STT	Mã MH	Tên môn học	Nhóm thi	Tổ thi	Ngày thi	Giờ thi
1	EE010IU	Electromagnetic Theory	01	001	14/06/2021	10g15
2	EE072IU	Computer and Communication Networks	01	001	14/06/2021	10g15
3	EE055IU	Principles of EE2	01	001	15/06/2021	08g00
4	EE055IU	Principles of EE2	02	001	15/06/2021	08g00
5	EE092IU	Digital Signal Processing	01	001	15/06/2021	15g15
6	EE092IU	Digital Signal Processing	02	001	15/06/2021	15g15
7	EE051IU	Principles of EE1	01	001	16/06/2021	08g00
8	EE051IU	Principles of EE1	02	001	16/06/2021	08g00
9	EEAC008IU	Sensors and Instrumentation	01	001	16/06/2021	08g00
10	EEAC004IU	PC Based Control and SCADA System	01	001	16/06/2021	10g15
11	EE050IU	Introduction to Computer for Engineers	01	001	16/06/2021	13g00
12	EE103IU	Image Processing	01	001	16/06/2021	13g00
13	EE105IU	Antennas and Microwave Engineering	01	001	16/06/2021	13g00
14	EEAC010IU	Electric Machine	01	001	16/06/2021	13g00
15	EE090IU	Electronics Devices	01	001	17/06/2021	08g00
16	EE104IU	Embedded Real-time System	01	001	17/06/2021	08g00
17	EE128IU	Internet of Things (IoT)	01	001	17/06/2021	08g00
18	EE068IU	Principles of Communication Systems	01	001	17/06/2021	10g15
19	EE083IU	Micro-processing Systems	01	001	17/06/2021	10g15
20	EEAC006IU	Programmable Logic Control (PLC)	01	001	17/06/2021	10g15
21	EE070IU	Wireless Communication Systems	01	001	18/06/2021	08g00
22	EE133IU	Emerging Engineering Technologies	01	001	18/06/2021	08g00
23	EE088IU	Signals & Systems	01	001	21/06/2021	08g00
24	EE114IU	Entrepreneurship	01	001	21/06/2021	10g15
25	EEAC020IU	Theory of Automatic Control	01	001	21/06/2021	10g15
26	EE053IU	Digital Logic Design	01	001	21/06/2021	13g00
27	EE130IU	Capstone Design 1	01	001	22/06/2021	10g15
28	EE057IU	Programming for Engineers (C)	01	001	22/06/2021	13g00
29	EE129IU	Internet of Things Lab (IoT Lab)	01	001	23/06/2021	08g00
30	EE049IU	Introduction to Electrical Engineering	01	001	01/07/2021	08g00

Discrete-Time Fourier Series (DTFS)

INTRODUCTION

- The Fourier series is a representation for ***periodic*** sequences.
- With a Fourier series, a sequence is represented as a ***linear combination of complex sinusoids***.
- The use of complex sinusoids is desirable due to their numerous attractive properties.
- Perhaps, most importantly, complex sinusoids are ***eigensequences*** of (DT) LTI systems.

HARMONICALLY RELATED COMPLEX SINUSOIDS

- A set of periodic complex sinusoids is said to be **harmonically related** if there exists some constant $\frac{2\pi}{N}$ such that the fundamental frequency of each complex sinusoid is an integer multiple of $\frac{2\pi}{N}$.
- Consider the set of harmonically-related complex sinusoids given by

$$\phi_k(n) = e^{j(2\pi/N)kn} \quad \text{for all integer } k.$$

- In the above set $\{\phi_k\}$, only N elements are distinct, since

$$\phi_k = \phi_{k+N} \quad \text{for all integer } k.$$

- Since the fundamental frequency of each of the harmonically-related complex sinusoids is an integer multiple of $\frac{2\pi}{N}$, a linear combination of these complex sinusoids must be N-periodic.

DT FOURIER SERIES

- An N-periodic complex-valued sequence x can be represented as a linear combination of harmonically-related complex sinusoids as

$$x(n) = \sum_{k=\langle N \rangle} a_k e^{j(2\pi/N)kn},$$

where $\sum_{k=\langle N \rangle}$ denotes summation over any N consecutive integers (e.g., $[0..N-1]$). (The summation can be taken over any N consecutive integers, due to the N-periodic nature of x and $e^{j(2\pi/N)kn}$.)

- The above representation of x is known as the (DT) **Fourier series** and the a_k are called **Fourier series coefficients**.
- The above formula for x is often called the **Fourier series synthesis equation**.
- To denote that the sequence x has the Fourier series coefficient sequence a, we write

$$x(n) \overset{\text{DTFS}}{\longleftrightarrow} a_k.$$

DT FOURIER SERIES

- A periodic sequence x with fundamental period N has the Fourier series coefficient sequence a given by

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-j(2\pi/N)kn}.$$

  (The summation can be taken over any N consecutive integers due to the N-periodic nature of x and $e^{-j(2\pi/N)kn}$.)
- The above equation for a_k is often referred to as the **Fourier series analysis equation**.
- Due to the N-periodic nature of x and $e^{-j(2\pi/N)kn}$, the sequence a is also N-periodic.

Properties of Fourier Series

PROPERTIES OF DT FOURIER SERIES

$$x(n) \overset{\text{DTFS}}{\longleftrightarrow} a_k \quad \text{and} \quad y(n) \overset{\text{DTFS}}{\longleftrightarrow} b_k$$

Property	Time Domain	Fourier Domain
Linearity	$\alpha x(n) + \beta y(n)$	$\alpha a_k + \beta b_k$
Translation	$x(n - n_0)$	$e^{-jk(2\pi/N)n_0} a_k$
Modulation	$e^{j(2\pi/N)k_0 n} x(n)$	a_{k-k_0}
Reflection	$x(-n)$	a_{-k}
Conjugation	$x^*(n)$	a^*_{-k}
Duality	a_n	$\frac{1}{N} x(-k)$
Periodic Convolution	$x \circledast y(n)$	$N a_k b_k$
Multiplication	$x(n)y(n)$	$a \circledast b_k$

Property	
Parseval's Relation	$\frac{1}{N} \sum_{n=\langle N \rangle} \lvert x(n) \rvert^2 = \sum_{k=\langle N \rangle} \lvert a_k \rvert^2$
Even Symmetry	x is even $\Leftrightarrow$ a is even
Odd Symmetry	x is odd $\Leftrightarrow$ a is odd
Real / Conjugate Symmetry	x is real $\Leftrightarrow$ a is conjugate symmetric

PARSEVAL'S RELATION

- A sequence x and its Fourier series coefficient sequence a satisfy the following relationship:

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x(n)|^2 = \sum_{k=\langle N \rangle} |a_k|^2 .$$

- The above relationship is simply stating that the amount of energy in a single period of x and the amount of energy in a single period of a are equal up to a scale factor.
- In other words, the transformation between a sequence and its Fourier series coefficient sequence preserves energy (up to a scale factor).

TRIGONOMETRIC FORM OF A FOURIER SERIES

- Consider the N-periodic sequence x with Fourier series coefficient sequence a.
- If x is real, then its Fourier series can be rewritten in trigonometric form as shown below.
- The **trigonometric form** of a Fourier series has the appearance

$$x(n) = \begin{cases} \alpha_0 + \sum_{k=1}^{N/2-1} \left[\alpha_k \cos\left(\frac{2\pi kn}{N}\right) + \beta_k \sin\left(\frac{2\pi kn}{N}\right)\right] + \\ \quad \alpha_{N/2} \cos(\pi n) & N \text{ even} \\ \alpha_0 + \sum_{k=1}^{(N-1)/2} \left[\alpha_k \cos\left(\frac{2\pi kn}{N}\right) + \beta_k \sin\left(\frac{2\pi kn}{N}\right)\right] & N \text{ odd,} \end{cases}$$

where $\alpha_0 = a_0$, $\alpha_{N/2} = a_{N/2}$, $\alpha_k = 2\operatorname{Re} a_k$, and $\beta_k = -2\operatorname{Im} a_k$.

- Note that the above trigonometric form contains only ***real*** quantities.

Chapter 8
DISCRETE TIME FOURIER TRANSFORM (DTFT)

MOTIVATION FOR THE FOURIER TRANSFORM

- The (DT) Fourier series provide an extremely useful representation for periodic sequences.
- Often, however, we need to deal with sequences that are not periodic.
- A more general tool than the Fourier series is needed in this case.
- The (DT) Fourier transform can be used to represent both periodic and aperiodic sequences.
- Since the (DT) Fourier transform is essentially derived from (DT) Fourier series through a limiting process, the Fourier transform has many similarities with Fourier series.

