Chapter 5

Schwarzschild Solution

Problem Set #5: 5.3, 5.4, 5.5 (Due Monday Dec. 2nd)

5.1 Birkhoff's theorem

There are very few exact solutions of the Einstein equations, but perhaps the most well known solution was first derived by Schwarzschild. One can check that the **Schwarzschild metric**

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right), \quad (5.1)$$

is a solution of the vacuum Einstein equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0 \quad (5.2)$$

by direct substitution.

According to **Birkhoff's theorem** the Schwarzschild solution is also a unique spherical symmetry solution of the vacuum Einstein equation. By spherical symmetry we mean that there is a set of three Killing vectors with following commutation relations,

$$\begin{aligned} \left[V^{(1)}, V^{(2)}\right] &= V^{(3)} && (5.3)\\ \left[V^{(2)}, V^{(3)}\right] &= V^{(1)} && (5.4)\\ \left[V^{(3)}, V^{(1)}\right] &= V^{(2)}. && (5.5) \end{aligned}$$

Then the Forbenius's Theorem implies that the integral curves of these vector fields are constraint to sub-manifolds. In the case of the vector fields $V^{(1)}$, $V^{(2)}$ and $V^{(3)}$ the submanifolds will be two-spheres that would foliate (almost) all of the manifold into two-spheres. For example, $\mathbb{R}^3$ can be foliated with concentric two-spheres centered in the origin. Then the Killing vector fields represent rotations around x, y and z axis respectively. These rotations move points around but the point remain at the same sphere at a fixed distance from origin. Does this foliation cover all of the $\mathbb{R}^3$? Could you think of other manifolds that can be foliated with two-spheres?

The foliation based on symmetries of the manifold can be used to put a coordinate system. If a manifold is n-dimentional and it is foliated with m-dimantional submanifolds then we can use coordinates u^i (where $i = 1...m$) to move around on a given submanifolds and coordinates v^I (where $i = 1...n$) to move between submanifolds. If the submanifolds are maximally symmetric (e.g. two spheres) then one can show that

$$ds^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = g_{IJ}(v)\mathrm{d}v^I \mathrm{d}v^J + f(v)\gamma_{ij}(u)\mathrm{d}u^i \mathrm{d}u^j. \tag{5.6}$$

All it says is that the metric of submanifolds $\gamma_{ij}(u)$ is the same on different submanifolds (since $g_{IJ}(v)$ and $f(v)$ are not functions of u) and that the cross-terms $\mathrm{d}v^I \mathrm{d}u^i$ can always be eliminated by redefining coordinates such that $\partial/\partial v^I$ is orthogonal to the submanifolds. For a spherically symmetric four dimensional manifold we get

$$ds^2 = g_{aa}(a,b)\mathrm{d}a^2 + g_{ab}(a,b)\left(\mathrm{d}a\mathrm{d}b + \mathrm{d}b\mathrm{d}a\right) + g_{bb}(a,b)\mathrm{d}b^2 + r^2(a,b)\mathrm{d}\Omega^2 \tag{5.7}$$

where

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2. \tag{5.8}$$

Our remaining task is to change the coordinates so that (5.7) takes the form of Schwarzschild metric (5.1). The first step is to change coordinates from a, b to a, r by inverting $r(a, b)$, i.e.

$$ds^2 = g_{aa}(a,r)\mathrm{d}a^2 + g_{ar}(a,r)\left(\mathrm{d}a\mathrm{d}r + \mathrm{d}r\mathrm{d}a\right) + g_{rr}(a,r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{5.9}$$

The second step is to change coordinates a, r to t, r so that there are no cross-terms $\mathrm{d}t\mathrm{d}r + \mathrm{d}r\mathrm{d}t$, or

$$ds^2 = m(t,r)\mathrm{d}t^2 + n(t,r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{5.10}$$

