

---

---

# REAL ANALYSIS

---

---

MTH102

EDITED BY  
ADITYA DEV

# Contents

<b>1</b>	<b>The set of <math>\mathbb{R}</math> and some of it's properties</b>	<b>2</b>
1.1	The set of $\mathbb{R}$ . . . . .	2
<b>2</b>	<b>Sequences</b>	<b>4</b>
2.1	Convergence . . . . .	4
2.2	Limit Theorem for Sequences . . . . .	5
2.3	Monotone and Cauchy Sequences . . . . .	9
2.4	Subsequences . . . . .	12
2.4.1	Subsequential Limits . . . . .	12
<b>3</b>	<b>Series</b>	<b>14</b>
3.1	Sum to infinity? . . . . .	14
3.2	Convergence Tests for Series . . . . .	14
3.3	Alternating Series Test . . . . .	17
3.4	Integral Test . . . . .	18
<b>4</b>	<b>Continuity</b>	<b>21</b>
4.1	Continuity and Functions . . . . .	21
4.1.1	Properties of Continuous Functions . . . . .	21
4.2	Uniform Continuity . . . . .	24
4.3	Limits of Functions . . . . .	27
<b>5</b>	<b>Sequence and Series of Functions</b>	<b>29</b>
5.1	Power Series . . . . .	29
5.2	Uniform Convergence . . . . .	30
<b>6</b>	<b>Differentiation</b>	<b>34</b>
6.1	Why differentiation?	
	Better ask Newton. . . . .	34
6.2	Mean Value Theorem . . . . .	35
6.3	Taylor Theorem . . . . .	37
<b>7</b>	<b>Die Theorie der Integration</b>	<b>38</b>
7.1	Integrals . . . . .	38
7.1.1	Fundamental Theorem of Calculus . . . . .	40
7.1.2	Intermediate Value Theorem for Integrals. . . . .	40
<b>8</b>	<b>Differentiation and Integration of Power Series</b>	<b>42</b>
	<b>Appendices</b>	<b>43</b>
<b>A</b>	<b>Power series of some common functions</b>	<b>44</b>

# Chapter 1

## The set of $\mathbb{R}$ and some of it's properties

### §1.1 The set of $\mathbb{R}$

#### Definition 1.1.1: Completeness Axiom

Every nonempty subset  $S$  of  $\mathbb{R}$  that is bounded above has a least upper bound. In other words,  $\sup S$  exists and is a real number

#### Corollary 1.1.1

Every nonempty subset  $S$  of  $\mathbb{R}$  that is bounded below has a greatest lower bound. In other words,  $\inf S$  exists and is a real number.

**Proof.** Let  $S$  be a set which is bounded below i.e.  $\forall s \in S, s \geq m$  for some  $m \in \mathbb{R}$ . Take the set  $-S = \{-s \mid s \in S\}$ . Therefore  $\forall u \in -S \Rightarrow \{u \leq -m\}$  or

$$\Rightarrow u \leq \sup(-S) \leq -m \quad \{\text{completeness axiom}\}$$

$$\Rightarrow -s \leq \sup(-S) \leq -m$$

$$\Rightarrow m \leq -\sup(-S) \leq s$$

Let  $\exists \lambda \in \mathbb{R}$  such that  $-\sup(-S) \leq \lambda \leq s$  which implies  $-s \leq -\lambda \leq \sup(-S)$ . Since,  $-\lambda$  cannot be the supremum of  $S$ .

Therefore,  $\boxed{\inf(S) = -\sup(-S)}$ . ■

#### Theorem 1.1.1: Archimedean Property

If  $a \geq 0$  and  $b \geq 0$ , then  $\exists n \in \mathbb{N}$  such that  $na \geq b$ . Or in other words we can say that the set of natural numbers is not bounded above.

**Proof.** Assume the Archimedean property fails. Then there exist  $a \geq 0$  and  $b \geq 0$  such that  $na \leq b \forall n \in \mathbb{N}$ . In particular,  $b$  is an upper bound for the set  $S = \{na \mid n \in \mathbb{N}\}$ . Let  $s_0 = \sup S$ ; this is where we are using the completeness

axiom. Since  $a \geq 0$ , we have  $s_0 \leq s_0 + a$ , so  $s_0 - a \leq s_0$ . Since  $s_0$  is the least upper bound for  $S$ ,  $s_0 - a$  cannot be an upper bound for  $S$ . It follows that  $s_0 - a \leq n_0 a$  for some  $n_0 \in \mathbb{N}$ . This implies  $s_0 \leq (n_0 + 1)a$ . Since  $(n_0 + 1)a$  is in  $S$ ,  $s_0$  is not an upper bound for  $S$  and we have reached a contradiction. Our assumption that the Archimedean property fails was wrong. ■

# Chapter 2

## Sequences

### §2.1 Convergence

#### Definition 2.1.1: Convergence

A sequence  $\{s_n\}$  is said to *converge* to a real number “s” provided that for each  $\epsilon > 0$  there exist as number N such that

$$n > N \text{ implies } |s_n - s| < \epsilon \quad (2.1)$$

If  $(s_n)$  converges to s, we will write  $\lim_{n \rightarrow \infty} s_n = s$ , or  $s_n \rightarrow s$ . The number s is called the limit of the sequence  $(s_n)$ . A sequence that does not converge to some real number is said to diverge (Is it?)

**Problem 1.** Let  $(s_n)$  be a sequence of nonnegative real numbers and suppose  $s = \lim s_n$ . Note  $s \geq 0$ . Prove  $\lim \sqrt[n]{s_n} = \sqrt[n]{s}$

**Solution. Case 1:** ( $s > 0$ ) Let  $\epsilon > 0$ . Since  $\lim s_n = s \Rightarrow \exists N$  such that

$$n > N \text{ implies } |s_n - s| < \epsilon \quad (2.2)$$

Now,  $n > N$  implies

$$|\sqrt{s_n} - \sqrt{s}| = \frac{|s_n - s|}{\sqrt{s_n} + \sqrt{s}} \leq \frac{|s_n - s|}{\sqrt{s}} < \frac{\sqrt{s}\epsilon}{\sqrt{s}} = \epsilon$$

**Case 2:** ( $s = 0$ ) Since  $s_n > 0 \Rightarrow |s_n - 0| < \epsilon \forall \epsilon > 0$ . Take  $\epsilon$  to be  $\epsilon^2$ .

$$\begin{aligned} \Rightarrow n > N \text{ implies } |s_n| < \epsilon^2 &\Rightarrow |\sqrt{s_n}| < \epsilon \\ &\Rightarrow |\sqrt{s_n} - 0| < \epsilon \end{aligned}$$

So,  $\lim \sqrt{s_n} = 0$  ■

**Problem 2.** Prove that:

$$1. \lim[\sqrt{n^2 + 1} - n] = 0$$

$$2. \lim[\sqrt{4n^2 + n} - 2n] = 1/4$$

**Solution.** I think it's enough to discuss the strategy because the reader should be able to proceed backwards.

1. Since,  $\sqrt{n^2 + 1} - n > 0$  we can simply remove the modulus sign and write  $\sqrt{n^2 + 1} - n < \epsilon$

$$\begin{aligned} &\Rightarrow n^2 + 1 < (\epsilon + n)^2 \\ &\Rightarrow n^2 + 1 < \epsilon^2 + n^2 + 2n\epsilon \\ &\Rightarrow \frac{1 - \epsilon^2}{2\epsilon} < n \end{aligned}$$

So, take your  $N = \frac{1-\epsilon^2}{2\epsilon}$  and proceed backwards

2. We can show that  $\frac{1}{4} > \sqrt{4n^2 + n} - 2n$  by simply assuming the inequality and we'll get the result that  $1/4 > 0$  which is indeed true or if we assume other inequality it'll lead to the contradiction.

Since,  $\frac{1}{4} - (\sqrt{4n^2 + n} - 2n) > 0$  we can write

$$\begin{aligned} &\Rightarrow \frac{1}{4} - (\sqrt{4n^2 + n} - 2n) < \epsilon \\ &\Rightarrow \frac{1}{4} - \epsilon + 2n < \sqrt{4n^2 + n} \end{aligned}$$

To square both the sides  $\frac{1}{4} - \epsilon + 2n > 0, \forall n \in \mathbb{N}$  which is if  $\epsilon < 9/4$ . Squaring both sides and cancelling the similar terms we get

$$\Rightarrow (\epsilon - 1/4)^2 < n(1 - 4(\epsilon - 1/4)^2)$$

Divide both side by  $(1 - 4(\epsilon - 1/4)^2)$  for that  $1 - 4(\epsilon - 1/4)^2 > 0$  or  $\epsilon < 3/4$  which also satisfies the above condition of  $\epsilon < 9/4$ . Since, we have to prove it for small enough  $\epsilon$ . So, for bigger epsilon it's automatically true i.e. if we prove for  $\epsilon < 3/4$  then it is true for  $\epsilon > 9/4$ . So,

$$\Rightarrow \frac{(\epsilon - 1/4)^2}{(1 - 4(\epsilon - 1/4)^2)} < n$$

Hence, to write a formal proof take your  $N = \frac{(\epsilon - 1/4)^2}{(1 - 4(\epsilon - 1/4)^2)}$

■

## §2.2 Limit Theorem for Sequences

### Definition 2.2.1

A sequence  $(s_n)$  is said to be bounded if there exist a real number  $M$ , such that  $|s_n| \leq M, \forall n \in \mathbb{N}$

Geometrically, this means that we can find an interval  $[-M, M]$  that contains every term in the sequence  $(x_n)$ .

**Theorem 2.2.1**

Convergent sequences are bounded

**Proof.** Let  $(s_n)$  be a convergent sequence of real numbers, let  $\lim s_n = s$ . Applying 2.1.1 with  $\epsilon = 1$  we obtain  $N$  in  $\mathbb{N}$  such that

$$\Rightarrow n > N \text{ implies } |s_n - s| < 1$$

From triangular inequality it implies  $|s_n| < |s| + 1$  for  $n > N$ . Take  $M = \max\{|s_1|, |s_2|, |s_3|, \dots, |s_N| + 1\}$ . Then we have  $|s_n| \leq M$  for all  $n \in \mathbb{N}$ , so  $(s_n)$  is bounded.

The choice of  $\epsilon$  is arbitrary ■

**Theorem 2.2.2**

If the sequence  $(s_n)$  converges to  $s$  and  $k$  is in  $\mathbb{R}$ , then the sequence  $(ks_n)$  converges to  $(ks)$ . That is,  $\lim(ks_n) = k \cdot \lim s_n$ .

**Theorem 2.2.3**

If  $s_n$  converges to  $s$  and  $t_n$  converges to  $t$ , then  $(s_n + t_n)$  converges to  $(s + t)$

$$\lim(s_n + t_n) = (s + t)$$

**Theorem 2.2.4**

If  $s_n$  converges to  $s$  and  $t_n$  converges to  $t$ , then  $(s_n \cdot t_n)$  converges to  $(s \cdot t)$

$$\lim(s_n \cdot t_n) = (s \cdot t)$$

The theorem can be proved using the identity  $(a + b)^2 = a^2 + b^2 + 2ab$  take  $(s_n + t_n)$  and  $(s_n - t_n)$  and proceed. (Wait did I prove  $\lim(a_n)^2 = a^2$  if  $a_n \rightarrow a$ . Well, if not then it's easy to prove.)