DEVELOPMENT OF THE DTFT (APERIODIC CASE)

- The (DT) Fourier series is an extremely useful signal representation.
- Unfortunately, this signal representation can only be used for periodic sequences, since a Fourier series is inherently periodic.
- Many sequences are not periodic, however.
- Rather than abandoning Fourier series, one might wonder if we can somehow use Fourier series to develop a representation that can also be applied to aperiodic sequences.
- By viewing an aperiodic sequence as the limiting case of an N-periodic sequence where $N \to \infty$, we can use the Fourier series to develop a signal representation that can be used for aperiodic sequences, known as the Fourier transform.

DEVELOPMENT OF THE DTFT (APERIODIC CASE)

- Recall that the Fourier series representation of an N-periodic sequence x is given by

$$x(n) = \sum_{k=\langle N\rangle} \underbrace{\left(\frac{1}{N}\sum_{\ell=\langle N\rangle} x(\ell)e^{-j(2\pi/N)k\ell}\right)}_{c_k} e^{j(2\pi/N)kn}.$$

- In the above representation, if we take the limit as $N \to \infty$, we obtain

$$x(n) = \frac{1}{2\pi}\int_{2\pi} \underbrace{\left(\sum_{\ell=-\infty}^{\infty} x(\ell)e^{-j\Omega\ell}\right)}_{X(\Omega)} e^{j\Omega n}d\Omega$$

  (i.e., as $N \to \infty$, the two finite summations become an integral and infinite summation, $\frac{1}{N}$ becomes $\frac{1}{2\pi}d\Omega$, and $\left(\frac{2\pi}{N}\right)k$ becomes Ω).

- This representation for aperiodic sequences is known as the Fourier transform representation.

GENERALIZED DTFT

- The classical Fourier transform for aperiodic sequences does not exist (i.e., $\sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}$ fails to converge) for some sequences of great practical interest, such as:
  - a nonzero constant sequence;
  - a periodic sequence (e.g., a real or complex sinusoid); and
  - the unit-step sequence (i.e., u).
- Fortunately, the Fourier transform can be extended to handle such sequences, resulting in what is known as the **generalized Fourier transform**.
- For our purposes, we can think of the classical and generalized Fourier transforms as being defined by the same formulas.
- Therefore, in what follows, we will not typically make a distinction between the classical and generalized Fourier transforms.

DTFT

- The **Fourier transform** of the sequence x, denoted $\mathcal{F}x$ or X, is given by

$$\mathcal{F}x(\Omega) = X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}.$$

- The preceding equation is sometimes referred to as **Fourier transform analysis equation** (or **forward Fourier transform equation**).
- The **inverse Fourier transform** of X, denoted $\mathcal{F}^{-1}X$ or x, is given by

$$\mathcal{F}^{-1}X(n) = x(n) = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega.$$

- The preceding equation is sometimes referred to as the **Fourier transform synthesis equation** (or **inverse Fourier transform equation**).
- As a matter of notation, to denote that a sequence x has the Fourier transform X, we write $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$.
- A sequence x and its Fourier transform X constitute what is called a **Fourier transform pair**.

EXAMPLE

Find the Fourier transform X of the sequence

$$x(n) = A\delta(n - n_0),$$

where A is a real constant and n_0 is an integer constant.

Solution. From the definition of the Fourier transform, we can write

$$X(\Omega) = \sum_{n=-\infty}^{\infty} A\delta(n - n_0)e^{-j\Omega n}$$

$$= A \sum_{n=-\infty}^{\infty} \delta(n - n_0)e^{-j\Omega n}.$$

Using the sifting property of the delta sequence, we can simplify the above result to obtain

$$X(\Omega) = Ae^{-j\Omega n_0}.$$

Thus, we have shown that

$$A\delta(n - n_0) \stackrel{\text{DTFT}}{\longleftrightarrow} Ae^{-j\Omega n_0}.$$

EXAMPLE

Find the Fourier transform X of the sequence $x(n) = u(n-a) - u(n-b)$,

where a and b are integer constants such that $a < b$.

Solution. To begin, we observe that

$$x(n) = \begin{cases} 1 & n \in [a \,.\,.\, b) \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} \\ &= \sum_{n=a}^{b-1} e^{-j\Omega n} \\ &= \sum_{n=a}^{b-1} \left(e^{-j\Omega}\right)^n \\ &= e^{-ja\Omega} \sum_{n=0}^{b-a-1} \left(e^{-j\Omega}\right)^n. \end{aligned}$$

EXAMPLE

The summation on the right-hand side corresponds to the sum of a geometric sequence.

The sum of the arithmetic sequence $a, a+d, a+2d, \ldots, a+(n-1)d$ is given by

$$\sum_{k=0}^{n-1}(a+kd) = \frac{n[2a+d(n-1)]}{2}.$$

The sum of the geometric sequence $a, ra, r^2a, \ldots, r^{n-1}a$ is given by

$$\sum_{k=0}^{n-1} r^k a = a\frac{r^n-1}{r-1} \quad \text{for } r \neq 1.$$

The sum of the infinite geometric sequence $a, ra, r^2a, \ldots$ is given by

$$\sum_{k=0}^{\infty} r^k a = \frac{a}{1-r} \quad \text{for } |r| < 1.$$

EXAMPLE

$$X(\Omega) = e^{-j\Omega a}\frac{\left(e^{-j\Omega}\right)^{b-a}-1}{e^{-j\Omega}-1}$$

$$= \frac{e^{-jb\Omega}-e^{-ja\Omega}}{e^{-j\Omega}-1}$$

$$= \frac{e^{-ja\Omega}-e^{-jb\Omega}}{1-e^{-j\Omega}}.$$

$$= e^{-j(a+b-1)\Omega/2}\left(\frac{\sin\left[\frac{(b-a)\Omega}{2}\right]}{\sin\left[\frac{\Omega}{2}\right]}\right).$$

$$u(n-a)-u(n-b) \xleftrightarrow{\text{DTFT}} e^{-j(a+b-1)\Omega/2}\left(\frac{\sin\left[\frac{1}{2}(b-a)\Omega\right]}{\sin\left[\frac{1}{2}\Omega\right]}\right).$$

EXAMPLE

Find the Fourier transform X of the sequence

$$x(n) = a^n u(n),$$

where a is a real constant satisfying $|a| < 1$.

EXAMPLE

Solution. From the definition of the Fourier transform, we can write

$$\begin{aligned} X(\Omega) &= \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} a^n e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} \left(a e^{-j\Omega}\right)^n. \end{aligned}$$

we can simplify the preceding equation (for $|a| < 1$) to obtain

$$\begin{aligned} X(\Omega) &= \frac{1}{1 - a e^{-j\Omega}} \\ &= \frac{e^{j\Omega}}{e^{j\Omega} - a}. \end{aligned}$$

$$a^n u(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{e^{j\Omega}}{e^{j\Omega} - a} \quad \text{for } |a| < 1.$$

EXAMPLE

Find the Fourier transform X of the sequence

$$x(n) = a^{|n|},$$

where a is a real constant satisfying $|a| < 1$.

EXAMPLE

Solution. From the definition of the Fourier transform, we can write

$$
\begin{aligned}
X(\Omega) &= \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\Omega n} \\
&= \sum_{n=-\infty}^{-1} a^{-n} e^{-j\Omega n} + \sum_{n=0}^{\infty} a^{n} e^{-j\Omega n} \\
&= \sum_{n=1}^{\infty} a^{n} e^{j\Omega n} + \sum_{n=0}^{\infty} a^{n} e^{-j\Omega n} \\
&= \sum_{n=1}^{\infty} \left(ae^{j\Omega}\right)^{n} + \sum_{n=0}^{\infty} \left(ae^{-j\Omega}\right)^{n}.
\end{aligned}
$$

EXAMPLE

we can simplify the preceding equation (for $|a| < 1$) to obtain

$$\begin{aligned}
X(\Omega) &= \frac{ae^{j\Omega}}{1-ae^{j\Omega}} + \frac{1}{1-ae^{-j\Omega}} \\
&= \frac{1-ae^{j\Omega}+ae^{j\Omega}(1-ae^{-j\Omega})}{(1-ae^{j\Omega})(1-ae^{-j\Omega})} \\
&= \frac{1-ae^{j\Omega}+ae^{j\Omega}-a^2}{1-ae^{-j\Omega}-ae^{j\Omega}+a^2} \\
&= \frac{1-a^2}{1-a(e^{j\Omega}+e^{-j\Omega})+a^2} \\
&= \frac{1-a^2}{1-2a\cos\Omega+a^2}.
\end{aligned}$$

$$a^{|n|} \overset{\text{DTFT}}{\longleftrightarrow} \frac{1-a^2}{1-2a\cos\Omega+a^2} \quad \text{for } |a| < 1.$$