Let $t(a, r)$ be the desired coordinate, then

$$\mathrm{d}t = \frac{\partial t}{\partial a}\mathrm{d}a + \frac{\partial t}{\partial r}\mathrm{d}r \tag{5.11}$$

and

$$\mathrm{d}t^2 = \left(\frac{\partial t}{\partial a}\right)^2 \mathrm{d}a^2 + \left(\frac{\partial t}{\partial a}\right)\left(\frac{\partial t}{\partial r}\right)\left(\mathrm{d}t\mathrm{d}r + \mathrm{d}r\mathrm{d}t\right) + \left(\frac{\partial t}{\partial r}\right)^2 \mathrm{d}r^2 \tag{5.12}$$

or

$$ds^2 = m(t,r)\left(\left(\frac{\partial t}{\partial a}\right)^2 \mathrm{d}a^2 + \left(\frac{\partial t}{\partial a}\right)\left(\frac{\partial t}{\partial r}\right)(\mathrm{d}t\mathrm{d}r + \mathrm{d}r\mathrm{d}t) + \left(\frac{\partial t}{\partial r}\right)^2 \mathrm{d}r^2\right) + n(t,r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{5.13}$$

By matching the coefficients of (5.9) and (5.13) we get three equations

$$\begin{aligned} m\left(\frac{\partial t}{\partial a}\right)^2 &= g_{aa} \\ n + m\left(\frac{\partial t}{\partial r}\right)^2 &= g_{rr} \\ m\left(\frac{\partial t}{\partial a}\right)\left(\frac{\partial t}{\partial r}\right) &= g_{ar} \end{aligned} \tag{5.14}$$

that can always be solved for three unknown functions $t(a,r)$, $m(a,r)$ and $n(a,r)$. Therefore we can always put a spherically symmetric metric into the form of (5.10). But since the signature of the metric is Lorentzian either m or n must be negative and thus without loss of generality we can assume that m is negative. Then (5.10) can be rewritten in terms of exponentials as

$$ds^2 = -e^{2\alpha(t,r)}\mathrm{d}t^2 + e^{2\beta(t,r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{5.15}$$

The final step is to determine α and β by solving the vacuum Einstein equation. If we label (t, r, θ, ϕ) as $(0, 1, 2, 3)$ then non-vanishing Christoffel symbols,

$$\begin{aligned} \Gamma^0_{00} &= \partial_0\alpha \\ \Gamma^0_{01} = \Gamma^0_{10} &= \partial_1\alpha \\ \Gamma^0_{11} &= e^{2(\beta-\alpha)}\partial_0\beta \\ \Gamma^1_{11} &= \partial_1\beta \\ \Gamma^1_{01} = \Gamma^1_{10} &= \partial_0\beta \\ \Gamma^1_{00} &= e^{2(\alpha-\beta)}\partial_1\alpha \\ \Gamma^1_{22} &= -re^{-2\beta} \\ \Gamma^2_{21} = \Gamma^2_{12} &= \frac{1}{r} \\ \Gamma^1_{33} &= -re^{-2\beta}\sin^2\theta \\ \Gamma^3_{31} = \Gamma^3_{13} &= \frac{1}{r} \\ \Gamma^2_{33} &= -\sin\theta\cos\theta \\ \Gamma^3_{32} = \Gamma^3_{23} &= \frac{\cos\theta}{\sin\theta}. \end{aligned} \tag{5.16}$$

and the non-vanishing component of Riemann tensor,

$$
\begin{aligned}
R^0{}_{101} &= e^{2(\beta-\alpha)}\left(\partial_0^2\beta+(\partial_0\beta)^2-\partial_0\alpha\partial_0\beta\right)+\left(\partial_1\alpha\partial_1\beta-\partial_1^2\alpha-(\partial_1\alpha)^2\right)\\
R^0{}_{202} &= -re^{-2\beta}\partial_1\alpha\\
R^0{}_{303} &= -re^{-2\beta}\sin^2\theta\partial_1\alpha\\
R^0{}_{212} &= -re^{-2\alpha}\partial_0\beta\\
R^0{}_{313} &= -re^{-2\alpha}\sin^2\theta\partial_0\beta\\
R^1{}_{212} &= -re^{-2\beta}\partial_1\beta\\
R^1{}_{313} &= -re^{-2\beta}\sin^2\theta\partial_1\beta\\
R^2{}_{323} &= \left(1-e^{-2\beta}\right)\sin^2\theta\partial_1\beta
\end{aligned}
\tag{5.17}
$$

