# FINAL REVIEW

Calculus I - Semester 2

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# REVIEW SECTIONS

1. ## Chapter 3: APPLICATIONS OF DIFFERENTIATION

   Related Rates, Maxima & Minima, Optimization problems, Mean Value Theorem, First and Second derivative tests, Concavity, Shape of curves, L'Hospital's Rule, Newton's method, Antiderivative/ Indefinite Integral

2. ## Chapter 4: INTEGRATION

   Definition of Definite Integral, Properties of Definite Integral, Fundamental Theorem of Calculus, Techniques of integration, Numerical (approximation) Integrals, Improper Integrals

3. ## Chapter 5: APPLICATIONS OF INTEGRATION

   Areas between curves, Volumes of Solid by revolution, Arc length, Average value of a function, Applications of Integration to Engineering, Economics, and Science

# CHAPTER 3

# APPLICATIONS OF DIFFERENTIATION

# CHAPTER 3: APPLICATIONS OF DIFFERENTIATION

## 1. Related Rates

- **Step 1:** Assign all variables
- **Step 2:** Find equation related to all variables
- **Step 3:** Use implicit differentiation to obtain equation of rates
- **Step 4:** Use the given information to find the unknown variable

. (10 points) The radius $r$ of a cylinder is increasing at a rate of 4 meters per hour, and the height $h$ of the cyclinder is decreasing at a rate of 4 meters per hour. At a certain instant, the radius is 5 meters and the height is 8 meters. What is the rate of change of the volume $V$ of the cylinder at that instant? Note that

$$V = \pi r^2 h.$$

Denote $r(t)$ = radius of the cylinder
$h(t)$ = height of the cylinder
$V(t)$ = volume of the cylinder

* Radius is increasing at a rate of 4 meters per hour
$\Rightarrow r'(t) = 4\, m/h, \quad \forall t \geqslant 0$

* Height is decreasing at a rate of 4 meters per hour
$\Rightarrow h'(t) = -4\, m/h, \quad \forall t \geqslant 0$

* Denote by T the instant that the radius is 5m and height is 8m
$\Rightarrow r(T) = 5m, \quad h(T) = 8m$ $\boxed{\text{Find } V'(T)?}$

* We have $V(T) = \pi [r(T)]^2 h(T)$

$$\Rightarrow V'(T) = 2\pi r'(T)\, r(T)\, h(T) + \pi [r(T)]^2 h'(T)$$

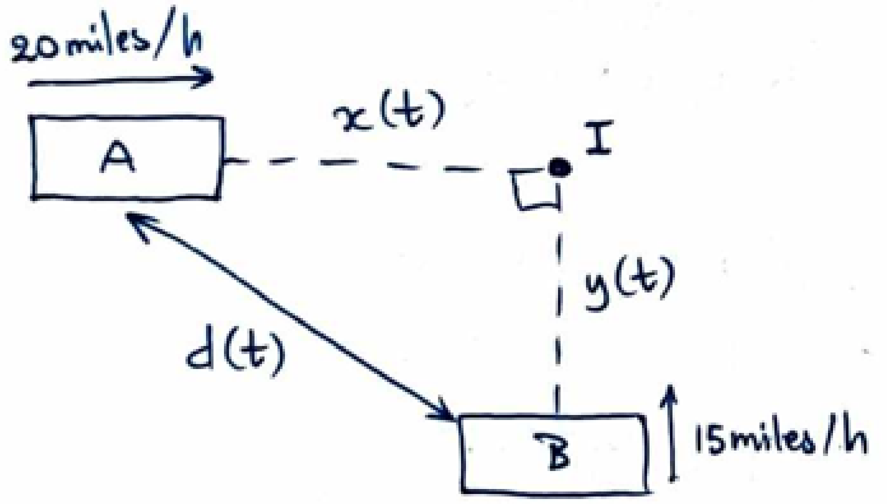
$$\Rightarrow V'(T) = (2\pi)(4)(5)(8) + \pi (5^2)(-4)$$

$$\Rightarrow V'(T) = 220\pi \quad (m^3/h)$$

# CHAPTER 3: APPLICATIONS OF DIFFERENTIATION

## 1. Related Rates

**Exercise 2.50.** Two ships are sailing along straight-line courses that intersect at right angles. Ship A is approaching the intersection point at a speed of 20 knots (nautical miles per hour). Ship B is approaching the intersection at 15 knots. At what rate is the distance between the ships changing when A is 5 nautical miles from the intersection point and B is 12 nautical miles from the intersection point?




$x'(t) = 20$ miles/h

$y'(t) = 15$ miles/h

Denote $x(t)$ = distance between A and intersection point I

$y(t)$ = distance between B and intersection point I

$d(t)$ = distance between A and B

* Denote by T the instant that A is 5 nautical miles and B is 12 nautical miles from intersection point I.

$\Rightarrow x(T) = 5$ miles, $y(T) = 12$ miles

$\boxed{\text{Find } d'(T)?}$

* Pythagorean Theorem :

$$[d(t)]^2 = [x(t)]^2 + [y(t)]^2$$

$$\Rightarrow [d(T)]^2 = [x(T)]^2 + [y(T)]^2 \quad (*)$$

Differentiate both sides of (*) with respect to t :

$$2d(T)\,d'(T) = 2x(T)\,x'(T) + 2y(T)\,y'(T)$$

$$\Rightarrow d'(T) = \frac{x(T)\,x'(T) + y(T)\,y'(T)}{d(T)} = \frac{(5)(20) + (12)(15)}{\sqrt{5^2 + 12^2}}$$

$$\Rightarrow d'(T) = 21.5 \text{ miles/h}$$

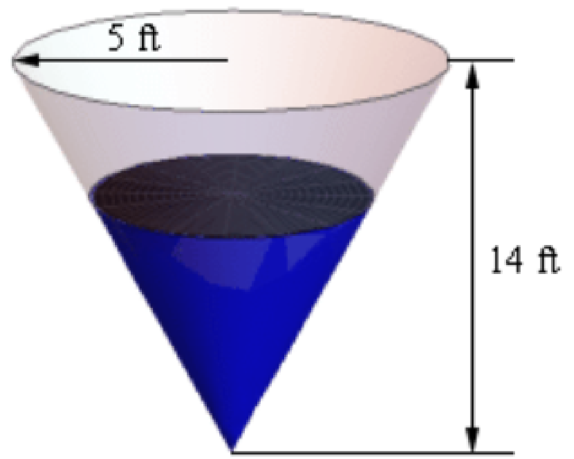
# CHAPTER 3: APPLICATIONS OF DIFFERENTIATION

## 1. Related Rates

***Example 4*** A tank of water in the shape of a cone is leaking water at a constant rate of $2\,\mathrm{ft}^3/\mathrm{hour}$. The base radius of the tank is 5 ft and the height of the tank is 14 ft.

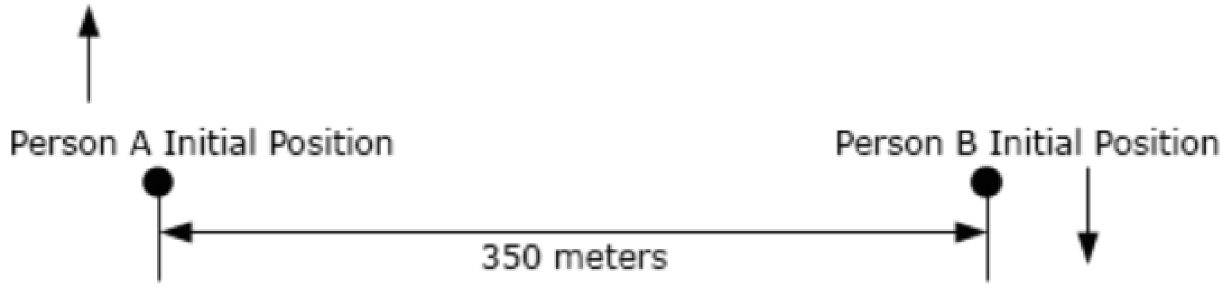
**(a)** At what rate is the depth of the water in the tank changing when the depth of the water is 6 ft? $-0.1386$

**(b)** At what rate is the radius of the top of the water in the tank changing when the depth of the water is 6 ft? $-0.04951$




***Example 8*** Two people on bikes are separated by 350 meters. Person A starts riding north at a rate of 5 m/sec and 7 minutes later Person B starts riding south at 3 m/sec. At what rate is the distance separating the two people changing 25 minutes after Person A starts riding?

$7.9958\mathrm{m}/\mathrm{sec}$



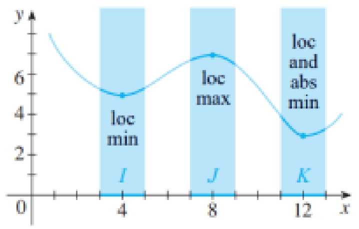
# CHAPTER 3: APPLICATIONS OF DIFFERENTIATION

**1 Definition** Let $c$ be a number in the domain $D$ of a function $f$. Then $f(c)$ is the

- **absolute maximum** value of $f$ on $D$ if $f(c) \geqslant f(x)$ for all $x$ in $D$.
- **absolute minimum** value of $f$ on $D$ if $f(c) \leqslant f(x)$ for all $x$ in $D$.

**2 Definition** The number $f(c)$ is a

- **local maximum** value of $f$ if $f(c) \geqslant f(x)$ when $x$ is near $c$.
- **local minimum** value of $f$ if $f(c) \leqslant f(x)$ when $x$ is near $c$.




**4 Fermat's Theorem** If $f$ has a local maximum or minimum at $c$, and if $f'(c)$ exists, then $f'(c) = 0$.

**6 Definition** A **critical number** of a function $f$ is a number $c$ in the domain of $f$ such that either $f'(c) = 0$ or $f'(c)$ does not exist.

**7** If $f$ has a local maximum or minimum at $c$, then $c$ is a critical number of $f$.

**The Closed Interval Method** To find the *absolute* maximum and minimum values of a continuous function $f$ on a closed interval $[a, b]$:

1. Find the values of $f$ at the critical numbers of $f$ in $(a, b)$.
2. Find the values of $f$ at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

## 2. Maxima & Minima

**Question 2.** [15 marks] Find the absolute maximum and absolute minimum values of the following function:

$$f(x) = \frac{x}{x^2+1}, \quad x \in [0, 2].$$

- $f'(x) = \dfrac{1-x^2}{(x^2+1)^2} = 0$

$\Leftrightarrow \left[\begin{array}{l} x = -1 \text{ (remove)} \\ x = 1 \end{array}\right.$

- $f(0) = 0$

$f(1) = \frac{2}{5}$

$f(2) = \frac{1}{2}$

Therefore, $\max\limits_{[0,2]} f(x) = f(2) = \frac{1}{2}$

$\min\limits_{[0,2]} f(x) = f(0) = 0$

# CHAPTER 3: APPLICATIONS OF DIFFERENTIATION

## 3. Optimization problems

Example 1: On a certain day in HCM city, the temperature T (in °C) could be modeled by the function $T(t) = 0.0004t^4 - 0.12t^2 + 33, \quad -12 \leq t \leq 12$ where $t$ is the time in hours and $t = 0$ at noon. What is the maximum temperature during the day?

$T'(t) = 0.0016t^3 - 0.24t$

Consider $T'(t) = 0 \Leftrightarrow 0.0016t^3 - 0.24t = 0$

$\Rightarrow \left[\begin{array}{l} t = -12.25 \text{ (remove)} \\ t = 0 \\ t = 12.25 \text{ (remove)} \end{array}\right.$

- $T(0) = 33$

  $T(-12) = 24.01$

  $T(12) = 24.01$

$\Rightarrow \max\limits_{[-12,12]} T(t) = T(0) = 33$

Therefore, the maximum temperature during the day is 33°C at noon.

**Examples 2.** Find the global maximum of the profit function (from one factory):

$$P(x) = R(x) - C(x), \quad x \in [0, 30, 000].$$

where $R(x) = 200x$ is the revenue function (when selling $x$ unit of product); and $C(x) = 500, 000 + 80x + 0.003x^2$ is the cost function (total production cost).

$700, 000(USD).$

# CHAPTER 3: APPLICATIONS OF DIFFERENTIATION

## 4. Mean Value Theorem

**The Mean Value Theorem** Let $f$ be a function that satisfies the following hypotheses:

1. $f$ is continuous on the closed interval $[a, b]$.
2. $f$ is differentiable on the open interval $(a, b)$.

Then there is a number $c$ in $(a, b)$ such that

$$\boxed{1} \qquad f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$\boxed{2} \qquad f(b) - f(a) = f'(c)(b - a)$$

Suppose $3 \le f'(x) \le 5$ for all values of x.