EXAMPLE

Find the inverse Fourier transform x of the (2π-periodic) sequence

$$X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k).$$

Solution. From the definition of the inverse Fourier transform, we can write

$$\begin{aligned} x(n) &= \frac{1}{2\pi} \int_{2\pi} \left[2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k) \right] e^{j\Omega n} d\Omega \\ &= \int_{2\pi} \delta(\Omega) e^{j\Omega n} d\Omega \\ &= \int_{-\pi}^{\pi} \delta(\Omega) e^{j\Omega n} d\Omega. \end{aligned}$$

Using the sifting property of the delta function,

$$x(n) = e^0 = 1.$$

$$1 \xleftrightarrow{\text{DTFT}} 2\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k).$$

- **Equivalence property.** For any continuous function x and any real constant t_0,

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0).$$

- **Sifting property.** For any continuous function x and any real constant t_0,

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = x(t_0).$$

Convergence Properties of the Fourier Transform

CONVERGENCE OF THE FOURIER TRANSFORM

- For a sequence x, the Fourier transform analysis equation (i.e., $X(\Omega) = \sum_{-\infty}^{\infty} x(n)e^{-j\Omega n}$) converges ***uniformly*** if

$$\sum_{k=-\infty}^{\infty} |x(k)| < \infty$$

(i.e., x is ***absolutely summable***).

- For a sequence x, the Fourier transform analysis equation (i.e., $X(\Omega) = \sum_{-\infty}^{\infty} x(n)e^{-j\Omega n}$) converges in the ***MSE sense*** if

$$\sum_{k=-\infty}^{\infty} |x(k)|^2 < \infty$$

(i.e., x is ***square summable***).

- For a bounded Fourier transform X, the Fourier transform synthesis equation (i.e., $x(n) = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega$) will always converge, since the integration interval is finite.

Properties of the Fourier Transform

PROPERTIES OF DTFT

Property	Time Domain	Frequency Domain
Linearity	$a_1x_1(n)+a_2x_2(n)$	$a_1X_1(\Omega)+a_2X_2(\Omega)$
Translation	$x(n-n_0)$	$e^{-j\Omega n_0}X(\Omega)$
Modulation	$e^{j\Omega_0 n}x(n)$	$X(\Omega-\Omega_0)$
Conjugation	$x^*(n)$	$X^*(-\Omega)$
Time Reversal	$x(-n)$	$X(-\Omega)$
Upsampling	$(\uparrow M)x(n)$	$X(M\Omega)$
Downsampling	$(\downarrow M)x(n)$	$\frac{1}{M}\sum_{k=0}^{M-1} X\left(\frac{\Omega-2\pi k}{M}\right)$
Convolution	$x_1 * x_2(n)$	$X_1(\Omega)X_2(\Omega)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{2\pi} X_1(\theta)X_2(\Omega-\theta)d\theta$
Freq.-Domain Diff.	$nx(n)$	$j\frac{d}{d\Omega}X(\Omega)$
Differencing	$x(n)-x(n-1)$	$\left(1-e^{-j\Omega}\right)X(\Omega)$
Accumulation	$\sum_{k=-\infty}^{n} x(k)$	$\frac{e^{j\Omega}}{e^{j\Omega}-1}X(\Omega)+\pi X(0)\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$

PROPERTIES OF DTFT

Property	
Periodicity	$X(\Omega) = X(\Omega + 2\pi)$
Parseval's Relation	$\sum_{n=-\infty}^{\infty} \lvert x(n) \rvert^2 = \frac{1}{2\pi} \int_{2\pi} \lvert X(\Omega) \rvert^2 \, d\Omega$
Even Symmetry	x is even $\Leftrightarrow$ X is even
Odd Symmetry	x is odd $\Leftrightarrow$ X is odd
Real / Conjugate Symmetry	x is real $\Leftrightarrow$ X is conjugate symmetric

DTFT PAIRS

Pair	$x(n)$	$X(\Omega)$
1	$\delta(n)$	1
2	1	$2\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)$
3	$u(n)$	$\frac{e^{j\Omega}}{e^{j\Omega}-1}+\sum_{k=-\infty}^{\infty}\pi\delta(\Omega-2\pi k)$
4	$a^n u(n), \|a\|<1$	$\frac{e^{j\Omega}}{e^{j\Omega}-a}$
5	$-a^n u(-n-1), \|a\|>1$	$\frac{e^{j\Omega}}{e^{j\Omega}-a}$
6	$a^{\|n\|}, \|a\|<1$	$\frac{1-a^2}{1-2a\cos\Omega+a^2}$
7	$\cos(\Omega_0 n)$	$\pi\sum_{k=-\infty}^{\infty}[\delta(\Omega-\Omega_0-2\pi k)+\delta(\Omega+\Omega_0-2\pi k)]$
8	$\sin(\Omega_0 n)$	$j\pi\sum_{k=-\infty}^{\infty}[\delta(\Omega+\Omega_0-2\pi k)-\delta(\Omega-\Omega_0-2\pi k)]$
9	$\cos(\Omega_0 n)u(n)$	$\frac{e^{j2\Omega}-e^{j\Omega}\cos\Omega_0}{e^{j2\Omega}-2e^{j\Omega}\cos\Omega_0+1}+\frac{\pi}{2}\sum_{k=-\infty}^{\infty}[\delta(\Omega-2\pi k-\Omega_0)+\delta(\Omega-2\pi k+\Omega_0)]$
10	$\sin(\Omega_0 n)u(n)$	$\frac{e^{j\Omega}\sin\Omega_0}{e^{j2\Omega}-2e^{j\Omega}\cos\Omega_0+1}+\frac{\pi}{2j}\sum_{k=-\infty}^{\infty}[\delta(\Omega-2\pi k-\Omega_0)-\delta(\Omega-2\pi k+\Omega_0)]$
11	$\frac{B}{\pi}\operatorname{sinc}(Bn), 0<B<\pi$	$\sum_{k=-\infty}^{\infty}\operatorname{rect}\left(\frac{\Omega-2\pi k}{2B}\right)$
12	$u(n)-u(n-M)$	$e^{-j\Omega(M-1)/2}\left(\frac{\sin(M\Omega/2)}{\sin(\Omega/2)}\right)$
13	$na^n u(n), \|a\|<1$	$\frac{ae^{j\Omega}}{(e^{j\Omega}-a)^2}$

PERIODICITY

- Recall the definition of the Fourier transform X of the sequence x:

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n}.$$

- For all integer k, we have that

$$\begin{aligned} X(\Omega + 2\pi k) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j(\Omega+2\pi k)n} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j(\Omega n+2\pi kn)} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n} \\ &= X(\Omega). \end{aligned}$$

- Thus, the Fourier transform X of the sequence x is always 2π-*periodic*.

LINEARITY

- If $x_1(n) \xleftrightarrow{\text{DTFT}} X_1(\Omega)$ and $x_2(n) \xleftrightarrow{\text{DTFT}} X_2(\Omega)$, then

$$a_1x_1(n) + a_2x_2(n) \xleftrightarrow{\text{DTFT}} a_1X_1(\Omega) + a_2X_2(\Omega),$$

  where a_1 and a_2 are arbitrary complex constants.

- This is known as the **linearity property** of the Fourier transform.

EXAMPLE

Using the Fourier transform pairs

$$\delta(n) \xleftrightarrow{\text{DTFT}} 1 \quad \text{and} \quad u(n) \xleftrightarrow{\text{DTFT}} \frac{e^{j\Omega}}{e^{j\Omega}-1} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k),$$

find the Fourier transform X of the sequence

$$x(n) = 2\delta(n) - u(n).$$

EXAMPLE

Solution. Taking the Fourier transform of x, we trivially have

$$X(\Omega) = \mathcal{F}\{2\delta(n) - u(n)\}(\Omega).$$

Using the linearity property of the Fourier transform, we can write

$$X(\Omega) = 2\mathcal{F}\delta(\Omega) - \mathcal{F}u(\Omega).$$

Using the given Fourier transform pairs, we obtain

$$\begin{aligned} X(\Omega) &= 2(1) - \left[\frac{e^{j\Omega}}{e^{j\Omega}-1} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)\right] \\ &= \frac{2e^{j\Omega} - 2 - e^{j\Omega}}{e^{j\Omega}-1} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k) \\ &= \frac{e^{j\Omega}-2}{e^{j\Omega}-1} + \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k). \end{aligned}$$

TRANSLATION

- If $x(n) \xleftrightarrow{\text{DTFT}} X(\Omega)$, then

$$x(n-n_0) \xleftrightarrow{\text{DTFT}} e^{-j\Omega n_0} X(\Omega),$$

  where n_0 is an arbitrary integer.

