire election model has been mechanically formalized. We do not have the space to review this entire first-order model, as it is nearly 1,000 lines of Alloy. The interested reader can download the specifications from an online resource we cannot mention here due to anonymity.

The simplicity of our model and the underlying approach should not color the novelty of the approach nor the impact of the work. In fact, we believe that a simple model and algorithm are a strength of the work, insofaras one need not be a logician or an expert in interactive theorem proving to understand and apply the results to new electoral schemes or to validate existing tally systems.

2.1 Mathematical Models

In this example, the core concepts of elections must all be modeled: *ballots*, *ballot boxes*, *candidates*, and *election results*.

Candidates Citizens running for an election are identified by (distinct) names. The set of all candidates is denoted $\mathcal{C}$.

Ballot An ordinal or preference ballot b is a strict total order on a set of candidates $\mathcal{C}$. The length of a ballot, $|b|$, is the number of preferences expressed. The minimum number of preferences is one, except in systems like that used in Australia where all preferences must be used. In a plurality voting scheme the maximum number of preferences is one. Otherwise, the maximum length of a ballot is the number of candidates in the election.

Ballot Box An *unordered ballot box* is a bag (multiset) of ballots; an *ordered ballot box* is a vector of ballots, $[b_1 b_2 \ldots]$. Both are *ballot boxes*, denoted $\mathcal{B}$. As a bag can be modeled by a vector where order does not matter, we only use the latter formalization in the following. An ordered ballot box is used to model voting schemes in which surplus ballots are chosen randomly.

As a ballot is a vector, a *ballot box* is encoded as a matrix, where each column represents a single ballot. In such a representation, the top row of the matrix identifies the first preference candidate for each ballot. Each following row contains either a dash ('-'), meaning no preference, or the identifier of the next preference candidate.

2.2 Methodology

Here we describe the methodology we used to write the formal specification of PR-STV using the formalized candidate, ballot, and ballot box datatypes.

To write such a formal model one must be precise and meticulous. Our method is to go through the law, line by line, and identify every definition, algorithmic step, and claim therein. Definitions are mapped to datatype definitions (above). Informal algorithms are mapped to abstract state machines using these datatypes. And claims are mapped to theorems. All artifacts in the formal specification are carefully annotated with comments providing traceability to and from the law text.

Alloy permits one to specify formal models using a concise, typed first-order language. Theorems are written as assertions and are checked by the Alloy model finder by exploring the explicit state space of the model in a breadth-first fashion. The author of the specification stipulates the size of the primitive datatypes involved (e.g., the number of bits in an integer) so as to restrict the state space of exploration. For this case study, around 1,000 lines of Alloy specifications were written.

Consequently, by the time the specification is complete, the Alloy system has both guaranteed that the model is well-typed and gives strong evidence that it is sound because all theorems are checked using the model finder. Note that this automated consistency checking is *not* the same as providing a full interactive proof of a soundness theorem in a higher-order logical framework. Such formalization is an interesting and useful exercise, but we did not do it for this case study. Instead, checking the dozens of theorem stipulated in law text is more akin to the kind of validation that we are advocating in this work. It gives us high confidence, but not a proof, that the mechanical formalization is sound and complete.

3 Election Outcomes

A naive approach to validating/testing electoral systems (if they are tested at all) is to randomly generate enormous numbers of random ballot boxes and then to compare the results of executing two or more different implementations of the same voting scheme. If different results are found, then the ballots are counted manually to determine which result is correct [2]. Ironically, it is common in industry to generate 10^4 tests; in academia one generates at least 10^7 tests.

This methodology is inadequate because, even if one generates billions of ballots in non-trivial election schemes, the fraction of the state space explored is vanishingly small. To make this fact clear, we analyze below the number of distinct ballot boxes in various schemes. Further examples are found in the Appendices.

3.1 Last Two Continuing Candidates

When there are just two continuing candidates and one remaining seat, the algorithm reduces to single winner plurality (first-past-the-post). In this case there are six possible election results (*candidate outcome events*) for each candidate.

