Applied Linear Algebra
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Homework

Chapter 4: Vector space

1. Determine whether the set, together with standard operations

a) $S = \{(x,y) : x \geq 0, y \in R\}$

Vector addition $(x_1, y_1), (x_2, y_2) \in S$

$\Rightarrow x_1, x_2 \geq 0 \quad , y_1, y_2 \in R$

$\Rightarrow (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in S$

Because of $x_1, x_2 \geq 0$; $x_1 + x_2 \geq 0$ and $y_1 + y_2 \in R$

Scalar multiplication

Let $(x,y) \in S \rightarrow x \geq 0, y \in R$

• $\alpha = -1$

$\Rightarrow \alpha(x,y) = -1(x,y) = (-x,y) \notin S$

$\Rightarrow$ S is not a vector space

b)

$S = \{(x, \frac{x}{2}) : x \in R\}$

Vector addition $(x_1, y_1), (x_2, y_2) \in S$

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$\Rightarrow (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in S$

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• Scalar multiplication

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$\Rightarrow$ S is not a vector space

2.

Is not a vector space $(x_1, y_1) + (x_2, y_2) = (x_1 y_1, x_2 y_2)$

$\neq$

$(x_2, y_2) + (x_1, y_1) = (x_2 y_2, x_1 y_1)$

3.

a) $W = \{(0, x_2, x_3) : x_2; x_3$ are real numbers $\}$

Proof: See $(0,0,0) \in W \Rightarrow W \neq \emptyset$

Let $u, v \in W$

$\Rightarrow u = (0, x_2, x_3)$ and $v = (0, y_2, y_3)$ with $x_2 x_3, y_2, y_3$ are real

$\Rightarrow u + v = (0, x_2, x_3) + (0, y_2, y_3)$

$= (0, x_2 + y_2, x_3 + y^3) \in W$

$\Rightarrow u + v \in W$ (1)

Next, take any $\alpha \in R \Rightarrow \alpha u = \alpha(0, x_2, x_3)$

$= (0, \alpha x_2, \alpha x_3) \in W$

$\Rightarrow \alpha u \in W$ (2)

From (1) and (2), W is subspace of R^3

b) $W = \{(x_1, x_2, 4) : x_1$ and x_2 are real numbers $\}$

Let $u, v \in W$

$u = (x_1, x_2, 4)$ and $v = (y_1, y_2, 4)$; x_1, x_2, y_1, y_2 are real num

$\Rightarrow u + v = (x_1 + y_1, x_2 + y_2 + 8) \notin W$ (W is a unclosed wides addition)

$\Rightarrow u + v \notin W$ (1)

- Take any $\alpha \in R$

$\Rightarrow \alpha u = \alpha(x_1, x_2, 4)$

$= (\alpha x_1, \alpha x_2, \alpha 4) \notin W$

$\Rightarrow \alpha u \notin W$ (2)

From (1), (2) $\Rightarrow$ W is not a subspace of R^3

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4. Write each vector as a linear combimition of the vectors

$$S = \{(2,0,7), (2,4,5), (2,-12,13)\}$$

a/ $u = (-1,5,-6)$

Let $u = c_1(2,0,7) + c_2(2,4,5) + c_3(2,-12,13)$

$\Rightarrow (-1,5,6) = (2c_1 + 2c_2 + 2c_3,\ 4c_2 - 12c_3,\ 7c_1 + 5c_2 + 13c_3)$

$$\begin{cases} 2c_1 + 2c_2 + 2c_3 = -1 \\ 4c_2 - 12c_3 = 5 \\ 7c_1 + 5c_2 + 13c_3 = -6 \end{cases} \quad (1)$$

The above system of equations (1) can be written as :

$$\underbrace{\begin{bmatrix} 2 & 2 & 2 \\ 0 & 4 & -12 \\ 7 & 5 & 13 \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 5 \\ -6 \end{bmatrix}}_{B}$$

Let

Then, the system $Ax = B$ is Consistent if $\operatorname{rank}([AB]) = r(A)$

where $[AB]$ denote the augmented matrix

$$[Ab] = \left[\begin{array}{ccc|c} 2 & 2 & 2 & -1 \\ 0 & 4 & -12 & 5 \\ 7 & 5 & 13 & -6 \end{array}\right] \xrightarrow[R_2/4]{R_1/2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & \frac{1}{2} \\ 0 & 1 & -3 & 5/4 \\ 7 & 5 & 13 & -6 \end{array}\right]$$

$$\xrightarrow{R_3 - 7R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -\frac{1}{2} \\ 0 & 1 & -3 & 5/4 \\ 0 & -2 & 6 & -5/2 \end{array}\right] \xrightarrow{R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1/2 \\ 0 & 1 & -3 & 5/4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence $\operatorname{rank}([AB]) = \operatorname{rank}(A) = 2$