and the non-vanishing component of the Ricci tensor,

$$
\begin{aligned}
R_{00} &= \left(\partial_0^2\beta+(\partial_0\beta)^2-\partial_0\alpha\partial_0\beta\right)+e^{2(\alpha-\beta)}\left(\partial_1^2\alpha+(\partial_1\alpha)^2-\partial_1\alpha\partial_1\beta+\frac{2}{r}\partial_1\alpha\right)\\
R_{11} &= -\left(\partial_1^2\alpha+(\partial_1\alpha)^2-\partial_1\alpha\partial_1\beta-\frac{2}{r}\partial_1\beta\right)+e^{2(\beta-\alpha)}\left(\partial_0^2\beta+(\partial_0\beta)^2-\partial_0\alpha\partial_0\beta\right)\\
R_{01}=R_{10} &= \frac{2}{r}\partial_0\beta\\
R_{22} &= e^{-2\beta}\left(r\left(\partial_1\beta-\partial_1\alpha\right)-1\right)+1\\
R_{33} &= \left(e^{-2\beta}\left(r\left(\partial_1\beta-\partial_1\alpha\right)-1\right)+1\right)\sin^2\theta.
\end{aligned}
\tag{5.18}
$$

The requirement that our metric solves the vacuum Einstein equation is equivalent to

$$R_{\mu\nu}=0 \tag{5.19}$$

for all μ and ν. In particular

$$R_{01}=0 \quad\Rightarrow\quad \partial_0\beta=0 \quad\Rightarrow\quad \beta=\beta(r) \tag{5.20}$$

and

$$R_{22}=0 \;\Rightarrow\; \partial_0 R_{22}=0 \;\Rightarrow\; \partial_0\partial_1\alpha=0 \;\Rightarrow\; \alpha=f(r)+g(t). \tag{5.21}$$

This implies that

$$-e^{2\alpha(t,r)}=-e^{2f(r)}e^{2g(t)} \tag{5.22}$$

and by redefining

$$\mathrm{d}t\to e^{-g(t)}\mathrm{d}t \tag{5.23}$$

the metric (5.15) takes the following form

$$ds^2=-e^{2\alpha(r)}\mathrm{d}t^2+e^{2\beta(r)}\mathrm{d}r^2+r^2\mathrm{d}\Omega^2. \tag{5.24}$$

Since all of the metric component are independent of time this proves that the spherically symmetric solution of vacuum Einstein equation must possesses a time-like Killing vector. Such metrics are called **stationary**. If in addition the time-like Killing vector is orthogonal to a family of space-like hypersurfaces, then the metric is call **static**.

To find $\alpha(r)$ and $\beta(r)$ we note that

$$R_{00} = 0\,, R_{11} = 0 \;\Rightarrow\; e^{2(\alpha-\beta)}R_{00}+R_{11} = 0 \;\Rightarrow\; \frac{2}{r}\left(\partial_1\alpha + \partial_1\beta\right) \;\Rightarrow\; \alpha = -\beta+\text{const.} \tag{5.25}$$

and by once again redefining time coordinates

$$ds^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + e^{-2\alpha(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{5.26}$$

But since

$$R_{22} = 0 \;\Rightarrow\; e^{2\alpha}\left(2r\partial_1\alpha + 1\right) = 1 \;\Rightarrow\; \partial_1\left(re^{2\alpha}\right) = 1 \;\Rightarrow\; e^{2\alpha} = 1 + \frac{\mu}{r} \tag{5.27}$$

we arrive at our final expression

$$ds^2 = -\left(1+\frac{\mu}{r}\right)\mathrm{d}t^2 + \left(1+\frac{\mu}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{5.28}$$

