Chapter 6
Laplace Transform

MOTIVATION BEHIND THE LAPLACE TRANSFORM

- Another important mathematical tool in the study of signals and systems is known as the Laplace transform.
- The Laplace transform can be viewed as a ***generalization of the (classical) Fourier transform***.
- Due to its more general nature, the Laplace transform has a number of ***advantages*** over the (classical) Fourier transform.
- First, the Laplace transform representation ***exists for some functions that do not have a Fourier transform representation***. So, we can handle some functions with the Laplace transform that cannot be handled with the Fourier transform.
- Second, since the Laplace transform is a more general tool, it can provide ***additional insights*** beyond those facilitated by the Fourier transform.

Laplace Transform

BILATERAL LAPLACE TRANSFORM

- The (bilateral) **Laplace transform** of the function x, denoted $\mathcal{L}x$ or X, is defined as

$$\mathcal{L}x(s) = X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt.$$

- The **inverse Laplace transform** of X, denoted $\mathcal{L}^{-1}X$ or x, is then given by

$$\mathcal{L}^{-1}X(t) = x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds,$$

where $\mathrm{Re}(s) = \sigma$ is in the ROC of X. (Note that this is a ***contour integration***, since s is complex.)

- We refer to x and X as a **Laplace transform pair** and denote this relationship as

$$x(t) \overset{\text{LT}}{\longleftrightarrow} X(s).$$

- In practice, we do not usually compute the inverse Laplace transform by directly using the formula from above. Instead, we resort to other means (to be discussed later).

BILATERAL AND UNILATERAL LAPLACE TRANSFORMS

- Two different versions of the Laplace transform are commonly used:
  1. the ***bilateral*** (or ***two-sided***) Laplace transform; and
  2. the ***unilateral*** (or ***one-sided***) Laplace transform.
- The unilateral Laplace transform is most frequently used to solve systems of linear differential equations with nonzero initial conditions.
- As it turns out, the only difference between the definitions of the bilateral and unilateral Laplace transforms is in the ***lower limit of integration***.
- In the bilateral case, the lower limit is $-\infty$, whereas in the unilateral case, the lower limit is 0 (i.e., $\int_{-\infty}^{\infty} x(t)e^{-st}dt$ versus $\int_{0^-}^{\infty} x(t)e^{-st}dt$).
- For the most part, we will focus our attention primarily on the bilateral Laplace transform.
- We will, however, briefly introduce the unilateral Laplace transform as a tool for solving differential equations.
- Unless otherwise noted, all subsequent references to the Laplace transform should be understood to mean ***bilateral*** Laplace transform.

REMARKS ON OPERATOR NOTATION

- For a function x, the Laplace transform of x is denoted using operator notation as $\mathcal{L}x$.
- The Laplace transform of x evaluated at s is denoted $\mathcal{L}x(s)$.
- Note that $\mathcal{L}x$ is a function, whereas $\mathcal{L}x(s)$ is a number.
- Similarly, for a function X, the inverse Laplace transform of X is denoted using operator notation as $\mathcal{L}^{-1}X$.
- The inverse Laplace transform of X evaluated at t is denoted $\mathcal{L}^{-1}X(t)$.
- Note that $\mathcal{L}^{-1}X$ is a function, whereas $\mathcal{L}^{-1}X(t)$ is a number.
- With the above said, engineers often abuse notation, and use expressions like those above to mean things different from their proper meanings.
- Since such notational abuse can lead to problems, it is strongly recommended that one refrain from doing this.

RELATIONSHIP BETWEEN LAPLACE AND FOURIER TRANSFORMS

- Let X and X_{F} denote the Laplace and (CT) Fourier transforms of x, respectively.
- The function X evaluated at $j\omega$ (where ω is real) yields $X_{\mathsf{F}}(\omega)$. That is,

$$X(j\omega) = X_{\mathsf{F}}(\omega).$$

- Due to the preceding relationship, the Fourier transform of x is sometimes written as $X(j\omega)$.
- The function X evaluated at an arbitrary complex value $s = \sigma + j\omega$ (where $\sigma = \mathrm{Re}(s)$ and $\omega = \mathrm{Im}(s)$) can also be expressed in terms of a Fourier transform involving x. In particular, we have

$$X(\sigma + j\omega) = X'_{\mathsf{F}}(\omega),$$

  where X'_{F} is the (CT) Fourier transform of $x'(t) = e^{-\sigma t}x(t)$.
- So, in general, the Laplace transform of x is the Fourier transform of an exponentially-weighted version of x.
- Due to this weighting, the Laplace transform of a function may exist when the Fourier transform of the same function does not.

EXAMPLE

Find the Laplace transform X of the function

$$x(t) = e^{-at}u(t),$$

where a is a real constant.

EXAMPLE

Find the Laplace transform X of the function

$$x(t) = e^{-at}u(t),$$

where a is a real constant.

Solution. Let $s = \sigma + j\omega$, where σ and ω are real. From the definition of the Laplace transform, we have

$$\begin{aligned} X(s) &= \mathcal{L}\{e^{-at}u(t)\}(s) \\ &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\,dt \\ &= \int_{0}^{\infty} e^{-(s+a)t}\,dt \\ &= \left[\left(-\tfrac{1}{s+a}\right)e^{-(s+a)t}\right]\Big|_0^{\infty}. \end{aligned}$$

we substitute $s = \sigma + j\omega$ in order to more easily determine when the above expression converges

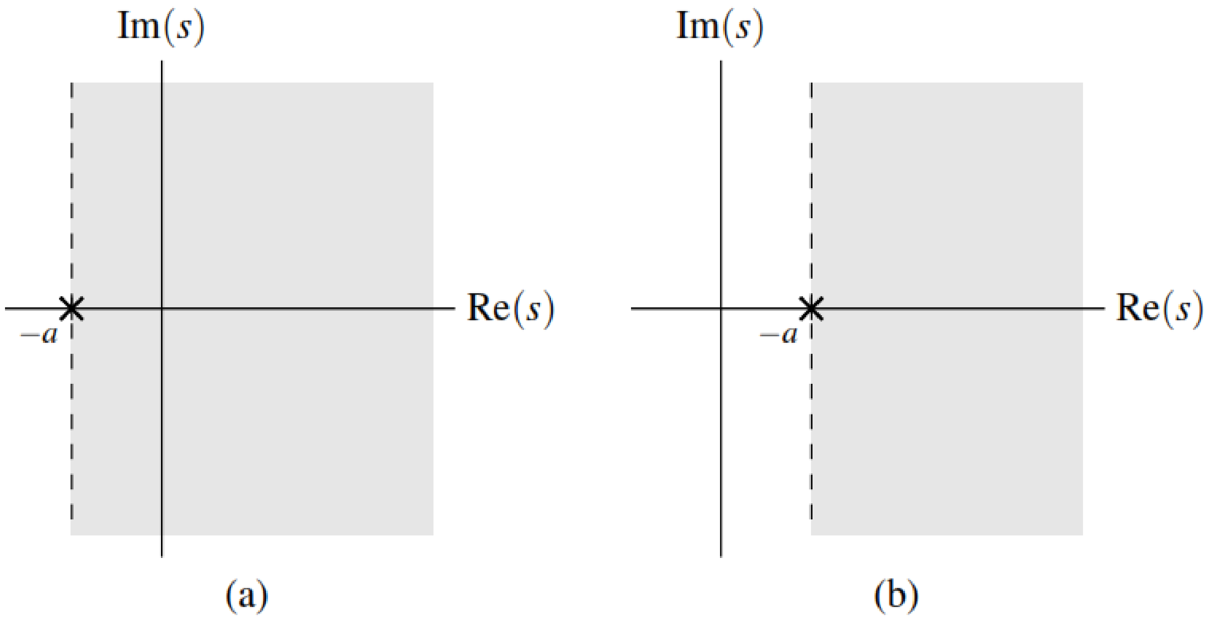
EXAMPLE

$$X(s) = \left[\left(-\frac{1}{\sigma+a+j\omega}\right)e^{-(\sigma+a+j\omega)t}\right]\Bigg|_0^\infty$$
$$= \left(\frac{-1}{\sigma+a+j\omega}\right)\left[e^{-(\sigma+a)t}e^{-j\omega t}\right]\Bigg|_0^\infty$$
$$= \left(\frac{-1}{\sigma+a+j\omega}\right)\left[e^{-(\sigma+a)\infty}e^{-j\omega\infty}-1\right].$$

we can see that the above expression only converges for $\sigma+a>0$ (i.e., $\text{Re}(s)>-a$).

$$X(s) = \left(\frac{-1}{\sigma+a+j\omega}\right)[0-1]$$
$$= \left(\tfrac{-1}{s+a}\right)(-1)$$
$$= \frac{1}{s+a}.$$

EXAMPLE



Region of convergence for the case that (a) $a > 0$ and (b) $a < 0$.

$$e^{-at}u(t) \stackrel{\text{LT}}{\longleftrightarrow} \frac{1}{s+a} \quad \text{for } \mathrm{Re}(s) > -a.$$

EXAMPLE

Find the Laplace transform X of the function

$$x(t) = -e^{-at}u(-t),$$

where a is a real constant.

EXAMPLE

Find the Laplace transform X of the function

$$x(t) = -e^{-at}u(-t),$$

where a is a real constant.

