Hypothesis testing

Outline

- Principle of hypothesis testing
- Test for population mean
  - when the population variance is known
  - when the population variance is unknown
- Test for population variance
- Test for comparing two means

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Introduction

- **Problem**: there are two competing claims about the value of a parameter, and the engineer must determine which claim is correct
- Procedure to make decision based on empirical data: **hypothesis testing**

Example

- The propellant in the rocket motor of an air crew escape system should have a mean burning rate of 50 cm/sec.
- If the burning rate is too low, the ejection seat may not function properly, leading to an unsafe ejection and possible injury of the pilot. Higher burning rates may imply instabilty in the propellant or an ejection seat that is too powerful, again leading to possible pilot injury.

In this example, we are interested in burning rate of a solide propellant used to power air crew system

- Burning rate is a random variable that is decribed by a distribution
- Our interest focus on the **mean** burning rate (*parameter of distribution*)
- Specifically, we are interested in **deciding whether or not the mean burning rate is 50 centimeters per second**.

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Statistical hypothesis

A statistical hypothesis is a statement about one or more populations.

Null Hypothesis H_0	Alternative Hypothesis H_a or H_1
assumption to test	an **opposite of null hypothesis**

Statistical hypothesis

A statistical hypothesis is a statement about one or more populations.

Null Hypothesis H_0

assumption to test

Alternative Hypothesis H_a or H_1

an **opposite of null hypothesis**

Remark

Hypothesis is a statement **about population**, not about sample statistic

Example

Want to decide whether the average number of TV set in U.S. Homes is equal to three

$$H_0 : \mu = 3 \qquad \cancel{H_0 : \overline{X} = 3}$$

Example

We are interested in deciding *whether or not* the $\underbrace{\textbf{mean burning rate}}_{\mu}$ is 50 centimeters per second.

We may express this formally as

$$H_0 : \mu = 50 \text{ centimeter per second}$$
$$H_1 : \mu \neq 50 \text{ centimeter per second}$$

Since the alternative hypothesis $H_1 : \mu \neq 50$ specifies values of μ that could be either greater or less than 50 centimeters per second, it is called a **two-sided alternative hypothesis**. In some situations, we may wish to formulate a **one-sided alternative hypothesis**, as in

$$H_0 : \mu = 50 \text{ vs } H_1 : \mu > 50$$

or

$$H_0 : \mu = 50 \text{ vs } H_1 : \mu < 50$$

- One-sided test
  - Upper-tail test

$$H_0 : \theta = \theta_0 \qquad H_1 : \theta > \theta_0$$

  - Lower-tail test

$$H_0 : \theta = \theta_0 \qquad H_1 : \theta < \theta_0$$

- Two-sided test

$$H_0 : \theta = \theta_0 \qquad H_1 : \theta \neq \theta_0$$

Remark

- H_0 always **contains** "=" or $\leq$ or $\geq$
- H_1: contains " $\neq$ ", " $>$ ", " $<$ "

The name of test depends on the sign in H_1.

Remark

- H_0 always **contains** "=" or $\leq$ or $\geq$
- H_1: contains " $\neq$ "," $>$ "," $<$ "

The name of test depends on the sign in H_1.

Example

- Producer advertises that

$$\underbrace{\text{their cable have an average breaking strength of at least 7,000 psi.}}_{\text{statement about qualify of cable which is described by the } \textbf{mean} \text{ - } \mu}$$

Express formally as

$$H_0 : \mu \geq 7000\text{psi} - \text{the statement}$$
$$H_1 : \mu < 7000\text{psi} - \text{opposite of the statement}$$

Example

Need to decide whether the new fuel injection system will provide a $\underbrace{\textbf{mean miles-per-gallon rating}}_{\mu}$ exceeding 24

Express formally as

$H_0 : \mu \leq 24$miles-per-gallon rating – opposite of the claim

$H_1 : \mu > 24$miles-per-gallon rating – the claim

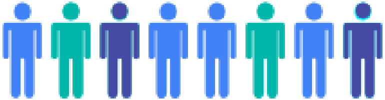
Test of a hypothesis

- A procedure leading to a decision about a particular hypothesis is called a **test of a hypothesis**.
- Hypothesis-testing procedures rely on using the evidence from information in a random sample from the population of interest.
  - reject H_0 in favor of H_1 if the evidence is inconsistent with H_0
  - fail to reject H_0 if the evidence is consistent with H_0

The truth or falsity of a particular hypothesis can never be known with certainty, unless we can examine the entire population.

Hypothesis Testing Process

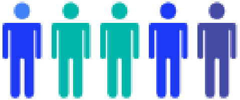
Claim: the population mean age is 50. (Null Hypothesis: $H_0: \mu = 50$)



Population

Now select a random sample

Is $\bar{X} = 20$ likely if $\mu = 50$?



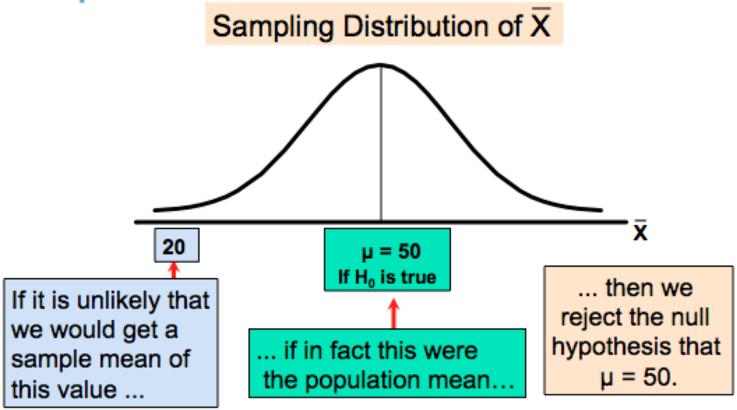
Sample

Suppose the sample mean age is 20: $\bar{X} = 20$

If not likely,

REJECT Null Hypothesis

Rejecting or Accepting H_0?