Event	Description
$\mathbb{W}$	The candidate is the poll-topper with the most votes.
$\underline{\mathbb{W}}$	The candidate is joint highest and only wins by tie-breaker.
$\mathbb{L}$	The candidate loses, but receives enough votes to reach the threshold.
$\underline{\mathbb{L}}$	The candidate is joint highest and only loses by tie-breaker.
$\mathbb{S}$	The candidate loses and does not reach the threshold.
$\underline{\mathbb{S}}$	The candidate is joint highest and loses by tie-breaker, but does not reach the threshold.

In plurality there is only one winner, so the winner is either in event $\mathbb{W}$ or $\underline{\mathbb{W}}$. If there is one loser, the 3 possible outcomes are:

Sub-Scenario	1^{st} Event	2^{nd} Event
1	$\mathbb{W}$	$\mathbb{L}$
2	$\mathbb{W}$	$\mathbb{S}$
3	$\underline{\mathbb{W}}$	$\underline{\mathbb{L}}$

Consequently, to test this particular election scenario, that of two continuing/remaining candidates and one remaining seat, there are three possible outcomes that must be exercised: one in which one candidate clearly wins and the other clearly looses, but showed well for himself ($\boxed{\mathbb{W}}\boxed{\mathbb{L}}$), another in which the loosing candidate was so unpopular as to not get his deposit back ($\boxed{\mathbb{W}}\boxed{\mathbb{S}}$), and the final outcome is when the two candidates tied and the outcome of the election was determined by a tie-breaking mechanism ($\boxed{\underline{\mathbb{W}}}\boxed{\underline{\mathbb{L}}}$).

Clearly, even when analyzing as simple an election scenario such as this one, hand-identifying each outcome is complex, and hand-writing a test for each outcome is foolhardly.

Now, lets turn our attention to a slightly more complex scenario to start to see how the number of events impacts the number of scenarios.

3.2 Filling of Last Seat

When there is one remaining seat, but at least three continuing candidates, then the algorithm reduces to Instant Runoff Voting (IRV). For each continuing candidate the following event outcomes are possible.

Event	Description
$\mathbb{H}$	The candidate is the poll-topper with a majority of the first preferences and is elected.
$\mathbb{Q}$	The candidate is elected during an intermediate round by receiving transfers.
$\mathbb{W}$	The candidate receives enough transfers to have a majority of the votes and is elected in the last round.
$\underline{\mathbb{W}}$	The candidate is elected by tie-breaker in last round.
$\mathbb{L}$	The candidate is defeated as the lowest candidate in any round but reached the threshold.
$\underline{\mathbb{L}}$	The candidate is defeated by tie-breaker in any round, but reached the threshold.
$\mathbb{S}$	The candidate is excluded as the lowest candidate in any round and did not reach the threshold.

Based upon these events, lets consider one simple scenario, focusing on two candidates, the winner and the highest loser (runner-up). In this scenario the following combinations of events are possible (the outcome $\mathbb{Q}$ is not possible because there are only two candidates and thus there will be no intermediate round).

1^{st} Event	2^{nd} Event	Description
$\mathbb{W}$	$\mathbb{L}$	The winner gets a majority and the loser reaches the threshold.
$\mathbb{W}$	$\mathbb{S}$	The winner gets a majority and loser does not reach the threshold.
$\underline{\mathbb{W}}$	$\underline{\mathbb{L}}$	The winner is elected by tie-breaker and the loser reaches the threshold.

Consequently, even though there are seven possible events, only three scenarios are possible. Thus, an increase in the number of possible events does not necessarily mean an increase in the number of scenarios.

3.3 PR-STV

PR-STV significantly complicates the picture of event types. For any winning candidate one of eight events can happen.

Event	Description
$\mathbb{N}$	The candidate is elected in the first round with a surplus containing at least one non-transferable vote
$\mathbb{T}$	The candidate is elected in the first round with at least one surplus vote
$\mathbb{H}$	The candidate is elected in the first round without surplus votes
$\mathbb{X}$	The candidate is elected after receiving vote transfers and then has a surplus with at least one non-transferable vote
$\mathbb{A}$	The candidate is elected during an intermediate round by receiving transfers and has a surplus to distribute
$\mathbb{Q}$	The candidate is elected during an intermediate round by receiving transfers, but without a surplus
$\mathbb{W}$	The candidate is elected as the highest continuing candidate on last round.
$\underline{\mathbb{W}}$	The candidate is elected by tie-breaker on the last round.

