# Complex Dynamical Systems and the Applications of Fractals

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## 1 Introduction

This paper is an undergraduate summary of topics regarding the study of Complex Dynamical Systems and the Applications of Fractals. The goal of the paper is to discuss the connection between iterative systems and fractal structures and to highlight some applications of fractals outside of mathematics.

## 2 What are Fractals?

In order to further discuss fractal structures, we must first establish what we consider a fractal. For the purpose of this paper we will consider two definitions of what a fractal is. The first definition comes more intuitively, being that a fractal is a geometric curve or structure that contains patterns that repeat at progressively smaller scales. These structures contain an implicit recurring self-symmetry. This definition gives a rough interpretation of fractals that is less mathematically concrete. However, we can use this definition to better understand the notion of fractals through more natural structures and patterns. Such natural patterns can be found throughout the world in places like snowflakes, rivers, and plant patterns as well as in the structure of the nervous and circulatory systems of many animals.The figures below display some natural fractal patterns.

Figure 1: Fractal patterns displayed on the leaves of a plant [4]

Figure 2: Fractals in the feather patterns of a peacock [5]

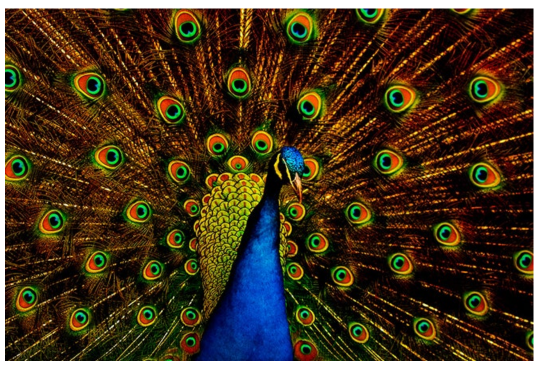
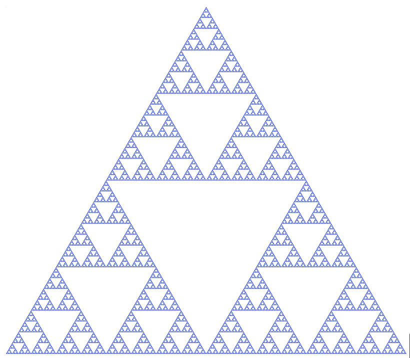


Figure 3: Fractal patterns in the branching network of a river [6]

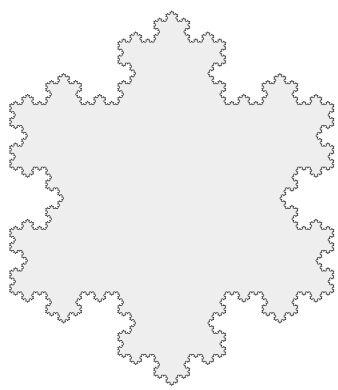
These natural patterns occur everywhere, and are typically considered to be quite aesthetic. Some studies have even shown that such highly structured natural patterns can lead to stress reduction. Along with naturally occurring patterns, there are also plenty of geometric figures that are fractals. Two examples of these figures are the Sierpinski Triangle and the Koch Snowflake. A computer generated image of these fractal is displayed below:

Figure 4: The Sierpinski Triangle



[7]

Figure 5 : The Koch Snowflake



[8]

The images that we have looked at thus far are examples of our common understanding of fractals. All of these examples contain a recursive self-similarity coupled with a very orderly structure. Next we will look at fractals through the lens of complex dynamical systems. In order to accomplish this we must move from our broader visual definition to a more mathematical definition of fractals. This definition stems from the concept of fractal dimension. For now we only require a brief understanding of fractal dimension. What is important is that we consider fractal dimension to be measure of how a structure's complexity varies with scale. For a more in depth look at fractal dimension, see section 6 below. Using this notion of fractal dimension, we define a fractal as a subset of Euclidean space with a fractal dimension that strictly exceeds its topological dimension.(1). This definition provides us with a mathematical representation of fractals in euclidean space upon which we can build. Looking at fractals from a mathematical perspective naturally leads us into the study of dynamical systems.