Solution. Let $s = \sigma + j\omega$, where σ and ω are real.

$$\begin{aligned}
X(s) &= \mathcal{L}\{-e^{-at}u(-t)\}(s) \\
&= \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}\,dt \\
&= \int_{-\infty}^{0} -e^{-at}e^{-st}\,dt \\
&= \int_{-\infty}^{0} -e^{-(s+a)t}\,dt \\
&= \left[\left(\tfrac{1}{s+a}\right)e^{-(s+a)t}\right]\Big|_{-\infty}^{0}.
\end{aligned}$$

EXAMPLE

we substitute $s = \sigma + j\omega$.

$$
\begin{aligned}
X(s) &= \left[\left(\frac{1}{\sigma+a+j\omega}\right) e^{-(\sigma+a+j\omega)t}\right]\Big|_{-\infty}^{0} \\
&= \left(\frac{1}{\sigma+a+j\omega}\right)\left[e^{-(\sigma+a)t} e^{-j\omega t}\right]\Big|_{-\infty}^{0} \\
&= \left(\frac{1}{\sigma+a+j\omega}\right)\left[1 - e^{(\sigma+a)\infty} e^{j\omega\infty}\right].
\end{aligned}
$$

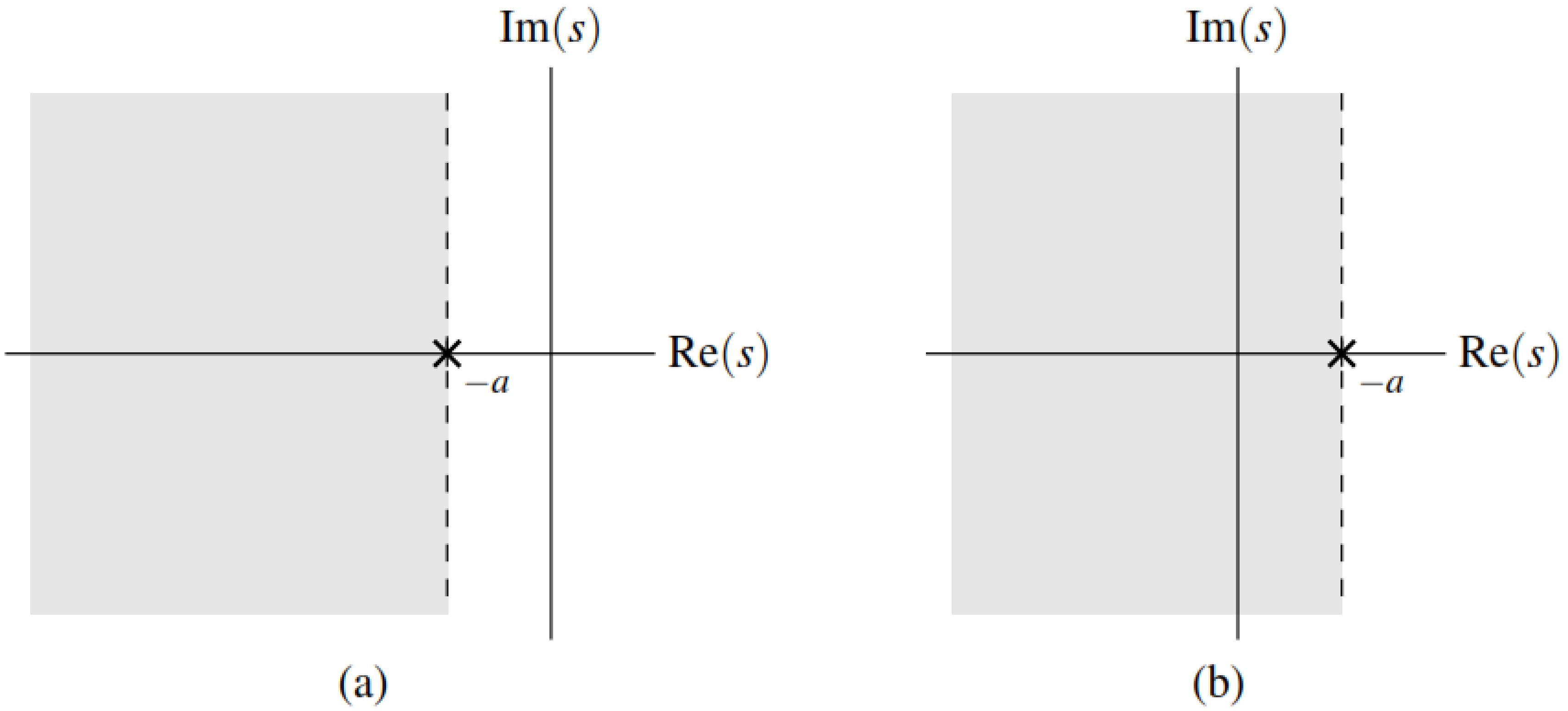
the above expression only converges for $\sigma + a < 0$ (i.e., $\mathrm{Re}(s) < -a$).

$$
\begin{aligned}
X(s) &= \left(\frac{1}{\sigma+a+j\omega}\right)[1-0] \\
&= \frac{1}{s+a}.
\end{aligned}
$$

Thus, we have that

$$
-e^{-at}u(-t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+a} \quad \text{for } \mathrm{Re}(s) < -a.
$$

EXAMPLE



Region of convergence for the case that (a) $a > 0$ and (b) $a < 0$.

Region of Convergence (ROC)

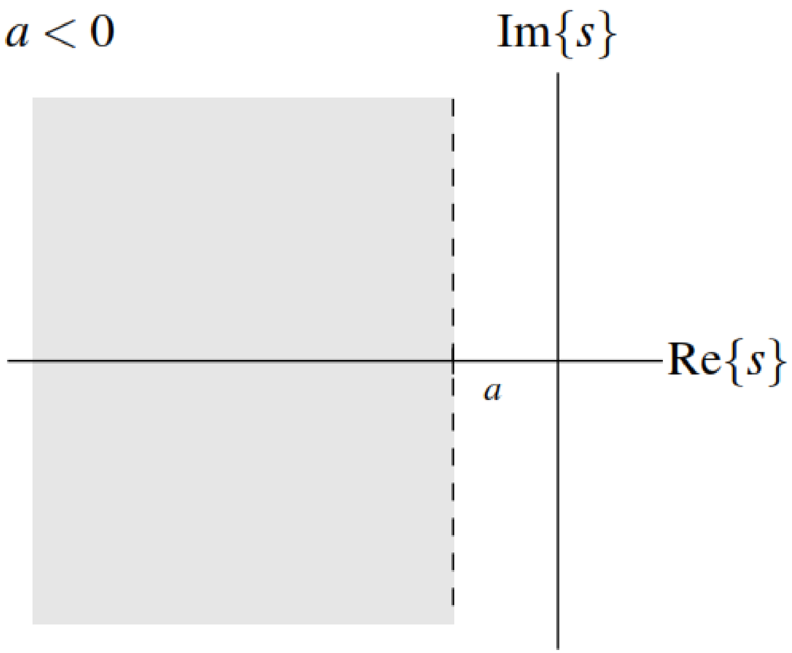
LEFT-HALF PLANE (LHP)

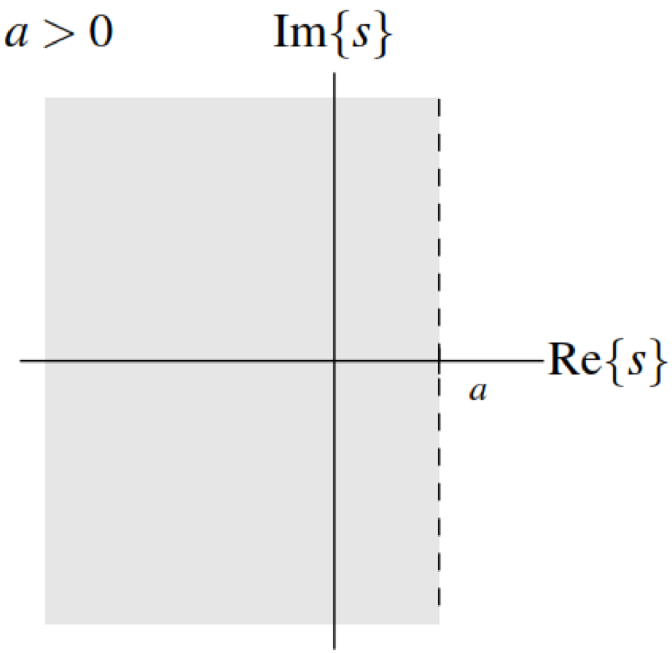
- The set R of all complex numbers s satisfying

$$\mathrm{Re}(s) < a$$

for some real constant a is said to be a **left-half plane (LHP)**.

- Some examples of LHPs are shown below.





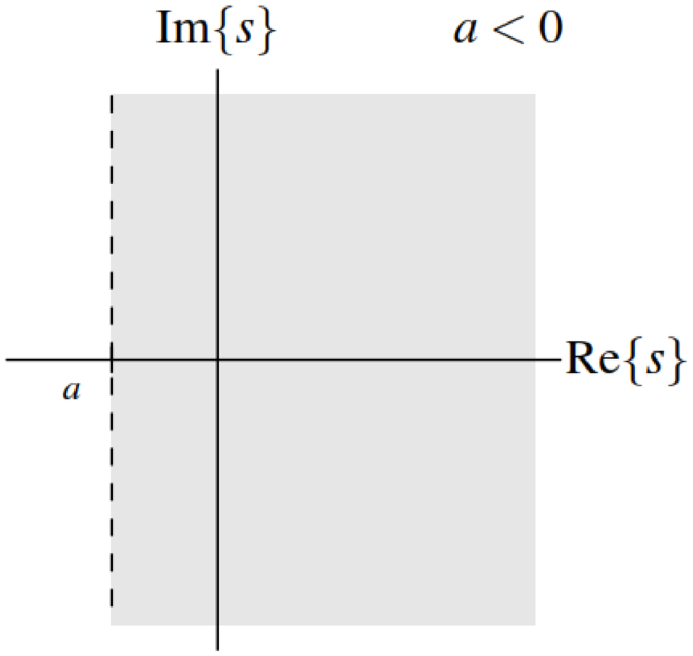
RIGHT-HALF PLANE (RHP)

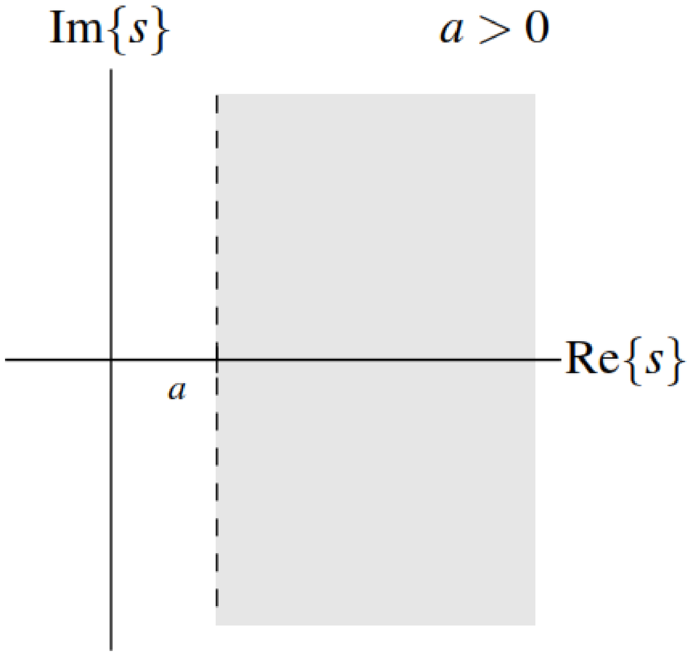
- The set R of all complex numbers s satisfying

$$\mathrm{Re}(s) > a$$

for some real constant a is said to be a **right-half plane (RHP)**.