The system $Ax = B$ has infinitely many solutions

The reduced system is

$$c_1 + c_2 + c_3 = -1/2$$
$$c_2 - 3c_3 = 5/4$$

Since $r(A) = 2$ and no. of variables $= 3$

we can choose $3 - 2 = 1$ variable indepently

Let $c_3 = 0$

Then $c_2 - 3c_3 = 5/4 \Rightarrow c_2 = 5/4$

$c_1 + c_2 + c_3 = \frac{-1}{2} \Rightarrow c_1 + \frac{5}{4} = \frac{-1}{2} \Rightarrow c_1 = \frac{-7}{4}$

$$\Rightarrow X = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -7/4 \\ 5/4 \\ 0 \end{bmatrix} \text{ is a solution}$$

b) $v = (-3, 15, 18)$

$= c_1(2,0,7) + c_2(2,4,5) + c_3(2,-12,13)$

$\Rightarrow (-3, 15, 18) = (2c_1 + 2c_2 + 2c_3, 4c_2 - 12c_3, 7c_1 + 5c_2 + 13c_3)$

$$\Rightarrow \begin{cases} 2c_1 + 2c_2 + 2c_3 = -3 \\ \quad\quad 4c_2 - 12c_3 = 15 \\ 7c_1 + 5c_2 + 13c_3 = 18 \end{cases} \quad (1)$$

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & -3 \\ 0 & 4 & -12 & 15 \\ 7 & 5 & 13 & 18 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & -\frac{1}{2} \\ 0 & 4 & -12 & 5 \\ 0 & 0 & 0 & 36 \end{array}\right]$$

$\rightarrow$ No solution, can't write v in S

5. Determine whether the set S spans R^3

a) $S = \{(4,7,3), (-1,2,6), (2,-3,5)\}$

$(x, y, z) = a(4,7,3) + b(-1,2,6) + c(2,-3,5)$

$$\begin{cases} 6a - 3b + c = 0 \\ 7a + 4b - 3c = 0 \\ a + 7b + 4c = 0 \end{cases} \Rightarrow a = b = c = 0$$

They are linearly independent $\rightarrow$ They span R^3

b) $S = \{(5,6,5), (2,1,-5), (0,-4,1)\}$

$a(5,6,5) + b(2,1,-5) + c(0,-4,1) = (0,0,0)$

$$\begin{cases} 5a + 2b = 0 \\ 6a + b - 4c = 0 \\ 5a - 5b + c = 0 \end{cases} \Rightarrow a = b = c = 0$$

$\Rightarrow$ They are linearly independent $\rightarrow$ They span R^3

6.

a/ $S = \{(-2,1,3), (2,9,-3), (2,3,-3)\}$

Let $A = (-2, 1, 3)$

$B = (2, 9, -3)$ The vector A, B, C are Linearly dependent it there

$C = (2, 3, -3)$ determinant is zero $|D| = 0$

$$|D| = \begin{bmatrix} -2 & 1 & 3 \\ 2 & 9 & -3 \\ 2 & 3 & -3 \end{bmatrix} = -2 \times (-27 + 9) - 1 \times (-6 + 6) + 3 \times (6 - 18)$$

$$= 36 - 0 - 36 = 0$$

Since $|D| = 0$, the vectors A, B, C are linearly dependent

b) $S = \{(-4,-3,4), (1,-2,3), (6,0,0)\}$

Let $A = (-4,-3,4)$
$B = (1,-2,3)$
$C = (6,0,0)$

$$|D| = \begin{vmatrix} -4 & -3 & 4 \\ 1 & -2 & 3 \\ 6 & 0 & 0 \end{vmatrix} = -6 \neq 0$$

Since $|D| \neq 0 \Rightarrow A, B, C$ are linearly independent.

7. For which values of t is each set linearly independent?

$S = \{(t,1,1), (1,t,1), (1,1,t)\}$

Let $S = \left\{ \begin{bmatrix} t \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ t \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ t \end{bmatrix} \right\}$

$$|D| = \begin{bmatrix} t & 1 & 1 \\ 1 & t & 1 \\ 1 & 1 & t \end{bmatrix} = t(t^2-1) - 1(t-1) + 1(1-t)$$
$$= t^3 - 3t + 2$$

The set linearly dependent when $D = 0$

$\Rightarrow t^3 - 3t + 2 = 0$

$$\left[\begin{array}{l} t = 1 \\ t = -2 \end{array}\right.$$

The set linearly independent when $t \neq -2$ and $t \neq 1$

$$B = \begin{pmatrix} 2 & -4 & 4 & -5 \\ 3 & 6 & -6 & -4 \\ -2 & -4 & 4 & 9 \end{pmatrix} \xrightarrow[R_2 - 3R_1]{R_1/2,\ R_2 - 3R_1} \begin{bmatrix} 1 & 2 & -2 & 5/2 \\ 0 & 0 & 0 & -23/2 \\ -2 & -4 & 4 & 9 \end{bmatrix}$$

$$\xrightarrow{R_3 + 2R_1} \begin{bmatrix} 1 & 2 & -2 & 5/2 \\ 0 & 0 & 0 & -23/2 \\ 0 & 0 & 0 & 14 \end{bmatrix} \xrightarrow[R_1 - \frac{5R_2}{2},\ R_3 - 14R_2]{R_2 = \frac{-2R_2}{23}} \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we take $x_2 = t$, $x_3 = S$

$x_1 = 2s - 2t$, $x_4 = 0$

$$\overline{X} = \begin{bmatrix} 2s - 2t \\ t \\ s \\ 0 \end{bmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} s$$

he nullity of the matrix is 2 A