Show that $18 \le f(8) - f(2) \le 30$

By the Mean Value Theorem, we have:

$$f'(c) = \frac{f(8) - f(2)}{8 - 2} = \frac{f(8) - f(2)}{6}$$

Then $3 \leqslant f'(x) \leqslant 5$

$$\Rightarrow 3 \leqslant \frac{f(8) - f(2)}{6} \leqslant 5$$

$$\Rightarrow 18 \leqslant f(8) - f(2) \leqslant 30 \quad \text{(proved)}$$

**Question 1.** (2014 – 2015, Semester 1)

**(b)** [15 marks] Show that $e^x < 1 + xe^x, \quad \forall x > 0.$

Denote $f(t) = 1 + te^t - e^t$

$\Rightarrow f'(t) = te^t$

Then $f$ is differentiable on $(-\infty, +\infty)$

Let $x \geqslant 0$ then $f(t)$ is continuous on $[0, x]$ and differentiable on $(0, x)$

* By the Mean Value Theorem, there is a $c \in (0, x)$ satisfying $f'(c) = \dfrac{f(x) - f(0)}{x - 0}$

$$\Rightarrow ce^c = \frac{1 + xe^x - e^x}{x} \qquad (1)$$

Since $c \in (0, x)$, $ce^c > 0$ and $x > 0$ (2)

From (1) & (2): $1 + xe^x - e^x > 0$

$$\Rightarrow e^x < 1 + xe^x, \quad x > 0 \quad \text{(proved)}$$

# CHAPTER 3: APPLICATIONS OF DIFFERENTIATION

## 4. Mean Value Theorem

3. Determine all the number(s) $c$ which satisfy the conclusion of Mean Value Theorem for $h(z) = 4z^3 - 8z^2 + 7z - 2$ on $[2, 5]$

$$\boxed{c = \frac{2 + \sqrt{79}}{3} = 3.6294}$$

5. Suppose we know that $f(x)$ is continuous and differentiable on the interval $[-7, 0]$, that $f(-7) = -3$ and that $f'(x) \leq 2$. What is the largest possible value for $f(0)$?

$$\boxed{f(0) \leq 11}$$

Use the Mean Value Theorem to show that

$$\left|\sin w - \sin z\right| \leq \left|w - z\right| \quad \text{for} \quad 0 \leq z < w \leq 2\pi .$$

Use the Mean Value Theorem to show that

$$\left|\ln w - \ln z\right| \leq 3\left|w - z\right| \quad \text{for} \quad 1/3 \leq z < w \leq 3$$

# CHAPTER 3: APPLICATIONS OF DIFFERENTIATION

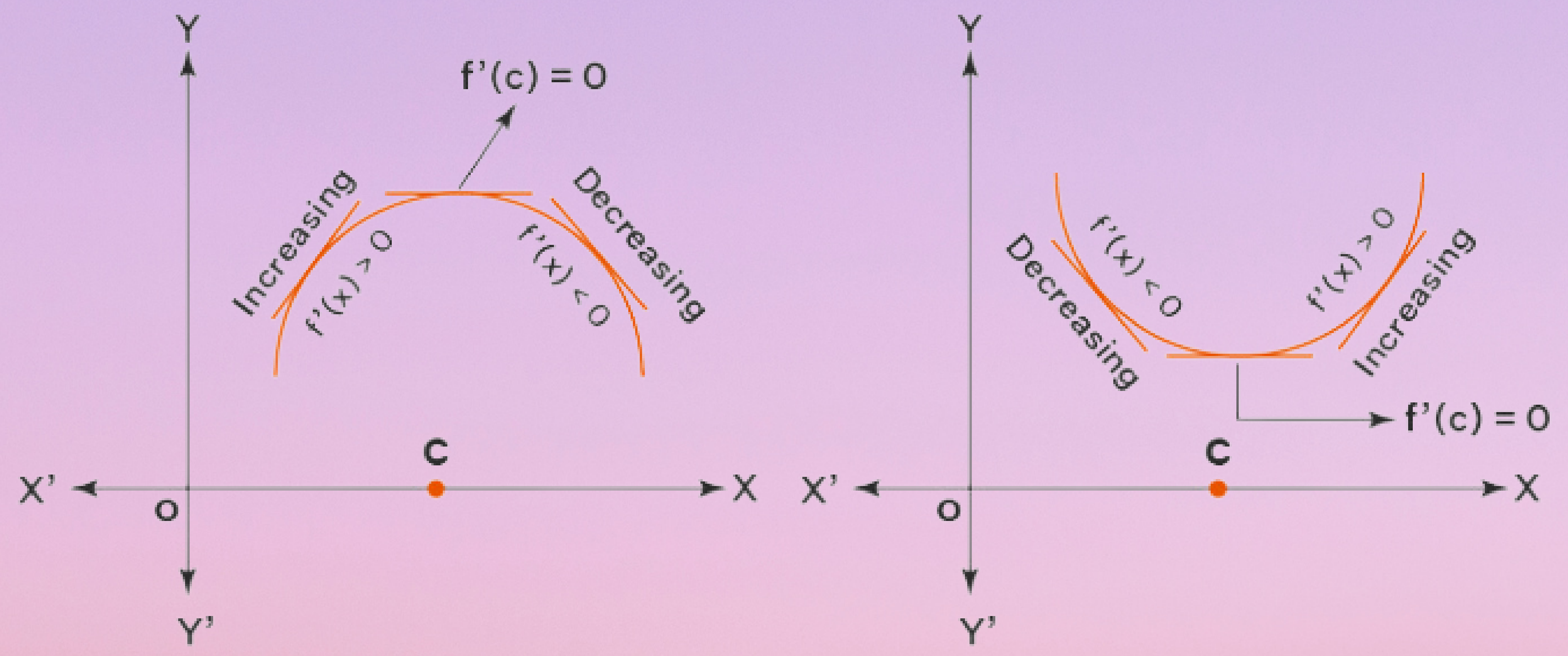
## 5. First and Second Derivative Tests

**The First Derivative Test** Suppose that $c$ is a critical number of a continuous function $f$.

(a) If $f'$ changes from positive to negative at $c$, then $f$ has a local maximum at $c$.

(b) If $f'$ changes from negative to positive at $c$, then $f$ has a local minimum at $c$.

(c) If $f'$ does not change sign at $c$ (for example, if $f'$ is positive on both sides of $c$ or negative on both sides), then $f$ has no local maximum or minimum at $c$.




Local Maxima        Local Minima

$$f(x) = x^3 - 12x + 5$$

Find the intervals of increase and decrease and identify any local extrema.

$f'(x) = 3x^2 - 12 = 0 \Rightarrow \left[\begin{array}{l} x = 2 \\ x = -2 \end{array}\right.$

| $x$ | $(-\infty,-2)$ | $(-2,2)$ | $(2,+\infty)$ |
|---|---|---|---|
| $f'(x)$ | + ↗ | ↘ − | + ↗ |
| $f(x)$ | − | − | − |

* $f(x)$ is increasing on $(-\infty,-2)$ and $(2,\infty)$, decreasing on $(-2,2)$
* $f$ has a local max at $(-2,21)$ and a local min at $(2,-11)$

# CHAPTER 3: APPLICATIONS OF DIFFERENTIATION

## 5. First and Second Derivative Tests

**The Second Derivative Test** Suppose $f''$ is continuous near $c$.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local minimum at $c$.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local maximum at $c$.

**Given Function:**

$$y = f(x)$$

**First Derivative:**

$$\frac{dy}{dx} = f'(x)$$

$$f'(x) = 0 \begin{array}{l} \nearrow x_1 \\ \searrow x_2 \end{array}$$

**Second Derivative:**

$$\frac{d}{dx} f'(x) = f''(x)$$

$$x_1 \longrightarrow f''(x) \longrightarrow f''(x_1) < 0$$

$x_1$ is Local Maxima

$$x_2 \longrightarrow f''(x) \longrightarrow f''(x_2) > 0$$

$x_2$ is Local Minima

## 6. Concavity

**Definition** If the graph of $f$ lies above all of its tangents on an interval $I$, then it is called **concave upward** on $I$. If the graph of $f$ lies below all of its tangents on $I$, it is called **concave downward** on $I$.

**Concavity Test**

(a) If $f''(x) > 0$ for all $x$ in $I$, then the graph of $f$ is concave upward on $I$.

(b) If $f''(x) < 0$ for all $x$ in $I$, then the graph of $f$ is concave downward on $I$.

Concave Upward

Also called "Convex" or "Convex Downward"

Concave Downward

Also called "Concave" or "Convex Upward"

# CHAPTER 3: APPLICATIONS OF DIFFERENTIATION

## 7. L'Hospital's Rule

**L'Hospital's Rule** Suppose $f$ and $g$ are differentiable and $g'(x) \neq 0$ on an open interval $I$ that contains $a$ (except possibly at $a$). Suppose that

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\infty/\infty$.) Then

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

if the limit on the right side exists (or is $\infty$ or $-\infty$).

$$\lim_{x \to 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6}$$

$\boxed{-\frac{6}{5}}$

$$\lim_{z \to 0} \frac{\sin(2z) + 7z^2 - 2z}{z^2(z+1)^2}$$

$\boxed{7}$

$$\lim_{z \to \infty} \frac{z^2 + \mathbf{e}^{4z}}{2z - \mathbf{e}^{z}}$$

$\boxed{-\infty}$

$$\lim_{w \to -4} \frac{\sin(\pi w)}{w^2 - 16}$$

$\boxed{-\frac{\pi}{8}}$

**Question 1.** (2014 – 2015, Semester 1)
**(a)** [10 marks] Find the limit $\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$.

Using L'Hospital Rule, we have:

$$\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x \to 0^+} \left(\frac{\sin x - x}{x \sin x}\right)$$

$$= \lim_{x \to 0^+} \left(\frac{\cos x - 1}{\sin x + x\cos x}\right) = \lim_{x \to 0^+} \left(\frac{-\sin x}{2\cos x - x\sin x}\right) = 0$$

| Determinate-Indeterminate Forms Table | |
|---|---|
| Indeterminate Forms | Determinate Forms |
| $0/0$ | $\infty + \infty = \infty$ |
| $\pm\infty/\pm\infty$ | $-\infty - \infty = -\infty$ |
| $\infty - \infty$ | $0^{\infty} = 0$ |
| $0(\infty)$ | $0^{-\infty} = \infty$ |
| $0^0$ | $(\infty)\cdot(\infty) = \infty$ |
| $1^{\infty}$ | |
| $\infty^0$ | |
| Use L'Hôpital's Rule | Do *Not* Use L'Hôpital's Rule |

# CHAPTER 3: APPLICATIONS OF DIFFERENTIATION

## 7. L'Hospital's Rule

$$\lim_{x\to a}[g(x)]^{h(x)}$$

① Denote $f(x) = [g(x)]^{h(x)}$

$\Rightarrow \ln[f(x)] = h(x)\ln[g(x)]$

② $\lim\limits_{x\to a}[\ln f(x)] = \lim\limits_{x\to a}[h(x)\ln g(x)]$

③ $\lim\limits_{x\to a}[g(x)]^{h(x)} = \lim\limits_{x\to a} e^{\ln f(x)} = e^{\lim\limits_{x\to a}\ln f(x)}$

$\lim\limits_{t\to\infty}\left[t\ln\left(1+\frac{3}{t}\right)\right]$

$\boxed{3}$

$\lim\limits_{x\to 1^+}\left[(x-1)\tan\left(\frac{\pi}{2}x\right)\right]$

$\boxed{-\frac{2}{\pi}}$

$\lim\limits_{y\to 0^+}[\cos(2y)]^{1/y^2}$

$\boxed{e^{-2}}$

**Ex:** Find $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

* Denote $f(x) = \left(1+\frac{1}{x}\right)^x$

$\Rightarrow \ln f(x) = x\ln\left(1+\frac{1}{x}\right)$

* $\lim\limits_{x\to\infty}\ln f(x) = \lim\limits_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)$ $\boxed{\infty.0}$

$= \lim\limits_{x\to\infty}\dfrac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$ $\boxed{\frac{0}{0}}$

By using L'Hospital's Rule:

$\lim\limits_{x\to\infty}\ln f(x) = \dfrac{\frac{-\frac{1}{x^2}}{1+\frac{1}{x}}}{-\frac{1}{x^2}} = \lim\limits_{x\to\infty}\dfrac{1}{1+\frac{1}{x}} = 1$

* $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim\limits_{x\to\infty} e^{\ln f(x)} = e^{\lim\limits_{x\to\infty}\ln f(x)} = e^1 = e$

# CHAPTER 3: APPLICATIONS OF DIFFERENTIATION

## 8. Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Use Newton's method to approximate a root of $f(x) = x^3 - 3x + 1$ in the interval $[1, 2]$. Let $x_0 = 2$ and find $x_1$, $x_2$, $x_3$, $x_4$, and $x_5$.

- $x_1 \approx 1.666666667$
- $x_2 \approx 1.548611111$
- $x_3 \approx 1.532390162$
- $x_4 \approx 1.532088989$
- $x_5 \approx 1.532088886$
- $x_6 \approx 1.532088886.$

V EXAMPLE 1 Starting with $x_1 = 2$, find the third approximation $x_3$ to the root of the equation $x^3 - 2x - 5 = 0$.