5.2 Schwarzschild metric

To give a physical interpretation to integration constant μ we consider the asymptotic limit

$$g_{00}(r \;\to\; \infty) = -1 - \frac{\mu}{r} \tag{5.29}$$

$$g_{rr}(r \;\to\; \infty) \approx 1 - \frac{\mu}{r} \tag{5.30}$$

which agrees with Newtonian (or weak gravitational) limit

$$g_{00} \;=\; -1 - 2\Phi \tag{5.31}$$

$$g_{rr} \;=\; 1 - 2\Phi \tag{5.32}$$

with potential

$$\Phi = -\frac{GM}{r} \tag{5.33}$$

if we set

$$\mu = -2GM. \tag{5.34}$$

This is in full agreement with Schwarzschild metric (5.1),

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right), \tag{5.35}$$

where M is the Newtonian mass that would be measured at large distances

$$r \gg 2GM. \tag{5.36}$$

Note that asymptotically $r \to \infty$ the Schwarzschild metric reduces to the Minkowski metric,

$$ds^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right). \tag{5.37}$$

It worth emphasizing that the Schwarzschild metric (5.36) has a coordinate singularity at $r = 0$ and at $r = 2GM$. Similar problem is present even in Minkowski metric (5.37) in spherical coordinates which is singular at $r = 0$. The coordinate singularity does not necessarily mean that the theory breaks down, and it is often possible to change the coordinates so that the resulting metric is finite everywhere. For example the Minkowski metric can be rewritten in Cartesian coordinates where the $r = 0$ is not any different from any other point.

However, there are situations in which the coordinate singularities lead to coordinate independent curvature singularities. For example, if the Ricci scalar R, or any other scalar formed from Riemann tensor (e.g. $R^{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$, etc.) divergencies at some point then such singularities cannot be removed by simply changing coordinates. This would be regarded as a break down of the theory. In the case of the Schwarzschild metric (5.36) one can check that at $r = 0$ there is a curvature singularity since

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{12G^2M^2}{r^6} \tag{5.38}$$

but none of the curvature scalars blow up at $r = 2GM$ and thus the point (actually a two-sphere) is non-singular. This suggests that we have chosen a poor coordinate system and one should try to find better coordinates to describe the geometry at $r = 2GM$, also known as **Schwarzschild radius**. Although there are objects (such as black holes) for which the full metric is required, many gravitation objects (such Sun) have radius many orders of magnitude larger than the Schwarzschild radius

$$R_\odot = 10^6 GM_\odot \gg 2GM \tag{5.39}$$

and the coordinate singularity of the vacuum Einstein equation at $r = 2GM$ becomes irrelevant.

5.3 Geodesics in Schwarzschild

The non-vanishing Christoffel symbols for the metric (5.35) are given by

$$\begin{aligned}
\Gamma^1_{00} &= \frac{GM}{r^3}(r-2GM) \\
\Gamma^1_{11} &= \frac{-GM}{r(r-2GM)} \\
\Gamma^0_{01} = \Gamma^0_{10} &= \frac{-GM}{r(r-2GM)} \\
\Gamma^2_{12} = \Gamma^2_{21} &= \frac{1}{r} \\
\Gamma^1_{22} &= -(r-2GM) \\
\Gamma^3_{13} = \Gamma^3_{31} &= \frac{1}{r} \\
\Gamma^1_{33} &= -(r-2GM)\sin^2\theta \\
\Gamma^2_{33} &= -\sin\theta\cos\theta \\
\Gamma^3_{23} = \Gamma^3_{32} &= \frac{\cos\theta}{\sin\theta}
\end{aligned} \tag{5.40}$$

leading to the following geodesic equations

$$\frac{d^2t}{d\lambda^2} + \frac{2GM}{r(r-2GM)}\frac{dr}{d\lambda}\frac{dt}{d\lambda} = 0 \tag{5.41}$$

$$\frac{d^2r}{d\lambda^2} + \frac{GM}{r^3}(r-2GM)\left(\frac{dt}{d\lambda}\right)^2 - \frac{GM}{r(r-2GM)}\left(\frac{dr}{d\lambda}\right)^2 - (r-2GM)\left(\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right) = (\tag{5.42}$$

$$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{d\theta}{d\lambda}\frac{d\theta}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0 \tag{5.43}$$

$$\frac{d^2\phi}{d\lambda^2} + \frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} + 2\frac{\cos\theta}{\sin\theta}\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} = 0. \tag{5.44}$$

To solve these equation we use the symmetry of the problem manifested by the four Killing vectors (three rotations and one time translation). The existence of Killing vectors imply that

$$K^\mu \frac{dx^\mu}{dt} = \text{const.} \tag{5.45}$$

is conserved along geodesics of the three components of angular momenta (one for magnitude and two for direction) and one component for the energy.