**Theorem 2.2.5**

If  $s_n$  converges to  $s$ . Then  $1/s_n$  converges to  $1/s$  for  $(s \neq 0)$

**Proof.** We begin by observing that

$$\left| \frac{1}{s_n} - \frac{1}{s} \right| = \frac{|s_n - s|}{|s_n s|}$$

Because  $(s_n) \rightarrow s$ , we can make the preceding numerator as small as we like by choosing  $n$  large. The problem comes in that we need a worst-case estimate on the size of  $1/(|s||s_n|)$ . Because the  $s_n$  terms are in the denominator, we are no longer interested in an upper bound on  $|s_n|$  but rather in an inequality of the form  $|s_n| \geq \delta > 0$ . This will then lead to a bound on the size of  $1/(|s||s_n|)$ . The trick is to look far enough out into the sequence  $(s_n)$  so that the terms are closer to  $s$  than

they are to 0. Consider the particular value  $\epsilon = |s|/2$ . Because  $(s_n) \rightarrow s$ , there exists an  $N_1$  such that  $|s_n - s| < |s|/2$  for all  $n \geq N_1$ . This implies  $|s_n| > |s|/2$ . Next, choose  $N_2$  so that  $n \geq N_2$  implies  $|s_n - s| < |s|$

$$|s_n - s| < \frac{\epsilon \cdot s^2}{2}$$

Finally, if we let  $N = \max\{N_1, N_2\}$ , then  $n \geq N$  implies.

$$\left| \frac{1}{s_n} - \frac{1}{s} \right| = |s_n - s| \frac{1}{|s_n s|} < \frac{\epsilon s^2}{2} \frac{1}{|s| \frac{|s|}{2}} = \epsilon$$

■

### Theorem 2.2.6

If  $s_n$  converges to  $s$  and  $t_n$  converges to  $t$  ( $t \neq 0$ ), then  $\frac{s_n}{t_n}$  converges to  $\left(\frac{s}{t}\right)$

$$\lim \frac{s_n}{t_n} = \frac{s}{t}$$

**Proof.** The proof is trivial and is left as an exercise for the readers ■

### Theorem 2.2.7

If  $s_n < t_n$  then  $\lim s_n \leq \lim t_n$ .

**Proof.** Let  $s = \lim s_n$  and  $t = \lim t_n$ . Let  $h_n = t_n - s_n > 0$ . So,  $h = \lim h_n \geq 0$  (Why?). Let's take  $h < 0 \Rightarrow -h > 0$ . So, there exists a  $N$  such that  $n > N$  implies  $|h_n - h| < -h \Rightarrow h_n < 0$ . Contradiction! So,  $h \geq 0 \Rightarrow t - s \geq 0$ . ■

### Theorem 2.2.8: (Basic Examples)

- a.  $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$  for  $p > 0$ .
- b.  $\lim_{n \rightarrow \infty} a^n = 0$  if  $|a| < 1$ .
- c.  $\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1$ .
- d.  $\lim_{n \rightarrow \infty} a^{\frac{1}{n}} = 1$ ,  $a > 0$ .

**Problem 3.** For a sequence  $(s_n)$  of positive real numbers, we have  $\lim s_n = +\infty$  if and only if  $\lim\left(\frac{1}{s_n}\right) = 0$ .

**Proof.** We need to prove that

$$\lim s_n = +\infty \Rightarrow \lim(1/s_n) = 0$$

and

$$\lim(1/s_n) = 0 \Rightarrow \lim s_n = +\infty$$



1. Since  $\lim s_n = +\infty$  so for every  $n > N$  there exist a  $M$  such that  $s_n > M$ .  
Take  $M = 1/\epsilon$ ,  $\epsilon > 0$

$$\Rightarrow s_n > M = 1/\epsilon$$

$$\Rightarrow \epsilon > \frac{1}{s_n} > 0$$

$$\Rightarrow \left| \frac{1}{s_n} - 0 \right| < \epsilon$$

2. Workout the above proof backwards by assuming  $\epsilon = 1/M$

■

**Problem 4.** Let  $s_1 = 1$  and  $s_{n+1} = \sqrt{1 + s_n}$ .

Assume that the sequence converge. Prove that the sequence converges to  $\frac{1}{2}(1 + \sqrt{5})$

**Proof.** Let the sequence converge to  $s$ . So, as  $n \rightarrow \infty$   $s_{n+1} = s_n = s$  therefore

$$\Rightarrow s_{n+1} = \sqrt{1 + s_n}$$

$$\Rightarrow s_{n+1}^2 - s_n - 1 = 0$$

$$\Rightarrow s^2 - s - 1 = 0 \text{ as } n \rightarrow \infty$$

$$\text{So, } s = \frac{1}{2}(1 + \sqrt{5})$$

In this problem we have assumed that the limit exist

■

### Theorem 2.2.9

Let  $\{a_n\}$  be a sequence of positive numbers such that  $\lim_{n \rightarrow \infty} a_n = L$ . Prove that  $\lim_{n \rightarrow \infty} \sqrt[n]{a_1 a_2 \cdots a_n} = L$

### Theorem 2.2.10

If  $\lim \left| \frac{a_{n+1}}{a_n} \right|$  exists [and equals  $L$ ], then  $\lim (a_n)^{\frac{1}{n}}$  exists [ and equals  $L$  ].  
Also, deduce  $\lim_{n \rightarrow \infty} \frac{n^n}{(n!)^{\frac{1}{n}}}$

**Proof.** Define the sequence  $\{b_n\}$  by  $b_1 = a_1$  and  $b_n = \frac{a_n}{a_{n-1}}$  for  $n \geq 2$ . Since  $\lim_{n \rightarrow \infty} \frac{a_n}{a_{n-1}} = L$ , we have  $\lim_{n \rightarrow \infty} b_n = L$ . Note that  $a_n = (b_1 b_2 \cdots b_n)$ . Applying the above Theorem to the sequence  $b_n$ , we get :

$$\lim_{n \rightarrow \infty} (a_n)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} (b_1 b_2 \cdots b_n)^{\frac{1}{n}} = L$$

Now, let  $a_n = \frac{n^n}{n!}$ . Note that

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \lim_{n \rightarrow \infty} \frac{(n+1)^{n+1}}{n^n} = \lim_{n \rightarrow \infty} \left( \frac{n+1}{n} \right)^n = \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^n = e$$

By the conclusion above, we have:

$$\lim_{n \rightarrow \infty} \frac{n^n}{(n!)^{\frac{1}{n}}} = \lim_{n \rightarrow \infty} a_n^{\frac{1}{n}} = e$$

■

## §2.3 Monotone and Cauchy Sequences

### Theorem 2.3.1

All bounded monotone sequences are convergent

**Proof.** I'll prove it for monotonically decreasing sequences

Let  $(s_n)$  be a bounded decreasing sequence i.e.  $\forall n \in \mathbb{N} [s_{n+1} < s_n]$  &  $[s_n] < M]$  for some  $M \in \mathbb{R}$ .

Since,  $|s_n| < M \Rightarrow -M < s_n < M$ . Applying Corollary 1.1.1 there exist a  $\lambda \in \mathbb{R}$  such that  $-M < \lambda < s_n$  for all  $n \in \mathbb{N}$ . Since,  $\lambda$  is the greatest lower-bound, therefore for any  $\epsilon > 0$  there exist a  $N$  such that  $s_N < \lambda + \epsilon$ . Also, the sequence is decreasing so  $\lambda - \epsilon < \lambda \leq s_{n+1} \leq s_n \leq \lambda < \lambda + \epsilon$

$$\Rightarrow -\epsilon < s_n - \lambda < \epsilon$$

$$\Rightarrow |s_n - \lambda| < \epsilon$$

So,  $\inf(s_n) = \lim s_n$  in case of decreasing sequences

■

### Definition 2.3.1: $\limsup' s$ & $\liminf' s$

For some sequence  $s_n$  we define:

$$\limsup s_n = \lim_{N \rightarrow \infty} \sup\{s_n : n > N\}$$

and

$$\liminf s_n = \lim_{N \rightarrow \infty} \inf\{s_n : n > N\}$$

**Problem 5.** Calculate the  $\limsup a_n$  and  $\liminf a_n$  for  $a_n = (-1)^n \frac{(n+5)}{n}$

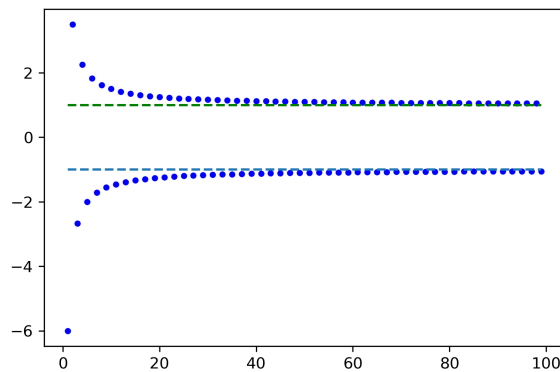


Figure 2.1:  $(-1)^n \frac{(n+5)}{n}$  v/s  $n$

**Solution.** Since in the set of subsequences of  $a_n$  the least element will be the  $\liminf$  and the greatest term would be  $\limsup$  so. Since the -ve values are less than the +ve ones and in case of  $-1 - \frac{5}{n}$  the least value for  $n$  gives the least of all so, we can say the first negative term of each subsequence will be the element of subsequential infimum. Therefore,  $\lim(-1 - \frac{5}{n}) = -1$  is the  $\liminf$ . Similar argument for  $\limsup$  as the greatest of all will be sup of each subsequence and the least value of  $n$  will give the largest of all. So, the sequence  $s_n = \frac{(n+5)}{n}$  will be sequence of subsequential supremum. And  $\lim \frac{(n+5)}{n} = 1$ . ■

### Theorem 2.3.2

Let  $(s_n)$  be a sequence  $\mathbb{R}$

(i) If  $\lim s_n$  is defined then:

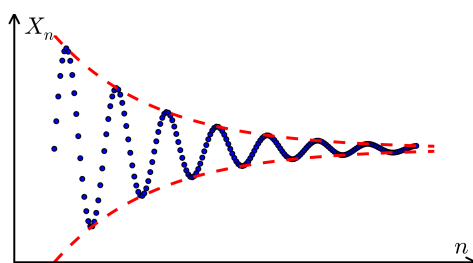
$$\liminf s_n = \limsup s_n = \lim s_n$$

(ii) If  $\liminf s_n = \limsup s_n$  then  $\lim s_n$  is defined and  $\liminf s_n = \limsup s_n = \lim s_n$

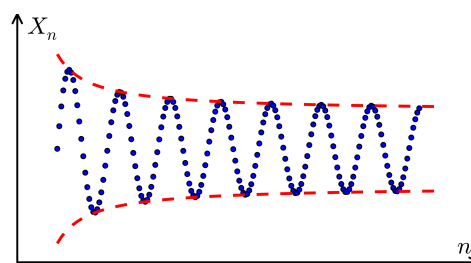
### Definition 2.3.2: CAUCHY SEQUENCE

[4] A sequence  $(s_n)$  of real numbers is said to be Cauchy if:  
for each  $\epsilon > 0$  there exist a  $N$  such :

$$\forall n, m > N \text{ implies } |s_n - s_m| < \epsilon$$



(a) The plot of a Cauchy sequence  $(x_n)$ , shown in blue, as  $(x_n)$  versus  $n$



(b) A sequence that is not Cauchy. But is bounded

Figure 2.2: Illustration for Cauchy sequences

### Lemma 2.3.1

Convergent sequence are Cauchy

### Lemma 2.3.2

Cauchy sequence are bounded

### Theorem 2.3.3

A sequence is a convergent sequence if and only if it is a Cauchy sequence.

**Problem 6.** Prove that  $|s_{n+1} - s_n| < 2^{-n}$  is cauchy. And hence convergent.

**Proof.** We've to prove that  $\forall \epsilon > 0$  there exist a  $N$  such that  $\forall n, m > N$ ,  $|s_m - s_n| < \epsilon$ .