And for any losing candidate one of eight events can happen.

Event	Description
$\mathbb{L}$	The candidate is defeated as the lower continuing candidate on the last round.
$\underline{\mathbb{L}}$	The candidate is defeated by tie-breaker on last round.
$\mathbb{E}$	The candidate is excluded as the lowest candidate in an earlier round but reached the threshold, all ballots are transferable
$\mathbb{D}$	The candidate is excluded in an earlier round and is below the threshold, all ballots are transferable
$\mathbb{S}$	The candidate is defeated in the last round and is below the threshold.
$\underline{\mathbb{S}}$	The candidate is excluded by tie-breaker and is below the threshold.
$\mathbb{F}$	The candidate is excluded as the lowest candidate in an earlier round but reached the threshold, with at least one non-transferable ballot
$\mathbb{U}$	The candidate is excluded in an earlier round and is below the threshold with at least one non-transferable ballot

Deriving all possible outcomes for a simple election (say, five candidates running for three seats) is now a seriously non-trivial exercise. Thus, we need some means by which to automatically generate all possible legal outcomes of an election, and from that outcome, derive test data (i.e., a concrete ballot box) to exercise this particular corner of the election algorithm. This is the purpose of our formalization and algorithm, as presented in the sequel.

4 Procedure for Automated Test Generation

The question arises, how do we find witnesses for each outcome? That is, how do we find the smallest set of test ballots required for each outcome, while also showing that the system can scale to accept larger numbers of test ballots. Such stress testing can be achieved by running one test with the maximum number of ballots, but otherwise we would prefer to find the smallest sample ballot box for each outcome. When stress or performance tests are required then the same approach could be used to find both the smallest and largest set of input data for each outcome.

Ballot counting system tests are identified and generated in a complete and formal way, complementing existing hand-written unit tests [11]. To accomplish this task, one needs to be able to generate the ballots in each distinct kind of ballot box identified using the results of the earlier sections of this paper. Effectively, the question is one of, "Given the election outcome R, what is a legal set of ballots B that guarantees R holds?"

What follows is a fairly straightforward exercise of applying model finding to the problem of test case generation. While this idea, at its core, is not novel [19], the use of Alloy for such generation, particularly for critical systems such as election tally systems, is completely novel. The automatic generation of system tests in this space is far beyond what any research group or e-voting corporation has accomplished to-date.

4.1 Generation of Ballot Boxes

We outline a simple example to show how it is possible to derive test data from the equivalence class of ballot boxes.

Based upon the mechanized model of PR-STV, we used the SAT4J solver with Alloy running concurrently in a thread pool to perform this test generation. We suspect that a native solver would be faster, but might not be thread safe; see `http://alloy.mit.edu/community/node/1080` for an explanation of why JNI solvers might not be thread safe.

Recall that each election outcome $\mathcal{O}$ is described by a single *election scenario*, $\mathcal{S}$, as described by a vector of *candidate outcome events*. We must derive from an outcome $\mathcal{O}$ a vector of ballots $\mathcal{B}$ that guarantees, when counted using the ballot counting algorithm of the election, exactly $\mathcal{O}$, assuming that ties are broken in a deterministic way. We write $\mathcal{B} \vdash_{\mathcal{S}} \mathcal{O}$ to mean counting $\mathcal{B}$ results in outcome $\mathcal{O}$

under scenario $\mathcal{S}$. Such a combination of ballots, outcome, and scenario is called an *election outcome configuration*.

In general, there are a large number of vectors of ballots that guarantee an election outcome. For practical reasons in validation, we wish to find the *smallest* vector that guarantees the outcome; i.e., given $\mathcal{O}$ and $\mathcal{S}$, find $\mathcal{B}$ such that $\forall b.b \vdash_{\mathcal{S}} \mathcal{O}.|\mathcal{B}| \leq |b|$.

For a given outcome $\mathcal{O}$, the conditions that a vector of ballots $\mathcal{B}$ must meet to fulfill scenario $\mathcal{S}$ is described using a first-order logical formula whose validity indicates $\mathcal{B} \vdash_{\mathcal{S}} \mathcal{O}$ holds. We denote this description Φ. Thus, $\mathcal{B} \vdash_{\mathcal{S}} \mathcal{O} \Leftrightarrow \Phi(\mathcal{B})$, or alternatively, $\Phi(\mathcal{B})\mathcal{B} \vdash_{\mathcal{S}} \mathcal{O}$.