Sample mean $\bar{X}$ can take many values

Evidence support H_0?

- Even H_0 is true, the sample mean $\bar{x}$ can be close or far from 50. We are more confident that the true mean is 50 if the sample mean $\bar{x}$ is "close" to 50
- "How close is enough". For example, if the sample mean is between 48.5 and 51.5 then we can consider that H_0 is acceptable
- Decide to reject H_0 when either $\bar{x} > 51.5$ or $\bar{x} < 48.5$

Critical values - Rejection Region

- The values 48.5 and 51.5 are called **critical values**
- The interval $[48.5; 51.5]$ is called **acceptance region**
- The interval $(-\infty, 48.5) \cup (51.5, \infty)$ is called **rejection region** or **critical region**

What happens if true population mean age $\mu = 50$ but the observed sample mean is 58?

We can reject H_0 even when H_0 is true

Face to a wrong decision making

Error in Decision Making

- **Type I error** Rejecting the null hypothesis H_0 when it is true
- **Type II error** Failing to reject the null hypothesis when it is false

	H_0 **is true**	H_0 **is false**
Do not reject H_0	Correct decision	Type II error
Reject H_0	Type I error	Correct decision

Level of significance

Because our decision is based on random variables, **probability of type I error**

$$\alpha = P(\text{reject } H_0 \text{ when } H_0 \text{ is true})$$

is called to be **level of significance**

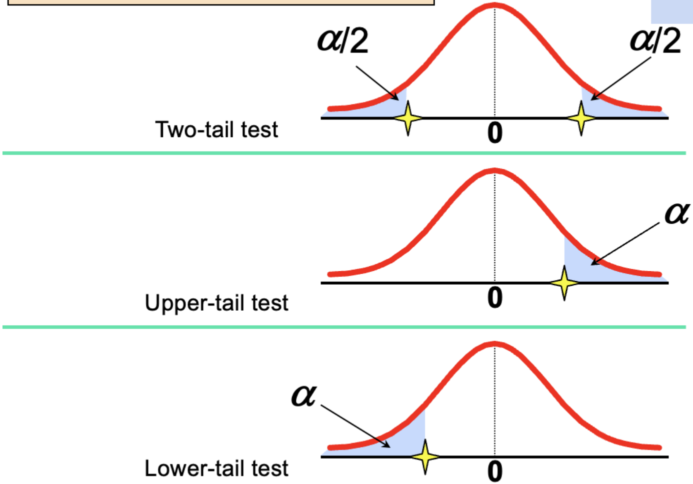
Fixed level of significance

- Typical value for level of significance α are 1%, 5%, 10%
- is selected by the researcher at the beginning
- provides critical values for the test
- defines rejection region of the sampling distribution

Level of significance = α

Represents critical value

Rejection region is shaded



Disadvantage of fixed significant level testing

- the null hypothesis was or was not rejected at a specified level of significance $\Rightarrow$ no idea about the statistic is just barely in or far in rejection region
- Level of significance is not usually set up in advance but rather look at in data to determine a level

P - value *is the smallest level of significance that would lead to rejection of null hypothesis with a given data.*

P-value is the observed significance level when we use value of evidence that we observe from data as critical value

General Procedure for Hypothesis Testing (1)

1. Determine parameter of interest
2. State null and alternative hypothesis
3. Choose a test statistic and compute value of the test statistic v_{obs}.

Critical value approach

4. Determine critical value and reject region W_α based on level of significane α, distribution of test statistic and type of test
5. Draw conlusion
   - $v_{obs} \in W_\alpha$: reject H_0
   - $v_{obs} \notin W_\alpha$: fail to reject H_0

P- value approach

4. Compute the P - value based on the value of test statistic v_{obs}
5. Draw conclusion
   - P-value $< \alpha$: reject H_0
   - P-value $> \alpha$: fail to reject H_0

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Tests for Single Mean μ when Population Variance σ^2 Known

Test

$$H_0 : \mu = \mu_0$$

vs

$$\begin{aligned} & H_1 : \mu \neq \mu_0 && \text{(two-sided test)} \\ \text{or} \quad & H_1 : \mu > \mu_0 && \text{(upper tail test)} \\ \text{or} \quad & H_1 : \mu < \mu_0 && \text{(lower tail test)} \end{aligned}$$

Z-test

Normal population or large sample size $n \geq 30$

If $H_0 : \mu = \mu_0$ is true then by central limit theorem or property of sample mean for normal population, **test statistics**

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \hookrightarrow N(0,1) \text{ called Z-test}$$

The value of Z computed from observed data $z_{obs} = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$

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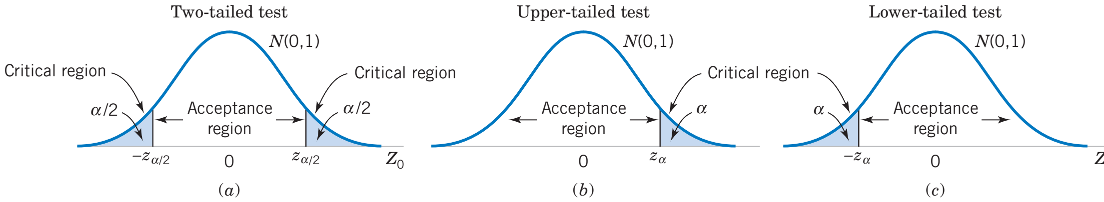
Conclusion with Critical value(s)



Reject H_0 if z_{obs} is in the reject (critical) region

$$|z_{obs}| > z_{\frac{\alpha}{2}}$$

Two - sided test

$$|z_{obs}| > z_{\alpha}$$

One - sided test

Conclusion with P-value

- *Two-sided test*: P-value = $2P(Z > |z_0|)$
- *One-sided test*: P-value = $P(Z > |z_0|)$

Reject H_0at level of significance α if $P_{value} < \alpha$

Example

A random sample of 100 recorded deaths in the United States during the past year showed an average life span of 71.8 years. Assuming a **population standard deviation of 8.9 years**, does this seem to indicate that the **mean life span today** is greater than 70 years? Use a 0.05 level of significance.