If we choose the coordinates so that the direction of angular momenta is in $\theta = 0$, then the conservation of the direction implies that the motion is in the plane of

$$\theta = \frac{\pi}{2}. \tag{5.46}$$

Then we are left with only two Killing vectors

$$K_\mu = \left(-\left(1 - \frac{2GM}{r}\right), 0, 0, 0\right) \tag{5.47}$$

and

$$L_\mu = \left(0, 0, 0, r^2 \sin^2\theta\right) = \left(0, 0, 0, r^2\right). \tag{5.48}$$

Then the conservation equations (5.45) take the forms of corresponding to conservations of energy

$$\left(1 - \frac{2GM}{r}\right)\frac{dt}{d\lambda} = E \tag{5.49}$$

and (the magnitude of) angular momentum

$$r^2 \frac{d\phi}{d\lambda} = L. \tag{5.50}$$

There are also an additional constants of motion

$$\epsilon = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{5.51}$$

since

$$\frac{dx^\lambda}{d\lambda}\nabla_\lambda\left(-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\right) = -g_{\mu\nu}\frac{dx^\nu}{d\lambda}\left(\frac{dx^\lambda}{d\lambda}\nabla_\lambda\frac{dx^\mu}{d\lambda}\right) - g_{\mu\nu}\frac{dx^\nu}{d\lambda}\left(\frac{dx^\lambda}{d\lambda}\nabla_\lambda\frac{dx^\nu}{d\lambda}\right) = 0. \tag{5.52}$$

For null geodesics $\epsilon = 0$ (e.g. trajectories of massless particles), for time-like geodesics it is convenient to use proper time parametrization so that $\epsilon = +1$ (e.g. trajectories of massive particle), and for space-like trajectories it is convenient to choose parametrization so that $\epsilon = -1$.

For the Schwarzschild metric the conservation equation (5.51) is

$$-\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 = -\epsilon. \tag{5.53}$$

and from (5.49) and (5.50) we get

$$-E^2 + \left(\frac{dr}{d\lambda}\right)^2 + \left(\frac{L^2}{r^2} + \epsilon\right)\left(1 - \frac{2GM}{r}\right) = 0 \tag{5.54}$$

which can be rewritten as

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + V(r) = \frac{1}{2}E^2 \tag{5.55}$$

where

$$V(r) = \frac{1}{2}\epsilon - \frac{\epsilon GM}{r} + \frac{L^2}{2r^2} - \frac{L^2 GM}{r^3}. \tag{5.56}$$

The first two terms in the potential are the same as in Newtonian mechanics, but the last term is a new term due to the effects of general relativity. Note that in the limit of small masses and large distances the correction is negligible.

To find circular orbits we can differentiate the potential and set it to zero. In Newtonian gravity

$$\frac{dV(r)}{dr} = \frac{1}{r^4}\left(\epsilon GMr^2 - L^2 r\right) = 0 \quad \Rightarrow \quad r = \frac{L^2}{\epsilon GM} \tag{5.57}$$

and in general relativity

$$\frac{dV(r)}{dr} = \frac{1}{r^4}\left(\epsilon GMr^2 - L^2 r + 3GML^2\right) = 0 \quad \Rightarrow \quad r = \frac{L^2 \pm \sqrt{L^4 - 12\epsilon G^2 M^2 L^2}}{2\epsilon GM}. \tag{5.58}$$

