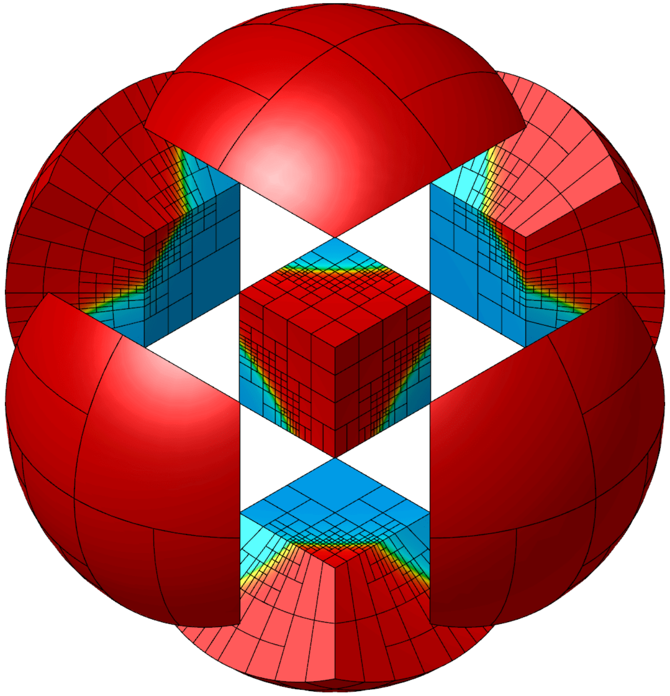

MFEM Tutorial

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Weak Formulation

Consider the Poisson equation with homogeneous Dirichlet BC

$$\begin{aligned} -\Delta u &= f \qquad &&\text{in } \Omega, \\ u &= 0 \qquad &&\text{on } \partial\Omega. \end{aligned}$$

The corresponding weak formulation is: Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) := (\nabla u, \nabla v) = (f, v) =: b(v) \quad \forall v \in H_0^1(\Omega).$$

Discrete Space

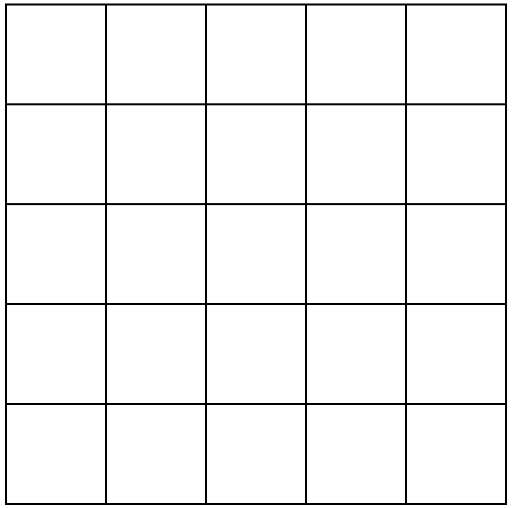
Let $\mathcal{T}_h$ be a given mesh, and

$$V_h = \{v \in C^0(\Omega) : v|_T \in \mathbb{P}_p(T)\ \forall T \in \mathcal{T}_h\}.$$

We impose the boundary condition by setting

$$V_h^g = \{v \in V_h : v|_e = \Pi_h^e g\ \forall e \in \mathcal{E}_h\}$$

where Π_h^e is a projection onto $\mathbb{P}_p(e)$ and globally continuous if necessary.



Discrete Formulation

The corresponding discrete problem is: Find $u_h \in V_h^g$ such that

$$a_h(u_h, v) := \sum_{T \in \mathcal{T}_h} (\nabla u_h, \nabla v)_T = \sum_{T \in \mathcal{T}_h} (f, v)_T =: b_h(v) \quad \forall v \in V_h^0.$$

Here, $a_h(\cdot,\cdot)$ is a bilinear form, and $b_h(\cdot)$ is a linear form so that

- $a_h(u + w, cv) = ca_h(u, v) + ca_h(w, v)$ for all $u, w, v \in V_h$ and $c \in \mathbb{R}$,
- $a_h(cu, v + w) = ca_h(u, v) + ca_h(u, w)$ for all $u, v, w \in V_h$ and $c \in \mathbb{R}$,
- $b_h(cv + w) = cb_h(v) + b_h(w)$ for all $v, w \in V_h$ and $c \in \mathbb{R}$.
- Note that $a_h(\cdot,\cdot) = a(\cdot,\cdot),\ b_h(\cdot) = b(\cdot)$ for the most of conforming FEMs.
- But, $a_h(\cdot,\cdot) \neq a(\cdot,\cdot)$ when you consider, e.g., Discontinuous Galerkin method.