Solution

- **Parameter of interest**: mean life span today μ
- $H_0 : \mu = 70$, $H_1 : \mu > 70$
- population std $\sigma = 8.9$, sample size $n = 100$, sample mean $\bar{x} = 71.8$
- Z-test

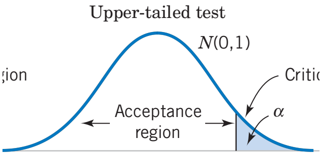
$$z_{obs} = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{71.8 - 70}{8.9/\sqrt{100}} = 2.02$$

Solution

- **Parameter of interest**: mean life span today μ
- $H_0 : \mu = 70$, $H_1 : \mu > 70$
- population std $\sigma = 8.9$, sample size $n = 100$, sample mean $\bar{x} = 71.8$
- Z-test

$$z_{obs} = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{71.8 - 70}{8.9/\sqrt{100}} = 2.02$$

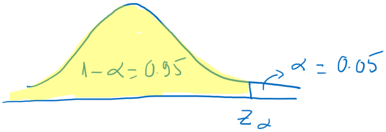
Approach 1 - Draw conclusion with critical value



- Level of significance $\alpha = .05 = 5\% \Rightarrow z_\alpha = 1.65$
- $z_{obs} > z_\alpha$: reject H_0 at 5% level of significance.

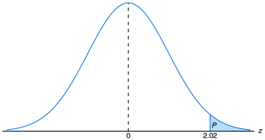
Find $z_\alpha = z_{0.05} = 1.65$

Table A3



	0	1	2	3	4	5	6	7	8	9
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767

Approach 2 - Draw conclusion with P-value



- P-value = $P(Z > 2.02) = .0217 = 2.17\%$
- Level of significance $\alpha = .05 = 5\%$
- P-value $< \alpha \Rightarrow$ reject H_0
- Comment: the evidence in favor of H_1 is even stronger than that suggested by a 0.05 level of significance.

Find P-value = $P(Z > 2.02) = .0217 = 2.17\%$

Table A3

P-value $= 1 - 0.9783$
$= 0.0217$

	0	1	2	3	4	5	6	7	8	9
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857

Remark

In order to draw conclusion, you can choose either critical value approach or P-value approach. They always prodive the same conclusion.

Example

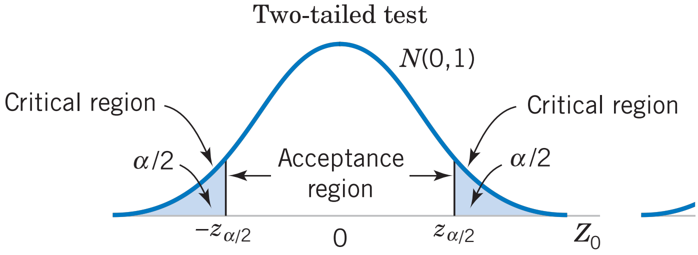
A manufacturer of sports equipment has developed a new synthetic fishing line that the company claims has a mean breaking strength of 8 kilograms with a standard deviation of .5 kilogram. Test the hypothesis that $\mu = 8$ kilograms against the alternative that $\mu \neq 8$ kilograms if a random sample of 50 lines is tested and found to have a mean breaking strength of 7.8 kilograms. Use a .01 level of significance

Solution

- $H_0 : \mu = 8, H_1 : \mu \neq 8$
- $\sigma = .5, n = 50, \bar{x} = 7.8$
- Z-test

$$z_{obs} = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{7.8 - 8}{.5/\sqrt{50}} = 2.83$$

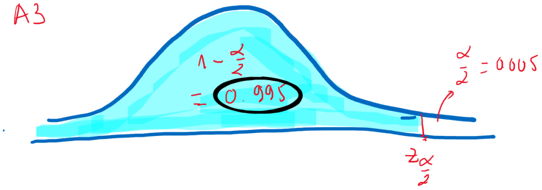
Critical value approach



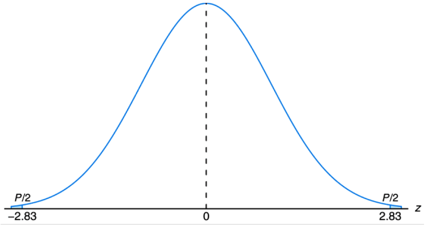
- Level of significance $\alpha = .01 = 1\% \Rightarrow z_{\frac{\alpha}{2}} = 2.58$
- $z_{obs} > z_{\frac{\alpha}{2}}$: reject H_0 at 1% level of significance.