## 3 Dynamical Systems

The study of fractals is inherently linked to the study of dynamical systems. That is to say that the study of fractals originates from the visualization of the behavior of dynamical systems. A dynamical system is a system in which a function describes the time dependence of a point in a geometrical space.(2). Dynamical systems are present in almost any model that involves time dependence. Some common examples of such systems are the motion of pendulums, population modeling, and flows within vector fields.

The study of fractals arises from the mapping of iterative dynamical systems. For the purpose of this paper we will be considering iterative mappings on the complex domain $C$. Consider a function $f : C \to C$ that maps points in the complex plane back into the complex plane. For some initial value $z_0$, we can find the iterations of $z_0$ by repeatedly applying $f$ to each iteration. For example, the first iterate $z_1$ is given by $z_1 = f(z_0)$. Similarly, $z_2 = f(z_1) = f \circ f(z_0)$. In generality, we can define the $n^{th}$ iterate as follows:

$$z_n = f(z_{n-1})$$

By the nature of how we form these iterates, each iterate has an innate dependence on the previous iterates. As a result of this it follows that each iterate depends solely on the initial value $z_0$, and the iteration number. This perspective gives us a similar evaluation for the $n^{th}$ iterate.

$$z_n = f \circ f \circ \cdots \circ f(z_0) = f^n(z_0)$$

Note that in the formula above $f^n$ is used to denote the $n^{th}$ degree composition of $f$ upon itself. Now that we have established our definition of an iterative function, we can explore some of the features of these functions. The first important feature of an iterative function $f$ is the fixed points. Given an iterative function $f$, a **fixed point** of $f$ is a point $z$ that is stationary under the mapping $f$. That is to say that a point $z \in C$ is a fixed point of $f$ if $f(z) = z$. From this we can see that the fixed points for a function $f$ are the roots to equation $f(z) - z = 0$. When considering the dynamics of $f$ on the entire complex plane, these fixed points typically define points of interest in the resulting fractal geometry.

For an example of a fixed point, consider the mapping $f : C \to C$ given by $f(z) = 2z - 2$. In order to find the fixed points of $f$ we must find the roots to $f(z) - z = 0$. Given our definition of $f$, we get the equation $z - 2 = 0$, so we can see that $z = 2$ is the only solution. This result is quite nice as we can easily see that $f(2) = 2$, confirming that $z = 2$ is a fixed point of $f$.

There are two major types of fixed points, attracting fixed points and repelling fixed points. A fixed point $z$ is an **attracting fixed point** if for $z'$ sufficiently close to $z$, the iteration of $z'$ under $f$ leads to eventual convergence to $z$. On the contrary, a fixed point $z$ is a **repelling fixed point** if for points $z'$ close to $z$, the iteration of $z'$ eventually diverges from $z$. This concept of attracting and repelling fixed points allows us to find potential regions of convergence and divergence in the complex plane under the iterative mapping $f$. Along with the study of fractals and physical systems, the idea of attractive fixed points plays an integral role in some roles of numerical computing methods such as Newton's method and fixed point iterations.

There are some nice results that can be shown regarding the attractiveness of a fixed point. If $z$ is a fixed point of $f$, it can be shown that $z$ is an **attracting** fixed point if $|f'(z)| < 1$. Similarly, $z$ is a **repelling** fixed point if $|f'(z)| > 1$.

The next important feature of iterative systems is the concept of cycles. For an iterative mapping $f$, a point $z$ is contained in a cycle on length $k$ if $f^k(z) = z$. A cycle is an ordered set of iterates that form a closed loop under the mapping $f$. Thus a cycle of length $k$ is a set of iterates $\{z_0, z_1, \ldots, z_{k-1}\}$ such that $f(z_{k-1}) = z_0$, or similarly that $f^k(z_0) = z_0$. Since we are describing a set of ordered set of iterates it must also be the case that $z_n = f(z_{n-1})$. We can see that the repeated iteration of any value within a cycle will remain within the cycle and that the starting point $z_0$ can be arbitrarily selected from any point in the cycle.

