Oxford Physics Department

Notes on General Relativity

S. Balbus

Recommended Texts

Weinberg, S. 1972, *Gravitation and Cosmology. Principles and applications of the General Theory of Relativity,* (New York: John Wiley)

What is now the classic reference, but lacking any physical discussions on black holes, and almost nothing on the geometrical interpretation of the equations. The author is explicit in his aversion to anything geometrical in what he views as a field theory. Alas, there is no way to make sense of equations, in any profound sense, without geometry! I also find that calculations are often performed with far too much awkwardness and unnecessary effort. Sections on physical cosmology are its main strength. To my mind, a much better pedagogical text is ...

Hobson, M. P., Efstathiou, G., and Lasenby, A. N. 2006, *General Relativity: An Introduction for Physicists,* (Cambridge: Cambridge University Press)

A very clear, very well-blended book, admirably covering the mathematics, physics, and astrophysics. Excellent coverage on black holes and gravitational radiation. The explanation of the geodesic equation is much more clear than in Weinberg. My favourite. (The metric has a different overall sign in this book compared with Weinberg and this course, so be careful.)

Misner, C. W., Thorne, K. S., and Wheeler, J. A. 1972, *Gravitation,* (New York: Freeman)

At 1280 pages, don't drop this on your toe. Even the paperback version. MTW, as it is known, is often criticised for its sheer bulk, its seemingly endless meanderings and its laboured strivings at building mathematical and physical intuition at every possible step. But I must say, in the end, there really is a lot of very good material in here, much that is difficult to find anywhere else. It is the opposite of Weinberg: geometry is front and centre from start to finish, and there is lots and lots of black hole physics. I very much like its discussion on gravitational radiation, though this is not part of the syllabus. (Learn it anyway!) There is a Track 1 and Track 2 for aid in navigation, Track 1 being the essentials.

Hartle, J. B. 2003, *Gravity: An Introduction to Einstein's General Theory of Relativity,* (San Francisco: Addison Wesely)

This is GR Lite, at a very different level from the previous three texts. But for what it is meant to be, it succeeds very well. Coming into the subject cold, this is not a bad place to start to get the lay of the land, to understand the issues in their broadest sense, and to be treated to a very accessible presentation. There will be times in your study of GR when it will be difficult to see the forests for the trees, when you will fell awash in a sea of indices and formalism. That will be a good moment to spend time with this text.

Notational Conventions & Miscellany

Space-time dimensions are labelled $0, 1, 2, 3$ or (Cartesian) t, x, y, z or (spherical) t, r, θ, ϕ. Time is always the 0-component.

Repeated indices are summed over, unless otherwise specified. (Einstein summation convention.)

The Greek indices $\kappa, \lambda, \mu, \nu$ etc. are uses to represent arbitrary space-time components in all general relativity calculations.

The Greek indices α, β, etc. are used to represent arbitrary space-time components in special relativity calculations (Minkowski space-time).

The Roman indices i, j, k are used to represent purely spatial components in any space-time.

The Roman indices a, b, c, d are used to represent fiducial space-time components for mnemonic aids, and in discussions of how to perform generic index-manipulations and permutations where Greek indices may cause confusion.

$*$ is used as a generic dummy index, summed over.

The tensor $\eta^{\alpha\beta}$ is numerically identical to $\eta_{\alpha\beta}$ with $-1, 1, 1, 1$ corresponding to the $00, 11, 22, 33$ diagonal elements.

Viewed as matrices, the metric tensors $g_{\mu\nu}$ and $g^{\mu\nu}$ are inverses. For diagonal metrics, their respective elements are therefore reciprocals.

c almost always denotes the speed of light. It is very occassionally used as an obvious tensor index. c is never set to unity unless explicitly stated to the contrary. (Relativity texts often set $c = 1$.) G is *never* unity no matter what. And don't even think of setting 2π to unity.

Notice that it is "Lorentz invariance," but "Lorenz gauge." Not a typo, two different blokes.

Really Useful Numbers

$c = 2.99792458 \times 10^8$ m s^{-1} (Exact speed of light.)

$c^2 = 8.9875517873681764 \times 10^{16}$ m^2 s^{-2} (Exact!)

$G = 6.67384 \times 10^{-11}$ m^3 kg^{-1} s^{-2} (Newton's G.)

$M_\odot = 1.98855 \times 10^{30}$ kg (Mass of the Sun.)

$r_\odot = 6.955 \times 10^8$ m (Radius of the Sun.)

$GM_\odot = 1.32712440018 \times 10^{20}$ m^3 s^{-2} (Solar gravitational parameter; more accurate than either G or $M_\odot$ separately.)

$2GM_\odot/c^2 = 2.9532500765 \times 10^3$ m (Solar Schwarzschild radius.)

$GM_\odot/c^2 r_\odot = 2.1231 \times 10^{-6}$ (Solar relativity parameter.)

$M_\oplus = 5.97219 \times 10^{24}$ kg (Mass of the Earth)

$r_\oplus = 6.371 \times 10^6$ m (Mean Earth radius.)

$GM_\oplus = 3.986004418 \times 10^{14}$ m^3 s^{-2}(Earth gravitational parameter.)

$2GM_\oplus/c^2 = 8.87005608 \times 10^{-3}$ m (Earth Schwarzschild radius.)

$GM_\oplus/c^2 r_\oplus = 6.961 \times 10^{-10}$ (Earth relativity parameter.)

For diagonal $g_{\mu\nu}$:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2g_{\lambda\lambda}} \left(\frac{\partial g_{\lambda\mu}}{\partial x^\nu} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \right) \quad \text{NO SUM OVER } \lambda$$

$$R_{\mu\kappa} = \frac{1}{2} \frac{\partial^2 \ln|g|}{\partial x^\kappa \partial x^\mu} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\lambda} + \Gamma^\eta_{\mu\lambda} \Gamma^\lambda_{\kappa\eta} - \frac{\Gamma^\eta_{\mu\kappa}}{2} \frac{\partial \ln|g|}{\partial x^\eta} \quad \text{FULL SUMMATION}$$

Contents

*Most of the fundamental ideas of
science are essentially simple, and
may, as a rule, be expressed in a
language comprehensible to everyone.*

— *Albert Einstein*

1 An overview

1.1 The legacy of Maxwell

We are told by the historians that the greatest Roman generals would have their most
important victories celebrated with a triumph. The streets would line with adoring crowds,
cheering wildly in support of their hero as he passed by in a grand procession. But the
Romans astutely realised the need for a counterpoise, so a slave would ride with the general,
whispering in his ear, "All glory is fleeting."

All glory is fleeting. And never more so than in theoretical physics. No sooner is a triumph
hailed, but unforseen puzzles emerge that couldn't possibly have been anticipated before the
breakthrough. The mid-nineteenth century reduction of all electromagnetic phenomena to
four equations, the "Maxwell Equations," is very much a case in point.

Maxwell's equations united electricity, magnetism, and optics, showing them to be differ-
ent manifestations of the same field. The theory accounted for the existence of electromag-
netic waves, explained how they propagate, and that the propagation velocity is $1/\sqrt{\epsilon_0\mu_0}$ (ϵ_0
is the permitivity, and μ_0 the permeability, of free space). This combination is numerically
precisely equal to the speed of light. Light is electromagnetic radiation! The existence of
electromagnetic radiation was then verified by brilliant experiments carried out by Heinrich
Hertz in 1887, in which the radiation was directly generated and detected.

But Maxwell's theory, for all its success, had disquieting features when one probed. For
one, there seemed to be no provision in the theory for allowing the velocity of light to change
with the observer's velocity. The speed of light is aways $1/\sqrt{\epsilon_0\mu_0}$. A related point was
that simple Galilean invariance was not obeyed, i.e. absolute velocities seemed to affect the
physics, something that had not been seen before. Lorentz and Larmor in the late nineteenth
century discovered that Maxwell's equations did have a simple mathematical velocity trans-
formation that left them invariant, but it was *not* Galilean, and most bizarrely, it involved
changing the *time.* The non-Galilean character of the transformation equation relative to
the "aetherial medium" hosting the waves was put down, a bit vaguely, to electromagnetic
interactions between charged particles that truly changed the length of the object. As to the
time change, well, one would have to put up with it as an aetherial formality.

All was resolved in 1905 when Einstein showed how, by adopting as a postulates (i) that
the speed of light was constant in all frames (as had already been indicated by a body of
irrefutable experiments, including the famous Michelson-Morley investigation); (ii) the aban-
donment of the increasingly problematic aether medium that supposedly hosted these waves;
and (iii) reinstating the truly essential Galilean notion that relative uniform velocity cannot
be detected by any physical experiment, that the "Lorentz transformations" (as they had
become known) must follow. *All* equations of physics, not just electromagnetic phenomena,
had to be invariant in form under these Lorentz transformations, even with its peculiar rela-
tive time variable. These ideas and the consequences that ensued collectively became known
as *relativity theory*, in reference to the invariance of form with respect to relative velocities.

The relativity theory stemming from Maxwell's equations is rightly regarded as one of the crown jewels of 20th century physics. In other words, a triumph.

1.2   The legacy of Newton

Another triumph, another problem. If indeed, all of physics had to be compatible with relativity, what of Newtonian gravity? It works incredibly well, yet it is manifestly *not* compatible with relativity, because Poisson's equation

$$\nabla^2 \Phi = 4\pi G \rho \tag{1}$$

implies instantaneous transmission of changes in the gravitational field from source to potential. (Here Φ is the Newtonian potential function, G the Newtonian gravitational constant, and ρ the mass density.) Wiggle the density locally, and throughout all of space there must instantaneously be a wiggle in Φ, as given by equaton (1).

In Maxwell's theory, the *electrostatic* potential satisfies its own Poisson equation, but the appropriate time-dependent potential obeys a wave equation:

$$\nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}, \tag{2}$$

and solutions of this equation propagate signals at the speed of light c. In retrospect, this is rather simple. Mightn't it be the same for gravity?

No. The problem is that the source of the signals for the electric potential field, i.e. the charge density, behaves differently from the source for the gravity potential field, i.e. the mass density. The electrical charge of an individual bit of matter does not change when the matter is viewed in motion, but the mass does: the mass increases with velocity. This seemingly simple detail complicates everything. Moreover, in a relativisitic theory, energy, like matter, is a source of a gravitational field, including the distributed energy of the gravitational field itself! A relativisitic theory of gravity would have to be nonlinear. In such a time-dependent theory of gravity, it is not even clear a priori what the appropriate mathematical objects should be on either the right side or the left side of the wave equation. Come to think of it, should we be using a wave equation at all?

1.3   The need for a geometrical framework

In 1908, the mathematician Hermann Minkowski came along and argued that one should view the Lorentz transformations not merely as a set of rules for how coordinates (including a time coordinate) change from one constant-velocity reference frame to another, but that these coordinates should be regarded as living in their own sort of pseudo-Euclidian geometry—a *space-time,* if you will: "Minkowski space."

To understand the motivation for this, start simply. We know that in ordinary Euclidian space we are free to choose any coordinates we like, and it can make no difference to the description of the space itself, for example, in measuring how far apart objects are. If (x, y) is a set of Cartesian coordinates for the plane, and (x', y') another coordinate set related to the first by a rotation, then

$$dx^2 + dy^2 = dx'^2 + dy'^2 \tag{3}$$

i.e., the distance between two closely spaced points is the same number, regardless of the coordinates used. $dx^2 + dy^2$ is said to be an "invariant."

Now, an abstraction. There is nothing special from a mathematical viewpoint about the use of $dx^2 + dy^2$ as our so-called metric. Imagine a space in which the metric invariant was $dy^2 - dx^2$. From a purely mathematical point of view, we needn't worry about the plus/minus sign. An invariant is an invariant. However, with $dy^2 - dx^2$ as our invariant, we are describing a Minkowski space, with $dy = cdt$ and dx an ordinary space interval, just as before. The fact that $c^2dt^2 - dx^2$ is an invariant quantity is precisely what we need in order to guarantee that the speed of light is always constant—an invariant! In this case, $c^2dt^2 - dx^2$ is always zero for light propagation along x, whatever coordinates (read "observers") are involved, and more generally,

$$c^2dt^2 - dx^2 - dy^2 - dz^2 = 0 \tag{4}$$

will guarantee the same in any direction. We have thus taken a kinematical requirement—that the speed of light be a universal constant—and given it a geometrical interpretation in terms of an invariant quantity (a "quadratic form" as it is sometimes called) in Minkowski space.

Pause. As the French would say, "Bof." And so what? Call it whatever you like. Who needs obfuscating mathematical pretence? Eschew obfuscation! The Lorentz transform stands on its own! That was very much Einstein's initial take on Minkowski's pesky little meddling with his theory.

But Einstein soon changed his tune, for it is the geometrical viewpoint that is the more fundamental. Einstein's great revelation, his big idea, was that *gravity arises because the effect of the presence of matter in the universe is to* distort *Minkowski's space-time. The distortions manifest themselves as what we view as the force of gravity, and thus these same distortions must become, in the limit of weak gravity, familiar Newtonian theory. Gravity itself is purely geometrical.*

Now that is one big idea. It is an idea that will take the rest of this course—and beyond—to explain. How did Einstein make this leap? Why did he change his mind? Where did this notion of geometry come from?

From a simple observation. In a freely falling elevator, or more safely in an aircraft executing a ballistic parabolic arch, one feels "weightless." That is, the effect of gravity can be made to locally disappear in the appropriate reference frame—the right coordinates. This is due to the fact that gravity has exactly the same effect on all mass, regardless of its composition, which is just what we would expect if objects were responding to background geometrical distortions instead of an applied force. In the effective absence of gravity, we locally return to the environment of undistorted ("flat" in mathematical parlance) Minkowski space-time, much as a flat Euclidian tangent plane is an excellent local approximation to the surface of a curved sphere. (Which is of course why it is easy to be fooled that the earth is globally flat.) The tangent plane coordinates locally "eliminate" spherical geometry complications. Einstein's notion that the effects of gravity are to cause a distortion of Minkowski space-time, and that it is always possible to find coordinates in which the local distortions may be similarly eliminated to leading order, is the foundational insight of general relativity. It is known as the *Equivalence Principle.* We will have much more to say on this topic.

Space-time. *Space-time.* Bringing in time, you see, is everything. Non-Euclidean geometry as developed by the great mathematician Bernhard Riemann begins with notion that any space looks *locally* "flat." Riemannian geometry is the language of gravitational theory, and Riemann himself had the notion that gravity might arise from a non-Euclidian curvature in three-dimensional space. He got nowhere, because time was not part of his geometry. It was the (underrated) genius of Minkowski to incorporate time into a purely geometrical theory that allowed Einstein to take the crucial next step, freeing himself to think of gravity solely in geometrical terms, without having to ponder over whether it made any sense to have time

as part of the geometrical framework. In fact, the Newtonian limit is reached *not* from the leading order curvature terms in the *spatial* part of the geometry, but from the leading order "curvature" (if that is the word) of the time.

Riemann created the mathematics of non-Euclidian geometry. Minkoswki realised that natural language of the Lorentz transformations was geometrical, including time as a key component of the geometrical interpretation. Einstein took the great leap of realising that gravity arises from the distortions of Minkowski's flat space-time created by the existence of matter.

Well done. You now understand the conceptual framework of general relativity, and that is itself a giant leap. From here on, it is just a matter of the technical details. But then, you and I also can paint like Leonardo da Vinci. It is just a matter of the technical details.

From henceforth, space by itself and time by itself, have vanished into the merest shadows, and only a blend of the two exists in its own right.

— *Hermann Minkowski*

2 The toolbox of geometrical theory: special relativity

In what sense is general relativity "general?" In the sense that since we are dealing with an abstract space-time geometry, the essential mathematical description must be the same in *any* coordinate system at all, not just those related by constant velocity reference frame shifts, or even just those coordinate transformations that make tangible physical sense. *Any coordinates at all.* Full stop.

We need the coordinates for our description of the structure of space-time, but somehow the essential physics (and other mathematical properties) must not depend on them, and it is no easy business to formulate a theory which satisfies this restriction. We owe a great deal to Bernhard Riemann for coming up with a complete mathematical theory for these non-Euclidian geometries. The sort of geometry in which it is always possible to find coordinates in which the space looks locally smooth is known as a *Riemannian manifold.* Mathematicians would say that an n-dimensional manifold is *homeomorphic* to n-dimensional Euclidian space. Actually, since our invariant interval $c^2 dt^2 - dx^2$ is not a simple sum of squares, but contains a minus sign, the manifold is said to be *pseudo-Riemannian.* Pseudo or no, the descriptive mathematical machinery is the same.

The objects that geometrical theories work with are scalars, vectors, and higher order tensors. You have certainly seen scalars and vectors before in your other physics courses, and you may have encountered tensors as well. We will need to be very careful how we define these objects, and very careful to distinguish them from objects that look like vectors and tensors (because they have the appropriate number of components) but actually are not.

To set the stage, we begin with the simplest geometrical objects of Minkowski space-time that are not just simple scalars: the 4-vectors.

2.1 The 4-vector formalism

In their most elementary form, the familiar Lorentz transformations from "fixed" laboratory coordinates (t, x, y, z) to moving frame coordinates (t', x', y', z') take the form

$$ct' = \gamma(ct - vx/c) = \gamma(ct - \beta x) \tag{5}$$

$$x' = \gamma(x - vt) = \gamma(x - \beta ct) \tag{6}$$

$$y' = y \tag{7}$$

$$z' = z \tag{8}$$

where v is the relative velocity (taken along the x axis), c the speed of light, $\beta = v/c$ and

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \equiv \frac{1}{\sqrt{1 - \beta^2}} \tag{9}$$

is the *Lorentz factor*. The primed frame can be thought of as the frame moving with an object we are studying, that is to say the object's rest frame. To go backwards to find (x, t) as a function (x', t'), just interchange the primed and unprimed coordinates in the above equations, and then flip the sign of v. Do you understand why this works?

Exercise. Show that in a coordinate free representation, the Lorentz transformations are

$$ct' = \gamma(ct - \boldsymbol{\beta} \cdot \boldsymbol{x}) \tag{10}$$

$$\boldsymbol{x'} = \boldsymbol{x} + \frac{(\gamma - 1)}{\beta^2}(\boldsymbol{\beta} \cdot \boldsymbol{x})\boldsymbol{\beta} - \gamma ct\boldsymbol{\beta} \tag{11}$$

where $c\boldsymbol{\beta} = \boldsymbol{v}$ is the vector velocity and boldface x's are spatial vectors. (Hint: This is not *nearly* as scary as it looks! Note that $\boldsymbol{\beta}/\beta$ is just a unit vector in the direction of the velocity and sort out the components of the equation.)

Exercise. The Lorentz transformation can be made to look more rotation-like by using hyperbolic trigonometry. The idea is to place equations (5)–(8) on the same footing as the transformation of Cartesian position vector components under a simple rotation, say about the z axis:

$$x' = x\cos\theta + y\sin\theta \tag{12}$$

$$y' = -x\sin\theta + y\cos\theta \tag{13}$$

$$z' = z \tag{14}$$

Show that if we define

$$\beta \equiv \tanh\zeta, \tag{15}$$

then

$$\gamma = \cosh\zeta, \quad \gamma\beta = \sinh\zeta, \tag{16}$$

and

$$ct' = ct\cosh\zeta - x\sinh\zeta, \tag{17}$$

$$x' = -ct\sinh\zeta + x\cosh\zeta. \tag{18}$$

What happens if we apply this transformation twice, once with "angle" ζ from (x, t) to (x', t'), then with angle ξ from (x', t') to (x'', t'')? How is (x, t) related to (x'', t'')?

Following on, rotations can be made to look more Lorentz-like by introducing

$$\alpha \equiv \tan\theta, \quad \Gamma \equiv \frac{1}{\sqrt{1 + \alpha^2}} \tag{19}$$

Then show that (12) and (13) become

$$x' = \Gamma(x + \alpha y) \tag{20}$$

$$y' = \Gamma(y - \alpha x) \tag{21}$$

Thus, while a having a different appearance, the Lorentz and rotational transformations have mathematical structures that are similar.

Of course lots of quantities besides position are vectors, and it is possible (indeed desirable) just to *define* a quantity as a vector if its individual components satisfy equations (12)–(14). Likewise, we find that many quantities in physics obey the transformation laws of

equations (5–8), and it is therefore natural to give them a name and to probe their properties more deeply. We call these quantities 4-vectors. They consist of an ordinary vector $\boldsymbol{V}$, together with an extra component —a "time-like" component we will designate as V^0. (We use superscripts for a reason that will become clear later.) The "space-like" components are then V^1, V^2, V^3. The generic form for a 4-vector is written V^α, with α taking on the values 0 through 3. Symbolically,

$$V^\alpha = (V^0, \boldsymbol{V}) \tag{22}$$

We have seen that $(ct, \boldsymbol{x})$ is one 4-vector. Another, you may recall, is the *4-momentum*,

$$p^\alpha = (E/c, \boldsymbol{p}) \tag{23}$$

where $\boldsymbol{p}$ is the ordinary momentum vector and E is the total energy. Of course, we speak of relativisitic momentum and energy:

$$\boldsymbol{p} = \gamma m \boldsymbol{v}, \quad E = \gamma m c^2 \tag{24}$$

where m is a particle's rest mass. Just as

$$(ct)^2 - x^2 \tag{25}$$

is an invariant quantity under Lorentz transformations, so to is

$$E^2 - (pc)^2 = m^2 c^4 \tag{26}$$

A rather plain 4-vector is p^α without the coefficient of m. This is the 4-velocity U^α,

$$U^\alpha = \gamma(c, \boldsymbol{v}) \tag{27}$$

Note that in the rest frame of a particle, $U^0 = c$ (a constant) and the ordinary 3-velocity components $\boldsymbol{U} = 0$. To get to any other frame, just use ("boost with") the Lorentz transformation. (Be careful with the sign of v). We don't have to worry that we boost along one axis only, whereas the velocity has three components. If you wish, just rotate the axes, after we've boosted. This sorts out all the 3-vector components the way you'd like, and leaves the time ("0") component untouched.

Humble in appearance, the 4-velocity is a most important 4-vector. Via the simple trick of boosting, the 4-velocity may be used as the starting point for constructing many other important physical 4-vectors. Consider, for example, a charge density ρ_0 which is at rest. We may create a 4-vector which, in the rest frame, has only one component: $\rho_0 c$ is the lonely time component and the ordinary spatial vector components are all zero. It is just like U^α, only with a different normalisation constant. Now boost! The resulting 4-vector is denoted

$$J^\alpha = \gamma(c\rho_0, \boldsymbol{v}\rho_0) \tag{28}$$

The time component give the charge density in any frame, and the 3- vector components are the corresponding standard current density $\boldsymbol{J}$! This 4-current is the fundamental 4-vector of Maxwell's theory. As the source of the fields, this 4-vector source current is the basis for Maxwell's electrodynamics being a fully relativistic theory. J^0 is the source of the electric field potential function Φ, and and $\boldsymbol{J}$ is the source of the magnetic field vector potential $\boldsymbol{A}$, and, as we will shortly see,

$$A^\alpha = (\Phi, \boldsymbol{A}/c) \tag{29}$$

is itself a 4-vector! Then, we can generate the fields themselves from the potentials by constructing a tensor...well, we are getting a bit ahead of ourselves.

13

2.2 More on 4-vectors

2.2.1 Transformation of gradients

We have seen how the Lorentz transformation express x'^{α} as a function of the x coordinates. It is a simple linear transformation, and the question naturally arises of how the partial derivatives, $\partial/\partial t$, $\partial/\partial x$ transform, and whether a 4-vector can be constructed from these components. This is a simple exercise. Using

$$ct = \gamma(ct' + \beta x') \tag{30}$$

$$x = \gamma(x' + \beta ct') \tag{31}$$

we find

$$\frac{\partial}{\partial t'} = \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} + \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} = \gamma\frac{\partial}{\partial t} + \gamma\beta c\frac{\partial}{\partial x} \tag{32}$$

$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t} = \gamma\frac{\partial}{\partial x} + \gamma\beta\frac{1}{c}\frac{\partial}{\partial t} \tag{33}$$

In other words,

$$\frac{1}{c}\frac{\partial}{\partial t'} = \gamma\left(\frac{1}{c}\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial x}\right) \tag{34}$$

$$\frac{\partial}{\partial x'} = \gamma\left(\frac{\partial}{\partial x} + \beta\frac{1}{c}\frac{\partial}{\partial t}\right) \tag{35}$$

and for completeness,

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y} \tag{36}$$

$$\frac{\partial}{\partial z'} = \frac{\partial}{\partial z}. \tag{37}$$

This is *not* the Lorentz transformation (5)–(8); it differs by the sign of v. By contrast, *coordinate differentials* dx^{α} transform, of course, just like x^{α}:

$$cdt' = \gamma(cdt - \beta dx), \tag{38}$$

$$dx' = \gamma(dx - \beta cdt), \tag{39}$$

$$dy' = dy, \tag{40}$$

$$dz' = dz. \tag{41}$$

This has a very important consequence:

$$dt'\frac{\partial}{\partial t'} + dx'\frac{\partial}{\partial x'} = \gamma^2\left[(dt - \beta\frac{dx}{c})\left(\frac{\partial}{\partial t} + \beta c\frac{\partial}{\partial x}\right) + (dx - \beta cdt)\left(\frac{\partial}{\partial x} + \beta\frac{1}{c}\frac{\partial}{\partial t}\right)\right], \tag{42}$$

or simplifying,

$$dt'\frac{\partial}{\partial t'} + dx'\frac{\partial}{\partial x'} = \gamma^2(1 - \beta^2)\left(dt\frac{\partial}{\partial t} + dx\frac{\partial}{\partial x}\right) = dt\frac{\partial}{\partial t} + dx\frac{\partial}{\partial x} \tag{43}$$

Adding y and z into the mixture changes nothing. Thus, a scalar product exists between dx^{α} and $\partial/\partial x^{\alpha}$ that yields a *Lorentz scalar,* much as $\boldsymbol{dx}\cdot\boldsymbol{\nabla}$, the ordinary complete differential, is a rotational scalar. It is the fact that only certain combinations of 4-vectors and 4-gradients appear in the equations of physics that allows these equations to remain invariant in form from one reference frame to another.

It is time to approach this topic, which is the mathematical foundation on which special and general relativity is built, on a firmer and more systematic footing.

2.2.2 Transformation matrix

We begin with a simple but critical notational convention: repeated indices are summed over, unless otherwise explicitly stated. This is known as the *Einstein summation convention,* invented to avoid tedious repeated summation Σ's. For example:

$$dx^\alpha \frac{\partial}{\partial x^\alpha} = dt \frac{\partial}{\partial t} + dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} \tag{44}$$

I will often further shorten this to $dx^\alpha \partial_\alpha$. This brings us to another important notational convention. I was careful to write ∂_α, not ∂^α. Superscripts will be reserved for vectors, like dx^α which transform like (5) through (8) from one frame to another (primed) frame moving a relative velocity v along the x axis. Subscripts will be used to indicate vectors that transfrom like the gradient components in equations (34)–(37). Superscipt vectors like dx^α are referred to as *contravariant* vectors; subscripted vectors as *covariant.* (The names will acquire significance later.) The co- contra- difference is an important distinction in general relativity, and we begin by respecting it here in special relativity.

Notice that we can write equations (38) and (39) as

$$[-cdt'] = \gamma([-cdt] + \beta dx) \tag{45}$$

$$dx' = \gamma(dx + \beta[-cdt]) \tag{46}$$

so that the 4-vector $(-cdt, dx, dy, dz)$ is covariant, like a gradient! We therefore have

$$dx^\alpha = (cdt, dx, dy, dz) \tag{47}$$

$$dx_\alpha = (-cdt, dx, dy, dz) \tag{48}$$

It is easy to go between covariant and contravariant forms by flipping the sign of the time component. We are motivated to formalise this by introducing a matrix $\eta_{\alpha\beta}$ defined as

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{49}$$

Then $dx_\alpha = \eta_{\alpha\beta} dx^\beta$ "lowers the index." We will write $\eta^{\alpha\beta}$ to raise the index, though it is a numerically identical matrix. Note that the invariant space-time interval may be written

$$c^2 d\tau^2 \equiv c^2 dt^2 - dx^2 - dy^2 - dz^2 = -\eta_{\alpha\beta} dx^\alpha dx^\beta \tag{50}$$

The time interval $d\tau$ is just the "proper time," the time shown ticking on the clock in the rest frame moving with the object of interest (since in this frame all spatial differentials dx^i are zero). Though introduced as a bookkeeping device, $\eta_{\alpha\beta}$ is an important quantity: it goes from being a constant matrix in special relativity to a function of coordinates in general relativity, mathematically embodying the departures of space-time from simple Minkowski form when matter is present.

The standard Lorentz transformation may now be written as a matrix equation, $dx'^\alpha = \Lambda^\alpha_\beta dx^\beta$, where

$$\Lambda^\alpha_\beta \, dx^\beta = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{pmatrix} \tag{51}$$

This is symmetric in α and β. (A possible notational ambiguity is difficult to avoid here: β used as a subscript or superscript of course never means v/c. Used in this way it is just a space-time index.) Direct matrix multiplication gives (do it, and notice that the η matrix must go in the middle...why?):

$$\Lambda^\alpha_\beta \Lambda^\nu_\mu \eta_{\alpha\nu} = \eta_{\beta\mu} \tag{52}$$

Then, if V^α is any contravariant vector and W_α any covariant vector, $V^\alpha W_\alpha$ must be an invariant (or "scalar") because

$$V'^\alpha W'_\alpha = V'^\alpha W'^\beta \eta_{\beta\alpha} = \Lambda^\alpha_\mu V^\mu \Lambda^\beta_\nu W^\nu \eta_{\beta\alpha} = V^\mu W^\nu \eta_{\mu\nu} = V^\mu W_\mu \tag{53}$$

For covariant vectors, for example ∂_α, the transformation is $\partial'_\alpha = \tilde{\Lambda}^\beta_\alpha \partial_\beta$, where $\tilde{\Lambda}^\beta_\alpha$ is the same as Λ^β_α, but the sign of β reversed:

$$\tilde{\Lambda}^\alpha_\beta = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{54}$$

Note that

$$\tilde{\Lambda}^\alpha_\beta \Lambda^\beta_\mu = \delta^\alpha_\mu, \tag{55}$$

where δ^α_μ is the Kronecker delta function. This leads immediately once again to $V'^\alpha W'_\alpha = V^\alpha W_\alpha$.

Notice that equation (38) says something rather interesting in terms of 4-vectors. The right side is just proportional to $-dx^\alpha U_\alpha$, where U_α is the (covariant) 4-vector corresponding to ordinary velocity v. Consider now the case $dt' = 0$, a surface in t, x, y, z, space-time corresponding to simultaneity in the frame of an observer moving at velocity $\boldsymbol{v}$. The equations of constant time in this frame are given by the requirement that dx^α and U_α are orthogonal.

Exercise. Show that the general Lorentz transformation matrix is:

$$\Lambda^\alpha_\beta = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+(\gamma-1)\beta_x^2/\beta^2 & (\gamma-1)\beta_x\beta_y/\beta^2 & (\gamma-1)\beta_x\beta_z/\beta^2 \\ -\gamma\beta_y & (\gamma-1)\beta_x\beta_y/\beta^2 & 1+(\gamma-1)\beta_y^2/\beta^2 & (\gamma-1)\beta_y\beta_z/\beta^2 \\ -\gamma\beta_z & (\gamma-1)\beta_x\beta_z/\beta^2 & (\gamma-1)\beta_y\beta_z/\beta^2 & 1+(\gamma-1)\beta_z^2/\beta^2 \end{pmatrix} \tag{56}$$

Hint: Keep calm and use (10) and (11).

2.2.3 Tensors

There is more to relativistic life than vectors and scalars. There are objects called tensors, with more that one indexed component. But possessing indices isn't enough! All tensor components must transform in the appropriate way under a Lorentz transformation. Thus, a tensor $T^{\alpha\beta}$ transforms according to the rule

$$T'^{\alpha\beta} = \Lambda^\alpha_\mu \Lambda^\beta_\nu T^{\mu\nu}, \tag{57}$$

while

$$T'_{\alpha\beta} = \tilde{\Lambda}^\mu_\alpha \tilde{\Lambda}^\nu_\beta T_{\mu\nu}, \tag{58}$$

and of course
$$T'^{\alpha}_{\beta} = \Lambda^{\alpha}_{\mu}\tilde{\Lambda}^{\nu}_{\beta}T^{\mu}_{\nu}, \tag{59}$$

You get the idea. Contravariant superscript use Λ, covariant subscript use $\tilde{\Lambda}$.