Basis of V_h

- Let $\{\phi_i\}_{i=1}^N$ be a basis of V_h. Then for given $v \in V_h$, there exists $(c_1, \cdots, c_N)$ such that

$$v = \sum_{j=1}^{N} c_j \phi_i$$

- Then we can rewrite the equation as: Find $u_h = \sum_{j=1}^N c_j \phi_j$ or find $\{c_j\}_{j=1}^N$ such that

$$a_h(\sum_{j=1}^{N} c_j \phi_j, v) = \sum_{j=1}^{N} c_j a_h(\phi_j, v) = b_h(v) \quad \forall v \in V_h.$$

- Since V_h is finite dimensional and $a_h(\cdot, \cdot)$ is a bilinear form, it is enough to test the equation against basis functions.

$$\sum_{j=1}^{N} c_j a_h(\phi_j, \phi_i) = b_h(\phi_i) \quad \forall i \in 1, \cdots, N.$$

Linear System

- Now we obtained a set of equations

$$\sum_{j=1}^{N} c_j a_h(\phi_j, \phi_i) = b_h(\phi_i) \quad \forall i \in 1, \cdots, N.$$

- This can be rewritten as

$$Ax = b$$

  where

$$A_{ij} = a_h(\phi_j, \phi_i), x_i = c_i, b_i = b_h(\phi_i)$$

Summary

- For given finite element space V_h with $dim(V_h) = N$, we find solution $u_h \in V_h$ by

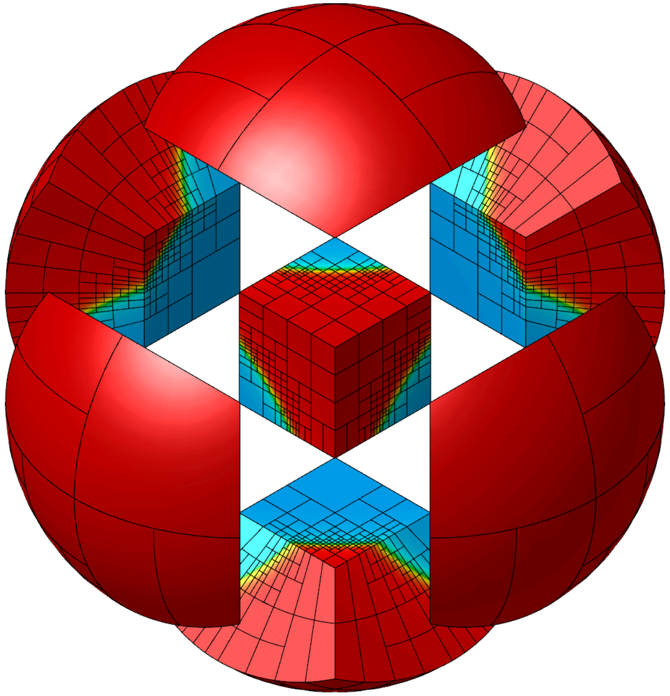
$$a_h(u_h, v) := \sum_{T \in \mathcal{T}_h} (\nabla u_h, \nabla v)_T = \sum_{T \in \mathcal{T}_h} (f, v)_T =: b_h(v) \quad \forall v \in V_h$$

- Using the basis representation, we obtain the corresponding linear system

$$Ax = b$$

where

$$A_{ij} = a_h(\phi_j, \phi_i),\ x_i = c_i,\ b_i = b_h(\phi_i)$$



Finite Element Space

Discrete Space

FiniteElementCollection

Mesh

FiniteElementSpace

GridFunction / Coefficient

Discrete Space

- Recall that a finite space is defined by

$$V_h = \{v \in V : v|_T \in V_T \text{ for all } T \in \mathcal{T}_h\}.$$

- We need three components to define V_h
  - Local space: V_T
    - Polynomial space $P_k(T)$, Raviart-Thomas space $RT_k(T)$, etc.
  - Mesh: $\mathcal{T}_h$
    - Triangular mesh, Quadrilateral mesh, Tetahedral mesh, etc.
  - Global continuity or regularity V
    - $L^2(\Omega)$, $C^0(\Omega)$, $C^1(\Omega)$, etc.
- The local space is `FiniteElementCollection`
- The mesh is `(Par)Mesh`
- Global continuity is encoded in `FiniteElementCollection`