In the limit of large angular momentum the solutions are given by

$$r = \begin{cases} \frac{L^2}{\epsilon GM} & \text{stable orbit} \\ 3GM & \text{unstable orbit} \end{cases} \tag{5.59}$$

and the two orbits coincide at

$$r = 6GM \tag{5.60}$$

when

$$L = \sqrt{12\epsilon} GM \tag{5.61}$$

which is the lowest bound of the radii of a stable orbits in Schwarzschild metric. Evidently, the possible range of closed stable orbits

$$r > 6GM \tag{5.62}$$

and closed unstable orbits

$$3GM < r < 6GM. \tag{5.63}$$

The situation with non-circular orbits is much more subtle since one needs to solve an equation for $\phi(\lambda)$ which is usually done using power series approximation. The solutions are not ellipses as in Newtonian mechanics and would experience precession. To find the angular solutions we make use of (5.50) or

$$\left(\frac{d\phi}{d\lambda}\right)^{-2} = \frac{r^4}{L^2}. \tag{5.64}$$

to rewrite (5.55) as

$$\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{L^2}r^4 - \frac{2GM}{rL^2}r^3 + r^2 - 2GM\,r = \frac{E^2}{L^2}r^4 \tag{5.65}$$

where we have chosen to work with proper time by setting $\epsilon = 1$. Then in a new variable

$$x = \frac{L^2}{GMr} \tag{5.66}$$

the equation is

$$\left(\frac{dx}{d\phi}\right)^2 + \frac{L^2}{G^2M} - 2x + x^2 - \frac{2G^2M^2}{L^2}x^3 = \frac{2E^2L^2}{G^2M^2}. \tag{5.67}$$

By differentiating it with respect to ϕ we get

$$\frac{d^2x}{d\phi^2} - 1 + x = \frac{3G^2M^2}{L^2}x^2. \tag{5.68}$$

since $\frac{dx}{d\phi} \neq 0$ for non-circular orbits. In Newtonian limit the right hand side would be zero and the solution would be given by

$$x_0 = 1 + e\cos\phi \tag{5.69}$$

where

$$e = \sqrt{1 - \frac{b^2}{a^2}} \tag{5.70}$$

is the eccentricity where a and b are the semi-major and semi-minor axes. To study the leading correction due to effects of general relativity we expand

$$x \approx x_0 + x_1 \tag{5.71}$$

to obtain

$$\frac{d^2x_1}{d\phi^2} + x_1 \approx \frac{3G^2M^2}{L^2}(1 + e\cos\phi)^2 = \frac{3G^2M^2}{L^2}\left(\left(1 + \frac{1}{2}e^2\right) + 2e\cos\phi + \frac{1}{2}e^2\cos 2\phi\right). \tag{5.72}$$

where we used $\cos^2\phi = (\cos 2\phi + 1)/2$. But since

$$\frac{d^2}{d\phi^2}(\phi\sin\phi) + \phi\sin\phi = 2\cos\phi \tag{5.73}$$

$$\frac{d^2}{d\phi^2}(\cos 2\phi) + \cos 2\phi = -3\cos 2\phi \tag{5.74}$$

we can combine the two solutions to match the right hand side of (5.72),

$$x_1 = \frac{3G^2M^2}{L^2}\left(\left(1+\frac{1}{2}e^2\right) + e\phi\sin\phi - \frac{1}{6}e^2\cos 2\phi\right). \tag{5.75}$$

where only the second term is important for large ϕ, and thus,

$$\begin{aligned} x &\approx 1 + e\cos\phi + \frac{3G^2M^2}{L^2}e\phi\sin\phi \\ &\approx 1 + e\cos\left[\left(1 - \frac{3G^2M^2}{L^2}\right)\phi\right]. \end{aligned} \tag{5.76}$$

Therefore during each orbit perihelion advances by

$$\Delta\phi = 2\pi\alpha = \frac{6\pi G^2M^2}{L^2}. \tag{5.77}$$

But from (5.66) and (5.69)

$$L^2 = GMxr \approx GM(1+e\cos\phi)\frac{(1-e^2)a}{1+e\cos\phi} = GM(1-e^2)a \tag{5.78}$$

and thus

$$\Delta\phi = \frac{6\pi GM}{(1-e^2)a}. \tag{5.79}$$

5.4 Black-Holes

Consider a stationary observer with respect to Schwarzschild coordinates with four-velocity

$$U = \left(\left(1 - \frac{2GM}{r}\right)^{-1/2}, 0, 0, 0\right). \tag{5.80}$$

Then the observer would measure frequency of a photon give by

$$\omega = -g_{\mu\nu}U^\mu\frac{dx^\nu}{d\lambda} = \left(1 - \frac{2GM}{r}\right)^{1/2}\frac{dt}{d\lambda} \tag{5.81}$$

and from conservation condition (5.49) along photon's trajectory,

$$\omega = -g_{\mu\nu}U^{\mu}\frac{dx^{\nu}}{d\lambda} = \left(1-\frac{2GM}{r}\right)^{1/2}\frac{dt}{d\lambda} = \left(1-\frac{2GM}{r}\right)^{-1/2}E. \tag{5.82}$$