Tensors are not hard to find. Remember equation (52)?

$$\Lambda^{\alpha}_{\beta}\Lambda^{\nu}_{\mu}\eta_{\alpha\nu} = \eta_{\beta\mu} \tag{60}$$

So $\eta_{\alpha\beta}$ is a tensor, with the same components in any frame! The same is true of δ^{α}_{β}, a *mixed* tensor (which is the reason for writing its indices as we have), that we must transform as follows:

$$\Lambda^{\nu}_{\mu}\tilde{\Lambda}^{\alpha}_{\beta}\delta^{\mu}_{\alpha} = \Lambda^{\nu}_{\alpha}\tilde{\Lambda}^{\alpha}_{\beta} = \delta^{\beta}_{\nu}. \tag{61}$$

Here is another tensor, slightly less trivial:

$$W^{\alpha\beta} = U^{\alpha}U^{\beta} \tag{62}$$

where the $U's$ are 4-velocities. This obviously transforms as tensor, since each U obeys its own vector transformation law. Consider next the tensor

$$T^{\alpha\beta} = \rho\langle u^{\alpha}u^{\beta}\rangle \tag{63}$$

where the $\langle\ \rangle$ notation indicates an average of all the 4-velocity products $u^{\alpha}u^{\beta}$ taken over a whole swarm of little particles, like a gas. (An average of 4-velocities is certainly itself a 4-velocity, and an average of all the little particle tensors is itself a tensor.) ρ is the local rest density.

The component T^{00} is just ρc^2, the energy density of the swarm. Moreover, if, as we shall assume, the particle velocities are isotropic, then $T^{\alpha\beta}$ vanishes if $\alpha \neq \beta$. When $\alpha = \beta \neq 0$, then $T^{\alpha\beta}$ is by definition the pressure P of the swarm. Hence, in the frame in which the swarm has no net bulk motion,

$$T^{\alpha\beta} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \tag{64}$$

This is, in fact, the most general form for the so-called *energy-momentum stress tensor* for an isotropic fluid in the rest frame of the fluid.

To find $T^{\alpha\beta}$ in any frame with 4-velocity U^{α} we could adopt a brute force method and apply the Λ matrix twice to the rest frame form, but what a waste of effort of that would be! If we can find *any* true tensor that agrees with our result in the rest frame, then that tensor is *the* unique tensor. Proof: if a tensor is zero in any frame, then it is zero in all frames, as a trivial consequence of the transformation law. Suppose the tensor I construct, which is designed to match the correct rest frame value, may not be (you think) correct in all frames. Hand me your tensor, what you think is the correct choice. Now, the two tensors by definition match in the rest frame. I'll subtract one from the other to form the difference between my tensor and the true tensor. The difference is also a tensor, but it vanishes in the rest frame by construction. Hence this "difference tensor" must vanish in all frames, so your tensor and mine are identical after all! Corollary: if you can prove that the two tensors are the same in any one particular frame, then they are the same in all frames. This is a very useful ploy.

17

The only two tensors we have at our disposal to construct $T^{\alpha\beta}$ are $\eta^{\alpha\beta}$ and $U^{\alpha}U^{\beta}$, and there is only one linear superposition that matches the rest frame value and does the trick:

$$T^{\alpha\beta} = P\eta^{\alpha\beta} + (\rho + P/c^2)U^{\alpha}U^{\beta} \tag{65}$$

This is the general form of energy-momentum stress tensor appropriate to an ideal fluid.

2.2.4 Conservation of $T^{\alpha\beta}$

One of the most salient properties of $T^{\alpha\beta}$ is that it is *conserved*, in the sense of

$$\frac{\partial T^{\alpha\beta}}{\partial x^{\alpha}} = 0 \tag{66}$$

Since gradients of tensors transform as tensors, this must be true in all frames. So what, exactly, are we conserving?

First, the time-like 0-component of this equation is

$$\frac{\partial}{\partial t}\left[\gamma^2\left(\rho + \frac{Pv^2}{c^4}\right)\right] + \boldsymbol{\nabla\cdot}\left[\gamma^2\left(\rho + \frac{P}{c^2}\right)\boldsymbol{v}\right] = 0 \tag{67}$$

which is the relativistic version of mass conservation,

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla\cdot}(\rho\boldsymbol{v}) = 0. \tag{68}$$

Elevated in special relativity, it becomes a statement of *energy conservation.* So one of the things we are conserving is energy. This is good.

The spatial part of the conservation equation reads

$$\frac{\partial}{\partial t}\left[\gamma^2\left(\rho + \frac{P}{c^2}\right)v_i\right] + \left(\frac{\partial}{\partial x^j}\right)\left[\gamma^2\left(\rho + \frac{P}{c^2}\right)v_iv_j\right] + \frac{\partial P}{\partial x^i} = 0 \tag{69}$$

You may recognise this as Euler's equation of motion, a statement of momentum conservation, upgraded to special relativity. Conserving momentum is also good.

What if there are other external forces? The idea is that these are included by expressing them in terms of the divergence of their own stress tensor. Then it is the total $T^{\alpha\beta}$ including, say, electromagnetic fields, that comes into play. What about the force of gravity? *That,* it will turn out, is on an all-together different footing.

You start now to gain a sense of the difficulty in constructing a theory of gravity compatible with relativity. The density ρ is part of the stress tensor, and it is the entire stress tensor in a relativistic theory that would have to be the source of the gravitational field, just as the entire 4-current J^{α} is the source of electromangetic fields. No fair just picking the component you want. Relativistic theories work with scalars, vectors and tensors to preserve their invariance properties from one frame to another. This insight is already an achievement: we can, for example, expect pressure to play a role in generating gravitational fields. Would you have guessed that? Our relativistic gravity equation maybe ought to look something like :

$$\nabla^2 G^{\mu\nu} - \frac{1}{c^2}\frac{\partial^2 G^{\mu\nu}}{\partial t^2} = T^{\mu\nu} \tag{70}$$

18

where $G^{\mu\nu}$ is some sort of, I don't know, a conserved tensor guy for the...space-time geometry and stuff? In Maxwell's theory we had a 4-vector (A^α) operated on by the so-called "d'Alembertian operator" $\nabla^2 - (1/c)^2 \partial^2/\partial t^2$ on the left side of the equation and a source (J^α) on the right. So now we just need to find a $G^{\mu\nu}$ tensor to go with $T^{\mu\nu}$. Right?

Actually, this really is not too bad a guess, but...well...patience. One step at a time.

Then there occurred to me the 'glücklichste Gedanke meines Lebens,' the happiest thought of my life, in the following form. The gravitational field has only a relative existence in a way similar to the electric field generated by magnetoelectric induction. Because [1] for an observer falling freely from the roof of a house there exists—*at least in his immediate surroundings*—no gravitational field.

— *Albert Einstein*

3 The effects of gravity

The central idea of general relativity is that presence of mass (more precisely the presence of any stress-energy tensor component) causes departures from flat Minkowski space-time to appear, and that other matter (or radiation) responds to these distortions in some way. There are then really two questions: (i) How does the affected matter/radiation move in the presence of a distorted space-time?; and (ii) How does the stress-energy tensor distort the space-time in the first place? The first question is purely computational, and fairly straightforward to answer. It lays the groundwork for answering the much more difficult second question, so let us begin here.

3.1 The Principle of Equivalence

We have discussed the notion that by going into a frame of reference that is in free-fall, the effects of gravity disappear. In this time of space travel, we are all familiar with astronauts in free fall orbits, and the sense of weightlessness that is produced. This manifestation of the Equivalence Principle is so palpable that hearing mishmashes like "In orbit there is no gravity" from an over eager science correspondent is a common experience. (Our own BBC correspondent, Prof. Chris Lintott, would certainly never say such a thing.)

The idea behind the equivalence principle is that the m in $F = ma$ and the m in the force of gravity $F_g = mg$ are the same m and thus the acceleration caused by gravity, g, is invariant for any mass. We could imagine that $F = m_I a$ and $F_g = m_g g$, in which the acceleration is $m_g g / m_I$, i.e., it varies with the ratio of inertial to gravitational mass, m_g/m_I. How well can we actually measure this ratio, or what is more key, how well do we know that it is truly a universal constant for all types of matter?

The answer is very well indeed. We don't of course do anything as crude as directly measuring the rate at which objects fall to the ground any more, à la Galileo and the tower of Pisa. As with all classic precision gravity experiments (including those of Galileo!) we use a pendulum. The first direct measurement of the gravitational to inertial mass actually predates relativity, the so-called Eötvös experiment (after Baron Lorànd Eötvös, 1848-1919).

The idea is shown in schematic form in figure [1]. Hang a pendulum from a string, but

---

[1]With apologies to any readers who may actually have fallen off the roof of a house—safe space statement.
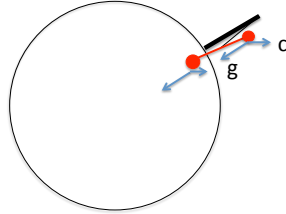
Figure 1: Schematic diagram of the Eötvös experiment. A barbell shape, the red object above, is hung from a pendulum on the Earth's surface (big circle) with two different material masses. Each mass is affected by gravity pulling it to the centre of the earth (g) and a centrifugal force due to the earth's rotation (c) shown as blue arrows. Any difference between the inertial mass and the gravitational mass will produce an unbalanced torque about the axis of the suspending fibre of the barbell.

instead of hanging a big mass, hang a rod, and put two masses at either end. There is a force of gravity toward the center of the earth (g in the figure), and a centrifugal force (c) due to the earth's rotation. The net force is the vector sum of these two, and if the components of the acceleration perpendicular to the string of each mass do not precisely balance, there will be a net torque twisting the masses about the string (a quartz fibre in the actual experiment). The absence of this twist is then a measurement of the lack of variability of m_I/m_g. In practice, to achieve high accuracy, the pendulum rotates with a tightly controlled period, so that the masses are sometimes hindered by any torque, sometimes pushed forward by the torque. This implants a frequency dependence onto the motion, and by using fourier signal processing, the resulting signal at a particular frequency can be tightly contrained. The ratio between any difference in the twisting accelerations on either mass and the average acceleration must be less than a few parts in 10^{12} (Su et al. 1994, Phys Rev D, 50, 3614). With direct laser ranging experiments to track the Moon's orbit, it is possible, in effect, to use the Moon and Earth as the masses on the pendulum as they orbit around the Sun! This gives an accuracy an order of magnitude better, a part in 10^{13} (Williams et al. 2012, Class. Quantum Grav., 29, 184004), an accuracy comparable to measuring the distance to the Sun within 1 cm.

There are two senses in which the Equivalence Principle may be used, a strong sense and weak sense. The weak sense is that it is not possible to detect the effects of gravity locally in a freely falling coordinate system, that all matter behaves identically in a gravitational field independent of its composition. Experiments can test this form of the Principle directly. The strong, much more powerful sense, is that *all physical laws,* gravitational or not, behave as they do in Minkowski space-time in a freely falling coordinate frame. In this sense the Principle is a postulate which appears to be true.

If going into a freely falling frame eliminates gravity locally, then going from an inertial frame to an accelerating frame reverses the process and mimics the effect of gravity—again,

locally. After all, if in an inertial frame

$$\frac{d^2x}{dt^2} = 0, \tag{71}$$

and we transform to the accelerating frame x' by $x = x' + gt^2/2$, where g is a constant, then

$$\frac{d^2x'}{dt^2} = -g, \tag{72}$$

which looks an awful lot like motion in a gravitational field.

One immediate consequence of this realisation is of profound importance: gravity affects light. In particular, if we are in an elevator of height h in a gravitational field of local strength g, *locally the physics is exactly the same as if we were accelerating upwards at g.* But the effect of this on light is then easily analysed: a photon released upwards reaches a detector at height h in a time h/c, at which point the detector is moving at a velocity gh/c relative to the bottom of the elevator (at the time of release). The photon is measured to be redshifted by an amount gh/c^2, or Φ/c^2 with Φ being the gravitational potential per unit mass at h. This is the classical gravitational redshift, the simplest nontrivial prediction of general relativity. The gravitational redshift has been measured accurately using changes in gamma ray energies (RV Pound & JL Snider 1965, *Phys. Rev.*, **140 B,** 788).

The gravitational redshift is the critical link between Newtonian theory and general relativity. It is not the distortion of space that gives rise to Newtonian gravity at the level we are familiar with, it is the distortion of the flow of time.

3.2   The geodesic equation

We denote by ξ^α our freely falling inertial coordinate frame in which the effects of gravity are locally absent. In this frame, the equation of motion for a particle is

$$\frac{d^2\xi^\alpha}{d\tau^2} = 0 \tag{73}$$

with

$$c^2 d\tau^2 = -\eta_{\alpha\beta} d\xi^\alpha d\xi^\beta \tag{74}$$

being the invariant time interval. (If we are doing light, then $d\tau = 0$, but ultimately it doesn't really matter. Either take a limit from finite $d\tau$, or use any other parameter you fancy, like your wristwatch. In the end, we won't use τ or your watch. As for $d\xi^\alpha$, it is just the freely-falling guy's ruler and *his* wristwatch.) Next, write this equation in any other set of coordinates you like, and call them x^μ. Our inertial coordinates ξ^α will be some function or other of the x^μ so

$$0 = \frac{d^2\xi^\alpha}{d\tau^2} = \frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) \tag{75}$$

where we have used the chain rule to express $d\xi^\alpha/d\tau$ in terms of $dx^\mu/d\tau$. Carrying out the differentiation,

$$0 = \frac{\partial\xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{76}$$

22

where now the chain rule has been used on $\partial \xi^\alpha / \partial x^\mu$. This may not look very promising. But if we multiply this equation by $\partial x^\lambda / \partial \xi^\alpha$, and remember to sum over α now, then the chain rule in the form

$$\frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{\partial \xi^\alpha}{\partial x^\mu} = \delta^\lambda_\mu \tag{77}$$

rescues us. (We are using the chain rule repeatedly and will certainly continue to do so, again and again. Make sure you understand this, *and* that you understand what variables are being held constant when the partial derivatives are taken. Deciding what is constant is just as important as doing the differentiation!) Our equation becomes

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0, \tag{78}$$

where

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \tag{79}$$

is known as the *affine connection,* and is a quantity of central importance in the study of Riemannian geometry and relativity theory in particular. You should be able to prove, using the chain rule of partial derivatives, an identity for the second derivatives of ξ^α that we will use shortly:

$$\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} = \frac{\partial \xi^\alpha}{\partial x^\lambda} \Gamma^\lambda_{\mu\nu} \tag{80}$$

(How does this work out when used in equation [76]?)

No need to worry, despite the funny notation. (Early relativity texts liked to use 𝔊𝔬𝔱𝔥𝔦𝔠 𝔉𝔬𝔫𝔱 for the affine connection, which added to the terror.) There is nothing especially mysterious about the affine connection. You use it all the time, probably without realising it. For example, in cylindrical (r, θ) coordinates, when you use the combinations $\ddot{r} - r\dot{\theta}^2$ or $r\ddot{\theta} + 2\dot{r}\dot{\theta}$ for your radial and tangential accelerations, you are using the affine connection and the geodesic equation.

Exercise. Prove the last statement using $\xi^x = r\cos\theta, \xi^y = r\sin\theta$.

Next, on the surface of a unit-radius sphere, choose any point as your North Pole, work in colatitude θ and azimuth ϕ coordinates, and show that *locally* near the North Pole $\xi^x = \theta\cos\phi$, $\xi^y = \theta\sin\phi$. It is in this sense that the ξ^α coordinates are tied to a local region of the space. In our free-fall coordinates, it is local to a point in space-time.

3.3 The metric tensor

In our locally inertial coordinates, the invariant space-time interval is

$$c^2 d\tau^2 = -\eta_{\alpha\beta} d\xi^\alpha d\xi^\beta, \tag{81}$$

so that in any other coordinates, $d\xi^\alpha = (\partial \xi^\alpha / \partial x^\mu) dx^\mu$ and

$$c^2 d\tau^2 = -\eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} dx^\mu dx^\nu \equiv -g_{\mu\nu} dx^\mu dx^\nu \tag{82}$$

where

$$g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} \tag{83}$$

23

is known as the *metric tensor*. The metric tensor embodies the information of how coordinate differentials combine to form the invariant interval of our space-time, and once we know $g_{\mu\nu}$, we know everything, including (as we shall see) the affine connections $\Gamma^\lambda_{\mu\nu}$. The object of general relativity theory is to compute $g_{\mu\nu}$, and a key goal of this course is to find the field equations that enable us to do so.

3.4 The relationship between the metric tensor and affine connection

Because of their reliance of the *local* freely falling inertial coordinates ξ^α, the $g_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$ quantities are awkward to use in their present formulation. Fortunately, there is a direct relationship between $\Gamma^\lambda_{\mu\nu}$ and the first derivatives of $g_{\mu\nu}$ that will allow us to become free of local bondage, and all us to dispense with the ξ^α altogether. Though their *existence* is crucial to formulate the mathematical structure, the practical need of the ξ's for actual calculations is minimal.

Differentiate equation (83):

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \eta_{\alpha\beta} \frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu} + \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial^2 \xi^\beta}{\partial x^\lambda \partial x^\nu} \tag{84}$$

Now use (80) for the second derivatives of ξ:

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\rho} \frac{\partial \xi^\beta}{\partial x^\nu} \Gamma^\rho_{\lambda\mu} + \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\rho} \Gamma^\rho_{\lambda\nu} \tag{85}$$

All remaining ξ derivatives may be absorbed as part of the metric tensor, leading to

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = g_{\rho\nu} \Gamma^\rho_{\lambda\mu} + g_{\mu\rho} \Gamma^\rho_{\lambda\nu} \tag{86}$$

It remains only to unweave the Γ's from the cloth of indices. This is done by first adding $\partial g_{\lambda\nu}/\partial x^\mu$ to the above, then subtracting it with indices μ and ν reversed.

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu} = g_{\rho\nu} \Gamma^\rho_{\lambda\mu} + \cancel{g_{\rho\mu} \Gamma^\rho_{\lambda\nu}} + g_{\rho\nu} \Gamma^\rho_{\mu\lambda} + \cancel{g_{\rho\lambda} \Gamma^\rho_{\mu\nu}} - \cancel{g_{\rho\mu} \Gamma^\rho_{\nu\lambda}} - \cancel{g_{\rho\lambda} \Gamma^\rho_{\nu\mu}} \tag{87}$$

Remembering that Γ is symmetric in its bottom indices, only the $g_{\rho\nu}$ terms survive, leaving

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu} = 2 g_{\rho\nu} \Gamma^\rho_{\mu\lambda} \tag{88}$$

Our last step is to mulitply by the inverse matrix $g^{\nu\sigma}$, defined by

$$g^{\nu\sigma} g_{\rho\nu} = \delta^\sigma_\rho, \tag{89}$$

leaving us with the pretty result

$$\Gamma^\sigma_{\mu\lambda} = \frac{g^{\nu\sigma}}{2} \left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu} \right). \tag{90}$$

Notice that there is no mention of the ξ's. The affine connection is completely specified by $g^{\mu\nu}$ and the derivatives of $g_{\mu\nu}$ in whatever coordinates you like. In practice, the inverse matrix is not difficult to find, as we will usually work with metric tensors whose off diagonal terms vanish. (Gain confidence once again by practicing the geodesic equation with cylindrical coordinates $g_{rr} = 1$, $g_{\theta\theta} = r^2$ and using [90.]) Note as well that with some very simple index relabeling, we have the mathematical identity

$$g_{\rho\nu}\Gamma^\rho_{\mu\lambda}\frac{dx^\mu}{d\tau}\frac{dx^\lambda}{d\tau} = \left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} - \frac{1}{2}\frac{\partial g_{\lambda\mu}}{\partial x^\nu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\lambda}{d\tau}. \tag{91}$$

We'll use this in a moment.

Exercise. Prove that $g^{\nu\sigma}$ is given explicitly by

$$g^{\nu\sigma} = \eta^{\alpha\beta}\frac{\partial x^\nu}{\partial \xi^\alpha}\frac{\partial x^\sigma}{\partial \xi^\beta}$$

3.5 Variational calculation of the geodesic equation

The physical significance of the relationship between the metric tensor and affine connection may be understood by a variational calculation. Off all possible paths in our spacetime from some point A to another B, which leaves the proper time an extremum (in this case, a maximum)? We describe the path by some external parameter p, which could be anything, perhaps the time on your wristwatch. Then the proper time from A to B is

$$T_{AB} = \int_A^B \frac{d\tau}{dp}\,dp = \frac{1}{c}\int_A^B \left(-g_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp}\right)^{1/2}dp \tag{92}$$

Next, vary x^λ to $x^\lambda + \delta x^\lambda$ (we are regarding x^λ as a function of p remember), with δx^λ vanishing at the end points A and B. We find

$$\delta T_{AB} = \frac{1}{2c}\int_A^B \left(-g_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp}\right)^{-1/2}\left(-\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\delta x^\lambda\frac{dx^\mu}{dp}\frac{dx^\nu}{dp} - 2g_{\mu\nu}\frac{d\delta x^\mu}{dp}\frac{dx^\nu}{dp}\right)dp \tag{93}$$

(Do you understand the final term in the integral?)

Since the leading inverse square root in the integrand is just $dp/d\tau$, δT_{AB} simplifies to

$$\delta T_{AB} = \frac{1}{2c}\int_A^B \left(-\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\delta x^\lambda\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - 2g_{\mu\nu}\frac{d\delta x^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)d\tau, \tag{94}$$

and p has vanished from sight. We now integrate the second term by parts, noting that the contribution from the endpoints has been specified to vanish. Remembering that

$$\frac{dg_{\lambda\nu}}{d\tau} = \frac{dx^\sigma}{d\tau}\frac{\partial g_{\lambda\nu}}{\partial x^\sigma}, \tag{95}$$

we find

$$\delta T_{AB} = \frac{1}{c}\int_A^B \left(-\frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \frac{\partial g_{\lambda\nu}}{\partial x^\sigma}\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau} + g_{\lambda\nu}\frac{d^2x^\nu}{d\tau^2}\right)\delta x^\lambda\,d\tau \tag{96}$$

25

or

$$\delta T_{AB} = \frac{1}{c} \int_A^B \left[\left(-\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} \right) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + g_{\lambda\nu} \frac{d^2 x^\nu}{d\tau^2} \right] \delta x^\lambda \, d\tau \qquad (97)$$

Finally, using equation (91), we obtain

$$\delta T_{AB} = \frac{1}{c} \int_A^B \left[\left(\frac{dx^\mu}{d\tau} \frac{dx^\sigma}{d\tau} \Gamma^\nu_{\mu\sigma} + \frac{d^2 x^\nu}{d\tau^2} \right) g_{\lambda\nu} \right] \delta x^\lambda \, d\tau \qquad (98)$$

Thus, if the geodesic equation (78) is satisfied, $\delta T_{AB} = 0$ is satisfied, and the proper time is an extremum. The very name "geodesic" is used in geometry to describe the path of minimum distance between two points in a manifold, and it is therefore of interest to see that there is a correspondence between a local "straight line" with zero curvature, and the local elimination of a gravitational field with the resulting zero acceleration. In the first case, the proper choice of local coordinates results in the second derivative with respect to an invariant spatial interval vanishing; in the second case, the proper choice of coordinates means that the second derivative with respect to an invariant time interval vanishes, but the essential mathematics is the same.

There is a very practical side to working with the variational method: it is often much easier to obtain the equations of motion for a given $g_{\mu\nu}$ this way than it is to construct them directly. In addition, the method quickly produces all the non-vanishing affine connection components, as the coefficients of $(dx^\mu/d\tau)(dx^\nu/d\tau)$. These quantities are then available for any variety of purposes (and they are needed for many).

In classical mechanics, we know that the equations of motion may be derived from a Lagrangian variational principle of least action, which doesn't seem geometrical at all. What is the connection with what we've just done? How do we make contact with Newtonian mechanics from the geodesic equation?

3.6 The Newtonian limit

We consider the case of a slowly moving mass ("slow" of course means relative to c, the speed of light) in a weak gravitational field ($GM/rc^2 \ll 1$). Since $cdt \gg |d\boldsymbol{x}|$, the geodesic equation greatly simplfies:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{00} \left(\frac{cdt}{d\tau} \right)^2 = 0. \qquad (99)$$

Now

$$\Gamma^\mu_{00} = \frac{1}{2} g^{\mu\nu} \left(\frac{\partial g_{0\nu}}{\partial(cdt)} + \frac{\partial g_{0\nu}}{\partial(cdt)} - \frac{\partial g_{00}}{\partial x^\nu} \right) \qquad (100)$$

In the Newtonian limit, the largest of the g derivatives is the spatial gradient, hence

$$\Gamma^\mu_{00} \simeq -\frac{1}{2} g^{\mu\nu} \frac{\partial g_{00}}{\partial x^\nu} \qquad (101)$$

Since the gravitational field is weak, $g_{\alpha\beta}$ differs very little from the Minkoswki value:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \quad h_{\alpha\beta} \ll 1, \qquad (102)$$

and the $\mu = 0$ geodesic equation is

$$\frac{d^2 t}{d\tau^2} + \frac{1}{2} \frac{\partial h_{00}}{\partial t} \left(\frac{dt}{d\tau} \right)^2 = 0 \qquad (103)$$

Clearly, the second term is zero for a static field, and will prove to be tiny when the gravitational field changes with time under nonrelativistic conditions—we are, after all, calculating the difference between proper time and observer time! Dropping this term we find that t and τ are linearly related, so that the spatial components of the geodesic equation become

$$\frac{d^2\boldsymbol{x}}{dt^2} - \frac{c^2}{2}\boldsymbol{\nabla}h_{00} = 0 \tag{104}$$

Isaac Newton would say:

$$\frac{d^2\boldsymbol{x}}{dt^2} + \boldsymbol{\nabla}\Phi = 0, \tag{105}$$

with Φ being the classical gravitational potential. The two views are consistent if

$$h_{00} \simeq -\frac{2\Phi}{c^2}, \quad g_{00} \simeq -\left(1 + \frac{2\Phi}{c^2}\right) \tag{106}$$

The quantity h_{00} is a dimensionless number of order v^2/c^2, where v is a velocity typical of the system, say an orbital speed. Note that h_{00} is determined by the dynamical equations only up to an additive constant, here chosen to make the geometry Minkowskian at large distances from the matter creating the gravitational field. At the surface of a spherical object of mass M and radius R,

$$h_{00} \simeq 2 \times 10^{-6} \left(\frac{M}{M_\odot}\right)\left(\frac{R_\odot}{R}\right) \tag{107}$$

where $M_\odot$ is the mass of the sun (about 2×10^{30} kg) and $R_\odot$ is the radius of the sun (about 7×10^8 m). As an exercise, you may want to look up masses of planets and other types of stars and evaluate h_{00}. What is its value at the surface of a white dwarf (mass of the sun, radius of the earth)? What about a neutron star (mass of the sun, radius of Oxford)?

We are now able to relate the geodesic equation to the principle of least action in classical mechanics. In the Newtonian limit, our variational integral becomes

$$\int \left[c^2(1 + 2\Phi/c^2)dt^2 - d|\boldsymbol{x}|^2\right]^{1/2} \tag{108}$$

Expanding the square root,

$$\int c\left(1 + \frac{\Phi}{c^2} - \frac{v^2}{2c^2} + ...\right) dt \tag{109}$$

where $v^2 \equiv d|\boldsymbol{x}|^2/dt$. Thus, minimising the Lagrangian (kinetic energy minus potential energy) is the same as maximising the proper time interval! What an unexpected and beautiful connection.

What we have calculated in this section is nothing more than our old friend the gravitational redshift, with which we began our formal study of general relativity. The invariant spacetime interval $d\tau$, the proper time, is given by

$$c^2 d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu \tag{110}$$

For an observer at rest at location x, the time interval registered on a clock will be

$$d\tau(x) = [-g_{00}(x)]^{1/2}dt \tag{111}$$

where dt is the time interval registered at infinity, where $-g_{00} \to 1$. (Compare: the "proper length" on the unit sphere for an interval at constant θ is $\sin \theta d\phi$, where $d\phi$ is the length registered by an equatorial observer.) If the interval between two wave crest crossings is found to be $d\tau(y)$ at location y, it will be $d\tau(x)$ when the light reaches x and it will be dt at infinity. In general,

$$\frac{d\tau(y)}{d\tau(x)} = \left[\frac{g_{00}(y)}{g_{00}(x)} \right]^{1/2}, \tag{112}$$

and in particular

$$\frac{d\tau(R)}{dt} = \frac{\nu(\infty)}{\nu} = [-g_{00}(R)]^{1/2} \tag{113}$$

where $\nu = 1/d\tau(R)$ is, for example, an atomic transition frequency measured at rest at the surface R of a body, and $\nu(\infty)$ the corresponding frequency measured a long distance away. Interestingly, the value of g_{00} that we have derived in the Newtonian limit is, in fact, the *exact* relativisitic value of g_{00} around a point mass M! (A black hole.) The precise redshift formula is

$$\nu_\infty = \left(1 - \frac{2GM}{Rc^2} \right)^{1/2} \nu \tag{114}$$

The redshift as measured by wavelength becomes infinite from light emerging from radius $R = 2GM/c^2$, the so-called Schwarzschild radius (about 3 km for a point with the mass of the sun!).

Historically, general relativity theory was supported in its infancy by the reported detection of a gravitational redshift in a spectral line observed from the surface of the white dwarf star Sirius B in 1925 by W.S. Adams. It "killed two birds with one stone," as the leading astronomer A.S. Eddington remarked. For it not only proved the existence of white dwarf stars (at the time controversial since the mechanism of pressure support was unknown), the measurement also confirmed an early and important prediction of general relativity theory: the redshift of light due to gravity.

Alas, the modern consensus is that the actual measurements were flawed! Adams knew what he was looking for and found it. Though he was premature, the activity this apparently positive observation imparted to the study of white dwarfs and relativity theory turned out to be very fruitful indeed. But we were lucky. Incorrect but highly regarded single-investigator observations have in the past caused much confusion and needless wrangling, as well as years of wasted effort.

The first definitive test for gravitational redshift came much later, and it was terrestrial: the 1959 Pound and Rebka experiment performed at Harvard University's Jefferson Tower measured the frequency shift of a 14.4 keV gamma ray falling (if that is the word for a gamma ray) 22.6 m. Pound & Rebka were able to measure the shift in energy—just a few parts in 10^{14}—by the then new technique of Mössbauer spectroscopy.

Exercise. A novel application of the gravitational redshift is provided by Bohr's refutation of an argument put forth by Einstein purportedly showing that an experiment could in principle be designed to bypass the quantum uncertainty relation $\Delta E \, \Delta t \geq h$. The idea is to hang a box containing a photon by a spring suspended in a gravitational field g. At some precise time a shutter is opened and the photon leaves. You weigh the box before and after the photon. There is in principle no interference between the arbitrarily accurate change in box weight and the arbitrarily accurate time at which the shutter is opened. Or is there?

1.) Show that box apparatus satisfies an equation of the form

$$M\ddot{x} = -Mg - kx$$

where M is the mass of the apparatus, x is the displacement, and k is the spring constant. Before release, the box is in equilibrium at $x = -gM/k$.

2.) Show that the momentum of the box apparatus after a short time interval Δt from when the photon escapes is

$$\delta p = -\frac{g\delta m}{\omega}\sin(\omega\Delta t) \simeq= -g\delta m\Delta t$$

where δm is the (uncertain!) photon mass and $\omega^2 = k/M$. With $\delta p \sim g\delta m\Delta t$, the uncertainty principle then dictates an uncertain location of the box position δx given by $g\delta m\,\delta x\Delta t \sim h$. But this is location uncertainty, not time uncertainty.

3.) Now the gravitational redshift comes in! Show that if there is an uncertainty in position δx, there is an uncertainty in the time of release: $\delta t \sim (g\delta x/c^2)\Delta t$.

4.) Finally use this in part (2) to establish $\delta E\,\delta t \sim h$ with $\delta E = \delta mc^2$.

Why does general relativity come into *nonrelativistic* quantum mechanics in such a fundamental way? Because the gravitational redshift is relativity theory's point-of-contact with classical Newtonian mechanics, and Newtonian mechanics when blended with the uncertainty principle is the start of nonrelativistic quantum mechanics.

4   Tensor Analysis

Further, the dignity of the science seems to require that every possible means be explored itself for the solution of a problem so elegant and so celebrated.

— Carl Friedrich Gauss

A mathematical equation is valid in the presence of general gravitational fields when

i.) It is a valid equation in the absence of gravity and respects Lorentz invariance.

ii.) It preserves its form, not just under Lorentz transformations, but under any coordinate transformation, $x \to x'$.