We can notice the similarity between this formula and the way we described fixed points. With this new definition we can observe that a fixed point is a cycle of length 1. For example, consider the mapping $f : C \to C$ given by $f(z) = z^2$. By solving $z^2 - z = 0$, we can see that $f$ has fixed points at $z = 0$ and $z = 1$. Therefore we have that the sets $\{0\}$ and $\{1\}$ are cycles of length 1 on $f$.

Now that we have defined cycles, we will begin to discuss how to find cycles for a given function. Let $f$ be an iterative mapping and assume that $z$ is a point in a cycle of length $k$ on $f$. We know that we must satisfy the condition that $f^k(z) = z$. Therefore we can say that each point along a cycle must be a root to $f^k(z) - z = 0$.

For example, consider the mapping $f : C \to C$ given by $f(z) = z^2 - 1$. We can find the points that are contained in cycles of length 2 by solving for the roots of $f^2(z) - z = 0$. The first step here is to compute $f^2(z) = f \circ f(z)$.

$$\begin{aligned}
f^2(z) = f \circ f(z) &= f(z^2 - 1) \\
f \circ f(z) &= (z^2 - 1)^2 - 1 \\
f \circ f(z) &= (z^4 - 2z^2 + 1) - 1 \\
f \circ f(z) &= z^4 - 2z^2
\end{aligned}$$

Now that we have computed $f^2(z)$, we can consider finding the roots of $f^2(z) - z = 0$.

$$\begin{aligned}
&f^2(z) - z = 0 \\
&z^4 - 2z^2 - z = 0 \\
&z(z^3 - 2z - 1) = 0 \\
&z(z+1)(z^2 - z - 1) = 0 \\
&z(z+1)(z - \frac{1+\sqrt{5}}{2})(z - \frac{1-\sqrt{5}}{2}) = 0
\end{aligned}$$

Thus we have that the points contained in 2-cycles of $f$ are $\{0, -1, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\}$. The final step is to determine which points for cycles. We can do this by evaluating $f$ at any of the cyclic points to determine the iterates. We will perform this example starting with the root $z = 0$.

$$\begin{aligned}
&f^1(0) = (0)^2 - 1 = -1 \\
&f^2(0) = f(-1) = (-1)^2 - 1 = 0
\end{aligned}$$

We can see that after two iterations, the point $z = 0$ was mapped back to itself, confirming our assumption that it was a point along a 2-cycle. The iterates of $z = 0$ will be given by $\{0, -1, 0, -1, \dots\}$, from which we can say that $\{0, -1\}$ forms a cycle of length 2 on $f$. From the properties of cycles, we know that no point contained in a cycle on $f$ has any iterates not contained within that cycle. This also implies that there cannot be any overlap between any cycles in $f$. Thus we can consider the iterates of $z = \frac{1+\sqrt{5}}{2}$.

$$f^1(\frac{1+\sqrt{5}}{2}) = (\frac{1+\sqrt{5}}{2})^2 - 1$$
$$f^1(\frac{1+\sqrt{5}}{2}) = \frac{6+2\sqrt{5}}{4} - 1$$
$$f^1(\frac{1+\sqrt{5}}{2}) = \frac{2+2\sqrt{5}}{4}$$
$$f^1(\frac{1+\sqrt{5}}{2}) = \frac{1+\sqrt{5}}{2}$$

This yields an interesting result. We have found that the point $z = \frac{1+\sqrt{5}}{2}$ is actually a fixed point of $f$. In a similar fashion, it can be shown that $\frac{1-\sqrt{5}}{2}$ is also a fixed point of $f$. This is because of the repeating pattern of cycles on iterative functions.

Since fixed points are cycles that repeat every iteration, we can deduce the neat result that fixed points are cyclic points for cycles of any size on $f$. We can also find a greater generalization of this result on non-fixed point cycles.