Take  $m > n$  and let  $m = n + k$ . So, by triangular inequality

$$|s_m - s_n| \leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \dots + |s_{n+1} - s_n| < \frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots + \frac{1}{2^m}$$

Since,  $m = n+k$  and if we add more +ve terms then we will get a G.P. with ratio  $\frac{1}{2}$  which look like this:

$$\frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots + \frac{1}{2^m} = \frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots + \frac{1}{2^{n+k}} = \frac{1}{2^{n-1}} \left(1 - \frac{1}{2^k}\right)$$

So,

$$|s_m - s_n| \leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| + \dots + |s_{n+1} - s_n| < \frac{1}{2^{n-1}} \left(1 - \frac{1}{2^k}\right) < \frac{1}{2^{n-1}}$$

And applying Theorem 2.2.8, we can say  $\frac{1}{2^{n-1}}$  is convergent. Hence, the same  $N$  will work for the sequence  $s_n$ . ■

We have not talked about Series / summation till now. Series, will be discussed in the next Section(3)

**Problem 7.** Let  $s_n$  be an increasing sequence. Prove that  $\sigma_n = \frac{1}{n}(s_1 + s_2 + \dots + s_n)$  is an increasing sequence.

**Proof.** Since, since  $\sigma_1 < \sigma_2$ . Let us suppose  $\sigma_n > \sigma_{n-1}$ . Now, we have to prove that it is true for the term  $\sigma_{n+1}$ .

Suppose, it's not true i.e.  $\sigma_{n+1} < \sigma_n$

$$\frac{1}{n+1}(s_1 + s_2 + \dots + s_{n+1}) < \frac{1}{n}(s_1 + s_2 + \dots + s_n)$$

If we further simplify the inequality, we get  $n s_{n+1} < (s_1 + s_2 + \dots + s_n)$  which is false. So, the assumption was false. Hence,  $\sigma_{n+1} > \sigma_n$ . ■

**Problem 8.** Define  $x_1 = 2$  and:

$$x_{n+1} = \frac{1}{2} \left( x_n + \frac{2}{x_n} \right)$$

Prove that the  $\lim_{n \rightarrow \infty} x_n = \sqrt{2}$ .

**Proof.** Observe that  $x_2 < x_1 = 2$ . So, assume  $x_{n+1} < x_n < x_{n-1} \dots < x_1 = 2$ . Suppose  $x_{n+2} > x_{n+1}$  (it would lead to a contradiction), it implies:

$$\begin{aligned} \Rightarrow \frac{1}{2} \left( x_{n+1} + \frac{2}{x_{n+1}} \right) &> x_{n+1} \\ \Rightarrow (x_{n+1})^2 &< 2 \end{aligned}$$

Substitute  $x_{n+1} = \frac{1}{2} \left( x_n + \frac{2}{x_n} \right)$  and solve. We get:

$$(x_n^2 - 2)^2 < 0$$

which is a contradiction. So, the series is monotonically decreasing and bounded (because each element is greater than zero and less than two). Hence, convergent. To prove  $\lim_{n \rightarrow \infty} x_n = \sqrt{2}$ , put  $x_{n+1} = x_n = x$  in the definition. ( $\because \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} x_{n+1}$ ) ■

## §2.4 Subsequences

**An informal definition of subsequence:** A **Subsequence** is a sequence that can be derived from another sequence by deleting some or no elements without changing the order of the remaining elements

### Theorem 2.4.1

If the sequence  $(s_n)$  converges, then every subsequence converges to the same limit.

### Theorem 2.4.2

Every sequence  $(s_n)$  has a monotonic subsequence.

### Theorem 2.4.3: Bolzano-Weierstrass Theorem

[5] Every bounded sequence has a convergent subsequence.

**Proof.** It's easy to prove that every bounded sequence has a convergent subsequence. Since, every sequence has a monotonic subsequence and since the sequence is bounded implies subsequence is bounded. And every bounded monotonic sequence is convergent. ■

### §2.4.1 Subsequential Limits

#### Definition 2.4.1

Let  $(s_n)$  be a sequence in  $\mathbb{R}$ . A subsequential limit is any real number or symbol  $+\infty$  or  $-\infty$  that is the limit of some subsequence of  $(s_n)$ .

When a sequence has a limit  $s$ , then all subsequences have limit  $s$ , so  $\{s\}$  is the set of subsequential limits

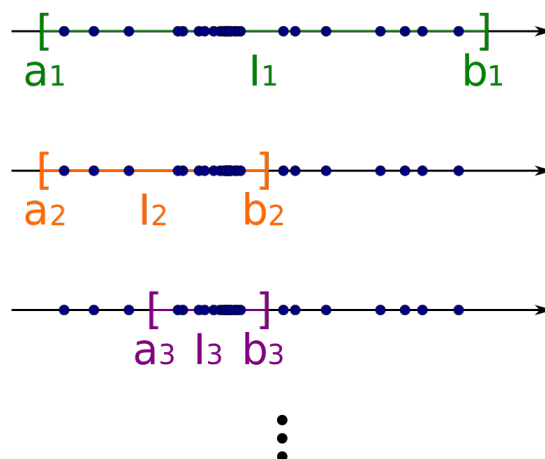


Figure 2.3: Bolzano-Weierstrass Theorem

**Theorem 2.4.4**

Let  $(s_n)$  be any sequence. There exists a monotonic subsequence whose limit is  $\limsup s_n$ , and there exists a monotonic subsequence whose limit is  $\liminf s_n$ .

**Recall** Let  $(s_n)$  be any sequence of real numbers, and let  $S$  be the set of subsequential limits of  $(s_n)$ . Recall

$$\liminf s_n = \lim_{N \rightarrow \infty} \inf\{s_n : n > N\} = \inf S$$

and

$$\limsup s_n = \lim_{N \rightarrow \infty} \sup\{s_n : n > N\} = \sup S$$

# Chapter 3

## Series

### §3.1 Sum to infinity?

#### Definition 3.1.1: Summation Notation

$$\sum_n^m a_k = a_n + a_{n+1} + \dots + a_m$$

2. To assign meaning to  $\sum_{n=m}^{\infty} a_n$ , we consider the sequences  $(s_n)_{n=m}^{\infty}$  of partial sums:

$$s_n = a_m + a_{m+1} + \dots + a_n = \sum_{k=m}^n a_k$$

The infinite series  $\sum_{n=m}^{\infty} a_n$  is said to converge provided the sequence  $(s_n)$  of partial sums converges to a real number  $S$ , in which case we define

$$\sum_{n=m}^{\infty} a_n = S$$

#### Definition 3.1.2: Cauchy Criterion for Series Convergence

We say a series  $\sum a_n$  satisfies the Cauchy criterion if its sequence  $(s_n)$  of partial sums is a Cauchy sequence i.e.:

for each  $\epsilon > 0$  there exists a number  $N$  such that:

$$n \geq m > N \text{ implies } |s_n - s_{m-1}| < \epsilon$$

And,  $s_n - s_{m-1} = \sum_n^m a_k$

A series converges iff it satisfies Cauchy criterion

#### Corollary 3.1.1

If  $\sum a_n$  converges then  $\lim a_n = 0$

### §3.2 Convergence Tests for Series

**§Comparison Test** Let  $\sum a_n$  be a series where  $a_n \geq 0$  for all  $n$ .

1. If  $\sum a_n$  converges and  $|b_n| \leq a_n$  for all  $n$ , then  $\sum b_n$  converges.
2. If  $\sum a_n = +\infty$  and  $b_n \geq a_n$  for all  $n$ , then  $\sum b_n = +\infty$

**Problem 9.** Show that the series  $s_n = \sum \frac{1}{n^2}$  converges

**Solution.** Observation:

$$\begin{aligned} \Rightarrow \frac{1}{n^2} &< \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n} \\ \Rightarrow \sum_2^n \frac{1}{n^2} &< \sum_2^n \left( \frac{1}{n-1} - \frac{1}{n} \right) \end{aligned}$$

Or

$$\begin{aligned} \Rightarrow 1 + \sum_2^n \frac{1}{n^2} &< 1 + \sum_2^n \left( \frac{1}{n-1} - \frac{1}{n} \right) \\ \Rightarrow \sum_2^n \frac{1}{n^2} &< 2 - \frac{1}{n} \end{aligned}$$

As  $n \rightarrow \infty$  we get:

$$\sum_2^\infty \frac{1}{n^2} < 2$$

Hence, it converges. ■

**§ Ratio Test:** [6] A series  $\sum a_n$  of nonzero terms. The usual form of the test makes use of the limit:

$$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

- (i) if  $L < 1$  then the series converges absolutely;
- (ii) if  $L > 1$  then the series diverges;
- (iii) if  $L = 1$  or the limit fails to exist, then the test is inconclusive, because there exist both convergent and divergent series that satisfy this case.

It is possible to make the ratio test applicable to certain cases where the limit  $L$  fails to exist, if limit superior and limit inferior are used. The test criteria can also be refined so that the test is sometimes conclusive even when  $L = 1$ . More specifically, let

- $R = \limsup \left| \frac{a_{n+1}}{a_n} \right|$
- $r = \liminf \left| \frac{a_{n+1}}{a_n} \right|$

Then the ratio test states that:

- if  $R < 1$ , the series converges absolutely;
- if  $r > 1$ , the series diverges;
- if  $\left| \frac{a_{n+1}}{a_n} \right| \geq 1$  for all large  $n$  (regardless of the value of  $r$ ), the series also diverges; this is because  $|a_n|$  is nonzero and increasing and hence  $a_n$  does not approach zero;
- the test is otherwise inconclusive



**§ Root Test:** [7] Let  $\sum a_n$  be a series and let  $\alpha = \limsup |a_n|^{1/n}$ . The series  $\sum a_n$

- (i) converges absolutely if  $\alpha < 1$ .
- (ii) diverges if  $\alpha > 1$ .
- (iii) Otherwise  $\alpha = 1$  and the test gives no information

Note that if:

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}$$

converges then it equals  $\alpha$  and may be used in the root test instead.

**Lemma 3.2.1**

Let  $|s_n|$  be a sequence of non-zero real numbers,  
Then,

$$\liminf \left| \frac{s_{n+1}}{s_n} \right| \leq \liminf \left| |s_n|^{\frac{1}{n}} \right| \leq \limsup \left| |s_n|^{\frac{1}{n}} \right| \leq \limsup \left| \frac{s_{n+1}}{s_n} \right|$$

**Problem 10.** Prove that  $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$

**Solution.** a. Given:

$$\begin{aligned} \Rightarrow a_n &= \frac{x^n}{n!} \\ \Rightarrow a_{n+1} &= \frac{x^{n+1}}{n+1!} \\ \Rightarrow \frac{a_{n+1}}{a_n} &= \frac{x}{n+1} \end{aligned}$$

As  $n \rightarrow \infty$  ratio  $\frac{a_{n+1}}{a_n} \rightarrow 0$ . Thus it converges<sup>1</sup>. That means the series  $\sum \frac{x^n}{n!}$  converges therefore  $\frac{x^n}{n!}$  converges to zero.

b. The series  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  converges. Hence  $\frac{x^n}{n!} \rightarrow 0$ .

c. [MathStackexchange](#)

■

**Problem 11.** Let  $\limsup |a_n| > 0$ . Then prove that  $\limsup |a_n|^{\frac{1}{n}} \geq 1$ .

**Proof.** Assume that  $\limsup |a_n| > 0$  but  $\limsup |a_n|^{\frac{1}{n}} < 1$ . We also conclude that  $\limsup |a_n|^{\frac{1}{n}} < 1$  implies  $\sum a_n$  converges absolutely (by Root test). So,  $\lim |a_n| = \limsup |a_n| = 0$ . Contradiction!. So,  $\limsup |a_n|^{\frac{1}{n}} \geq 1$ . ■

**Theorem 3.2.1**

Let  $\sum |a_n|$  be a convergent series & let  $(b_n)$  be a bounded sequence. Then,  $\sum a_n b_n$  is also convergent.

