HO CHI MINH CITY INTERNATIONAL UNIVERSITY

FINAL EXAMINATION

Academic year 2011-2012, Semester 1

Duration: 120 minutes

SUBJECT: **Differential Equations**	
Chair of Department of Mathematics	Lecturer:
Signature:	Signature:
Full name: Prof. Phan Quoc Khanh	Full name: Dr. Pham Huu Anh Ngoc

Instructions:

- *Open-book examination. Laptops are NOT allowed.*

Question 1. (15 marks) At any time t, the rate of growth of the population N of deer in a state park is proportional to the product of N and $L-N$, that is

$$\frac{dN}{dt} = kN(L-N)$$

where k is the proportionality constant and $L = 500$ is the maximum number of deer the park can maintain. When $t = 0, N = 100$ and when $t = 4, N = 200$. Write N as a function of t.

Question 2. (a) (10 marks) Find a real number A such that $y(x) := Ax^2$ is a solution of the linear differential equation

$$x^2y'' - 3xy' + 4y = 0.$$

(b) (15 marks) Solve the following differential equation

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0, \qquad x \in (0, \infty).$$

Question 3. (25 marks) Find the general solution of the following differential equation

$$y''' + y'' + y' + y = e^{-x} + 4x.$$

Question 4. (20 marks) Solve the linear system of differential equations

$$\begin{cases} \frac{dx}{dt} &= x + 2y \\ \frac{dy}{dt} &= -2x + y. \end{cases}$$

Question 5. (15 marks) Use the method of variation of parameters to find a particular solution of the following differential equation

$$y'' - y = \frac{e^x}{e^x + 1}.$$

Solve the given differential equation.

End.

SOLUTIONS:

Question 1. Let N be the population N of deer at time t. Then we have

$$\frac{dN}{dt} = kN(L-N) = kN(500-N), \qquad t \geq 0,$$

where k is a constant. Solving the above equation, we have

$$\frac{N}{500-N} = Ce^{500kt} \qquad t \geq 0.$$

By the given assumptions, $N(0) = 100$ and $N(4) = 200$. Then we have $C = \frac{1}{4}$ and $k = \frac{1}{2000}\ln\frac{8}{3}$. Therefore,

$$N(t) = \frac{500(\frac{8}{3})^{\frac{t}{4}}}{4+(\frac{8}{3})^{\frac{t}{4}}}, \qquad t \geq 0.$$

Question 2. a) $A = 1$.

b) By a), $y_1(x) = x^2$ is a solution of $y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$.

The second solution of $y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$ is given by $y_2(x) = x^2\ln x$. Thus, the general solution is

$$y(x) = C_1x^2 + C_2x^2\ln x$$

Question 3.

The general solution of the corresponding homogeneous equation is

$$y(x) = c_1e^{-x} + c_2\cos x + c_3\sin x.$$

A particular solution of $y''' + y'' + y' + y = e^{-x} + 4x$ is

$$y_p(x) = \frac{1}{2}xe^{-x} + 4(x-1).$$

Thus, the general solution of $y''' + y'' + y' + y = e^{-x} + 4x$ is

$$y(x) = \frac{1}{2}xe^{-x} + 4(x-1) + c_1e^{-x} + c_2\cos x + c_3\sin x.$$

Question 4. The general solution of the given system is

$$x(t) = e^t(c_1\cos 2t + c_2\sin 2t); \qquad y(t) = e^t(-c_1\sin 2t + c_2\cos 2t).$$

Question 5. The general solution of $y'' - y = 0$ is

$$y(x) = C_1e^x + C_2e^{-x}.$$

Using the method of variation of parameters, we get a particular solution of $y'' - y = \frac{e^x}{e^x+1}$:

$$y_p = \frac{1}{2}(e^x[x - \ln(e^x+1)] + e^{-x}[\ln(e^x+1) - e^x]).$$

Thus, the general solution of $y'' - y = \frac{e^x}{e^x+1}$ is given by

$$y(x) = \frac{1}{2}(e^x[x - \ln(e^x+1)] + e^{-x}[\ln(e^x+1) - e^x]) + C_1e^x + C_2e^{-x}.$$