Let $S = \left\{ \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 1 \end{pmatrix} \right\}$

and $\left\{ \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right\}$ are linear of R^2

$T = \{ w_1, w_2 \}, \quad S = \{ u_1, u_2 \}$

$[v]_T = c_1 w_1 + c_2 w_2$

$\Rightarrow [v]_S = [c_1 w_1 + c_2 w_2]_S = c_1 [w_1]_S + c_2 [w_2]_S$

$= \begin{pmatrix} [w_1]_S & [w_2]_S \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Transition matrix from T to S.

Denote $[w_1]_S = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, [w_2]_S = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$[w_1]_S = x_1 u_1 + x_2 u_2 = x_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ (1)

$[w_2]_u = y_1 u_1 + y_2 u_2 = y_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + y_2 \begin{pmatrix} -4 \\ 1 \end{pmatrix}$ (2)

From (1), (2) we have:

$x_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} -3x_1 - 4x_2 \\ x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

$\Leftrightarrow$

$y_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + y_2 \begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} -3y_1 - 4y_2 \\ y_1 + y_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

$\begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 1 & 1 \end{pmatrix}$

$\Leftrightarrow AX = B$

where $A = \begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix} = (u_1, u_2)$

$$X = ([w_1]_S \; [w_2]_S) = P_T$$

$B = (w_1 \; w_2)$ we can write

$$(u_1 \; u_2)([w_1]_S \; [w_2]_S) = (w_1 \; w_2)$$

$$(u_1 \; u_2) P_{T\to S} = (w_1 \; w_2)$$

from equation : $AX = B$

Consider $\overline{A} = (A \mid B) \xrightarrow{\text{Row trans}} (I \mid X)$

we have:

$$\overline{A} = \left(\begin{array}{cc|cc} -3 & -4 & 5 & 4 \\ 1 & 1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} -3 & -4 & 5 & 4 \\ 0 & -1/3 & 8/3 & 7/3 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} -3 & -4 & 5 & 4 \\ 0 & -1 & 8 & 7 \end{array}\right) \to \left(\begin{array}{cc|cc} -3 & 0 & -27 & -24 \\ 0 & -1 & 8 & 7 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & 0 & 9 & 8 \\ 0 & 1 & -8 & -7 \end{array}\right) \text{ So, } X = \begin{pmatrix} 9 & 8 \\ -8 & -7 \end{pmatrix}$$

or $P_{T\to S} = \begin{pmatrix} 9 & 8 \\ -8 & -7 \end{pmatrix}$

b) we have $[v]_S = P_{T\to S} \cdot [v]_T$

Since $[v]_T = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, then $[v]_S = \begin{pmatrix} 9 & 8 \\ -8 & -7 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = u_1 - u_2$

$$[v]_T = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = w_1 - w_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$[v]_S = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = u_1 - u_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix} - \begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Let $S = \left\{ \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \end{pmatrix} \right\}$

$T = \left\{ \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} -2 \\ 5 \end{pmatrix} \right\}$ R^2

a) Find $P_{T \to S}$!

b) If $[v]_T = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Find v_S