- This is known as the **translation (or time-domain shifting) property** of the Fourier transform.

EXAMPLE

Using the Fourier transform pair

$$a^n u(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{e^{j\Omega}}{e^{j\Omega} - a} \quad \text{for } |a| < 1,$$

find the Fourier transform X of the sequence

$$x(n) = a^n u(n-3),$$

where a is a complex constant satisfying $|a| < 1$.

Solution. To begin, we observe that

$$x(n) = a^3 a^{n-3} u(n-3)$$

Define the sequence

$$v_1(n) = a^n u(n)$$

Using the definition of v, we can rewrite x as

$$x(n) = a^3 v_1(n-3).$$

EXAMPLE

Taking the Fourier transforms of v_1 and x, we obtain

$$V_1(\Omega) = \frac{e^{j\Omega}}{e^{j\Omega} - a} \quad \text{and}$$

$$X(\Omega) = a^3 e^{-j3\Omega} V_1(\Omega).$$

Substituting the formula for V_1 into the formula for X, we obtain

$$\begin{aligned} X(\Omega) &= a^3 e^{-j3\Omega} V_1(\Omega) \\ &= a^3 e^{-j3\Omega} \left(\frac{e^{j\Omega}}{e^{j\Omega} - a} \right) \\ &= \frac{a^3 e^{-j2\Omega}}{e^{j\Omega} - a}. \end{aligned}$$

MODULATION

- If $x(n) \xleftrightarrow{\text{DTFT}} X(\Omega)$, then

$$e^{j\Omega_0 n}x(n) \xleftrightarrow{\text{DTFT}} X(\Omega - \Omega_0),$$

  where Ω_0 is an arbitrary real constant.

- This is known as the **modulation (or frequency-domain shifting) property** of the Fourier transform.

CONJUGATION

- If $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, then

$$x^*(n) \overset{\text{DTFT}}{\longleftrightarrow} X^*(-\Omega).$$

- This is known as the **conjugation property** of the Fourier transform.

TIME REVERSAL

- If $x(n) \xleftrightarrow{\text{DTFT}} X(\Omega)$, then

$$x(-n) \xleftrightarrow{\text{DTFT}} X(-\Omega).$$

- This is known as the **time-reversal property** of the Fourier transform.

UPSAMPLING

- If $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, then

$$(\uparrow M)x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(M\Omega).$$

- This is known as the **upsampling property** of the Fourier transform.

DOWNSAMPLING

- If $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, then

$$(\downarrow M)x(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{1}{M}\sum_{k=0}^{M-1} X\left(\frac{\Omega - 2\pi k}{M}\right).$$

- This is known as the **downsampling property** of the Fourier transform.

CONVOLUTION

- If $x_1(n) \overset{\text{DTFT}}{\longleftrightarrow} X_1(\Omega)$ and $x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} X_2(\Omega)$, then

$$x_1 * x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} X_1(\Omega)X_2(\Omega).$$

- This is known as the **convolution (or time-domain convolution) property** of the Fourier transform.
- In other words, a convolution in the time domain becomes a multiplication in the frequency domain.
- This suggests that the Fourier transform can be used to avoid having to deal with convolution operations.

MULTIPLICATION

- If $x_1(n) \overset{\text{DTFT}}{\longleftrightarrow} X_1(\Omega)$ and $x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} X_2(\Omega)$, then

$$x_1(n)x_2(n) \overset{\text{DTFT}}{\longleftrightarrow} \frac{1}{2\pi}\int_{2\pi} X_1(\theta)X_2(\Omega-\theta)d\theta.$$

- This is known as the **multiplication (or time-domain multiplication) property** of the Fourier transform.
- Do not forget the factor of $\frac{1}{2\pi}$ in the above formula!
- This property of the Fourier transform is often tedious to apply (in the forward direction) as it turns a multiplication into a convolution.

FREQUENCY DOMAIN DIFFERANTIATION

- If $x(n) \xleftrightarrow{\text{DTFT}} X(\Omega)$, then

$$nx(n) \xleftrightarrow{\text{DTFT}} j\frac{d}{d\Omega}X(\Omega).$$

- This is known as the **frequency-domain differentiation property** of the Fourier transform.

DIFFERNCING

- If $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, then

$$x(n) - x(n-1) \overset{\text{DTFT}}{\longleftrightarrow} \left(1 - e^{-j\Omega}\right) X(\Omega).$$

- This is known as the **differencing property** of the Fourier transform.
- Note that this property follows quite trivially from the linearity and translation properties of the Fourier transform.

ACCUMULATION

- If $x(n) \xleftrightarrow{\text{DTFT}} X(\Omega)$, then

$$\sum_{k=-\infty}^{n} x(k) \xleftrightarrow{\text{DTFT}} \frac{e^{j\Omega}}{e^{j\Omega}-1} X(\Omega) + \pi X(0) \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k).$$

- This is known as the **accumulation property** of the Fourier transform.

PARSEVAL'S RELATION

- If $x(n) \overset{\text{DTFT}}{\longleftrightarrow} X(\Omega)$, then

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{2\pi} |X(\Omega)|^2 \, d\Omega$$

  (i.e., the energy of x and energy of X are equal up to a factor of 2π).

- This is known as **Parseval's relation**.
- Since energy is often a quantity of great significance in engineering applications, it is extremely helpful to know that the Fourier transform ***preserves energy*** (up to a scale factor).

EVEN AND ODD SYMMETRY

- For a sequence x with Fourier transform X, the following assertions hold:
  1. x is even $\Leftrightarrow$ X is even; and
  2. x is odd $\Leftrightarrow$ X is odd.
- In other words, the forward and inverse Fourier transforms preserve even/odd symmetry.

REAL SEQUENCES

- A sequence x is ***real*** if and only if its Fourier transform X satisfies

$$X(\Omega) = X^*(-\Omega) \text{ for all } \Omega$$

  (i.e., X is ***conjugate symmetric***).

- Thus, for a real-valued sequence, the portion of the graph of a Fourier transform for negative values of frequency Ω is ***redundant***, as it is completely determined by symmetry.
- From properties of complex numbers, one can show that $X(\Omega) = X^*(-\Omega)$ is equivalent to

$$|X(\Omega)| = |X(-\Omega)| \quad \text{and} \quad \arg X(\Omega) = -\arg X(-\Omega)$$

  (i.e., $|X(\Omega)|$ is ***even*** and $\arg X(\Omega)$ is ***odd***).

- Note that x being real does ***not*** necessarily imply that X is real.

Fourier Transform of Periodic Sequences

FT OF PERIODIC SEQUENCES

- The Fourier transform can be generalized to also handle periodic sequences.
- Consider an N-periodic sequence x.
- Define the sequence x_N as

$$x_N(n) = \begin{cases} x(n) & 0 \le n < N \\ 0 & \text{otherwise.} \end{cases}$$

(i.e., $x_N(n)$ is equal to $x(n)$ over a single period and zero elsewhere).

- Let a denote the Fourier series coefficient sequence of x.
- Let X and X_N denote the Fourier transforms of x and x_N, respectively.
- The following relationships can be shown to hold:

$$X(\Omega) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} X_N\left(\frac{2\pi k}{N}\right) \delta\left(\Omega - \frac{2\pi k}{N}\right),$$

$$a_k = \frac{1}{N} X_N\left(\frac{2\pi k}{N}\right), \quad \text{and} \quad X(\Omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\Omega - \frac{2\pi k}{N}\right).$$

FT OF PERIODIC SEQUENCES

- The Fourier series coefficient sequence a is produced by sampling X_N at integer multiples of the fundamental frequency $\frac{2\pi}{N}$ and scaling the resulting sequence by $\frac{1}{N}$.
- The Fourier transform of a periodic sequence can only be nonzero at integer multiples of the fundamental frequency.