**Proof.** By triangular inequality we can show that:

$$\left| \sum_{k=m}^n a_k b_k \right| \leq \sum_{k=m}^n |a_k b_k|$$

Given,  $|b_n| \leq M$  (it's bounded), implies:

$$\Rightarrow |a_k| |b_k| \leq |a_k| M$$

$$\Rightarrow \sum |a_k| |b_k| \leq \sum |a_k| M$$

Since,  $\sum a_n$  converges; therefore by cauchy criterion  $\exists N \in \mathbb{N}$  such that  $\forall n \geq m > N$ :

$$\begin{aligned} \Rightarrow \sum_{k=m}^n |a_k| &< \frac{\epsilon}{M} \\ \Rightarrow \sum_{k=m}^n M |a_k| &< \epsilon \end{aligned}$$

Or

$$\Rightarrow \left| \sum_{k=m}^n a_k b_k \right| \leq \sum |a_k| |b_k| \leq \sum_{k=m}^n M |a_k| < \epsilon$$

Hence,  $\sum a_n b_n$  is convergent by comparison test. ■

**Corollary 3.2.1**

Let  $a_n \geq 0$  &  $\sum a_n$  converges. Then  $\sum (a_n)^p$  converges  $\forall p > 1$

**Proof.** The above expression can be rewritten as:

$$\sum_{k=m}^n |a_k|^p = \sum_{k=m}^n |a_k| |a_k|^{p-1}$$

Since,  $\sum a_n$  converges, therefore  $a_n \rightarrow 0$ . So, sequence  $a_n$  is convergent, hence bounded. So,  $a_k^{p-1}$  is bounded.

$\therefore$  By previous theorem, we can say  $\sum (a_k)^p$  converges. ■

### §3.3 Alternating Series Test

**Theorem 3.3.1**

If  $a_1 \geq a_2 \geq \dots \geq a_n \geq \dots \geq 0$  and  $\lim a_n = 0$ , then the alternating series  $\sum (-1)^{n+1} a_n$  converges. Moreover, the partial sums  $s_n = \sum_{k=1}^n (-1)^{k+1} a_k$  satisfy  $|s - s_n| \leq a_n$  for all  $n$ .

**Proof.** We need to show that the sequence  $(s_n)$  converges. Note that the subsequence  $(s_{2n})$  is increasing because  $s_{2n+2} - s_{2n} = -a_{2n+2} + a_{2n+1} \geq 0$ . Similarly, the subsequence  $(s_{2n-1})$  is decreasing since  $s_{2n+1} - s_{2n-1} = a_{2n+1} - a_{2n} \leq 0$ . We claim:

$$s_{2m} \leq s_{2n+1} \quad \text{for all } m, n \in \mathbb{N}$$

First note that  $s_{2n} \leq s_{2n+1}$  for all  $n$ , because  $s_{2n+1} - s_{2n} = a_{2n+1} \geq 0$ . If  $m \leq n$ , then the above equation holds because  $s_{2m} \leq s_{2n} \leq s_{2n+1}$ . If  $m \geq n$ , then equation holds because  $s_{2n+1} \geq s_{2m+1} \geq s_{2m}$ . We see that  $(s_{2n})$  is an increasing subsequence of  $(s_n)$  bounded above by each odd partial sum, and  $(s_{2n+1})$  is a decreasing subsequence of  $(s_n)$  bounded below by each even partial sum. By Theorem (2.3.1), these subsequences converge, say to  $s$  and  $t$ . Now

$$t - s = \lim_{n \rightarrow \infty} s_{2n+1} - \lim_{n \rightarrow \infty} s_{2n} = \lim_{n \rightarrow \infty} (s_{2n+1} - s_{2n}) = \lim_{n \rightarrow \infty} a_{2n+1} = 0$$

. so  $s = t$ , follows that  $\lim_n s_n = s$ .

To check the last claim, note that  $s_{2k} \leq s \leq s_{2k+1}$ , so both  $s_{2k+1} - s$  and  $s - s_{2k}$  are clearly bounded by  $s_{2k+1} - s_{2k} = a_{2k+1} \leq a_{2k}$ . So, whether  $n$  is even or odd, we have  $|s - s_n| \leq a_n$ . ■

### §3.4 Integral Test

#### Theorem 3.4.1

Consider an integer  $N$  and a non-negative function  $f$  defined on the unbounded interval  $[N, \infty)$ , on which it is monotone decreasing. Then the infinite series:

$$\sum_n^{\infty} f(n)$$

converges to a real number if and only if the improper integral

$$\int_N^{\infty} f(x) dx$$

is finite. In other words, if the integral diverges, then the series diverges as well.

**Problem 12.** Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence such that  $\liminf |a_n| = 0$ . Prove there is a subsequence  $(a_{n_k})_{k \in \mathbb{N}}$  such that  $\sum_{k=1}^{\infty} (a_{n_k})$  converges.

**Proof.** We first set  $n_0 = 1$  and  $c_1 = 1$ . By the property of  $\liminf$ , there exists  $n_1 > n_0 = 1$  such that  $|a_{n_1}| < c_1 = 1$ . Why does such an  $n_1$  exist? For sake of contradiction, assume there is no  $n \in \mathbb{N}$  such that  $|a_n| < 1$ , then 1 would be a lower bound for  $\{a_n : n \in \mathbb{N}\}$ . In which case, for all  $N$ , 1 would be a lower bound for  $\{a_n : n > N\}$  so that  $1 \leq \inf\{a_n : n > N\}$ , in which case  $1 \leq \lim_{N \rightarrow \infty} \inf\{a_n : n > N\}$ , so we would get  $1 \leq 0$ . A contradiction.

Then we set  $c_2 = 1/4$ . Again, there exists  $n_2 > n_1$  such that  $|a_{n_2}| < c_2 = 1/4$ . Why does such an  $n_2$  exist? (the same argument can be used here as above)

We can continue this fashion. At the  $k$ -th step, we set  $c_k = 1/k^2$  and we can then find some  $n_k > n_{k-1}$  such that  $|a_{n_k}| < c_k = 1/k^2$ .

Note that

$$\sum_{k=1}^{\infty} |a_{n_k}| < \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Since the RHS converges, the LHS converges, i.e.,  $\sum_{k=1}^{\infty} a_{n_k}$  converges absolutely. ■

**Problem 13.** Prove if  $(a_n)$  is a decreasing sequence of real numbers and if  $\sum a_n$  converges, then  $\lim n \cdot a_n = 0$ .

**Proof.** Since it's given that  $\sum a_n$  converges. Therefore by cauchy criteria  $\forall \epsilon > 0$   $\exists N \in \mathbb{N}$  such that  $\forall n > m + 1 > N$ : (we'll be using the fact that for  $n > m$  we have  $a_n < a_m$ )

$$\sum_{m+1}^n a_k < \epsilon$$

$$(n - m)a_n \leq a_{m+1} + a_{m+2} \cdots + a_n < \epsilon$$

Hence,  $\lim_{n \rightarrow \infty} (n - m)a_n = 0$ . Further, we see that :

$$\lim_{n \rightarrow \infty} na_n = \lim_{n \rightarrow \infty} (n - m)a_n + \lim_{n \rightarrow \infty} ma_n = 0$$

we didn't use a mod because observe that all terms are positive

■

**Problem 14.** Show that  $\sum \frac{1}{n^p}$  converges iff  $p > 1$

**Proof.** 1. In particular, for  $p \leq 1$ , we can write  $\sum \frac{1}{n^p} = +\infty$ . For  $p = 2$  we have already proved it in Problem 9.

For  $p > 2$ , we can prove it by the comparison test as :

$$\left\{ \frac{1}{n^p} < \frac{1}{n^2} \right\}, \quad \forall \quad p > 0$$

Since,  $\frac{1}{n^2}$  converges. Then  $\frac{1}{n^p}$  converges by comparison test.

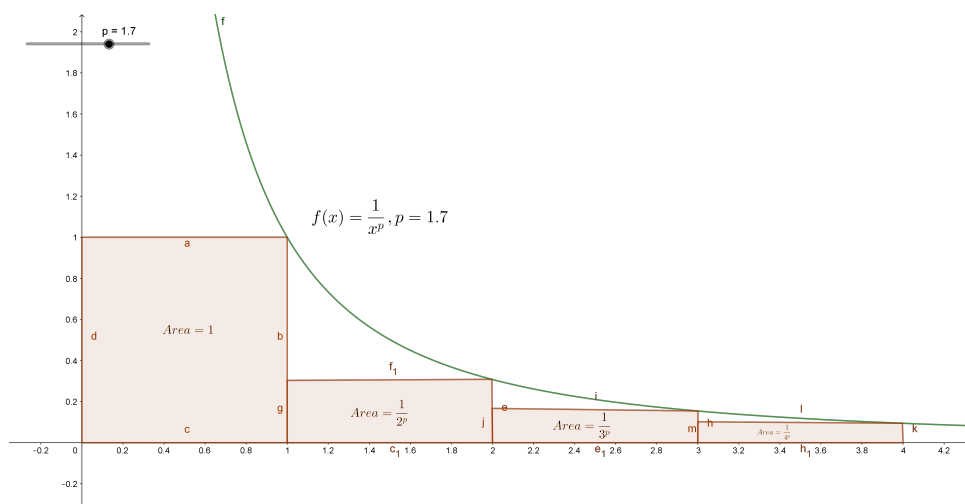
Or

The above proposition can be used to prove the result for  $p > 2$

For  $1 < p < 2$

We have,

$$\underbrace{\sum_{k=1}^n \frac{1}{k^p}}_{\text{sum of rectangles going from } 1 \rightarrow n} \leq \underbrace{1 + \int_1^n \frac{1}{x^p} dx}_{\text{area under the curve}}$$



2.

Figure 3.1: Geometric intuition for  $\sum \frac{1}{n^p}$ 

$$\begin{aligned} \Rightarrow \sum_{k=1}^n \frac{1}{k^p} &\leq 1 + \left[ \frac{x^{1-p}}{1-p} \right]_1^n \\ \Rightarrow \sum_{k=1}^n \frac{1}{k^p} &\leq 1 + \left[ \frac{n^{1-p}}{1-p} - \frac{1}{1-p} \right] \\ \Rightarrow \sum_{k=1}^n \frac{1}{k^p} &\leq \left[ \frac{p - n^{1-p}}{p-1} \right] \end{aligned}$$

As  $n \rightarrow \infty$  we get(  $\because 1-p < 0$ ),

$$\Rightarrow \sum_{k=1}^n \frac{1}{k^p} \leq \left[ \frac{p}{p-1} \right]$$

■

# Chapter 4

## Continuity

### §4.1 Continuity and Functions

#### Definition 4.1.1

A function  $f$  whose domain is defined over  $\mathbb{R}$  is said to be continuous at a point  $x_0 \in \text{dom}(f)$  iff:

for each  $\epsilon > 0$  there exist a  $\delta > 0$  such that:

$$x \in \text{dom}(f) \text{ and } |x - x_0| < \delta \text{ imply } |f(x) - f(x_0)| < \epsilon$$

.

Or

Let  $f$  be a real-valued function whose domain is a subset of  $\mathbb{R}$ . The function  $f$  is continuous at  $(x_0) \in \text{dom}(f)$  if, for every sequence  $(x_n) \in \text{dom}(f)$  converging to  $x_0$ , we have  $\lim_n f(x_n) = f(x_0)$  i.e.:

if for every sequence in domain if  $x_n \rightarrow x_0$  we have  $f(x_n) \rightarrow f(x_0)$

#### Theorem 4.1.1

Let  $f$  be a real-valued function with  $\text{dom}(f) \subseteq \mathbb{R}$ . If  $f$  is continuous at  $x_0$  in  $\text{dom}(f)$ , then  $|f|$  and  $kf$ ,  $k \in \mathbb{R}$ , are continuous at  $x_0$

#### §4.1.1 Properties of Continuous Functions

**§ Bounded Function** A real-valued function  $f$  is said to be bounded if  $\{f(x) : x \in \text{dom}(f)\}$  is a bounded set, i.e., if there exists a real number  $M$  such that  $|f(x)| \leq M$  for all  $x \in \text{dom}(f)$ .