Thus the frequency shift as measured by two observers at rest with respect to Schwarzschild coordinates is given by

$$\frac{\omega_2}{\omega_1} = \sqrt{\frac{1-\frac{2GM}{r_1}}{1-\frac{2GM}{r_2}}} \approx 1 - \frac{GM}{r_1} + \frac{GM}{r_2} = 1 + \Phi_1 - \Phi_2. \tag{5.83}$$

Thus the photons moving away from the origin (or larger values of Newtonian potential) are red-shifted (or loose energy), and the photons moving towards the origin (or smaller values of Newtonian potential) are blue-shifted (or gain energy).

Consider the behavior of light cones in Schwarzschild coordinates for constant ϕ and $\theta = 0$, i.e.

$$ds^2 = 0 = -\left(1-\frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 \tag{5.84}$$

Then

$$\left(\frac{dt}{dr}\right)^2 = \left(1-\frac{2GM}{r}\right)^{-2} \tag{5.85}$$

or

$$\frac{dt}{dr} = \pm\left(1-\frac{2GM}{r}\right)^{-1}. \tag{5.86}$$

In the limit $r \to 2GM$ the two solutions close up as

$$\lim_{r\to 2GM}\frac{dt}{dr} = \infty. \tag{5.87}$$

This means that it would take an infinite coordinate time to reach the surface $r = 2GM$ for an arbitrary observer, but it does not mean that the proper time to reach the surface is infinite. The main problem is that the coordinate time runs too fast near $r = 2GM$. To fix the problem we can introduce the so called **tortoise coordinates**

$$r^* = r + 2GM\ln\left(\frac{r}{2GM} - 1\right) \tag{5.88}$$

in which (5.86) has a simple solution

$$t = \pm r^* + \text{const}. \tag{5.89}$$

In the tortoise coordinates the Schwarzschild solution takes the following form

$$ds^2 = \left(1 - \frac{2GM}{r}\right)\left(-\mathrm{d}t^2 + \mathrm{d}r^{*2}\right) + r^2 d\Omega^2. \tag{5.90}$$

Evidently the light-cone do not "close up", but the surface $r = 2GM$ was pushed to $r^* = -\infty$. Since our objective is to study the region $r < 2GM$ the tortoise coordinates are not sufficient.

The solution is to consistently follow the future directed curves by replacing t with a null coordinate

$$v = t + r^*. \tag{5.91}$$

Then is we also switch back to the radial coordinate r we get

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}v^2 + (\mathrm{d}v\mathrm{d}r + \mathrm{d}r\mathrm{d}v) + r^2 d\Omega^2. \tag{5.92}$$

The new coordinates are known as **Eddington-Fikelstein** in which

$$\frac{dv}{dr} = 0 \tag{5.93}$$

and

$$\frac{dv}{dr} = 2\left(1 - \frac{2GM}{r}\right)^{-1} \tag{5.94}$$

describe respectively the in-falling and outgoing future directed null trajectories. Note that all of the in-falling light rays are described by horizontal lines

$$v = \text{const} \tag{5.95}$$

and at $r = 2GM$ the outgoing light ray is a vertical line

$$r = \text{const} \tag{5.96}$$

which describes the so-called **event horizon**.

Alternatively one could have chosen to follow the past directed curves by defining

$$u = t - r^* \tag{5.97}$$

so that the metric is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}u^2 + (\mathrm{d}u\mathrm{d}r + \mathrm{d}r\mathrm{d}u) + r^2 d\Omega^2. \tag{5.98}$$

In these coordinates

$$\frac{du}{dr} = 0 \tag{5.99}$$

and

$$\frac{du}{dr} = -2\left(1 - \frac{2GM}{r}\right)^{-1} \tag{5.100}$$

describe respectively the outgoing and in-falling null trajectories.