Encoding in Alloy Modeling Language Formally this is achieved using bounded checks in the Alloy Analyser [10].

Informally, to find the minimal sized $\mathcal{B}$, we iteratively describe election configurations $\mathcal{B} \vdash_{\mathcal{S}} \mathcal{O}$ with monotonically increasing numbers of ballots, starting with a ballot box of size one. These descriptions consist of a set of definitions that describe the outcome and a single theorem that states that $\mathcal{O}$ is *not* possible. If the number of ballots is too small to produce the desired outcome, then the formulation of $\mathcal{B} \vdash_{\mathcal{S}} \mathcal{O}$ will be inconsistent, and Alloy will return a satisfiable solution.

Alternatively, if the ballot box size is just large enough, Alloy will insist that the predicate is *invalid* and provide a counterexample proof context, whose values indicate the necessary values of all of the ballots in $\mathcal{B}$.

Example: Instant Runoff Voting Consider 3 candidate IRV. Two possible outcome classes are QLE and CLE–no candidate has a majority so one is eliminated and then in the next round, one candidate has a majority. These are two distinct cases: firstly a ballot box of 3 ballots for A, 2 ballots for B, 1 ballot for C, and secondly a ballot box of 2 ballots for A, 2 ballots for B, and 1 ballot with (1st=C 2nd=A).

In both cases, no one has a majority, C is eliminated, and then A wins with a 3 to 2 majority. In both cases the threshold would be one vote. In both cases C is an Early Loser ($\mathbb{E}$) and B is a Loser ($\mathbb{L}$).

An Election Configuration Example Consider a plurality election with two candidates ($|\mathcal{C}| = 2$). As discussed in the earlier examples, there are three scenarios associated with this election configuration: $[\mathbb{WL}]$, $[\mathbb{WS}]$, and $[\underline{\mathbb{WL}}]$.

In the following, let T be a tiebreaker function that chooses a winner from a set of candidates.

As earlier, let $\mathcal{B}$ denote a ballot box and b a ballot. Let $b[n]$ be the n^{th} preference of ballot b. Finally, as earlier, let τ be the threshold of votes for a given electoral system.

Formalization Each candidate outcome is described by an definition that expresses the relationship between the number of votes that candidate receives and the outcome. Since most first-order theorem provers do not provide native support for the generalized summation quantifier, we use a generic encoding described by Leino and Monahan [15].

Axiomatization We first need definitions that stipulate the well-formedness of ballots.

$$\forall b \in \mathcal{B} \;.\; b[1] \in |\mathcal{C}|$$

$$(\sum_{\mathcal{B}} b[1] = A) + (\sum_{\mathcal{B}} b[1] = B) = |\mathcal{B}|$$

Definition wf_b describes the well-formedness of ballots, while definition $\mathrm{wf}_{\mathcal{B}}$ describes the well-formedness of the ballot box. If an electoral system permits empty preferences then this latter definition is modified to accommodate such.

Formalizing Scenarios Next, we need to formalize the scenarios of this particular two candidate plurality election as follows, where the label of each formula indicates the semantics of event of the same name e.g., formula $\mathbb{W}$ describes the meaning of event $\mathbb{W}$.

As we commonly quantify over all ballots in $\mathcal{B}$, we write the quantifications over $\mathcal{B}$ rather than the more wordy $b \in \mathcal{B}$. Finally, we encode the set of ballots as the first index in the map b i.e., the second ballot's third preference is $b[2][3]$. Note that these summations are generalized quantifiers: $\sum(b[1] = A)$ means "count the number of ballots whose first preference is candidate A."

$$\sum_{\mathcal{B}} (b[1] = A) > \sum_{\mathcal{B}} (b[1] = B) \tag{$\mathbb{W}$}$$

$$\sum_{\mathcal{B}} (b[1] = A) = \sum_{\mathcal{B}} (b[1] = B) \wedge (T = A) \tag{$\underline{\mathbb{W}}$}$$

$$\tau \leq \sum_{\mathcal{B}} (b[1] = B) \tag{$\mathbb{L}$}$$

$$\sum_{\mathcal{B}} (b[1] = B) < \tau \tag{$\mathbb{S}$}$$

Note that the rightmost clause of formula $\underline{\mathbb{W}}$ states that the coin-flip function picked candidate one as the winner. Also remember that these are axioms of our theory of PR-STV elections, and thus redundant clauses repeating earlier axioms are unnecessary (e.g., repeating $\mathbb{W}$'s inequality in $\mathbb{L}$'s definition).