$f'(x) = 3x^2 - 2$ , $x_1 = 2$

By using Newton's method, we have:

$$\begin{cases} x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 2 - \dfrac{f(2)}{f'(2)} = 2.1 \\ x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = 2.1 - \dfrac{f(2.1)}{f'(2.1)} = 2.0946 \end{cases}$$

# CHAPTER 4

# INTEGRATION

# CHAPTER 4: INTEGRATION

## 1. Definition & Properties

**2 Definition of a Definite Integral** If $f$ is a function defined for $a \leqslant x \leqslant b$, we divide the interval $[a, b]$ into $n$ subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 (= a), x_1, x_2, \dots, x_n (= b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \dots, x_n^*$ be any **sample points** in these subintervals, so $x_i^*$ lies in the $i$th subinterval $[x_{i-1}, x_i]$. Then the **definite integral of $f$ from $a$ to $b$** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

### Properties of the Definite Integral

- **Một số định nghĩa**

$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha\int_a^b f(x)dx + \beta\int_a^b g(x)dx$$

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

If $f_o$ is an odd function then $\int_{-a}^a f_o(x)dx = 0$

If $f_e$ is an even function then $\int_{-a}^a f_e(x)dx = 2\int_0^a f_e(x)dx$

- Comparison Properties

If $f(x) \geq 0$ for $a \leq x \leq b$ then $\int_a^b f(x)dx \geq 0.$

If $f(x) \leq g(x)$ and $a \leq b$ then $\int_a^b f(x)dx \leq \int_a^b g(x)dx.$

If $a \leq b$ then $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$

If $m \leq f(x) \leq M$ for $a \leq x \leq b$ then

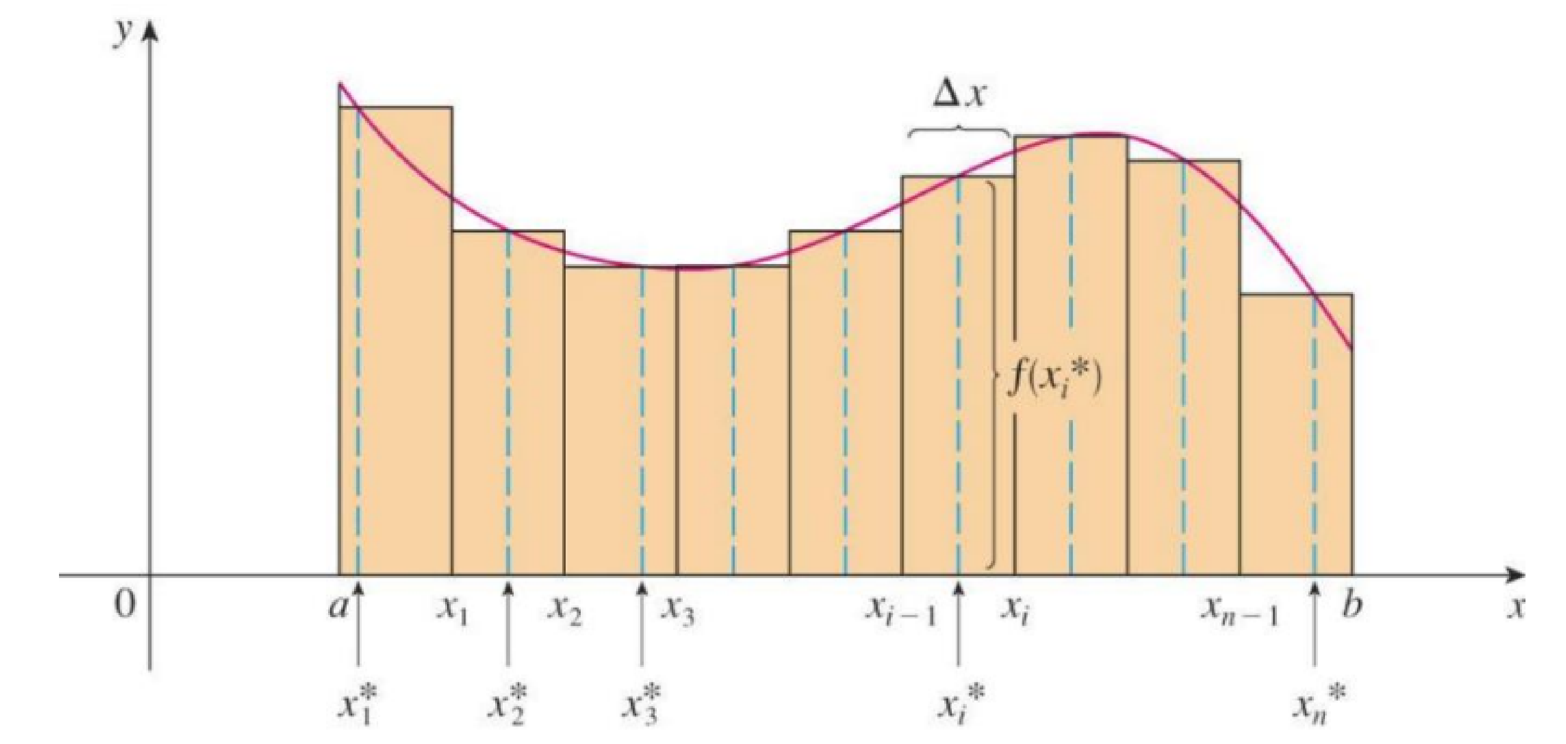
$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a).$$

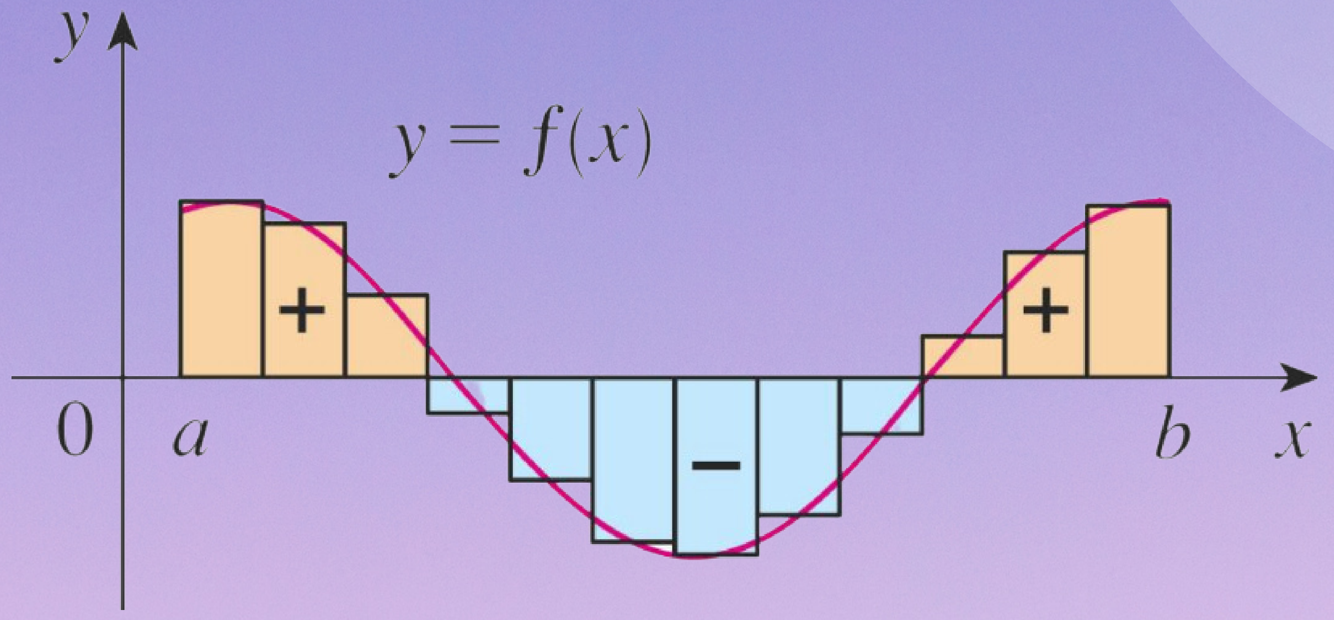
# CHAPTER 4: INTEGRATION

## 1. Definition & Properties

Then the Riemann sum of $f(x)$ on $[a, b]$ is

$$R(f,P)=\sum_{i=1}^{n} f(x_i^*)\Delta x_i \quad \text{where} \quad \Delta x_i = x_i - x_{i-1}.$$




**The Riemann sum** = (the sum of the areas above the x-axis) minus (the sum of the areas below the x-axis)

$$I=\int_a^b f(x)dx=\lim_{n\to\infty} R(f,P)=\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

(provided also that as $n \to \infty$, max $\Delta x_i \to 0$)

$\Rightarrow$ *An **integral** is the **limit** of a Riemann **sum**.*

# CHAPTER 4: INTEGRATION

## 2. Fundamental Theorem of Calculus

**The Fundamental Theorem of Calculus** Suppose $f$ is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F$ is any antiderivative of $f$, that is, $F' = f$.

We noted that Part 1 can be rewritten as

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$$

which says that if $f$ is integrated and then the result is differentiated, we arrive back at the original function $f$. Since $F'(x) = f(x)$, Part 2 can be rewritten as

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

**Exercise 3.27.** Evaluate $\lim_{x\to 0} \dfrac{\int_0^x \cos(t^2)dt}{x}$

By using L'Hospital's Rule:

$$\lim_{x\to 0} \frac{\int_0^x \cos(t^2)\,dt}{x} = \lim_{x\to 0} \frac{\frac{d}{dt}\left(\int_0^x \cos(t^2)dt\right)}{1}$$

$$= \lim_{x\to 0} \frac{\cos(x^2)}{1} = 1$$

# CHAPTER 4: INTEGRATION

## 2. Fundamental Theorem of Calculus

$$\frac{d}{dx}\left(\int_a^{u(x)} f(t)\,dt\right) = f(u(x))\,u'(x)$$

Let $F(x) = \int_x^{2x} t^3\,dt$. Find $F'(x)$

$15x^3$

Let $F(x) = \int_1^{\sqrt{x}} \sin t\,dt$. Find $F'(x)$.

$\frac{\sin\sqrt{x}}{2\sqrt{x}}$

**Exercise 3.29.** The limit $\lim\limits_{x\to 0} \frac{\int_{x^2}^{0} \sqrt{1+t^3}\,dt}{x^2}$. is

$$\lim_{x\to 0} \frac{\int_{x^2}^{0}\sqrt{1+t^3}\,dt}{x^2} = \lim_{x\to 0} \frac{-\int_0^{x^2}\sqrt{1+t^3}\,dt}{x^2}$$

By using L'Hospital's Rule :

$$\lim_{x\to 0} \frac{\int_{x^2}^{0}\sqrt{1+t^3}\,dt}{x^2} = \lim_{x\to 0} \frac{\frac{d}{dt}\left(-\int_0^{x^2}\sqrt{1+t^3}\,dt\right)}{2x}$$

$$= \lim_{x\to 0} \frac{\left(-\sqrt{1+x^3}\right)(2x)}{2x} = \lim_{x\to 0}\left(-\sqrt{1+x^3}\right) = -1$$

# CHAPTER 4: INTEGRATION

## 3. Techniques of Integration

### SUBSTITUTION

$$\int f(g(x))g'(x)dx = \int f(u)du \quad \text{where } u = g(x)$$

3. $\displaystyle\int (3-4w)\left(4w^2-6w+7\right)^{10} dw$ $\quad -\frac{1}{22}\left(4w^2-6w+7\right)^{11}+c$

10. $\displaystyle\int \left(\cos(3t)-t^2\right)\left(\sin(3t)-t^3\right)^5 dt$ $\quad \frac{1}{18}\left(\sin(3t)-t^3\right)^6+c$

15. $\displaystyle\int \frac{6}{7+y^2}dy$ $\quad \frac{6}{\sqrt{7}}\tan^{-1}\left(\frac{y}{\sqrt{7}}\right)+c$

1) Find $I = \int_0^1 e^x \sin(e^x)\,dx$

Denote $u = e^x$
$\Rightarrow du = e^x dx$

| $x$ | 0 | 1 |
|---|---|---|
| $u$ | 1 | $e$ |

$$I = \int_1^e \sin(u)\,du = -\cos u\Big|_{u=1}^{u=e} = -\cos e + \cos 1$$

2) Find $I = \int_1^e \frac{\ln x}{x}dx$

Denote $u = \ln x$
$\Rightarrow du = \frac{dx}{x}$

| $x$ | 1 | $e$ |
|---|---|---|
| $u$ | 0 | 1 |

$$I = \int_0^1 u\,du = \frac{u^2}{2}\Big|_{u=0}^{u=1} = \frac{1}{2}$$

# CHAPTER 4: INTEGRATION

## 3. Techniques of Integration

## INTEGRATION BY PARTS

$$\int U\,dV = UV - \int V\,dU$$

**(u)** – *nhất lô, nhì đa, tam lượng, tứ mũ*

2. $\displaystyle\int_6^0 (2+5x)\,\mathbf{e}^{\frac{1}{3}x}\,dx$

3. $\displaystyle\int (3t+t^2)\sin(2t)\,dt$

4. $\displaystyle\int 6\tan^{-1}\left(\frac{8}{w}\right)dw$

5. $\displaystyle\int \mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right)dz$

6. $\displaystyle\int_0^{\pi} x^2\cos(4x)\,dx$

7. $\displaystyle\int t^7\sin(2t^4)\,dt$

8. $\displaystyle\int y^6\cos(3y)\,dy$

9. $\displaystyle\int (4x^3-9x^2+7x+3)\,\mathbf{e}^{-x}\,dx$

https://tutorial.math.lamar.edu/problems/calcii/integrationbyparts.aspx

3) Find $I = \displaystyle\int_1^3 \ln x\,dx$

Denote $\begin{cases} u = \ln x \\ dv = dx \end{cases} \Rightarrow \begin{cases} du = \dfrac{dx}{x} \\ v = x \end{cases}$

$\Rightarrow I = x\ln x \Big|_{x=1}^{x=3} - \displaystyle\int_1^3 dx$

$= 3\ln 3 - x\Big|_{x=1}^{x=3}$

$= 3\ln 3 - 3 + 1 = 3\ln 3 - 2$

4) Find $\displaystyle\int_0^1 f(x)g'(x)\,dx$ if $\displaystyle\int_0^1 f'(x)g(x)\,dx = 5$, $f(0) = 2$, $f(1) = 4$, $g(0) = -4$, $g(1) = 3$

Denote $\begin{cases} u = f(x) \\ dv = g'(x)\,dx \end{cases} \Rightarrow \begin{cases} du = f'(x)\,dx \\ v = g(x) \end{cases}$

$\Rightarrow I = \displaystyle\int_0^1 f(x)g'(x)\,dx = f(x)g(x)\Big|_{x=0}^{x=1} - \int_0^1 f'(x)g(x)\,dx$

$= [f(1)g(1) - f(0)g(0)] - 5$

$= 15$

# CHAPTER 4: INTEGRATION

## 3. Techniques of Integration

### TRIGONOMETRIC INTEGRALS

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$
$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

$$\int \sin^3\left(\frac{2}{3}x\right)\cos^4\left(\frac{2}{3}x\right)\,dx \qquad \frac{3}{14}\cos^7\left(\frac{2}{3}x\right) - \frac{3}{10}\cos^5\left(\frac{2}{3}x\right) + c$$

$$\int_{\pi}^{2\pi} \cos^3\left(\frac{1}{2}w\right)\sin^5\left(\frac{1}{2}w\right)\,dw \qquad -\frac{1}{12}$$

$$\int \frac{2 + 7\sin^3(z)}{\cos^2(z)}\,dz \qquad 2\tan(z) + 7\frac{1}{\cos(z)} + 7\cos(z) + c = 2\tan(z) + 7\sec(z) + 7\cos(z) + c$$