Fourier Transform and Frequency Spectra of Sequences

FREQUENCY SPECTRA OF SEQUENCES

- Like Fourier series, the Fourier transform also provides us with a frequency-domain perspective on sequences.
- That is, instead of viewing a sequence as having information distributed with respect to ***time*** (i.e., a function whose domain is time), we view a sequence as having information distributed with respect to ***frequency*** (i.e., a function whose domain is frequency).
- The Fourier transform X of a sequence x provides a means to ***quantify*** how much information x has at different frequencies.
- The distribution of information in a sequence over different frequencies is referred to as the ***frequency spectrum*** of the sequence.

FT AND FREQUENCY SPECTRA

- To gain further insight into the role played by the Fourier transform X in the context of the frequency spectrum of x, it is helpful to write the Fourier transform representation of x with $X(\Omega)$ expressed in ***polar form*** as follows:

$$x(n) = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_{2\pi} |X(\Omega)|\, e^{j[\Omega n + \arg X(\Omega)]}d\Omega.$$

- In effect, the quantity $|X(\Omega)|$ is a ***weight*** that determines how much the complex sinusoid at frequency Ω contributes to the integration result $x(n)$.
- Perhaps, this can be more easily seen if we express the above integral as the ***limit of a sum***, derived from an approximation of the integral using the area of rectangles, as shown on the next slide. [Recall that $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x$ where $\Delta x = \frac{b-a}{n}$ and $x_k = a + k\Delta x$.]

FT AND FREQUENCY SPECTRA

- Expressing the integral (from the previous slide) as the *limit of a sum*, we obtain

$$x(n) = \lim_{\ell \to \infty} \frac{1}{2\pi} \sum_{k=1}^{\ell} \Delta\Omega \left| X(\Omega') \right| e^{j[\Omega' n + \arg X(\Omega')]},$$

  where $\Delta\Omega = \frac{2\pi}{\ell}$ and $\Omega' = k\Delta\Omega$.

- In the above equation, the kth term in the summation corresponds to a complex sinusoid with fundamental frequency $\Omega' = k\Delta\Omega$ that has had its *amplitude scaled* by a factor of $|X(\Omega')|$ and has been *time shifted* by an amount that depends on $\arg X(\Omega')$.
- For a given $\Omega' = k\Delta\Omega$ (which is associated with the kth term in the summation), the larger $|X(\Omega')|$ is, the larger the amplitude of its corresponding complex sinusoid $e^{j\Omega' n}$ will be, and therefore the larger the contribution the kth term will make to the overall summation.
- In this way, we can use $|X(\Omega')|$ as a *measure* of how much information a sequence x has at the frequency Ω'.

FT AND FREQUENCY SPECTRA

- The Fourier transform X of the sequence x is referred to as the **frequency spectrum** of x.
- The magnitude $|X(\Omega)|$ of the Fourier transform X is referred to as the **magnitude spectrum** of x.
- The argument $\arg X(\Omega)$ of the Fourier transform X is referred to as the **phase spectrum** of x.
- Since the Fourier transform is a function of a real variable, a sequence can potentially have information at any real frequency.
- Earlier, we saw that for periodic sequences, the Fourier transform can only be nonzero at integer multiples of the fundamental frequency.
- So, the Fourier transform and Fourier series give a consistent picture in terms of frequency spectra.
- Since the frequency spectrum is complex (in the general case), it is ***usually represented using two plots***, one showing the magnitude spectrum and one showing the phase spectrum.

FREQUENCY SPECTRA OF REAL SEQUENCES

- Recall that, for a ***real*** sequence x, the Fourier transform X of x satisfies

$$X(\Omega) = X^*(-\Omega)$$

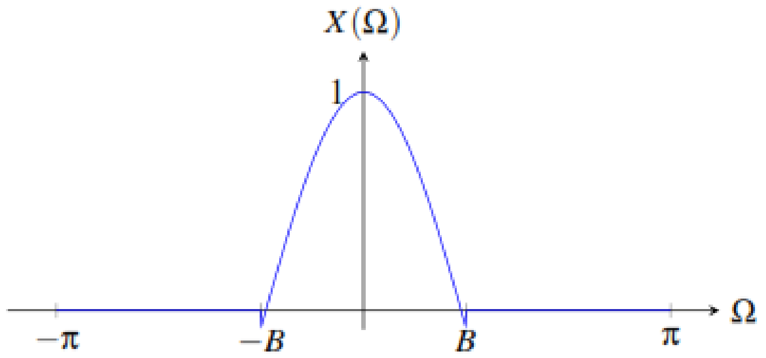
(i.e., X is ***conjugate symmetric***), which is equivalent to

$$|X(\Omega)| = |X(-\Omega)| \quad \text{and} \quad \arg X(\Omega) = -\arg X(-\Omega).$$

- Since $|X(\Omega)| = |X(-\Omega)|$, the magnitude spectrum of a real sequence is always ***even***.
- Similarly, since $\arg X(\Omega) = -\arg X(-\Omega)$, the phase spectrum of a real sequence is always ***odd***.
- Due to the symmetry in the frequency spectra of real sequences, we typically ***ignore negative frequencies*** when dealing with such sequences.
- In the case of sequences that are complex but not real, frequency spectra do not possess the above symmetry, and ***negative frequencies become important***.

BANDWIDTH

- A sequence x with Fourier transform X satisfying $X(\Omega) = 0$ for all Ω in $(-\pi, \pi]$ except for some interval I is said to be **bandlimited** to frequencies in I.
- The **bandwidth** of a sequence x with Fourier transform X is the length of the interval in $(-\pi, \pi]$ over which X is nonzero.
- For example, the sequence x with the Fourier transform X shown below is bandlimited to frequencies in $[-B, B]$ and has bandwidth $B - (-B) = 2B$.



- Since x is real in the above example (as X is conjugate symmetric), we might choose to ignore negative frequencies, in which case x would be deemed to be bandlimited to frequencies in $[0, B]$ and have bandwidth $B - 0 = B$.

ENERGY DENSITY SPECTRA

- By Parseval's relation, the energy E in a sequence x with Fourier transform X is given by

$$E = \frac{1}{2\pi} \int_{2\pi} E_x(\Omega) d\Omega,$$

where

$$E_x(\Omega) = |X(\Omega)|^2 .$$

- We refer to E_x as the **energy-density spectrum** of the sequence x.
- The function E_x indicates how the energy in x is distributed with respect to frequency.
- For example, the energy contributed by frequencies in the range $[\Omega_1, \Omega_2]$ is given by

$$\frac{1}{2\pi} \int_{\Omega_1}^{\Omega_2} E_x(\Omega) d\Omega.$$

Fourier Transform and LTI Systems

FREQUENCY RESPONSE OF LTI SYSTEMS

- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the Fourier transforms of x, y, and h, respectively.
- Since $y(n) = x * h(n)$, we have that

$$Y(\Omega) = X(\Omega)H(\Omega).$$

- The function H is called the **frequency response** of the system.
- A LTI system is ***completely characterized*** by its frequency response H.
- The above equation provides an alternative way of viewing the behavior of a LTI system. That is, we can view the system as operating in the frequency domain on the Fourier transforms of the input and output signals.
- The frequency spectrum of the output is the product of the frequency spectrum of the input and the frequency response of the system.

FREQUENCY RESPONSE OF LTI SYSTEMS

- In the general case, the frequency response H is a complex-valued function.
- Often, we represent $H(\Omega)$ in terms of its magnitude $|H(\Omega)|$ and argument $\arg H(\Omega)$.
- The quantity $|H(\Omega)|$ is called the **magnitude response** of the system.
- The quantity $\arg H(\Omega)$ is called the **phase response** of the system.
- Since $Y(\Omega) = X(\Omega)H(\Omega)$, we trivially have that

$$|Y(\Omega)| = |X(\Omega)|\,|H(\Omega)| \quad \text{and} \quad \arg Y(\Omega) = \arg X(\Omega) + \arg H(\Omega).$$

- The magnitude spectrum of the output equals the magnitude spectrum of the input times the magnitude response of the system.
- The phase spectrum of the output equals the phase spectrum of the input plus the phase response of the system.

FREQUENCY RESPONSE OF LTI SYSTEMS

- Since the frequency response H is simply the frequency spectrum of the impulse response h, if h is ***real***, then

$$|H(\Omega)| = |H(-\Omega)| \quad \text{and} \quad \arg H(\Omega) = -\arg H(-\Omega)$$

(i.e., the magnitude response $|H(\Omega)|$ is ***even*** and the phase response $\arg H(\Omega)$ is ***odd***).