What does "preserves its form" mean? It means that the equation must be written in terms of quantities that transform as scalars, vectors, and higher ranked tensors under general coordinate transformations. From (ii), we see that if we can find one coordinate system in which our equation holds, it will hold in any set of coordinates. But by (i), the equation *does* hold in locally freely falling coordinates, in which the effect of gravity is locally absent. The effect of gravity is strictly embodied in the two key quantities that emerge from the calculus of coordinate transformations: the metric tensor $g_{\mu\nu}$ and its first derivatives in $\Gamma^{\lambda}_{\mu\nu}$. This approach is known as the *Principle of General Covariance,* and it is a very powerful tool indeed.

4.1   Transformation laws

The simplest vector one can write down is the ordinary coordinate differential dx^{μ}. If $x'^{\mu} = x'^{\mu}(x)$, there is no doubt how the dx'^{μ} are related to the dx^{μ}. It is called the chain rule, and it is by now very familiar:

$$dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \, dx^{\nu} \tag{115}$$

Any set of quantities V^{μ} that transforms in this way is known as a *contravariant vector*:

$$V'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \, V^{\nu} \tag{116}$$

A *covariant vector,* by contrast, transforms as

$$U'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \, U_{\nu} \tag{117}$$

"CO LOW, PRIME BELOW." (Sorry. Maybe you can do better.) These definitions of contravariant and covariant vectors are consistent with those we first introduced in our discussions of the Lorentz matrices Λ^{α}_{β} and $\tilde{\Lambda}^{\beta}_{\alpha}$ in Chapter 2, but now generalised from specific linear transformations to arbitrary transformations.

The simplest covariant vector is the gradient $\partial/\partial x^\mu$ of a scalar Φ. Once again, the chain rule tells us how to transform from one set of coordinates to another—we've no choice:

$$\frac{\partial \Phi}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial \Phi}{\partial x^\nu} \tag{118}$$

The generalisation to tensor transformation laws is immediate. A contravariant tensor $T^{\mu\nu}$ transforms as

$$T'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} T^{\rho\sigma} \tag{119}$$

a covariant tensor $T_{\mu\nu}$ as

$$T'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} T_{\rho\sigma} \tag{120}$$

and a mixed tensor T^μ_ν as

$$T'^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x^\sigma}{\partial x'^\nu} T^\rho_\sigma \tag{121}$$

The generalisation to mixed tensors of arbitrary rank should be self-evident.

By this definition the metric tensor $g_{\mu\nu}$ really *is* a covariant tensor, just as its notation would lead you to believe, because

$$g'_{\mu\nu} \equiv \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x'^\mu} \frac{\partial \xi^\beta}{\partial x'^\nu} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\lambda} \frac{\partial \xi^\beta}{\partial x^\rho} \frac{\partial x^\lambda}{\partial x'^\mu} \frac{\partial x^\rho}{\partial x'^\nu} \equiv g_{\lambda\rho} \frac{\partial x^\lambda}{\partial x'^\mu} \frac{\partial x^\rho}{\partial x'^\nu} \tag{122}$$

and the same for the *contravariant* $g^{\mu\nu}$. But the gradient of a vector is *not*, in general, a tensor or a vector:

$$\frac{\partial V'^\lambda}{\partial x'^\mu} = \frac{\partial}{\partial x'^\mu} \left(\frac{\partial x'^\lambda}{\partial x^\nu} V^\nu \right) = \frac{\partial x'^\lambda}{\partial x^\nu} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial V^\nu}{\partial x^\rho} + \frac{\partial^2 x'^\lambda}{\partial x^\rho \partial x^\nu} \frac{\partial x^\rho}{\partial x'^\mu} V^\nu \tag{123}$$

The first term is just what we would have wanted if we were searching for a tensor transformation law. But oh those pesky second order derivatives—the final term spoils it all. This of couse vanishes when the coordinate transformation is linear (as when we found that vector derivatives are perfectly good tensors under the Lorentz transformations), but not in general. We will show in the next section that while the gradient of a vector is in general not a tensor, there is an elegant solution around this problem.

Tensors can be created and manipulated in many ways. Direct products of tensors are tensors:

$$W^{\mu\nu}_{\rho\sigma} = T^{\mu\nu} S_{\rho\sigma} \tag{124}$$

for example. Linear combinations of tensors multiplied by scalars of the same rank are obviously tensors of the same rank. A tensor can lower its index by multiplying by $g_{\mu\nu}$ or raise it with $g^{\mu\nu}$:

$$T'^\rho_\mu \equiv g'_{\mu\nu} T'^{\nu\rho} = \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial x^\lambda}{\partial x'^\nu} \frac{\partial x'^\nu}{\partial x^\kappa} \frac{\partial x'^\rho}{\partial x^\tau} g_{\sigma\lambda} T^{\kappa\tau} = \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial x'^\rho}{\partial x^\tau} g_{\sigma\kappa} T^{\kappa\tau} \tag{125}$$

which indeed transforms as a tensor of mixed second rank, T^ρ_μ. To clarify: multiplying $T^{\mu\nu}$ by *any* covariant tensor $S_{\rho\mu}$ generates a mixed tensor M^ν_ρ, but we adopt the convention of keeping the name T^ν_ρ when multiplying by $S_{\rho\mu} = g_{\rho\mu}$, and thinking of the index as being lowered. (And of course index-raising for multiplication by $g^{\rho\mu}$.)

Mixed tensors can "contract" to scalars. Start with T^μ_ν. Then consider the transformation of T^μ_μ:

$$T'^\mu_\mu = \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x^\rho}{\partial x'^\mu} T^\nu_\rho = \delta^\rho_\nu T^\nu_\rho = T^\nu_\nu \tag{126}$$

i.e., T^μ_μ is a scalar T. Exactly the same type of calculation shows that $T^{\mu\nu}_\mu$ is a vector T^ν, and so on. Remember to contract "up–down:" $T^\mu_\mu = T$, *not* $T^{\mu\mu} = T$.

4.2 The covariant derivative

Recall the geodesic equation

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0. \tag{127}$$

The left hand side has one free component, and the right hand side surely is a vector: the trivial zero vector. If this equation is to be general, the left side needs to transform as a vector. Neither of the two terms by itself is a vector, yet somehow their sum transforms as a vector.

Rewrite the geodesic equation as follows. Denote $dx^\lambda/d\tau$, a true vector, as U^λ. Then

$$U^\mu \left[\frac{\partial U^\lambda}{\partial x^\mu} + \Gamma^\lambda_{\mu\nu} U^\nu \right] = 0 \tag{128}$$

Ah ha! Since the left side must be a vector, the stuff in square brackets must be a tensor: it is contracted with a vector U^μ to produce a vector—namely zero. The square brackets must contain a mixed tensor of rank two. Now, $\Gamma^\lambda_{\mu\nu}$ vanishes in locally free falling coordinates, in which we know that simple partial derivatives of vectors are indeed tensors. So this prescription tells us how to generalise this: to make a real tensor out of an ordinary partial derivative, form the quantity

$$\frac{\partial U^\lambda}{\partial x^\mu} + \Gamma^\lambda_{\mu\nu} U^\nu \equiv U^\lambda_{;\mu} \tag{129}$$

the so called *covariant derivative.* We use a semi-colon to denote covariant differentiation following convention. (Some authors get tired of writing out partial derivatives and so use a comma (e.g $V^\nu_{,\mu}$), but it is more clear to use full partial derivative notation, and we shall abide by this.) The covariant derivative reverts to an ordinary partial derivative in local freely falling coordinates, but it is a true tensor. We therefore have at hand our partial derivative in tensor form.

You know, this is too important a result not to check in detail. Perhaps you think there is something special about the gesodesic equation. Moreover, we need to understand how to construct the covariant derivative of covariant vectors and more general tensors. (Notice the use of the word "covariant" twice in that last statement in two different senses. Apologies for the awkward but standard mathematical nomenclature.)

The first thing we need to do is to establish the transformation law for $\Gamma^\lambda_{\mu\nu}$. This is just repeated application of the chain rule:

$$\Gamma'^\lambda_{\mu\nu} \equiv \frac{\partial x'^\lambda}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x'^\mu \partial x'^\nu} = \frac{\partial x'^\lambda}{\partial x^\rho} \frac{\partial x^\rho}{\partial \xi^\alpha} \frac{\partial}{\partial x'^\mu} \left(\frac{\partial x^\sigma}{\partial x'^\nu} \frac{\partial \xi^\alpha}{\partial x^\sigma} \right) \tag{130}$$

Carrying through the derivative,

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial \xi^{\alpha}}\left(\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial^2 \xi^{\alpha}}{\partial x^{\tau}\partial x^{\sigma}} + \frac{\partial^2 x^{\sigma}}{\partial x'^{\mu}\partial x'^{\nu}}\frac{\partial \xi^{\alpha}}{\partial x^{\sigma}}\right) \tag{131}$$

Cleaning up, and recognising an affine connection when we see one, helps to rid us of the ξ's:

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\Gamma^{\rho}_{\tau\sigma} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial^2 x^{\rho}}{\partial x'^{\mu}\partial x'^{\nu}} \tag{132}$$

This may also be written

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\Gamma^{\rho}_{\tau\sigma} - \frac{\partial x^{\rho}}{\partial x'^{\nu}}\frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial^2 x'^{\lambda}}{\partial x^{\sigma}\partial x^{\rho}} \tag{133}$$

Do you see why? (Hint: Either integrate $\partial/\partial x'^{\mu}$ by parts or differentiate the identity

$$\frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\rho}}{\partial x'^{\nu}} = \delta^{\lambda}_{\nu}.)$$

Hence

$$\Gamma'^{\lambda}_{\mu\nu}V'^{\nu} = \left(\frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\frac{\partial x^{\sigma}}{\partial x'^{\nu}}\Gamma^{\rho}_{\tau\sigma} - \frac{\partial x^{\rho}}{\partial x'^{\nu}}\frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial^2 x'^{\lambda}}{\partial x^{\rho}\partial x^{\sigma}}\right)\frac{\partial x'^{\nu}}{\partial x^{\eta}}V^{\eta}, \tag{134}$$

and spotting the tricky "sum-over-x'^{ν}" Kronecker delta functions,

$$\Gamma'^{\lambda}_{\mu\nu}V'^{\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}}\frac{\partial x^{\tau}}{\partial x'^{\mu}}\Gamma^{\rho}_{\tau\sigma}V^{\sigma} - \frac{\partial x^{\sigma}}{\partial x'^{\mu}}\frac{\partial^2 x'^{\lambda}}{\partial x^{\rho}\partial x^{\sigma}}V^{\rho} \tag{135}$$

Finally, adding this to (123), the unwanted terms cancel just as they should. We obtain

$$\frac{\partial V'^{\lambda}}{\partial x'^{\mu}} + \Gamma'^{\lambda}_{\mu\nu}V'^{\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\nu}}\frac{\partial x^{\rho}}{\partial x'^{\mu}}\left(\frac{\partial V^{\nu}}{\partial x^{\rho}} + \Gamma^{\nu}_{\rho\sigma}V^{\sigma}\right), \tag{136}$$

as desired. This combination really does transform as a tensor ought to.

It is now a one-step process to deduce how covariant derivatives work for covariant vectors. Consider

$$V_{\lambda}V^{\lambda}_{;\mu} = V_{\lambda}\frac{\partial V^{\lambda}}{\partial x^{\mu}} + \Gamma^{\lambda}_{\mu\nu}V^{\nu}V_{\lambda} \tag{137}$$

which is a perfectly good covariant vector. Integrating by parts on the first term on the right, and then switching λ and ν in the final term, this expression is identical to

$$\frac{\partial(V^{\lambda}V_{\lambda})}{\partial x^{\mu}} - V^{\lambda}\left[\frac{\partial V_{\lambda}}{\partial x^{\mu}} - \Gamma^{\nu}_{\mu\lambda}V_{\nu}\right]. \tag{138}$$

Since the first term is the covariant gradient of a scalar, and the entire expression must be a good covariant vector, the term in square brackets must be a purely covariant tensor of rank two. We have very quickly found our generalisation for the covariant derivative of a covariant vector:

$$V_{\lambda;\mu} = \frac{\partial V_{\lambda}}{\partial x^{\mu}} - \Gamma^{\nu}_{\mu\lambda}V_{\nu} \tag{139}$$

That this really *is* a vector can also be directly verified via a calculation exactly similar to our previous one for the covariant derivative of a contravariant vector.

Covariant derivatives of *tensors* are now simple to deduce. The tensor $T^{\lambda\kappa}$ must formally transform like a contravariant *vector* if we "freeze" one of its indices at some particular component and allow the other to take on all component values. Since the formula must be symmetric in the two indices,

$$T^{\lambda\kappa}_{;\mu} = \frac{\partial T^{\lambda\kappa}}{\partial x^\mu} + \Gamma^\lambda_{\mu\nu} T^{\nu\kappa} + \Gamma^\kappa_{\nu\mu} T^{\lambda\nu} \tag{140}$$

and then it should also follow

$$T_{\lambda\kappa;\mu} = \frac{\partial T_{\lambda\kappa}}{\partial x^\mu} - \Gamma^\nu_{\lambda\mu} T_{\nu\kappa} - \Gamma^\nu_{\kappa\mu} T_{\lambda\nu} \tag{141}$$

and of course

$$T^\lambda_{\kappa;\mu} = \frac{\partial T^\lambda_\kappa}{\partial x^\mu} + \Gamma^\lambda_{\nu\mu} T^\nu_\kappa - \Gamma^\nu_{\mu\kappa} T^\lambda_\nu \tag{142}$$

The generalisation to tensors of arbitrary rank should now be self-evident. To generate the affine connection terms, freeze all indices in your tensor, then unfreeze them one-by-one, treating each unfrozen index as either a covariant or contravariant vector, depending upon whether it is down or up. Practise this until it is second-nature.

We now can give a precise rule for how to take an equation that is valid in special relativity, and generalise it to the general relativistic theory of gravity. Work exclusively with 4-vectors and 4-tensors. Replace $\eta_{\alpha\beta}$ with $g_{\mu\nu}$. Take ordinary derivatives and turn them into covariant derivatives. Voilà, your equation is set for the presence of gravitational fields.

It will not have escaped your attention, I am sure, that applying (141) to $g_{\mu\nu}$ produces

$$g_{\mu\nu;\lambda} = \frac{\partial g_{\mu\nu}}{\partial x^\lambda} - g_{\rho\nu}\Gamma^\rho_{\mu\lambda} - g_{\mu\rho}\Gamma^\rho_{\nu\lambda} = 0 \tag{143}$$

where equation (86) has been used for the last equality. *The covariant derivatives of $g_{\mu\nu}$ vanish.* This is exactly what we would have predicted, since the ordinary derivatives of $\eta_{\alpha\beta}$ vanish in special relativity, and thus the covariant derivative of $g_{\mu\nu}$ should vanish in the presence of gravitational fields.

Here are two important technical points that are easily shown. (You should do so explicitly.)

i.) The covariant derivative obeys the Leibniz rule for direct products. For example:

$$(T^{\mu\nu} U_{\lambda\kappa})_{;\rho} = T^{\mu\nu}_{;\rho} U_{\lambda\kappa} + T^{\mu\nu} U_{\lambda\kappa;\rho}$$

ii.) The operation of contracting two tensor indices commutes with covariant differentiation. It does not matter which you do first.

4.3   The affine connection and basis vectors

The reader may be wondering how this all relates to our notions of, say, spherical geometry and its associated set of unit vectors and coordinates. The answer is: very simply. Our discussion will be straightforward and intuitive, rather than rigorous.

A vector $\boldsymbol{V}$ may be expanded in a set of basis vectors,

$$\boldsymbol{V} = V^a \boldsymbol{e}_a \tag{144}$$

where we sum over the repeated a, but a here on a **bold-faced** vector refers to a particular vector in the basis set. The V^a are the usual vector contravariant components, old friends, just numbers. Note that the sum is not a scalar formed from a contraction! We've used roman letters here to help avoid that pitfall.

The covariant components are associated with what mathematicians are pleased to call a *dual* basis:

$$\boldsymbol{V} = v_b \boldsymbol{e}^b \tag{145}$$

Same $\boldsymbol{V}$, just different ways of representing its components. If the $\boldsymbol{e}$'s seem a little abstract, don't worry, just take them at a formal level for the moment.

The basis and the dual basis are related by a dot product rule,

$$\boldsymbol{e}_a \cdot \boldsymbol{e}^b = \delta_a^b \tag{146}$$

where the dot product, though formal, has the properties of relating orthonormal bases. The basis vectors transform just as good vectors should:

$$\boldsymbol{e}'_a = \frac{\partial x^b}{\partial x'^a} \boldsymbol{e}_b, \quad \boldsymbol{e}'^a = \frac{\partial x'^a}{\partial x^b} \boldsymbol{e}^b. \tag{147}$$

Note that the dot product rule gives

$$\boldsymbol{V} \cdot \boldsymbol{V} = V^a V_b \boldsymbol{e}_a \cdot \boldsymbol{e}^b = V^a V_b \delta_a^b = V^a V_a, \tag{148}$$

as we would expect. On the other hand, expanding the differential line element $\boldsymbol{ds}$,

$$ds^2 = \boldsymbol{e}_a dx^a \cdot \boldsymbol{e}_b dx^b = \boldsymbol{e}_a \cdot \boldsymbol{e}_b dx^a dx^b \tag{149}$$

so that we recover the metric tensor

$$g_{ab} = \boldsymbol{e}_a \cdot \boldsymbol{e}_b \tag{150}$$

Exactly the same style calculation gives

$$g^{ab} = \boldsymbol{e}^a \cdot \boldsymbol{e}^b \tag{151}$$

These last two equations tell us first, that the g_{ab} is the coefficient of $\boldsymbol{e}^a$ in an expansion of the vector $\boldsymbol{e}_b$ in the usual basis:

$$\boldsymbol{e}_b = g_{ab} \boldsymbol{e}^a, \tag{152}$$

and second, that g^{ab} is the coefficient of $\boldsymbol{e}_a$ in an expansion of the vector $\boldsymbol{e}^b$ in the dual basis:

$$\boldsymbol{e}^b = g^{ab} \boldsymbol{e}_a \tag{153}$$

We've recovered the rules for raising and lowering indices, in this case for the entire basis vector.

Basis vectors change with coordinate position, as vectors do. We *define* Γ^b_{ac} by

$$\frac{\partial \boldsymbol{e}_a}{\partial x^c} = \Gamma^b_{ac} \boldsymbol{e}_b \tag{154}$$

so that
$$\Gamma^b_{ac} = \boldsymbol{e}^b \!\cdot\! \partial_c \boldsymbol{e}_a \equiv \partial_c(\boldsymbol{e}_a \!\cdot\! \boldsymbol{e}^b) - \boldsymbol{e}_a \!\cdot\! \partial_c \boldsymbol{e}^b = -\boldsymbol{e}_a \!\cdot\! \partial_c \boldsymbol{e}^b \tag{155}$$
(in the obvious shorthand notation $\partial/\partial x^c = \partial_c$.) The last equality gives the expansion

$$\frac{\partial \boldsymbol{e}^b}{\partial x^c} = -\Gamma^b_{ac}\boldsymbol{e}^a \tag{156}$$

Consider $\partial_c g_{ab} = \partial_c(\boldsymbol{e}_a \!\cdot\! \boldsymbol{e}_b)$. Using (154),

$$\partial_c g_{ab} = (\partial_c \boldsymbol{e}_a) \!\cdot\! \boldsymbol{e}_b + \boldsymbol{e}_a \!\cdot\! (\partial_c \boldsymbol{e}_b) = \Gamma^d_{ac}\boldsymbol{e}_d \!\cdot\! \boldsymbol{e}_b + \boldsymbol{e}_a \!\cdot\! \Gamma^d_{bc}\boldsymbol{e}_d, \tag{157}$$

or finally
$$\partial_c g_{ab} = \Gamma^d_{ac}g_{db} + \Gamma^d_{bc}g_{ad}, \tag{158}$$
exactly what we found in (86)! This leads, in turn, precisely to (90), the equation for the affine connection in terms of the g partial derivatives. We now have a more intuitive understanding of what the Γ's really represent: they are expansion coefficients for the derivatives of basis vectors, which is how we are used to thinking of the extra acceleration terms in non Cartesian coordinates when we first encounter them.

4.4   Volume element

The transformation of the metric tensor $g_{\mu\nu}$ may be thought of as a matrix equation:

$$g'_{\mu\nu} = \frac{\partial x^\kappa}{\partial x'^\mu}g_{\kappa\lambda}\frac{\partial x^\lambda}{\partial x'^\nu} \tag{159}$$

Remembering that the determinant of the product of matrices is the product of the determinants, we find
$$g' = \left|\frac{\partial x}{\partial x'}\right|^2 g \tag{160}$$

where g is the determinant of $g_{\mu\nu}$ (just the product of the diagonal terms for the diagonal metrics we will be using), and the notation $|\partial x'/\partial x|$ indicates the Jacobian of the transformation $x \to x'$. The significance of this result is that there is another quantity that also transforms with a Jacobian factor: the volume element d^4x.

$$d^4x' = \left|\frac{\partial x'}{\partial x}\right| d^4x. \tag{161}$$

This means
$$\sqrt{-g'}\, d^4x' = \sqrt{-g}\left|\frac{\partial x}{\partial x'}\right|\left|\frac{\partial x'}{\partial x}\right| d^4x = \sqrt{-g}\, d^4x. \tag{162}$$

In other words, $\sqrt{-g}\, d^4x$ is the invariant volume element of curved space-time. The minus sign is used merely as an absolute value to keep the quantities positive. In flat Minkowski space time, d^4x is invariant by itself. (Euclidian example: in going from Cartesian ($g = 1$) to cylindrical polar ($g = R^2$), to spherical coordinates ($g = r^4 \sin^2\theta$); we have $dx\,dy = R\,dR\,dz\,d\phi = r^2\sin\theta\,dr\,d\theta\,d\phi$. You knew that.)

4.5 Covariant div, grad, curl, and all that

The ordinary partial derivative of a scalar transforms generally as covariant vector, so in this case there is no distinction between a covariant and standard partial derivative. Another easy result is

$$V_{\mu;\nu} - V_{\nu;\mu} = \frac{\partial V_\mu}{\partial x^\nu} - \frac{\partial V_\nu}{\partial x^\mu}. \tag{163}$$

(The affine connection terms are symmetric in the two lower indices, so they cancel.) More interesting is

$$V^\mu_{;\mu} = \frac{\partial V^\mu}{\partial x^\mu} + \Gamma^\mu_{\mu\lambda} V^\lambda \tag{164}$$

where by definition

$$\Gamma^\mu_{\mu\lambda} = \frac{g^{\mu\rho}}{2} \left(\frac{\partial g_{\rho\mu}}{\partial x^\lambda} + \frac{\partial g_{\rho\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\rho} \right) \tag{165}$$

Now, $g^{\mu\rho}$ is symmetric in its indices, whereas the last two g derivatives combined are antisymmetric in the same indices, so that combination disappears entirely. We are left with

$$\Gamma^\mu_{\mu\lambda} = \frac{g^{\mu\rho}}{2} \frac{\partial g_{\rho\mu}}{\partial x^\lambda} \tag{166}$$

In this course, we will be dealing entirely with diagonal metric tensors, in which $\mu = \rho$ for nonvanishing entries, and $g^{\mu\rho}$ is the reciprocal of $g_{\mu\rho}$. In this simple case,

$$\Gamma^\mu_{\mu\lambda} = \frac{1}{2} \frac{\partial \ln |g|}{\partial x^\lambda} \tag{167}$$

where g is as usual the determinant of $g_{\mu\nu}$, here just the product of the diagonal elements. Though our result seems specific to diagonal $g_{\mu\nu}$, Weinberg, pp. 106-7, shows that this result is true for *any* $g_{\mu\nu}$.[2]

The covariant divergence (164) becomes

$$V^\mu_{;\mu} = \frac{1}{\sqrt{|g|}} \frac{\partial(\sqrt{|g|} V^\mu)}{\partial x^\mu} \tag{168}$$

a neat and tidy result. Note that

$$\int \sqrt{|g|} d^4x \, V^\mu_{;\mu} = 0 \tag{169}$$

if V^μ vanishes sufficiently rapidl) at infinity. (Why?)

We cannot leave the covariant derivative without discussing $T^{\mu\nu}_{;\mu}$. Conserved stress tensors are general relativity's "coin of the realm." Write this out:

$$T^{\mu\nu}_{;\mu} = \frac{\partial T^{\mu\nu}}{\partial x^\mu} + \Gamma^\mu_{\mu\lambda} T^{\lambda\nu} + \Gamma^\nu_{\mu\lambda} T^{\mu\lambda}, \tag{170}$$

---

[2]Sketchy proof for the mathematically inclined: For matrix M, trace Tr, differential δ, to first order in δ we have $\delta \ln \det M = \ln \det(M + \delta M) - \ln \det M = \ln \det M^{-1}(M + \delta M) = \ln \det(1 + M^{-1}\delta M) = \ln(1 + \text{Tr} \, M^{-1}\delta M) = \text{Tr} \, M^{-1}\delta M$. Can you supply the missing details?

and using (167), we may condense this to

$$T^{\mu\nu}_{;\mu} = \frac{1}{\sqrt{|g|}}\frac{\partial(\sqrt{|g|}T^{\mu\nu})}{\partial x^\mu} + \Gamma^\nu_{\mu\lambda}T^{\mu\lambda}. \tag{171}$$

For an antisymmetric tensor, call it $A^{\mu\nu}$, the last term drops out because Γ is symmetric in its lower indices:

$$A^{\mu\nu}_{;\mu} = \frac{1}{\sqrt{|g|}}\frac{\partial(\sqrt{|g|}A^{\mu\nu})}{\partial x^\mu} \tag{172}$$

4.6 Hydrostatic equilibrium

You have been patient and waded through a sea of indices, and it is time to be rewarded. We will do our first real physics problem in general relativity: hydrostatic equilibrium.

In Newtonian mechanics, you will recall that hydrostatic equilibrium represents a balance between a pressure gradient and the force of gravity. In general relativity this is completely encapsulated in the condition

$$T^{\mu\nu}_{;\mu} = 0$$

applied to the energy-momentum stress tensor (65), updated to covariant status:

$$T^{\mu\nu} = Pg^{\mu\nu} + (\rho + P/c^2)U^\mu U^\nu \tag{173}$$

Our conservation equation is

$$0 = T^{\mu\nu}_{;\mu} = g^{\mu\nu}\frac{\partial P}{\partial x^\mu} + \left[(\rho + P/c^2)U^\mu U^\nu\right]_{;\mu} \tag{174}$$

where we have made use of the fact that the $g_{\mu\nu}$ covariant derivative vanishes. Using (171):

$$0 = g^{\mu\nu}\frac{\partial P}{\partial x^\mu} + \frac{1}{|g|^{1/2}}\frac{\partial}{\partial x^\mu}\left[|g|^{1/2}(\rho + P/c^2)U^\mu U^\nu\right] + \Gamma^\nu_{\mu\lambda}(\rho + P/c^2)U^\mu U^\lambda \tag{175}$$

In static equilibrium, all the U components vanish except U^0. To determine this, we use

$$g_{\mu\nu}U^\mu U^\nu = -c^2 \tag{176}$$

the upgraded version of special relativity's $\eta_{\alpha\beta}U^\alpha U^\beta = -c^2$. Thus,

$$(U^0)^2 = -\frac{c^2}{g_{00}}, \tag{177}$$

and with

$$\Gamma^\nu_{00} = -\frac{g_{\mu\nu}}{2}\frac{\partial g_{00}}{\partial x^\mu}, \tag{178}$$

our equation reduces to

$$0 = g^{\mu\nu}\left[\frac{\partial P}{\partial x^\mu} + \left(\rho c^2 + P\right)\frac{\partial \ln|g_{00}|^{1/2}}{\partial x^\mu}\right] \tag{179}$$

38

Since $g_{\mu\nu}$ has a perfectly good inverse, the term in square brackets must be zero:

$$\frac{\partial P}{\partial x^\mu} + \left(\rho c^2 + P\right) \frac{\partial \ln |g_{00}|^{1/2}}{\partial x^\mu} = 0 \tag{180}$$

This is the general relativistic equation of hydrostatic equilibrium. Compare this with the Newtonian counterpart:

$$\boldsymbol{\nabla} P + \rho \boldsymbol{\nabla} \Phi = 0 \tag{181}$$

The difference for a static problem is the replacement of ρ by $\rho + P/c^2$ for the inertial mass density, and the use of $\ln |g_{00}|^{1/2}$ for the potential (to which it reduces in the Newtonian limit).

If $P = P(\rho)$, $P' \equiv dP/d\rho$, equation (180) may be formally integrated:

$$\int \frac{P'(\rho)\, d\rho}{P(\rho) + \rho c^2} + \ln |g_{00}|^{1/2} = \text{constant.} \tag{182}$$

Exercise. Solve this equation exactly for the case $|g_{00}| = (1 - 2GM/rc^2)^{1/2}$ (e.g., near the surface of a neutron star) and $P = K\rho^\gamma$ for $\gamma \geq 1$.

4.7 Covariant differentiation and parallel transport

In this section, we view covariant differentiation in a different light. We make no new technical developments, rather we understand the content of the geodesic equation in a different way. Start with a by now old friend,

$$\frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0. \tag{183}$$

Writing $dx^\lambda/d\tau$ as the vector it is, V^λ, to help our thinking a bit,

$$\frac{dV^\lambda}{d\tau} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} V^\nu = 0, \tag{184}$$

a covariant formulation of the statement that the vector V^λ is conserved along a geodesic path. But the covariance property of this statement has nothing to do with the specific identity of V^λ with $dx^\lambda/d\tau$. The full left-side would of this equation is a covariant vector for any V^λ as long as V^λ itself is a bona fide contravariant vector. The right side simply tells us that the covariant left side is zero (because in this case momentum is conserved.) Therefore, just as we "upgrade" from special to general relativity the partial derivative,

$$\frac{\partial V^\alpha}{\partial x^\beta} \to \frac{\partial V^\lambda}{\partial x^\mu} + \Gamma^\lambda_{\mu\nu} V^\nu \equiv V^\lambda_{;\mu} \tag{185}$$

we upgrade the derivative along a path $x(\tau)$ in the same way by multiplying by $dx^\mu/d\tau$ and summing over the index μ:

$$\frac{dV^\alpha}{d\tau} \to \frac{dV^\lambda}{d\tau} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} V^\nu \equiv \frac{DV^\lambda}{D\tau} \tag{186}$$

$DV^\lambda/D\tau$ is a true vector; the transformation

$$\frac{DV'^\lambda}{D\tau} = \frac{\partial x'^\lambda}{\partial x^\mu}\frac{DV^\mu}{D\tau} \tag{187}$$

may be verified directly. (The inhomogeneous contributions from the Γ transformation and the derivatives of the derivatives of the coordinate transformation coefficients cancel in a manner exactly analogous to our original covariant partial derviative calculation.)

Exactly the same reasoning is used to define the covariant derivative for a covariant vector,

$$\frac{dV_\lambda}{d\tau} - \Gamma^\nu_{\mu\lambda}\frac{dx^\mu}{d\tau}V_\nu \equiv \frac{DV_\lambda}{D\tau}. \tag{188}$$

and for tensors, e.g.:

$$\frac{dT^\sigma_\lambda}{d\tau} + \Gamma^\sigma_{\mu\nu}\frac{dx^\nu}{d\tau}T^\mu_\lambda - \Gamma^\nu_{\lambda\mu}\frac{dx^\mu}{d\tau}T^\sigma_\nu \equiv \frac{DT^\sigma_\lambda}{D\tau}. \tag{189}$$

When a vector or tensor quantity is carried along a particle path does not change in a locally inertially reference frame ($d/d\tau = 0$) becomes in arbitrary coordinates ($D/D\tau = 0$), the same physical result expressed in a covariant language. (Once again this works because of manifest agreement in the inertial coordinates, and then zero is zero in any coordinate frame.) The condition $D/D\tau = 0$ is known as *parallel transport*. A vector, for example, may always point along the y axis as we move it around in the xy plane, but its r and θ components will have constantly to change to keep this true! This is the content of the parallel transport equation.

If we do a round trip and come back to our starting point, does a vector have to have the same value it began with? You might think that the answer must be yes, but it turns out to be more complicated than that. Indeed, it is a most interesting question...

The stage is now set to introduce the key tensor embodying the gravitational distortion of space-time.