FiniteElementCollection

- We use `FiniteElementCollection` to represent a local finite element space.
- Examples are: (P_k: Complete polynomial, Q_k: Tensor product polynomial)
  - `H1_FECollection` - Continuous P_k/Q_k space
  - `DG_FECollection/L2_FECollection` - Discontinuous P_k/Q_k space
  - `RT_FECollection` - $H(div)$-conforming Raviart-Thomas space
  - `ND_FECollection` - $H(curl)$-conforming Nedelec space
  - see, FiniteElementCollection for exhaustive list.
- Code Example:

```
FiniteElementCollection *fec = new DG_FECollection(order, dim);
FiniteElementCollection *fec = new H1_FECollection(order, dim);
```

Mesh

- We use `Mesh` class to represent meshes.
- A `Mesh` object can be created either from a file

```
Mesh mesh("path/to/meshfile.mesh");
```

- or

```
Mesh mesh = Mesh::MakeCartesian2D(nx, ny, Element::QUADRILATERAL); // Uniform 2D rectangular mesh
```

- You can also perform refinements or save the mesh

```
Mesh mesh = Mesh::MakeCartesian2D(nx, ny, Element::TRIANGLE);
mesh.UniformRefinement();     // Uniform refine
mesh.RefineByError(err, tol); // Refine where err > tol
mesh.Save("path/to/meshfile.mesh");
```

- It supports nonconforming and/or curved meshes.
- The mesh file can be created 1) manually, 2) Gmsh, and so on. See, this guide.

Finite Element Space

- With `FiniteElementCollection` and `Mesh`, we can define a finite element space V_h.
- We use a `FiniteElementSpace` class to store the information
- For example, $P^1 - C^0$ finite element space associated with a uniform rectangular mesh

```
const int dim = 2;
const int order = 1;
const int nx = ny = 10;
FiniteElementCollection *fec = new H1_FECollection(order, dim);
Mesh mesh = Mesh::MakeCartesian2D(nx, ny, Element::QUADRILATERAL);
FiniteElementSpace fes(&mesh, fec); // P1-C0 Finite element space
```

- A vector finite element space can be constructed either
  - with vector `FiniteElementCollection`, e.g., `RT_FECollection`
  - with tensor product space with a scalar `FiniteElementCollection`

```
const int vdim = 2;
FiniteElementSpace vfes(&mesh, fec, vdim); // P1-C0 vector FE space
```

GridFunction

- A discrete function $u_h \in V_h$ can be created using

```
GridFunction u(&fes);
```

- For a discrete function $u_h \in V_h$, there exists a unique $\{c_i\}_{i=1}^N \subset \mathbb{R}$ such that

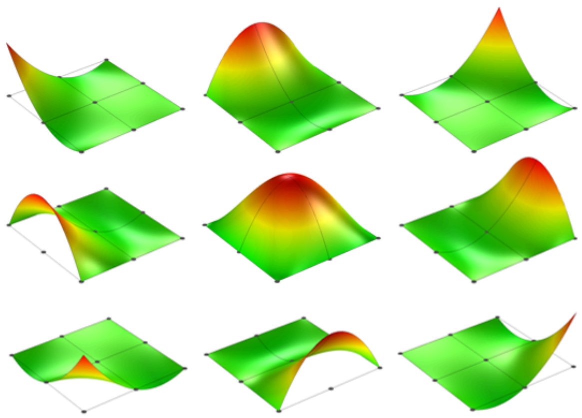
$$u_h(x) = \sum_{i=1}^N c_i \phi_i(x) \text{ where } V_h = \operatorname{span}(\{\phi_i\}_{i=1}^N).$$

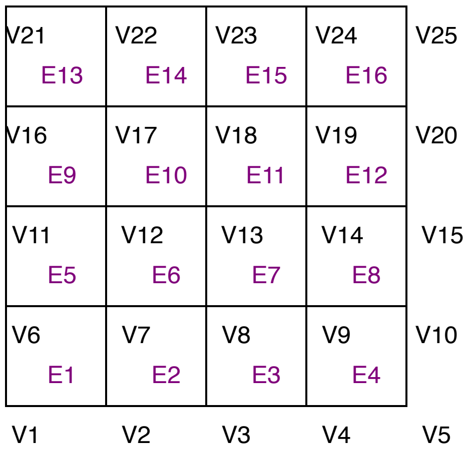
- A `GridFunction` stores the primal vector, `[c_1, c_2, ..., c_N]` associated with u_h and V_h (`fes`).
- If we use nodal basis functions $\phi_i(x_j) = \delta_{ij}$, then $c_i = u_h(x_i)$ for each node x_i.