5) Find $\int \sin^3 x \cos^8 x \, dx$

$$I = \int \sin^3 x \cos^8 x \, dx$$
$$= \int \sin x . \sin^2 x \cos^8 x \, dx$$
$$= \int \sin x \,(1 - \cos^2 x)\cos^8 x \, dx$$

Denote $u = \cos x \Rightarrow du = -\sin x \, dx$

$$I = \int u^8 (u^2 - 1)\,du = \int (u^{10} - u^8)\,du$$
$$= \frac{u^{11}}{11} - \frac{u^9}{9} + C$$
$$= \frac{1}{11}\cos^{11} x - \frac{1}{9}\cos^9 x + C$$

# CHAPTER 4: INTEGRATION

## 3. Techniques of Integration

### TRIGONOMETRIC INTEGRALS

- For integrals involving $\sqrt{a^2 + x^2}$ → using x = $a\tan\theta$
- For integrals involving $\sqrt{a^2 - x^2}$ → using x = $a\sin\theta$

6) Find $I = \int_0^1 \frac{dx}{(x^2+1)\sqrt{x^2+1}}$

Denote $x = \tan\theta$

$$\Rightarrow dx = \frac{1}{\cos^2\theta}\,d\theta = (1+\tan^2\theta)\,d\theta$$

| $x$ | $0$ | $1$ |
| --- | --- | --- |
| $\theta$ | $0$ | $\frac{\pi}{4}$ |

$$I = \int_0^{\frac{\pi}{4}} \frac{(1+\tan^2\theta)\,d\theta}{(\tan^2x+1)\sqrt{\tan^2x+1}}$$

$$= \int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\tan^2x+1}} = \int_0^{\frac{\pi}{4}} \frac{d\theta}{\sqrt{\frac{1}{\cos^2\theta}}}$$

$$= \int_0^{\frac{\pi}{4}} \frac{d\theta}{\frac{1}{|\cos\theta|}} = \int_0^{\frac{\pi}{4}} |\cos\theta|\,d\theta = \int_0^{\frac{\pi}{4}} \cos\theta\,d\theta$$

$$= \sin\theta \Big|_{\theta=0}^{\theta=\frac{\pi}{4}} = \boxed{\frac{\sqrt{2}}{2}}$$

# CHAPTER 4: INTEGRATION

## 3. Techniques of Integration

### PARTIAL FRACTIONS

8) Find $I = \int \frac{x^3+x}{x-1}\,dx$

$$\int \frac{x^4 - 5x^3 + 6x^2 - 18}{x^3 - 3x^2}\,dx$$

$$\frac{1}{2}x^2 - 2x + 2\ln|x| - \frac{6}{x} - 2\ln|x-3| + c$$

Long division:

$$\begin{array}{r|l} x^3+x & x-1 \\ \hline -\;(x^3-x^2) & x^2+x+2 \\ \hline x+x^2 & \\ -\;(x^2-x) & \\ \hline 2x & \\ -\;(2x-2) & \\ \hline 2 & \end{array}$$

$$\frac{x^3+x}{x-1} = x^2+x+2+\frac{2}{x-1}$$

$$\Rightarrow I = \int\left(x^2+x+2+\frac{2}{x-1}\right)dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C$$

# CHAPTER 4: INTEGRATION

## 3. Techniques of Integration

## PARTIAL FRACTIONS

9) Find $I=\int_2^3 \frac{x-8}{3x^2+2x-8}\,dx$

$$\int \frac{3x+11}{x^2-x-6}\,dx$$

$4\ln|x-3|-\ln|x+2|+c$

$$\int \frac{x^2+4}{3x^3+4x^2-4x}\,dx$$

$-\ln|x|+\frac{1}{2}\ln|x+2|+\frac{5}{6}\ln|3x-2|+c$

$$\int \frac{x^2-29x+5}{(x-4)^2(x^2+3)}\,dx$$

$\ln|x-4|+\frac{5}{x-4}-\frac{1}{2}\ln|x^2+3|+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+c$

$$I=\int_2^3 \frac{x-8}{(3x-4)(x+2)}\,dx$$

Assume that $\frac{x-8}{(3x-4)(x+2)}=\frac{A}{3x-4}+\frac{B}{x+2}$

$\Rightarrow x-8=A(x+2)+B(3x-4)$

$\Rightarrow x-8=(A+3B)x+(2A-4B)$

$\Rightarrow \begin{cases} A+3B=1 \\ 2A-4B=-8 \end{cases} \Rightarrow \begin{cases} A=-2 \\ B=1 \end{cases}$

$$I=\int_2^3\left(\frac{-2}{3x-4}+\frac{1}{x+2}\right)dx$$

$=\left(-\frac{2}{3}\ln|3x-4|+\ln|x+2|\right)\Big|_{x=2}^{x=3}$

$=-\frac{2}{3}\ln 5+\ln 5+\frac{2}{3}\ln 2-\ln 4$

$=\frac{1}{3}\ln 5-\frac{4}{3}\ln 2=\frac{1}{3}\ln\frac{5}{2^4}$

# CHAPTER 4: INTEGRATION

## 4. Numerical (Approximation) Integrals

## MIDPOINT RULE

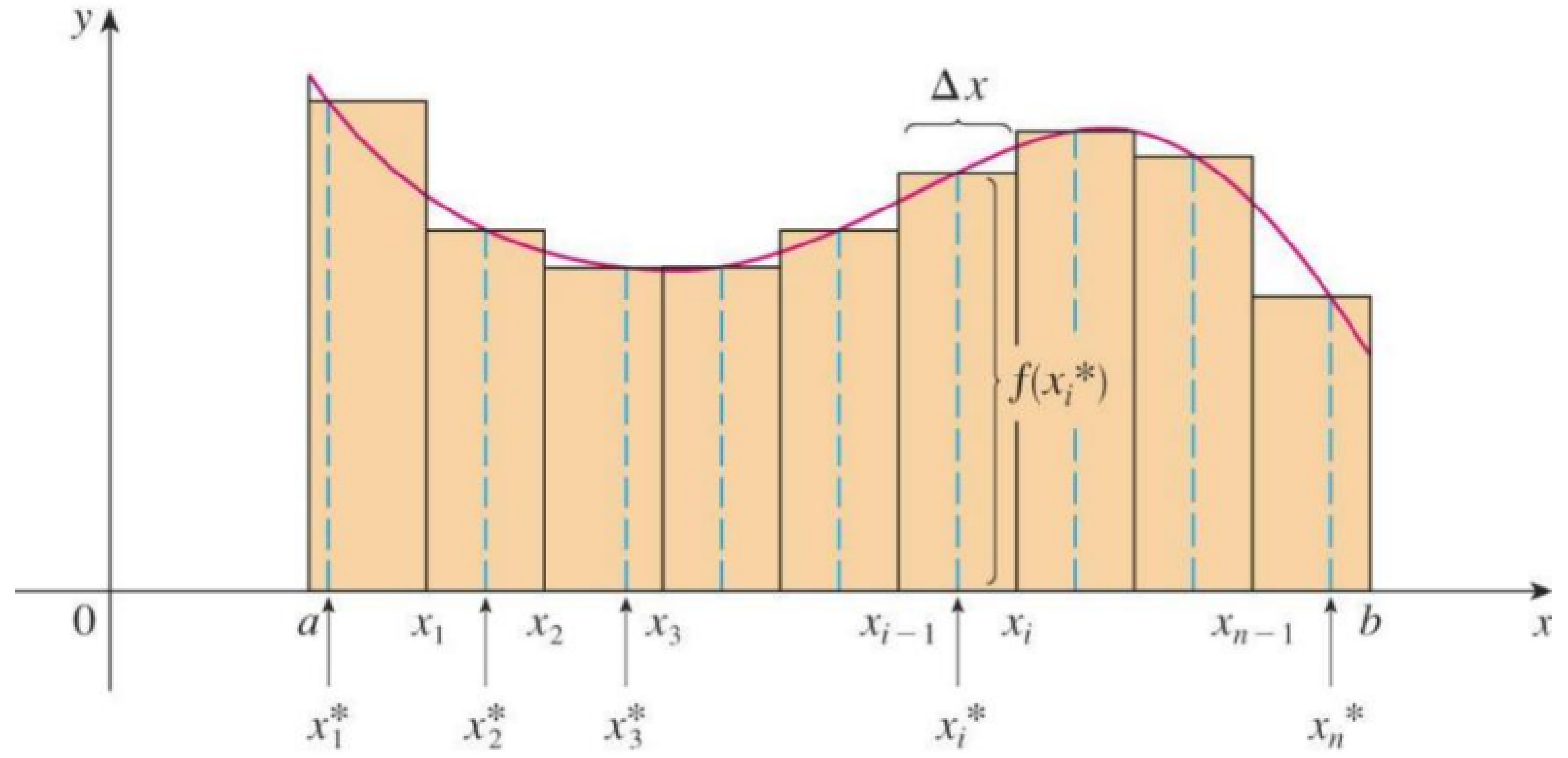
$\int_a^b f(x)dx \approx \Delta x \sum_{i=1}^{N} f\left(\frac{x_{i-1}+x_i}{2}\right)$ where $x_i^* = \frac{x_{i-1}+x_i}{2}$

## TRAPEZOIDAL RULE

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}[f(x_0)+2f(x_1)+2f(x_2)+\dots+2f(x_{N-1})+f(x_N)]$$

## SIMPSON RULE

$\int_a^b f(x)dx \approx \frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+\dots\ 2f(x_{N-2})+4f(x_{N-1})+f(x_N)]$ **(N is even)**




$\int_a^b f(x)\,dx$ ⇨ Divide [a,b] into n equal subintervals of equal length $\Delta x = \frac{b-a}{n}$

$X_0 = a$

$X_1 = a + \Delta x$

$X_2 = a + 2\,\Delta x$

$X_i = a + i\,\Delta x$

...

$X_n = b$

## 4. Numerical (Approximation) Integrals

## MIDPOINT RULE

$$\int_a^b f(x)dx \approx \Delta x \sum_{i=1}^{N} f\left(\frac{x_{i-1}+x_i}{2}\right) \quad \text{where } x_i^* = \frac{x_{i-1}+x_i}{2}$$

1. $\displaystyle\int_1^7 \frac{1}{x^3+1}\,dx$ using $n=6$   $0.33197137$

2. $\displaystyle\int_{-1}^{2} \sqrt{e^{-x^2}+1}\,dx$ using $n=6$   $3.70700857$

3. $\displaystyle\int_0^4 \cos\left(1+\sqrt{x}\right)dx$ using $n=8$   $-2.51625938$

10) Find an approximate value of the integral $\displaystyle\int_1^2 \frac{dx}{x}$ by using the Midpoint Rule with $N=5$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{5} = \frac{1}{5} = 0,2$$

$x_0 = 1$

$x_1 = 1 + 0,2 = 1,2$

$x_2 = 1,4$

$x_3 = 1,6$

$x_4 = 1,8$

$x_5 = 2,0$

$\rightarrow$ Midpoints:

$$\begin{cases} x_1^* = \dfrac{x_0+x_1}{2} = 1,1 \\ x_2^* = 1,3 \\ x_3^* = 1,5 \\ x_4^* = 1,7 \\ x_5^* = 1,9 \end{cases}$$

$$I = \int_1^2 \frac{dx}{x} \approx \Delta x \sum_{i=1}^{5} f(x_i^*)$$

$$= 0,2\left[f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)\right]$$

$$= 0.2\left[\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{1.7} + \frac{1}{1.9}\right]$$

$$\approx 0.691908$$

## 4. Numerical (Approximation) Integrals

### TRAPEZOIDAL RULE

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{N-1})+f(x_N)\right]$$

11) Estimate $\int_0^{30} f(x)dx$ using Trapezoidal Rule.

| | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|---|
| $x$ | 0 | 5 | 10 | 15 | 20 | 25 | 30 |
| $f(x)$ | 1 | 1.7 | 2.2 | 1.9 | 1.6 | 1.1 | 0.7 |

$\Delta x = 5$

$$\int_0^{30} f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+2f(x_4)+2f(x_5)+f(x_6)\right]$$

$$= \frac{5}{2}\left(1+2\times 1.7+2\times 2.2+2\times 1.9+2\times 1.6+2\times 1.1+0.7\right)$$

$$= 46.75$$

1. $\displaystyle\int_1^7 \frac{1}{x^3+1}\,dx$ using $n=6$ $\boxed{0.42620830}$

2. $\displaystyle\int_{-1}^2 \sqrt{e^{-x^2}+1}\,dx$ using $n=6$ $\boxed{3.69596543}$

3. $\displaystyle\int_0^4 \cos\left(1+\sqrt{x}\right)dx$ using $n=8$ $\boxed{-2.43000475}$

# CHAPTER 4: INTEGRATION

## 4. Numerical (Approximation) Integrals

### SIMPSON RULE

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+\dots 2f(x_{N-2})+4f(x_{N-1})+f(x_N)\right]$$ **(N is even)**

1. $\displaystyle\int_1^7 \frac{1}{x^3+1}\,dx$ using $n=6$ $\boxed{0.37154155}$

2. $\displaystyle\int_{-1}^2 \sqrt{\mathrm{e}^{-x^2}+1}\,dx$ using $n=6$ $\boxed{3.70358145}$

3. $\displaystyle\int_0^4 \cos\left(1+\sqrt{x}\right)dx$ using $n=8$ $\boxed{-2.47160136}$