Let $z_9$ be a cyclic point of length $k$ under a mapping $f$. Let $m \geq k$ such that $m = nk$ for some $n \in N$. Now consider the mapping $g(z) = f^k(z)$. We know that $z_0$ is a cyclic point of $f$ in a cycle that repeats every $k$ iterations. Thus it follows that $f^k(z_0) = z_0$, which implies that $z_0$ is a fixed point of $g$. Since $z_0$ is a fixed point on $g$, it follows clearly that $g^n(z_0) = z_0$. Lastly, given our definition of $g$ we have that $g^n(z) = f^{nk}(z) = f^m(z)$. Therefore we have the result that $f^m(z_0) = z_0$, so $z_0$ is contained in a cycle of length $m$.

This gives us the generalization that if $z$ is contained in a cycle of length $k$ under a mapping $f$, then $z$ is also contained in a cycle of length $m$ for any $m$ that is a multiple of $k$. This concludes our discussion of cyclic points, but there are many other interesting results that can be found regarding cycles of iterative maps.

# 4 Chaos

Now that we have discussed the major features of iterative systems, we can begin to explore the geometry that arises from these systems. While our typical intuition of fractals is of figures with very intricate and orderly geometry, it seems quite contradictory that most fractals stem from chaos. Within dynamical systems, we consider a seed value $z_0$ to be chaotic under a mapping $f$ if it has a neighborhood that behaves chaotically. That is to say that a small pertubation from $z_0$ leads to drastically different results under repeated iteration of $f$. In generality, a chaotic point is a point that is highly sensitive to initial conditions. This idea of chaos and chaotic points is critical in forming the structures we associate with fractals.

Let $f : C_\infty \to C$ be an iterative map. We define the **Julia Set** to be the set of points in $C_\infty$ that behave chaotically. We then define the **Fatou Set** to be the set of non-chaotic points. Equivalently, we can define the Fatou set as the compliment of the Julia set.

Using these two definitions, we can divide the extended complex plane into two subsets. This natural division provides us with a geometric structure in the Julia set with which we can produce very interesting images. While this is a binary division of the extended complex plane, more sophisticated models use convergence rates in order to form a color gradient for even greater contrast. These images can be formed from any iterative function mapping. However one of the most widely studied mappings in fractal dynamics is mappings of the form $f(z) = z^2 + c$ for $c \in C$. This class of functions is known to produce fascinating results, and is also the class of functions studied under the famous Mandelbrot Set. For more on the Mandelbrot Set, see section 5 below.

Below are some computer generated images of the Julia Sets of different iterative functions in the complex plane. These images were produced using applets that were created in conjunction with the paper "Complex Analysis Topics for Undergraduates and Beginning Researchers: an Exploration with Unsolved Problems"[3]:

Figure 6: Julia set for $f(z) = z^2 - 1$

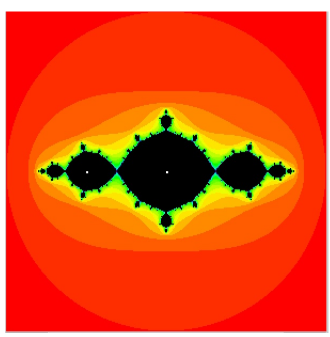


Figure 7: A Dragon curve, the Julia set for $f(z) = z^2 + 0.36 + 0.1i$

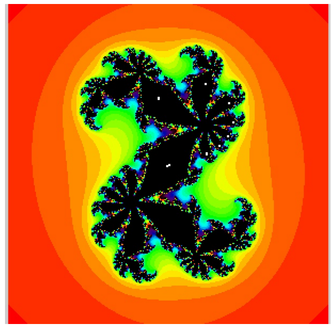


Figure 8: An Elephant curve, the Julia set for $f(z) = z^2 - 0.77 + 0.173i$

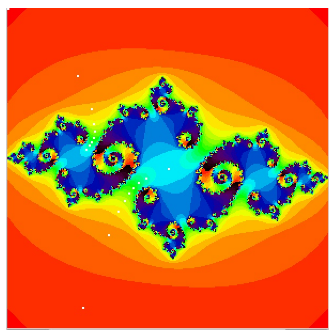
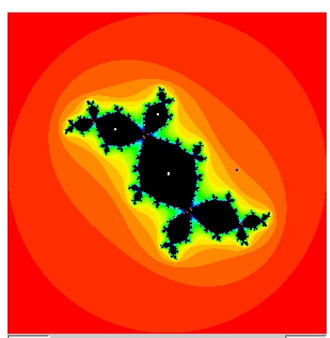