#### Theorem 4.1.2

Let  $f$  be a continuous real-valued function on a closed interval  $[a, b]$ . Then  $f$  is a bounded function. Moreover,  $f$  assumes its maximum and minimum values

on  $[a, b]$ ; that is, there exist  $x_0, y_0 \in [a, b]$  such that  $f(x_0) \leq f(x) \leq f(y_0)$  for all  $x \in [a, b]$ .

### Theorem 4.1.3: Intermediate value theorem

If  $f$  is a continuous real-valued function on an interval  $I$ , then  $f$  has the intermediate value property on  $I$ : Whenever  $a, b \in I$ ,  $a < b$  and  $y$  lies between  $f(a)$  and  $f(b)$  i.e.  $[f(a) < y < f(b) \text{ or } f(b) < y < f(a)]$ , there exists at least one  $x \in (a, b)$  such that  $f(x) = y$ .

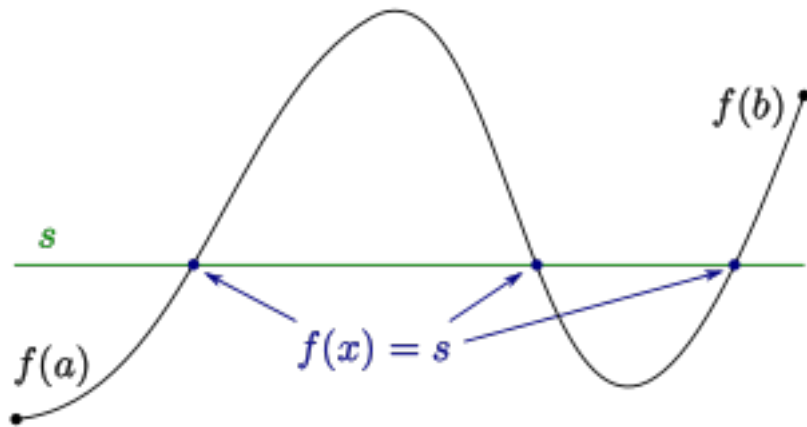


Figure 4.1: Intermediate value theorem

**Problem 15.** Prove that  $\lim_{x \rightarrow a} |x + 2| = |a + 2|$

**Proof.** Simply take,  $\delta = \epsilon$ . We have  $|x - a| < \delta$ .

$$||x + 2| - |a + 2|| \leq |x - a| < \delta = \epsilon \quad (\text{By triangular inequality})$$

■

**Problem 16.** Prove that  $\lim_{x \rightarrow a} (4 + x - 3x^2) = (4 + a - 3a^2)$

**Proof.** We need to prove that

$$|4 + x - 3x^2 - (4 + a - 3a^2)| = |(x - a) - 3(x - a)(x + a)| < \epsilon$$

Or

$$|x - a||1 - 3(x + a)| < \epsilon$$

Since, we are trying to find a  $\delta$  we can apply the triangular inequality on the above expression and we get.

$$|x - a||1 - 3(x + a)| \leq |x - a|(1 + 3|x + a|) < \epsilon$$

We need to find an upper bound for the above expression ( $|1 + 3(x + a)|$ ) which is not dependent on  $x$ .

If we choose  $|x - a| < \delta < \frac{\epsilon}{|1 + 3(x + a)|}$  then it would imply  $|x - a||1 + 3(x + a)| < \epsilon$ . But when we have one  $\delta > 0$  that works, any smaller value will also work. By choosing  $\delta < 1$  we would have:

$$|x - a| < 1 \quad \text{when ever} \quad |x - a| < \delta$$

$$\Rightarrow ||x| - |a|| \leq |x - a| < 1$$

Or

$$\Rightarrow |x| < 1 + |a|$$

$$\Rightarrow |x + a| \leq |x| + |a| < |2a| + 1$$

$$\Rightarrow [1 + 3|x + a|] < [1 + 3(1 + 2|a|)]$$

Since, we need both the above assumptions i.e.  $|x - a| < \delta$  &  $|x - a| < 1$  to be satisfied. So, take :

$$\delta = \min \left( 1, \frac{\epsilon}{1 + 3(1 + 2|a|)} \right)$$

And, whole discussion above proves that it would work. ■

**Problem 17.** Is  $\lim_{x \rightarrow 4} \frac{4x+1}{3x-4}$  is continuous for  $x \neq 4/3$ ?

**Problem 18.** Let  $f$  and  $g$  be continuous functions on  $[a, b]$  such that  $f(a) \geq g(a)$  and  $f(b) \leq g(b)$ . Prove  $f(x_0) = g(x_0)$  for at least one  $x_0$  in  $[a, b]$

**Proof.** Define  $h(x) = f(x) - g(x)$ . It's given that  $f(a) - g(a) \geq 0$  and  $f(b) - g(b) \leq 0$ . So, by intermediate value theorem there exists a  $x_0 \in [a, b]$ . Such that  $h(x_0) = 0$ . ■

**Problem 19.** Suppose  $f$  is continuous on  $[0, 2]$  and  $f(0) = f(2)$ . Prove there exist  $x, y \in [0, 2]$  such that  $|y - x| = 1$  and  $f(x) = f(y)$ .

**Proof.** Define  $g(x) = f(x + 1) - f(x)$ . Let  $|x - y| = 1$  implies either  $x = y + 1$  or  $y = x + 1$ . Take anyone of them, it won't matter as we can exchange  $y$  with  $x$  for another case.

Also,  $g(0) = f(1) - f(0)$  and  $g(1) = f(2) - f(1)$ . Adding both the equations  $g(0) + g(1) = 0$ , which implies that one of them is negative of other i.e. if any one of the value is +ve the other is -ve. So, there exists a value of  $x \in [0, 1]$  such that  $g(x) = 0$  or  $f(x + 1) = f(x) \equiv f(y) = f(x)$ . ■



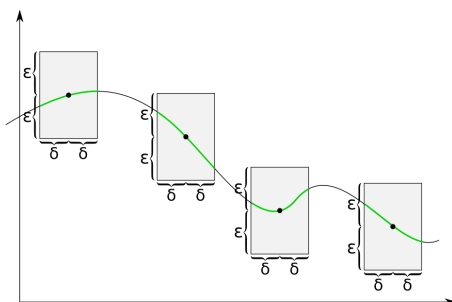
## §4.2 Uniform Continuity

### Definition 4.2.1: Uniform Continuity

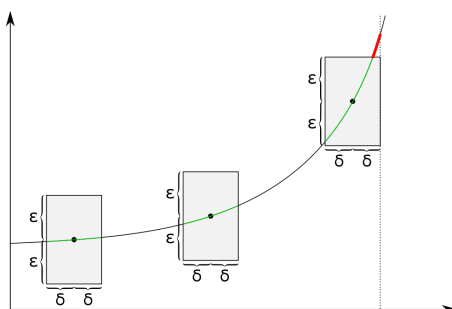
Let  $f$  be a real-valued function defined on a set  $S \subseteq \mathbb{R}$ . Then  $f$  is uniformly continuous on  $S$  if for each  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$x, y \in S \text{ and } |x - y| < \delta \text{ imply } |f(x) - f(y)| < \epsilon$$

We will say  $f$  is uniformly continuous if  $f$  is uniformly continuous on  $\text{dom}(f)$ .



(a) For uniformly continuous functions, there is for each  $\epsilon > 0$  a  $\delta > 0$  such that when we draw a rectangle around each point of the graph with width  $2\delta$  and height  $2\epsilon$ , the graph lies completely inside the rectangle.



(b) For functions that are not uniformly continuous, there is an  $\epsilon > 0$  such that regardless of the  $\delta > 0$  here are always points on the graph, when we draw a  $2\epsilon - 2\delta$  rectangle around it, there are values directly above or below the rectangle.

Figure 4.2: Uniform Continuity

If your function happens to satisfy  $0 < |f'(x)| < M$  for every  $x$ , then something like  $\delta = \epsilon/M$  will probably work to show uniform continuity.

### Theorem 4.2.1

If  $f$  is continuous on a closed interval  $[a, b]$ , then  $f$  is uniformly continuous on  $[a, b]$ .

**Theorem 4.2.2**

If  $f$  is uniformly continuous on a set  $S$  and  $(s_n)$  is a Cauchy sequence in  $S$ , then  $(f(s_n))$  is a Cauchy sequence.

**Problem 20.** Prove that if  $f$  is uniformly continuous on a bounded set  $S$ , then  $f$  is a bounded function on  $S$ .

**Proof.** Since, uniformly continuous functions are cauchy on a closed interval (and if the interval is not closed then we can close it! We have that power.; and cauchy sequences are bounded. ■

**Definition 4.2.2**

Let  $f$  be a function defined domain of  $f$ . Another function  $\tilde{f}$  is called an extension of  $f$  if.

$$\left\{ \begin{array}{l} \text{dom}(f) = \text{dom}(\tilde{f}) \\ f(x) = \tilde{f}(x) \end{array} \right\} \forall x \in \text{dom}(f)$$

Let  $f$  be defined on  $(a, b)$ , if  $f$  is uniformly continuous. Then,  $f$  can be extended to  $\tilde{f}$  on  $[a, b]$ .

**Theorem 4.2.3**

A real-valued function  $f$  on  $(a, b)$  is uniformly continuous on  $(a, b)$  if and only if it can be extended to a continuous function  $\tilde{f}$  on  $[a, b]$ .

**Theorem 4.2.4**

Let  $f$  be a continuous function on an interval  $I$  [ $I$  may be bounded or unbounded]. Let  $I^\circ$  be the interval obtained by removing from  $I$  any endpoints that happen to be in  $I$ . If  $f$  is differentiable on  $I^\circ$  and if  $f'$  is bounded on  $I^\circ$ , then  $f$  is uniformly continuous on  $I$ .

**Problem 21.** 1. Let  $f$  be a continuous real-valued function with domain  $(a, b)$ . Show that if  $f(r) = 0$  for each rational number  $r$  in  $(a, b)$ , then  $f(x) = 0$  for all  $x \in (a, b)$

2. Let  $f$  and  $g$  be continuous real-valued functions on  $(a, b)$  such that  $f(r) = g(r)$  for each rational number  $r$  in  $(a, b)$ . Prove  $f(x) = g(x)$  for all  $x \in (a, b)$

**Proof.** 1. Suppose, towards a contradiction, that there is an  $x \in [a, b]$  with .  
 As  $f$  is continuous, there is for every  $\epsilon = |f(x)|/2$  some  $\delta > 0$  such that for all  $x' \in [a, b]$  with  $|x' - x| < \delta$  it follows that  $|f(x) - f(x')| < \epsilon$ . As , there is however a rational  $x' \in [a, b]$  with  $|x - x'| < \delta$  . But now  $|f(x) - \underbrace{f(x')}_0| < |f(x)| < |f(x)|/2$  . Contradiction!

Or

Let  $x \in [a, b]$ , and let  $q_n$  be a sequence of rational numbers, such that  $q_n \rightarrow x$ . By continuity of  $f$ , we have:

$$f(x) = f(\lim_{n \rightarrow \infty} q_n) = \lim_{n \rightarrow \infty} f(q_n) = 0$$

Using the above result we can define a function  $f(x) - g(x)$ , and  $f(x) = g(x)$  or  $f(x) - g(x) = 0$  for all rationals. ■

**Problem 22.** Let

$$f(x) = \begin{cases} 0; & \text{for } x \text{ irrational} \\ \frac{1}{q}; & \text{for } x = \frac{p}{q} \\ 1, & \text{for } x = 0 \end{cases}$$

Show  $f$  is continuous at each point of  $\mathbb{R} \setminus \mathbb{Q}$  and discontinuous at each point of  $\mathbb{Q}$ .