# CHAPTER 4: INTEGRATION

## 5. Improper Integrals

**1.** $\displaystyle\int_0^{\infty} (1+2x)\,\mathbf{e}^{-x}\,dx$ 3

**6.** $\displaystyle\int_2^{\infty} \frac{9}{(1-3z)^4}\,dz$ $\frac{1}{125}$

**8.** $\displaystyle\int_{-\infty}^{\infty} \frac{6w^3}{(w^4+1)^2}\,dw$ 0

**10.** $\displaystyle\int_{-\infty}^{0} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx$ 1

12) Evaluate $I = \int_0^{\infty} x e^{-x^2+1}\,dx$

$$I = \int_0^{\infty} x e^{-x^2+1}\,dx = \lim_{t\to+\infty} \underbrace{\int_0^{t} x e^{-x^2+1}\,dx}_{J} \qquad (1)$$

Denote $u = -x^2+1$

$\Rightarrow du = -2x\,dx$

| $x$ | $0$ | $t$ |
|---|---|---|
| $u$ | $1$ | $-t^2+1$ |

$$J = \int_1^{-t^2+1} \frac{e^u\,du}{-2} = -\frac{1}{2}e^u \Big|_{u=1}^{u=-t^2+1}$$

$$\Rightarrow J = -\frac{1}{2}\left(e^{-t^2+1} - e\right) \qquad (2)$$

From (1) and (2):

$$I = \lim_{t\to+\infty}\left[-\frac{1}{2}\left(e^{-t^2+1} - e\right)\right]$$

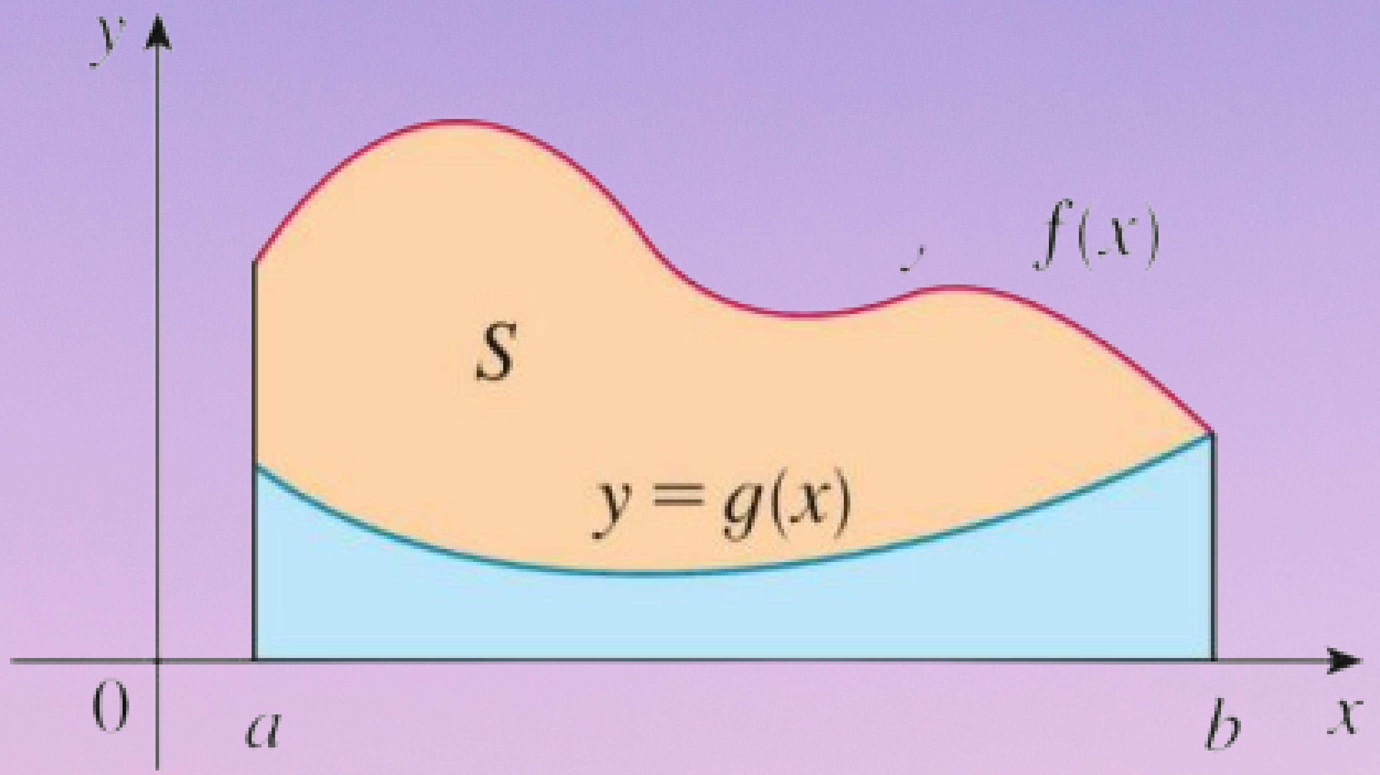
$$= -\frac{1}{2}\left(e^{-\infty} - e\right) = \boxed{\frac{e}{2}}$$

# CHAPTER 5

# APPLICATIONS OF INTEGRATION

# CHAPTER 5: APPLICATIONS OF INTEGRATION

## 1. Area between curves




$$A = \int_a^b \left| f(x) - g(x) \right| dx.$$

Determine the area of the region bounded by the given set of curves:

**4.** $y = \dfrac{8}{x}$, $y = 2x$ and $x = 4$ — $6.4548$

**5.** $x = 3 + y^2$, $x = 2 - y^2$, $y = 1$ and $y = -2$ — $\boxed{9}$

**6.** $x = y^2 - y - 6$ and $x = 2y + 4$ — $\boxed{\frac{343}{6}}$

**7.** $y = x\sqrt{x^2+1}$, $y = \mathbf{e}^{-\frac{1}{2}x}$, $x = -3$ and the y-axis. — $17.17097$

**8.** $y = 4x + 3$, $y = 6 - x - 2x^2$, $x = -4$ and $x = 2$ — $\boxed{\frac{343}{12}}$

**9.** $y = \dfrac{1}{x+2}$, $y = (x+2)^2$, $x = -\dfrac{3}{2}$, $x = 1$ — $7.9695$

**10.** $x = y^2 + 1$, $x = 5$, $y = -3$ and $y = 3$ — $\boxed{\frac{46}{3}}$

**11.** $x = \mathbf{e}^{1+2y}$, $x = \mathbf{e}^{1-y}$, $y = -2$ and $y = 1$ — $\boxed{22.9983}$

# CHAPTER 5: APPLICATIONS OF INTEGRATION

## 2. Volumes of Solid of Revolution

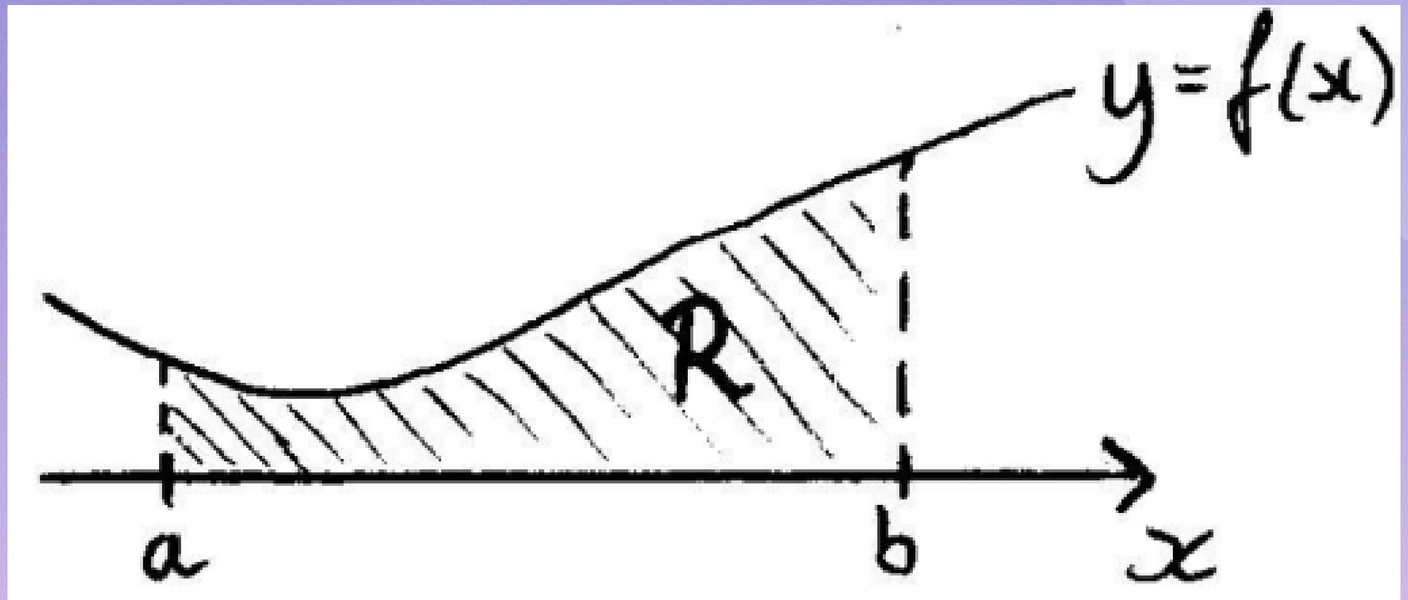
### DISK METHOD

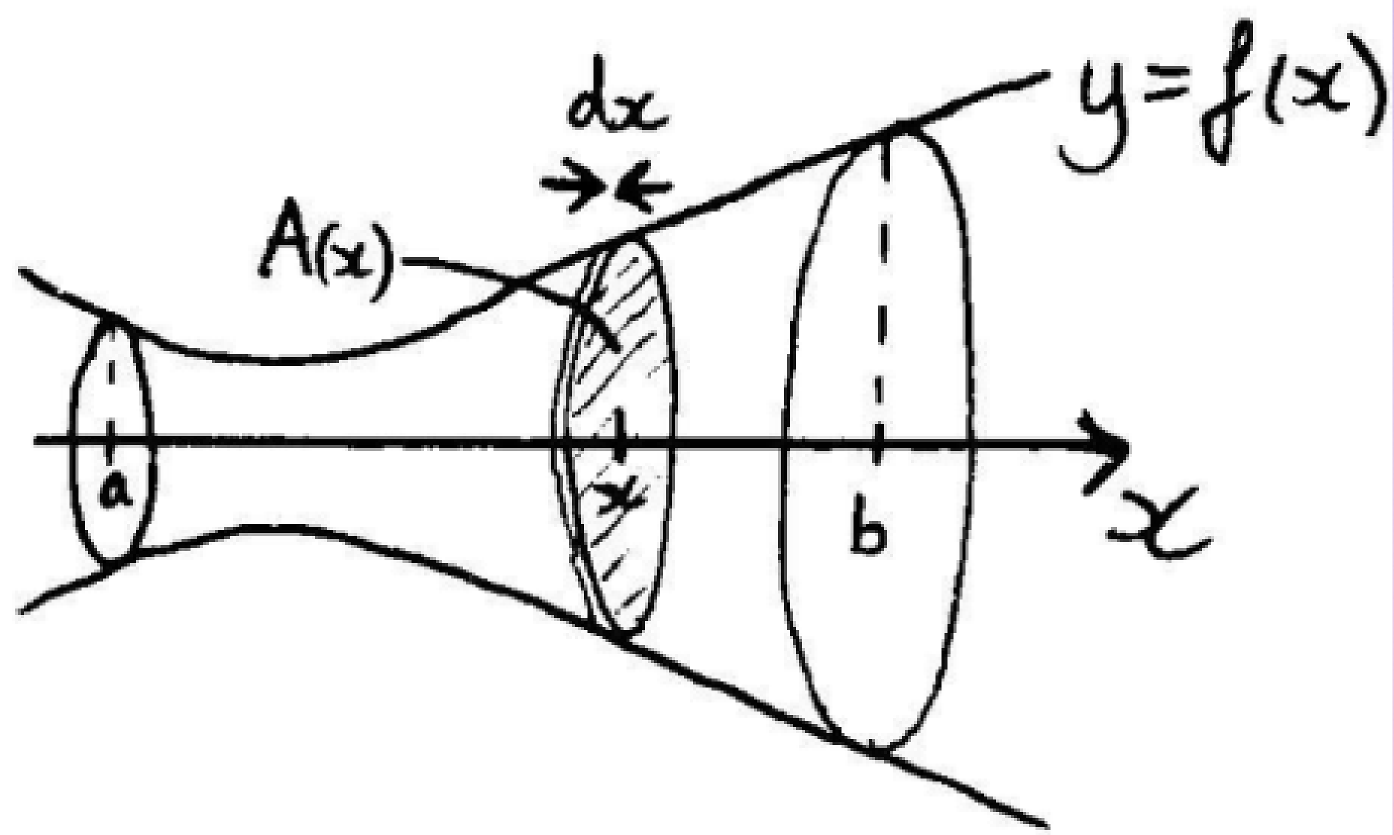
**Volumes by Slicing**

$$V = \int_a^b dV = \int_a^b A(x)dx$$

**Volumes found by Slicing**

$$V = \int_a^b A(x)dx = \pi\int_a^b \left(f(x)\right)^2 dx$$



# CHAPTER 5: APPLICATIONS OF INTEGRATION

## 2. Volumes of Solid of Revolution

$$V = \int_a^b A(x)dx = \pi\int_a^b \left(f(x)\right)^2 dx$$

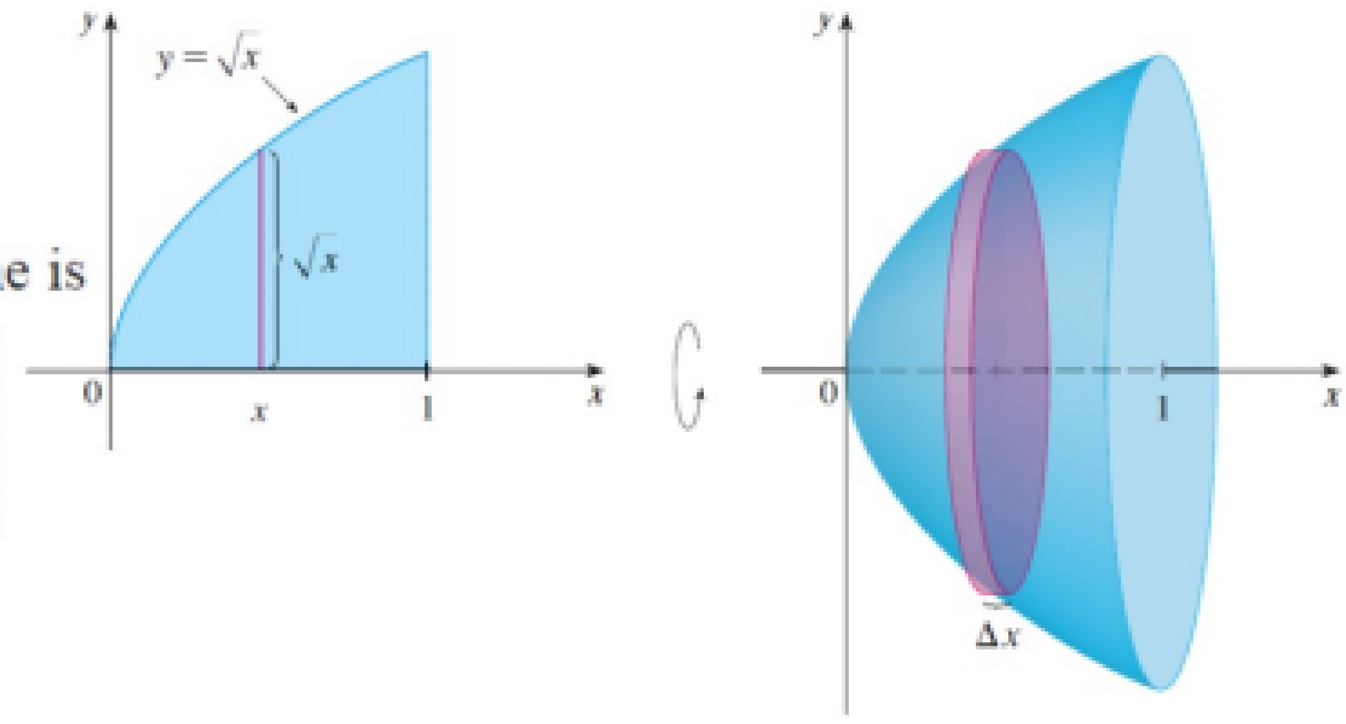
**V** **EXAMPLE 2** Find the volume of the solid obtained by rotating about the $x$-axis the region under the curve $y = \sqrt{x}$ from 0 to 1. Illustrate the definition of volume by sketching a typical approximating cylinder.

SOLUTION The region is shown in Figure 6(a). If we rotate about the $x$-axis, we get the solid shown in Figure 6(b). When we slice through the point $x$, we get a disk with radius $\sqrt{x}$. The area of this cross-section is

$$A(x) = \pi\left(\sqrt{x}\right)^2 = \pi x$$

The solid lies between $x = 0$ and $x = 1$, so its volume is

$$\boxed{V = \int_0^1 A(x)\,dx = \int_0^1 \pi x\,dx = \pi \frac{x^2}{2}\bigg]_0^1 = \frac{\pi}{2}}$$




(a) (b)

# CHAPTER 5: APPLICATIONS OF INTEGRATION

## 2. Volumes of Solid of Revolution

$$V = \int_a^b A(x)dx = \pi \int_a^b \left(f(x)\right)^2 dx$$

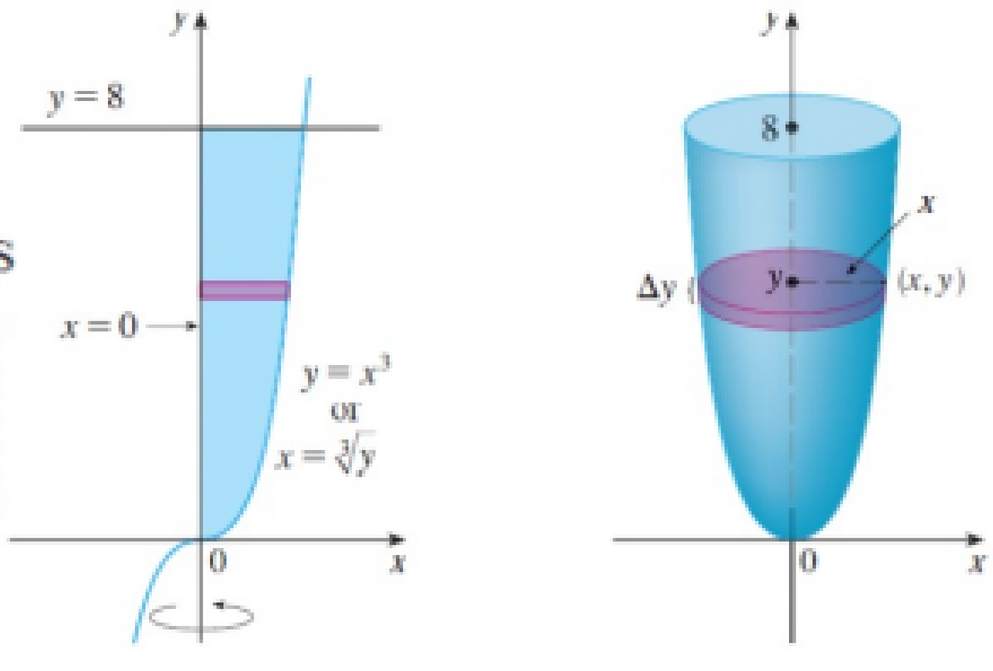
**V** **EXAMPLE 3** Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the $y$-axis.

SOLUTION Because the region is rotated about the $y$-axis, it makes sense to slice the solid perpendicular to the $y$-axis and therefore to integrate with respect to $y$. If we slice at height $y$, we get a circular disk with radius $x$, where $x = \sqrt[3]{y}$. So the area of a cross-section through $y$ is

$$A(y) = \pi x^2 = \pi(\sqrt[3]{y})^2 = \pi y^{2/3}$$

Since the solid lies between $y = 0$ and $y = 8$, its volume is

$$\boxed{V = \int_0^8 A(y)\,dy = \int_0^8 \pi y^{2/3}\,dy = \pi\left[\tfrac{3}{5}y^{5/3}\right]_0^8 = \frac{96\pi}{5}}$$



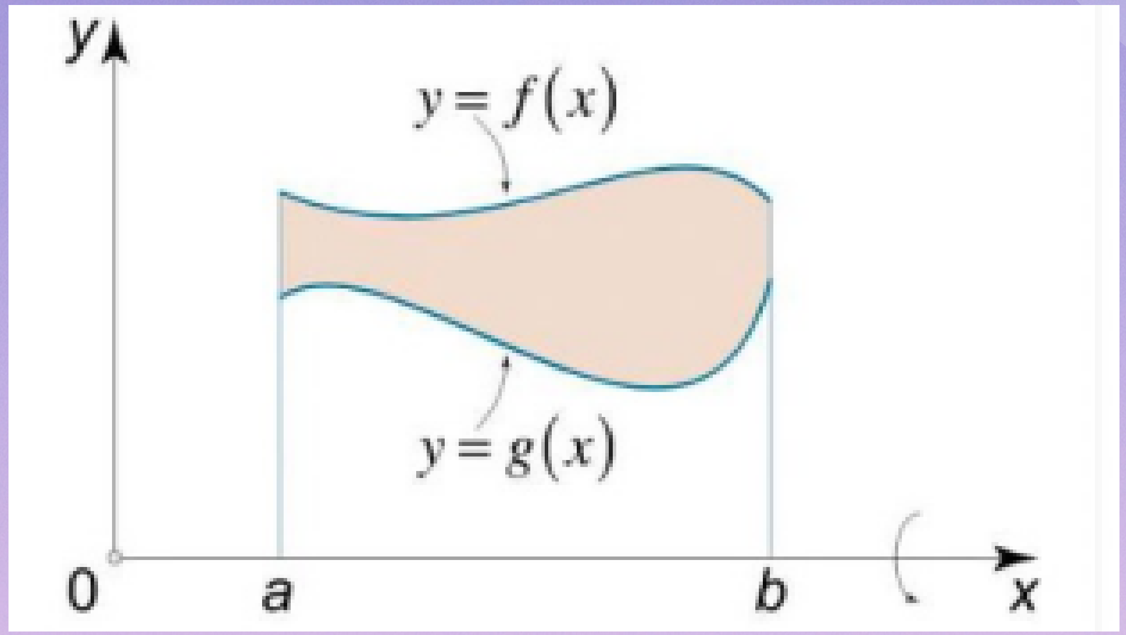

(a) (b)

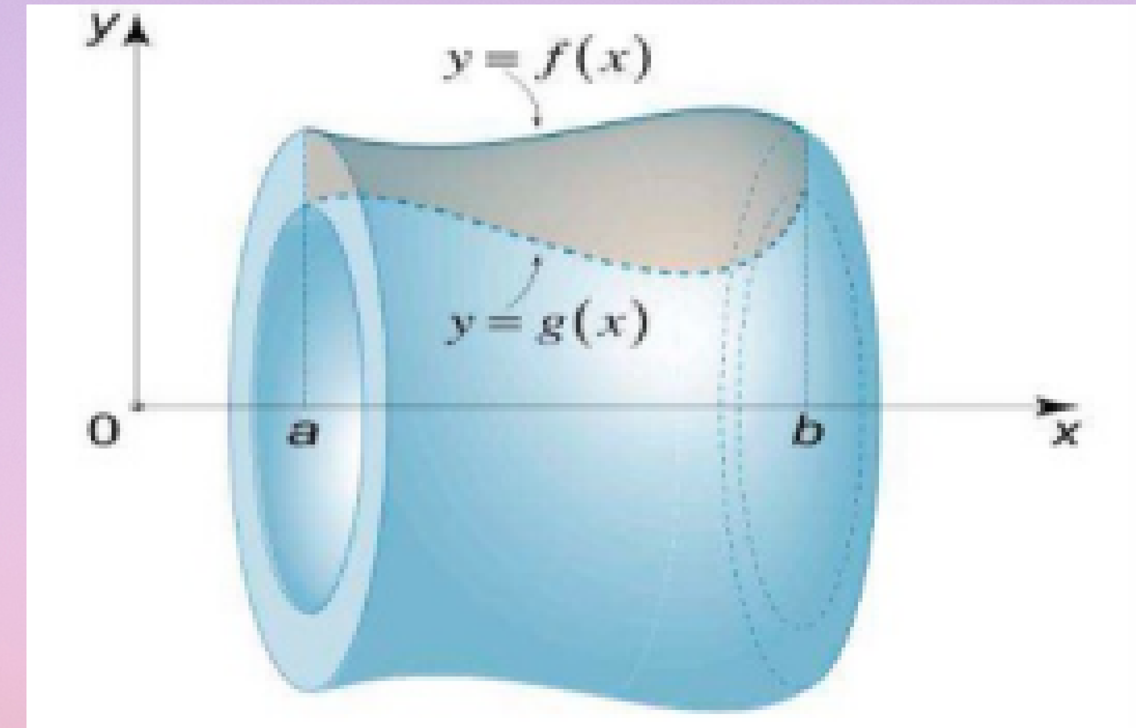
# CHAPTER 5: APPLICATIONS OF INTEGRATION

## 2. Volumes of Solid of Revolution

### WASHER METHOD

$$V = \pi \int_a^b \left( [f(x)]^2 - [g(x)]^2 \right) dx.$$



# CHAPTER 5: APPLICATIONS OF INTEGRATION

## 2. Volumes of Solid of Revolution

$$V = \pi \int_a^b \left( [f(x)]^2 - [g(x)]^2 \right) dx.$$

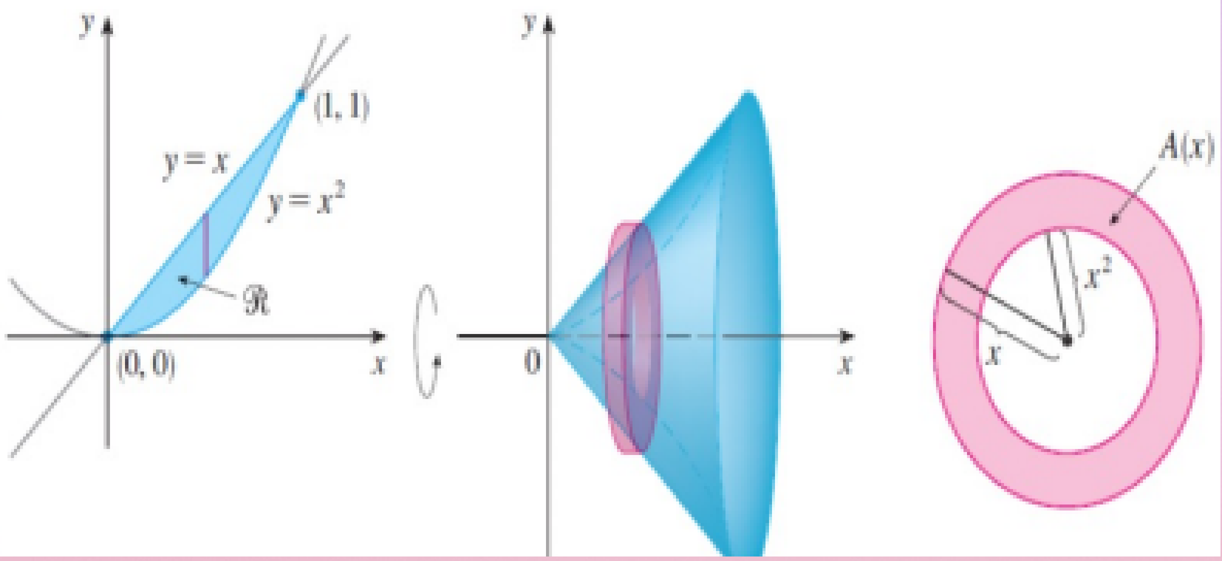
**EXAMPLE 4** The region $\mathcal{R}$ enclosed by the curves $y = x$ and $y = x^2$ is rotated about the $x$-axis. Find the volume of the resulting solid.