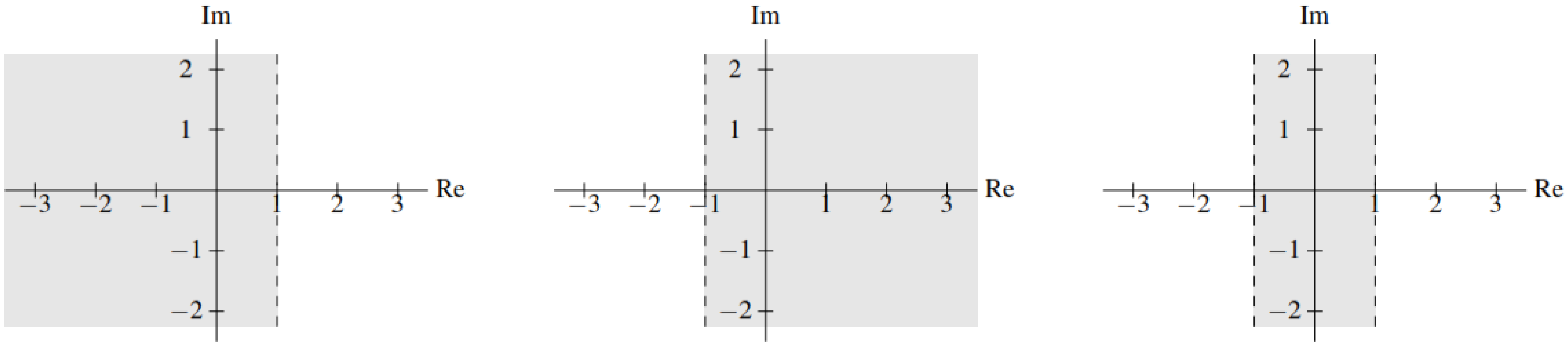
- Some examples of RHPs are shown below.





INTERSECTION OF SETS

- For two sets A and B, the **intersection** of A and B, denoted $A \cap B$, is the set of all points that are in both A and B.
- An illustrative example of set intersection is shown below.



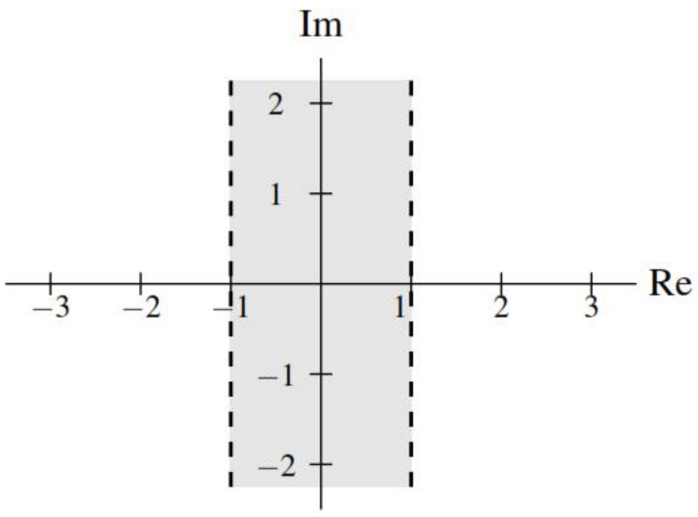
R_1 $\qquad$ R_2 $\qquad$ $R_1 \cap R_2$

ADDING A SCALAR TO A SET

- For a set S and a scalar constant a, $S+a$ denotes the set given by

$$S+a=\{z+a : z\in S\}$$

  (i.e., $S+a$ is the set formed by adding a to each element of S).
- Effectively, adding a scalar to a set applies a translation (i.e., shift) to the region associated with the set.
- An illustrative example is given below.



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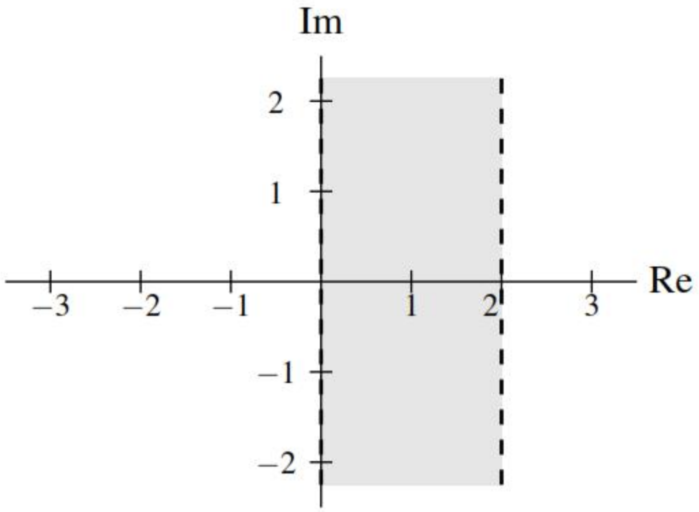


$R+1$

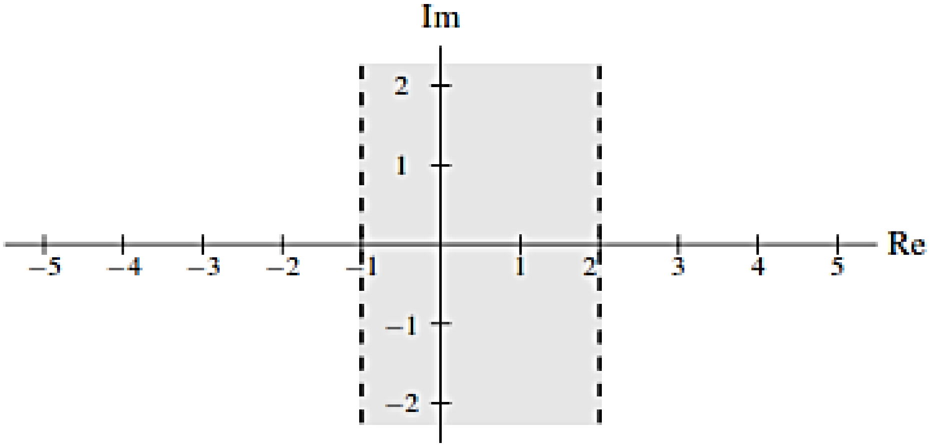
MULTIPLYING A SET BY A SCALAR

- For a set S and a scalar constant a, aS denotes the set given by

$$aS = \{az : z \in S\}$$

  (i..e, aS is the set formed by multiplying each element of S by a).
- Multiplying z by a affects z by: scaling by $|a|$ and rotating about the origin by $\arg a$.
- So, effectively, multiplying a set by a scalar applies a scaling and/or rotation to the region associated with the set.
- An illustrative example is given below.

MULTIPLYING A SET BY A SCALAR



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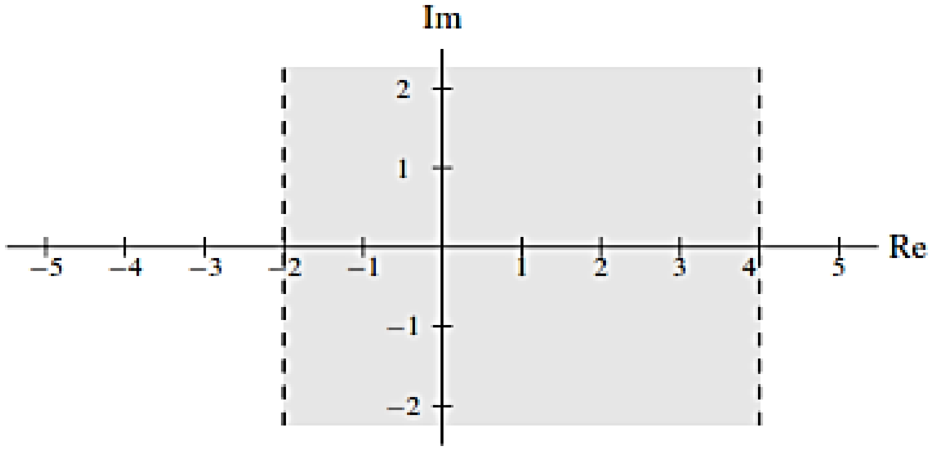
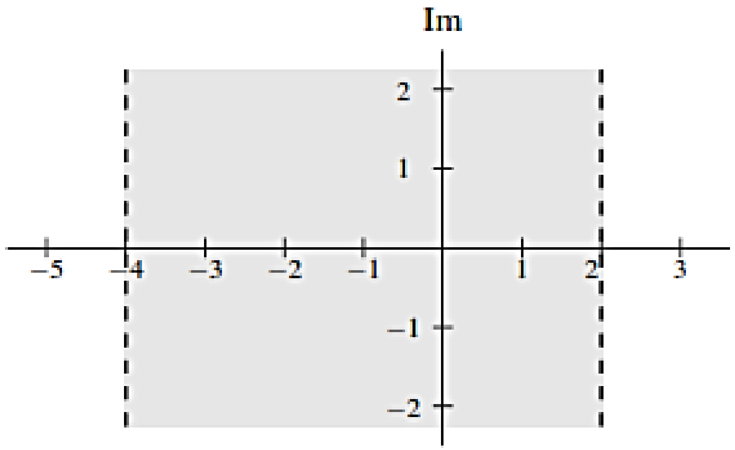