5 The curvature tensor

The properties which distinguish space from other conceivable triply-extended magnitudes are only to be deduced from experience...At every point the three-directional measure of curvature can have an arbitrary value if only the effective curvature of every measurable region of space does not differ noticeably from zero.

— G. F. B. Riemann

5.1 Commutation rule for covariant derivatives

The covariant derivative shares many properties with the ordinary partial derivative: it is a linear operator, it obeys the Leibniz rule, and it allows true tensor status to be bestowed upon partial derivatives under any coordinate transformation. A natural question arises. Ordinary partial derivatives commute: the order in which they are taken does not matter, provided suitable smoothness conditions are present. Is the same true of covariant derivatives? Does $V^{\mu}_{;\sigma;\tau}$ equal $V^{\mu}_{;\tau;\sigma}$?

Just do it.

$$V^{\mu}_{;\sigma} = \frac{\partial V^{\mu}}{\partial x^{\sigma}} + \Gamma^{\mu}_{\nu\sigma} V^{\nu} \equiv T^{\mu}_{\sigma} \tag{190}$$

Then

$$T^{\mu}_{\sigma;\tau} = \frac{\partial T^{\mu}_{\sigma}}{\partial x^{\tau}} + \Gamma^{\mu}_{\nu\tau} T^{\nu}_{\sigma} - \Gamma^{\nu}_{\sigma\tau} T^{\mu}_{\nu}, \tag{191}$$

or

$$T^{\mu}_{\sigma;\tau} = \frac{\partial^{2} V^{\mu}}{\partial x^{\tau} \partial x^{\sigma}} + \frac{\partial}{\partial x^{\tau}} \left(\Gamma^{\mu}_{\lambda\sigma} V^{\lambda} \right) + \Gamma^{\mu}_{\nu\tau} \left(\frac{\partial V^{\nu}}{\partial x^{\sigma}} + \Gamma^{\nu}_{\lambda\sigma} V^{\lambda} \right) - \Gamma^{\nu}_{\sigma\tau} \left(\frac{\partial V^{\mu}}{\partial x^{\nu}} + \Gamma^{\mu}_{\lambda\nu} V^{\lambda} \right) \tag{192}$$

The first term and the last group (proportional to $\Gamma^{\nu}_{\sigma\tau}$) are manifestly symmetric in σ and τ, and so will vanish when the same calculation is done with the indices reversed and then subtracted off. A bit of inspection shows that the same is true for all the remaining terms proportional to the partial derivatives of V^{μ}. The residual terms from taking the covariant derivative commutator are

$$T^{\mu}_{\sigma;\tau} - T^{\mu}_{\tau;\sigma} = \left[\frac{\partial \Gamma^{\mu}_{\lambda\sigma}}{\partial x^{\tau}} - \frac{\partial \Gamma^{\mu}_{\lambda\tau}}{\partial x^{\sigma}} + \Gamma^{\mu}_{\nu\tau} \Gamma^{\nu}_{\lambda\sigma} - \Gamma^{\mu}_{\nu\sigma} \Gamma^{\nu}_{\lambda\tau} \right] V^{\lambda}, \tag{193}$$

which we may write as

$$T^{\mu}_{\sigma;\tau} - T^{\mu}_{\tau;\sigma} = R^{\mu}_{\lambda\sigma\tau} V^{\lambda} \tag{194}$$

Now the right side of this equation must be a tensor, and V^{λ} is an arbitrary vector, which means that $R^{\mu}_{\lambda\sigma\tau}$ needs to transform its coordinates as a tensor. That it does so may also be

verified explicitly in a nasty calculation (if you want to see it spelt out in detail, see Weinberg pp.132-3). We conclude that

$$R^{\mu}_{\lambda\sigma\tau} = \frac{\partial \Gamma^{\mu}_{\lambda\sigma}}{\partial x^{\tau}} - \frac{\partial \Gamma^{\mu}_{\lambda\tau}}{\partial x^{\sigma}} + \Gamma^{\mu}_{\nu\tau}\Gamma^{\nu}_{\lambda\sigma} - \Gamma^{\mu}_{\nu\sigma}\Gamma^{\nu}_{\lambda\tau} \tag{195}$$

is indeed a true tensor, and it is called the *curvature tensor.* In fact, it may be shown (Weinberg p. 134) that this is the only tensor that is *linear* in the second derivatives of $g_{\mu\nu}$ and contains only its first and second derivatives.

Why do we refer to this mixed tensor as the "curvature tensor?" Well, clearly it vanishes in ordinary flat Minkowski space-time—we simply choose Cartesian coordinates to do our calculation. Then, because $R^{\mu}_{\lambda\sigma\tau}$ is a tensor, if it is zero in one set of coordinates, it is zero in all. Commuting covariant derivatives makes sense in this case, since they amount to ordinary derivatives. So distortions from Minkowski space are essential.

Our intuition sharpens with the yet more striking example of parallel transport. Consider a vector V_{λ} whose covariant derivative along a curve $x(\tau)$ vanishes. Then,

$$\frac{dV_{\lambda}}{d\tau} = \Gamma^{\mu}_{\lambda\nu}\frac{dx^{\nu}}{d\tau}V_{\mu} \tag{196}$$

Consider next a tiny round trip journey over a closed path in which V_{λ} is changing by the above prescription. If we remain in the neighbourhood of some point X^{ρ}, with x^{ρ} passing through X^{ρ} at some instant τ_0, $x^{\rho}(\tau_0) = X^{\rho}$, we Taylor expand as follows:

$$\Gamma^{\mu}_{\lambda\nu}(x) = \Gamma^{\mu}_{\lambda\nu}(X) + (x^{\rho} - X^{\rho})\frac{\partial \Gamma^{\mu}_{\lambda\nu}}{\partial X^{\rho}} + ... \tag{197}$$

$$V_{\mu}(x) = V_{\mu}(X) + (x^{\rho} - X^{\rho})\frac{dV_{\mu}}{dX^{\rho}} + ... = V_{\mu}(X) + (x^{\rho} - X^{\rho})\Gamma^{\sigma}_{\mu\rho}(X)V_{\sigma}(X) + ... \tag{198}$$

(we have used the parallel transport equation for the derivative term in V_{μ}), whence

$$\Gamma^{\mu}_{\lambda\nu}(x)V_{\mu}(x) = \Gamma^{\mu}_{\lambda\nu}V_{\mu} + (x^{\rho} - X^{\rho})V_{\sigma}\left(\frac{\partial \Gamma^{\sigma}_{\lambda\nu}}{\partial X^{\rho}} + \Gamma^{\sigma}_{\mu\rho}\Gamma^{\mu}_{\lambda\nu}\right) + ... \tag{199}$$

where all quantities on the right (except x!) are evaluated at X. Integrating

$$dV_{\lambda} = \Gamma^{\mu}_{\lambda\nu}(x)V_{\mu}(x)\,dx^{\nu} \tag{200}$$

around a tiny closed path $\oint$, and using (200) and (199), we find that there is a change in the starting value ΔV_{λ} arising from the term linear in x^{ρ} given by

$$\Delta V_{\lambda} = \left(\frac{\partial \Gamma^{\sigma}_{\lambda\nu}}{\partial X^{\rho}} + \Gamma^{\sigma}_{\mu\rho}\Gamma^{\mu}_{\lambda\nu}\right)V_{\sigma}\oint x^{\rho}\,dx^{\nu} \tag{201}$$

The integral $\oint x^{\rho}dx^{\nu}$ certainly doesn't vanish. (Try integrating it around a unit square in the xy plane), but it *is* antisymmetric in ρ and ν. (Integrate by parts and note that the integrated term vanishes, being an exact differential.) That means the part of the Γ terms that survives the summation is the part antisymmetric in (ρ, ν). Since any object depending on two indices, say $A(\rho, \nu)$, can be written as a symmetric part plus an antisymmetric part,

$$\frac{1}{2}[A(\rho, \nu) + A(\nu, \rho)] + \frac{1}{2}[A(\rho, \nu) - A(\nu, \rho)],$$

we find

$$\Delta V_\lambda = \frac{1}{2} R^\sigma_{\lambda\nu\rho} V_\sigma \oint x^\rho \, dx^\nu \tag{202}$$

where

$$R^\sigma_{\lambda\nu\rho} = \left(\frac{\partial \Gamma^\sigma_{\lambda\nu}}{\partial X^\rho} - \frac{\partial \Gamma^\sigma_{\lambda\rho}}{\partial X^\nu} + \Gamma^\sigma_{\mu\rho}\Gamma^\mu_{\lambda\nu} - \Gamma^\sigma_{\mu\nu}\Gamma^\mu_{\lambda\rho} \right) \tag{203}$$

is precisely the curvature tensor.

Exercise. **A laboratory demonstration.** Take a pencil and move it round the surface of a flat desktop without rotating the pencil. Moving the pencil around a closed path, *always parallel to itself,* will not change its orientation. Now do the same on the surface of a spherical globe. Take a small pencil, pointed poleward, and move it from the equator along the 0° meridian through Greenwich till you hit the north pole. Now, once again parallel to itself, move the pencil down the 90°E meridian till you come to the equator. Finally, once again parallel to itself, slide the pencil along the equator to return to the starting point at the prime meridian.

Has the pencil orientation changed from its initial one? Explain.

Curvature[3], or more precisely the departure of space-time from Minkowskii structure, reveals itself through the existence of the curvature tensor $R^\sigma_{\lambda\nu\rho}$. If space-time is Minkowskii-flat, every component of the curvature tensor vanishes. An important consequence is that parallel transport around a closed loop can result in a vector or tensor not returning to its orginal value, if the closed loop encompasses matter (or its energy equivalent). An experiment was proposed in the 1960's to measure the precession of a gyroscope orbiting the earth due to the effects of the space-time curvature tensor. This eventually evolved into a satellite known as Gravity Probe B, a 750 million USD mission, launched in 2004. Alas, it was plagued by technical problems for many years, and its results were controversial because of unexpectedly high noise levels (solar activity). A final publication of science results in 2011 claims to have verified the predictions of general relativity to high accuracy, including an even smaller effect known as "frame dragging" from the earth's rotation, but my sense is that there is lingering uneasiness in the physics community regarding the handling of the noise. Do an internet search on Gravity Probe B and judge for yourself!

When GPB was first proposed in the early 1960's, tests of general relativity were very few and far between. Since that time, experimental GR has evolved tremendously, with gravitational lenses, the so-called Shapiro time delay effect, and a beautiful indirect confirmation of the existence of gravitational radiation. (All these will be discussed in later chapters.) There is no serious doubt that the leading order general relativity parallel transport prediction must be right—indeed, it appears that we have actually seen this effect directly in close binary pulsar systems. Elaborate artificial gyroscopes precessing in earth orbit seem somehow less exciting to many than perhaps they once were.

---

[3] "Curvature" is one of these somewhat misleading mathematical labels that has stuck, like "imaginary" numbers. The name implies an external dimension into which the space is curved or embedded, an unnecessary complication. The space is simply *distorted.*

5.2 Algebraic identities of $R^{\sigma}_{\nu\lambda\rho}$

5.2.1 Remembering the curvature tensor formula.

It is helpful to have a mnemonic for generating the curvature tensor. The hard part is keeping track of the indices. Remember that the tensor itself is just a sum of derivatives of Γ and quadratic products of Γ. That part is easy to remember, since the curvature tensor has "dimensions" of $1/x^2$ where x represents a coordinate. For the coordinate juggling of R^a_{bcd} start with:

$$\frac{\partial \Gamma^a_{bc}}{\partial x^d} + \Gamma^*_{bc}\Gamma^a_{d*}$$

where the first $abcd$ ordering is simple to remember since it follows the same placement in R^a_{bcd}, and $*$ is a dummy variable. For the second $\Gamma\Gamma$ term, remember to just write out the lower bcd indices straight across, making the last unfilled space a dummy index $*$. The counterpart dummy index that is summed over must then be the upper slot on the other Γ, since there is no self-contracted Γ in the full curvature tensor. There is then only one place left for upper a. To finish off, just subtract the same thing with c and d reversed. Think of it as swapping your CD's. We arrive at:

$$R^a_{bcd} = \frac{\partial \Gamma^a_{bc}}{\partial x^d} - \frac{\partial \Gamma^a_{bd}}{\partial x^c} + \Gamma^*_{bc}\Gamma^a_{d*} - \Gamma^*_{bd}\Gamma^a_{c*} \tag{204}$$

5.3 $R_{\lambda\mu\nu\kappa}$: fully covariant form

The fully covariant form of the stress tensor involves both second order derivatives of $g_{\mu\nu}$, which will be our point of contact with Newtonian theory and the full field equations. It is also important for the theory of gravitational radiation. So hang on, we have some heavy weather ahead.

We define

$$R_{\lambda\mu\nu\kappa} = g_{\lambda\sigma}R^{\sigma}_{\mu\nu\kappa} \tag{205}$$

or

$$R_{\lambda\mu\nu\kappa} = g_{\lambda\sigma}\left[\frac{\partial \Gamma^{\sigma}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\sigma}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\sigma}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\sigma}_{\nu\eta}\right] \tag{206}$$

Remembering the definition of the affine connection (90), the right side of (206) is

$$\frac{g_{\lambda\sigma}}{2}\frac{\partial}{\partial x^{\kappa}}\left[g^{\sigma\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^{\nu}} + \frac{\partial g_{\rho\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\rho}}\right)\right] - \frac{g_{\lambda\sigma}}{2}\frac{\partial}{\partial x^{\nu}}\left[g^{\sigma\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^{\kappa}} + \frac{\partial g_{\rho\kappa}}{\partial x^{\mu}} - \frac{\partial g_{\mu\kappa}}{\partial x^{\rho}}\right)\right]$$

$$+g_{\lambda\sigma}\left(\Gamma^{\mu}_{\nu\lambda}\Gamma^{\sigma}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\sigma}_{\nu\eta}\right) \tag{207}$$

The κ and ν partial x derivatives will operate on the $g^{\sigma\rho}$ term and the g-derivative terms. Let us begin with the second group, the $\partial g/\partial x$ derivatives, as it is simpler. With $g_{\lambda\sigma}g^{\sigma\rho} = \delta^{\lambda}_{\rho}$, the terms that are linear in the second order g derivatives are

$$\frac{1}{2}\left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^{\kappa}\partial x^{\mu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\kappa}\partial x^{\lambda}} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^{\nu}\partial x^{\mu}} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^{\nu}\partial x^{\lambda}}\right) \tag{208}$$

If you can sense the beginnings of the classical wave equation lurking in these linear second order derivatives, the leading terms when $g_{\mu\nu}$ departs only a little from $\eta_{\mu\nu}$, then you are very much on the right track.

44

We are not done of course. We have the terms proportional to the κ and ν derivatives of $g^{\sigma\rho}$, which certainly do not vanish in general. But the covariant derivative of the metric tensor $g_{\lambda\sigma}$ *does* vanish, so invoke this sleight-of-hand integration by parts:

$$g_{\lambda\sigma}\frac{\partial g^{\sigma\rho}}{\partial x^\kappa} = -g^{\sigma\rho}\frac{\partial g_{\lambda\sigma}}{\partial x^\kappa} = -g^{\sigma\rho}\left(\Gamma^\eta_{\kappa\lambda}g_{\eta\sigma} + \Gamma^\eta_{\kappa\sigma}g_{\eta\lambda}\right) \tag{209}$$

where in the final equality, equation (141) has been used. By bringing $g^{\sigma\rho}$ out from the partial derivative, it recombines to form affine connections once again. All the remaining terms of $R_{\lambda\mu\nu\kappa}$ from (207) are now of the form $g\Gamma\Gamma$:

$$-\left(\Gamma^\eta_{\kappa\lambda}g_{\eta\sigma} + \cancel{\Gamma^\eta_{\kappa\sigma}g_{\eta\lambda}}\right)\Gamma^\sigma_{\mu\nu} + \left(\Gamma^\eta_{\nu\lambda}g_{\eta\sigma} + \cancel{\Gamma^\eta_{\nu\sigma}g_{\eta\lambda}}\right)\Gamma^\sigma_{\mu\kappa} + g_{\lambda\sigma}\left(\cancel{\Gamma^\eta_{\mu\nu}\Gamma^\sigma_{\kappa\eta}} - \cancel{\Gamma^\eta_{\mu\kappa}\Gamma^\sigma_{\nu\eta}}\right), \tag{210}$$

It is not obvious at first, but four of these six $g\Gamma\Gamma$ terms cancel out— the second group with the fifth, the fourth group with the sixth—leaving only the first and third terms:

$$g_{\eta\sigma}\left(\Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\kappa} - \Gamma^\eta_{\kappa\lambda}\Gamma^\sigma_{\mu\nu}\right) \tag{211}$$

Adding together the terms in (208) and (211), we arrive at

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa\partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa\partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu\partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu\partial x^\lambda}\right) + g_{\eta\sigma}\left(\Gamma^\eta_{\nu\lambda}\Gamma^\sigma_{\mu\kappa} - \Gamma^\eta_{\kappa\lambda}\Gamma^\sigma_{\mu\nu}\right) \tag{212}$$

Note the following important symmetry properties for the indices of $R_{\lambda\mu\nu\kappa}$. Because they may be expressed as vanishing tensor equations, they may be established in any coordinate frame, so we choose a local frame in which the Γ vanish. They are then easily verified from the terms linear in the g derivatives in (212):

$$R_{\lambda\mu\nu\kappa} = R_{\nu\kappa\lambda\mu} \qquad \text{(symmetry)} \tag{213}$$

$$R_{\lambda\mu\nu\kappa} = -R_{\mu\lambda\nu\kappa} = -R_{\lambda\mu\kappa\nu} = R_{\mu\lambda\kappa\nu} \qquad \text{(antisymmetry)} \tag{214}$$

$$R_{\lambda\mu\nu\kappa} + R_{\lambda\kappa\mu\nu} + R_{\lambda\nu\kappa\mu} = 0 \qquad \text{(cyclic)} \tag{215}$$

5.4   The Ricci Tensor

The Ricci tensor is the curvature tensor contracted on its (raised) first and third indices, R^a_{bad}. In terms of the covariant curvature tensor:

$$R_{\mu\kappa} = g^{\lambda\nu}R_{\lambda\mu\nu\kappa} = g^{\lambda\nu}R_{\nu\kappa\lambda\mu} \text{ (by symmetry) } = g^{\nu\lambda}R_{\nu\kappa\lambda\mu} = R_{\kappa\mu} \tag{216}$$

so that the Ricci tensor is symmetric.

The Ricci tensor is an extremely important tensor in general relativity. Indeed, we shall very soon see that $R_{\mu\nu} = 0$ is Einstein's Laplace equation. There is enough information here to calculate the deflection of light by a gravitating body or the advance of a planet's orbital perihelion! What is tricky is to guess the general relativistic version of the Poisson equation, and no, it is not $R_{\mu\nu}$ proportional to the stress energy tensor $T_{\mu\nu}$! Notice that while $R^\lambda_{\mu\nu\kappa} = 0$ implies that the Ricci tensor vanishes, the converse does not follow: $R_{\mu\nu} = 0$ does not necessarily mean that the full curvature tensor (covariant or otherwise) vanishes.

Exercise. **Fun with the Ricci tensor.** Prove that

$$R_{\mu\kappa} = -g^{\lambda\nu}R_{\mu\lambda\nu\kappa} = -g^{\lambda\nu}R_{\lambda\mu\kappa\nu} = g^{\lambda\nu}R_{\mu\lambda\kappa\nu}$$

45

and that $g^{\lambda\mu}R_{\lambda\mu\nu\kappa} = g^{\nu\kappa}R_{\lambda\mu\nu\kappa} = 0$. Why does this mean that $R_{\mu\kappa}$ is the only second rank covariant tensor that can be formed from contracting $R_{\lambda\mu\nu\kappa}$? The stage is then set for an examination of the algebraic properties of $R^{\lambda}_{\mu\nu\kappa}$, its symmetries, and the Royal Road to GR via the Bianchi Identities.

We are not quite through contracting. We may form the curvature scalar

$$R \equiv R^{\mu}_{\mu} \tag{217}$$

another very important quantity in general relativity.

Exercise. **The curvature scalar is unique.** Prove that

$$R = g^{\nu\lambda}g^{\mu\kappa}R_{\lambda\mu\nu\kappa} = -g^{\nu\lambda}g^{\mu\kappa}R_{\mu\lambda\nu\kappa}$$

and that

$$g^{\lambda\mu}g^{\nu\kappa}R_{\lambda\mu\nu\kappa} = 0.$$

Justify the title of this exercise.

5.5   The Bianchi Identities

The covariant curvature tensor obeys a very important differential identity, analogous to **div(curl)=0.** These are the Bianchi identities. We prove the Bianchi identities in our favourite freely falling inertial coordinates with $\Gamma = 0$, and since we will be showing that a tensor is zero in these coordinates, it is zero in all coordinates. In $\Gamma = 0$ coordinates,

$$R_{\lambda\mu\nu\kappa;\eta} = \frac{1}{2}\frac{\partial}{\partial x^{\eta}}\left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^{\kappa}\partial x^{\mu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\kappa}\partial x^{\lambda}} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^{\mu}\partial x^{\nu}} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^{\nu}\partial x^{\lambda}}\right) \tag{218}$$

The Bianchi identities follow from cycling: ν replaces $\to \kappa$, κ replaces $\to \eta$, η replaces $\to \nu$. Leave λ and μ alone. This gives

$$R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0 \tag{219}$$

An easy way to do the bookkeeping on this is just to pay attention to the g's: once you've picked a particular value of $\partial^2 g_{ab}$ in the numerator, the other ∂x^c indices downstairs are unambiguous, since as coordinate derivatives their order is immaterial. The first term in (219) is then just shown: $(g_{\lambda\nu}, -g_{\mu\nu}, -g_{\lambda\kappa}, g_{\mu\kappa})$. Cycle to get the second group for the second Bianchi term, $(g_{\lambda\eta}, -g_{\mu\eta}, -g_{\lambda\nu}, g_{\mu\nu})$. The final term then is $(g_{\lambda\kappa}, -g_{\mu\kappa}, -g_{\lambda\eta}, g_{\mu\eta})$. Look: every g has its opposite when you add these all up, so the sum is clearly zero.

We would like to get equation (219) into the form of a single vanishing covariant tensor divergence, for reasons that will soon become very clear. Contract λ with ν, remembering the symmetries. (In the second term on the left side of [219], swap ν and η before contracting, changing the sign.)

$$R_{\mu\kappa;\eta} - R_{\mu\eta;\kappa} + R^{\nu}_{\mu\kappa\eta;\nu} = 0 \tag{220}$$

Next, contract μ with κ:

$$R_{;\eta} - R^{\mu}_{\eta;\mu} - R^{\nu}_{\eta;\nu} = 0 \tag{221}$$

(Did you understand the manipulations to get that final term on the left?

$$R^{\nu}_{\mu\kappa\eta;\nu} = g^{\nu\sigma}R_{\sigma\mu\kappa\eta;\nu} = -g^{\nu\sigma}R_{\mu\sigma\kappa\eta;\nu}$$

Now it is easy to raise μ and contract with κ:

$$-g^{\nu\sigma} R^{\mu}_{\sigma\mu\eta;\nu} = -g^{\nu\sigma} R_{\sigma\eta;\nu} = -R^{\nu}_{\eta;\nu})$$

Cleaning up, our contracted identity (221) becomes:

$$(\delta^{\mu}_{\eta} R - 2R^{\mu}_{\eta})_{;\mu} = 0. \tag{222}$$

Raising η (we can of course bring $g^{\nu\eta}$ inside the covariant derivative to do this—why?), and dividing by -2 puts this identity into its classic form:

$$\left(R^{\mu\nu} - g^{\mu\nu}\frac{R}{2} \right)_{;\mu} = 0 \tag{223}$$

Einstein did not know this identity when he was struggling mightily with his theory, but to be fair neither did most mathematicians! The identities were actually first discovered by the German mathematician A. Voss in 1880, then independently in 1889 by Ricci. These results were then quickly forgotten, even, it seems, by Ricci himself. Bianchi then rediscovered them on his own in 1902, but they were still not widely known in the mathematics community in 1915. This was a pity, because the Bianchi identities have been called the "royal road to the Gravitational Field Equations " by Einstein's biographer A. Pais. It seems to have been the mathematician H. Weyl who in 1917 first recognised the importance of the Bianchi identitites for relativity, but the particular derivation we have followed here was not formulated until 1922, by Harward. The reason for the identities' importance is precisely analogous to Maxwell's understanding of the restrictions that the curl operator imposes on the field it generates, and to why the displacement current needs to be added to the equation $\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$: taking the divergence, the right hand source term must be *physically* conserved. Maxwell needed and invoked a physical displacement current, $(1/c^2)\partial \boldsymbol{E}/\partial t$, added to the right side of the equation. Here, we shall apply the Bianchi identities to guarantee the analogue (and it really is a precise mathematical analogue) of "the divergence of the curl is zero," a *geometrical* constraint[4] that ensures that the Field Equations have conservation of the stress energy tensor automatically built into their fundamental formulation.

---

[4]Chapter 15 of MTW presents a discussion in the language of differential forms of how to interpret the Bianchi identities as the notion that "the boundary of a boundary is zero."

6 The Einstein Field Equations

6.1 Formulation

We will now apply the principle of general covariance to the gravitational field itself. What is the relativistic analogue of $\nabla^2\Phi = 4\pi G\rho$? We have now built up a sufficiently strong mathematical arsenal from Riemannian geometry to be able to give a satisfactory answer to this question.

We know that we must work with vectors and tensors to maintain general covariance, and that the Newtonian-Poisson source, ρ, is a mere component of a more general stress-energy tensor $T_{\mu\nu}$ (in covariant tensor form) in relativity. We expect therefore that the gravitional field equations will take the form

$$G_{\mu\nu} = CT_{\mu\nu} \tag{224}$$

where $G_{\mu\nu}$ is a tensor comprised of $g_{\mu\nu}$ and its second derivatives, or products of the first derivatives of $g_{\mu\nu}$. We guess this since we know that in the Newtonian limit the largest component of $g_{\mu\nu}$ is the $g_{00} \simeq -1 - 2\Phi/c^2$ component, we need to recover Poisson, and we are seeking a theory of gravity that does not change its character with scale: that is, it has no characteristic length associated with it where things start to change. The last condition may strike you as a bit too restrictive at this stage. And, umm..well, we know it is actually wrong when applied to the Universe at large! But it is the simplest assumption that we can make that will satisfy all the basic requirements of a good theory. We'll come back to the updates once we have version 1.0.

Next, we know that the stress energy tensor is conserved in the sense of $T^{\mu\nu}_{;\nu} = 0$. We know from our work with the Bianchi identities of the previous section that this will automatically be satisfied if we take $G_{\mu\nu}$ to be proportional to the particular linear combination

$$G_{\mu\nu} \propto R_{\mu\nu} - \frac{g_{\mu\nu}R}{2}$$

(Notice that there is no difficulty shifting indices up or down as considerations demand: our index shifters $g_{\mu\nu}$ and $g^{\mu\nu}$ all have vanishing covariant derivatives and can moved inside and outside of semi-colons.) We have determined the field equations of gravity up to an overall

normalisation:

$$R_{\mu\nu} - \frac{g_{\mu\nu}R}{2} = CT_{\mu\nu} \tag{225}$$

The final step is to recover the Newtonian limit. In this limit, $T_{\mu\nu}$ is dominated by T_{00}, and $g_{\mu\nu}$ can be replaced by $\eta_{\alpha\beta}$ when shifting indices. The leading order derivative of $g_{\mu\nu}$ that enters into the field equations comes from

$$g_{00} \simeq -1 - \frac{2\Phi}{c^2}$$

where Φ is the usual Newtonian potential. In what follows, we use i, j, k to indicate spatial indices, and 0 will always be reserved for time.

The exact 00 component of the field equation reads,

$$R_{00} - \frac{1}{2}g_{00}R = CT_{00} \tag{226}$$

To determine R, notice that while T_{ij} is small, R_{ij} is *not* small! In fact, precisely because T_{ij} is small, R_{ij} must nearly cancel with the R term in the ij component of our equation:

$$R_{ij} - \frac{1}{2}g_{ij}R = \cancel{CT_{ij}} \quad \text{(small)}, \tag{227}$$

so that in the Newtonian limit, with $g_{ij} \simeq \eta_{ij}$,

$$R_{ij} \simeq \frac{1}{2}\eta_{ij}R \rightarrow R^i_j \simeq \frac{1}{2}\delta^i_j R. \tag{228}$$

Taking the trace of (228),

$$\frac{3}{2}R \simeq R^k_k \equiv R - R^0_0 \simeq R + R_{00} \rightarrow R \simeq 2R_{00}. \tag{229}$$

Therefore, returning to (226) with $g_{00} \simeq \eta_{00}$,

$$CT_{00} = R_{00} - \frac{1}{2}\eta_{00}R = R_{00} + \frac{1}{2}(2R_{00}) = 2R_{00}. \tag{230}$$

Calculating R_{00} explicitly,

$$R_{00} = R^\lambda_{0\lambda 0} = \eta^{\lambda\rho}R_{\rho 0\lambda 0} = -R_{0000} + R_{1010} + R_{2020} + R_{3030}. \tag{231}$$

Now, we need only the linear part of $R_{\lambda\mu\nu\kappa}$ in the weak field limit:

$$R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\kappa \partial x^\lambda} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\nu \partial x^\mu} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\nu \partial x^\lambda} \right), \tag{232}$$

and in the static limit only the final term on the right side of this equation survives:

$$R_{0000} \simeq 0, \qquad R_{i0j0} = \frac{1}{2}\frac{\partial^2 g_{00}}{\partial x^i \partial x^j}. \tag{233}$$

49

Finally,

$$CT_{00} = C\rho c^2 = 2R_{00} = 2 \times \frac{1}{2}\nabla^2 g_{00} = -\frac{2}{c^2}\nabla^2\Phi \qquad (234)$$

This happily agrees with the Poisson equation if $C = -8\pi G/c^4$. We therefore arrive at the *Einstein Field Equations*:

$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^4}T_{\mu\nu}} \qquad (235)$$

The Field Equations first appeared in Einstein's notes on 25 November 1915, just over a hundred years ago, after an inadvertent competition with the mathematician David Hilbert, triggered by an Einstein colloquium at Göttingen. (Talk about being scooped! Hilbert actually derived the Field Equations first, by a variational method, but rightly insisted on giving Einstein full credit for the physical theory.)

It is useful to have these equation in a slightly different form. Contracting μ and ν, we obtain

$$R = \frac{8\pi G}{c^4}T \qquad (236)$$

Thus, we may rewrite the original equation with only the Ricci tensor on the right:

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) \equiv -\frac{8\pi G}{c^4}S_{\mu\nu} \qquad (237)$$

where we have introduced the *source term*,

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T \qquad (238)$$

In vacuum, the Field Equations reduce to their Laplace form,

$$R_{\mu\nu} = 0 \qquad (239)$$

One final point. If we allow the possibility that gravity could change its form on different scales, it is always possible to add a term of the form $\Lambda g_{\mu\nu}$ to $G_{\mu\nu}$, where Λ is a constant, without violating the conservation of $T_{\mu\nu}$ condition. This is because the covariant derivatives of $g_{\mu\nu}$ vanish identically and $T_{\mu\nu}$ is still conserved. Einstein, pursuing the consequences of his theory for cosmology, realised that his Field Equations did not produce a static universe. This is bad, he thought, everyone knows the Universe is static. So he sought a source of static stabilisation, added the Λ term back into the Field Equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu}, \qquad (240)$$

and dubbed Λ the *cosmological constant.* Had he not done so, he could have made a spectacular prediction: the universe is dynamic, a player in its own game, and must be either expanding or contracting.[5] With the historical discovery of an expanding univese, Einstein retracted the Λ term, calling it "the biggest mistake of my life." It seems not to have damaged his career.

---

[5]Even within the context of straight Euclidian geometry and Newtonian dynamics, uniform expansion of an infinite space avoids the self-consistency problems associated with a static model. I've never understood why this simple point is not emphasised more.

Surprise. We now know that this term is, in fact, present on the largest cosmological scales, and on these scales it is not a small effect. It mimics (and may well be) an energy density of the vacuum itself. It is measured to be 70% of the effective energy density in the Universe. It is to be emphasised that Λ must be taken into account only on the largest scales when the locally much higher baryon and dark matter densities are lowered by effective smoothing; it is negligible otherwise. The so-called biggest mistake of Einstein's life was therefore two-fold: first, introducing Λ for the wrong reason, and then retracting it for the wrong reason!

Except for cosmological problems, we will always assume $\Lambda = 0$.