Find $z_{\frac{\alpha}{2}} = z_{.005} = 2.58$

Table A3



	0	1	2	3	4	5	6	7	8	9
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974



- For two - sided test:

$$\text{P-value} = 2P(Z > 2.83) = .0046$$

- Level of significance $\alpha = .01 = 1\%$
- P-value $< \alpha \Rightarrow$ reject H_0 that $\mu = 8$ kg

Find $\frac{P}{2} = .0023$

Table A3

$P/2 = 1 - 0.9977$
$= 0.0023$

	0	1	2	3	4	5	6	7	8	9
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986

Practice

A researcher wishes to test the claim that the average cost of tuition and fees at a four year public college is greater than \$5700. She selects a random sample of 36 four-year public colleges and finds the mean to be \$5950. The population standard deviation is \$659. Is there evidence to support the claim at $\alpha = 0.05$?

Test for mean μ when population variance σ^2 is unknown

Test

$$H_0 : \mu = \mu_0$$

vs

$$\begin{aligned} & H_1 : \mu \neq \mu_0 && \text{(two-sided test)} \\ \text{or} \quad & H_1 : \mu > \mu_0 && \text{(upper tail test)} \\ \text{or} \quad & H_1 : \mu < \mu_0 && \text{(lower tail test)} \end{aligned}$$

If $H_0 : \mu = \mu_0$ is true then **test statistics**

$$T = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} \hookrightarrow T_{n-1}$$

called T-test

t - distribution with $n - 1$ degree of freedom

The value of T computed from data $t_{obs} = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$

If $H_0 : \mu = \mu_0$ is true then **test statistics**

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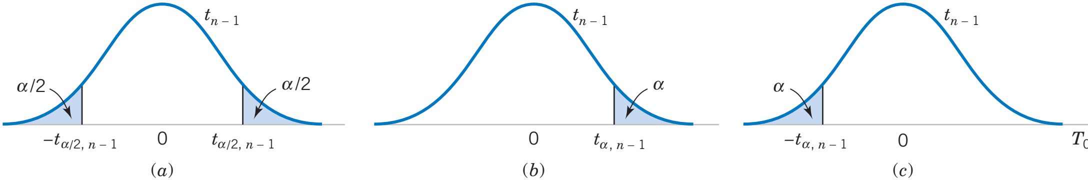
called T-test

t - distribution with $n - 1$ degree of freedom

The value of T computed from data $t_{obs} = \frac{\bar{x}-\mu_0}{s/\sqrt{n}}$

Conclusion with Critical value



Reject H_0 if t_{obs} is in the reject (critical) region

$$|t_{obs}| > t_{\frac{\alpha}{2},n-1} \qquad\qquad |t_{obs}| > t_{\alpha,n-1}$$

Two - sided test One - sided test

Conclusion with P-value

- *Two-sided test*: P-value = $2P(T > |t_{obs}|)$
- *One-sided test*: P-value = $P(T > |t_{obs}|)$

Conclusion

- Reject H_0 if P-value $< \alpha$
- Fail to reject H_0 if P-value $> \alpha$

Example

If a random sample of 12 homes included in a planned study indicates that vacuum cleaners use an average of 42 kilowatt hours per year with a standard deviation of 11.9 kilowatt hours. Does this suggest at the .05 level of significance that vacuum cleaners use, on average, less than 46 kilowatt hours annually?

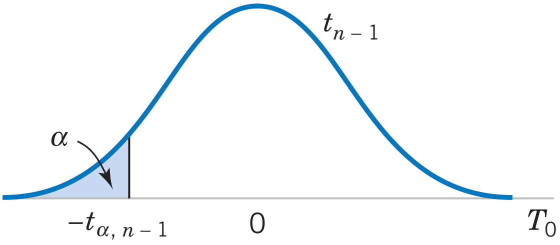
Assume the **population of kilowatt hours to be normal.**

Solution

- **Parameter of interest**: average kwh that vacuum cleaners use μ
- $H_0 : \mu = 46, H_1 : \mu < 46$
- Given $n = 12, \bar{x} = 42, s = 11.9$
- T-test

$$t_{obs} = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{42 - 46}{11.9/\sqrt{12}} = -1.16$$

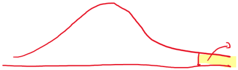
Draw conclusion with Critical value - rejection region



- $\alpha = 5\% \Rightarrow t_{\alpha,n-1} = t_{.05,11} = 1.796$
- $t_{obs} > -t_{\alpha,n-1}$: fail to reject H_0

Find $t_{\alpha,n-1} = t_{.05,11} = 1.796$

Table A4



$t_{\alpha,\nu}$

$\alpha = 0.05$

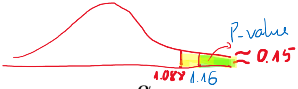
$\nu = n-1 = 12-1 = 11$

v	α 0.40	0.30	0.20	0.15	0.10	0.05	0.025
1	0.325	0.727	1.376	1.963	3.078	6.314	12.706
2	0.289	0.617	1.061	1.386	1.886	2.920	4.303
3	0.277	0.584	0.978	1.250	1.638	2.353	3.182
4	0.271	0.569	0.941	1.190	1.533	2.132	2.776
5	0.267	0.559	0.920	1.156	1.476	2.015	2.571
6	0.265	0.553	0.906	1.134	1.440	1.943	2.447
7	0.263	0.549	0.896	1.119	1.415	1.895	2.365
8	0.262	0.546	0.889	1.108	1.397	1.860	2.306
9	0.261	0.543	0.883	1.100	1.383	1.833	2.262
10	0.260	0.542	0.879	1.093	1.372	1.812	2.228
11	0.260	0.540	0.876	1.088	1.363	1.796	2.201

- P-value = $P(T > |t_{obs}|) = P(T > 1.16) = .135$
- Level of significance $\alpha = .05$
- P-value $> \alpha$
- **Decision**: fail to reject H_0 and conclude that the average number of kilowatt hours used annually by home vacuum cleaners is not significantly less than 46.