Figure 9: The Douady Rabbit, the Julia set for $f(z) = z^2 - 0.123 + 0.745i$



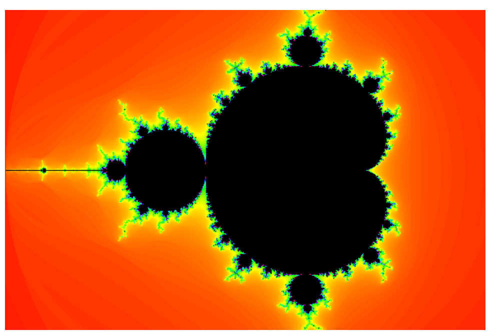
# 5 The Mandelbrot Set

One of the most widely known topics within the study of Fractals is the Mandelbrot Set. As mentioned in Section 4, the Mandelbrot revolves around the class of iterative functions of the form $f(z) = z^2 + c$ for $c \in C$. Previously when forming fractal structures, we considered the geometry formed by the Julia and Fatou sets. The Mandelbrot set takes a different approach to forming fractal structures. Instead the Mandelbrot set is determined by the parameter $c$ that defines the functions of the class $f(z) = z^2 + c$. This is an interesting perspective to take on a class of functions, and illuminates the infinite number of possible intriguing structures that can be generated from a single class of functions.

A value $c$ is in the Mandelbrot set if and only if the iteration of $f(z) = z^2 + c$ does not diverge for an initial value of $z_0 = 0$. Similarly, a point is in the Mandelbrot set if and only if $f^n(0)$ is bounded for all $n \in N$.

The most fascinating part about the Mandelbrot set comes in the visualization of the Mandelbrot set. The set contains an intricate and elaborate boundary that contains a beautiful self symmetry at finer scales of magnification. The figure below displays the Mandelbrot set. The two primary features of the set are the main cardioid (the rounded feature on the right) and the smaller left cardioid. It can be seen through magnification that the small bulbs around the perimeter of the set are all self similar, and even contain more self similar bulbs on their perimeters. This pattern repeats infinitely throughout finer and finer scales, and even more fractal patterns begin to arise through magnification of the set.

Figure 10: The Mandelbrot Set[9]



# 6 Fractal Dimension

We have discussed the underlying complex dynamics of fractal structures. While these figures may provide some stunning results, we must consider the question of how we can apply the field of fractals to solve real world problems. The primary basis upon which this is done is through the concept of fractal dimension. The **fractal dimension** of a structure is a measure of how the structure's complexity varies with scale. With this measure, we have a strong tool for analyzing the complexity of geometric figures.

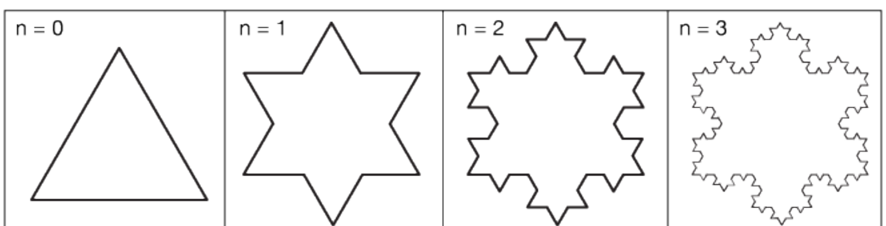
There are many applications of fractal dimension throughout multiple fields such as engineering, geography, and most interestingly biology. Some of these applications are discussed in more detail in Section 6. First we will roughly discuss some of the methods used to estimate the fractal dimension of a given structure. We will consider simpler fractals that have a defined construction pattern in order to gain an understanding of fractal dimension. Such fractals will be composed of line segments of equal length, and each iteration will be generated by a predetermined construction rule. Then we will analyze how this can be extrapolated to fractals that do not follow a concise construction pattern.