$2R$



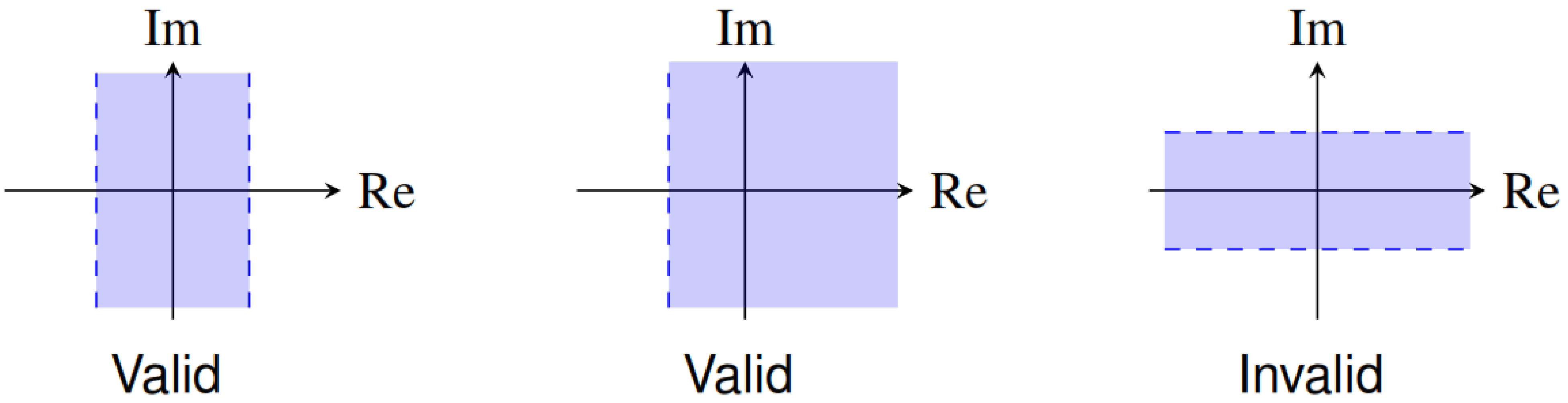
$-2R$

REGION OF CONVERGENCE (ROC)

- As we saw earlier, for a function x, the complete specification of its Laplace transform X requires not only an algebraic expression for X, but also the ROC associated with X.
- Two very different functions can have the same algebraic expressions for X.
- On the slides that follow, we will examine a number of key properties of the ROC of the Laplace transform.

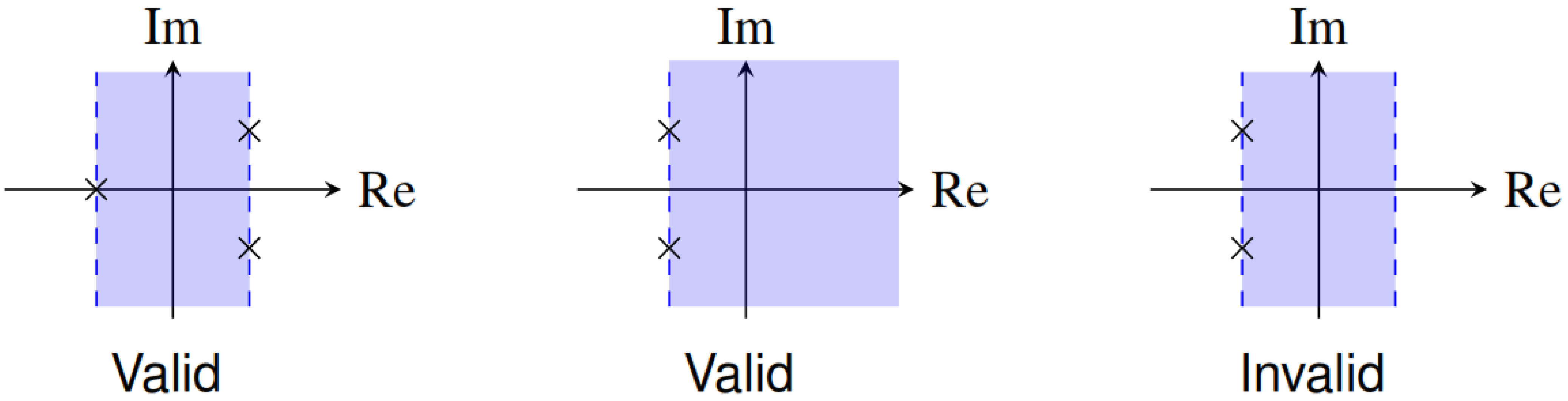
ROC PROPERTY 1: GENERAL FORM

- The ROC of a Laplace transform consists of ***strips parallel to the imaginary axis*** in the complex plane.
- That is, if a point s_0 is in the ROC, then the vertical line through s_0 (i.e., $\mathrm{Re}(s) = \mathrm{Re}(s_0)$) is also in the ROC.
- Some examples of sets that would be either valid or invalid as ROCs are shown below.



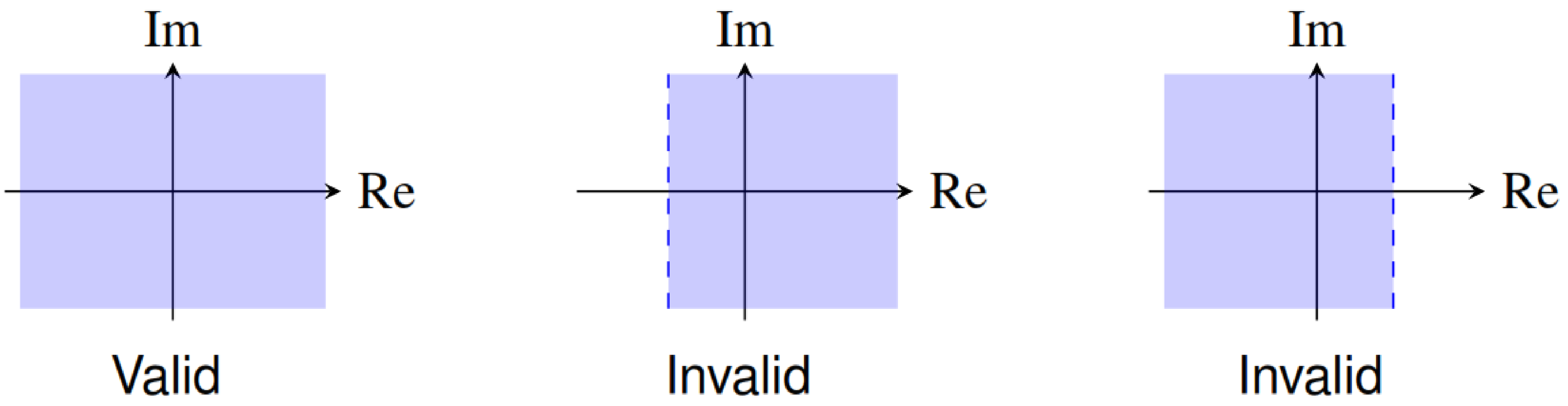
ROC PROPERTY 2: RATIONAL LAPLACE TRANSFORM

- If a Laplace transform X is a ***rational*** function, the ROC of X ***does not contain any poles*** and is ***bounded by poles or extends to infinity***.
- Some examples of sets that would be either valid or invalid as ROCs of rational Laplace transforms are shown below.



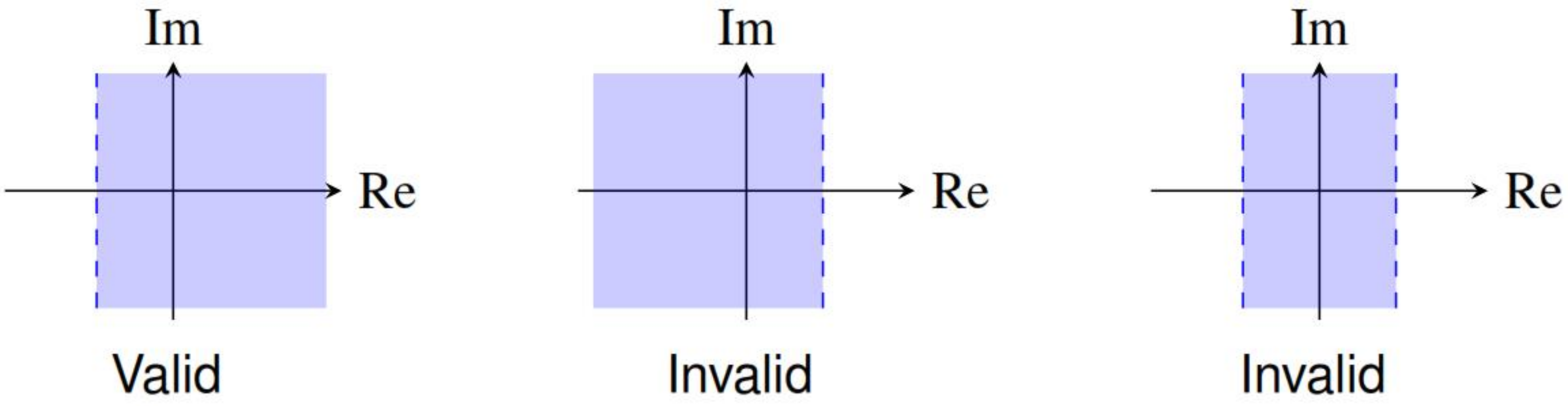
ROC PROPERTY 3: FINITE-DURATION FUNCTIONS

- If a function x is ***finite duration*** and its Laplace transform X converges for at least one point, then X converges for ***all*** points in the complex plane (i.e., the ROC is the entire complex plane).
- Some examples of sets that would be either valid or invalid as ROCs for X, if x is finite duration, are shown below.