SOLUTION The curves $y = x$ and $y = x^2$ intersect at the points $(0, 0)$ and $(1, 1)$. The region between them, the solid of rotation, and a cross-section perpendicular to the $x$-axis are shown in Figure 8. A cross-section in the plane $P_x$ has the shape of a *washer* (an annular ring) with inner radius $x^2$ and outer radius $x$, so we find the cross-sectional area by subtracting the area of the inner circle from the area of the outer circle:

$$A(x) = \pi x^2 - \pi(x^2)^2 = \pi(x^2 - x^4)$$

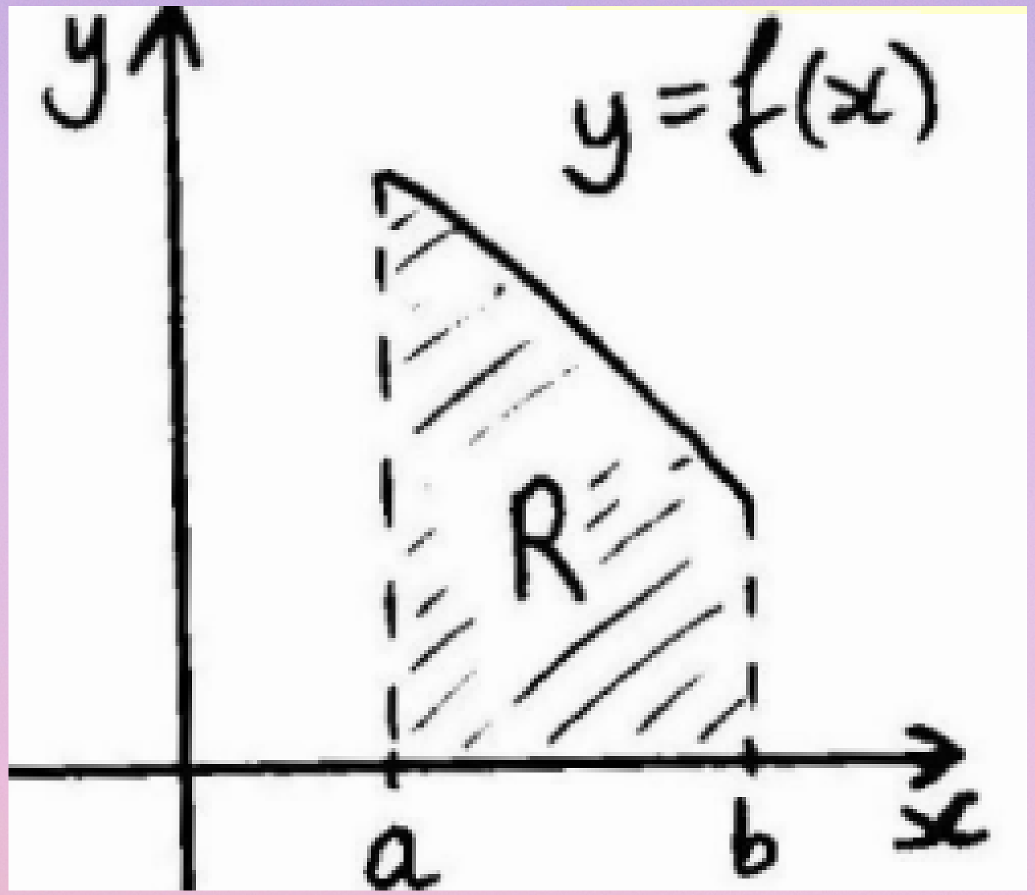
$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^2 - x^4)\,dx$$

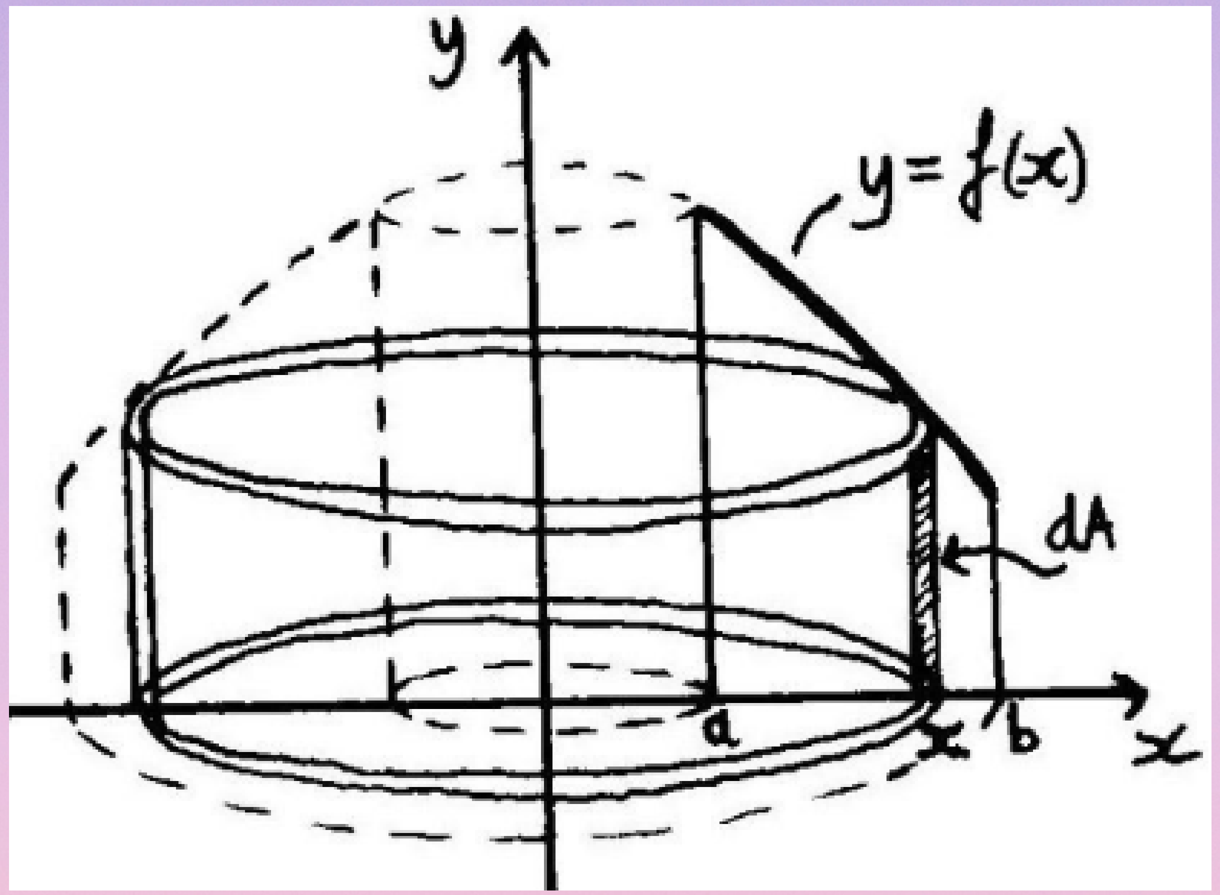
$$= \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{2\pi}{15}$$



## 2. Volumes of Solid of Revolution

### CYLINDRICAL SHELLS METHOD




$$V = 2\pi \int_a^b x\, f(x)\, dx$$

# CHAPTER 5: APPLICATIONS OF INTEGRATION

## 2. Volumes of Solid of Revolution

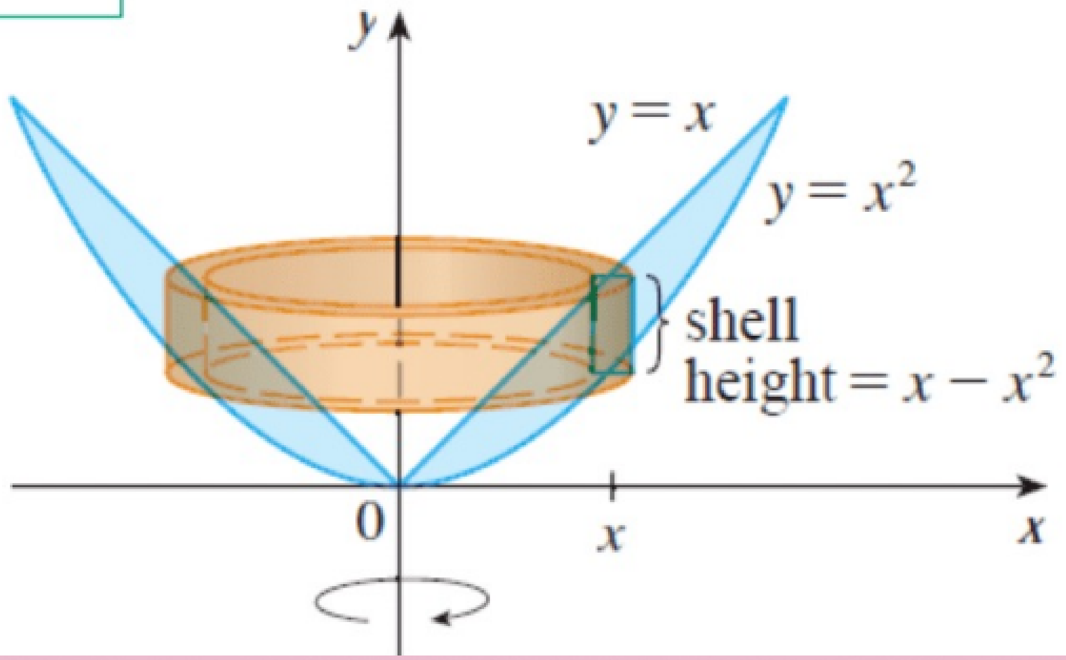
$$V = 2\pi \int_a^b x\, f(x)\, dx$$

**V EXAMPLE 2** Find the volume of the solid obtained by rotating about the $y$-axis the region between $y = x$ and $y = x^2$.

**SOLUTION** The region and a typical shell are shown in Figure 8. We see that the shell has radius $x$, circumference $2\pi x$, and height $x - x^2$. So the volume is

$$\boxed{V = \int_0^1 (2\pi x)(x - x^2)\, dx = 2\pi \int_0^1 (x^2 - x^3)\, dx}$$

$$= 2\pi \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{\pi}{6}$$



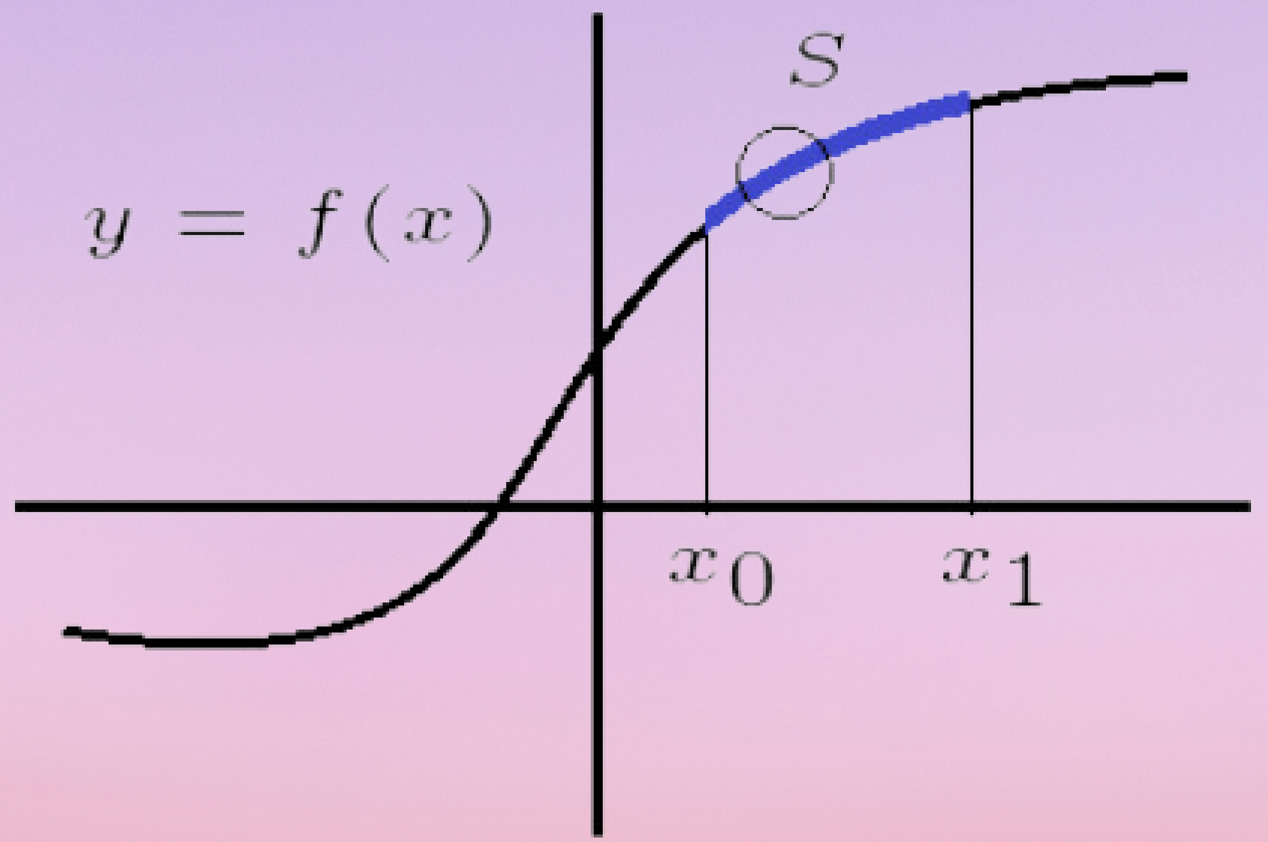
# CHAPTER 5: APPLICATIONS OF INTEGRATION