The following methods describing fractal dimension are from the paper "Practical Application of Fractal Analysis: Problems and Solutions" [1].Consider a fractal that is constructed iteratively through a repeated construction pattern from some base shape. Let $N_l$ denote the number of line segments at construction level $l$, and let $L_l$ denote the length of each line segment along the curve. We then define the fractal dimension $D$ to be given by the following:

$$D = \frac{\log(N_l/N_{l+1})}{\log(L_{l+1}/L_l)}$$

The first iterative fractal we will consider is the Koch Snowflake. This geometric fractal is generated by a repeated construction pattern. From the figure below, we can see that the construction is straightforward, as each new construction level is generated by adding triangles of $\frac{1}{3}$ the size to each of the edges in the previous iteration.

Figure 11: The Koch Snowflake Construction Pattern




| | | | | |
|---|---|---|---|---|
| Number of sides (N) | 3 | 12 | 48 | 192 |
| Side length (S) | 1 | 1/3 | 1/9 | 1/27 |
| Perimeter length (P) | 3 | 4 | 5.33 | 9.11 |

[10]

As an example, we will estimate the fractal dimension of this figure using construction level $l = 0$. For this case we see that $N_0 = 3$ and $N_1 = 12$. Similarly we have that $L_0 = 1$ and $L_1 = \frac{1}{3}$. This gives us the following result:

$$D = \frac{\log(3/12)}{\log((1/3)/1)} = \frac{\log(1/4)}{\log(1/3)} = 1.2618$$

From this we have that the fractal dimension of the Koch Snowflake is roughly 1.2618. One of the challenges with fractal dimension is that is it a somewhat abstract measurement that is typically unfamiliar. That is to say that we have far less intuition as to what types of values to expect and what values represent very extreme complexity. Typically a fractal dimension near 1.0 represents a smooth surface, which follows from our formula. Similarly values around and above 2.0 typically represent high complexity. For reference, the boundary of the Mandelbrot set has a fractal dimension of 2.0. Below are some other famous fractals and their fractal dimensions for even further reference:

Figure 12: The Sierpinski Triangle [11], fractal dimension of 1.5850

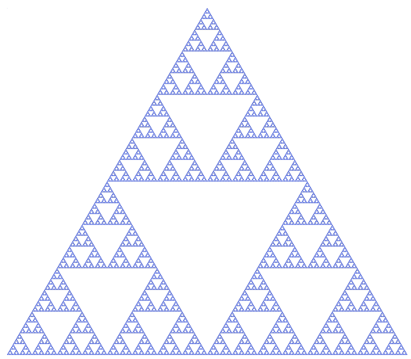


Figure 13: The Apollonian Gasket [11], fractal dimension of 1.3057

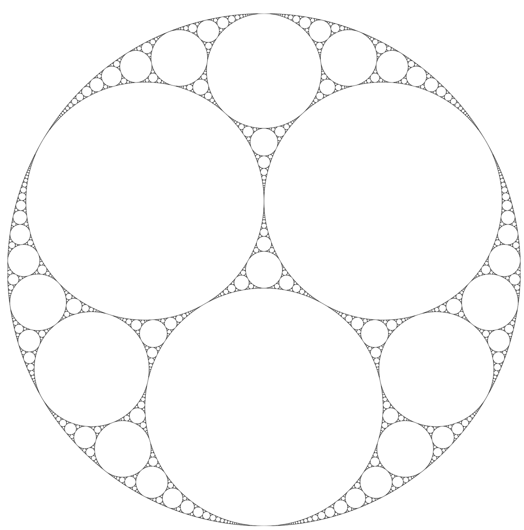


Figure 14: The Dragon Curve [11], fractal dimension of 1.5236

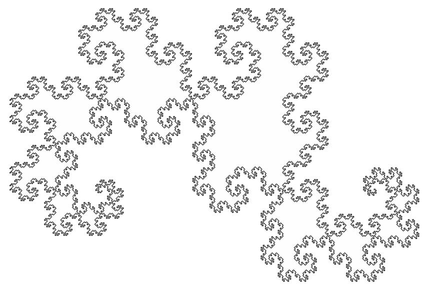
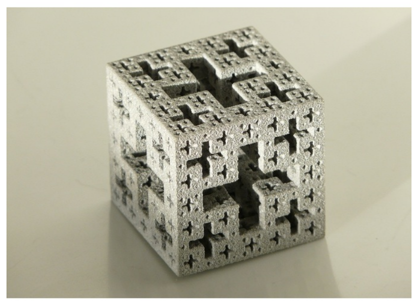