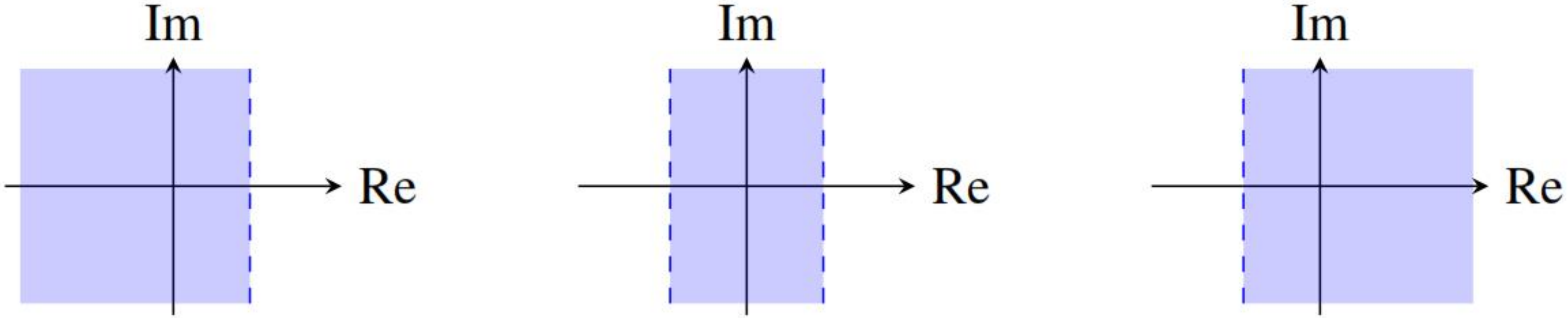
ROC PROPERTY 4: RIGHT-SIDED FUNCTIONS

- If a function x is ***right sided*** and the (vertical) line $\mathrm{Re}(s) = \sigma_0$ is in the ROC of the Laplace transform $X = \mathcal{L}x$, then all values of s for which $\mathrm{Re}(s) > \sigma_0$ must also be in the ROC (i.e., the ROC includes a RHP containing $\mathrm{Re}(s) = \sigma_0$).
- Thus, if x is ***right sided but not left sided***, the ROC of X is a ***RHP***.
- Some examples of sets that would be either valid or invalid as ROCs for X, if x is right sided but not left sided, are shown below.



ROC PROPERTY 5: LEFT-SIDED FUNCTIONS

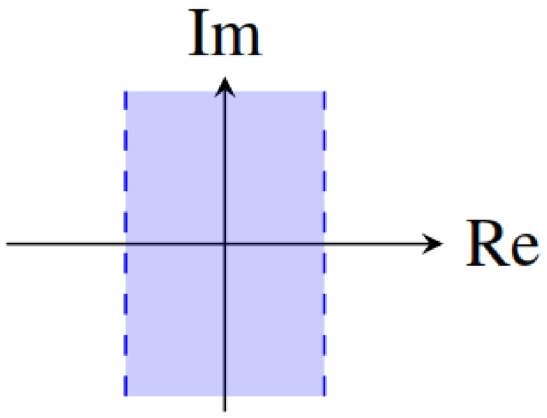
- If a function x is ***left sided*** and the (vertical) line $\mathrm{Re}(s) = \sigma_0$ is in the ROC of the Laplace transform $X = \mathcal{L}x$, then all values of s for which $\mathrm{Re}(s) < \sigma_0$ must also be in the ROC (i.e., the ROC includes a ***LHP*** containing $\mathrm{Re}(s) = \sigma_0$).
- Thus, if x is ***left sided but not right sided***, the ROC of X is a ***LHP***.
- Some examples of sets that would be either valid or invalid as ROCs for X, if x is left sided but not right sided, are shown below.



Valid Invalid Invalid

ROC PROPERTY 6: TWO-SIDED FUNCTIONS

- If a function x is ***two sided*** and the (vertical) line $\mathrm{Re}(s) = \sigma_0$ is in the ROC of the Laplace transform $X = \mathcal{L}x$, then the ROC will consist of a ***strip*** in the complex plane that includes the line $\mathrm{Re}(s) = \sigma_0$.
- Some examples of sets that would be either valid or invalid as ROCs for X, if x is two sided, are shown below.



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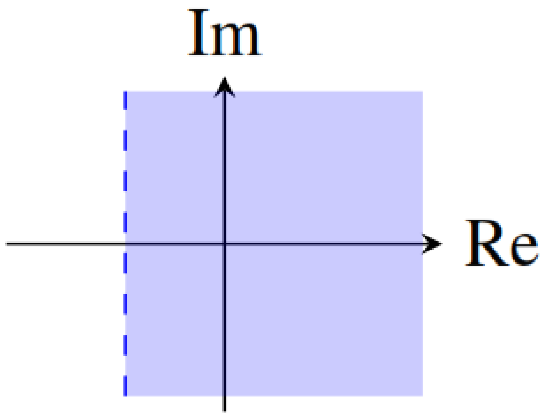

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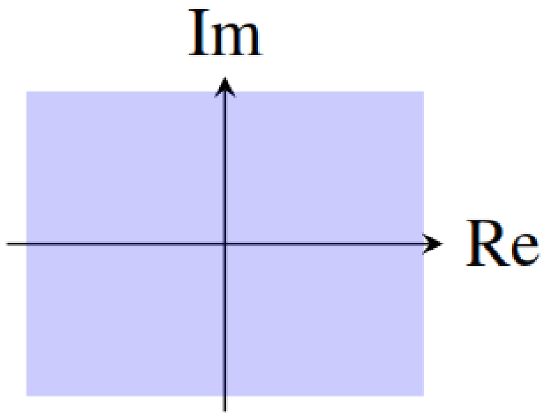

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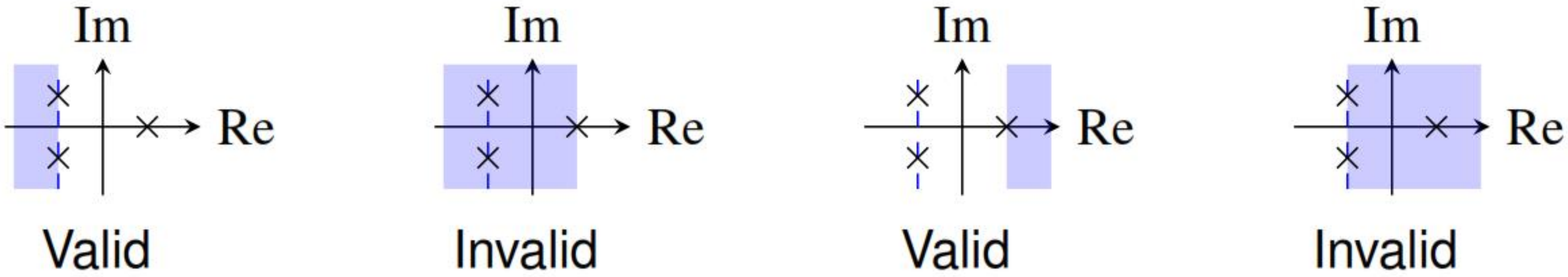
ROC PROPERTY 7: MORE ON RATIONAL LAPLACE TRANSFORMS

- If the Laplace transform X of a function x is ***rational*** (with at least one pole), then:
  1. If x is ***right sided***, the ROC of X is to the right of the rightmost pole of X (i.e., the ***RHP*** to the ***right of the rightmost pole***).
  2. If x is ***left sided***, the ROC of X is to the left of the leftmost pole of X (i.e., the ***LHP*** to the ***left of the leftmost pole***).
- This property is implied by properties 1, 2, 4, and 5.
- Some examples of sets that would be either valid or invalid as ROCs for X, if X is rational and x is left/right sided, are given below.

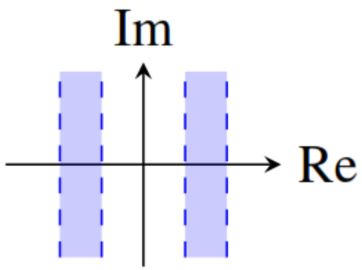


GENERAL FORM OF THE ROC

- To summarize the results of properties 3, 4, 5, and 6, if the Laplace transform X of the function x exists, the ROC of X depends on the left- and right-sidedness of x as follows:

x		
left sided	right sided	ROC of X
no	no	strip
no	yes	RHP
yes	no	LHP
yes	yes	everywhere

- Thus, we can infer that, if X exists, its ROC can only be of the form of a LHP, a RHP, a vertical strip, or the entire complex plane.
- For example, the sets shown below would not be valid as ROCs.



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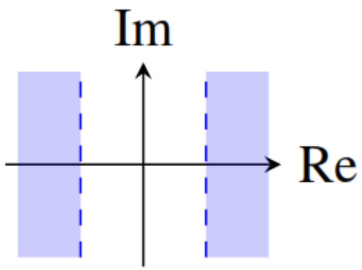

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Properties of the Laplace Transform

PROPERTIES OF THE LAPLACE TRANSFORM

Property	Time Domain	Laplace Domain	ROC
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(s)+a_2X_2(s)$	At least $R_1 \cap R_2$
Time-Domain Shifting	$x(t-t_0)$	$e^{-st_0}X(s)$	R
Laplace-Domain Shifting	$e^{s_0t}x(t)$	$X(s-s_0)$	$R+\mathrm{Re}(s_0)$
Time/Laplace-Domain Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	aR
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Time-Domain Convolution	$x_1 * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Laplace-Domain Differentiation	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Time-Domain Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R \cap \{\mathrm{Re}(s)>0\}$

Property	
Initial Value Theorem	$x(0^+)=\lim_{s\to\infty} sX(s)$
Final Value Theorem	$\lim_{t\to\infty} x(t)=\lim_{s\to 0} sX(s)$

LAPLACE TRANSFORM PAIRS

Pair	$x(t)$	$X(s)$	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re}(s) < 0$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}(s) > 0$
5	$-t^n u(-t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}(s) < 0$
6	$e^{-at} u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) > -a$
7	$-e^{-at} u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) < -a$

LAPLACE TRANSFORM PAIRS

Pair	$x(t)$	$X(s)$	ROC
8	$t^n e^{-at} u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}(s) > -a$
9	$-t^n e^{-at} u(-t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}(s) < -a$
10	$\cos(\omega_0 t) u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
11	$\sin(\omega_0 t) u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
12	$e^{-at} \cos(\omega_0 t) u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > -a$
13	$e^{-at} \sin(\omega_0 t) u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > -a$

LINEARITY

- If $x_1(t) \overset{\text{LT}}{\longleftrightarrow} X_1(s)$ with ROC R_1 and $x_2(t) \overset{\text{LT}}{\longleftrightarrow} X_2(s)$ with ROC R_2, then

  $$a_1x_1(t) + a_2x_2(t) \overset{\text{LT}}{\longleftrightarrow} a_1X_1(s) + a_2X_2(s) \text{ with ROC } R \text{ containing } R_1 \cap R_2,$$

  where a_1 and a_2 are arbitrary complex constants.
- This is known as the **linearity property** of the Laplace transform.
- The ROC R always contains $R_1 \cap R_2$ but can be larger (in the case that pole-zero cancellation occurs).