6.2   Coordinate ambiguities

There is no unique solution to the Field Equation because of the fact that they have been constructed to admit a new solution by a transformation of coordinates. To make this point as clear as possible, imagine that we have solved for the metric $g_{\mu\nu}$, and in turns out to be plain old Minkowski space. Denote the coordinates as t for the time dimension and α, β, γ for the spatial dimensions. Even if we restrict ourselves to diagonal $g_{\mu\nu}$, we might have found that the diagonal entries are $(-1, 1, 1, 1)$ or $(-1, 1, \alpha^2, 1)$ or $(-1, 1, \alpha^2, \alpha^2 \sin^2 \beta)$ depending upon whether we happen to be using Cartesian, cylindrical, or spherical spatial coordinate systems. Thus, we always have the freedom to work with coordinates that simplify our equations or that make physical properties of our solutions more transparent.

This is particularly useful for gravitational radiation. You may remember when you studied electromagnetic radiation that the equations for the potentials (both $\boldsymbol{A}$ and Φ) simplified considerably when a particular gauge was used—the Lorenz gauge. A different gauge could have been used and the potential would have looked different, but the fields would have been the same. The same is true for gravitational radiation, in which a coordinate transformation plays this role.

For the problem of determining $g_{\mu\nu}$ around a point mass—the Schwarzschild black hole—we will choose to work with coordinates that look as much as possible like standard spherical coordinates.

6.3   The Schwarzschild Solution

We wish to determine the form of the metric tensor $g_{\mu\nu}$ for the space-time surrounding a point mass M by solving the equation $R_{\mu\nu} = 0$, subject to the appropriate boundary conditions.

Because the space-time is static and spherically symmetric, we expect the invariant line element to take the form

$$-c^2 d\tau^2 = -B\, c^2 dt^2 + A\, dr^2 + C\, d\Omega^2 \tag{241}$$

where $d\Omega$ is the (undistorted) solid angle,

$$d\Omega^2 = d\theta^2 + \sin^2 \theta\, d\phi^2$$

and A, B, and C are all functions of the radial variable. We may choose our coordinates so that C is defined to be r^2 (if it is not already, do a coordinate transformation $r'^2 = C(r)$ and then drop the $'$). A and B will then be some unknown functions of r to be determined. Our metric is now in "standard form:"

$$-c^2 d\tau^2 = -B(r)\, c^2 dt^2 + A(r)\, dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta\, d\phi^2\right) \tag{242}$$

We may now read the components of $g_{\mu\nu}$:

$$g_{tt} = -B(r) \qquad g_{rr} = A(r) \qquad g_{\theta\theta} = r^2 \qquad g_{\phi\phi} = r^2 \sin^2\theta \tag{243}$$

and its inverse $g^{\mu\nu}$,

$$g^{tt} = -B^{-1}(r) \qquad g^{rr} = A^{-1}(r) \qquad g^{\theta\theta} = r^{-2} \qquad g^{\phi\phi} = r^{-2}(\sin\theta)^{-2} \tag{244}$$

The determinant of $g_{\mu\nu}$ is $-g$, where

$$g = r^4 AB \sin^2\theta \tag{245}$$

The affine connection for a *diagonal* metric tensor reads

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2g_{\lambda\lambda}} \left(\frac{\partial g_{\lambda\mu}}{\partial x^\nu} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda} \right) \qquad \text{NO SUM OVER } \lambda. \tag{246}$$

Obviously, only Γ's with at least one repeated index will be present. (Why?) The nonvanishing components follow straightforwardly:

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{B'}{2B}$$

$$\Gamma^r_{tt} = \frac{B'}{2A} \quad \Gamma^r_{rr} = \frac{A'}{2A} \quad \Gamma^r_{\theta\theta} = -\frac{r}{A} \quad \Gamma^r_{\phi\phi} = -\frac{r\sin^2\theta}{A}$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r} \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\,\cos\theta$$

$$\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = \frac{1}{r} \qquad \Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \cot\theta \tag{247}$$

where $A' = dA/dr$, $B' = dB/dr$. We will also make use of this table to compute the orbits in a Schwarzschild geometry.

Next, we need the Ricci Tensor:

$$R_{\mu\kappa} \equiv R^\lambda_{\mu\lambda\kappa} = \frac{\partial \Gamma^\lambda_{\mu\lambda}}{\partial x^\kappa} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\lambda} + \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\lambda\eta} \tag{248}$$

Remembering equation (167), this may be written

$$R_{\mu\kappa} = \frac{1}{2}\frac{\partial^2 \ln g}{\partial x^\kappa \partial x^\mu} - \frac{\partial \Gamma^\lambda_{\mu\kappa}}{\partial x^\lambda} + \Gamma^\eta_{\mu\lambda}\Gamma^\lambda_{\kappa\eta} - \frac{\Gamma^\eta_{\mu\kappa}}{2}\frac{\partial \ln g}{\partial x^\eta} \tag{249}$$

Right. First R_{tt}. Remember, static fields.

$$R_{tt} = -\frac{\partial \Gamma^r_{tt}}{\partial r} + \Gamma^\eta_{t\lambda}\Gamma^\lambda_{t\eta} - \Gamma^\eta_{tt}\Gamma^\lambda_{\lambda\eta}$$

$$= -\frac{\partial}{\partial r}\left(\frac{B'}{2A}\right) + \Gamma^t_{t\lambda}\Gamma^\lambda_{tt} + \Gamma^r_{t\lambda}\Gamma^\lambda_{tr} - \Gamma^r_{tt}\Gamma^\lambda_{\lambda r}$$

$$= -\frac{\partial}{\partial r}\left(\frac{B'}{2A}\right) + \Gamma^t_{tr}\Gamma^r_{tt} + \Gamma^r_{tt}\Gamma^t_{tr} - \frac{\Gamma^r_{tt}}{2}\frac{\partial \ln g}{\partial r}$$

$$= -\left(\frac{B''}{2A}\right) + \frac{B'A'}{2A^2} + \frac{B'^2}{4AB} + \frac{\cancel{B'^2}}{\cancel{4AB}} - \frac{B'}{4A}\left(\frac{A'}{A} + \frac{\cancel{B'}}{\cancel{B}} + \frac{4}{r}\right)$$

This gives

$$R_{tt} = -\frac{B''}{2A} + \frac{B'}{4A}\left(\frac{B'}{B} + \frac{A'}{A}\right) - \frac{B'}{rA} \tag{250}$$

Next, R_{rr}:

$$R_{rr} = \frac{1}{2}\frac{\partial^2 \ln g}{\partial r^2} - \frac{\partial \Gamma^r_{rr}}{\partial r} + \Gamma^\eta_{r\lambda}\Gamma^\lambda_{r\eta} - \frac{\Gamma^r_{rr}}{2}\frac{\partial \ln g}{\partial r}$$

$$= \frac{1}{2}\frac{\partial}{\partial r}\left(\frac{\cancel{A'}}{\cancel{A}} + \frac{B'}{B} + \frac{4}{r}\right) - \frac{\partial}{\cancel{\partial r}}\cancel{\left(\frac{A'}{2A}\right)} + \Gamma^\eta_{r\lambda}\Gamma^\lambda_{r\eta} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B} + \frac{4}{r}\right)$$

$$= \frac{B''}{2B} - \frac{1}{2}\left(\frac{B'}{B}\right)^2 - \frac{2}{r^2} + \left(\Gamma^t_{rt}\right)^2 + \left(\Gamma^r_{rr}\right)^2 + \left(\Gamma^\theta_{r\theta}\right)^2 + \left(\Gamma^\phi_{r\phi}\right)^2 - \frac{1}{4}\left(\frac{A'}{A}\right)^2 - \frac{A'B'}{4AB} - \frac{A'}{rA}$$

$$= \frac{B''}{2B} - \frac{1}{2}\left(\frac{B'}{B}\right)^2 - \frac{\cancel{2}}{\cancel{r^2}} + \frac{B'^2}{4B^2} + \frac{\cancel{A'^2}}{\cancel{4A^2}} + \frac{\cancel{1}}{\cancel{r^2}} + \frac{\cancel{1}}{\cancel{r^2}} - \frac{1}{4}\cancel{\left(\frac{A'}{A}\right)^2} - \frac{A'B'}{4AB} - \frac{A'}{rA}$$

So that finally

$$R_{rr} = \frac{B''}{2B} - \frac{1}{4}\frac{B'}{B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rA} \tag{251}$$

Tired? Well, here is a spoiler: all we will need for the problem at hand is R_{tt} and R_{rr}, so you can now skip to the end of the section. For the true fanatics, we are just getting warmed up! On to $R_{\theta\theta}$:

$$R_{\theta\theta} = \frac{\partial \Gamma^\lambda_{\theta\lambda}}{\partial \theta} - \frac{\partial \Gamma^\lambda_{\theta\theta}}{\partial x^\lambda} + \Gamma^\eta_{\theta\lambda}\Gamma^\lambda_{\theta\eta} - \Gamma^\eta_{\theta\theta}\Gamma^\lambda_{\lambda\eta}$$

$$= \frac{1}{2}\frac{\partial^2 \ln g}{\partial \theta^2} - \frac{\partial \Gamma^r_{\theta\theta}}{\partial r} + \Gamma^\eta_{\theta\lambda}\Gamma^\lambda_{\theta\eta} - \Gamma^r_{\theta\theta}\Gamma^\lambda_{\lambda r}$$

$$= \frac{d(\cot\theta)}{d\theta} + \frac{d}{dr}\left(\frac{r}{A}\right) + \Gamma^\eta_{\theta\lambda}\Gamma^\lambda_{\theta\eta} + \frac{r}{2A}\frac{\partial \ln g}{\partial r}$$

$$= -\frac{1}{\sin^2\theta} + \frac{1}{A} - \frac{rA'}{A^2} + \Gamma^r_{\theta\lambda}\Gamma^\lambda_{\theta r} + \Gamma^\theta_{\theta\lambda}\Gamma^\lambda_{\theta\theta} + \Gamma^\phi_{\theta\lambda}\Gamma^\lambda_{\theta\phi} + \frac{r}{2A}\left(\frac{A'}{A} + \frac{B'}{B} + \frac{4}{r}\right)$$

$$= -\frac{1}{\sin^2\theta} + \frac{3}{A} - \frac{rA'}{2A^2} + \Gamma^r_{\theta\theta}\Gamma^\theta_{\theta r} + \Gamma^\theta_{\theta r}\Gamma^r_{\theta\theta} + \left(\Gamma^\phi_{\theta\phi}\right)^2 + \frac{rB'}{2AB}$$

$$= -\frac{1}{\sin^2\theta} + \frac{3}{A} - \frac{rA'}{2A^2} - \frac{2}{A} + \cot^2\theta + \frac{rB'}{2AB}$$

The trigonometric terms add to -1. We finally obtain

$$R_{\theta\theta} = -1 + \frac{1}{A} + \frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) \tag{252}$$

$R_{\phi\phi}$ is the last nonvanishing Ricci component. No whining now! The first term in (248) vanishes, since nothing in the metric depends on ϕ. Then,

$$R_{\phi\phi} = -\frac{\partial \Gamma^\lambda_{\phi\phi}}{\partial x^\lambda} + \Gamma^\eta_{\phi\lambda}\Gamma^\lambda_{\phi\eta} - \frac{\Gamma^\eta_{\phi\phi}}{2}\frac{\partial \ln|g|}{\partial x^\eta}$$

$$= -\frac{\partial \Gamma^r_{\phi\phi}}{\partial r} - \frac{\partial \Gamma^\theta_{\phi\phi}}{\partial \theta} + \Gamma^r_{\phi\lambda}\Gamma^\lambda_{\phi r} + \Gamma^\theta_{\phi\lambda}\Gamma^\lambda_{\phi\theta} + \Gamma^\phi_{\phi\lambda}\Gamma^\lambda_{\phi\phi} - \frac{1}{2}\Gamma^r_{\phi\phi}\frac{\partial \ln|g|}{\partial r} - \frac{1}{2}\Gamma^\theta_{\phi\phi}\frac{\partial \ln|g|}{\partial \theta}$$

$$= \frac{\partial}{\partial r}\left(\frac{r\sin^2\theta}{A}\right) + \frac{\partial}{\partial \theta}(\sin\theta\cos\theta) + \Gamma^r_{\phi\phi}\Gamma^\phi_{\phi r} + \Gamma^\theta_{\phi\phi}\Gamma^\phi_{\phi\theta} + \Gamma^\phi_{\phi r}\Gamma^r_{\phi\phi} + \Gamma^\phi_{\phi\theta}\Gamma^\theta_{\phi\phi}$$

$$+ \frac{1}{2}\sin\theta\cos\theta\frac{\partial\ln\sin^2\theta}{\partial\theta} + \frac{1}{2}\left(\frac{r\sin^2\theta}{A}\right)\left(\frac{A'}{A} + \frac{B'}{B} + \frac{4}{r}\right)$$

$$= \frac{\cancel{\sin^2\theta}}{\cancel{A}} - \frac{rA'\sin^2\theta}{A^2} + \cancel{\cos^2\theta} - \sin^2\theta - \frac{\cancel{\sin^2\theta}}{\cancel{A}} - \cancel{\cos^2\theta} - \frac{\sin^2\theta}{A} - \cancel{\cos^2\theta} + \cancel{\cos^2\theta} + \frac{r\sin^2\theta}{2A}\left(\frac{A'}{A} + \frac{B'}{B} + \frac{4}{r}\right)$$

$$= \sin^2\theta\left[\frac{r}{2A}\left(-\frac{A'}{A} + \frac{B'}{B}\right) + \frac{1}{A} - 1\right] = \sin^2\theta R_{\theta\theta}$$

The fact that $R_{\phi\phi} = \sin^2\theta R_{\theta\theta}$ and that $R_{\mu\nu} = 0$ if μ and ν are not equal are a consequence of the spherical symmetry and time reversal symmetry of the problem. If the first relation did not hold, then an ordinary rotation of the axes would change the form of the tensor despite the spherical symmetry, which is impossible. If $R_{ti} \equiv R_{it}$ were present (i is a spatial index), the coordinate transformation $t' = -t$ would change the components of the Ricci tensor. But $R_{\mu\nu}$ must be invariant to this form of time reversal coordinate change. (Why?) Note that this argument is not true for R_{tt}. (Why not?)

Learn to think like a mathematical physicist in this kind of a calculation, taking into account the symmetries that are present, and you can save a lot of work.

Exercise. **Self-gravitating masses in general relativity.** We are solving in this section the vacuum equations $R_{\mu\nu} = 0$, but it is of great interest for stellar structure and cosmology to have a set of equations for a self-gravitating spherical mass. Toward that end, we recall equation (237):

$$R_{\mu\nu} = -\frac{8\pi G}{c^4}S_{\mu\nu} \equiv -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{g_{\mu\nu}}{2}T^\lambda_\lambda\right)$$

Let us evaluate $S_{\mu\nu}$ for the case of an isotropic stress energy tensor of an ideal gas in its rest frame. With

$$g_{tt} = -B, \quad g_{rr} = A, \quad g_{\theta\theta} = r^2, \quad g_{\phi\phi} = r^2\sin^2\theta,$$

the stress-energy tensor

$$T_{\mu\nu} = Pg_{\mu\nu} + (\rho + P/c^2)U_\mu U_\nu,$$

where U_μ is the 4-velocity, show that, in addition to the trivial condition

$$U_r = U_\theta = U_\phi = 0,$$

we must have $U_t = -c\sqrt{B}$ (remember equation [176]) and that

$$S_{tt} = \frac{B}{2}(3P + \rho c^2), \quad S_{rr} = \frac{A}{2}(\rho c^2 - P), \quad S_{\theta\theta} = \frac{r^2}{2}(\rho c^2 - P)$$

We will develop the solutions of $R_{\mu\nu} = -8\pi G S_{\mu\nu}/c^4$ shortly.

Enough. We have more than we need to solve the problem at hand. To solve the equations $R_{\mu\nu} = 0$ is now a rather easy task. Two components will suffice (we have only A and B to solve for after all), all others then vanish indentically. In particular, work with R_{rr} and R_{tt}, both of which must separately vanish, so

$$\frac{R_{rr}}{A} + \frac{R_{tt}}{B} = -\frac{1}{rA}\left(\frac{A'}{A} + \frac{B'}{B}\right) = 0 \tag{253}$$

whence we find

$$AB = \text{constant} = 1 \tag{254}$$

where the constant must be unity since A and B go over to their Minkowski values at large distances. The condition that $R_{tt} = 0$ is now from (250) simply

$$B'' + \frac{2B'}{r} = 0, \tag{255}$$

which means that B is a linear superposition of a constant plus another constant times $1/r$. But B must approach unity at large r, so the first constant is one, and we know from long ago that the next order term at large distances must be $2\Phi/c^2$ in order to recover the Newtonian limit. Hence,

$$B = 1 - \frac{2GM}{rc^2}, \qquad A = \left(1 - \frac{2GM}{rc^2}\right)^{-1} \tag{256}$$

The *Schwarzschild Metric* for the space-time around a point mass is exactly

$$\boxed{-c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2} \tag{257}$$

This remarkable, simple and critically important exact solution of the Einstein Field Equation was obtained in 1916 by Karl Schwarzschild from the horrors of the trenches of World War I. Tragically, Schwarzschild did not survive the war,[6] dying from a skin infection five months after finding his marvelous solution. He managed to communicate his result fully in a letter to Einstein. His last letter to Einstein was dated December 22, 1915, some 28 days after the formulation of the Field Equations.

Exercise. **The Tolman-Oppenheimer-Volkoff Equation.** Let us strike again while the iron is hot. Referring back to Exercise (11), we repeat part of our Schwarzschild calculation, but with the source terms $S_{\mu\nu}$ retained. Form a familiar combination once again:

$$\frac{R_{rr}}{A} + \frac{R_{tt}}{B} = -\frac{1}{rA}\left(\frac{A'}{A} + \frac{B'}{B}\right) = -\frac{8\pi G}{c^4}\left(\frac{S_{tt}}{B} + \frac{S_{rr}}{A}\right) = -\frac{8\pi G}{c^4}(P + \rho c^2)$$

Show now that adding $2R_{\theta\theta}/r^2$ eliminates the B dependence:

$$\frac{R_{rr}}{A} + \frac{R_{tt}}{B} + \frac{2R_{\theta\theta}}{r^2} = -\frac{2A'}{rA^2} - \frac{2}{r^2} + \frac{2}{Ar^2} = -\frac{16\pi G\rho}{c^2}.$$

---

[6]The WWI deaths of Karl Schwarzschild for the Germans and of Henry Moseley for the British were incalculable losses for science. Schwarzschild's son Martin also became a great astronomer, developing much of the modern theory of stellar evolution.

Solve this equation for A and show that the solution with finite $A(0)$ is

$$A(r) = \left(1 - \frac{2G\mathcal{M}(r)}{r}\right)^{-1}, \quad \mathcal{M}(r) = \int_0^r 4\pi\rho(r')\, r'^2 \, dr'$$

Finally, use the equation $R_{\theta\theta} = -8G\pi S_{\theta\theta}/c^4$ together with hydrostatic equilibrium (180) (for the term B'/B in $R_{\theta\theta}$) to obtain the celebrated Tolman-Oppenheimer-Volkoff equation for the interior structure of general relativistic stars:

$$\frac{dP}{dr} = -\frac{G\mathcal{M}(r)\,\rho}{r^2}\left(1 + \frac{P}{\rho c^2}\right)\left(1 + \frac{4\pi r^3 P}{\mathcal{M}(r)\,c^2}\right)\left(1 - \frac{2G\mathcal{M}(r)}{rc^2}\right)^{-1}$$

This is a rather long, but completely straightforward, exercise.

Students of stellar structure will recognise the classical equation hydrostatic equilibrium equation for a Newtonian star, with three correction terms. The final factor on the right is purely geometrical, the radial curvature term A from the metric. The corrective replacement of ρ by $\rho + P/c^2$ arises even in the special relativistic equations of motion for the inertial density; for inertial purposes P/c^2 is an effective density. Finally the modification of the gravitating $\mathcal{M}(r)$ term also includes a contribution from the density, as though an additional effective mass density $P(r)/3c^2$ were spread throughout the interior spherical volume within r, even though $P(r)$ is just the local pressure. In massive stars, this pressure could be radiative.

6.4   The Schwarzschild Radius

It will not have escaped the reader's attention that at

$$r = \frac{2GM}{c^2} \equiv R_S \tag{258}$$

the metric becomes singular in appearance. R_S is known as the Schwarzschild radius. Numerically, normalising M to one solar mass $M_\odot$,

$$R_S = 2.95\,(M/M_\odot)\text{ km}, \tag{259}$$

which is *well* inside any normal star! The Schwarzschild radius is part of the external vacuum space-time only for black holes. Indeed, it is what makes black holes black. At least it was *thought* to be the feature that made black holes truly black, until Hawking came along in 1974 showed us that quantum field theory changes the behaviour of black holes. But as usual, we are getting ahead of ourselves. More on "Hawking radiation" later. Let us stick to classical theory.

I have been careful to write "singular in appearance" because in fact, the space-time is perfectly well behaved at $r = R_S$. It is only the coordinates that become strained at this point, and these coordinates have been introduced, you will recall, so that they would be familiar to us, we happy band of observers at in finity, as ordinary spherical coordinates. The curvature scalar R, for example, remains zero without a ripple as we pass through $r = R_S$. We can see this coordinate effect happening if we start with the ordinary metric on the unit sphere,

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

and changing to coordinates to $x = \sin\theta$:

$$ds^2 = \frac{dx^2}{1-x^2} + x^2 d\phi^2$$

This looks horrible at $x = 1$, but in reality nothing is happening. Since x is just the distance from the z-axis to spherical surface (i.e. cylindrical radius), the "singularity" simply reflects the fact that at the equator x has reached its maximum value 1. So, dx must be zero at this point. x is just a bad coordinate at the equator; ϕ is a bad coordinate at the pole. Bad coordinates happen to good spacetimes. Get over it.

The physical interpretation of the first two terms of the metric (257) is that the proper time interval at a fixed spatial location is given by

$$dt \left(1 - \frac{2GM}{rc^2}\right)^{1/2} \qquad \text{(proper time interval at fixed location)}. \qquad (260)$$

The proper radial distance interval at a fixed angular location and time is

$$dr \left(1 - \frac{2GM}{rc^2}\right)^{-1/2} \qquad \text{(proper radial distance interval at fixed time \& angle)}. \qquad (261)$$

Exercise. **Getting rid of the Schwarzschild coordinate singularity. A challenge problem for the adventurous student only. Make sure you want to do this before you start.** Consider the rather unusual coordinate transformation found Martin Kruskal. Start with our standard spherical coordinates t, r, θ, ϕ and introduce new r' and t' coordinates:

$$r'^2 - c^2 t'^2 = c^2 T^2 \left(\frac{rc^2}{2GM} - 1\right) \exp\left(\frac{rc^2}{2GM}\right)$$

$$\frac{2r'ct'}{r'^2 + c^2 t'^2} = \tanh\left(\frac{c^3 t}{2GM}\right)$$

where T is an arbitrary constant. Show that the Schwarzschild metric transforms to

$$-c^2 d\tau^2 = \left(\frac{32 G^3 M^3}{c^8 r T^2}\right) \exp\left(\frac{-rc^2}{2GM}\right) (c^2 dt'^2 - dr'^2) - r^2 d\Omega^2$$

where T is arbitrary constant with dimensions of time, and r is the implicit solution of our first equation for $r'^2 - c^2 t'^2$. The right side of this equation has a minimum of $-c^2 T^2$ at $r = 0$, hence we must have

$$r'^2 > c^2 (t'^2 - T^2)$$

always. When $t' < T$ there is no problem. But when $t' > T$ there are two distinct regions: $r' = \pm c\sqrt{t'^2 - T^2}$! Then the metric has a real singularity at either of these values of r' (which is just $r = 0$), but still no singularity at $r' = \pm ct'$, the value $r = R_S$.

6.5 Schwarzschild spacetime.

6.5.1 Radial photon geodesic

This doesn't mean that there is nothing of interest happening at $r = R_S$.

For starters, the gravitational redshift recorded by an observer at infinity relative to someone at rest at location r in the Schwarzschild space-time is given (we now know) *precisely* by

$$dt = \frac{d\tau}{(1 - 2GM/rc^2)^{1/2}} \quad \text{(Exact.)} \tag{262}$$

so that at $r \to R_S$, signals arrive at a distant observer's post infinitely redshifted. What does this mean?

Comfortably sitting in the DWB whilst monitoring the radio signals my hardworking graduate student is sending me en route from a thesis mission to take measurements of the $r = R_S$ tidal forces in a nearby black hole, I grow increasingly impatient. Not only are the complaints becoming progressively more torpid and drawn out, the transmission frequency keeps shifting to longer and longer wavelengths, out of my receiver's bandpass. (Most irritating.) Eventually, of course, all contact is lost. I never receive any signal of any kind from within R_S. R_S is said to be the location of the *event horizon.* The singularity at $r = 0$ is present, but completely hidden from the outside world at $R = R_S$ within an event horizon. It is what Roger Penrose has aptly named "cosmic censorship."

The time coordinate change for light to travel from r_A to r_B following its geodesic path is given by setting

$$-(1 - 2GM/rc^2)c^2 dt^2 + dr^2/(1 - 2GM/rc^2) = 0$$

and then computing

$$t_{AB} = \int_A^B dt = \frac{1}{c} \int_{r_A}^{r_B} \frac{dr}{(1 - 2GM/rc^2)} = \frac{r_B - r_A}{c} + \frac{R_S}{c} \ln\left(\frac{r_B - R_S}{r_A - R_S}\right) \tag{263}$$

which will be recognised as the Newtonian time interval plus a logarithmic correction proporitional to the Schwarzschild radius R_S. Note that our expression become infinite when a path endpoint includes R_S. When R_S may be considered small over the entire integration path, to leading order

$$t_{AB} \simeq \frac{r_B - r_A}{c} + \frac{R_S}{c} \ln\left(\frac{r_A}{r_B}\right) = \frac{r_B - r_A}{c}\left(1 + \frac{R_S \ln(r_A/r_B)}{r_B - r_A}\right) \tag{264}$$

A GPS satellite orbits at an altitude of 20,200 km, and the radius of the earth is 6370 km. R_S for the earth is only 9mm! (Make a fist. Squeeze the entire earth inside it. You're not even close to making a black hole.)

$$\frac{R_S}{r_B - r_A} \simeq \frac{9 \times 10^{-3}}{(20,200 - 6370) \times 10^3} = 6.5 \times 10^{-10}$$

This level of accuracy, about a part in 10^9, is needed for determining positions on the surface of the earth to a precision of a few meters (as when your GPS intones "Turn right onto the Lon-don Road."). How does the gravitational effect compare with the *second order* kinematic time dilation due to the satellite's motion? You should find them comparable.

6.5.2 Orbital equations

Start with the geodesic equation, written in terms of an arbitrary time parameter p:

$$\frac{d^2 x^\lambda}{dp^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{dp} \frac{dx^\nu}{dp} = 0 \tag{265}$$

It doesn't matter what p is, just use your watch. Using the table of equation (247), it is very easy to write down the equations for the orbits in a Schwarzschild geometry:

$$\frac{d^2(ct)}{dp^2} + \frac{B'}{B}\frac{dr}{dp}\frac{d(ct)}{dp} = 0, \tag{266}$$

$$\frac{d^2r}{dp^2} + \frac{B'}{2A}\left(\frac{cdt}{dp}\right)^2 + \frac{A'}{2A}\left(\frac{dr}{dp}\right)^2 - \frac{r}{A}\left(\frac{d\theta}{dp}\right)^2 - \frac{r\sin^2\theta}{A}\left(\frac{d\phi}{dp}\right)^2 = 0, \tag{267}$$

$$\frac{d^2\theta}{dp^2} + \frac{2}{r}\frac{dr}{dp}\frac{d\theta}{dp} - \sin\theta\cos\theta\left(\frac{d\phi}{dp}\right)^2 = 0, \tag{268}$$

$$\frac{d^2\phi}{dp^2} + \frac{2}{r}\frac{dr}{dp}\frac{d\phi}{dp} + 2\cot\theta\frac{d\theta}{dp}\frac{d\phi}{dp} = 0. \tag{269}$$

Obviously, it is silly to keep θ as a variable. The orbit may be set to the $\theta = \pi/2$ plane. Then, our equations become:

$$\frac{d^2(ct)}{dp^2} + \frac{B'}{B}\frac{dr}{dp}\frac{d(ct)}{dp} = 0, \tag{270}$$

$$\frac{d^2r}{dp^2} + \frac{B'}{2A}\left(\frac{cdt}{dp}\right)^2 + \frac{A'}{2A}\left(\frac{dr}{dp}\right)^2 - \frac{r}{A}\left(\frac{d\phi}{dp}\right)^2 = 0, \tag{271}$$

$$\frac{d^2\phi}{dp^2} + \frac{2}{r}\frac{dr}{dp}\frac{d\phi}{dp} = 0. \tag{272}$$

Remember that A and B are functions of r! Then, the first and last of these equations are particularly simple:

$$\frac{d}{dp}\left(B\frac{cdt}{dp}\right) = 0 \tag{273}$$

$$\frac{d}{dp}\left(r^2\frac{d\phi}{dp}\right) = 0 \tag{274}$$

It is convenient to choose our parameter p to be close to the time:

$$\frac{dt}{dp} = B^{-1}, \tag{275}$$

and of course general relativity conserves angular momentum for a spherical geometry:

$$r^2\frac{d\phi}{dp} = J \qquad \text{(constant)} \tag{276}$$

Finally, just as we may form an energy integration constant from the radial motion equation in Newtonian theory, so too in Schwarzschild geometry. Multiplying (271) by $2Adr/dp$, and using our results for dt/dp and $d\phi/dp$, we find:

$$\frac{d}{dp}\left[A\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{c^2}{B}\right] = 0 \tag{277}$$

59

or
$$A \left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{c^2}{B} = -E \quad \text{(constant.)} \tag{278}$$

For $\theta = \pi/2$ orbits,

$$-c^2 \left(\frac{d\tau}{dp}\right)^2 = -B\,c^2 \left(\frac{dt}{dp}\right)^2 + A \left(\frac{dr}{dp}\right)^2 + r^2 \left(\frac{d\phi}{dp}\right)^2 = -\frac{c^2}{B} + A \left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} = -E, \tag{279}$$

using our results for dt/dp, dr/dp and $d\phi/dp$. Hence for matter, $E > 0$, while $E = 0$ for photons. To leading Newtonian order $E \simeq c^2$, i.e. the rest mass energy per unit mass! Substituting for B in (278), we find that extremal values of orbital r locations correspond to

$$\left(1 - \frac{2GM}{rc^2}\right) \left(\frac{J^2}{r^2} + E\right) - c^2 = 0 \tag{280}$$

for matter, and thus to

$$\left(1 - \frac{2GM}{rc^2}\right) \frac{J^2}{r^2} - c^2 = 0 \tag{281}$$

for photons.

The radial equation of motion may be written for $dr/d\tau$, dr/dt, or $dr/d\phi$ respectively (we use $AB = 1$):

$$\left(\frac{dr}{d\tau}\right)^2 + \frac{c^2}{A}\left(1 + \frac{J^2}{Er^2}\right) = \frac{c^4}{E} \tag{282}$$

$$\left(\frac{dr}{dt}\right)^2 + \frac{B^2}{A}\left(E + \frac{J^2}{r^2}\right) = \frac{Bc^2}{A} \tag{283}$$

$$\left(\frac{dr}{d\phi}\right)^2 + \frac{r^2}{A}\left(1 + \frac{Er^2}{J^2}\right) = \frac{c^2 r^4}{J^2} \tag{284}$$

From here, it is simply a matter of evaluating a (perhaps complicated) integral over r to obtain a solution.