The Outcome Theorem Now, we wish to try to prove a theorem that stipulates that, for a given scenario, an expected outcome is *not* possible for a given number of ballots.

After asserting to the theorem prover the above definitions (accomplished with the `BG_PUSH` command in the Alloy SAT4J solver and either the `definition` attribute or a `push` command in SMT-LIB [4,21]), we ask the prover to check the validity of the following theorem (by simply stating the theorem in Alloy or using the `check-sat` command in SMT-LIB), that captures the meaning of scenario [$\mathbb{WL}$]:

$$|\mathcal{B}| = 1 \Rightarrow \neg(\mathbb{W} \wedge \mathbb{L})$$

If the prover responds with "valid," then we know that we need more than one ballot, and we make a new attempt:

$$|\mathcal{B}| = 2 \Rightarrow \neg(\mathbb{W} \wedge \mathbb{L})$$

Consequently, if that attempt also fails, we attempt to prove the theorem with three ballots:

$$|\mathcal{B}| = 3 \Rightarrow \neg(\mathbb{W} \wedge \mathbb{L})$$

at which time the prover returns an "invalid" response with a counterexample. The counterexample for this particular theorem will be of the form

$$b[1][1] = A \wedge b[2][1] = A \wedge b[3][1] = B$$

thereby providing a minimal ballot box that guarantees election outcome [$\mathbb{WL}$]. Note that to check minimality we can attempt to prove the theorem $(\mathbb{W} \wedge \mathbb{L}) \Rightarrow 3 \leq |\mathcal{B}|$, though such a theorem is quite difficult for automated solvers to prove give the implicit quantification over ballot boxes and is, in general, can only be proven with an interactive theorem prover.

4.2 Open Source Implementation

The source code for all software and all mechnized theory is available under the terms of the MIT open source license and can be found in an online resource we cannot mention here due to anonymity.

5 Evaluation and Threats to Validity

To test our approach, we have used our methodology to test Vótáil, the aforementioned rigorously engineered tally system for Irish PR-STV. Vótáil was developed using a rigorous methodology with the application of several formal methods tools for design and implementation formal verification.

To test Vótáil (and other Irish PR-STV tally systems) we executed our election generator (whose name is not mentioned to maintain anonymity) on a large sixteen core system for nearly one month. The specific algorithm we used gradually added candidates and seats to the election definition, essentially exploring all elections scenarios in a breadth-first fashion. The resulting log file tracing every election generation is over 700MB in size and we generated 137,000 elections. We terminated our test generation after generating elections with seven candidates vying for three seats as generating more complex election scenarios becomes increasingly computationally expensive.

To evaluate the quality of our system tests we executed all tests on Vótáil using Emma to perform coverage analysis [5]. Recall that Vótáil was developed using a rigorous development method including several static checkers and had already been lightly verified using ESC/Java2. Consequently, it would be somewhat surprising to find errors in the implementation.

With our original model Emma reported that executing this test suite resulted a fraction below 100% statement coverage and 100% condition coverage. In order to achieve full statement coverage, the original Alloy model was expanded to include extra outcomes; in particular, we needed to model the possibility that a winning candidate might have no surplus votes.

Using this test suite we discovered two errors in its implementation, namely a null pointer exception and possible non-termination of a loop. On closer examination we discovered that both of these errors were not caught during the original formal verification of Vótáil due to under-specification (a missing loop invariant).

Of course, this level of coverage (100% statement and condition coverage) does *not* prove that the system is error-free. One could easily take the fixed set of system tests and code around its model coverage with sufficient effort. But what it does do is (a) provide strong evidence, *especially when combined with a rigorous development method and formal verification*, that the system is correct, and (b) raise the state-of-the-art for election tally system testing enormously.