EXAMPLE

Find the Laplace transform X of the function $x = x_1 + x_2,$

$$x_1(t) = e^{-t}u(t) \quad \text{and} \quad x_2(t) = e^{-t}u(t) - e^{-2t}u(t).$$

EXAMPLE

Find the Laplace transform X of the function $x = x_1 + x_2,$

$$x_1(t) = e^{-t}u(t) \quad \text{and} \quad x_2(t) = e^{-t}u(t) - e^{-2t}u(t).$$

Solution. Using Laplace transform pairs

$$\begin{aligned} X_1(s) &= \mathcal{L}\{e^{-t}u(t)\}(s) \\ &= \frac{1}{s+1} \quad \text{for } \operatorname{Re}(s) > -1 \quad \text{and} \\ X_2(s) &= \mathcal{L}\{e^{-t}u(t) - e^{-2t}u(t)\}(s) \\ &= \mathcal{L}\{e^{-t}u(t)\}(s) - \mathcal{L}\{e^{-2t}u(t)\}(s) \\ &= \frac{1}{s+1} - \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -1 \\ &= \frac{1}{(s+1)(s+2)} \quad \text{for } \operatorname{Re}(s) > -1. \end{aligned}$$

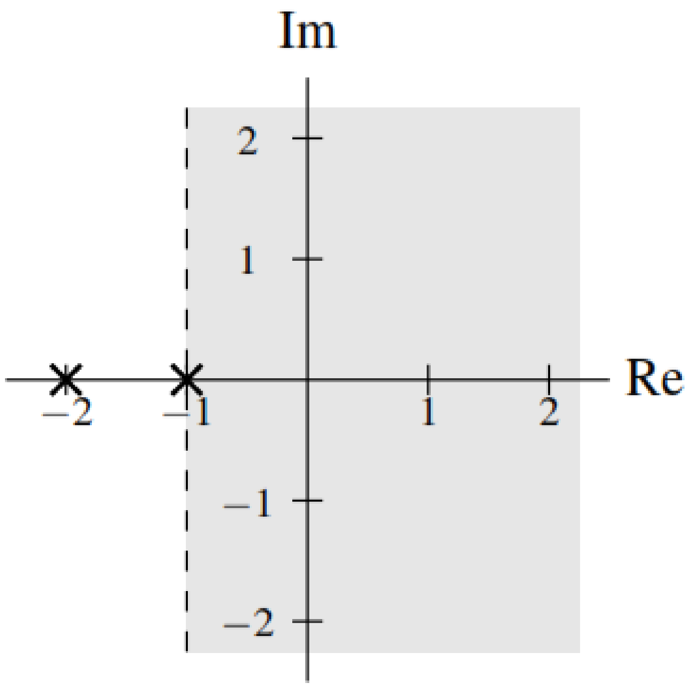
$$\begin{aligned} X(s) &= \mathcal{L}\{x_1 + x_2\}(s) \\ &= X_1(s) + X_2(s) \\ &= \frac{1}{s+1} + \frac{1}{(s+1)(s+2)} \\ &= \frac{s+2+1}{(s+1)(s+2)} \\ &= \frac{s+3}{(s+1)(s+2)}. \end{aligned}$$

EXAMPLE

Now, we must determine the ROC of X.

ROC of X must contain the intersection of the ROCs of X_1 and X_2. So, the ROC must contain $\operatorname{Re}(s) > -1$.

$$X(s) = \frac{s+3}{(s+1)(s+2)} \quad \text{for } \operatorname{Re}(s) > -1.$$



TIME-DOMAIN SHIFTING

- If $x(t) \xleftrightarrow{\text{LT}} X(s)$ with ROC R, then

$$x(t-t_0) \xleftrightarrow{\text{LT}} e^{-st_0}X(s) \text{ with ROC } R,$$

  where t_0 is an arbitrary real constant.

- This is known as the **time-domain shifting property** of the Laplace transform.

EXAMPLE

Find the Laplace transform X of

$$x(t) = u(t-1).$$

EXAMPLE

Find the Laplace transform X of

$$x(t) = u(t-1).$$

Solution.

$$u(t) \overset{\text{LT}}{\longleftrightarrow} 1/s \text{ for } \operatorname{Re}(s) > 0.$$

Using the time-domain shifting property, we can deduce

$$x(t) = u(t-1) \overset{\text{LT}}{\longleftrightarrow} X(s) = e^{-s}\left(\tfrac{1}{s}\right) \text{ for } \operatorname{Re}(s) > 0.$$

Therefore, we have

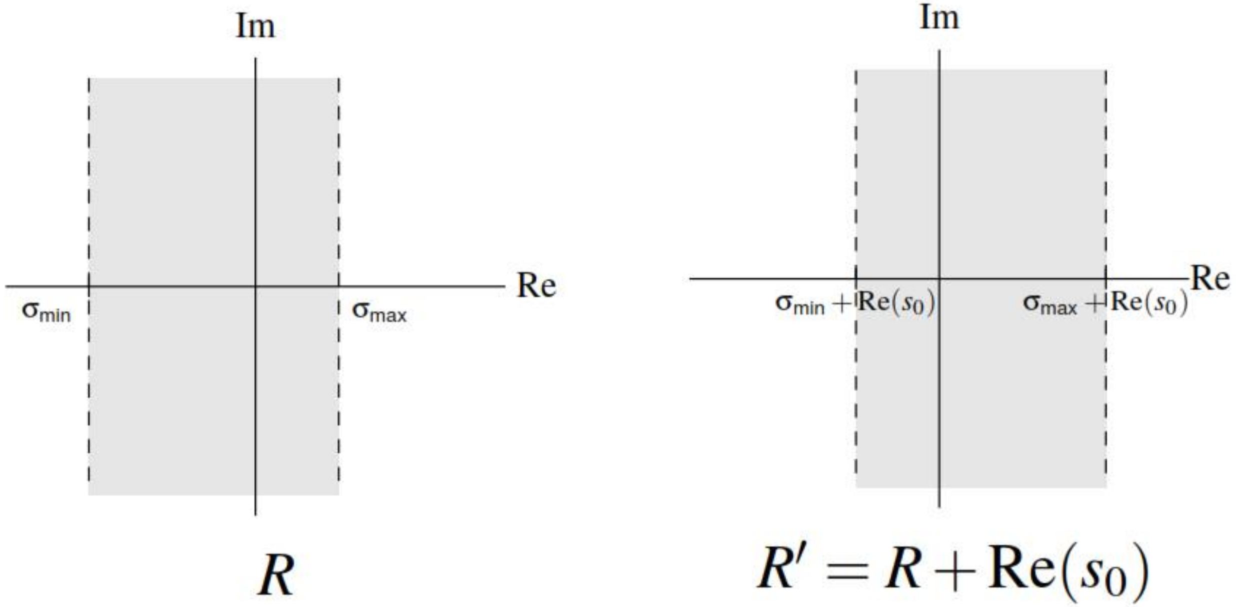
$$X(s) = \frac{e^{-s}}{s} \text{ for } \operatorname{Re}(s) > 0.$$

LAPLACE-DOMAIN SHIFTING

- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

$$e^{s_0 t}x(t) \overset{\text{LT}}{\longleftrightarrow} X(s-s_0) \text{ with ROC } R' = R + \mathrm{Re}(s_0),$$

  where s_0 is an arbitrary complex constant.
- This is known as the **Laplace-domain shifting property** of the Laplace transform.
- As illustrated below, the ROC R is ***shifted*** right by $\mathrm{Re}(s_0)$.



EXAMPLE

$$e^{-|t|} \xleftrightarrow{\text{LT}} \frac{2}{1-s^2} \quad \text{for } -1 < \operatorname{Re}(s) < 1,$$

find the Laplace transform X of

$$x(t) = e^{5t}e^{-|t|}.$$

EXAMPLE

$$e^{-|t|} \overset{\text{LT}}{\longleftrightarrow} \frac{2}{1-s^2} \quad \text{for } -1 < \text{Re}(s) < 1,$$

find the Laplace transform X of

$$x(t) = e^{5t}e^{-|t|}.$$

Solution.

Using the Laplace-domain shifting property, we can deduce

$$x(t) = e^{5t}e^{-|t|} \overset{\text{LT}}{\longleftrightarrow} X(s) = \frac{2}{1-(s-5)^2} \quad \text{for } -1+5 < \text{Re}(s) < 1+5,$$

$$X(s) = \frac{2}{1-(s-5)^2} \quad \text{for } 4 < \text{Re}(s) < 6.$$

$$X(s) = \frac{2}{1-(s-5)^2} = \frac{2}{1-(s^2-10s+25)} = \frac{2}{-s^2+10s-24} = \frac{-2}{s^2-10s+24} = \frac{-2}{(s-6)(s-4)}.$$

Therefore, we have

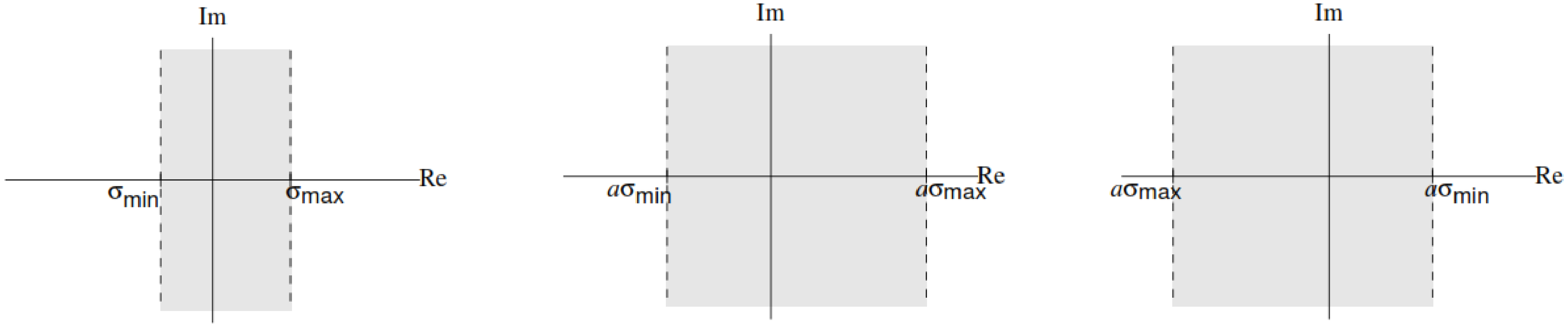
$$X(s) = \frac{-2}{(s-4)(s-6)} \quad \text{for } 4 < \text{Re}(s) < 6.$$

TIME-DOMAIN/LAPLACE-DOMAIN SHIFTING

- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

$$x(at) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right) \quad \text{with ROC } R' = aR,$$

  where a is a nonzero real constant.
- This is known as the **(time-domain/Laplace-domain) scaling property** of the Laplace transform.
- As illustrated below, the ROC R is ***scaled*** and ***possibly flipped*** left to right.