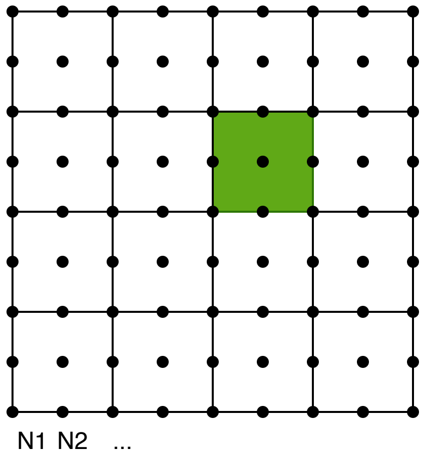
Summary

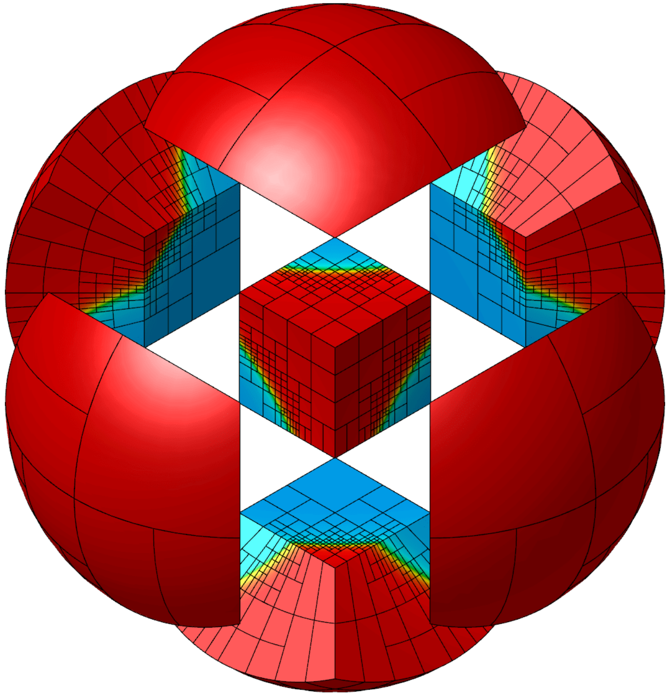
```
const int order = 2; // order of polynomial
const int nx = ny = 4; // number of elements in each axis

FiniteElementCollection *fec = new H1_FECollection(order, dim); // C0, Quadratic Polynomial
Mesh mesh = Mesh::MakeCartesian2D(nx, ny, Element::QUADRILATERAL); // Uniform quadrilateral mesh
const int dim  = mesh.Dimension(); // spatial dimension, 2
// You can also define a vector FE space
const int vdim = 2; // vector dimension
FiniteElementSpace fespace(&mesh, fec, vdim); // [C0-Q2 space]^2
GridFunction u(&fespace);
```









BilinearForm and LinearForm

BilinearForm

LinearForm

NonlinearForm

FiniteElementSpace FiniteElementCollection

Mesh

Find $u_h \in V_h = \{v \in V : v|_T \in V_T,\ \forall T \in \mathcal{T}_h\}$ such that

GridFunction

$$\sum_{T\in\mathcal{T}_h} (\nabla u_h, \nabla v)_T =: a_h(u_h, v) = b(v) := \sum_{T\in\mathcal{T}_h} (f, v)$$

Find $u_h \in V_h = \{v \in V : v|_T \in V_T,\ \forall T \in \mathcal{T}_h\}$ such that

$$\sum_{T \in \mathcal{T}_h} \underbrace{(\nabla u_h, \nabla v)_T}_{\text{BilinearFormIntegrator}} =: \overset{\text{BilinearForm}}{\overset{\downarrow}{a_h}}(u_h, v) = \overset{\text{LinearForm}}{\overset{\downarrow}{b}}(v) := \sum_{T \in \mathcal{T}_h} \underbrace{(f, v)}_{\text{LinearFormIntegrator}}$$