6.6   The deflection of light by an intervening body.

The first prediction made by General Relativity Theory that could be tested was that starlight passing by the limb of the sun would be slightly but measurably deflected by the gravitational field. This type of measurement can only be done, of course, when the sun is completely eclipsed by the moon. Fortunately, the timing of the appearance of the theory with an eclipse was ideal. One of the longest total solar eclipses of the century occured in 29 May 1919. The path of totality extended from a strip in South America, to central Africa. An expedition headed by A.S. Eddington observed the eclipse from the island of Principe, off the west coast of Africa. Measurements of thirteen stars confirmed that not only did gravity affect the propagation of light, it did so by an amount that was in much better accord with general relativity theory rather than Newtonian theory with the velocity set equal to c. (The latter gives a deflection angle half as large as GR, in essence because the $2GM/rc^2$ terms in both the dt and dx metric coefficients contribute equally to the photon deflection, whereas in the Newtonian limit only the dt modification is retained—as we know.) This success earned
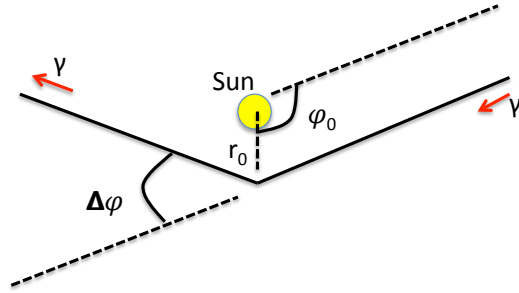
Figure 2: Bending of light by the gravitational field of the sun. The angle ϕ_0 is the azimuth of the photon (denoted γ) orbit at the point of closest approach r_0. The total change in ϕ is then $2\phi_0$, and the deflection angle $\Delta\phi$ from a straightline orbit is $2\phi_0 - \pi$.

Einstein press coverage that today is normally reserved for rock stars. *Everybody* knew who Albert Einstein was!

Today, not only deflection, but "gravitational lensing" across the electromagnetic spectrum has become a standard astronomical technique to discover and probe dark matter in all its forms: from small planets to huge and diffuse cosmological agglomerations.

Let us return to the classic test. Refer to fig. [1]. The path of a photon is bent as it encounters the gravitational field of the a body, here the sun. If the asymptotic azimuthal angle ϕ is taken to be zero radians when the photon is at infinity, then equation (284) tells us that at radius r, ϕ is

$$\phi(r) = \int_r^\infty \frac{A^{1/2} dr'}{r'\left(\dfrac{c^2 r'^2 A}{J^2} - 1\right)^{1/2}} \tag{285}$$

The trick to make our life a bit simpler here is realise that the point of closest approach, denoted r_0 and calculated from equation (281), is obviously a zero of the denominator of (285). So forget about c and J, we can rewrite the above as

$$\phi(r) = \int_r^\infty \frac{A^{1/2} dr'}{r'\left(\dfrac{r'^2 A(r')}{r_0^2 A(r_0)} - 1\right)^{1/2}} \tag{286}$$

The angle we want is $\Delta\phi = 2\phi(r_0) - \pi$. It remains only to calculate $\phi(r_0)$, a straightforward enough exercise, since we seek only the leading correction from $A = 1$. It is more convenient to set $r = r_0$ after the calculation, to avoid worrying over diverging integrals. With

$$\frac{1}{A(r')} = 1 - \frac{2GM}{r'c^2} \equiv 1 - \frac{\epsilon}{r'} \tag{287}$$

To leading order in ϵ (ϵ is of course just R_S, but this notation reminds us it is small!):

$$\left(\frac{r'^2 A(r')}{r_0^2 A(r_0)} - 1\right)^{1/2} = \frac{r'}{r_0}\left(1 - \frac{r_0^2}{r'^2} - \frac{\epsilon}{r_0} + \frac{\epsilon}{r'}\right)^{1/2} = \frac{r'}{r_0}\left(\frac{r'^2 - r_0^2}{r'^2} - \epsilon\frac{r' - r_0}{r'r_0}\right)^{1/2}$$

$$= \left(\frac{r'^2 - r_0^2}{r_0^2}\right)^{1/2}\left(1 - \frac{\epsilon r'}{r_0(r' + r_0)}\right)^{1/2}$$

and thus

$$\left(\frac{r'^2 A(r')}{r_0^2 A(r_0)} - 1\right)^{-1/2} = \frac{r_0}{(r'^2 - r_0^2)^{1/2}}\left(1 + \frac{\epsilon r'}{2r_0(r' + r_0)} + ...\right) \tag{288}$$

Finally, $\phi(r)$ becomes

$$\int_r^\infty \frac{A^{1/2} dr'}{r'\left(\dfrac{r'^2 A(r')}{r_0^2 A(r_0)} - 1\right)^{1/2}} = \int_r^\infty \frac{r_0 dr'}{r'(r'^2 - r_0^2)^{1/2}}\left(1 + \frac{\epsilon}{2r'} + \frac{\epsilon r'}{2r_0(r' + r_0)} + ...\right) \tag{289}$$

The integrals in question are

$$\int_r^\infty \frac{r_0 dr'}{r'(r'^2 - r_0^2)^{1/2}} = \sin^{-1}(r_0/r) \tag{290}$$

$$\frac{\epsilon}{2} \int_r^\infty \frac{r_0 dr'}{r'^2(r'^2 - r_0^2)^{1/2}} = \frac{\epsilon}{2r_0} \left[1 - \left(1 - \frac{r_0^2}{r^2}\right)^{1/2}\right] \tag{291}$$

$$\frac{\epsilon}{2} \int_r^\infty \frac{dr'}{(r' + r_0)(r'^2 - r_0^2)^{1/2}} = \frac{\epsilon}{2r_0} + \frac{\epsilon}{2(r^2 - r_0^2)^{1/2}} \left(1 - \frac{r}{r_0}\right) \tag{292}$$

Einstein's prediction of the deflection angle of a passing photon due to the presence of a spherical gravitating mass M is therefore:

$$\Delta\phi = 2\phi(r_0) - \pi = \frac{2\epsilon}{r_0} = \frac{4GM}{r_0 c^2} = 1.75'' \; (M/M_\odot)(R_\odot/r_0) \tag{293}$$

Happily, arcsecond deflections were just about at the limit of reliable photographic methods of measurement in 1919. Those arcsecond deflections unleashed a true revolutionary paradigm shift. For once, the words are not an exaggeration.

6.7 The advance of the perihelion of Mercury

For Einstein, the revolution had started earlier, even before he had his Field Equations. The vacuum form of the Field Equations is, as we know, sufficient to describe gravitational fields outside of the gravitating bodies themselves, and working with $R_{\mu\nu} = 0$, Einstein found, and on November 18, 1915 presented, the explanation of a 60 year old astronomical puzzle: what was the cause of Mercury's excess perihelion advance of $43''$ per century? The actual measured perihelion advance is much larger, but after the interactions from all the planets are taken into account, the $43''$ is a residual 7.5% of the total, that is not explained. According to Einstein's biographer A. Pais, the discovery that this perihelion advance emerged from general relativity was

"...by far the strongest emotional experience in Einstein's scientific life, perhaps in all his life. Nature had spoken to him. He had to be right."

6.7.1 Newtonian orbits

Interestingly, the perihelion first-order GR calculation is not much more difficult than straight Newtonian. GR introduces a $1/r^2$ term in the effective gravitational potential, but there is already a $1/r^2$ term from the centrifugal term! Other corrections do not add substantively to the difficulty. We thus begin with a detailed review of the Newtonian problem, and we will play off this solution for the perihelion advance.

Conservation of energy is

$$\frac{v_r^2}{2} + \frac{J^2}{2r^2} - \frac{GM}{r} = \mathcal{E} \tag{294}$$

where J is the (constant) specific angular momentum $r^2 d\phi/dt$ and $\mathcal{E}$ is the constant energy per unit mass. This is just the low energy limit of (282), whose exact form we may write as

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{c^2}{E}\left(\frac{J^2}{2r^2}\right) - \frac{GM}{r}\left(1 + \frac{GMJ^2}{r^2 E}\right) = \left(\frac{c^2 - E}{2E}\right)c^2. \tag{295}$$

We now identify E with c^2 to leading order, and to next order $(c^2 - E)/2$ with $\mathcal{E}$ (i.e. the mechanical energy above and beyond the rest mass energy). The Newtonian equation may
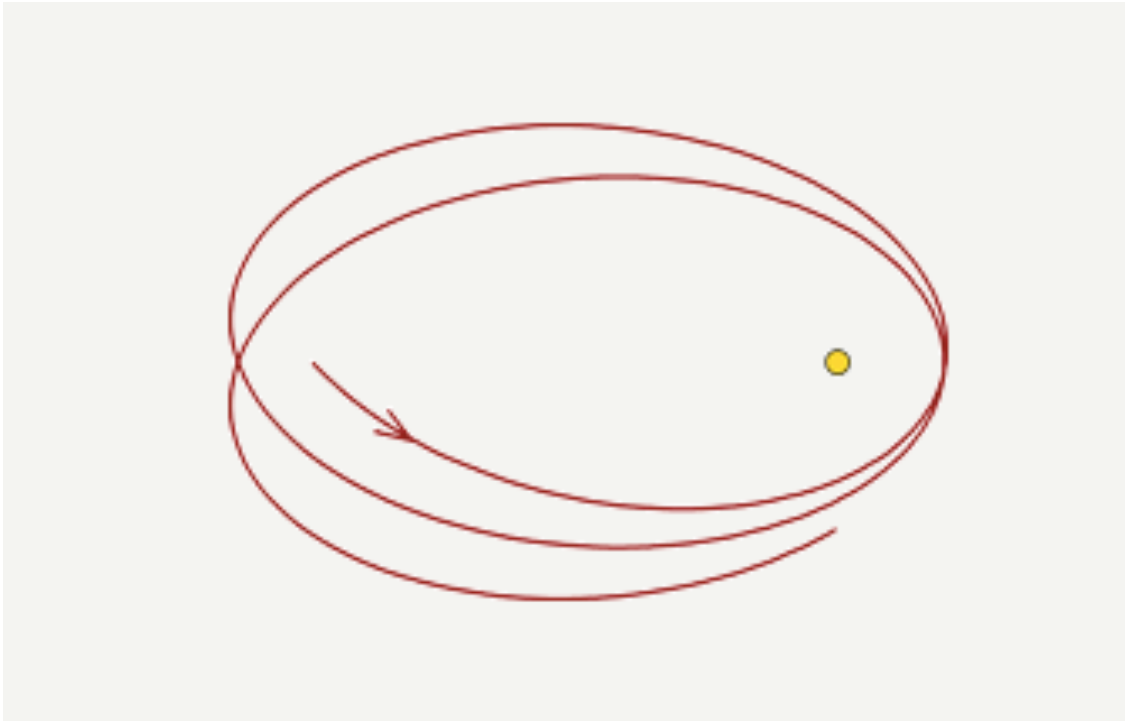
Figure 3: Departures from a $1/r$ gravitational potential cause elliptical orbits not to close. In the case of Mercury, the perihelion advances by 43 seconds of arc per century. The effect is shown here, *greatly* exaggerated.

be written

$$v_r = \frac{dr}{d\phi}\frac{d\phi}{dt} = \frac{J}{r^2}\frac{dr}{d\phi} = \pm\left(2\mathcal{E} + \frac{2GM}{r} - \frac{J^2}{r^2}\right)^{1/2} \tag{296}$$

and thence separated:

$$\int \frac{J\,dr}{r^2\left(2\mathcal{E} + \frac{2GM}{r} - \frac{J^2}{r^2}\right)^{1/2}} = \pm\phi \tag{297}$$

With $x = 1/r$,

$$\int \frac{dx}{\left(\frac{2\mathcal{E}}{J^2} + \frac{2GMx}{J^2} - x^2\right)^{1/2}} = \mp\phi \tag{298}$$

or

$$\int \frac{dx}{\left[\frac{2\mathcal{E}}{J^2} + \frac{G^2M^2}{J^4} - \left(x - \frac{GM}{J^2}\right)^2\right]^{1/2}} = \mp\phi \tag{299}$$

The integral is standard trigonometric:

$$\cos^{-1}\left[\frac{x - \frac{GM}{J^2}}{\left(\frac{2\mathcal{E}}{J^2} + \frac{G^2M^2}{J^4}\right)^{1/2}}\right] = \pm\phi \tag{300}$$

In terms of $r = 1/x$ this equation unfolds and simplifies to

$$r = \frac{J^2/GM}{1 + \epsilon\cos\phi}, \qquad \epsilon^2 \equiv 1 + \frac{2\mathcal{E}J^2}{G^2M^2} \tag{301}$$

With $\mathcal{E} < 0$ we find that $\epsilon < 1$, and that (301) is just the equation for a classical elliptical orbit of eccentricity ϵ. We identify the *semi-latus rectum,*

$$L = J^2/GM \tag{302}$$

the perihelion r_- and the aphelion r_+,

$$r_- = \frac{L}{1 + \epsilon}, \qquad r_+ = \frac{L}{1 - \epsilon}, \qquad \frac{1}{L} = \frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right) \tag{303}$$

and the semi-major axis

$$a = \frac{1}{2}(r_+ + r_-), \qquad L = a(1 - \epsilon^2) \tag{304}$$

Notice that the zeros of the denominator in the integral (299) occur at $x_- = 1/r_-$ and $x_+ = 1/r_+$, corresponding properly in our arccosine function to ϕ equals 0 and π respectively. The relativistic correction will turn out to be the following: when the arccosine advances in this same way by π, we will find that ϕ advances by a bit more!

One last technical point. A "lemma" we will shortly make use of is

$$\int_{x_+}^{x_-} dx \, \frac{x - GM/J^2}{\left[\dfrac{2\mathcal{E}}{J^2} + \dfrac{G^2 M^2}{J^4} - \left(x - \dfrac{GM}{J^2} \right)^2 \right]^{1/2}} = 0 \tag{305}$$

since the elementary indefinite integral is equal to (-1) times the denominator, and when evaluated as a definite integral this denominator vanishes at each end point.

Exercise.) **The Shows must go on.** Show that the semi-minor axis of an ellipse is $b = a\sqrt{1 - \epsilon^2}$. Show that the area of an ellipse is πab. Show that the total energy of a two-body bound system is $-Gm_1 m_2/2a$, independent of ϵ. Show that the period of a two-body bound system is $2\pi\sqrt{a^3/GM}$, independent of ϵ. (There is a very simple way to do the latter!)

6.7.2 Schwarzschild orbits

We begin with equation (284) for Schwarzschild orbits:

$$\left(\frac{dr}{d\phi} \right)^2 + \frac{r^2}{A} \left(1 + \frac{Er^2}{J^2} \right) = \frac{c^2 r^4}{J^2} \tag{306}$$

As with our Newtonian calculation, this separates nicely, leading to

$$\int \frac{dr \, A^{1/2}}{r^2 \left(\dfrac{c^2 A}{J^2} - \dfrac{1}{r^2} - \dfrac{E}{J^2} \right)^{1/2}} = \phi \tag{307}$$

(With no loss of generality, we regard ϕ as increasing over the course of its orbit.) In the denominator, we next expand $A(r)$:

$$A \simeq 1 + \frac{2GM}{rc^2} + \frac{4G^2 M^2}{r^2 c^4} \tag{308}$$

Notice the expansion through second order here: first order only brings us to Newtonian gravity! The terms in the integral's square root denominator are then

$$\frac{c^2 - E}{J^2} + \frac{2GM}{rJ^2} - \frac{1}{r^2} \left(1 - \frac{4G^2 M^2}{c^2 J^2} \right) \equiv \frac{2\mathcal{E}}{J^2} + \frac{2GM}{rJ^2} - \frac{\alpha}{r^2} \tag{309}$$

where we have identified the Newtonian-like energy constant $2\mathcal{E}$ with $c^2 - E$, and α is quantity nearly equal to unity. This gives the entire integral a Newtonian appearance that will facilitate interpretation. Our integration task is

$$\alpha^{-1/2} \int \frac{dr \, A^{1/2}}{r^2 \left(\dfrac{2\mathcal{E}}{J'^2} + \dfrac{2GM}{rJ'^2} - \dfrac{1}{r^2} \right)^{1/2}} = \phi \tag{310}$$

where J' is a redefined angular momentum variable,

$$J'^2 = \alpha J^2 \tag{311}$$

and

$$\alpha^{-1/2} \simeq 1 + \frac{2G^2 M^2}{c^2 J^2} \tag{312}$$

(Note that it does not matter whether we use J' or J in the small $1/c^2$ term since the difference is yet higher order.) We thus have

$$\left(1 + \frac{2G^2 M^2}{c^2 J^2}\right) \int \frac{dr\,(1 + GM/c^2 r)}{r^2 \left(\frac{2\mathcal{E}}{J'^2} + \frac{2GM}{r J'^2} - \frac{1}{r^2}\right)^{1/2}} = \phi \tag{313}$$

Once again we change variables to $r = 1/x$,

$$\left(1 + \frac{2G^2 M^2}{c^2 J^2}\right) \int \frac{dx\,(1 + GMx/c^2)}{\left(\frac{2\mathcal{E}}{J'^2} + \frac{2GMx}{J'^2} - x^2\right)^{1/2}} = \phi \tag{314}$$

where the overall sign can be accounted for by integrating from smaller to larger x. The first integral in (314)

$$\left(1 + \frac{2G^2 M^2}{c^2 J^2}\right) \int \frac{dx}{\left(\frac{2\mathcal{E}}{J'^2} + \frac{2GMx}{J'^2} - x^2\right)^{1/2}} \tag{315}$$

is identical in form to (298), so we may immediately write down the result

$$\left(1 + \frac{2G^2 M^2}{c^2 J^2}\right) \cos^{-1}\left[\frac{x - \frac{GM}{J'^2}}{\left(\frac{2\mathcal{E}}{J'^2} + \frac{G^2 M^2}{J'^4}\right)^{1/2}}\right] \tag{316}$$

The second integral in (314) may be written

$$(1 + ...) \int \frac{dx\,(x - GM/J'^2)(GM/c^2) + (GM/cJ')^2}{\left(\frac{2\mathcal{E}}{J'^2} + \frac{2GMx}{J'^2} - x^2\right)^{1/2}} \tag{317}$$

The ..., indicates a neglected term of higher order. We will be integrating between x_+ and x_- and so by our lemma (305) the term in $(x - GM/J'^2)$ vanishes upon integration. The final remaining integral in $(GM/cJ')^2$ is identical to the one we have just done. Putting the two results together,

$$\left(1 + \frac{3G^2 M^2}{c^2 J^2}\right) \cos^{-1}\left[\frac{x - \frac{GM}{J'^2}}{\left(\frac{2\mathcal{E}}{J'^2} + \frac{G^2 M^2}{J'^4}\right)^{1/2}}\right] = \phi \tag{318}$$

(Again: we need not make the distinction bewteen J and J' in small correction terms!) In advancing along the orbit from aphelion x_+ to perihelion x_-, the arccosine advances dutifully

by π. And, in the days of Isaac Newton, so would ϕ! But now Herr Einstein has given us a bit more of an advance in ϕ: π plus $3\pi G^2 M^2 / c^2 J^2$, or from one perihelion to the next, the advance per orbit is

$$\Delta\phi = \frac{6\pi G^2 M^2}{c^2 J^2} = \frac{6\pi GM}{c^2 L}, \tag{319}$$

using our expression for the semilatus rectum L. With $L = 5.546 \times 10^{10}$ m for Mercury and an orbital period of 7.6×10^6 s, this value of $\Delta\phi$ works out to be precisely $43''$ per century, which is precisely the anomalous astronomical measurement. Until the yet more stunning measurements of orbital changes from gravitational radiation energy losses from the binary pulsar 1913+16 announced in 1982, the perihelion advance of Mercury was relativity's theory greatest observational success.

6.7.3 Perihelion advance: another route

The perihelion advance of Mercury is important enough that another derivation is worthwhile and enlightening. Equation (284) may be written in terms of $u = 1/r$ as

$$\left(\frac{du}{d\phi}\right)^2 + \left(1 - \frac{2GMu}{c^2}\right)\left(u^2 + \frac{E}{J^2}\right) = \frac{c^2}{J^2}. \tag{320}$$

Now differentiate with respect to ϕ and simplify. The resulting equation is remarkably elegant:

$$u'' + u = \frac{GME}{c^2 J^2} + \frac{3GMu^2}{c^2} \simeq \frac{GM}{J^2} + \frac{3GMu^2}{c^2} \tag{321}$$

since E is very close to c^2 for Mercury and the difference here is immaterial. The Newtonian limit corresponds to dropping the final term on the right side of the equation; the resulting solution is

$$u = \frac{GM}{J^2}(1 + \epsilon\cos\phi) \;\; \text{from} \;\; r = \frac{J^2/GM}{1 + \epsilon\cos\phi} \tag{322}$$

where ϵ is an arbitrary constant. This is just the classic equation for a conic section, with hyperbolic ($\epsilon > 1$), parabolic ($\epsilon = 1$) and ellipsoidal ($\epsilon < 1$) solutions. For ellipses, ϵ is the eccentricity.

The general relativistic term $3GMu^2/c^2$ is of course very small, so we are entirely justified in using the Newtonian solution for u^2 in this higher order term. Writing $u = u_N + \delta u$ with u_N given by (322), the differential equation becomes

$$\frac{d^2\delta u}{d\phi^2} + \delta u = \frac{3(GM)^3}{c^2 J^4}(1 + 2\epsilon\cos\phi + \epsilon^2\cos^2\phi). \tag{323}$$

In Problem Set 2, you will be asked to solve this equation. The resulting solution for $u = u_N + \delta u$ may be written

$$u \simeq \frac{GM}{J^2}\left(1 + \epsilon\cos[\phi(1 - \alpha)]\right) \tag{324}$$

where $\alpha = 3(GM/Jc)^2$. Thus, the perihelion occurs not with a ϕ-period of 2π, but with a period of

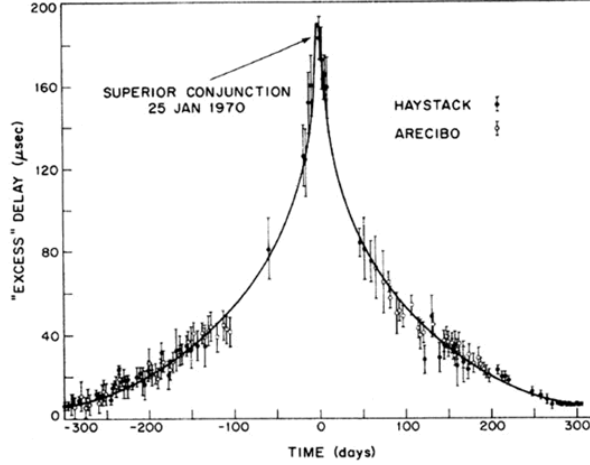$$\frac{2\pi}{1 - \alpha} \simeq 2\pi + 2\pi\alpha, \tag{325}$$

Figure 4: Radar echo delay from Venus as a function of time, fit with general relativistic prediction.

i.e. an advance of

$$\Delta\phi = 2\pi\alpha = 6\pi \left(\frac{GM}{Jc} \right)^2 \tag{326}$$

in precise agreement with (319).

6.8   Shapiro delay: the fourth protocol

For many years, the experimental foundation of general relativity consisted of the three tests we have described that were first proposed by Einstein: the gravitational red shift, the bending of light by gravitational fields, and the advance of Mercury's perihelion. In 1964, nearly a decade after Einstein's passing, a fourth test was proposed: the time delay by radio signals when crossing near the sun in the inner solar system. The idea, proposed and carried out by Irwin Shapiro, is that a radio signal is sent from earth, bounces off Mercury, and returns. One does the experiment when Mercury is at its closest point to the earth, then repeats the experiment when the planet is on the far side of orbit. There should be an additional delay of the pulses when Mercury is on the far side of the sun because of the traversal of the radio waves across the sun's Schwarzschild geometry. It is this delay that is measured.

Recall equation (283), using the "ordinary" time parameter t for an observer at infinity, with $E = 0$ for radio waves:

$$\left(\frac{dr}{dt} \right)^2 + \frac{B^2}{A} \frac{J^2}{r^2} = \frac{Bc^2}{A} \tag{327}$$

It is convenient to evaluate the constant J in terms of r_0, the point of closest approach to the sun. With $dr/dt = 0$, we easily find

$$J^2 = \frac{r_0^2 c^2}{B_0} \tag{328}$$

where $B_0 \equiv B(r_0)$. The differential equation then separates and we find that the time $t(r, r_0)$

69

to traverse from r_0 to r (or vice-versa) is

$$t(r, r_0) = \frac{1}{c} \int_{r_0}^{r} \frac{A\, dr}{\left(1 - \frac{B}{B_0} \frac{r_0^2}{r^2}\right)^{1/2}}, \tag{329}$$

where we have made use of $AB = 1$. We do manipulations in the denominator similar to those done in section 6.6. Expanding to first order in $GM/c^2 r$ with $B = 1 - 2GM/c^2 r$:

$$1 - \frac{B}{B_0} \frac{r_0^2}{r^2} \simeq 1 - \left[1 + \frac{2GM}{c^2}\left(\frac{1}{r_0} - \frac{1}{r}\right)\right] \frac{r_0^2}{r^2}. \tag{330}$$

This may now be rewritten as:

$$1 - \frac{B}{B_0} \frac{r_0^2}{r^2} \simeq \left(1 - \frac{r_0^2}{r^2}\right)\left(1 - \frac{2GM r_0}{c^2 r(r + r_0)}\right) \tag{331}$$

Using this in our time integral for $t(r_0, r)$ and expanding,

$$t(r_0, r) = \frac{1}{c} \int_{r_0}^{r} dr \left(1 - \frac{r_0^2}{r^2}\right)^{-1/2}\left(1 + \frac{2GM}{rc^2} + \frac{GM r_0}{c^2 r(r + r_0)}\right) \tag{332}$$

The required integrals are

$$\frac{1}{c} \int_{r_0}^{r} \frac{r\, dr}{(r^2 - r_0^2)^{1/2}} = \frac{1}{c}(r^2 - r_0^2)^{1/2} \tag{333}$$

$$\frac{2GM}{c^3} \int_{r_0}^{r} \frac{dr}{(r^2 - r_0^2)^{1/2}} = \frac{2GM}{c^3} \cosh^{-1}\left(\frac{r}{r_0}\right) = \frac{2GM}{c^3} \ln\left(\frac{r}{r_0} + \sqrt{\frac{r^2}{r_0^2} - 1}\right) \tag{334}$$

$$\frac{GM r_0}{c^3} \int_{r_0}^{r} \frac{dr}{(r + r_0)(r^2 - r_0^2)^{1/2}} = \frac{GM}{c^3} \sqrt{\frac{r - r_0}{r + r_0}} \tag{335}$$

Thus,

$$t(r, r_0) = \frac{1}{c}(r^2 - r_0^2)^{1/2} + \frac{2GM}{c^3} \ln\left(\frac{r}{r_0} + \sqrt{\frac{r^2}{r_0^2} - 1}\right) + \frac{GM}{c^3} \sqrt{\frac{r - r_0}{r + r_0}} \tag{336}$$

We are interested in $2t(r_1, r_0) + 2t(r_2, r_0)$ for the path from the earth, by the sun, reflected from the planet, and back.

It may seem straightforward to plug in values appropriate to the earth's radial location and the planet's (either Mercury or Venus, in fact), compute the "expected" Newtonian time for transit (a sum of the first terms) and then measure the actual time for comparison with our formula. In practice to know what the delay is, we have to know what the Newtonian transit time is to fantastic accuracy! In fact, the way this is done is to treat the problem not as a measurement of a single delay time, but as an entire function of time given by our solution (336) with $r = r(t)$. Figure (3) shows such a fit near the passage of superior conjunction (i.e. the far side orbital near the sun in sky projection), in excellent agreement with theory. Exactly how the parameterisation is carried out would take us too far afield; there is some discussion in Weinberg's book, pp. 202–207, and an abundance of topical information on the internet under "Shapiro delay."

Modern applications of the Shaprio delay use pulsars as signal probes, whose time passage properties are altered by the presence of gravitational waves, a topic for the next chapter.

7   Gravitational Radiation

They are not objective, and (like absolute velocity) are not detectable by any conceivable experiment. They are merely sinuosities in the co-ordinate system, and the only speed of propagation relevant to them is "the speed of thought."

— A. S. Eddington writing in 1922 of Einstein's suspicions.

On September 14, 2015, at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational Wave Observatory simultaneously observed a transient gravitational wave signal. The signal sweeps upwards from 35 to 250 Hz with a peak gravitational wave strain of 1×10^{-21}. It matches the waveform predicted by general relativity for the inspiral and merger of a pair of black holes and the ringdown of the resulting single black hole.

— B. P. Abbott et al., 2016, Physical Review Letters, 116, 061102

Gravity is spoken in the three languages. First, there is traditional Newtonian potential theory, the language used by most practicing astrophysicists. Then, there is the language of Einstein's General Relativity Theory, the language of Riemannian geometry that we have been studying. Finally, there is the language of quantum field theory: gravity is a theory of the exchange of spin 2 particles, gravitons, much as electromagnetism is a theory arising from the exchange of spin 1 photons. Just as the starting point of quantum electrodynamics is the radiation theory of Maxwell, the starting point of quantum gravity must be a classical radiation theory of gravity. Unlike quantum electrodynamics, the most accurate physical theory ever created, there is no quantum theory of gravity at present, and there is not even a consensus approach. Quantum gravity is therefore very much an active area of ongoing research. For the theorist, this is reason enough to study the theory of gravitational radiation in general relativity. But there are good reasons for the practical astrophysicist to get involved. In Februrary 2016, the first detection of gravitational waves was announced. The event singal had been received and recorded on September 14, 2015, and is denoted G[ravitational]W[ave]150914. The detection was so clean, and matched the wave form predictions of general relativity in such detail, there can be no doubt that the detection was genuine. A new way to probe the most impenetrable parts of the universe is at hand.

The theory of General Relativity in the limit when $g_{\mu\nu}$ is very close to $\eta_{\mu\nu}$ *is* a classical theory of gravitational radiation, and not just Newtonian theory, in the same way that Maxwellian Electrodynamics is a classical radiation theory. The field equations for $g_{\mu\nu} - \eta_{\mu\nu}$

71

become in the weak field limit a set of rather ordinary looking wave equations with source terms, like Maxwell's Equations. The principal difference is that electrodynamics is sourced by a vector quantity (the vector potential $\boldsymbol{A}$ with the potential Φ combine to form a 4-vector), whereas gravitational fields in general relativity are sourced by a tensor quantity (the stress tensor $T_{\mu\nu}$). This becomes a major difference when we relax the condition that the gravity field be weak: the gravitational radiation itself makes a contribution to its own source, something electromagnetic radiation cannot do. But this is not completely unprecedented. We have seen this sort of thing before, in a classical context: sound waves can themselves generate acoustical disturbances, and one of the things that happens is a shock wave, or sonic boom. While a few somewhat pathological mathematical solutions for exact gravitational radiation waves are known, in general people either work in the weak field limit or resort to numerical solutions of the field equations. Even with powerful computers, however, precise numerical solutions of the field equations for astrophysically interesting problems—like merging black holes—have long been a major technical problem. In the last decade, a breakthrough has occurred, and it is now possible to compute highly accurate wave forms for these kinds of problems, with critically important predictions for the new generation of gravitational wave detectors.

As we have noted, astrophysicists have perhaps the most important reason of all to understand gravitational radiation: we are on the verge of what will surely be a golden age of gravitational wave astronomy. That gravitational radiation truly exists was established in 1974, when a close binary (7.75 hour period) system with a neutron star and a pulsar (PSR 1913+16) was discovered by Hulse and Taylor. So much orbital information could be extracted from this remarkable system that it was possible to predict, then measure, the rate of orbital decay (more precisely, the speed-up of the period of the decaying orbit) caused by the energy carried off by gravitational radiation. The resulting inspiral, though tiny in practical terms, was large enough to be cleanly and clearly measurable. General relativity turned out to be exactly correct (Taylor & Weisberg, ApJ, 1982, 253, 908), and the 1993 Nobel Prize in Physics was awarded to Hulse and Taylor for this historical achievement.

The September 2015 gravitational wave detection established that i) the reception and analysis of gravitational waves is feasible and will soon become a widely-used probe of the universe; ii) black holes exist beyond any doubt whatsoever, this is the proverbial "smoking-gun"; iii) the full dynamical content of strong field general relativity on the scales of stellar systems is completely correct. This achievement is a true historical milestone in physics. Some have speculated that its impact on astronomy will rival Galileo's introduction of the telescope. Perhaps Hertz's 1887 detection of electromagnetic radiation in the lab is another apt comparison. (Commerical exploitation of gravity waves is probably some ways off! Maybe it can be taxed.)

There is more to come. In the near future, extremely delicate pulsar timing experiments, in which arrival times of pulses are measured to fantastic precision, will come on line. In essence, this is a measure of the Shapiro delay, not caused by the Sun or a star, but by the passage of a gravitational wave.

The subject of gravitational radiation is complicated and computationally intensive. Even the basic basics involve a real effort. Although the topic lies outside the syllabus, I would like to present a discussion for the strongly motivated student. For the astrophysical applications relevant to this course, skip to page (83) and simply take the gravitational wave luminosity formula as given.