R $R' = aR,\ a > 0$ $R' = aR,\ a < 0$

EXAMPLE

Using only properties of the Laplace transform

$$e^{-|t|} \xleftrightarrow{\text{LT}} \frac{2}{1-s^2} \quad \text{for } -1 < \text{Re}(s) < 1,$$

find the Laplace transform X of the function

$$x(t) = e^{-|3t|}.$$

EXAMPLE

Using only properties of the Laplace transform

$$e^{-|t|} \xleftrightarrow{\text{LT}} \frac{2}{1-s^2} \quad \text{for } -1 < \text{Re}(s) < 1,$$

find the Laplace transform X of the function

$$x(t) = e^{-|3t|}.$$

Solution. We are given

$$e^{-|t|} \xleftrightarrow{\text{LT}} \frac{2}{1-s^2} \quad \text{for } -1 < \text{Re}(s) < 1.$$

Using the time-domain scaling property, we can deduce

$$x(t) = e^{-|3t|} \xleftrightarrow{\text{LT}} X(s) = \frac{1}{|3|} \frac{2}{1-(\frac{s}{3})^2} \quad \text{for } 3(-1) < \text{Re}(s) < 3(1).$$

EXAMPLE

Thus, we have

$$X(s) = \frac{2}{3\left[1-\left(\frac{s}{3}\right)^2\right]} \quad \text{for } -3 < \operatorname{Re}(s) < 3.$$

Simplifying, we have

$$X(s) = \frac{2}{3(1-\frac{s^2}{9})} = \frac{2}{3(\frac{9-s^2}{9})} = \frac{2(9)}{3(9-s^2)} = \frac{6}{9-s^2} = \frac{-6}{(s+3)(s-3)}.$$

Therefore, we have

$$X(s) = \frac{-6}{(s+3)(s-3)} \quad \text{for } -3 < \operatorname{Re}(s) < 3.$$

CONJUGATION

- If $x(t) \xleftrightarrow{\text{LT}} X(s)$ with ROC R, then

$$x^*(t) \xleftrightarrow{\text{LT}} X^*(s^*) \quad \text{with ROC } R.$$

- This is known as the **conjugation property** of the Laplace transform.

EXAMPLE

Using only properties of the Laplace transform and the transform pair

$$e^{(-1-j)t}u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+1+j} \text{ for } \operatorname{Re}(s) > -1,$$

find the Laplace transform X of

$$x(t) = e^{(-1+j)t}u(t).$$

EXAMPLE

Using only properties of the Laplace transform and the transform pair

$$e^{(-1-j)t}u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+1+j} \text{ for } \operatorname{Re}(s) > -1,$$

find the Laplace transform X of

$$x(t) = e^{(-1+j)t}u(t).$$

Solution. To begin, let $v(t) = e^{(-1-j)t}u(t)$

First, we determine the relationship between x and v. We have

$$\begin{aligned}
x(t) &= \left(\left(e^{(-1+j)t}u(t)\right)^*\right)^* \\
&= \left(\left(e^{(-1+j)t}\right)^* u^*(t)\right)^* \\
&= \left[e^{(-1-j)t}u(t)\right]^* \\
&= v^*(t).
\end{aligned}$$

EXAMPLE

Thus, $x = v^*$. Next, we find the Laplace transform of x. We are given

$$v(t) = e^{(-1-j)t}u(t) \overset{\text{LT}}{\longleftrightarrow} V(s) = \frac{1}{s+1+j} \text{ for } \operatorname{Re}(s) > -1.$$

Using the conjugation property, we can deduce

$$x(t) = e^{(-1+j)t}u(t) \overset{\text{LT}}{\longleftrightarrow} X(s) = \left(\frac{1}{s^*+1+j}\right)^* \text{ for } \operatorname{Re}(s) > -1.$$

Simplifying the algebraic expression for X, we have

$$X(s) = \left(\frac{1}{s^*+1+j}\right)^* = \frac{1^*}{(s^*+1+j)^*} = \frac{1}{s+1-j}.$$

Therefore, we can conclude

$$X(s) = \frac{1}{s+1-j} \text{ for } \operatorname{Re}(s) > -1.$$

TIME-DOMAIN CONVOLUTION

- If $x_1(t) \overset{\text{LT}}{\longleftrightarrow} X_1(s)$ with ROC R_1 and $x_2(t) \overset{\text{LT}}{\longleftrightarrow} X_2(s)$ with ROC R_2, then

$$x_1 * x_2(t) \overset{\text{LT}}{\longleftrightarrow} X_1(s)X_2(s) \text{ with ROC } R \text{ containing } R_1 \cap R_2.$$

- This is known as the **time-domain convolution property** of the Laplace transform.
- The ROC R always contains $R_1 \cap R_2$ but can be larger than this intersection (if pole-zero cancellation occurs).
- Convolution in the time domain becomes ***multiplication*** in the Laplace domain.
- Consequently, it is often much easier to work with LTI systems in the Laplace domain, rather than the time domain.

EXAMPLE

Find the Laplace transform X of the function $x(t) = x_1 * x_2(t)$,

$$x_1(t) = \sin(3t)u(t) \quad \text{and} \quad x_2(t) = tu(t).$$

EXAMPLE

Find the Laplace transform X of the function $x(t) = x_1 * x_2(t)$,

$$x_1(t) = \sin(3t)u(t) \quad \text{and} \quad x_2(t) = tu(t).$$

Solution.

$$x_1(t) = \sin(3t)u(t) \overset{\text{LT}}{\longleftrightarrow} X_1(s) = \frac{3}{s^2+9} \text{ for } \operatorname{Re}(s) > 0 \quad \text{and}$$

$$x_2(t) = tu(t) \overset{\text{LT}}{\longleftrightarrow} X_2(s) = \frac{1}{s^2} \text{ for } \operatorname{Re}(s) > 0.$$

Using the time-domain convolution property, we have

$$x(t) \overset{\text{LT}}{\longleftrightarrow} X(s) = \left(\frac{3}{s^2+9}\right)\left(\frac{1}{s^2}\right) \text{ for } \{\operatorname{Re}(s) > 0\} \cap \{\operatorname{Re}(s) > 0\}.$$

The ROC of X is $\{\operatorname{Re}(s) > 0\} \cap \{\operatorname{Re}(s) > 0\}$

$$X(s) = \frac{3}{s^2(s^2+9)} \text{ for } \operatorname{Re}(s) > 0.$$

TIME-DOMAIN DIFFERENTIATION

- If $x(t) \overset{\text{LT}}{\longleftrightarrow} X(s)$ with ROC R, then

$$\frac{dx(t)}{dt} \overset{\text{LT}}{\longleftrightarrow} sX(s) \text{ with ROC } R' \text{ containing } R.$$

- This is known as the **time-domain differentiation property** of the Laplace transform.
- The ROC R' always contains R but can be larger than R (if pole-zero cancellation occurs).
- Differentiation in the time domain becomes ***multiplication by s*** in the Laplace domain.
- Consequently, it can often be much easier to work with differential equations in the Laplace domain, rather than the time domain.

EXAMPLE

Find the Laplace transform X of the function

$$x(t) = \tfrac{d}{dt}\delta(t).$$

EXAMPLE

Find the Laplace transform X of the function

$$x(t) = \frac{d}{dt}\delta(t).$$

Solution.

we have that

$$\delta(t) \overset{\text{LT}}{\longleftrightarrow} 1 \text{ for all } s.$$

Using the time-domain differentiation property, we can deduce

$$x(t) = \frac{d}{dt}\delta(t) \overset{\text{LT}}{\longleftrightarrow} X(s) = s(1) \text{ for all } s.$$

Therefore, we have

$$X(s) = s \quad \text{for all } s.$$

LAPLACE-DOMAIN DIFFERENTIATION

- If $x(t) \xleftrightarrow{\text{LT}} X(s)$ with ROC R, then

$$-tx(t) \xleftrightarrow{\text{LT}} \frac{dX(s)}{ds} \quad \text{with ROC } R.$$

- This is known as the **Laplace-domain differentiation property** of the Laplace transform.

EXAMPLE

$$e^{-2t}u(t) \xleftrightarrow{\text{LT}} \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2,$$

find the Laplace transform X of the function $x(t) = te^{-2t}u(t)$.

$$x(t) = te^{-2t}u(t).$$

EXAMPLE

$$e^{-2t}u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2,$$

find the Laplace transform X of the function $x(t) = te^{-2t}u(t)$.

Solution. We are given

$$e^{-2t}u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s+2} \quad \text{for } \operatorname{Re}(s) > -2.$$

Using the Laplace-domain differentiation and linearity properties, we can deduce

$$x(t) = te^{-2t}u(t) \overset{\text{LT}}{\longleftrightarrow} X(s) = -\tfrac{d}{ds}\left(\frac{1}{s+2}\right) \quad \text{for } \operatorname{Re}(s) > -2.$$

Simplifying the algebraic expression for X, we have

$$X(s) = -\tfrac{d}{ds}\left(\frac{1}{s+2}\right) = -\tfrac{d}{ds}(s+2)^{-1} = (-1)(-1)(s+2)^{-2} = \frac{1}{(s+2)^2}.$$

Therefore, we conclude

$$X(s) = \frac{1}{(s+2)^2} \quad \text{for } \operatorname{Re}(s) > -2.$$

TIME-DOMAIN INTEGRATION

- If $x(t) \xleftrightarrow{\text{LT}} X(s)$ with ROC R, then

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\text{LT}} \frac{1}{s}X(s) \text{ with ROC } R' \text{ containing } R \cap \{\mathrm{Re}(s) > 0\}.$$

- This is known as the **time-domain integration property** of the Laplace transform.
- The ROC R' always contains at least $R \cap \{\mathrm{Re}(s) > 0\}$ but can be larger (if pole-zero cancellation occurs).
- Integration in the time domain becomes ***division by s*** in the Laplace domain.
- Consequently, it is often much easier to work with integral equations in the Laplace domain, rather than the time domain.