MAGNITUDE AND PHASE RESPONSE

- Recall that a LTI system $\mathcal{H}$ with frequency response H is such that
$$\mathcal{H}\left\{e^{j\Omega n}\right\}(n) = H(\Omega)e^{j\Omega n}.$$
- Expressing $H(\Omega)$ in polar form, we have
$$\begin{aligned}\mathcal{H}\{e^{j\Omega n}\}(n) &= |H(\Omega)|\, e^{j\arg H(\Omega)}e^{j\Omega n}\\ &= |H(\Omega)|\, e^{j[\Omega n+\arg H(\Omega)]}\\ &= |H(\Omega)|\, e^{j\Omega(n+\arg[H(\Omega)]/\Omega)}.\end{aligned}$$
- Thus, the response of the system to the sequence $e^{j\Omega n}$ is produced by applying two transformations to this sequence:
  - (amplitude) scaling by $|H(\Omega)|$; and
  - translating by $-\frac{\arg H(\Omega)}{\Omega}$ (using bandlimited interpolation if $-\frac{\arg H(\Omega)}{\Omega} \notin \mathbb{Z}$).
- Therefore, the magnitude response determines how different complex sinusoids are ***scaled*** (in amplitude) by the system.
- Similarly, the phase response determines how different complex sinusoids are ***translated*** (i.e., delayed/advanced) by the system.

MAGNITUDE DISTORTION

- Recall that a LTI system $\mathcal{H}$ with frequency response H is such that

$$\mathcal{H}\{e^{j\Omega n}\}(n) = |H(\Omega)|\, e^{j\Omega(n+\arg[H(\Omega)]/\Omega)}.$$

- If $|H(\Omega)|$ is a constant (for all Ω), every complex sinusoid is scaled by the same amount when passing through the system.
- A system for which $|H(\Omega)| = 1$ (for all Ω) is said to be **allpass**.
- In the case of an allpass system, the magnitude spectra of the system's input and output are identical.
- If $|H(\Omega)|$ is not a constant, different complex sinusoids are scaled by different amounts, resulting in what is known as **magnitude distortion**.

PHASE DISTORTION

- Recall that a LTI system $\mathcal{H}$ with frequency response H is such that

$$\mathcal{H}\{e^{j\Omega n}\}(n) = |H(\Omega)|\, e^{j\Omega(n+\arg[H(\Omega)]/\Omega)}.$$

- The preceding equation can be rewritten as

$$\mathcal{H}\{e^{j\Omega n}\}(n) = |H(\Omega)|\, e^{j\Omega[n-\tau_p(\Omega)]} \quad \text{where} \quad \tau_p(\Omega) = -\frac{\arg H(\Omega)}{\Omega}.$$

- The function τ_p is known as the **phase delay** of the system.
- If $\tau_p(\Omega) = n_d$ (where n_d is a constant), the system shifts all complex sinusoids by the same amount n_d.
- Since $\tau_p(\Omega) = n_d$ is equivalent to the (unwrapped) phase response being of the form $\arg H(\Omega) = -n_d\Omega$ (which is a linear function with a zero constant term), a system with a constant phase delay is said to have **linear phase**.
- In the case that $\tau_p(\Omega) = 0$, the system is said to have **zero phase**.
- If $\tau_p(\Omega)$ is not a constant, different complex sinusoids are shifted by different amounts, resulting in what is known as **phase distortion**.

DISTORTIONLESS TRANSMISSION

- Consider a LTI system $\mathcal{H}$ with input x and output y given by

$$y(n) = x(n - n_0),$$

  where n_0 is an integer constant.
- That is, the output of the system is simply the input delayed by n_0.
- This type of behavior is the ideal for which we strive in real-world communication systems (i.e., the received signal y equals a delayed version of the transmitted signal x).
- Taking the Fourier transform of the preceding equation, we have

$$Y(\Omega) = e^{-j\Omega n_0} X(\Omega).$$

- Thus, the system has the frequency response H given by

$$H(\Omega) = e^{-j\Omega n_0}.$$

- Since the phase delay of the system is $\tau_p(\Omega) = -\left(\frac{-\Omega n_0}{\Omega}\right) = n_0$, the phase delay is constant and the system has linear phase.

Fourier Transform Relationships

DUALITY BETWEEN DTFT AND CTFS

- The DTFT analysis and synthesis equations are, respectively, given by

$$X(\Omega) = \sum_{k=-\infty}^{\infty} x(k) e^{-jk\Omega} \quad \text{and} \quad x(n) = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{jn\Omega} d\Omega.$$

- The CTFS synthesis and analysis equations are, respectively, given by

$$x_c(t) = \sum_{k=-\infty}^{\infty} a(k) e^{jk(2\pi/T)t} \quad \text{and} \quad a(n) = \frac{1}{T} \int_{T} x_c(t) e^{-jn(2\pi/T)t} dt,$$

which can be rewritten, respectively, as

$$x_c(t) = \sum_{k=-\infty}^{\infty} a(-k) e^{-jk(2\pi/T)t} \quad \text{and} \quad a(-n) = \frac{1}{T} \int_{T} x_c(t) e^{jn(2\pi/T)t} dt.$$

- The CTFS synthesis equation with $T = 2\pi$ corresponds to the DTFT analysis equation with $X = x_c$, $\Omega = t$, and $x(n) = a(-n)$.
- The CTFS analysis equation with $T = 2\pi$ corresponds to the DTFT synthesis equation with $X = x_c$ and $x(n) = a(-n)$.
- Consequently, the DTFT X of the sequence x can be viewed as a CTFS representation of the 2π-periodic spectrum X.

RELATIONSHIP BETWEEN DTFT AND CTFT

- Let x be a bandlimited function and let T denote a sampling period for x that satisfies the Nyquist condition.
- Let $\tilde{y}$ be the function obtained by impulse sampling x with sampling period T. That is,

$$\tilde{y}(t) = \sum_{n=-\infty}^{\infty} x(Tn)\delta(t - Tn).$$

- Let y denote the sequence obtaining by sampling x with sampling period T. That is,

$$y(n) = x(Tn).$$

- Let $\tilde{Y}$ denote the (CT) Fourier transform of $\tilde{y}$ and let Y denote the (DT) Fourier transform of y.
- Then, the following relationship holds:

$$Y(\Omega) = \tilde{Y}\left(\tfrac{\Omega}{T}\right) \quad \text{for all } \Omega \in \mathbb{R}.$$

RELATIONSHIP BETWEEN DTFT AND CTFT

- Let x be a sequence with (DT) Fourier transform X such that

$$x(n) = 0 \quad \text{for all } n \notin [0\,..\,M-1].$$

- Let $\tilde{X}$ denote the N-point DFT of X. That is,

$$\tilde{X}(k) = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)kn} \quad \text{for } k \in [0\,..\,N-1].$$

- Suppose now that $N \geq M$.
- Then, the following relationship holds:

$$X\left(\tfrac{2\pi}{N}k\right) = \tilde{X}(k) \quad \text{for } k \in [0\,..\,N-1].$$

- In other words, the elements of the sequence $\tilde{X}$ correspond to uniformly-spaced samples of the function X.