Find P-value = $P(T > |t_{obs}|) = P(T > 1.16) = .135$ (= tdist(1.16, 11, 1) in Excel)

Table A4



$\nu = n - 1$
$= 12 - 1$
$= 11$

	α						
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closest value to 1.16

Example

A public health official claims that the mean home water use is 350 gallons a day. 20 randomly selected homes was investigated with the average daily water uses as follows:

340	344	362	375
356	386	354	364
332	402	340	355
362	322	372	324
318	360	338	370

Do the data contradict the claim with level of significant $\alpha = 1\%$? Assume that the population has normal distribution.

Solution

- **Parameter of interest:** mean home water use μ
- Hypothesis $H_0 : \mu = 350$, $H_1 : \mu \neq 350$
- $\bar{x} = 353.8,\ s = 21.8478,\ n = 20$
- T-test

$$t_{obs} = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{353.8 - 350}{\frac{21.8478}{20}} = .7778$$

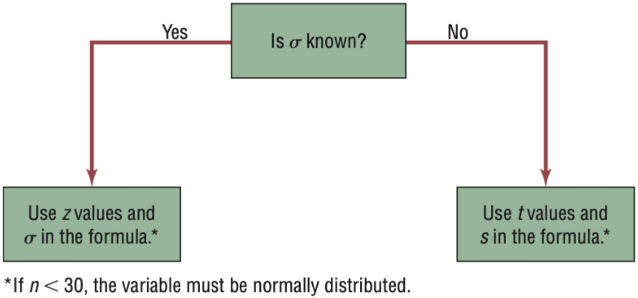
- Critical value for two - sided test $t_{\alpha/2,n-1} =???$
  (look up table value of t-distribution with 19 degree of freedom)

- Conclusion??

Practice

A medical investigation claims that the average number of infections per week at a hospital in southwestern Pennsylvania is 16.3. A random sample of 10 weeks had a mean number of 17.7 infections. The sample standard deviation is 1.8. Is there enough evidence to reject the investigator's claim at $\alpha = 0.05$? Suppose that population has normal distribution

Test for population mean - z - test or t-test



$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

$$T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$$

normal population

Test for variance σ^2 of normal population

$$H_0 : \sigma^2 = \sigma_0^2$$

vs

- $H_1 : \sigma^2 \neq \sigma_0^2$ (Two-sided test)
- $H_1 : \sigma^2 > \sigma_0^2$ (Upper tail test)
- $H_1 : \sigma^2 < \sigma_0^2$ (Lower tail test)

χ^2 - test

normal population

If $H_0 : \sigma^2 = \sigma_0^2$ is true then **test statistic**

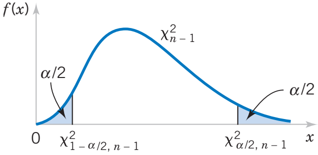
$$\chi^2 = \frac{(n-1)S^2}{\sigma_0^2} \sim \chi^2_{n-1}$$

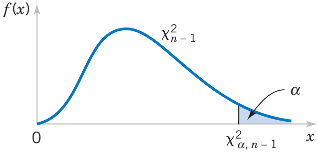
distribution with $n - 1$ degree of freedom

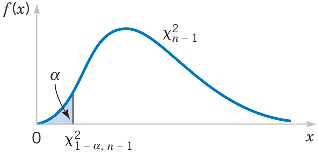
Observed value of χ^2 computed from data $\chi^2_{obs} = \frac{(n-1)s^2}{\sigma_0^2}$

Conclusion with critical value(s)

Reject H_0 if χ^2_{obs} is in the reject region







$$\chi^2_{obs} < \chi^2_{1-\frac{\alpha}{2},n-1}$$

or $\chi^2_{obs} > \chi^2_{\frac{\alpha}{2},n-1}$

Two - sided test

$$\chi^2_{obs} > \chi^2_{\alpha,n-1}$$

Upper - tail test

$$\chi^2_{obs} < \chi^2_{1-\alpha,n-1}$$

Lower - tail test

Example

A manufacturer of car batteries claims that the life of the company's batteries is approximately normally distributed with a standard deviation equal to 0.9 year.If a random sample of 10 of these batteries has a standard deviation of 1.2 years, do you think that $\sigma > 0.9$ year? Use a 0.05 level of significance.

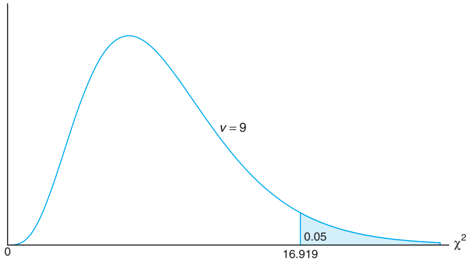
Solution

- Hypothesis $H_0 : \sigma^2 = 0.9^2 = 0.81$ vs $H_1 : \sigma^2 > 0.81$
- $n = 10, s = 1.2$ or $s^2 = 1.2^2 = 1.44, \sigma_0^2 = 0.81$
- χ^2 - test value

$$\chi^2_{obs} = \frac{(n-1)s^2}{\sigma_0^2} = 16.0$$

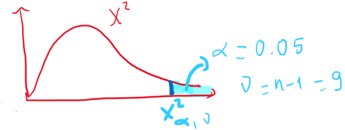
Draw conclusion with Critical value - rejection region

- Critical value for upper tail test
  $\alpha = .05 \Rightarrow \chi^2_{\alpha,n-1} = \chi^2_{0.05,9} = 16.916$
- $\chi^2_{obs} < \chi^2_{.05,9} \Rightarrow$ accept H_0. There is not enough evidence that $\sigma > 0.9$