EXAMPLE

Find the Laplace transform X of the function $x(t) = \int_{-\infty}^{t} e^{-2\tau} \sin(\tau) u(\tau) d\tau.$

EXAMPLE

Find the Laplace transform X of the function $x(t) = \int_{-\infty}^{t} e^{-2\tau} \sin(\tau) u(\tau) d\tau.$

Solution.

$$e^{-2t} \sin(t) u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{(s+2)^2+1} \text{ for } \operatorname{Re}(s) > -2.$$

Using the time-domain integration property, we can deduce

$$x(t) = \int_{-\infty}^{t} e^{-2\tau} \sin(\tau) u(\tau) d\tau \overset{\text{LT}}{\longleftrightarrow} X(s) = \frac{1}{s} \left[\frac{1}{(s+2)^2+1} \right] \text{ for } \{\operatorname{Re}(s) > -2\} \cap \{\operatorname{Re}(s) > 0\}.$$

The ROC of X is $\{\operatorname{Re}(s) > -2\} \cap \{\operatorname{Re}(s) > 0\}$ (as opposed to a superset thereof), since no pole-zero cancellation takes place. Simplifying the algebraic expression for X, we have

$$X(s) = \frac{1}{s} \left[\frac{1}{(s+2)^2+1} \right] = \frac{1}{s} \left(\frac{1}{s^2+4s+4+1} \right) = \frac{1}{s} \left(\frac{1}{s^2+4s+5} \right).$$

Therefore, we have

$$X(s) = \frac{1}{s(s^2+4s+5)} \text{ for } \operatorname{Re}(s) > 0.$$

INITIAL VALUE THEOREM

- For a function x with Laplace transform X, if x is ***causal*** and contains ***no impulses or higher order singularities at the origin***, then

$$x(0^+) = \lim_{s\to\infty} sX(s),$$

  where $x(0^+)$ denotes the limit of $x(t)$ as t approaches zero from positive values of t.
- This result is known as the **initial value theorem**.
- In situations where X is known but x is not, the initial value theorem eliminates the need to explicitly find x by an inverse Laplace transform calculation in order to evaluate $x(0^+)$.
- In practice, the values of functions at the origin are frequently of interest, as such values often convey information about the initial state of systems.
- The initial value theorem can sometimes also be helpful in checking for errors in Laplace transform calculations.

FINAL VALUE THEOREM

- For a function x with Laplace transform X, if x is ***causal*** and $x(t)$ has a ***finite limit*** as $t \to \infty$, then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s).$$

- This result is known as the **final value theorem**.
- In situations where X is known but x is not, the final value theorem eliminates the need to explicitly find x by an inverse Laplace transform calculation in order to evaluate $\lim_{t\to\infty} x(t)$.
- In practice, the values of functions at infinity are frequently of interest, as such values often convey information about the steady-state behavior of systems.
- The final value theorem can sometimes also be helpful in checking for errors in Laplace transform calculations.

Determination of Inverse Laplace Transform

FINDING INVERSE LAPLACE TRANSFORM

- Recall that the inverse Laplace transform x of X is given by

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} \, ds,$$

  where $\mathrm{Re}(s) = \sigma$ is in the ROC of X.
- Unfortunately, the above contour integration can often be ***quite tedious*** to compute.
- Consequently, we do not usually compute the inverse Laplace transform directly using the above equation.
- For rational functions, the inverse Laplace transform can be more easily computed using ***partial fraction expansions***.
- Using a partial fraction expansion, we can express a rational function as a sum of lower-order rational functions whose inverse Laplace transforms can typically be found in tables.

Laplace Transform and LTI Systems

SYSTEM FUNCTION OF LTI SYSTEMS

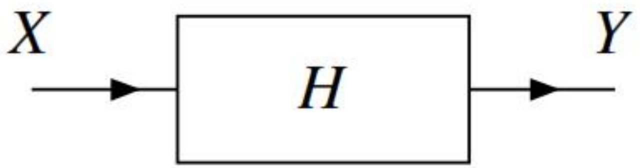
- Consider a LTI system with input x, output y, and impulse response h. Let X, Y, and H denote the Laplace transforms of x, y, and h, respectively.
- Since $y(t) = x * h(t)$, the system is characterized in the Laplace domain by

$$Y(s) = X(s)H(s).$$

- As a matter of terminology, we refer to H as the **system function** (or **transfer function**) of the system (i.e., the system function is the Laplace transform of the impulse response).
- A LTI system is ***completely characterized*** by its system function H.
- When viewed in the Laplace domain, a LTI system forms its output by multiplying its input with its system function.
- If the ROC of H includes the imaginary axis, then $H(j\omega)$ is the ***frequency response*** of the LTI system.

BLOCK DIAGRAM REPRESENTATIONS OF LTI SYSTEMS

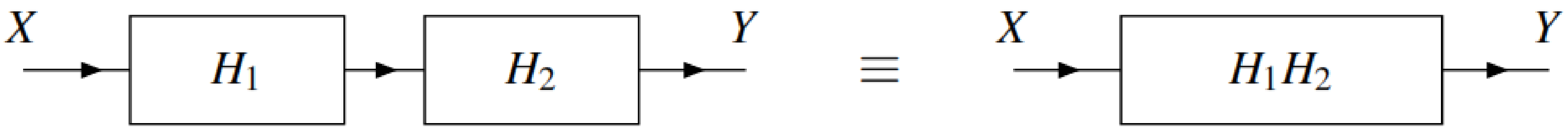
- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the Laplace transforms of x, y, and h, respectively.
- Often, it is convenient to represent such a system in block diagram form in the Laplace domain as shown below.



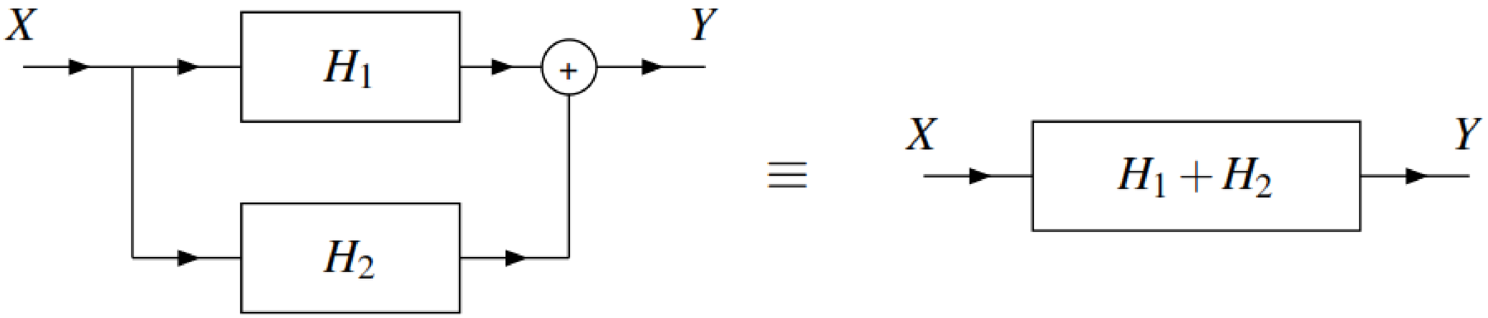
- Since a LTI system is completely characterized by its system function, we typically label the system with this quantity.

INTERCONNECTION OF LTI SYSTEMS

- The *series* interconnection of the LTI systems with system functions H_1 and H_2 is the LTI system with system function H_1H_2. That is, we have the equivalence shown below.



- The *parallel* interconnection of the LTI systems with system functions H_1 and H_2 is the LTI system with the system function $H_1 + H_2$. That is, we have the equivalence shown below.



CAUSALITY

- If a LTI system is ***causal***, its impulse response is causal, and therefore ***right sided***. From this, we have the result below.
- **Theorem.** The ROC associated with the system function of a ***causal*** LTI system is a ***RHP*** or the ***entire complex plane***.
- In general, the ***converse*** of the above theorem is ***not necessarily true***. That is, if the ROC of the system function is a RHP or the entire complex plane, it is not necessarily true that the system is causal.
- If the system function is ***rational***, however, we have that the converse does hold, as indicated by the theorem below.
- **Theorem.** For a LTI system with a ***rational*** system function H, ***causality*** of the system is ***equivalent*** to the ROC of H being the ***RHP to the right of the rightmost pole*** or, if H has no poles, the entire complex plane.

BIBO STABILITY

- Whether or not a system is BIBO stable depends on the ROC of its system function.
- **Theorem.** A LTI system is ***BIBO stable*** if and only if the ROC of its system function H contains the ***imaginary axis*** (i.e., $\mathrm{Re}(s) = 0$).
- **Theorem.** A ***causal*** LTI system with a (proper) ***rational*** system function H is BIBO stable if and only if all of the poles of H lie in the left half of the plane (i.e., all of the poles have ***negative real parts***).

INVERTIBILITY

- A LTI system $\mathcal{H}$ with system function H is invertible if and only if there exists another LTI system with system function H_{inv} such that

$$H(s)H_{\text{inv}}(s) = 1,$$

in which case H_{inv} is the system function of $\mathcal{H}^{-1}$ and

$$H_{\text{inv}}(s) = \frac{1}{H(s)}.$$

- Since distinct systems can have identical system functions (but with differing ROCs), the inverse of a LTI system is ***not necessarily unique***.
- In practice, however, we often desire a stable and/or causal system. So, although multiple inverse systems may exist, we are frequently only interested in ***one specific choice*** of inverse system (due to these additional constraints of stability and/or causality).

LTI SYSTEMS AND DIFFERENTIAL EQUATIONS

- Many LTI systems of practical interest can be represented using an ***Nth-order linear differential equation with constant coefficients***.
- Consider a system with input x and output y that is characterized by an equation of the form

$$\sum_{k=0}^{N} b_k \left(\frac{d}{dt}\right)^k y(t) = \sum_{k=0}^{M} a_k \left(\frac{d}{dt}\right)^k x(t),$$

  where the a_k and b_k are complex constants and $M \leq N$.
- Let h denote the impulse response of the system, and let X, Y, and H denote the Laplace transforms of x, y, and h, respectively.
- One can show that H is given by

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} a_k s^k}{\sum_{k=0}^{N} b_k s^k}.$$

- Observe that, for a system of the form considered above, the system function is always ***rational***.