BilinearForm

- We want to express our bilinear form $a_h(u,v) = \sum_{T\in T_h} a_T(u,v)$.
- This operator is $a_h(\cdot,\cdot) : V_h \times V_h \to \mathbb{R}$.
- We use `BilinearForm` to represent such operator

```
BilinearForm a_h(&fespace);
```

- If $a_h(\cdot,\cdot) : V_h \times W_h \to \mathbb{R}$, then

```
MixedBilinearForm a_h(&fespace1, &fespace2);
```

- We can add local bilinear forms to populate our bilinear form `a_h`.

```
a_h.AddDomainIntegrator(new DiffusionIntegrator()); // a_h += (∇u, ∇v)
a_h.AddDomainIntegrator(new MassIntegrator());      // a_h += (u, v)
```

- `DiffusionIntegrator` and `MassIntegrator` are `BilinearFormIntegrator`
- This corresponds to a local bilinear form $a_T(u,v)$. See, `fem/bilininteg.hpp`

LinearForm

- Similarly, we have a linear form $b_h(v) = \sum_{T \in T_h} b_T(v)$.
- This operator is $b_h(\cdot) : V_h \to \mathbb{R}$.
- We use `LinearForm` to represent such operator

```
LinearForm b_h(&fespace);
```

- We can add local linear forms to populate our linear form `b_h`.

```
b_h.AddDomainIntegrator(new DomainLFIntegrator(f)); // b_h += (f,v)
b_h.AddBoundaryIntegrator(new BoundaryLFIntegrator(g_N));// b_h += (g_N, v)_e
```

- `DomainLFIntegrator` and `BoundaryLFIntegrator` are `LinearFormIntegrator`
- This corresponds to a local linear form $b_T(v)$. See, `fem/lininteg.hpp`

Assembly

- With basis $\{\phi_i\}$ of V_h, $a_h(\cdot,\cdot)$ can be realized

$$a_h(\phi_i,\phi_j) = \sum_{T\in\mathcal{T}_h} a_T(\phi_i,\phi_j).$$

- For given T, we define

$$J_T = \{j \in \mathbb{N} : \phi_j|_T \neq 0\}$$

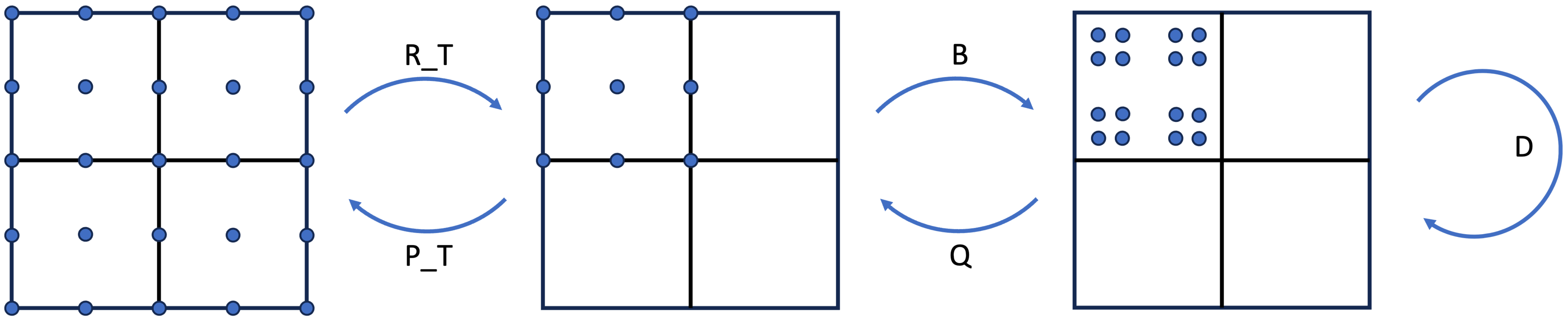
- If basis is local, $|J_T| \ll dim(V_h)$. Also, we have

$$a_T(\phi_i,\phi_j) = \begin{cases} c & \text{if } i,j \in J_T, \\ 0 & \text{otherwise.} \end{cases}$$

  Similar condition holds for $b_T(\phi_j)$.

- This implies sparsity of A, and provides an assembly procedure.