SPECTRAL SAMPLING EXAMPLE

- Consider the sequence

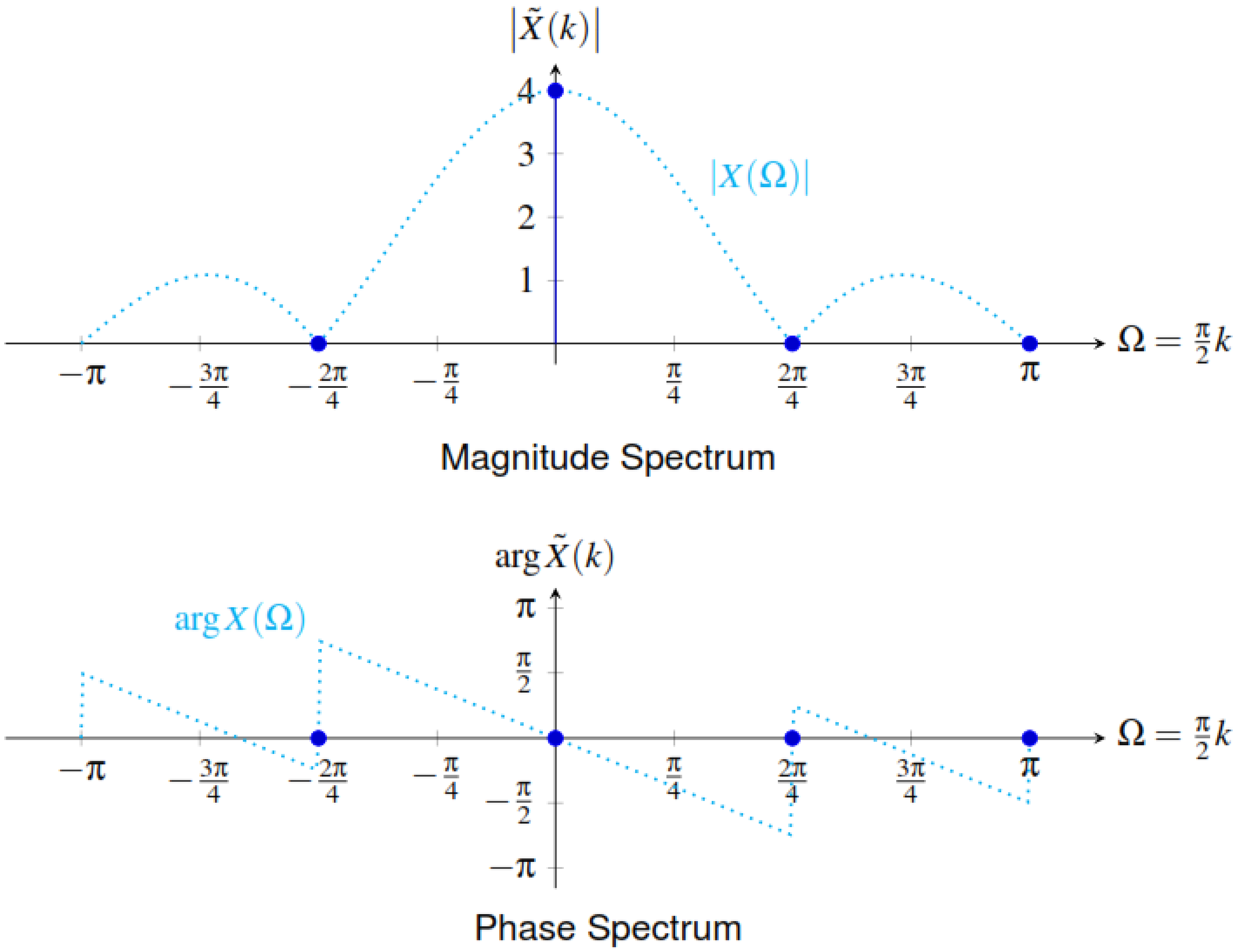
$$x(n) = u(n) - u(n-4).$$

- The Fourier transform X of x can be shown to be

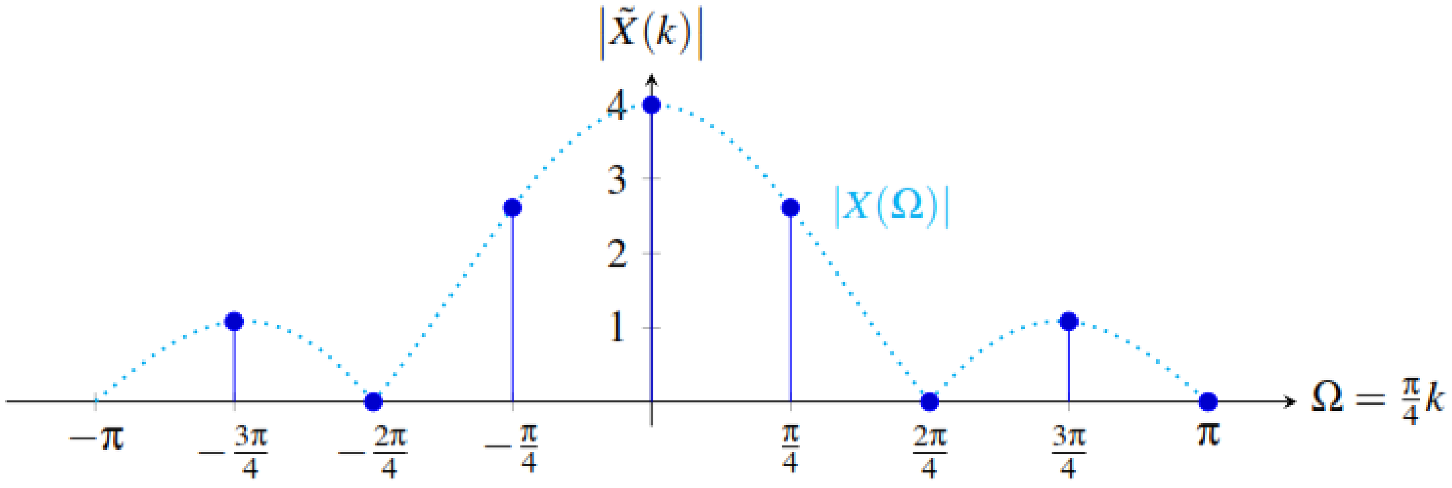
$$X(\Omega) = e^{-j(3/2)\Omega}\left[\frac{\sin(2\Omega)}{\sin\left(\frac{1}{2}\Omega\right)}\right].$$

- Clearly, $x(n) = 0$ for all $n \notin [0..3]$.
- Therefore, uniformly-spaced samples of X can be obtained from an N-point DFT $\tilde{X}$ of x, where $N \geq 4$.
- The subsequent slides show the sampled spectrum obtained by the DFT for several values of N.