To describe both the future directed, past directed and space-like geodesics all on the same chart we define

$$v' = \left(\frac{r}{2GM} - 1\right)^{1/2} \exp\left(\frac{r+t}{4GM}\right) \tag{5.101}$$

$$u' = -\left(\frac{r}{2GM} - 1\right)^{1/2} \exp\left(\frac{r-t}{4GM}\right). \tag{5.102}$$

In the new coordinates the metric

$$ds^2 = -\frac{16G^3M^3}{r} \exp\left(-\frac{r}{2GM}\right)(\mathrm{d}v'\mathrm{d}u' + \mathrm{d}u'\mathrm{d}v') + r^2 d\Omega^2 \tag{5.103}$$

does not have any coordinates singularities at the horizon for any geodesic passing through $r = 2GM$. We can now switch from two null coordinates v' and u' back having only one time coordinate

$$T = \frac{1}{2}(v' + u') = \left(\frac{r}{2GM} - 1\right)^{1/2} \exp\left(\frac{r}{4GM}\right) \sinh\left(\frac{t}{4GM}\right) \tag{5.104}$$

and one more spatial coordinate

$$R = \frac{1}{2}(v - u) = \left(\frac{r}{2GM} - 1\right)^{1/2} \exp\left(\frac{r}{4GM}\right) \cosh\left(\frac{t}{4GM}\right) \tag{5.105}$$

(known as **Kruskal coordinates**) in terms of which the metric becomes

$$ds^2 = \frac{32G^3M^3}{r} \exp\left(-\frac{r}{2GM}\right)\left(-\mathrm{d}T^2 + \mathrm{d}R^2\right) + r^2 d\Omega^2 \tag{5.106}$$

where

$$T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) \exp\left(\frac{r}{2GM}\right). \tag{5.107}$$

There are a number of useful properties of Kruskal coordinates:

- Null rays are at ±45% angles

$$T = \pm R + \mathrm{const.} \tag{5.108}$$

- Horizon is a null surface
$$T = \pm R. \tag{5.109}$$
- Constant r surfaces are hyperbolas
$$T^2 - R^2 = \text{const.} \tag{5.110}$$
- Constant t surfaces are straight lines
$$T = R \tanh\left(\frac{t}{4GM}\right). \tag{5.111}$$
- The range of coordinates
$$\begin{array}{ccc} -\infty & \leq R \leq & \infty \\ T^2 & < & R^2 + 1. \end{array} \tag{5.112}$$

The Kruskal diagram provides an easy way of analyzing the Schwarzschild solution where the coordinates have infinite range. Another diagram which is often used to study the causal structure of space-time is known as the conformal diagram (or Penrose diagram, or Carter-Penrose diagram) where the range of coordinates is finite by construction. In the null version the Kruskal coordinates are given by
$$ds^2 = -\frac{16G^3M^3}{r}\exp\left(-\frac{r}{2GM}\right)(\mathrm{d}v'\mathrm{d}u' + \mathrm{d}u'\mathrm{d}v') + r^2 d\Omega^2 \tag{5.113}$$
where
$$v'u' = -\left(\frac{r}{2GM} - 1\right)e^{r/2GM}. \tag{5.114}$$
The one can map the coordinates u' and v' with infinite range to new coordinates
$$v'' = \arctan\left(\frac{v'}{\sqrt{2GM}}\right) \tag{5.115}$$
$$u'' = \arctan\left(\frac{u'}{\sqrt{2GM}}\right) \tag{5.116}$$
with only finite range
$$\begin{array}{ccc} -\frac{\pi}{2} & < v'' < & +\frac{\pi}{2} \\ -\frac{\pi}{2} & < u'' < & +\frac{\pi}{2} \\ -\frac{\pi}{2} & < v'' + u'' < & +\frac{\pi}{2}. \end{array} \tag{5.117}$$
In these new coordinates the diagram for the maximally extended Schwarzschild solution is called a conformal diagram.