Assembly



- `R_T`, `P_T`: Mapping between the global and local
  - `R_T*A = A(J_T, J_T) = A_loc`
- `B`, `Q`: Mapping between basis and values at points.
  - `B` can be function value, derivative value, etc.
- `D`: Operation on quadrature values (dot products, weight, ...)
  - `D(U, Ux, Uy, V, Vx, Vy) = Ux*Vx + Uy*Vy`

Assembly - Pseudocode

Algorithm 2 Matrix Assembly

for all $T \in \mathcal{T}_h$ **do**
  $J_T \leftarrow$ local dofs related to T
  for all $a_T \in$ `BilinearForm` **do**
    for all x_q in `quadtrature_points` **do**
      for all ϕ_i, ϕ_j with $i, j \in J_T$ **do**
        ▷ $a_T(u,v) = (L_1(u), L_2(v))$ with some operators L_1, L_2
        `A_loc(i,j)+=` $w_q L_1(\phi_i(x_q)) L_2(\phi_j(x_q))$
      end for
    end for
  end for
  `A_glb(J_T,J_T)+=A_loc`
end for

Assembly - Pseudocode

Algorithm 2 Matrix Assembly

for all $T \in \mathcal{T}_h$ **do**
  $J_T \leftarrow$ local dofs related to T
  for all $a_T \in$ `BilinearForm` **do**
    for all x_q in `quadtrature_points` **do**
      for all ϕ_i, ϕ_j with $i, j \in J_T$ **do**
        $\triangleright$ $a_T(u, v) = (L_1(u), L_2(v))$ with some operators L_1, L_2
        `A_loc(i,j)+=` $w_q L_1(\phi_i(x_q)) L_2(\phi_j(x_q))$
      end for
    end for
  end for
  `A_glb(J_T,J_T)+=A_loc`
end for

FormIntegrator::ElementAssemble(...)

How to write FormIntegrator

```
void MassIntegrator::AssembleElementMatrix(
    const FiniteElement &el, ElementTransformation &Trans, DenseMatrix &elmat)
{
   // INPUT el: Local finite element: basis, dof, ...
   // INPUT Trans: Local mesh information. e.g., element location, Jacobian
   // OUTPUT elmat: resulting (ϕ_j, ϕ_i) where 0 ≤ i,j ≤ nd is local dof
   int nd = el.GetDof(); // the number of basis related to current element
   double w; // store weight

   Vector shape; // store basis function values at an integration point
   shape.SetSize(nd);
   elmat.SetSize(nd); // set output size

   // Determine integration rule in the location
   const IntegrationRule *ir = IntRule ? IntRule : &GetRule(el, el, Trans);

   elmat = 0.0; // initialize with 0
   for (int i = 0; i < ir->GetNPoints(); i++) // for each integration point
   {
      // Get current integration point (x_i, w_i)
      const IntegrationPoint &ip = ir->IntPoint(i);
      Trans.SetIntPoint (&ip);
      //
      el.CalcPhysShape(Trans, shape);
      // update weight based on the element jacobian and integration weight
      w = Trans.Weight() * ip.weight;
      // elmat = elmat + w * (shape * shape^T)
      // Why? DIY
      AddMult_a_VVt(w, shape, elmat);
   }
} // Also see, DiffusionIntegrator, MixedVectorDivergenceOperator, ...
```

Assembly Level

```
a_h.SetAssemblyLevel(AssemblyLevel::<LEVEL>)
```

- `FULL` (or `LEGACY`)
  - Construct global sparse matrix
- `ELEMENT`
  - Assemble only local matrices `A_loc`. `Ax` can be computed element-wisely,

$$\mathtt{A*x} = \sum_T P_T * (\mathtt{A\_loc} * \mathtt{x\_loc})$$

- `PARTIAL`
  - Assemble only on a "reference" element, and other computations are done on-the-fly. See, here
- `NONE`
  - Store nothing, everything is on-the-fly.