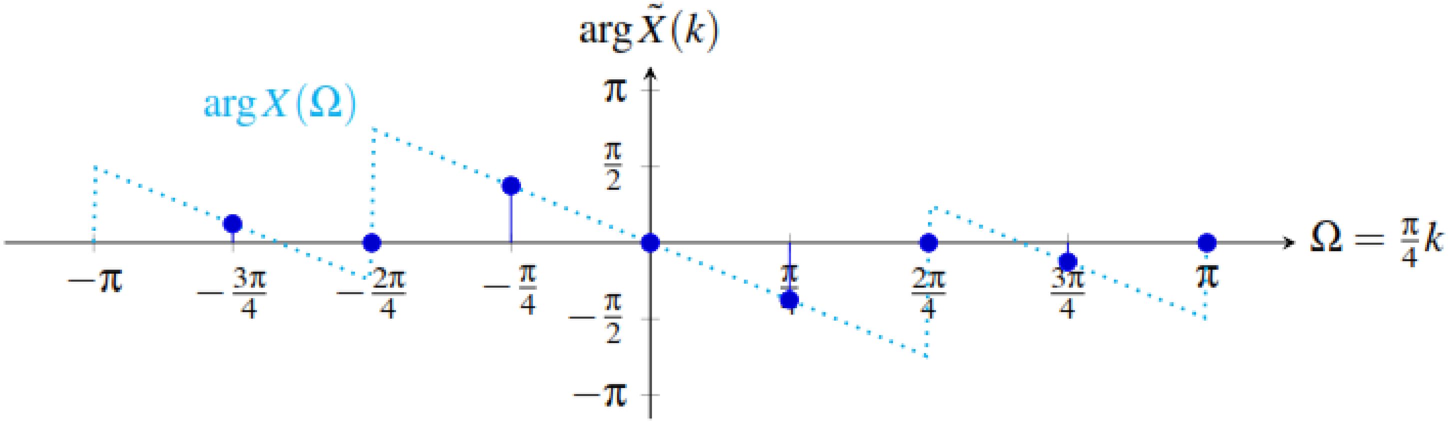
SPECTRAL SAMPLING EXAMPLE



SPECTRAL SAMPLING EXAMPLE

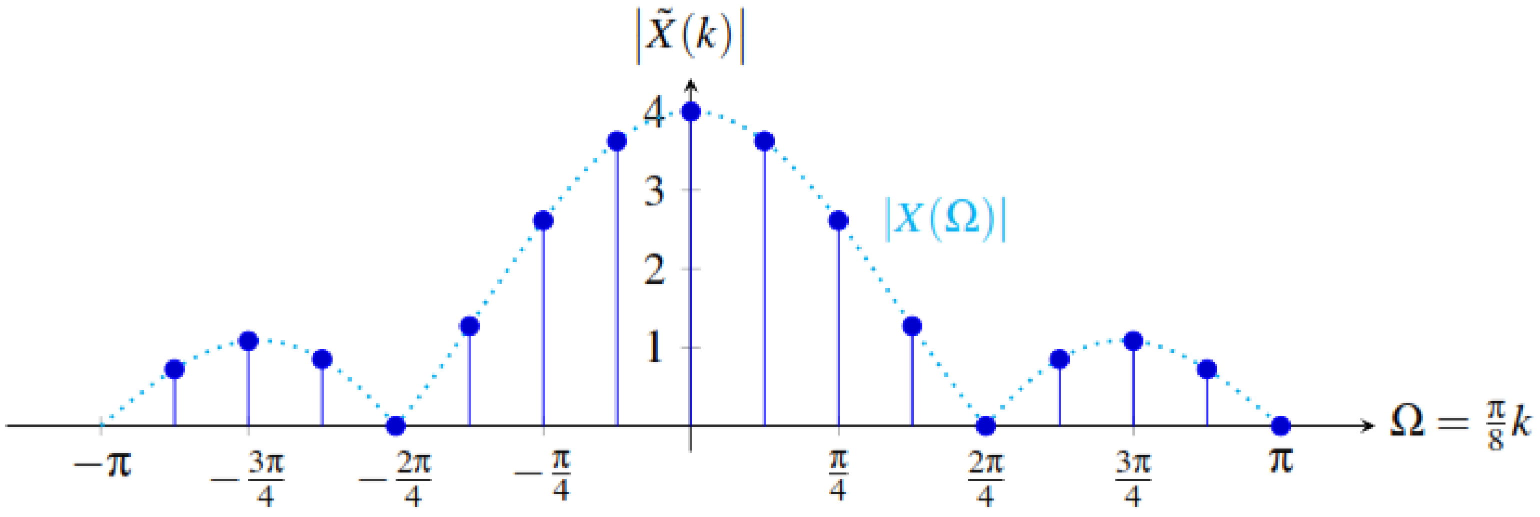


Magnitude Spectrum

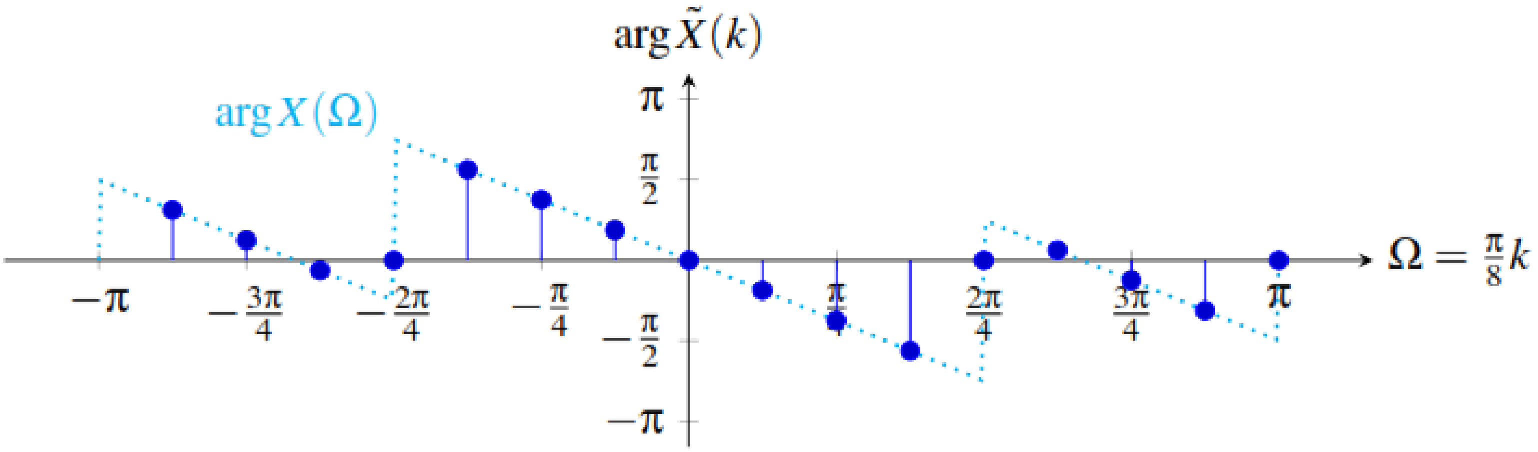


Phase Spectrum

SPECTRAL SAMPLING EXAMPLE

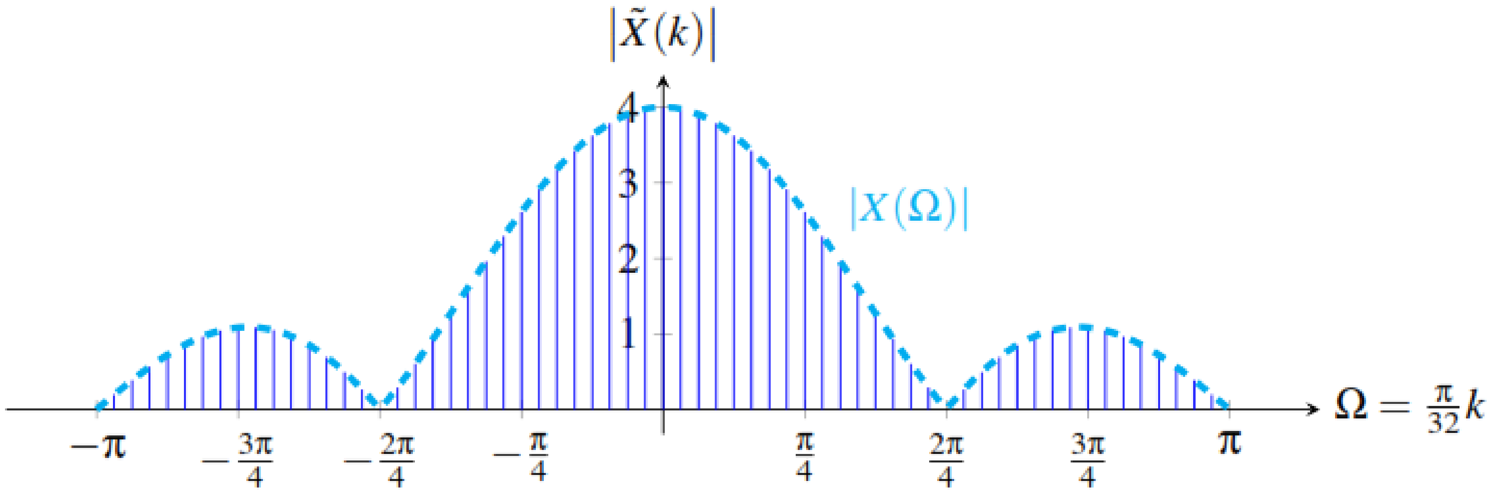


Magnitude Spectrum

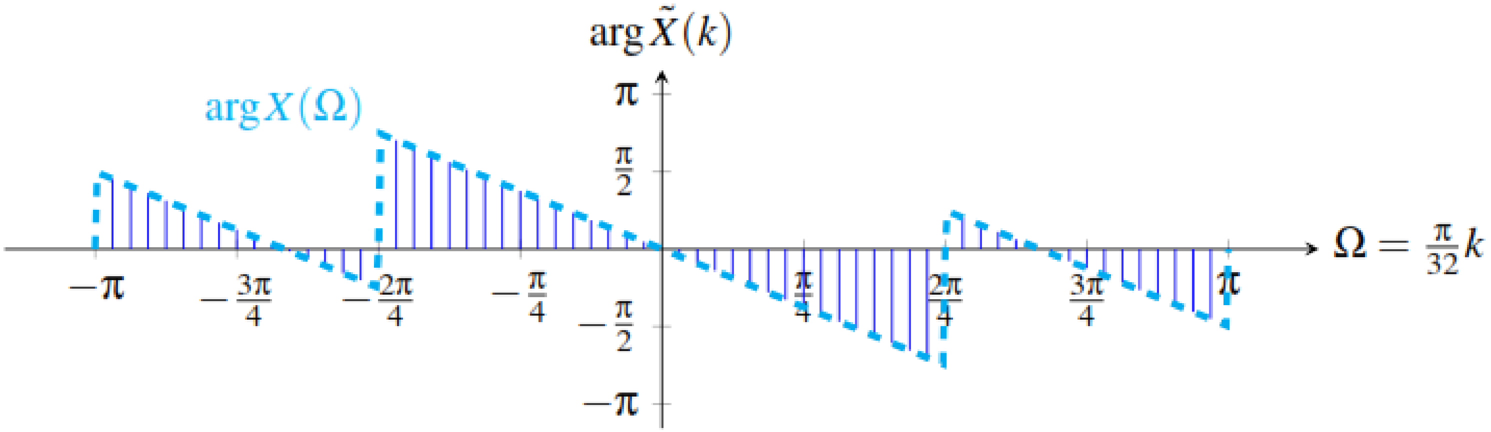


Phase Spectrum

SPECTRAL SAMPLING EXAMPLE



Magnitude Spectrum



Phase Spectrum

EXAMPLE