6 Conclusions

The fact that we found errors in a tally system that was engineered using EAL level 7 methods and tools strongly supports our hypothesis that this kind of automated, domain-specific validation is critical for digital electronic voting systems worldwide.

Moving forward, we believe that it would be of great value to democracies around the world to formalize the other large handful of popular election schemes worldwide using the same software framework. By doing so we can generate a complete set of system tests for every tally scheme in widespread use. The lack of a standardized format for election data from the IEEE or similar is unfortunate, so perhaps we can make recommendations in this regard.

Of course, having all of these system tests generated is a useful outcome for everyone building election systems, academic and industrial alike, but is not a panacea. As advocated by others using applied formal methods, verification and validation of mission- and safety-critical systems is mandatory. Techniques go hand-in-hand toward ensuring that our critical software systems, like those of election software, are correct. This work simply provides a strong touchstone for the runtime validation side of things, while much work remains to be done with regards to verification and certification.

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A Appendix: Voting Schemes

To analyze this challenge, a number of definitions are necessary to establish a clear nomenclature for the later formalisms of this paper. We focus first on voting schemes for context and later, in Section 2, we provide the formal definitions of all of the components informally mentioned here.

A *voting scheme* is an algorithm for counting ballots. A *preference voting scheme* requires the voter to rank two or more candidates ($\mathcal{C}$) in order of preference from first to last. A *plurality voting scheme* requires the voter to pick one candidate, and thus is equivalent to the preference scheme when the ranking list has unitary size.

The *election result* $(\mathcal{W}, \mathcal{L})$ consists of (1) the identification of the winner or winners of the election and (2) the identification of those candidates who achieved a certain *threshold* (denoted τ) of votes, e.g., 5 percent, needed either to qualify for public funding in future elections or to recoup a deposit paid. This threshold facet of our election model is not universal, but is a critical component in many electoral systems. Note that winners and losers are disjoint.

We denote a ballot box $\mathcal{B}$ as a set of ballots b. Mathematically, a voting scheme $\mathcal{E}$ is a function that takes a ballot box (a set of ballots) as its input, and produces an election result as its output. More formally, $\mathcal{E} : \mathcal{B} \rightarrow (\mathcal{W}, \mathcal{L})$ where $\mathcal{W} \subseteq \mathcal{C}$, $\mathcal{L} \subset \mathcal{C}$, and $\mathcal{W} \cap \mathcal{L} = \emptyset$.

Single Winner Plurality Voting Plurality voting is one of the simplest possible voting schemes. The candidate with the most votes is the winner. When there is only one remaining seat and just two continuing candidates, then PR-STV reduces to single-winner Plurality.

Instant Runoff Voting (IRV) IRV allows the voter to rank one or more candidates in order of relative preference, from first to last.

IRV usually has a single winner, but the candidate with the most votes must also have a majority of all votes, otherwise the candidate with least votes is excluded and each ballot for that candidate is transferred to the next candidate in order of preference. This evaluation-and-transfer continues until one of the candidates achieves an overall majority.

When there is just one remaining seat, or a special election to fill a vacancy in one seat, then PR-STV reduces to IRV.

Order of Elimination The candidate with the least number of votes credited to him or her in the curent round is selected for elimination. If there is an equality of votes, then previous rounds are considered. If two or more candidates have equal lowest votes in all rounds, then random selection is used.

Variants of PR-STV To highlight the complexities of election schemes, consider the following variants of PR-STV. As schemes vary, so must testing/validation strategies. For example, Australia, Ireland, Malta, Scotland, and Massachusetts use different variants of PR-STV for their elections [1].