7.1 The linearised gravitational wave equation

We will always assume that the metric is close to Minkowski space,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{337}$$

To leading order, when we raise and lower indices we may do so with $\eta_{\mu\nu}$. Be careful with $g^{\mu\nu}$ itself:

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{338}$$

to ensure $g_{\mu\nu}g^{\nu\kappa} = \delta_\mu^\kappa$. (You can raise the index of g with η only when approximating $g^{\mu\nu}$ as its leading order value, which is $\eta^{\mu\nu}$.) Note that

$$\eta^{\mu\nu}h_{\nu\kappa} = h_\kappa^\mu, \quad \eta^{\mu\nu}\frac{\partial}{\partial x^\nu} = \frac{\partial}{\partial x_\mu} \tag{339}$$

and that we can slide dummy indices "up-down" sometimes:

$$\frac{\partial h_{\mu\nu}}{\partial x_\mu} = \eta_{\mu\rho}\frac{\partial h_\nu^\rho}{\partial x_\mu} = \frac{\partial h_\nu^\rho}{\partial x^\rho} \equiv \frac{\partial h_\nu^\mu}{\partial x^\mu} \tag{340}$$

The story begins with the Einstein Field Equations cast in a form in which the "linearised Ricci tensor" is isolated. Specifically, we write

$$R_{\mu\nu} = R_{\mu\nu}^{(1)} + R_{\mu\nu}^{(2)} + ...\text{etc.} \tag{341}$$

and

$$G_{\mu\nu}^{(1)} = R_{\mu\nu}^{(1)} - \eta_{\mu\nu}\frac{R^{(1)}}{2} \tag{342}$$

where $R_{\mu\nu}^{(1)}$ is all the Ricci tensor terms linear in $h_{\mu\nu}$, $R_{\mu\nu}^{(2)}$ all terms quadratic in $h_{\mu\nu}$, and so forth. The linearised affine connection is

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2}\eta^{\lambda\rho}\left(\frac{\partial h_{\rho\nu}}{\partial x^\mu} + \frac{\partial h_{\rho\mu}}{\partial x^\nu} - \frac{\partial h_{\mu\nu}}{\partial x^\rho}\right) = \frac{1}{2}\left(\frac{\partial h_\nu^\lambda}{\partial x^\mu} + \frac{\partial h_\mu^\lambda}{\partial x^\nu} - \frac{\partial h_{\mu\nu}}{\partial x_\lambda}\right). \tag{343}$$

In terms of $h_{\mu\nu}$ and $h = h_\mu^\mu$, from equation (208) on page 44, we explicitly find

$$R_{\mu\nu}^{(1)} = \frac{1}{2}\left(\frac{\partial^2 h}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 h_\mu^\lambda}{\partial x^\nu \partial x^\lambda} - \frac{\partial^2 h_\nu^\lambda}{\partial x^\mu \partial x^\lambda} + \Box h_{\mu\nu}\right) \tag{344}$$

where

$$\Box \equiv \frac{\partial^2}{\partial x^\lambda \partial x_\lambda} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{345}$$

is the d'Alembertian (clearly a Lorentz invariant), making a most welcome appearance into the proceedings. Contracting μ with ν, we find that

$$R^{(1)} = \Box h - \frac{\partial^2 h^{\mu\nu}}{\partial x^\mu \partial x^\nu} \tag{346}$$

where we have made use of

$$\frac{\partial h_\mu^\lambda}{\partial x_\mu} = \frac{\partial h^{\lambda\mu}}{\partial x^\mu}.$$

Assembling $G_{\mu\nu}^{(1)}$, we find

$$2G_{\mu\nu}^{(1)} = \frac{\partial^2 h}{\partial x^\mu \partial x^\nu} - \frac{\partial^2 h_\mu^\lambda}{\partial x^\lambda \partial x^\nu} - \frac{\partial^2 h_\nu^\lambda}{\partial x^\lambda \partial x^\mu} + \Box h_{\mu\nu} - \eta_{\mu\nu}\left(\Box h - \frac{\partial^2 h^{\lambda\rho}}{\partial x^\lambda \partial x^\rho}\right). \tag{347}$$

The full, nonlinear Field Equations may then formally be written

$$G_{\mu\nu}^{(1)} = -\left(\frac{8\pi G T_{\mu\nu}}{c^4} + G_{\mu\nu} - G_{\mu\nu}^{(1)}\right) \equiv -\frac{8\pi G(T_{\mu\nu} + \tau_{\mu\nu})}{c^4}, \tag{348}$$

where

$$\tau_{\mu\nu} = \frac{c^4}{8\pi G}(G_{\mu\nu} - G_{\mu\nu}^{(1)}) \simeq \frac{c^4}{8\pi G}\left(R_{\mu\nu}^{(2)} - \eta_{\mu\nu}\frac{R^{(2)}}{2}\right) \tag{349}$$

Though composed of geometrical terms, the quantity $\tau_{\mu\nu}$ is written on the right side of the equation with the stress energy tensor $T_{\mu\nu}$, and is interpreted as the *stress energy contribution of the gravitational radiation itself.* We shall have more to say on this in section 7.4. In linear theory, $\tau_{\mu\nu}$ is neglected in comparison with the ordinary matter $T_{\mu\nu}$.

This is a bit disappointing to behold. Even the linearised Field Equations look to be a mess! But then you may have forgotten that Maxwell's wave equations for the potentials are not, at first, very pretty. Let me remind you. Here are the equations for the scalar potential Φ and vector potential $\boldsymbol{A}$:

$$\nabla^2\Phi + \frac{1}{c^2}\frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\boldsymbol{A}) = -\frac{\rho}{\epsilon_0} \tag{350}$$

$$\nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t}\right) = -\mu_0\boldsymbol{J} \tag{351}$$

But we know the story here well. Work in the Lorenz gauge, which we are always free to do:

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0 \tag{352}$$

In invariant 4-vector language, this is just $\partial_\alpha A^\alpha = 0$. Then, the dynamical equations simplify:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial\Phi}{\partial t^2} = \Box\Phi = -\frac{\rho}{\epsilon_0} \tag{353}$$

$$\nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = \Box\boldsymbol{A} = -\mu_0\boldsymbol{J} \tag{354}$$

and physically transparent Lorentz-invariant wave equations emerge. Might something similar happen for the Einstein Field Equations?

That the answer might be YES is supported by noticing that $G_{\mu\nu}^{(1)}$ can be written entirely in terms of the "Bianchi-like" quantity

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{\eta_{\mu\nu}h}{2}, \quad \text{or} \quad \bar{h}_\nu^\mu = h_\nu^\mu - \frac{\delta_\nu^\mu h}{2}. \tag{355}$$

74

Using this into (347), the Field Equation becomes

$$2G^{(1)}_{\mu\nu} = \Box \bar{h}_{\mu\nu} - \frac{\partial^2 \bar{h}^\lambda_\mu}{\partial x^\nu \partial x^\lambda} - \frac{\partial^2 \bar{h}^\lambda_\nu}{\partial x^\mu \partial x^\lambda} + \eta_{\mu\nu} \frac{\partial^2 \bar{h}^{\lambda\rho}}{\partial x^\lambda \partial x^\rho} = -\frac{16\pi G T_{\mu\nu}}{c^4}. \tag{356}$$

(It is easiest to verify this by starting with (356), substituting with (355), and showing that this leads to (347).)

Interesting. Except for $\Box \bar{h}_{\mu\nu}$, every term in this equation involves the divergence of $\bar{h}^\mu_\nu$ or $\bar{h}^{\mu\nu}$. Hmmm. Shades of Maxwell's $\partial A^\alpha / \partial x^\alpha$. In the Maxwell case, the freedom of gauge invariance allowed us to pick the gauge in which $\partial A^\alpha / \partial x^\alpha = 0$. Does our equation have a gauge invariance that will allow us to do the same for gravitational radiation so that we can set these $\bar{h}$-divergence derivatives to zero?

It does. Go back to equation (347) and on the right side, change $h_{\mu\nu}$ to $h'_{\mu\nu}$, where

$$h'_{\mu\nu} = h_{\mu\nu} - \frac{\partial \xi_\nu}{\partial x^\mu} - \frac{\partial \xi_\mu}{\partial x^\nu}, \tag{357}$$

and the ξ_μ represent any vector function. You will find that the form of the equation is completely unchanged, i.e. the ξ_μ terms cancel out identically! This is a true gauge invariance.

In this case, what is happening is that an infinitesimal coordinate transformation itself is *acting* as a gauge transformation. If

$$x'^\mu = x^\mu + \xi^\mu(x), \quad \text{or} \quad x^\mu = x'^\mu - \xi^\mu(x') \qquad \text{to lead order.} \tag{358}$$

then

$$g'_{\mu\nu} = \eta'_{\mu\nu} + h'_{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma} = \left(\delta^\rho_\mu - \frac{\partial \xi^\rho}{\partial x^\mu} \right) \left(\delta^\sigma_\nu - \frac{\partial x^\sigma}{\partial x^\nu} \right) (\eta_{\rho\sigma} + h_{\rho\sigma}) \tag{359}$$

With η' identical to η, we must have

$$h'_{\mu\nu} = h_{\mu\nu} - \frac{\partial \xi_\nu}{\partial x^\mu} - \frac{\partial \xi_\mu}{\partial x^\nu} \tag{360}$$

as before. But don't confuse *general covariance* under coordinate transformations with this gauge transformation. Unlike general covariance, the gauge transformation doesn't actually change the coordinates. We keep the same x's and add a group of certain functional derivatives to the $h_{\mu\nu}$ (analogous to adding a gradient $\boldsymbol{\nabla}\Phi$ to $\boldsymbol{A}$ in Maxwell's theory) and we find that the equations remain identical (just as we would find if we took $\boldsymbol{\nabla}\times[\boldsymbol{A} + \boldsymbol{\nabla}\Phi]$ in the Maxwell case).

Just as the Lorenz gauge $\partial_\alpha A^\alpha = 0$ was useful in the case of Maxwell's equations, so now is the so-called *harmonic gauge*:

$$\frac{\partial \bar{h}^\mu_\nu}{\partial x^\mu} = \frac{\partial h^\mu_\nu}{\partial x^\mu} - \frac{1}{2} \frac{\partial h}{\partial x^\nu} = 0 \tag{361}$$

For then, the Field Equations (356) take the "wave-equation" form

$$\boxed{\Box \bar{h}_{\mu\nu} = -\frac{16\pi G T_{\mu\nu}}{c^4}} \tag{362}$$

75

How we can be sure that even with our gauge freedom we can be certain of finding the right ξ^μ to ensure the emergence of (362). If we have been unfortunate enough to be working in a gauge in which equation (361) is not satisfied, then form $h'_{\mu\nu}$ à la equation (360) and demand that $\partial h'^\mu_\nu/\partial x^\mu = (1/2)\partial h'/\partial x^\nu$. We find that this implies

$$\Box \xi_\nu = \frac{\partial \bar{h}^\mu_\nu}{\partial x^\mu}, \tag{363}$$

a wave equation for ξ_ν identical in form to (362). For this, a solution certainly exists. Indeed, our experience with electrodynamics has taught us that the solution to the fundamental radiation equation (362) takes the form

$$\bar{h}_{\mu\nu}(\boldsymbol{r},t) = \frac{4G}{c^4} \int \frac{T_{\mu\nu}(\boldsymbol{r}',t-R/c)}{R} \, d^3r', \qquad R \equiv |\boldsymbol{r}-\boldsymbol{r}'| \tag{364}$$

and hence a similar solution exisits for (363). The $\bar{h}_{\mu\nu}$, like their electrodynamic counterparts, are determined at time t and location $\boldsymbol{r}$ by a source intergration over $\boldsymbol{r}'$ taken at the retarded times $t' \equiv t - R/c$. In other words, disturbances in the gravitional field travel at a finite speed, the speed of light c.

7.1.1 Come to think of it...

You may not have actually seen the solution (364) before, and it is important. Let's derive it.

Consider the equation

$$-\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} + \nabla^2\Psi = -4\pi f(\boldsymbol{r},t) \tag{365}$$

We specialise to the Green's function solution

$$-\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} + \nabla^2 G = -4\pi\delta(\boldsymbol{r})\delta(t) \tag{366}$$

Of course, our particular choice of origin is immaterial, as is our zero of time, so that we shall replace $\boldsymbol{r}$ by $\boldsymbol{R} \equiv \boldsymbol{r} - \boldsymbol{r}'$ and t by $\tau \equiv t - t'$ at the end of the calculation, with the primed values being fiducial reference points. The solution will still be valid with these shifts of space and time origins.

Fourier transform (366) by integrating over $\int e^{i\omega t}dt$ and denote the fourier transform of G by $\tilde{G}$:

$$k^2\tilde{G} + \nabla^2\tilde{G} = -4\pi\delta(\boldsymbol{r}) \tag{367}$$

where $k^2 = \omega^2/c^2$. Clearly $\tilde{G}$ is a function only of r, hence the solution to the homogenous equation away from the origin,

$$\frac{d^2(r\tilde{G})}{dr^2} + k^2(r\tilde{G}) = 0,$$

is easily found to be $\tilde{G} = e^{\pm ikr}/r$. The delta function behaviour is actually *already* included here, as can be seen by taking the limit $k \to 0$, in which we recover the correct potential of a point charge, with the proper normalisation already in place. The back transform gives

$$G = \frac{1}{2\pi r}\int_{-\infty}^{\infty} e^{\pm ikr - i\omega t} \, d\omega \tag{368}$$

76

which we recognise as a Dirac delta function (remember $\omega/k = c$):

$$G = \frac{\delta(t \pm r/c)}{r} \to \frac{\delta(t - r/c)}{r} \to \frac{\delta(\tau - R/c)}{R} \tag{369}$$

where we have selected the *retarded* time solution $t - r/c$ as a bow to causality, and moved thence to τ, R variables for arbitary locations and times. We thus see that a flash at $t = t'$ located at $\boldsymbol{r} = \boldsymbol{r'}$ produces an effect at a time R/c later, at a distance R from the flash. The general solution constructed from our Green's function is

$$\Psi = \int \frac{f(\boldsymbol{r'}, t')}{R} \delta(t - R/c - t') dt' \boldsymbol{dr'} = \int \frac{f(\boldsymbol{r'}, t')}{R} \boldsymbol{dr'} \tag{370}$$

where $t' = t - R/c$, the retarded time.

7.2 Plane waves

To understand more fully the solution (364), consider the problem in which $T_{\mu\nu}$ has an oscillatory time dependence, $e^{-i\omega t'}$. The source, say a binary star system, occupies a finite volume; we seek the solution for $\bar{h}_{\mu\nu}$ at distances huge compared with the scale of the source itself, i.e. $r \gg r'$. Then,

$$R \simeq r - \boldsymbol{e_r} \cdot \boldsymbol{r'} \tag{371}$$

where $\boldsymbol{e_r}$ is a unit vector in the $\boldsymbol{r}$ direction, and

$$\bar{h}_{\mu\nu}(\boldsymbol{r}, t) \simeq \exp[i(kr - \omega t)] \frac{4G}{rc^4} \int T_{\mu\nu}(\boldsymbol{r'}) \exp(-i\boldsymbol{k} \cdot \boldsymbol{r'}) \, d^3r \tag{372}$$

with $\boldsymbol{k} = (\omega/c)\boldsymbol{e_r}$ the wavenumber in the radial direction. Since r is huge, this has the asymptotic form of a plane wave. Hence, $\bar{h}_{\mu\nu}$ and thus $h_{\mu\nu}$ itself have the form of simple plane waves, travelling in the radial direction, at large distances from the source generating them. These waves turn out to have some remarkable polarisation properties, which we now discuss.

7.2.1 The transverse-traceless (TT) gauge

Consider a traveling plane wave for $h_{\mu\nu}$, orienting our z axis along $\boldsymbol{k}$, so that

$$k^0 = \omega/c, \; k^1 = 0, \; k^2 = 0, \; k^3 = \omega/c \qquad \text{and} \qquad k_0 = -\omega/c, \; k_i = k^i \tag{373}$$

where as usual we raise and lower indices with $\eta_{\mu\nu}$ or its numerical identical dual $\eta^{\mu\nu}$.

Then $h_{\mu\nu}$ takes the form

$$h_{\mu\nu} = e_{\mu\nu} a \, \exp(ik_\rho x^\rho) \tag{374}$$

where a is an amplitude and $e_{\mu\nu} = e_{\nu\mu}$ a polarisation tensor, again with the η's raising and lowering subscripts. Thus

$$e_{ij} = e^i_j = e^{ij} \tag{375}$$

$$e^{0i} = -e^i_0 = e^0_i = -e_{0i} \tag{376}$$

$$e^{00} = e_{00} = -e^0_0 \tag{377}$$

The harmonic constraint

$$\frac{\partial h^\mu_\nu}{\partial x^\mu} = \frac{1}{2}\frac{\partial h^\mu_\mu}{\partial x^\nu} \tag{378}$$

implies

$$k_\mu e^\mu_\nu = k_\nu e^\mu_\mu/2 \tag{379}$$

For $\nu = 0$ this means

$$k_0 e^0_0 + k_3 e^3_0 = k_0(e^i_i + e^0_0)/2, \tag{380}$$

or

$$-(e_{00} + e_{30}) = (e_{ii} - e_{00})/2. \tag{381}$$

When $\nu = j$ (a spatial index),

$$k_0 e^0_j + k_3 e^3_j = k_j(e_{ii} - e_{00})/2 \tag{382}$$

The $j = 1$ and $j = 2$ cases reduce to

$$e_{01} + e_{31} = e_{02} + e_{32} = 0, \tag{383}$$

while $j = 3$ yields

$$e_{03} + e_{33} = (e_{ii} - e_{00})/2 = -(e_{00} + e_{03}) \tag{384}$$

Equations (383) and the first=last equality of (384) yield

$$e_{01} = -e_{31}, \quad e_{02} = -e_{32}, \quad e_{03} = -(e_{00} + e_{33})/2 \tag{385}$$

Using the above expression for e_{03} in the first=second equality of (384) then gives

$$e_{22} = -e_{11} \tag{386}$$

Of the 10 independent components of the symmetric $e_{\mu\nu}$ the harmonic condition (378) thus enables us to express e_{0i} and e_{22} in terms of e_{3i}, e_{00}, and e_{11}. These latter 5 components plus a sixth, e_{12}, remain unconstrained for the moment.

But wait! We have not yet used the gauge freedom of equation (360) *within* the harmonic constraint. We can still continue to eliminate components of $e_{\mu\nu}$. In particular, let us choose

$$\xi_\mu(x) = i\epsilon_\mu \exp(ik_\rho x^\rho) \tag{387}$$

where the ϵ_μ are four constants to be chosen. This satisfies $\Box\xi_\mu=0$, and therefore does not change the harmonic coordinate condition, $\partial_\mu \bar{h}^\mu_\nu = 0$. Then following the prescription of (360), we generate a new, but physically equivalent polarisation tensor,

$$e'_{\mu\nu} = e_{\mu\nu} + k_\mu\epsilon_\nu + k_\nu\epsilon_\mu \tag{388}$$

and by choosing the ϵ_μ appropriately, we can eliminate all of the $e'_{\mu\nu}$ except for e'_{11}, $e'_{22} = -e'_{11}$, and e'_{12}. In particular, using (388),

$$e'_{11} = e_{11}, \qquad e'_{12} = e_{12} \tag{389}$$

unchanged. But with $k = \omega/c$,

$$e'_{13} = e_{13} + k\epsilon_1, \quad e'_{23} = e_{23} + k\epsilon_2, \quad e'_{33} = e_{33} + 2k\epsilon_3, \quad e'_{00} = e_{00} - 2k\epsilon_0, \tag{390}$$

so that these four components may be set to zero by a simple choice of the ϵ_μ. We may work in this gauge, which is *transverse* (since the only e_{ij} components that are present are transverse to the z direction of propagation) and traceless (since $e_{11} = -e_{22}$). Oddly enough, this gauge is named the transverse-traceless (TT) gauge. *Notice that in the TT gauge, $h_{\mu\nu}$ vanishes if any of its indices are 0, whether raised or lowered.*

78

7.3 The quadrupole formula

In the limit of large r ("compact source approximation"), equation (364) is:

$$\bar{h}^{\mu\nu}(\boldsymbol{r}, t) = \frac{4G}{rc^4} \int T^{\mu\nu}(\boldsymbol{r}', t') d^3 r', \tag{391}$$

where $t' = t - r/c$ is the retarded time. Moreover, for the TT gauge, we are interested in the spatial ij components of this equation, since all time indices vanish. (Also, because $\bar{h}_{\mu\nu}$ is traceless, we need not distinguish between h and $\bar{h}$.) The integral over T_{ij} may be cast in a very convenient form as follows.

$$0 = \int \frac{\partial(x'^j T^{ik})}{\partial x'^k} d^3 r' = \int \left(\frac{\partial T^{ik}}{\partial x'^k} \right) x'^j d^3 r' + \int T^{ij} d^3 r', \tag{392}$$

where the first equality follows because the first integral reduces to a surface integration of T^{ik} at infinity, where it is presumed to vanish. Thus

$$\int T^{ij} d^3 r' = -\int \left(\frac{\partial T^{ik}}{\partial x'^k} \right) x'^j d^3 r' = \int \left(\frac{\partial T^{i0}}{\partial x'^0} \right) x'^j d^3 r' = \frac{1}{c} \frac{d}{dt'} \int T^{i0} x'^j d^3 r' \tag{393}$$

where the second equality uses the conservation of $T^{\mu\nu}$. Remember that t' is the retarded time. As T_{ij} is symmetric in its indices,

$$\frac{d}{dt'} \int T^{i0} x'^j d^3 r' = \frac{d}{dt'} \int T^{j0} x'^i d^3 r' \tag{394}$$

Continuing in this same spirit,

$$0 = \int \frac{\partial(T^{0k} x'^i x'^j)}{\partial x'^k} d^3 r' = \int \left(\frac{\partial T^{0k}}{\partial x'^k} \right) x'^i x'^j d^3 r' + \int (T^{0i} x'^j + T^{0j} x'^i) d^3 r' \tag{395}$$

Using exactly the same reasoning as before,

$$\int (T^{0i} x'^j + T^{0j} x'^i) d^3 r' = \frac{1}{c} \frac{d}{dt'} \int T^{00} x'^i x'^j d^3 r' \tag{396}$$

Therefore,

$$\int T^{ij} d^3 r' = \frac{1}{2c^2} \frac{d^2}{dt'^2} \int T^{00} x'^i x'^j d^3 r' \tag{397}$$

Inserting this in (391), we obtain the *quadrupole formula* for gravitational radiation:

$$\boxed{\bar{h}^{ij} = \frac{2G}{c^6 r} \frac{d^2 I^{ij}}{dt'^2}} \tag{398}$$

where I^{ij} is the *quadrupole-moment tensor* of the energy density:

$$I^{ij} = \int T^{00} x'^i x'^j d^3 r' \tag{399}$$

To estimate this numerically, we write

$$\frac{d^2 I^{ij}}{dt'^2} \sim M a^2 c^2 \omega^2 \tag{400}$$

where M is the characteristic mass of the rotating system, a an internal separation, and ω a characteristic frequency, an orbital frequency for a binary say. Then

$$\bar{h}^{ij} \sim \frac{2GMa^2\omega^2}{c^4 r} \simeq 7 \times 10^{-22} (M/M_\odot)(a_{11}^2 \omega_7^2 / r_{100}) \tag{401}$$

where $M/M_\odot$ is the mass in solar masses, a_{11} the separation in units of 10^{11} cm (about a separation of one solar radius), ω_7 the frequency associated with a 7 hour orbital period (similar to PSR193+16) and r_{100} the distance in units of 100 parsecs, some 3×10^{20} cm. A typical rather large h one might expect at earth from a local astronomical source is then of order 10^{-21}.

What about the LIGO source, GW150914? How does our formula work in this case? The distance in this case is cosmological, not local, with $r = 1.2 \times 10^{22}$ km, or in astronomical parlance, about 400 megaparsecs (Mpc). In this case, we write (401) as

$$\bar{h}^{ij} \sim \frac{2GMa^2\omega^2}{c^4 r} = \left(\frac{2.9532}{r_{km}}\right)\left(\frac{M}{M_\odot}\right)\left(\frac{a\omega}{c}\right)^2 \simeq 1 \times 10^{-22} \frac{M/M_\odot}{r_{Gpc}}\left(\frac{a\omega}{c}\right)^2, \tag{402}$$

since $2GM_\odot/c^2$ is just the Sun's Schwarzschild radius. (One Gpc$=10^3$Mpc $= 3.0856 \times 10^{22}$km.) The point is that $(a\omega/c)^2$ is a number not very different from 1 for a relativistic source, perhaps 0.1 or so. Plugging in numbers with $M/M_\odot = 60$ and $(a\omega/c)^2 = 0.1$, we find $\bar{h}_{ij} = 1.5 \times 10^{-21}$, just about as observed at peak amplitude.

Exercise. Prove that $\bar{h}^{ij}$ given by (398) is an *exact* solution of $\Box \bar{h}^{ij} = 0$, for any r, even if r is not large.

7.4 Radiated Energy

We have yet to make the link between $h_{\mu\nu}$ and the actual energy flux that is carrried off by the time varying metric coefficients. Alas, properly deriving an expression for the energy flux carried by gravitational waves is a labourious task. Whereas we have thus far worked only to linear order in the $G_{\mu\nu}$ tensor, the stress energy tensor of (weak) gravitational waves is contained in the $G_{\mu\nu}^{(2)}$ terms, which are quadratic in the $h_{\mu\nu}$ amplitudes. That is, with

$$G_{\mu\nu} = G_{\mu\nu}^{(1)} + G_{\mu\nu}^{(2)} + ... = -\frac{8\pi G T_{\mu\nu}}{c^4} \tag{403}$$

we may also write the Field Equation in the form,

$$G_{\mu\nu}^{(1)} = -\frac{8\pi G(T_{\mu\nu} + \tau_{\mu\nu})}{c^4}, \tag{404}$$

where

$$\tau_{\mu\nu} = \frac{c^4}{8\pi G} G_{\mu\nu}^{(2)} \tag{405}$$

is to leading order the stress energy tensor of the gravitational waves themselves. The problem is that evaluating $G^{(2)}_{\mu\nu}$ is a long digression, and I will avoid it here. The motivated reader is refered to either Weinberg's text or Hobson, Efstahiou & Lasenby for the nitty-gritty details.

We can at least *motivate* the general form of the energy flux in gravitational waves, which in the transverse traceless (TT) in the k direction is given by:

$$\boxed{\mathcal{F}_k = -\frac{c^4}{32\pi G}\frac{\partial h_{ij}^{TT}}{\partial t}\frac{\partial h_{ij}^{TT}}{\partial x^k}} \tag{406}$$

To see why $\mathcal{F}_k$ has the general form it does, start with the fundamental wave equation in the harmonic gauge (not necessarily the TT gauge), written as

$$\frac{1}{c^2}\frac{\partial^2 \bar{h}_{\mu\nu}}{\partial t} - \nabla^2 \bar{h}_{\mu\nu} = \kappa T_{\mu\nu}, \quad \kappa = \frac{16\pi G}{c^4}. \tag{407}$$

Multiply by $\partial\bar{h}_{\mu\nu}/\partial t$:

$$\frac{1}{c^2}\frac{\partial\bar{h}_{\mu\nu}}{\partial t}\frac{\partial^2\bar{h}_{\mu\nu}}{\partial t} - \frac{\partial\bar{h}_{\mu\nu}}{\partial t}\nabla^2\bar{h}_{\mu\nu} = \kappa T_{\mu\nu}\frac{\partial\bar{h}_{\mu\nu}}{\partial t}. \tag{408}$$

This can be rearranged in the form of a conservation equation:

$$\frac{\partial}{\partial t}\left(\frac{\partial_t\bar{h}_{\mu\nu}\,\partial_t\bar{h}_{\mu\nu}}{2c^2} + \frac{\partial_k\bar{h}_{\mu\nu}\,\partial_k\bar{h}_{\mu\nu}}{2}\right) - \frac{\partial}{\partial x^k}\left(\partial_t\bar{h}_{\mu\nu}\,\partial_k\bar{h}_{\mu\nu}\right) = \kappa T_{\mu\nu}\frac{\partial\bar{h}_{\mu\nu}}{\partial t}. \tag{409}$$

If we mulitply by $1/2$,

$$\frac{\partial}{\partial t}\left(\frac{\partial_t\bar{h}_{\mu\nu}\,\partial_t\bar{h}_{\mu\nu}}{4\kappa c^2} + \frac{\partial_k\bar{h}_{\mu\nu}\,\partial_k\bar{h}_{\mu\nu}}{4\kappa}\right) - \frac{\partial}{\partial x^k}\left(\frac{1}{2\kappa}\partial_t\bar{h}_{\mu\nu}\,\partial_k\bar{h}_{\mu\nu}\right) = \frac{1}{2}T_{\mu\nu}\frac{\partial\bar{h}_{\mu\nu}}{\partial t}, \tag{410}$$

we have an equation that states that the time derivative of one quantity, plus the divergence of another, equals a source term, which is indeed the form of a conservation equation. In particular, we recognise

$$\mathcal{F}_k = -\frac{1}{2\kappa}\frac{\partial\bar{h}_{\mu\nu}}{\partial t}\frac{\partial\bar{h}_{\mu\nu}}{\partial x^k} = -\frac{c^4}{32\pi G}\frac{\partial\bar{h}_{\mu\nu}}{\partial t}\frac{\partial\bar{h}_{\mu\nu}}{\partial x^k} \tag{411}$$

as the energy flux associated with gravitational waves. The *form* of this flux, a product of t and x^k derivatives, is unique to the homogeneous wave equation, but only up to an overall normalisation constant. (This is why the energy flux for sound waves, for example, can be written the same form using the velocity potential function instead $\bar{h}_{\mu\nu}$.) The nomalisation constant, this factor of $1/2$ that was slipped in, is for now just cheat; we have not derived it of course. In reality, it is fixed by the identification of the right side of the energy equation (410) as *minus* the rate at which the gravitational field does work on the sources. This is a difficult calculation. Ultimately, it is the same factor of 2 in the relationship between h_{00} and the Newtonian potential, but it is not easy to show this. (See MTW, Chapter 36, for a discussion of the "radiation reaction" back on slow moving sources.) For our first glimpse at gravitational radiation, we will leave it at that. But at least you can get a sense for why the flux has the form it does.

7.5 The energy loss formula for gravitational waves

Our next step is to evaluate the h_{ij}^{TT} quantities in terms of the transverse and traceless components of I_{ij}. First, we work only with the traceless component, denoted J_{ij}:

$$J_{ij} = I_{ij} - \frac{\delta_{ij}}{3} I \tag{412}$$

where I is the trace of I_{ij}. Next, we address the transverse property. The projection of a vector $\boldsymbol{v}$ onto a plane perpendicular to a unit direction vector $\boldsymbol{n}$ is accomplished simply by removing the component of $\boldsymbol{v}$ along $\boldsymbol{n}$. Denoting the resulting vector as $\boldsymbol{w}$,

$$\boldsymbol{w} = \boldsymbol{v} - (\boldsymbol{n} \cdot \boldsymbol{v})\boldsymbol{n} \tag{413}$$

or

$$w_j = (\delta_{ij} - n_i n_j)v_i \equiv P_{ij}v_i \tag{414}$$

where we have introduced the projection tensor

$$P_{ij} = \delta_{ij} - n_i n_j,$$

$$n_i P_{ij} = 0, \qquad P_{ij}P_{jk} = P_{ik}, \qquad P_{ii} = 2. \tag{415}$$

Projecting tensor components presents no difficulties,

$$w_{ij} = P_{ik}P_{jl}v_{kl}, \tag{416}$$

nor does the extraction of a projected tensor that is both traceless and transverse:

$$w_{ij}^{TT} = \left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \right) v_{kl}, \quad \to w_{ii}^{TT} = (P_{ik}P_{il} - P_{kl}) = (P_{kl} - P_{kl}) = 0. \tag{417}$$

So with

$$J_{ij}^{TT} = \left(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \right) J_{kl}, \tag{418}$$

we have

$$h_{ij}^{TT} = \frac{2G}{c^6 r} \frac{d^2 J_{ij}^{TT}}{dt'^2} \tag{419}$$

Recalling that $t' = t - r/c$ and the J^{TT}'s are functions of t' (not t!),

$$\frac{\partial h_{ij}^{TT}}{\partial t} = \frac{2G}{c^6 r} \frac{d^3 J_{ij}^{TT}}{dt'^3}, \quad \frac{\partial h_{ij}^{TT}}{\partial r} = -\frac{2G}{c^7 r} \frac{d^3 J_{ij}^{TT}}{dt'^3} \tag{420}$$

where, in the second expression we retain only the dominant term in $1/r$. The radial flux of gravitational waves is then given by (406):

$$\mathcal{F}_r = \frac{G}{8\pi r^2 c^9} \frac{d^3 J_{ij}^{TT}}{dt'^3} \frac{d^3 J_{ij}^{TT}}{dt'^3} \tag{421}$$

Note the remarkable $1/c^9$ dependence!