**Proof.** It's easy to prove for  $x \in \mathbb{Q}$ . Take a sequence  $(x_n)$  of irrationals such that  $x_n \rightarrow p/q$ . But,  $f(x_n) = 0 \neq f(p/q) = 1/q$ . Also, observe that the function is periodic with period 1, i.e.  $f(x) = f(x+1)$ .

For  $x$  in set of irrational numbers. For a sequence of irrationals it's obvious that the difference b/w the function values will be 0 always. For a sequence of rationals we can think of any sequence of rationals converging to  $x$ , then the denominator value will approach infinity(it's not a formal proof). An example is shown below shown in the footnote below. ■

**Problem 23.** Let

$$f(x) = \begin{cases} 0; & \text{for } x \text{ irrational} \\ 1; & \text{for } x \text{ rational} \end{cases}$$

Show  $f$  is discontinuous at each point of  $\mathbb{R}$ .

**Proof.** We begin by considering a sequence of irrational numbers  $x_n$  converging to a rational  $x_0$ .  $x_n = x_0 + \frac{\lambda}{n}$  where  $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ . Since, every element of  $x_n$  is irrational implies  $f(x_n) = 0 \neq f(x_0) = 1$ . So, there exist a sequence in the domain such that  $x_n \rightarrow x_0$  but  $f(x_n) \not\rightarrow f(x_0)$  for rational  $x_0$ .

Similarly, using the density property of rational numbers <sup>1</sup>there exist a sequence of rational numbers  $(x_n)$  in  $\mathbb{R}$  such that  $x_n \rightarrow x_0$  for  $x_0$  irrational. But  $f(x_n) = 1 \neq f(x_0) = 0$ . Hence , it is not continuous. ■

<sup>1</sup>you need a sequence of rationals converging to the irrational  $x$ . In theory, we already know one: consider the decimal expansion of  $x$ . When  $x$  is irrational, the sequence is necessarily infinite, doesn't eventually repeat itself forever. Suppose

$$x = m + 0.d_1d_2\dots$$

**Problem 24.** Let  $f$  be a continuous function on  $[0, \infty)$ . Prove that if  $f$  is uniformly continuous on  $[k, \infty)$  for some  $k$ , then  $f$  is uniformly continuous on  $[0, \infty)$ .

**Problem 25.** Let  $f$  be a continuous function on  $[a, b]$ . Show that the function  $f^*$  defined as  $f^*(x) = \sup\{f(y) : a \leq y \leq x\}$ , for  $x \in [a, b]$ , is an increasing continuous function on  $[a, b]$ .

**Proof.** Take  $x_2 > x_1$ . Let  $S_1 = [a, x_1]$  &  $S_2 = [a, x_2]$ . Observe  $S_1 \subset S_2$ . By definition  $f^*(x_1) = \sup\{f(y) : a \leq y \leq x_1\}$  &  $f^*(x_2) = \sup\{f(y) : a \leq y \leq x_2\}$ . So,  $f(x_1) \geq f(x) \forall x \in S_1$  and  $f(x_2) \geq f(x) \forall x \in S_2$ . Since,  $S_1 \subset S_2$  so,  $x_1 \in S_2$  implies  $f(x_2) \geq f(x_1)$ . ■

### §4.3 Limits of Functions

In this section we'll formalize the notion of limit of a function and this will help us for a careful study of derivatives

#### Definition 4.3.1

Let  $S \subseteq \mathbb{R}$  and  $S \neq \emptyset$ , let  $a$  be a real number or symbol  $-\infty$  or  $\infty$  that is a limit of a sequence in  $S$ . And let  $L$  be a real number or symbol  $-\infty$  or  $\infty$ . We write  $\lim_{x \rightarrow a} f(x) = L$  if:

$f$  is a function defined on  $S$

and

for every sequence  $(x_n)$  in  $S$  converging to  $a$ , we have  $\lim_{n \rightarrow \infty} f(x_n) = L$

The expression " $\lim_{x \rightarrow a} f(x)$ " is read as "limit, as  $x$  tends to  $a$  along  $S$ , of  $f(x)$ ,"

#### Notations

$$S = (-\infty, b) : \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a^-} f(x)$$

and

$$S = (a, \infty) : \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a^+} f(x)$$

**Question:** Let  $a$  be a real number &  $S = (a, b) \subseteq \mathbb{R}$ . Does  $\lim_{x \rightarrow a} f(x)$  depends on  $S$ . If we take a different set  $T = (a, b_1) \subseteq \mathbb{R}$  then what's the relation b/w  $\lim_{x \rightarrow a} f(x)$  &  $\lim_{x \rightarrow a} f(x)$

where  $m$  is an integer. Let  $x_n = m + 0.d_1 \dots d_n$ . In other words,  $x_n$  is the decimal representation of  $x$  cut off at the  $n$ th digit after the decimal point ( $x_0 = m$ ). Then every  $x_n$  is rational, and:

$$\lim_{n \rightarrow \infty} x_n = x$$

**Answer:** Since, as the  $t_n \in T$  converge to  $a$ , then after some iterations the elements of sequence  $t_n$  will be the elements of set  $S$  (Assuming  $S \subseteq T$ ).then:

$$\lim_{x \rightarrow a^S} f(x) = \lim_{x \rightarrow a^T} f(x)$$

### Theorem 4.3.1

Let  $f$  be a function for which the limit  $L = \lim_{x \rightarrow a^S} f(x)$  exists and is finite. If  $g$  is a function defined on  $\{f(x) : x \in S\} \cup \{L\}$  that is continuous at  $L$ , then  $\lim_{x \rightarrow a^S} g \circ f(x)$  exists and equals  $g(L)$ .

### Theorem 4.3.2

Let  $f$  be a function defined on a subset  $S$  of  $\mathbb{R}$ , let  $a$  be a real number that is the limit of some sequence in  $S$ , and let  $L$  be a real number. then  $\lim_{x \rightarrow a^S} f(x) = L$  if and only if

for each  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$x \in S \text{ and } |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon. \quad (4.1)$$

**Proof.** Assuming the statement and proving (3) is trivial so I leave it.

Let  $\lim_{n \rightarrow \infty} f(x_n) = L$ :

Now assume  $\lim_{n \rightarrow \infty} x_n = a$ , but (3) fails. So, there exist a  $\epsilon > 0$  such that for all  $\delta > 0$   $|f(x_n) - f(a)| \geq \epsilon$ . Take  $\delta = 1/n$ . So, it implies for all  $n \in \mathbb{N}$  there exist  $x_n$  in  $S$  such that  $|x_n - a| < 1/n$  while  $|f(x_n) - f(a)| \geq \epsilon$ . Hence our assumption was wrong and there exist a sequence  $(x_n)$  which converges to  $a$ , but  $f(x_n)$  doesn't converge to  $f(a)$  ■

### Corollary 4.3.1

Let  $f$  be a function defined on  $J \setminus \{a\}$  for some open interval  $J$  containing  $a$ , and let  $L$  be a real number. Then  $\lim_{x \rightarrow a} f(x) = L$  iff:

for each  $\epsilon > 0$  there exists  $\delta > 0$  such that  
 $|x - a| < \delta$  implies  $|f(x) - L| < \epsilon$ .

## Chapter 5

# Sequence and Series of Functions

### §5.1 Power Series

#### Definition 5.1.1: Power Series

$\sum_0^\infty a_n x^n$  is called a Power Series.

#### Definition 5.1.2: Radius of Convergence

Let  $\beta = \limsup |a_n|^{\frac{1}{n}}$ . Then **Radius of convergence** is defined as:

$$R = \frac{1}{\beta}$$

#### Theorem 5.1.1

1.  $\sum_0^\infty a_n x^n$  converges for  $|x| < R$ .
2.  $\sum_0^\infty a_n x^n$  diverges for  $|x| > R$ .

**Proof.** Take  $t \in \mathbb{R}$ . Take  $r_t = \limsup |a_n t^n|^{\frac{1}{n}}$ . Then

$$r_t = \limsup |a_n t^n|^{\frac{1}{n}} = |t| \limsup |a_n|^{\frac{1}{n}} = |t| \beta$$

**Case 1.** Let  $0 < R < \infty$

$$r_t = \beta t = \frac{|t|}{R} = \begin{cases} |t| < R \Rightarrow \overbrace{r_t < 1}^{\text{converges by root test}} \\ |t| > R \Rightarrow \underbrace{r_t > 1}_{\text{diverges by root test}} \end{cases}$$

**Case 2.** Let  $R = \infty$

Here  $\beta = 0$  implies  $r_t = 0 < 1$ . So, converges for all  $x$  by root test.

**Case 3.** Let  $R < 0$

$\Rightarrow \beta = \infty$  implies  $r_t = \infty$  ; for all  $|t| \neq 0$ . Therefore Diverges. ■

**Example:**  $\sum x^n$  . Here  $a_n = 1$ .

$$\therefore \lim \left| \frac{a_{n+1}}{a_n} \right| = 1 \quad \therefore \beta = 1, R = 1$$

. Therefore the series converges for  $|x| < 1$ . A strict inequality!

**Example:** Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n$$

The radius of convergence for the series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (y)^n$  is  $R = 1$ , so it converges for  $|y| < 1$  or  $x \in (0, 2)$  at  $x = 0$  and  $y = -1$ , we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (-1)^n = - \sum_{n=1}^{\infty} \frac{1}{n}$$

therefore it diverges to  $-\infty$ . at  $x = 2$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = -\ln 2$$

Hence, it converges & the interval of convergence is  $(0, 2]$ .

**Problem 26.** Consider a power series  $\sum a_n x_n$  with radius of convergence  $R$ . Prove that if all the coefficients  $a_n$  are integers and if infinitely many of them are nonzero, then  $R \leq 1$ .

**Proof.** Given  $|a_n| \geq 1$  ( $\because a_n$  are integers). So,  $s_n = \sup\{|a_k| : k \geq n\} \geq 1$ . Further  $|s_n|^{\frac{1}{n}} \geq 1$  implies  $\lim |s_n|^{\frac{1}{n}} \geq 1$ . Or  $R \leq 1$  using 5.1.2. ■

**Problem 27.** Prove that if  $\limsup |a_n| > 0$ , then  $\limsup |a_n|^{\frac{1}{n}} \geq 1$ .

**Proof.** Let  $\limsup |a_n| > 0$  (notice the strict inequality!). But for sake of contradiction assume  $\limsup |a_n|^{\frac{1}{n}} < 1$ . Then  $\sum a_n$  converges, which implies  $\lim a_n = 0$  (See 3.1.1). A contradiction! ■

## §5.2 Uniform Convergence

### Definition 5.2.1: Pointwise Convergence

Let  $(f_n)$  be a sequence of real-valued functions defined on a set  $S \subseteq \mathbb{R}$ . The sequence  $(f_n)$  converges pointwise [i.e., at each point] to a function  $f$  defined on  $S$  if

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad \text{for all } x \in S$$

We often write  $\lim f_n = f$  pointwise [on  $S$ ] or  $f_n \rightarrow f$  pointwise [on  $S$ ].

Now observe  $f_n \rightarrow f$  pointwise on  $S$  means exactly the following:  
for each  $\epsilon > 0$  and  $x$  in  $S$  there exists  $N$  such that

$$|f_n(x) - f(x)| < \epsilon \quad \text{for } n > N$$

Note the value of  $N$  depends on both  $\epsilon$  and  $x$  in  $S$ .