Find $\chi^2_{.05,9} = 16.916$

Table A5



	α									
v	**0.30**	**0.25**	**0.20**	**0.10**	**0.05**	**0.025**	**0.02**	**0.01**	**0.005**	**0.001**
1	1.074	1.323	1.642	2.706	3.841	5.024	5.412	6.635	7.879	10.827
2	2.408	2.773	3.219	4.605	5.991	7.378	7.824	9.210	10.597	13.815
3	3.665	4.108	4.642	6.251	7.815	9.348	9.837	11.345	12.838	16.266
4	4.878	5.385	5.989	7.779	9.488	11.143	11.668	13.277	14.860	18.466
5	6.064	6.626	7.289	9.236	11.070	12.832	13.388	15.086	16.750	20.515
6	7.231	7.841	8.558	10.645	12.592	14.449	15.033	16.812	18.548	22.457
7	8.383	9.037	9.803	12.017	14.067	16.013	16.622	18.475	20.278	24.321
8	9.524	10.219	11.030	13.362	15.507	17.535	18.168	20.090	21.955	26.124
9	10.656	11.389	12.242	14.684	16.919	19.023	19.679	21.666	23.589	27.877
10	11.781	12.549	13.442	15.987	18.307	20.483	21.161	23.209	25.188	29.588

Practice

A machine that automatically controls the amount of ribbon on a tape has recently been installed. This machine will be judged to be effective if the standard deviation σ of the amount of ribbon on a tape is no greater than .15 cm. If **a sample of 20 tapes yields a sample variance of** $s^2 = .025\ cm^2$, are we justified in concluding that the machine is ineffective? Use a 0.05 level of significance.

Test for population proportion

Problem

$$H_0 : p = p_0$$

vs

$$H_1 : p \neq p_0$$
$$H_1 : p < p_0$$
$$H_1 : p > p_0$$

For one-sided test, the null hypothesis can be $p \geq p_0$ or $p \leq p_0$

Z-test for large sample size

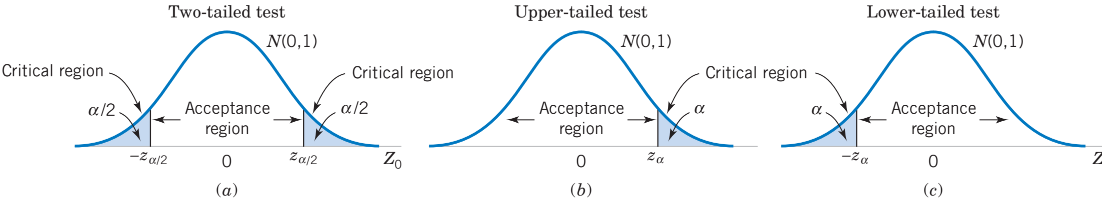
If $p = p_0$ is true then the number of success in the sample $X \sim Bin(n, p_0)$ is approximated by $\mathcal{N}(np_0, np_0(1-p_0))$.

***Z*-test**

$$Z = \frac{X - np_0(1-p_0))}{\sqrt{np_0(1-p_0)}} = \frac{\hat{P} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \sim \mathcal{N}(0,1)$$

with observed value $z_{obs} = \frac{\hat{p}-p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$ where $\hat{p} = \frac{x}{n}$ is the sample proportion

Conclusion with Critical value(s)



Reject H_0 if z_{obs} is in the reject (critical) region

$$|z_{obs}| > z_{\frac{\alpha}{2}}$$

Two - sided test

$$|z_{obs}| > z_{\alpha}$$

One - sided test

Conclusion with P-value

- *Two-sided test*: P-value = $2P(Z > |z_0|)$
- *One-sided test*: P-value = $P(Z > |z_0|)$

Reject H_0at level of significance α if $P_{value} < \alpha$

Example

A commonly prescribed drug for relieving nervous tension is believed to be only 60% effective. Experimental results with a new drug administered to a random sample of 100 adults who were suffering from nervous tension show that 70 received relief. Is this sufficient evidence to conclude that the new drug is superior to the one commonly prescribed? Use a 0.05 level of significance

Solution

- Problem statement

$$H_0 : p = 0.6 \;\; vs \;\; H_1 : p > 0.6$$

- Given: $n = 100$, $x = 70$, $\hat{p} = \frac{70}{100} = 0.7$.
- Test statistic
- $$z_{obs} = \frac{0.7 - 0.6}{\sqrt{\frac{(0.6)(1-0.6)}{100}}} = 2.04$$

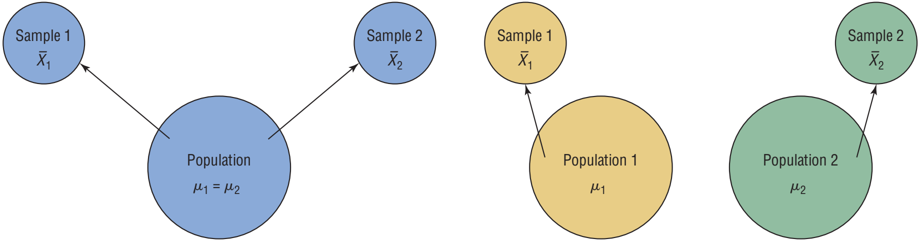
- with $\alpha = 0.5$ and lower tail-test, the critical value $z_\alpha = 1.65$
- Conclusion: $|z_{obs}| > z_\alpha$ implies to reject H_0 at 5%, that is the new drug is superior

Practice

According to the U.S. Bureau of the Census, 25.5 percent of the population of those age 18 or over smoked in 1990. A scientist has recently claimed that this percentage has since increased, and to prove her claim she randomly sampled 500 individuals from this population. If 138 of them were smokers, is her claim proved? Use the 5 percent level of significance.