The final steps are a matter of a straightforward but somewhat tedious calculation to write out the J_{ij}^{TT} in terms of the J_{ij} using the projection operator. (It is here that the fact

that J_{ij} is traceless is a computational advantage.) With $\dot{X}$ standing for dX/dt', you will find

$$\dddot{J}^{TT}_{ij} \dddot{J}^{TT}_{ij} = \dddot{J}_{ij} \dddot{J}_{ij} - 2\dddot{J}_{ij} \dddot{J}_{ik} n_j n_k + \frac{1}{2} \dddot{J}_{ij} \dddot{J}_{kl} n_i n_j n_k n_l \tag{422}$$

The gravitational wave luminosity is an integration of this distribution over all solid angles,

$$L_{GW} = \int r^2 \mathcal{F}_r \, d\Omega \tag{423}$$

You will need

$$\int n_i n_j \, d\Omega = \frac{4\pi}{3} \delta_{ij} \tag{424}$$

which is pretty simple: the two vector components are not the same it vanishes by symmetry (e.g. the average of xy over a sphere is zero), otherwise the average of x^2 or y^2 or z^2 on the unit sphere is clearly just $1/3$ of $x^2 + y^2 + z^2 = 1$. More scary is

$$\int n_i n_j n_k n_l \, d\Omega = \frac{4\pi}{15} (\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}), \tag{425}$$

but keep calm and think. The only way this thing cannot vanish is if two of the indices agree with one another and the remaining two indices also agree with one another. (Maybe the second pair is just the same pair as the first, maybe not.) But this index agreement requirement is precisely what the symmetric combination of delta functions ensures. To get the $4\pi/15$ factor, set $i = j$ and sum, and the same thing with $l = k$. The integral on the left is then trivially $\int n_i n_i n_l n_l \, d\Omega = \int d\Omega = 4\pi$. The combination of delta functions is $9 + 3 + 3 = 15$. Hence the normalisation factor $4\pi/15$. Putting this all together and carrying out the integral, the total gravitational luminosity is given by a beautifully simple formula, first derived by Albert Einstein in 1918:

$$\boxed{L_{GW} = \frac{G}{5c^9} \dddot{J}_{ij} \dddot{J}_{ij} = \frac{G}{5c^9} \left(\dddot{I}_{ij} \dddot{I}_{ij} - \frac{1}{3} \dddot{I}_{ii} \dddot{I}_{jj} \right)} \tag{426}$$

7.6 Gravitational radiation from binary stars

In Weinberg's 1972 text, gravitational radiation detection looms as a very distant possibility, and rightly so. The section covering this topic devotes its attention to the possibility that rapidly rotating neutron stars might just be a good source. But, alas, for this to occur the neutron star would have to possess a sizeable and rapidly varying quadrupole moment, and this neutron stars do not possess. Neutron stars are nearly exact spheres, even when rotating rapidly as pulsars. They are in essence perfectly axisymmetric; even were they to have quadrupole moment, it would not change with time.

The possibility that Keplerian orbits might be interesting from the point-of-view of measuring gravitational radiation is never mentioned in Weinberg. Certainly ordinary orbits involving ordinary stars are not a promising source. But compact objects (white dwarfs, neutron stars or black holes) in very close binaries, with orbital periods measured in hours, were discovered within two years of the book's publication, and these turn out to be *extremely* interesting. They are the central focus of modern day gravitational wave research. As we have noted earlier, the first confirmation of the existence of gravitational radiation came from the binary pulsar system 1913+16, in which the change in the orbital period from the loss of wave energy was inferred via the arrival times of the pulsar signal. The radiation

level itself was well below the threshold of direct detection. Over long enough time scales, a tight binary of compact objects may lose enough energy through gravitational radiation that the resulting inspiral goes all the way to completion and the system either coalesces or explodes in a supernova. There are predicted to be enough merging binaries in the universe that the detection rate should be interesting astrophysically. LIGO has already published its first detection, and given how quickly it was found when the critical upgrade was made, there are certainly grounds for optimsim. The final frenzied seconds of black holes coalescence will emit detectable gravitational wave signatures rich in physical content. Such waveforms can now be determined numerically to high precision (F. Pretorius 2005, Phys. Rev. Lett. 95, 121101). In the near future, the will almost certainly be measured on a regular basis.

Let us apply equation (426) to the case of two point masses in a classical Keplerian orbit. There is of course no contradiction between assuming a classical orbit and calculating its gravitational energy loss. We are working here in the regime in which the losses themselves exert only a tiny change on the orbit over one period, and the objects themselves, while close by ordinary astronomical standards, are separated by a distance well beyond their respective Schwarzschild radii. (Pretorius [2005] does not make this restriction, of course!)

The orbital elements are defined on page 65. The separation r of the two bodies is given as a function of azimuth ϕ as

$$r = \frac{L}{1 + \epsilon \cos \phi} \tag{427}$$

where L is the semilatus rectum and ϵ is the orbital eccentricity. With M being the total mass of the individual objects, $M = m_1 + m_2$, l the constant specific angular momentum, and a is the semi-major axis, we have

$$r^2 \frac{d\phi}{dt} = l, \qquad L = \frac{l^2}{GM} = a(1 - \epsilon^2) \tag{428}$$

and thus

$$\frac{d\phi}{dt} = \left(\frac{GM}{a^3(1 - \epsilon^2)^3} \right)^{1/2} (1 + \epsilon \cos \phi)^2 \qquad \frac{dr}{dt} = \left(\frac{GM}{a(1 - \epsilon^2)} \right)^{1/2} \epsilon \sin \phi \tag{429}$$

The distance from the center-of-mass of each body is denoted r_1 and r_2. Writing these as vector quantities,

$$\boldsymbol{r_1} = \frac{m_2 \boldsymbol{r}}{M}, \qquad \boldsymbol{r_2} = -\frac{m_1 \boldsymbol{r}}{M} \tag{430}$$

Thus the coordinates in the xy orbital plane are

$$\boldsymbol{r_1} = \frac{m_2 r}{M}(\cos \phi, \sin \phi), \qquad \boldsymbol{r_2} = \frac{m_1 r}{M}(-\cos \phi, -\sin \phi) \tag{431}$$

The nonvanishing moment tensors I_{ij} are then

$$I_{xx} = \frac{m_1 m_2^2 + m_1^2 m_2}{M^2} r^2 \cos^2 \phi = \mu r^2 \cos^2 \phi \tag{432}$$

$$I_{yy} = \mu r^2 \sin^2 \phi \tag{433}$$

$$I_{xy} = Iyx = \mu r^2 \sin \phi \cos \phi \tag{434}$$

$$I_{ii} = I_{xx} + I_{yy} = \mu r^2 \tag{435}$$

where μ is the reduced mass $m_1 m_2/M$. It is a now lengthy, but entirely straightforward task to differentiate each of these moments three times. You should begin with the relatively easy $\epsilon = 0$ case when reproducing the formulae below, though I present the results for finite ϵ here:

$$\frac{d^3 I_{xx}}{dt^3} = \alpha(1 + \epsilon \cos \phi)^2 (2 \sin 2\phi + 3\epsilon \sin \phi \cos^2 \phi), \tag{436}$$

$$\frac{d^3 I_{yy}}{dt^3} = -\alpha(1 + \epsilon \cos \phi)^2 [2 \sin 2\phi + \epsilon \sin \phi (1 + 3 \cos^2 \phi)], \tag{437}$$

$$\frac{d^3 I_{xy}}{dt^3} = \frac{d^3 I_{yx}}{dt^3} = -\alpha(1 + \epsilon \cos \phi)^2 [2 \cos 2\phi - \epsilon \cos \phi (1 - 3 \cos^2 \phi)], \tag{438}$$

where

$$\alpha^2 \equiv \frac{4G^3 m_1^2 m_2^2 M}{a^5 (1 - \epsilon^2)^5} \tag{439}$$

Now equation (426) yields after some assembling:

$$L_{GW} = \frac{32}{5} \frac{G^4}{c^5} \frac{m_1^2 m_2^2 M}{a^5 (1 - \epsilon^2)^5} (1 + \epsilon \cos \phi)^4 [(1 + \epsilon \cos \phi)^2 + (\epsilon^2 \sin^2 \phi)/12] \tag{440}$$

Our final step is to average L_{GW} over an orbit. This is *not* simply an integral over $d\phi/2\pi$. We must integrate over time, i.e., over $d\phi/\dot{\phi}$, and then divide by the orbital period to do a *time* average. Once again, the words "lengthy but straightforward" are a good description! The answer is

$$\langle L_{GW} \rangle = \frac{32}{5} \frac{G^4}{c^5} \frac{m_1^2 m_2^2 M}{a^5} f(\epsilon) = 1.00 \times 10^{25} \ m_{\odot 1}^2 m_{\odot 2}^2 M_\odot (a_\odot)^{-5} \ f(\epsilon) \ \text{Watts}, \tag{441}$$

where

$$f(\epsilon) = \frac{1 + (73/24)\epsilon^2 + (37/96)\epsilon^4}{(1 - \epsilon^2)^{7/2}} \tag{442}$$

and $\odot$ indicates solar units of mass (1.99×10^{30} kg) and length (one solar radius is 6.955×10^8 m). (Peters and Mathews 1963). Equations (441) and (442) give the famous gravitational wave energy loss formula for a classical Keplerian orbit. Notice the dramatic effect of finite eccentricity via the $f(\epsilon)$ function. The first binary pulsar to be discovered, PSR1913+16, has an eccentricity of about 0.62, and thus an enhancement of its gravitational wave energy loss that is boosted by more than an order of magnitude relative to a circular orbit.

This whole problem must have seemed like an utter flight of fancy in 1963: the concept of a neutron star was barely credible and not taken seriously; the notion of pulsar timing was simply beyond conceptualisation. A lesson, perhaps, that no good calculation of an interesting physical problem ever goes to waste!

Exercise. When we studied Schwarzschild orbits, there was an exercise to show that the total Newtonian orbital energy of a bound two body system is $-Gm_1 m_2/2a$ and that the system period is proportional to $a^{3/2}$, independent of the eccentricity. Use these results to show that the orbital period change due to the loss of gravitational radiation is given by

$$\dot{P} = -\frac{192\pi}{5} \left(\frac{m_1 m_2}{M^2} \right) \left(\frac{GM}{ac^2} \right)^{5/2} f(\epsilon)$$

This is a measurable quantity! Stay tuned.

7.7   Detection of gravitational radiation

7.7.1   Preliminary comments

The history of gravitational radiation has been somewhat checkered. Albert Einstein himself stumbled several times, both conceptually and computationally. Arguments of fundamental principle persisted through the early 1960's; technical arguments still go on.

At the core of the early controversy was the question of whether gravitational radiation existed at all! The now classic Peters and Mathews paper of 1963 begins with a disclaimer that they are assuming that the "standard interpretation" of the theory is correct. The confusion concerned whether the behaviour of $h_{\mu\nu}$ potentials were just some sort of mathematical coordinate effect, devoid of any actual physical consequences. For example, if we calculate the affine connection $\Gamma^\mu_{\nu\lambda}$ and apply the geodesic equation,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \qquad (443)$$

and ask what happens to a particle initially at rest with $dx^\nu/d\tau = (-c, \mathbf{0})$. The subsequent evolution of the spatial velocity components is then

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{00}c^2 = 0 \qquad (444)$$

But equation (343) clearly shows that $\Gamma^i_{00} = 0$ since any h with a zero index vanishes for our TT plane waves. The particle evidently remains at rest. Is there is no effect of gravitational radiation on ordinary matter?!

Coordinates, coordinates, coordinates. The point, once again, is that coordinates by themselves mean nothing, any more than does the statement "My house is located at the vector (2, 1.3)." By now we should have learned this lesson. We picked our gauge to make life simple, and we have simply found a coordinate system that is frozen to the individual particles. There is nothing more to it than that. The *proper* spatial separation between two particles with coordinate separation dx^i is $ds^2 = (\eta_{ij} - h_{ij})dx^i\,dx^j$, and that separation surely is not constant because h_{11}, h_{22}, and $h_{12} = h_{21}$ are wiggling even while the dx^i are fixed. It was Richard Feynman who in 1955 seems to have given the simplest and most convincing argument for the existence of graviational waves. If the separation is between two beads on a rigid stick and the beads are free to slide, they will oscillate with the tidal force of the wave. If there is now a tiny bit of stickiness, the beads will heat the stick. Where did that energy come from? It could only be the wave. The "sticky bead argument" became iconic in the relativity community.

The two independent states of linear polarisation of a gravitational wave are sometimes referred to as $+$ and $\times$, "plus" and "cross." The behave similarly, but rotated by 45°. The $+$ wave as it passes initially causes a prolate distortion along the vertical part of the plus sign, squeezes from prolate to oblate distorting along the vertical axis, then squeezes inward from oblate to prolate once again. The $\times$ wave shows the same oscillation pattern along a rotation pattern rotated by 45°. (An excellent animation is shown in the Wikipedia article "Gravitational Waves.") These are true physical distortions caused by the tidal force of the gravitational wave.

In the midst of what had been intensively theoretical investigations and debate surrounding of gravitational radiation, in 1968 a physicist named Joseph Weber suddenly announced that he had actually detected gravitational radiation experimentally in his lab coming in prodigious amounts from the centre of the Milk Way Galaxy! His technique was to use what

are now called "Weber bars", giant cylinders of aluminum fitted with special piezoelectric devices that can convert tiny mechanical oscillations into electrical signals. The gravitational waves distorted these great big bars by a tiny, tiny amount, and the signals were picked up. Or at least that was the idea. The dimensionless relative strain $\delta l/l$ of a bar of length l due to passing wave would be of order h_{ij}, or 10^{-21} by our optimistic estimate. (Demonstrate this last statement.) Too make a long, sad story very short, Weber had made elementary errors and was discredited. But his legacy was not wholly negative: the possibility of detecting gravitational waves gradually caught on and became part of mainstream physics. Fifty years later, LIGO has at last directly detected strains at the level of $h \sim 10^{-21}$. This is borders on magic: if l is 10 km, δl is 10^{-15} cm, one percent of the radius of a proton!

7.7.2   Indirect methods: orbital energy loss in binary pulsars

In 1974, a remarkable binary system was discovered by Hulse and Taylor (1975, ApJ (Letters), 195, L51). One of the stars was a pulsar with a pulse period of 59 milliseconds, i.e., a neutron star that rotates about 17 times a second. The orbital period was 7.75 hours, a very tight binary with a separation of about the radius of the Sun. The other star was not seen, only inferred, but the very small separation between the two stars together with the absence of any eclipse of the pulsar suggested that the companion was also a compact star. (If the binary orbital plane were close to being in the plane of the sky to avoid observed eclipses, then the pulsar pulses would show no Doppler shifts, in sharp contradiction to observations.)

What made this yet more extraordinary is that pulsars are the most accurate clocks in the universe, far more accurate than any earthbound atomic clock. The most accurately measured pulsar has a pulse period known to 17 significant figures! Indeed, pulsars can be calibrated only by ensemble averages of large numbers of atomic clocks. Nature has placed its most accurate clock in the middle of binary system in which fantastically precise timing is required. This then, is the ultimate general relativity laboratory.

Classic nonrelativistic binary observation techniques allow one to determine five parameters from observations of the pulsar: the semimajor axis projected against the plane of the sky $(a \sin i)$, the eccentricity e, the orbital period P, and two parameters related to the periastron (the point of closest separation): its angular position within the orbit and a time reference point for when it occurs.

Relativistic effects, something new and beyond standard analysis, give two more parameters. The first is the advance of the perihelion (exactly analogous to Mercury) which in the case of PSR 1913+16 is $4.2°$ per year. (Recall that Mercury's is only 43 arc seconds per century!) The second is the *second order* $(\sim v^2/c^2)$ Doppler shift of the pulse period from both the gravitational redshift of the combined system and the rotational kinematics. These seven parameters allow a complete determination of the masses and orbital components of the system, a neat achievement in itself. The masses of the neutron stars are $1.4414 \, M_\odot$ and $1.3867 \, M_\odot$, remarkably similar to one another and remarkably similar to the Chandrasekhar mass $1.42 \, M_\odot$[7]. (The digits in the neutron stars' masses are all significant!) More importantly, there is a third relativistic effect also present, and therefore the problem is over-constrained. That is to say, it is possible to make a *prediction.* The orbital period shortens due to the inspiraling caused by the loss of mechanical energy carried off by gravitational radiation, equation (441). Thus, by monitoring the precise arrival times of the pulsar signals coming from the slowly decaying orbit, the *existence* of gravitational radiation could be quantitatively confirmed, even though the radiation itself was not directly observable.

---

[7]This is the upper limit to the mass of a white dwarf star. If the mass exceeds this value, it collapses to either a neutron star or black hole, but cannot remain a white dwarf.
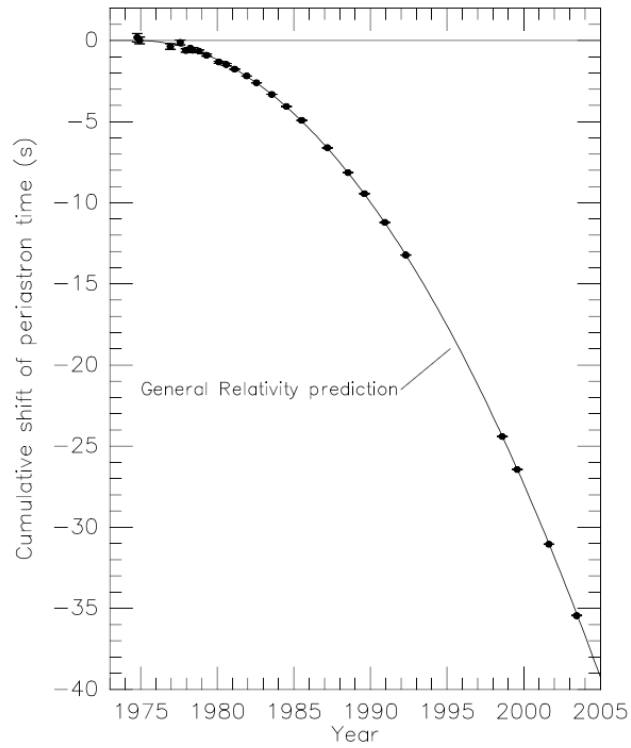
Figure 5: The cumulative change in the periastron event ("epoch") caused by the inspiral of the pulsar PSR1913+16. The dots are the data, the curve is the *prediction,* not the best fit! This prediction is confirmed to better than a fraction of a percent.

Figure [5] shows the results of many years of observations. The dots are the cumulative change in the time of periastron due to the more progressively more rapid orbital period as the neutron stars inspiral from gravitational radiation losses. Without the radiation losses, there would still be a perihelion advance of course, but the time between perihelia would not change–it would just be a bit longer than an orbital period. The cumulative change between perihelia is an indication of actual energy loss. The solid line is *not* a fit to the data. It is the prediction of general relativity of what the cumulative change in the "epoch of perihelion" (as it is called) should be, according to the energy loss formula of Peters and Mathews, (441). This beautiful precision fit leaves no doubt whatsover that the quadrupole radiation formula of Einstein is correct. For this achievement, Hulse and Taylor won a well-deserved Nobel Prize in 1993. It must be just a coincidence that this is about the time that the data points seem to become more sparse.

Direct detection of gravitational waves is a very recent phenomenon. There are two types of gravitational wave detectors currently in operation. The first is based on a classic 19th century laboratory apparatus: a Michelson interferometer. The second makes use of pulsar emission pulses—specifically their arrival times—as a probe of the $h_{\mu\nu}$ caused by gravitational waves as they propagate across our line of site to the pulsar. The interferometer detectors are designed for wave frequencies from ~ 10 Hz to 1000's of Hz. This is now up and running. By contrast, the pulsar measurements are sensitive to frequencies of tens to hundreds of *micro* Hz. A very different range, measuring physical processes on very different scales. This technique has yet to be demonstrated. The high frequency interferometers measure the gravitational radiation from stellar-mass black holes or neutron star binaries merging together. The low frequency pulsar timing will measure black holes merging, but with masses of order 10^9 solar masses. These are the masses of galactic core black holes in active galaxies.

7.7.3   Direct methods: LIGO

LIGO, or Laser Interferometer Gravitational-Wave Observatory, detects gravitational waves as described in figure (6). In the absence of a wave, the arms are set to destructively interfere, so that no light reaches the detector. The idea is that a gravitational wave passes through the apparatus from above or below, each period of oscillation slightly squeezing one arm, slightly extending the other. With coherent laser light traversing each arm, when it re-superposes at the centre, the phase will become *ever* so slightly out of precise cancellation, and photons will appear in the detector. In practice, the light makes many passages back and forth along a 4 km arm before analysis. The development of increased sensitivity comes from engineering greater and greater numbers of reflections, and thus a greater effective path length. There are two such interferometers, one in Livingston, Louisiana, the other in Hanford, Washington, a separation of 3000 km. Both must show a simultaneous wave passage (actually, with an offset of 10 milliseconds for speed of light travel time) for the signal to be verified.

This is a highly simplified description, of course. All kinds of ingenious amplification and noise suppression techniques go into this project, which is designed to measure induced strains at the incredible level of 10^{-21}. This detection is only possible because we measure not the flux of radiation, which would have a $1/r^2$ dependence with distance to the source, but the h_{ij} amplitude, which has a $1/r$ dependence.

Figure (7) shows a match of an accurate numerical simulation to the processed LIGO event GW150914. I have overlaid three measured wave periods P_1, P_2, and P_3, with each of their respective lengths given in seconds. (These were measured with a plastic ruler directly from the diagram!) The total duration of these three periods is 0.086 s. Throughout this time the black holes are separated by a distance in excess of of 4 R_S, so we are barely at the limit for which we can trust Newtonian orbit theory. Let's give it a try for a circular orbit.
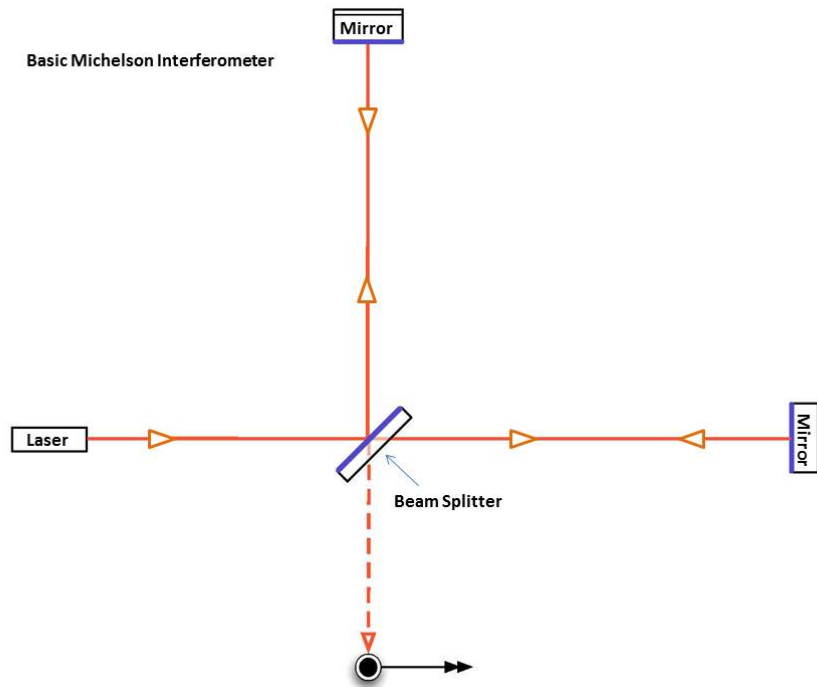
Figure 6: A schematic interferometer. Coherent light enters from the laser at the left. Half is deflected 45° upward by the beam splitter, half continues on. The two halves reflect from the mirrors. The beams re-superpose at the splitter, interfere, and are passed to a detector at the bottom. If the path lengths are identical or differ by an integral number of wavelengths they interfere constructively; if they differ by an odd number of half-wavelengths they cancel one another. In "null" mode, the two arms are set to destructively interfere so that no light whatsoever reaches the detector. A passing gravity wave just barely offsets this precise destructive interference and causes laser photons to appear in the detector.

(Circularity is not unexpected for the final throes of coalescence.)

Using the zero eccentricity orbital period decrease formula from the previous exercise, but remembering that the orbital period P is twice the gravitational wave period P_{GW},

$$\dot{P}_{GW} = -\frac{96\pi}{5}\left(\frac{m_1 m_2}{M^2}\right)\left(\frac{GM}{ac^2}\right)^{5/2}$$

We eliminate the semi-major axis a in favour of the measured period P_{GW},

$$P^2 = \frac{4\pi^2 a^3}{GM}, \quad \text{whence } P_{GW}^2 = \frac{\pi^2 a^3}{GM}$$

This gives

$$\dot{P}_{GW} = -\frac{96\pi^{8/3}}{5c^5}\left(\frac{GM_c}{P_{GW}}\right)^{5/3} \tag{445}$$

where we have introduced what is known as the "chirp mass" M_c,

$$M_c = \frac{(m_1 m_2)^{3/5}}{M^{1/5}} \tag{446}$$

The chirp mass (so-named because if the gravitational wave were audible at the same frequencies, it would indeed sound like a chirp!) is the above combination of m_1 and m_2, which is directly measurable from P_{GW} and its derivative. It can be shown (try it!) that $M = m_1 + m_2$ is a minimum when $m_1 + m_2$, in which case

$$m_1 = m_2 \simeq 1.15 M_c.$$

Putting numbers in (445),

$$M_{c\odot} = -5.522 \times 10^3 P_{GW} \dot{P}_{GW}^{3/5} \tag{447}$$

where $M_{c\odot}$ is the chirp mass in solar masses. From the GW150914 data, we estimate

$$\dot{P}_{GW} \simeq \frac{P_3 - P_1}{P_1 + P_2 + P_3} = \frac{-0.0057}{0.086} = -0.0663,$$

and for P_{GW} we use the midvalue $P_2 = 0.0283$. This yields

$$M_{c\odot} \simeq 30.7 \tag{448}$$

compared with "$M_c \simeq 30 M_\odot$" in Abbot et al (2016)! I'm sure this remarkable level of agreement is somewhat (but not entirely!) fortuitous. Even in this, its simplest presentation, the wave form presents a wealth of information. The "equal mass" coalescing black hole system comprises two $35 M_\odot$ black holes, and certainly at that mass a compact object can only be a black hole!

The two masses need not be equal of course, so is it possible that this is something other than a coalescing black hole binary? We can quickly rule out any other possibility, without a sophisticated analysis. It cannot be any combination of white dwarfs or neutron stars, because the chirp mass is too big. Could it be, say, a black hole plus a neutron star. With a fixed $M_c = 30 M_\odot$, and at most a $\sim 2 M_\odot$, the black hole would have to be some $1700 M_\odot$. So? Well, then the Schwarzschild radius would have to be very large, and coalescence would have occured at a separation distance too large for any high frequencies to be generated!
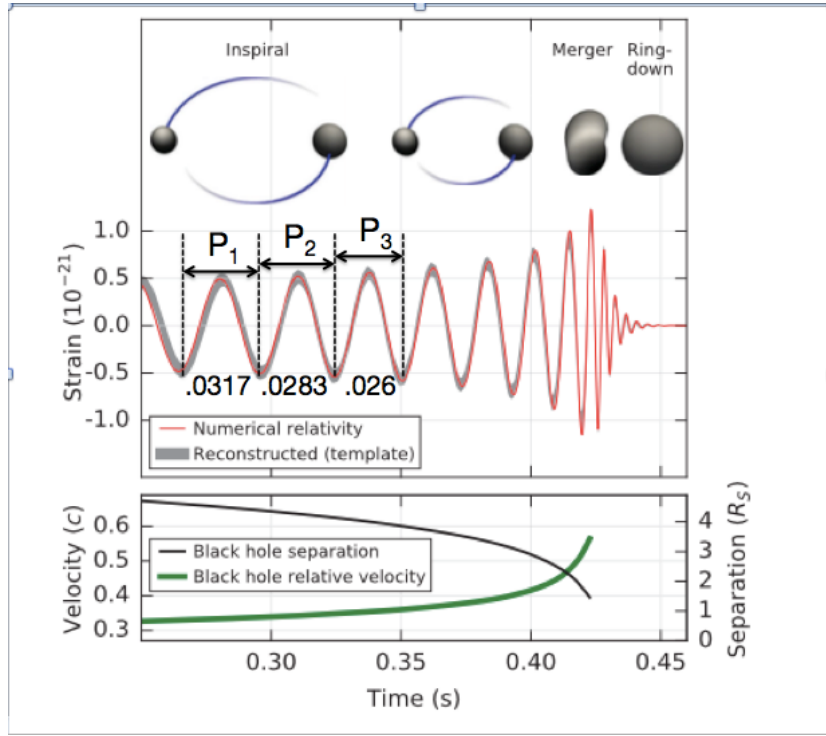
Figure 7: From Abbot et al. (2016). The upper diagram is a schematic rendering of the black hole inspiral process, from slowly evolution in a quasi-Newtonian regime, to a strongly interacting regime, followed by a coalescence and "ring-down," as the emergent single black hole settles down to its final, nonradiating geometry. The middle figure is the gravitational wave strain, overlaid with three identified periods discussed in the the text. The final bottom plot shows the separation of the system and the relative velocity as a function of time, from insprial just up to the moment of coalescence.

But there are frequencies present toward the end of the wave form event in excess of 75 Hz. This is completely incompatible with a black hole mass of this magnitude.

A sophisticated analysis using accurate first principle numerical simulations of gravitational wave from coalescing black holes tells an interesting history, though one rather well-captured by our naive efforts. Using a detailed match to the waveform, the following can be deduced. The system lies at a distance of some 400 Mpc, with significant uncertainties here of order 40%. At these distances, the wave form needs to be corrected for cosmological expansion effects, and the masses in the source rest frame are $36M_\odot$ and $29M_\odot$, with $\pm15\%$ uncertainties. The final mass, $62M_\odot$ is less than the sum of the two, $65M_\odot$: some $3M_\odot c^2$ worth of energy has disappeared in gravitational waves! At 5×10^{47}J, this is, I believe, the largest explosion of any kind every recorded. A billion years later some of that energy, in the form of ripples in space itself, tickles the interferometer arms in Louisiana and Texas.
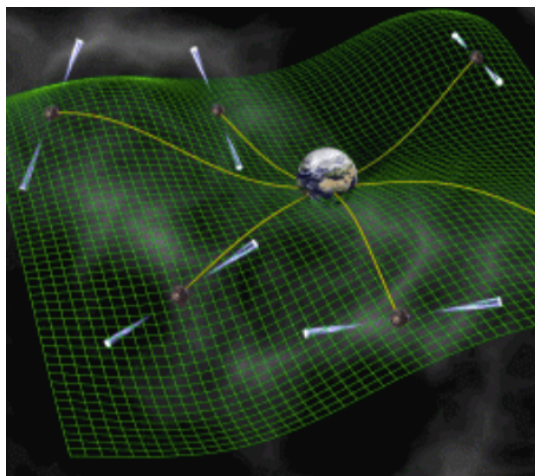
What a story.

Figure 8: A schematic view of a gravitational wave passing through an array of pulsar probes.

7.7.4 Direct methods: Pulsar timing array

Pulsars are, as we have noted, fantastically precise clocks. Within the pulsar cohort, those with millisecond periods are the most accurate of all. The period of PSR1937+21 is known to be 1.5578064688197945 milliseconds, an accuracy of one part in 10^{17}. One can then predict the arrival time of a pulse to this level of accuracy as well. By constraining variations in pulse arrival times from a single pulsar, we can set an upper limit to amount of gravitational radiation that the signal has traversed. But we don't just have one pulsar. But why settle for one pulsar and mere constraints? We have many, distributed more or less uniformly through the galaxy. If the arrival times from this "pulsar timing array" (PTA) were correlated with one another in a mathematically calculable manner, this would be a direct indication of the passage of a gravitational wave. This technique is sensitive to very long wavelength gravitational radiation, light-years in extent. At the time of this writing, there are only upper limits from the PTA measurements.