## 2. Volumes of Solid of Revolution

- Volume of the solid generated by rotating S about the x – axis is:
  $V_x = \pi \int_a^b ([f(x)]^2 - [g(x)]^2)dx$
- Volume of the solid generated by rotating S about the y – axis is:
  $V_y = 2\pi \int_a^b x(f(x) - g(x))dx$

## 3. Arc length

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$



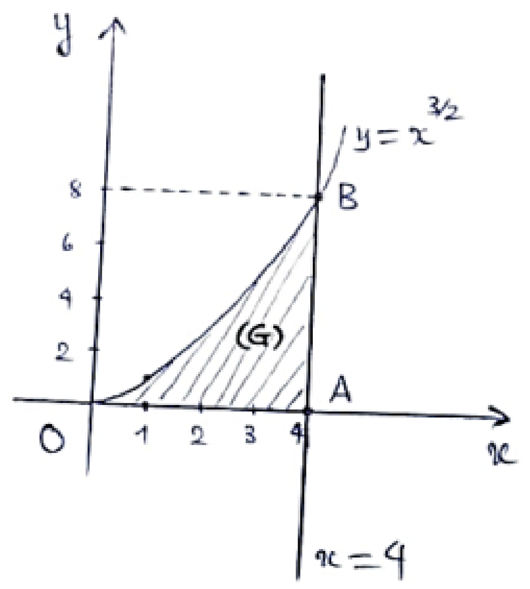
# CHAPTER 5: APPLICATIONS OF INTEGRATION

## 3. Arc length

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

14) Region G is enclosed by $y = x^{3/2}$, $x = 0$, $x = 4$ and $y = 0$

(a) Find the area of region G.




$$\text{Area }(G) = \int_0^4 (x^{3/2} - 0)\, dx = \frac{2}{5} x^{5/2} \Big|_{x=0}^{x=4} = \frac{64}{5}$$

(b) Find the volume of the solid generated by rotating the region G about the x-axis.

$$V_x = \pi \int_0^4 \left[ (x^{3/2})^2 - 0 \right] dx = \pi \int_0^4 x^3\, dx = \frac{\pi}{4} x^4 \Big|_{x=0}^{x=4} = 64\pi$$

(c) Find the circumference of the region G

* Circumference of G = length (OA) + length (AB) + length $(\overset{\frown}{BO})$ = 12 + length$(\overset{\frown}{BO})$

* Length$(\overset{\frown}{BO})$ $= \int_0^4 \sqrt{1 + \left[ (x^{3/2})'_x \right]^2}\, dx = \int_0^4 \sqrt{1 + \left( \frac{3}{2}\sqrt{x} \right)^2}\, dx = \int_0^4 \sqrt{1 + \frac{9}{4}x}\, dx$

Denote $u = 1 + \frac{9}{4}x$

$\Rightarrow du = \frac{9}{4} dx$

| x | 0 | 4 |
|---|---|---|
| u | 1 | 10 |

$\Rightarrow$ length$(\overset{\frown}{BO})$ $= \int_1^{10} \sqrt{u} \cdot \frac{4}{9} du = \frac{4}{9} \cdot \frac{2}{3} u^{3/2} \Big|_{u=1}^{u=10} = \frac{8}{27}(10^{2/3} - 1)$

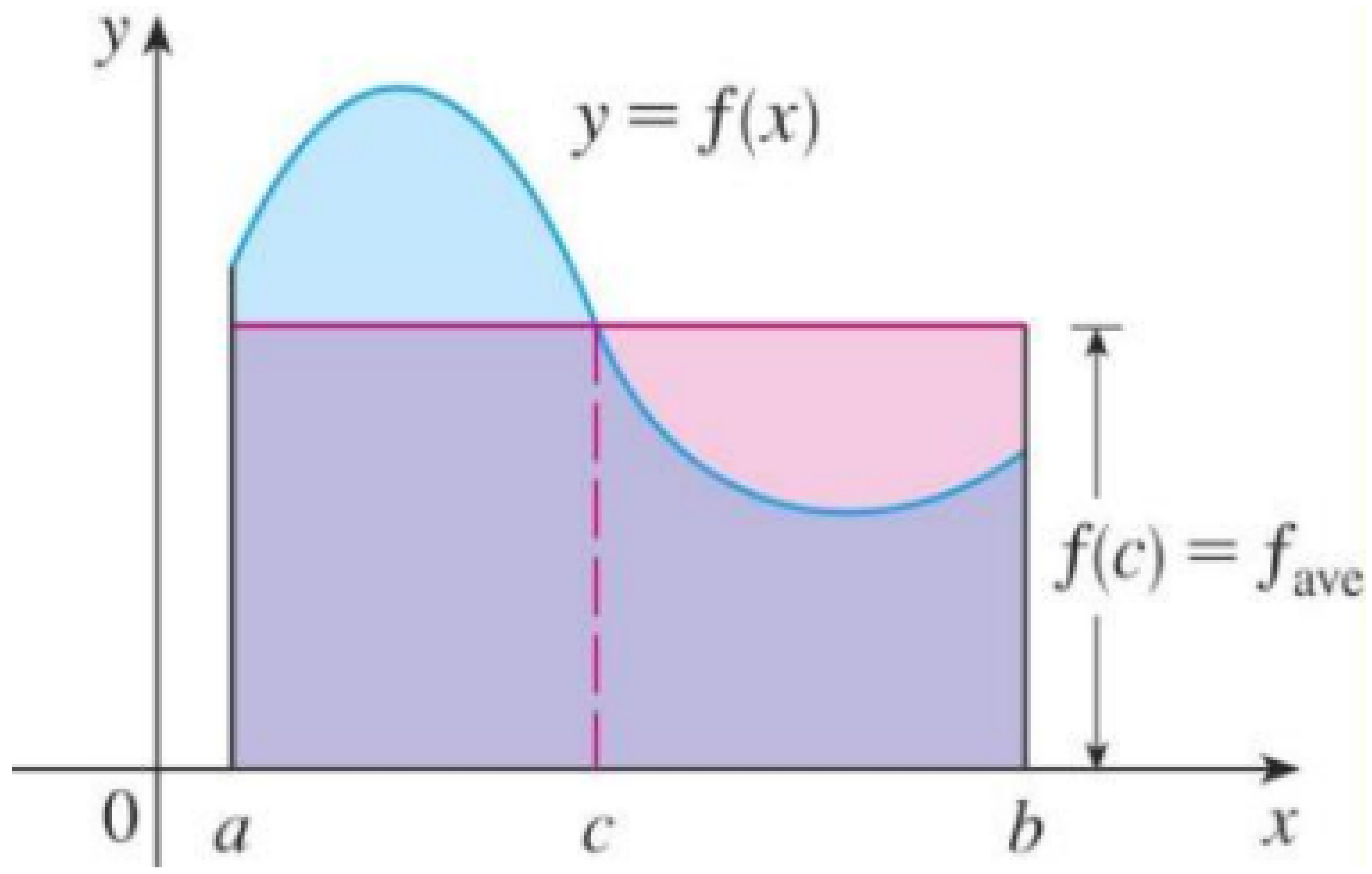
$\Rightarrow$ Circumference of G $= 12 + \frac{8}{27}(10^{2/3} - 1)$

# CHAPTER 5: APPLICATIONS OF INTEGRATION

## 4. Average Value

The average value of a funtion f on the interval [a,b] is:

$$\boxed{f_{av} = \frac{1}{b-a}\int_a^b f(x)dx}$$




**Exercise 3.43.** A cup of hot tea with initial temperature $90^o$ C is placed in a room where the temperature is $25^o$ C. According to Newton's Law of Cooling, the temperature of the tea after t minutes is

$$T(t) = 25 + 65e^{-\frac{t}{50}}.$$

Evaluate the average temperature of the tea during the first hour.

The average temperature of the tea during the first hour is:

$$\frac{1}{60-0}\int_0^{60} T(t)\,dt$$

$$= \frac{1}{60}\int_0^{60}\left(25 + 65e^{-t/50}\right)dt$$

$$= \frac{1}{60}\left(25t - 50\times 65e^{-t/50}\right)\Big|_{t=0}^{t=60}$$

$$= 62.85^\circ C$$

# CHAPTER 5: APPLICATIONS OF INTEGRATION

## 5. Applications

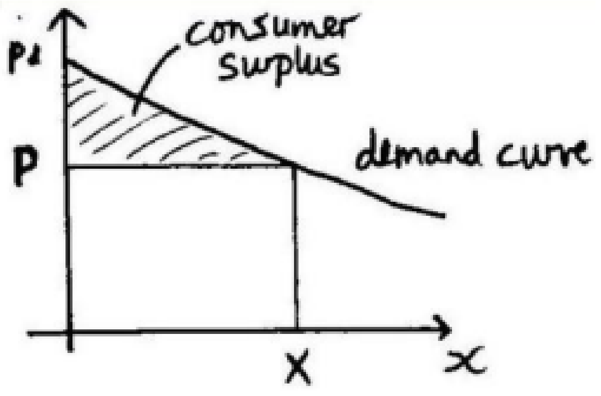
If X is the amount of product currently available (or production level), P = p(X) is the current selling price.

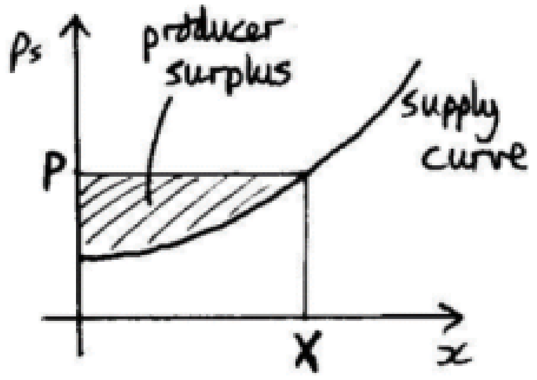
- **Consumer surplus**:

$$\boxed{\int_0^X [p_d(x) - P]dx}$$

- **Producer surplus**:

$$\boxed{\int_0^X [P - p_s(x)]dx}$$




16) A demand curve is given by $P_d = \frac{450}{x+8}$

Find the consumer surplus when the selling price x is \$10

Consumer surplus $= \int_0^X (P_d(x) - P)\,dx$

Selling price $p(X) = 10$

$\Rightarrow \frac{450}{X+8} = 10$

$\Rightarrow X = 37$

$\rightarrow$ Consumer surplus $= \int_0^{37} \left(\frac{450}{x+8} - 10\right) dx = A$

Denote $u = x+8$

$\Rightarrow du = dx$

| x | 0 | 37 |
|---|---|---|
| u | 8 | 45 |

$A = \int_8^{45} \frac{450}{u}\,du - \int_0^{37} 10\,dx$

$= 450 \ln|u| \Big|_{u=8}^{u=45} - 10x \Big|_{x=0}^{x=37}$

$= 450 \ln \frac{45}{8} - 370$

# CHAPTER 5: APPLICATIONS OF INTEGRATION

## 5. Applications

**Exercise 3.40.** An oil storage tank cracks at time $t = 0$ and oil leaks from the tank at a rate of $r(t) = e^{-0.1t}$ liters per minute. How much oil leaks out during the first ten minutes?

Denote $V(t)$ is the amount of oil that leaks out after $t$ min.

$\Rightarrow V'(t)$ is the rate at which oil is leaking out after $t$ min.

$\Rightarrow V'(t) = r(t) = e^{-0.1t}$ $(*)$

Integrate both sides of $(*)$ over $[0,10]$:

$$\int_0^{10} V'(t)\,dt = \int_0^{10} e^{-0.1t}\,dt$$

$$\Rightarrow V(t)\Big|_{t=0}^{t=10} = -\frac{1}{0.1} e^{-0.1t}\Big|_{t=0}^{t=10}$$

$$\Rightarrow V(10) - V(0) = -10\left(\frac{1}{e} - 1\right)$$

$$\Rightarrow V(10) = 10 - \frac{10}{e}$$

Therefore, $\left(10 - \frac{10}{e}\right)$ liters of oil leaks out after 10 min.

# CHAPTER 5: APPLICATIONS OF INTEGRATION

## 5. Applications

**Exercise 3.39.** A particle moves along a straight line and its velocity at time $t$ is given by $v(t) = 2t + \cos 2t$. If its initial position is $s(0) = 1$, then its position at $t = T$ is

$s'(t) = v(t)$
$v'(t) = a(t)$

Integrate both sides of $v(t) = 2t + \cos 2t$ over $[0,T]$:

$$\int_0^T v(t)\,dt = \int_0^T (2t + \cos 2t)\,dt$$

$$\Rightarrow s(t)\Big|_{t=0}^{t=T} = \left(t^2 + \frac{1}{2}\sin 2t\right)\Big|_{t=0}^{t=T}$$

$$\Rightarrow s(T) - s(0) = T^2 + \frac{1}{2}\sin 2T$$

$$\Rightarrow s(T) = 1 + T^2 + \frac{1}{2}\sin 2T$$

# THANK YOU!

Do you have any question?