**Example** Let  $f_n(x) = x^n$  for  $x \in [0, 1]$ . Then  $f_n \rightarrow f$  pointwise on  $[0, 1]$  where  $f(x) = 0$  for  $x \in [0, 1)$  and  $f(1) = 1$ . Or we can write:

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} x^n = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{else} \end{cases}$$

Consider,

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{else} \end{cases}$$

Then we can write  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ .

### Definition 5.2.2: Uniform Convergence

Let  $(f_n)$  be a sequence of real-valued functions defined on a set  $S \subseteq \mathbb{R}$ . The sequence  $(f_n)$  converges uniformly on  $S$  to a function  $f$  defined on  $S$  if  
for each  $\epsilon > 0$  there exists a number  $N$  such that

$$|f_n(x) - f(x)| < \epsilon \quad \text{for all } x \in S \quad \text{and } n > N$$

. We write  $\lim f_n = f$  uniformly on  $S$  or  $f_n \rightarrow f$  uniformly on  $S$ .

**Example:** Define,

$$f_n(x) = x^n \quad \text{for } x \in [0, 1]$$

$$f(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

then  $\epsilon = 1/2$ , then we consider for all  $x \in [0, 1]$  and all  $n > N$ .

$$|f_n(x) - f(x)| < \epsilon = 1/2$$

$$|x| \leq 1 \text{ \& } n > N$$

$$\Rightarrow |f_n - f| < 1/2$$

at  $x = 0$  it's not possible as  $|0 - 1| < 1/2$  is not true. So, not uniformly converging.



**Example** Let  $f(x) = \frac{1}{n} \sin(nx) \forall x \in \mathbb{R}$

for any  $x \lim_{n \rightarrow \infty} f_n(x) = 0$  pointwise on  $\mathbb{R}$ . Define  $f(x) = 0$  then  $f_n \rightarrow f$ . In fact  $f_n \rightarrow f$  uniformly on  $\mathbb{R}$ . Also, let  $N = 1/\epsilon$ . Then for  $n > N$  and all  $x \in \mathbb{R}$  we have

$$|f_n(x) - f(x)| = |f_n(x) - 0| = \left| \frac{1}{n} \sin(nx) \right| \leq \frac{1}{n} < \frac{1}{N} = \epsilon$$

Since,  $N$  is independent of  $x$ .

### Theorem 5.2.1

Uniform limit of continuous function is continuous. More precisely, let  $(f_n)$  be a sequence of functions on a set  $S \subseteq \mathbb{R}$ , suppose  $f_n \rightarrow f$  uniformly on  $S$ , and suppose  $S = \text{dom}(f)$ . If each  $f_n$  is continuous at  $x_0$  in  $S$ , then  $f$  is continuous at  $x_0$ . [So if each  $f_n$  is continuous on  $S$ , then  $f$  is continuous on  $S$ .]

**Remark** Remember  $f_n \rightarrow f$  on  $S$  uniformly iff,

$$\limsup_{n \rightarrow \infty} \{|f_n(x) - f(x)| : x \in S\} = 0$$

We can also consider  $\sum_{k=0}^{\infty} g_k(x)$ , where  $g_k$  is a function of  $S \subset \mathbb{R}$ .

### Theorem 5.2.2

If  $g_k$  is continuous  $\forall k$  and if  $\sum g_k$  converges uniformly, then it converges to a continuous function on  $S$ .

**Proof.** Let  $f_n = \sum_{k=0}^n g_k(x)$  is continuous on  $S$  and  $f_n(x) = \sum_{k=0}^{\infty} g_k(x)$  uniformly.

$\therefore f(x) = \sum_{k=0}^{\infty} g_k(x)$  is continuous. ■

### Corollary 5.2.1

$f(x) = \sum a_n x^n$  is a continuous function if the convergence is uniform.

**Problem 28.** Prove that if  $(f_n)$  is a sequence of uniformly continuous functions on an interval  $(a, b)$ , and if  $f_n \rightarrow f$  uniformly on  $(a, b)$ , then  $f$  is also uniformly continuous on  $(a, b)$ .

**Proof.** We need to show that  $\forall \delta > 0 \exists \epsilon > 0$  such that  $|x - y| < \delta$  implies  $|f(x) - f(y)| < \epsilon$ .

Using triangular inequality we can show that:

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$

Using the  $\epsilon/3$  argument we can show that, there exists a  $N$  such that  $n > N_1$  implies  $|f_n(x) - f_n(y)| < \epsilon/3$  (it's uniform cont.) and  $|x - y| < \delta$ . And we can apply

definition of uniform convergence to show  $|f_n(y) - f(y)| < \epsilon/3$  by choosing some  $N_2$  for  $x$  and then some  $N_3$  for  $y$ . So we have for  $n > N = \max(N_1, N_2, N_3)$ .

$$|f(x) - f(y)| < 3 \cdot \frac{\epsilon}{3} = \epsilon$$

■

## Chapter 6

# Differentiation

### §6.1 Why differentiation? Better ask Newton.

#### Definition 6.1.1

$f$  is a real-valued function on an open interval  $I$ . Let  $a \in I$ . Then  $f$  is differentiable at “a” if the limit :

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

exists & is finite. We also say  $f$  has a derivative at a.

$$f'(x) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

**Remark**  $f'(x)$  itself is a function with domain  $f \subseteq \text{dom} f$ .

#### Theorem 6.1.1

If  $f$  be diff. at  $x = a$ . Then  $f$  is continuous at a.

**Proof.**  $f(x) = (x - a) \frac{f(x) - f(a)}{x - a} + f(a)$

$$\therefore \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} (x - a) \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} + \lim_{x \rightarrow a} f(a) = 0 \cdot f'(a) + f(a) \quad (6.1)$$

(6.2)

■

#### Theorem 6.1.2: Properties

If  $f, g$  are diff. at  $x = a$  and  $c \in \mathbb{R}$  then

1.  $(cf)'(a) = cf'(a)$
2.  $(f + g)'(a) = f'(a) + g'(a)$

$$3. (fg)'(a) = f'(a)g(a) + f(a)g'(a)$$

$$4. \left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$$

**Theorem 6.1.3: Chain Rule**

$f$  be differentiable at  $x = a$  and  $g$  be differentiable at  $f(a)$ . Then  $(g \circ f)(x)$  is diff at  $x = a$ . And  $(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$

**§6.2 Mean Value Theorem****Theorem 6.2.1**

If  $f$  is defined on an open interval containing  $x_0$ , if  $f$  assumes its maximum or minimum at  $x_0$ , and if  $f$  is differentiable at  $x_0$ , then  $f'(x_0) = 0$ .

**Theorem 6.2.2: Rolle's Theorem**

Let  $f$  be a continuous function on  $[a, b]$  that is differentiable on  $(a, b)$  and satisfies  $f(a) = f(b)$ . There exists [at least one]  $x$  in  $(a, b)$  such that  $f'(x) = 0$ .

**Theorem 6.2.3: Mean Value Theorem**

Let  $f$  be a continuous function on  $[a, b]$  that is differentiable on  $(a, b)$ . Then there exists [at least one]  $x$  in  $(a, b)$  such that:

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

**Corollary 6.2.1**

Let  $f$  be a differentiable function on  $(a, b)$  such that  $f'(x) = 0$  for all  $x \in (a, b)$ . Then  $f$  is a constant function on  $(a, b)$ .

**Corollary 6.2.2**

Let  $f$  and  $g$  be differentiable functions on  $(a, b)$  such that  $f' = g'$  on  $(a, b)$ . Then there exists a constant  $c$  such that  $f(x) = g(x) + c$  for all  $x \in (a, b)$ .

**Corollary 6.2.3**

Let  $f$  be a differentiable function on interval  $(a, b)$ , then:

- (i.)  $f$  is strictly increasing if  $f'(x) > 0$  for all  $x \in (a, b)$ ;
- (ii.)  $f$  is strictly decreasing if  $f'(x) < 0$  for all  $x \in (a, b)$ ;

- (iii.)  $f$  is increasing if  $f'(x) \geq 0$  for all  $x \in (a, b)$ ;
- (iv.)  $f$  is decreasing if  $f'(x) \leq 0$  for all  $x \in (a, b)$ ;

**Theorem 6.2.4: Intermediate Value Theorem for Derivatives.**

Let  $f$  be a differentiable function on  $(a, b)$ . If  $a < x_1 < x_2 < b$ , and if  $c$  lies between  $f'(x_1)$  and  $f'(x_2)$ , there exists [at least one]  $x$  in  $(x_1, x_2)$  such that  $f'(x) = c$ .

**Problem 29.** Let  $f$  be differentiable on  $\mathbb{R}$  with  $a = \sup\{|f'(x)| : x \in \mathbb{R}\} < 1$ . Select  $x_0 \in \mathbb{R}$  and define  $x_n = f(x_{n-1})$  for  $n \geq 1$ . Thus  $x_1 = f(x_0)$ ,  $x_2 = f(x_1)$ , etc. Prove  $(x_n)$  is a convergent sequence.

**Proof.** Using the mean value theorem we can see  $f$  is a contraction, because

$$|f(x) - f(y)| = |f'(c)||x - y| \leq a|x - y|$$

With  $a < 1$ .

Now to see it is a convergent / cauchy sequence, select  $x_0$  and note  $x_n = f(x_{n-1})$ , what can you say of

$$|x_n - x_m|$$

for arbitrary  $m, n$ ?

HINT: Use triangle inequality and  $|x_n - x_{n+1}| \leq a^n|x_0 - x_1|$ . For alternate proof refer to Problem 6

Suppose  $m > n$  then:

$$\begin{aligned} |x_n - x_m| &\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \\ &\leq \sum_{k=n}^m a^k |x_0 - x_1| \leq \sum_{k=n}^{\infty} a^k |x_0 - x_1| < +\infty \end{aligned}$$

Now, because this last series converges ( $a < 1$ ), by the Cauchy criterion, given  $\varepsilon > 0$ , there is some  $N$  such that

$$\sum_{k=N}^{\infty} a^k |x_0 - x_1| < \varepsilon$$

If we pick  $n, m \geq N$  we're done, now we know  $x_n$  is a Cauchy sequence in  $\mathbb{R}$ , and thus has a limit  $x$ , now

$$x = \lim x_n = \lim f(x_{n-1}) = f(\lim x_{n-1}) = f(x)$$

hence  $x$  is a fixed point. To see its uniqueness suppose  $x_1, x_2$  are two fixed points, then:

$$|x_1 - x_2| = |f(x_1) - f(x_2)| \leq a|x_1 - x_2|$$

With  $a < 1$  this is only true if  $x_1 = x_2$ . ■

### §6.3 Taylor Theorem

#### Definition 6.3.1: Taylors Series

Let  $f$  be a function defined on some open interval containing  $c$ . If  $f^{(k)}(c)$  exists  $\forall k$ , then the series:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x - c)^k$$

is called as **Taylor Series** of function  $f(x)$  about  $c$ . For  $n \geq 1$ ; remainder  $R_n(x)$  is defined as:

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!} (x - c)^k$$

The remainder is important because, for any  $x$ ;

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x - c)^k \quad \text{if and only if} \quad \lim_{n \rightarrow \infty} R_n(x) = 0$$

#### Theorem 6.3.1

Let  $f$  be defined on  $(a, b)$  where  $a < c < b$ ; here we allow  $a = -\infty$  or  $b = \infty$ . Suppose the  $n^{\text{th}}$  derivative  $f^n$  exists on  $(a, b)$ . Then for each  $x \neq c$  in  $(a, b)$  there is some  $y$  between  $c$  and  $x$  such that

$$R_n(x) = \frac{f^n(y)}{n!} (x - c)^n$$

#### Corollary 6.3.1

Let  $f$  be defined on interval  $(a, b)$  and  $a < c < b$ . If  $f^{(k)}$  exists  $\forall k$ , & there exists a single constant  $C$  such that  $|f^{(k)}(c)| \leq C$ , then

$$\lim_{n \rightarrow \infty} R_n(x) = 0 \quad \forall x \in (a, b)$$

## Chapter 7

# Die Theorie der Integration

### §7.1 Integrals

#### Definition 7.1.1: The Darboux Integral

Let  $f$  be a bounded function on a closed interval  $[a, b]$ . For  $S \subseteq [a, b]$ , we adopt the notation:

- $M(f, S) = \sup\{f(x) : x \in S\}$
- $m(f, S) = \inf\{f(x) : x \in S\}$ .

