Visualization of orbits in general relativity

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Introduction

The orbit of some massive object, e.g.: a planet, around some other massive object, e.g.: a star, is obtained from solving the Einstein field equations.

Model equations

Massive objects orbiting other massive objects follow geodesics:

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{rs} \frac{dx^r}{dq\tau} \frac{dx^s}{d\tau} \tag{1}$$

Where x^λ are the coordinates, Γ^λ_{rs} the Christoffel symbols of the second kind and τ an affine parameter. The set of equations represented by (1) can be simplified by imposing conservation of energy and angular momentum. The equations can be simplified further by only considering orbits with a polar angle $\phi = \frac{\pi}{2}$. Conservation of energy leads to:

$$E = \left(1 - \frac{2GM}{r}\right) \frac{dt}{d\tau} \tag{2}$$

Where E is the energy, G the gravitational constant, M the mass of the gravitating object, r the radial coordinate and t the time coordinate. Conservation of angular momentum, along with $\phi = \frac{\pi}{2}$, gives:

$$L = r^2 \frac{d\theta}{d\tau} \tag{3}$$

Where L is the angular momentum and θ is the azimuthal angle. There is an additional quantity ϵ which is conserved according to:

$$\epsilon = -g_{rs} \frac{dx^r}{d\tau} \frac{dx^s}{d\tau} \tag{4}$$

Where g_{rs} is the Schwarzschild metric:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 \left(d\phi^2 + \sin^2(\phi) d\theta^2\right) \tag{5}$$

Combining (2), (3) and (4), along with the fact that $\phi = \frac{\pi}{2}$, gives a relation for r:

$$\left(\frac{dr}{d\tau}\right)^2 = E^2 - \left(\frac{L^2}{r^2} + \epsilon\right)\left(1 - \frac{2GM}{r}\right) \tag{6}$$

Substituting (3) in (6) and $\epsilon = 1$ for massive objects gives a relation between r and the azimuthal angle θ:

$$\left(\frac{dr}{d\theta}\right)^2 = \frac{(E^2 - 1)}{L^2} r^4 + \frac{2GM}{L^2} r^3 - r^2 + 2GMr \tag{7}$$

Defining a new variable:

$$u = \frac{L^2}{GMr} \tag{8}$$

Then equation (7) can be rewritten to:

$$\left(\frac{du}{d\theta}\right)^2 = \frac{(E^2-1)L^2}{G^2M^2} + 2u - u^2 + \frac{2G^2M^2}{L^2}u^3 \tag{9}$$

In order to keep track of the sign of $\frac{dx}{d\theta}$ the derivative of equation (9) with respect to θ is taken to obtain another relation between x and θ:

$$\frac{d^2u}{d\theta^2} = 1 - u + \frac{3G^2M^2}{L^2}u^2 \tag{10}$$

Visualization

An example of some simulation results of a test particle in a gravitational field is given in Figure 1. An orbital precession, such as that observed in Mercury's orbit, is clearly visible.

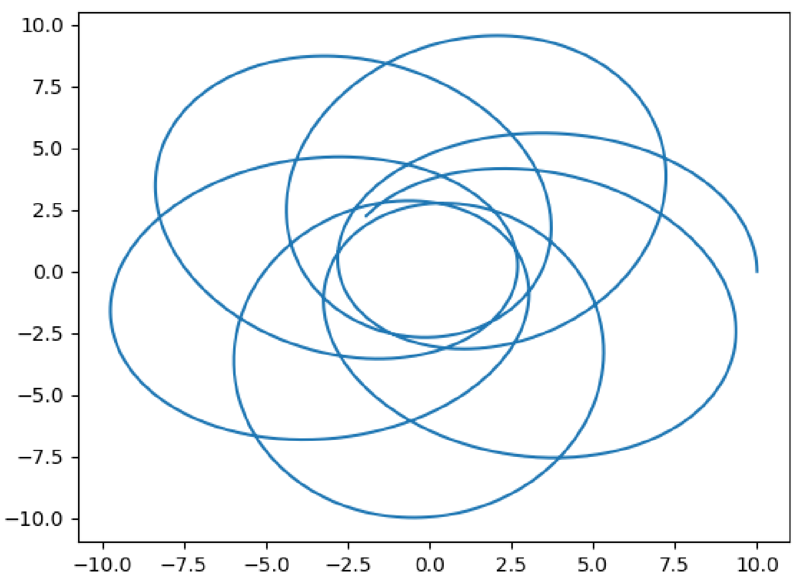

Figure 1: Orbit of test particle.