Table of contents

Comparison means of two populations - Known variances



(a) Difference is not significant **(b)** Difference is significant

Comparison problem

- Distribution of two population are $N(\mu_X, \sigma_X^2)$, and $N(\mu_Y, \sigma_Y^2)$ or large sample sizes
- Know σ_X, σ_X, don't know μ_X, μ_Y
- **Want to compare μ_X and μ_Y**

Problem statement

$$H_0 : \mu_X - \mu_Y = d_0$$

vs

$$H_1 : \mu_X - \mu_Y \neq d_0$$ **two -sided test**

$$H_1 : \mu_X - \mu_Y > d_0$$ **upper - tail test**

$$H_1 : \mu_X - \mu_Y < d_0$$ **lower - tail test**

Sampling Distribution

- Sample mean $\bar{X}$ population 1 is used to estimate μ_X
- Sample mean $\bar{X}$ population 2 is used to estimate μ_Y
- $\bar{X} - \bar{Y}$ is an estimator for $\mu_X - \mu_Y$
- If both population have normal distributions $\mathrm{N}(\mu_X, \sigma_X^2)$, and $\mathrm{N}(\mu_Y, \sigma_Y^2)$ or large sample sizes then $\bar{X} - \bar{Y} \sim \mathrm{N}(\mu_X - \mu_Y, \dfrac{\sigma_X^2}{n_X} + \dfrac{\sigma_Y^2}{n_Y})$ given H_0 is true

If $H_0 : \mu_X - \mu_Y = d_0$ is true then

Test statistics is Z-test

$$Z = \frac{\bar{X} - \bar{Y} - d_0}{\sqrt{\frac{\sigma_x^2}{n_X} + \frac{\sigma_Y^2}{n_Y}}} \sim \mathrm{N}(0, 1)$$

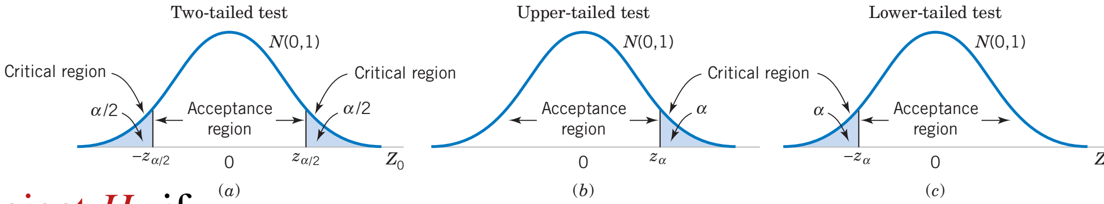
P-value

- *Two-sided test*: P-value = $2P(Z > |z_{obs}|)$
- *One-sided test*: P-value = $P(Z > |z_{obs}|)$

where z_{obs} is the value of Z computed from data

$$z_{obs} = \frac{\bar{x} - \bar{y} - d_0}{\sqrt{\frac{\sigma_x^2}{n_X} + \frac{\sigma_y^2}{n_Y}}}$$

Critical value - Reject region



Reject H_0 if

$$|z_{obs}| > z_{\frac{\alpha}{2}}$$

$$|z_{obs}| > z_{\alpha}$$

Two - sided test

One - sided test

Example

A company produces a sample of 10 tires using method 1 and a sample of 8 using method 2. They want to show that there is no difference in the average life time of tires. The first tire set is tested at location A where the standard deviation is known to be 4000 km, and second set is tested at location B where sd is 6000. What conclusion can be drawn with 5% level of significance from the following data? Suppose that both populations have normal distributions.

TABLE 8.3 *Tire Lives in Units of 100 Kilometers*

Tires Tested at A	Tires Tested at B
61.1	62.2
58.2	56.6
62.3	66.4
64	56.2
59.7	57.4
66.2	58.4
57.8	57.6
61.4	65.4
62.2	
63.6	

Solution

- $H_0 : \mu_A = \mu_B$ vs $H_1 : \mu_A \neq \mu_B$
- $n_A = 10, \bar{x}_A = 61.65, \sigma_A = 4000$
- $n_B = 8, \bar{x}_B = 60.025, \sigma_B = 6000$
- Z-test

$$z_{obs} = \frac{\bar{x}_A - \bar{x}_B}{\sqrt{\frac{\sigma_A^2}{n_A} + \frac{\sigma_B^2}{n_B}}} = .066$$

- Critical value?
- Rejection region?
- Conclusion?

A real estate agent compares the selling prices of homes in two municipalities in southwestern Pennsylvania to see if there is a difference. Is there enough evidence to reject the claim that the average cost of a home in both locations is the same? Use $\alpha = 0.01$.

Scott	Lionier
$\bar{x}_1 = \$93,430$	$\bar{x}_2 = \$98,043$
$\sigma_1 = \$5,602$	$\sigma_2 = \$4732$
$n_1 = 35$	$n_2 = 40$

Comparison means of two population - Unknown variance

Comparison problem

- Distribution of two population are $\mathrm{N}(\mu_X, \sigma_X^2)$, and $\mathrm{N}(\mu_Y, \sigma_Y^2)$ or large sample sizes
- **Don't know** σ_X, σ_X
- **Want to compare** μ_X **and** μ_Y