- **Australia** - Australia uses IRV to elect its House of Representatives and an open list system for its Senate, where voters can choose either to vote for individual candidates using PR-STV or to vote "above-the-line" for a party. If voters choose to use PR-STV then all available preferences must be used [6].
- **Ireland** - Ireland uses PR-STV for local, national and European elections. Transfers are rounded to the nearest whole ballot, so the order in which ballots are transferred makes a difference to the result [16]. Not all preferences need to be used, so voters may choose to use only one preference, as in Plurality voting, if desired.
- **Malta** - Malta uses PR-STV for local, national and European elections. For national elections Malta also adds additional members so that the party with the most first preference votes is guaranteed a majority of seats.
- **Scotland, UK** - Scotland uses PR-STV for local elections. Rather than randomly select which ballots to include in the surplus, fractions of each ballot are transferred, that gives a more accurate result but takes much longer to count if counted by hand [8].
- **Massachusetts, USA** - Cambridge in Massachusetts uses PR-STV for city elections. Candidates with less than fifty votes are eliminated in the first round and surplus ballots are chosen randomly.

The fact that a single complex voting scheme like PR-STV has this many variants in use highlights the challenges in reasoning about and validating a given software implementation. This fact makes our work that much more valuable, as each algorithm only need be analyzed *once* to derive a complete validation that may be used again and again over arbitrary implementations of a ballot counting algorithm.

Irish PR-STV To give context, we now discuss the mechanics of Irish PR-STV in more detail.

Preference Ballots The voter writes the number “1” beside his or her favorite candidate. There can only be one first preference.

The voter then considers which candidate would be his or her next preference if his or her favorite candidate is either excluded from the election or is elected with a surplus of votes.

The second preference is marked with “2” or some equivalent notation. The can be only one second preference; there cannot be a joint second preference. Likewise for third and subsequent preferences. Not all preferences need to be used.

Multi-seat constituencies Each constituency is represented by either three, four or five seats.

The Droop Quota The quota is calculated so that not all winners can reach the quota. The droop quota is $1 + \frac{V}{1+S}$, where V is the total number of valid votes cast and S is the number of vacancies (or seats) to be filled [7]. The quota is chosen so that any candidate reaching the quota is automatically elected, and so that the number of candidates that might reach the quota less than the number of seats.

For example, in a five-seat constituency a candidate needs just over one-sixth of the total vote to be assured of election.

Surplus The surplus for each candidate, is the number of ballots in excess of the quota (if any). The surplus ballots are then available for redistribution to other continuing candidates.

The selection of which ballots belong to the surplus is a complex issue, depending on the round of counting. In the first round of counting, any surplus is divided into sub-piles for each second preference, so that the distribution of the ballots in the surplus is proportional to the second-preferences. In later rounds the surplus is taken from the last parcel of ballots received from other candidates. This surplus is then sorted into sub-piles according to the next available preference.

For example, if the quota is 9,000 votes and candidate A receives 10,000 first preference votes. The surplus is 1,000 votes. Suppose 5,000 ballots had candidate B as next preference, 3,000 had candidate C and 2,000 had candidate D. Then the surplus consists of 500 ballots taken from the 5000 for candidate B, 300 from the 3000 for candidate C and 200 from the 2000 for candidate D. Ideally each subset would also be sorted according to third and subsequent preference, but this does not happen under the current procedure for counting by hand, nor was it mandated in the previous guidelines for electronic voting in Ireland.

Exclusion of weakest candidates When there are more candidates than available seats, and all surplus votes have been distributed, the continuing candidate with least votes is excluded. If two or more candidates have equal lowest votes (at all stages of the count) then one is chosen randomly for exclusion.

All ballots from the pile of the excluded candidate are then transferred to the next preference for a continuing candidate, or to the pile of non-transferable votes.

This continues until another candidate is elected with a surplus or until the number of continuing candidates equals the number of remaining seats.

Filling of Last Seat and Bye-elections When there is only one seat remaining to be filled, i.e., the number of candidates having so far reached the quota is one less than the number of seats, or in a bye-election for a single vacancy, then the algorithm becomes the same as Instant Runoff Voting; no more surplus distributions are possible, and candidates with least votes are excluded until only two remain.

Last Two Continuing Candidates When there are two continuing candidates and one remaining seat, then the algorithm becomes the same as single-seat first-past-the-post plurality; the candidate with more votes than the other is deemed elected to the remaining seat, without needing to reach the quota. If there is a tie then one candidate is chosen randomly.

B Appendix: Detailed Examples

This appendix contains some more detailed examples for estimation the number of possible ballots, number of possible outcomes, and the number of distinct permutations of ballot papers.