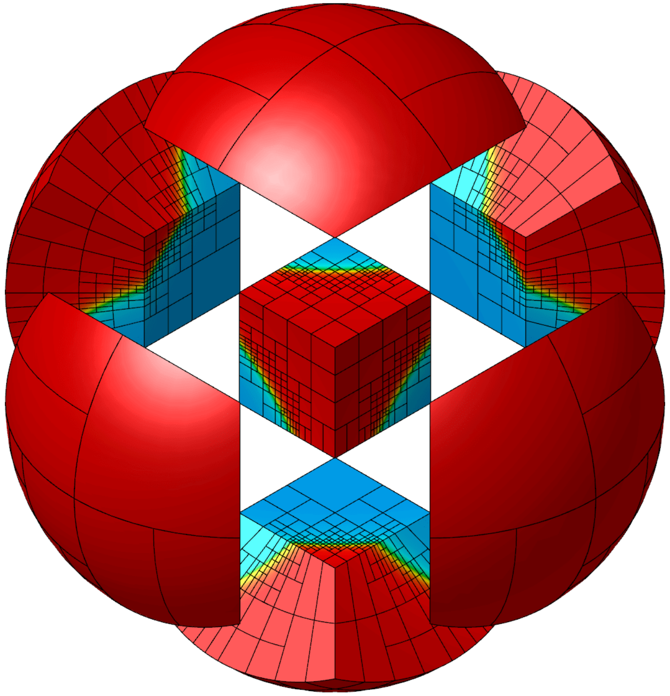
Coefficient

- Recall that we used `f` when we define local linear form

```
LinearForm b_h(&fespace);
b_h.AddDomainIntegrator(new DomainLFIntegrator(f)); // b_h += (f,v)
```

- Here, `f` is a `Coefficient`. More specifically, `FunctionCoefficient`.
- `Coefficient` represents a function

```
ConstantCoefficient one_cf(1.0);
FunctionCoefficient f([](Vector &x){return sin(pi*x[0])*sin(pi*x[1]);});
FunctionCoefficient g([](Vector &x, double t){return exp(-t)*sin(pi*x[0]);});
GridFunctionCoefficient u_cf(&u_h); // Use GridFunction as a function
u_h.ProjectCoefficient(f); // interpolate given function f
b_h.AddDomainIntegrator(new DomainLFIntegrator(f)); // b_h += (f,v)
```



Ax=b and Solve

Essential Boundary

FormLinearSystem

Solvers

RecoverFEMSolution

Visualization

$a_h(u, v) = b_h(v)$ and $Ax = b$

- Now we described our discrete problem $a_h(u, v) = b_h(v)$ in `MFEM`.
- Note that the essential BC has not been addressed yet.
- Recall that for C^0-finite element space,
  we impose Dirichlet boundary condition by
  $V_h^g = \{v \in V_h : v|_{\Gamma_D} = g_D\}$
- Degrees of freedom related to Γ_D are NOT unknowns.
- Therefore, we need to remove those degrees of freedom from the final linear system.

Essential Boundary

- In a mesh file, you can specify boundary attribute

```
# Mesh faces/edges on the boundary, e.g. triangles (2)
boundary
8 # 8 boundaries
1 1 0 1 # Bottom (1), Segment, vertex 0 - vertex 1
1 1 1 2 # Borrom (1), Segment, vertex 1 - vertex 2
2 1 2 4 # Right (2), Segment, vertex 2 - vertex 4
2 1 4 8 # Right (2), Segment, vertex 4 - vertex 8
...
```

- Suppose that you want to specify Dirichlet BC at bottom.

```
Mesh mesh(mesh_file);
// create an integer array of size, the number of boundary attributes
Array<int> ess_bdr(mesh.bdr_attributes.Max()); // in this case, 4
ess_bdr = 0; // initialize with 0
ess_bdr[0] = 1; // first boundary, bottom, is essential boundary
// stores all degrees of freedom on the Dirichlet Boundary
Array<int> ess_tdof_list;
fespace.GetEssentialTrueDofs(ess_bdr, ess_tdof_list); // Obtain related dofs
```

FormLinearSystem

- To obtain the linear system from `BilinearForm a_h` and `LinarForm b_h`, we can

  ```
  OperatorPtr A;
  Vector B, X;
  a_h.FormLinearSystem(ess_tdof_list, u, b_h, A, X, B);
  ```

- Here,
  - `u` : `GridFunction`, the solution with specified value on essential boundary
  - `A` : `OperatorPtr` or `SparseMatrix`, the resulting matrix A
  - `B` : `Vector`, the resulting load vector b (also called dual vector)
  - `X` : `Vector`, the vector containing coefficient, x (also called primal vector)

Solve Linear System

- To solve a system, there are many solvers in `MFEM`.
- Direct Sparse Solvers: `UMFPackSolver`: General matrix from `SuiteSparse`
- Iterative Solvers:
  - `CG`: (Preconditioned) Conjugate gradient method for SPD system