Problem statement

$$H_0 : \mu_X - \mu_Y = d_0$$

vs

$$H_1 : \mu_X - \mu_Y \neq d_0 \text{ two -sided test}$$
$$H_1 : \mu_X - \mu_Y > d_0 \text{ upper - tail test}$$
$$H_1 : \mu_X - \mu_Y < d_0 \text{ lower - tail test}$$

two normal populations and unknown variances but equal

If $H_0 : \mu_X - \mu_Y = d_0$ is true then

Test statistics is T-test

$$T = \frac{\bar{X} - \bar{Y} - d_0}{\sqrt{\frac{S_x^2}{n_X} + \frac{S_y^2}{n_Y}}} \sim T(n_x + n_y - 2)$$

where $S_x^2 = \frac{\sum(X_i - \bar{X})^2}{n_x - 1}$, $S_y^2 = \frac{\sum(Y_i - \bar{Y})^2}{n_y - 1}$ are sample variances of two samples

Pooled estimator for common variance $\sigma_x^2 = \sigma_y^2 = \sigma^2$

$$S_p^2 = \frac{(n_x - 1)S_x^2 + (n_y - 1)S_y^2}{n_x + n_y - 2}$$

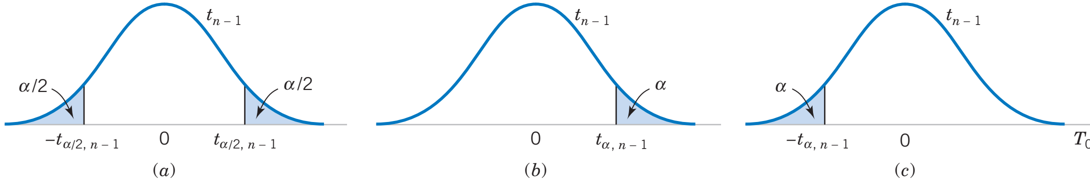
T-test with pooled estimator

$$T = \frac{\bar{X} - \bar{Y} - d_0}{\sqrt{S_p^2 \left(\frac{1}{n_x} + \frac{1}{n_y} \right)}} \sim T(n_x + n_y - 2)$$

with

$$t_{obs} = \frac{\bar{x} - \bar{y} - d_0}{\sqrt{s_p^2 \left(\frac{1}{n_x} + \frac{1}{n_y} \right)}} \sim T(n_x + n_y - 2)$$

Conclusion with Critical value



Reject H_0 if t_{obs} is in the reject (critical) region

$$|t_{obs}| > t_{\frac{\alpha}{2},n-1}$$

Two - sided test

$$|t_{obs}| > t_{\alpha,n-1}$$

One - sided test

Conclusion with P-value

- *Two-sided test*: P-value = $2P(T > |t_{obs}|)$
- *One-sided test*: P-value = $P(T > |t_{obs}|)$

Conclusion

- Reject H_0 if P-value $< \alpha$
- Fail to reject H_0 if P-value $> \alpha$

Duration for treatment cold

Treated with Vitamin C	Treated with Placebo
5.5	6.5
6.0	6.0
7.0	8.5
6.0	7.0
7.5	6.5
6.0	8.0
7.5	7.5
5.5	6.5
7.0	7.5
6.5	6.0
	8.5
	7.0

Do the data listed prove that taking 4 grams daily of vitamin C reduces the mean length of time a cold lasts?
Use $\alpha = 5\%$.
Assume normal populations, equal population variances.

Solution

Test

$$H_0 : \mu_c - \mu_p \geq 0 \quad vs \quad H_1 : \mu_c \mu_p < 0 \textbf{ lower-tail test}$$

where μ_c is the mean time a cold lasts when the vitamin C tablets are taken and μ_p is the mean time when the placebo is taken.

Given $n_x = 10$, $n_y = 12$, $\bar{x} = 6.450$, $\bar{y} = 7.125$, $s_x^2 = 0.581$, $s_y^2 = 0.778$. We have

$$s_p^2 = \frac{(10-1)s_x^2 + (12-1)s_y^2}{10+12-2} = 0.689$$

The value of the test statistic is

$$t_{obs} = \frac{\bar{x} - \bar{y}}{\sqrt{s_p^2\left(\frac{1}{n_x} + \frac{1}{n_y}\right)}} = \frac{6.450 - 7.125}{\sqrt{0.689\left(\frac{1}{10} + \frac{1}{12}\right)}} = -1.90$$

$$|t_{obs}| = 1.90 > t_{0.5,20} = 1.725$$

Conclusion: Reject H_0 at 5% level of significance. That is, at the 5% level of significance the evidence is significant in establishing that vitamin C reduces the mean time that a cold persists

Practice

The viscosity of two different brands of car oil is measured and the following data resulted:

Brand 1	10.62, 10.58, 10.33, 10.72, 10.44, 10.74
Brand 2	10.50, 10.52, 10.58, 10.62, 10.55, 10.51, 10.53

Test the hypothesis that the mean viscosity of the two brands is equal, assuming that the populations have normal distributions with equal variances.

Z- test

two normal populations and unknown and unequal variances but large sample sizes

If $H_0 : \mu_X - \mu_Y = d_0$ is true then **Test statistics** is Z-test

$$\boxed{Z = \frac{\bar{X} - \bar{Y} - d_0}{\sqrt{\frac{S_x^2}{n_X} + \frac{S_y^2}{n_Y}}} \sim \mathcal{N}(0, 1)}$$

where $S_x^2 = \frac{\sum(X_i - \bar{X})^2}{n_x - 1}, \qquad S_y^2 = \frac{\sum(Y_i - \bar{Y})^2}{n_y - 1}$

Practice

Sample weights (in pounds) of newborn babies born in two adjacent counties in Western Pennsylvania yielded the following data.

$$n_x = 53, \bar{x} = 6.8, s_x^2 = 5.2$$
$$n_y = 44, \bar{y} = 7.2, s_y^2 = 4.9$$

Consider a test of the hypothesis that the mean weight of newborns is the same in both counties. Use $\alpha = 5\%$