Figure 15: The Jerusalem Cube [11], fractal dimension of 2.529



While the formula discussed is very useful for calculating the fractal dimension of figures with concise construction patterns, this formula clearly does not hold for more general fractals. If we have a figure in which the line segments are not of even lengths and where there is no overarching self symmetry, we must find other methods to approximate the fractal dimension. The method we are about to discuss uses another interpretation of fractal dimension using the power law.

Let $L(\eta)$ denote the length of a curve that is made of line segments of equal length $\eta$. The power law gives us the following result regarding the fractal dimension $D$ of the curve:

$$L(\eta) = \eta^{1-D}$$

We can then apply a logarithmic transformation to this equation to obtain a similar equation that will be more useful for some future results. Note that this equation is linear in terms of the line segment length $\eta$.

$$\log(L(\eta)) = \alpha + (1 - D)\log(\eta)$$

The most common method for approximating the fractal dimension of a curve is known as the Walker's Ruler Algorithm [1]. In order to yield an accurate measure of fractal dimension with this method we must have that the curve is continuous, but it need not be differentiable. This algorithm works by imposing multiple fixed length rulers to approximate the curve. Each ruler will be a line segment length with a fixed length $\eta$, and we will measure the number of segments $n$ of length $\eta$ that are needed to span the entire curve. We require that the first segment used starts at one endpoint of the curve and has it's other endpoint along the curve. We can then connect each successive line segment to the end of the previous segment until we have spanned the entire fractal. If a ruler of length $\eta$ requires $n$ segments to cover the entire curve, then we can approximate the length of the curve to be $L(\eta) = n \times \eta$.

The algorithm is run by using $M$ different fixed length rulers $\eta_1, \eta_2, \ldots, \eta_M$ in order to gather $M$ different measurements $L(\eta_1), L(\eta_2), \ldots, L(\eta_M)$. With a sufficient number of data points $M$, we can then approximate the fractal dimension using regression. We do this by fitting our data points to the logarithmic equation derived from the power law. If we let $x_i = \log(\eta_i)$ and $y_i = \log(L(\eta_i))$, then we get the following system of equations:

$$y_i = \alpha + (1 - D)x_i$$

Therefore applying linear regression to these data points will give us an approximation to the slope $\lambda$ of the best fit line. From this, we can estimate the fractal dimension of the entire curve by letting $D = 1 - \lambda$.

This algorithm has many practical applications, and is commonly used in approximating the fractal dimension of real world curves. The other prominent method for approximating fractal dimension is known as the box-counting algorithm, but that will not be discussed in this paper. Now we will look into some applications of fractal dimension outside of traditional mathematical studies.