A partition of  $[a, b]$  is any finite ordered subset  $P$  having the form

$$P = \{a = t_0 < t_1 < \cdots < t_n = b\}.$$

The upper Darboux sum  $U(f, P)$  of  $f$  with respect to  $P$  is the sum:

$$U(f, P) = \sum \overbrace{M(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})}^{\text{Area of rect. above the curve}}$$

and the lower Darboux sum  $L(f, P)$  is:

$$L(f, P) = \sum \underbrace{m(f, [t_{k-1}, t_k]) \cdot (t_k - t_{k-1})}_{\text{Area of rect. below the curve}}$$

Note

$$U(f, P) \leq M(f, [a, b])(b - a);$$

and

$$L(f, P) \geq m(f, [a, b])(b - a);$$

The upper Darboux integral  $U(f)$  of  $f$  over  $[a, b]$  is defined by

$$U(f) = \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}$$

and the lower Darboux integral is

$$L(f) = \sup\{L(f, P) : P \text{ is a partition of } [a, b]\}$$

We say  $f(x)$  is integrable if on  $[a, b]$  if  $L(f) = U(f)$ .

**Observe:**  $L(f, \{a, b\}) \leq L(f, P) \leq U(f, P) \leq U(f, \{a, b\})$ .

**Lemma 7.1.1**

Let  $f$  be a bounded function on  $[a, b]$ . If  $P$  and  $Q$  are partitions of  $[a, b]$  and  $P \subseteq Q$ , then

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$$

**Lemma 7.1.2**

If  $f$  is a bounded function on  $[a, b]$ , and if  $P$  and  $Q$  are partitions of  $[a, b]$ , then  $L(f, P) \leq U(f, Q)$

**Lemma 7.1.3**

If  $f$  is a bounded function on  $[a, b]$ , then  $L(f) \leq U(f)$ .

**Theorem 7.1.1**

A bounded function  $f$  on  $[a, b]$  is integrable if and only if for each  $\epsilon > 0$  there exists a partition  $P$  of  $[a, b]$  such that

$$U(f, P) - L(f, P) < \epsilon$$

**Definition 7.1.2: Mesh**

The *mesh* of a partition  $P$  is the maximum length of the subintervals comprising  $P$ . Thus if:

$$P = \{a = t_0 < t_1 < t_2 \cdots < t_n = b\},$$

then

$$\text{mesh}(P) = \max\{t_k - t_{k-1} : k = 1, 2, 3, \dots, n\}$$

**Definition 7.1.3: Riemann Sum**

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function defined on a closed interval  $[a, b]$  of the real numbers,  $\mathbb{R}$ , and

$$P = \{a = t_0 < t_1 < t_2 \cdots t_n = b\}$$

be a partition of  $[a, b]$ . A Riemann sum of  $f$  associated with the partition  $P$  is a sum of the form

$$\sum_{k=1}^n f(x_k)(t_k - t_{k-1})$$

where  $x_k \in [t_{k-1}, t_k]$  for  $k = 1, 2, 3, \dots, n$ . The choice of  $x_k$  is arbitrary, so there are infinitely many Riemann sums associated with a single function and



partition.

#### Definition 7.1.4: Riemann Integral

The function  $f$  is Riemann integrable on  $[a, b]$  if there exists a number  $r$  such that for each  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$|S - r| < \delta$$

for every Riemann sum of  $f$  associated with the partition  $P$  having  $\text{mesh}(P) < \delta$ . The number  $r$  is the Riemann integral of  $f$  on  $[a, b]$  and will be provisionally written as  $\mathcal{R} \int_a^b f$ .

#### Theorem 7.1.2

A bounded function  $f$  on  $[a, b]$  is integrable if and only if for each  $\epsilon > 0$  there exists a  $\delta > 0$  such that  $\text{mesh}(P) < \delta$  implies

$$U(f, P) - L(f, P) < \epsilon$$

for all partitions  $P$  of  $[a, b]$ .

### §7.1.1 Fundamental Theorem of Calculus

#### Theorem 7.1.3

If  $g$  is a continuous function on  $[a, b]$  that is differentiable on  $(a, b)$ , and if  $g'$  is integrable on  $[a, b]$ , then

$$\int_a^b g' = g(b) - g(a)$$

#### Theorem 7.1.4

Let  $f$  be an integrable function on  $[a, b]$ . For  $x$  in  $[a, b]$ , let

$$F(x) = \int_a^x f(t) dt.$$

Then  $F$  is continuous on  $[a, b]$ . If  $f$  is continuous at  $x_0$  in  $(a, b)$ , then  $F$  is differentiable at  $x_0$  and

$$F'(x) = f(x)$$

### §7.1.2 Intermediate Value Theorem for Integrals.

#### Theorem 7.1.5: Intermediate Value Theorem for Integrals.

If  $f$  is a continuous function on  $[a, b]$ , then for at least one  $x$  in  $(a, b)$  we have:

$$f(x) = \int_a^b f$$

**Problem 30.** Let  $f(x) = x$  for rational  $x$  and  $f(x) = 0$  for irrational  $x$ .

1. Calculate the upper and lower Darboux integrals for  $f$  on the interval  $[0, b]$ .
2. Is  $f$  integrable on  $[0, b]$ ?

## Chapter 8

# Differentiation and Integration of Power Series

### Theorem 8.0.1: Weierstrass M-test

Let  $(M_k)$  be a sequence of non-negative real numbers where  $\sum M_k < \infty$ . If  $|g_k(x)| \leq M_k$  for all  $x$  in a set  $S$ , then  $\sum g_k$  converges uniformly on  $S$

**Proof.** Let  $\epsilon > 0$ ,  $\exists N$  such that  $n \geq m > N \Rightarrow \sum_{k=m}^{\infty} M_k < \epsilon$ . Then if  $n \geq m > N$  and  $x \in S$ ,

$$\left| \sum_{k=m}^n g_k(x) \right| \leq \sum_{k=m}^n |g_k(x)| \leq \sum_{k=m}^n M_k < \epsilon$$

$\therefore$  by cauchy criterion on uniform cont. on  $S$ ,  $\sum g_k$  converges uniformly on  $S$ . ■

**Problem 31.** Show that if the series  $\sum g_n$  converges uniformly on a set  $S$ , then

$$\lim_{n \rightarrow \infty} \sup\{|g_n(x)| : x \in S\} = 0$$

**Proof.** For  $\epsilon > 0$ ,  $\exists N$  such that  $n \geq m > N \Rightarrow \left| \sum_m^n g_k(x) \right| < \epsilon$ .  
In particular  $n > N$

$$\Rightarrow |g_n| < \epsilon \forall x \in S$$

$$\therefore \sup\{|g_n(x)| : x \in S\} \leq \epsilon$$

$$\lim_{n \rightarrow \infty} \sup\{|g_n(x)| : x \in S\} = 0$$

■

### Theorem 8.0.2

Let  $\sum a_n x^n$  be a power series with radius of convergence  $R > 0$ . If  $0 < R_1 < R$ , then the power series converges uniformly on  $[-R_1, R_1]$  to a continuous function.

**Lemma 8.0.1**

If the power series  $\sum a_n x^n$  has radius of convergence  $R$ , then the power series

$$\sum n a_n x^n \quad \& \quad \sum \frac{a_n}{n+1} x^{n+1}$$

have the same radius of convergence  $R$ .

**Theorem 8.0.3**

Suppose  $f(x) = \sum a_n x^n$  has radius of convergence  $R > 0$ . Then

$$\int_0^x f(t) dt = \sum \frac{a_n}{n+1} x^{n+1} \quad \forall x < R$$

**Theorem 8.0.4**

Let  $f(x) = \sum a_n x^n$  have radius of convergence  $R > 0$ . Then  $f$  is differentiable on  $(-R, R)$  and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \forall |x| < R$$

**Theorem 8.0.5: Abel's Theorem**

Let  $f(x) = \sum a_n x^n$  be a power series with  $0 < R < \infty$ . If the series converges at  $x = R$ , then  $f$  is continuous at  $x = R$ . If the series converges at  $x = -R$ , then  $f$  is continuous at  $x = -R$ .

## Appendix A

# Power series of some common functions

### Exponential Series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots .$$

It converges for all  $x \in \mathbb{R}$ .

### Natural logarithm

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots ,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots .$$

They converge for  $|x| < 1$

### Geometric series

$$\begin{aligned} \frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\ \frac{1}{(1-x)^2} &= \sum_{n=1}^{\infty} nx^{n-1} \\ \frac{1}{(1-x)^3} &= \sum_{n=2}^{\infty} \frac{(n-1)n}{2} x^{n-2} . \end{aligned}$$

All are convergent for  $|x| < 1$ . These are special cases of Binomial Series.

**Binomial series** The binomial series is a power series

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

whose coefficients are the generalized [binomial coefficients](#)

$$\binom{\alpha}{n} = \prod_{k=1}^n \frac{\alpha - k + 1}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} .$$

**Trigonometric Functions** The usual trigonometric functions and their inverses have the following power series:

$$\begin{aligned}
 \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots &\quad \text{for all } x \\
 \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots &\quad \text{for all } x \\
 \tan x &= \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots &\quad \text{for } |x| < \frac{\pi}{2} \\
 \sec x &= \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} &= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \cdots &\quad \text{for } |x| < \frac{\pi}{2} \\
 \arcsin x &= \sum_{n=0}^{\infty} \frac{(2n)!}{4^n(n!)^2(2n+1)} x^{2n+1} &= x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots &\quad \text{for } |x| \leq 1 \\
 \arccos x &= \frac{\pi}{2} - \arcsin x \\
 &= \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{4^n(n!)^2(2n+1)} x^{2n+1} &= \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} - \cdots &\quad \text{for } |x| \leq 1 \\
 \arctan x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots &\quad \text{for } |x| \leq 1, \ x \neq \pm i
 \end{aligned}$$

All angles are expressed in radians. The numbers  $B_k$  appearing in the expansions of  $\tan x$  are the [Bernoulli numbers](#). The  $E_k$  in the expansion of  $\sec x$  are [Euler numbers](#).

# Bibliography

- [1] Stephen Abbott. *Understanding analysis*, volume 2. Springer, 2001.
- [2] Kenneth A. Ross. *Elementary analysis*. Springer, 2013.
- [3] Wikipedia. Wikipedia: The free encyclopedia.
- [4] Wikipedia contributors. Cauchy sequence — Wikipedia, the free encyclopedia, 2019. [Online; accessed 7-May-2020].
- [5] Wikipedia contributors. Bolzano–weierstrass theorem — Wikipedia, the free encyclopedia, 2020. [Online; accessed 7-May-2020].
- [6] Wikipedia contributors. Ratio test — Wikipedia, the free encyclopedia, 2020. [Online; accessed 7-May-2020].
- [7] Wikipedia contributors. Root test — Wikipedia, the free encyclopedia, 2020. [Online; accessed 7-May-2020].
- [8] Wikipedia contributors. Riemann sum — Wikipedia, the free encyclopedia, 2020. [Online; accessed 8-May-2020].