B.1 Number of Distinct Ballots

The number of distinct permutations of non-empty preferences is $\sum_{l=1}^{C} (C)_l$, where $C = |\mathcal{C}|$ and partial ballots are allowed, so that the number of preferences used range in length from one to the number of candidates. For a ballot of length l, $(C)_l$ is the number of distinct preferences that can be expressed.

Examples and Encoding Ballots This distinct ballot count is best understood, particularly for those unexcited by combinatorics, by examining cases for small C and enumerating all possible ballots.

Two Candidates There are four different ways to vote for two candidates (named Alice and Bob): two ballots of length 1, and two ballots of length 2, that is $(2)_1 + (2)_2$:

Ballot	Alice	Bob	Encoding of Ballot
1	1^{st}	-	$\boxed{A}\boxed{-}$
2	-	1^{st}	$\boxed{B}\boxed{-}$
3	1^{st}	2^{nd}	$\boxed{A}\boxed{B}$
4	2^{nd}	1^{st}	$\boxed{B}\boxed{A}$

$\boxed{A}\boxed{-}$ has a different meaning than $\boxed{A}\boxed{B}$. If we had an election with two ballots $\boxed{B}\boxed{-}$ and $\boxed{A}\boxed{B}$, then Bob would be the winner.

Note the symmetry of these four ballots. There are effectively only two different ballots if the candidates cannot be differentiated.

Number of Distinct Outcomes. If B is the number of distinct non-empty ballots that can be cast, and $V = |\mathcal{B}|$ is the number of votes cast, then the number of possible combinations of ballots is B^V if the order of ballots is important, and $\frac{B^V}{V!}$ if not.

A typical electoral configuration in Ireland is a five seat constituency with a typical voting population of 100,000 and 24 candidates. Consequently, the number of possible ballot boxes is $(\sum_{l=1}^{24} (24)_l)^{100,000}$, an astronomical number of tests that would be impossible to run.

To avoid this explosion, we partition the set of all possible ballot boxes into equivalence classes with respect to the counting algorithm chosen. We consider the equivalence class of election results for all three counting schemes.

Each election outcome is described by an *election scenario* that is a vector of *candidate outcome events*. Both of these terms are defined in the following.

The key idea is that election scenarios represent an equivalence class of election outcomes, thereby letting us collapse the testing state space due to symmetries in candidates. We will return to this point in detail below in the early examples.

Three Candidates. There are 15 legal ways to vote for three candidates called Alice, Bob, and Charlie:

Ballot	Alice	Bob	Charlie	Encoding
1	1^{st}	-	-	A | $-$ | $-$
2	-	1^{st}	-	B | $-$ | $-$
3	-	-	1^{st}	C | $-$ | $-$
4	1^{st}	2^{nd}	-	A | B | $-$
5	1^{st}	-	2^{nd}	A | C | $-$
6	2^{nd}	1^{st}	-	B | A | $-$
7	-	1^{st}	2^{nd}	B | C | $-$
8	2^{nd}	-	1^{st}	C | A | $-$
9	-	2^{nd}	1^{st}	C | B | $-$
10	1^{st}	2^{nd}	3^{rd}	A | B | C
11	1^{st}	3^{rd}	2^{nd}	A | C | B
12	2^{nd}	1^{st}	3^{rd}	B | A | C
13	3^{rd}	1^{st}	2^{nd}	B | C | A
14	2^{nd}	3^{rd}	1^{st}	C | A | B
15	3^{rd}	2^{nd}	1^{st}	C | B | A

There are 3 ballots of length 1, 6 ballots of length 2 and 6 ballots of length 3, that totals $(3)_1 + (3)_2 + (3)_3 = 15$. Again, note the symmetry of these ballots, as there are only three different kinds of ballots in these fifteen ballots.

More than Three Candidates. Each additional candidate number n means one extra ballot of length 1, plus another C ballots in which the extra candidate is the last preference, plus every other way in which the candidate could be inserted into the existing set of ballots, in one of n positions along that ballot.

For example, when there are four candidates, the number of single preference ballots increases to 4, the number of length 2 ballots is $4 \times (4 - 1)$, the number of length 3 ballots is $4 \times (4 - 1) \times (4 - 2)$ and the number of full length ballots is 4!, for a total of 64 ballots, of which there are only three equivalence classes.