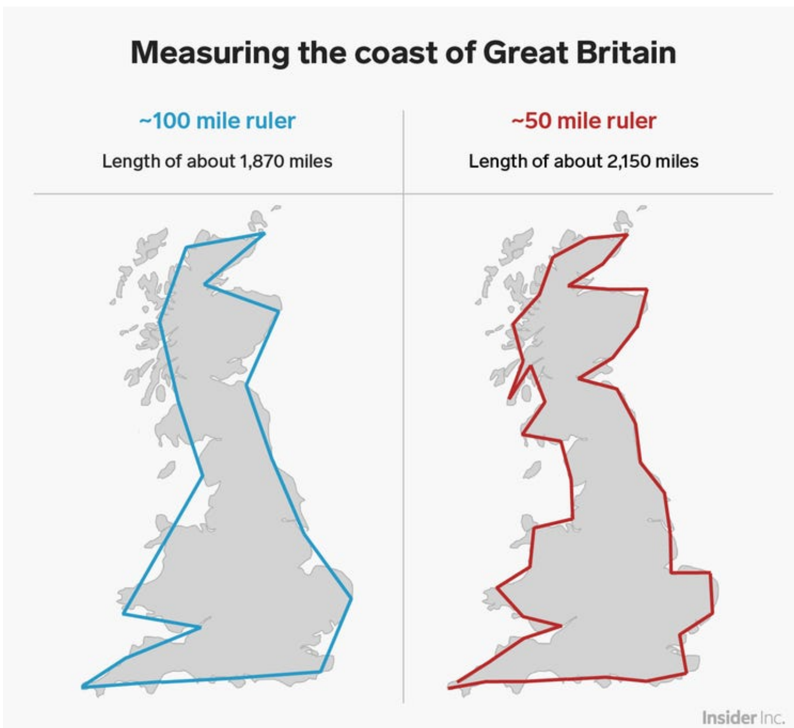
# 7 Applications of Fractals

### The Coastline Paradox

One interesting problem involving fractals is known as the coastline paradox. This paradox arises from attempting to measure the length of the coastline surrounding a landmass. The paradox is that coastlines typically are fractal-like and thus have a fractal dimension. Since coastlines behave like fractals, it follows that there will always be a degree of uncertainty in our measurement of the coastline length. This is because we must use some fixed ruler length in order to measure the length of the coastline. Since the coastline has a fractal dimension, this ensures that no matter our measurement length, we will always underestimate the length of the coastline. Using a smaller ruler length will increase our accuracy, but we know that fractals are infinitely repeating structures and thus our new measurement will always underestimate the true length. The smaller our measurements get, the more complexity we introduce, and in theory this would repeat down to the level of single grains of sand along the coastline. It can be shown that the measured length actually increases exponentially with reduced measurement size.

In some ways this paradox violates our expectations by suggesting that a coastline may have an immeasurable and thus infinite length. On a similar note, this suggests that all infinitely recurring fractals have a finite area but an infinite perimeter. Below is an example of this paradox, showing the length of the British coastline approximates using two different fixed length rulers.

Figure 16: British Coastline measurements with varying scale[12]




### Biomedical Systems

In many cases, fractals appear in systems we may not immediately expect. One of these areas is the study of biomedical systems. With all of the variability in stable living conditions, there is certainly a large amount of chaos that naturally drives these systems. Highly complex fractal structures can be found throughout the human body in areas such as neural networks, t. For example, consider the nervous and cardiovascular systems. These systems are comprised of large networks of neurons and blood vessels that cover the entire body. Along with being vast networks, these systems are known to have an orderly structure including a recursive self-symmetry. This is the baseline for our notion of a fractal and thus opens the door for the exploration of these systems, along with many other systems of the body.

In a 2019 study titled "A Healthy Dose of Chaos: Using fractal frameworks for engineering higher-fidelity biomedical systems"[2], researchers applied the concept of fractal dimension to measure the complexity of different human body systems. The results that they found were quite eye-opening and illuminate the potential for fractal applications in the future.

These researchers found that in nearly every system of the human body, there is a correlation between fractal dimension and chaos of the system and incidence of different health conditions. For example, there is a fractal structure of the fibers in the cardiovascular system that actually follows a characteristic power law. They found that individuals with Alzheimer's disease and stroke had a significantly low fractal dimension in the vasculature system. Using a similar method to look at the structure of the lungs, they found that individuals with asthma had a low fractal dimension, and individuals with Pulmonary Fibrosis had a significantly high fractal dimension in the tissue of their lungs. They analyzed multiple other systems of the body and found similar results for each, suggesting that there is a "natural dose of chaos to normal biological function".

This study shows evidence that not only are fractals the backbone of many anatomical structures, but there is also a natural force that favors certain optimal fractal dimensions. The applications of fractals within the biomedical field are limited and there is a great amount of room for future exploration. This study is one example of just how important fractals are to life and opens the door for the future study of fractals outside of traditional mathematics. Chaos can be found in almost every scientific field, and the study of fractals allows us to derive order from the chaos.

# 8 Resources


1. Gonzado, Guido, et al. "Practical Application of Fractal Analysis: Problems and Solutions." Oxford Academic, 1 Feb. 1998, academic.oup.com/gji /article/132/2/275/671597.

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