    ```
    CGSolver cg();
    cg.SetOperator(*A);
    cg.Mult(B, X);
    ```

  - `GMRES`: (Preconditioned) generalized minimal residual method for general system

    ```
    GMRESSolver gmres();
    gmres.SetOperator(*A);
    gmres.Mult(B, X);
    ```

Recover Solution

- Now, your solution is saved in `Vector X`.
- You can convert it back to a `GridFunction` by using

```
a_h.RecoverFEMSolution(X, b, u);
```

- Then you can save and visualize using

```
u.Save(solution.gf);
sout << "solution\n" << mesh << u; // plot solution
sout << "view 0 0\n"; // top view
sout << flush; // draw
```

- For more about visualization, see `ex1.cpp` and `glvis`.
- Note that you need to run `glvis` in the background!
- For 3D visualization or high-quality results, use `ParaView`.

Parallel Implementation

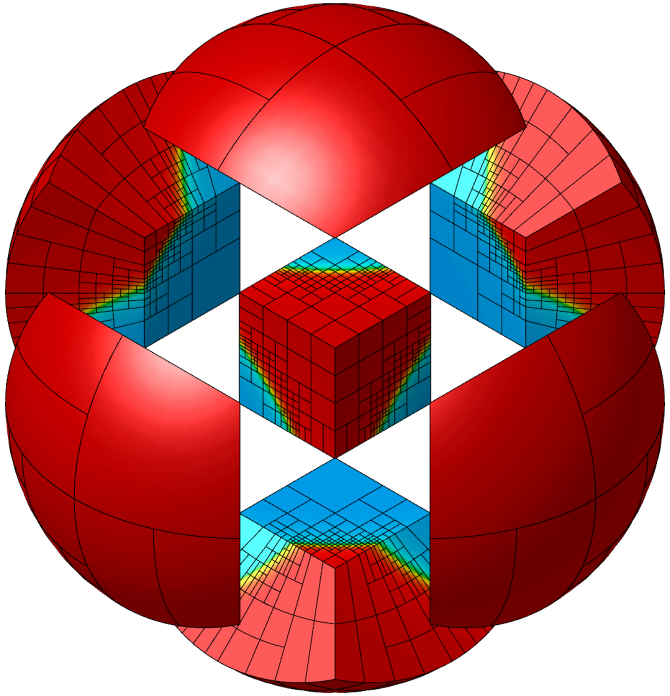
- `make parallel` - Requires additional dependencies, see instruction.
- Usually, you can just put `Par` before types. See, ex0p.cpp.

```
Mpi::Init(); // Initialize MPI
H1_FECollection fec(order, dim);
ParMesh mesh(mesh_file); // Elements are distributed in each process

ParFiniteElementSpace fes(&mesh, &fec);
ParBilinearForm a_h(&fes);
ParLinearForm b_h(&fes);
ParGridFunction u_h(&fes);

a_h.AddDomainIntegrator(new DiffusionIntegrator());
ConstantCoefficient one_cf(1.0);
b_h.AddDomainIntegrator(new DomainLFIntegrator(one_cf));
...
```

- Run with `mpirun -np <num_process> <file>`



Questions

FiniteElementSpace FiniteElementCollection Mesh

Find $u_h \in V_h = \{v \in V : v|_T \in V_T,\ \forall T \in \mathcal{T}_h\}$ such that

GridFunction BilinearForm LinearForm

$$\underbrace{\sum_{T\in\mathcal{T}_h} (\nabla u_h, \nabla v)_T}_{\text{BilinearFormIntegrator}} =: a_h(u_h, v) = b(v) := \underbrace{\sum_{T\in\mathcal{T}_h} (f, v)}_{\text{LinearFormIntegrator}}$$

https://dohyun-cse.github.io/mfem-tutorial

Appendix

- Mac OS need a patch and `patchelf`

```
brew install patchelf
git clone git@github.com:mfem/pymfem.git
git checkout rpath-patch
cd pymfem
python setup.py install ...
```

- `PyMFEM` needs `swig`

```
brew install swig
python -m pip install swig
python -m pip install mfem
python setup.py install
```

- `PyMFEM` with parallel

```
brew install open-mpi
python -m pip install mpi4py
python setup.py install --with-parallel
```