Chapter 5
Continuous Time Fourier Transform

MOTIVATION FOR THE FOURIER TRANSFORM

- The (CT) Fourier series provide an extremely useful representation for periodic functions.
- Often, however, we need to deal with functions that are not periodic.
- A more general tool than the Fourier series is needed in this case.
- The (CT) Fourier transform can be used to represent both periodic and aperiodic functions.
- Since the Fourier transform is essentially derived from Fourier series through a limiting process, the Fourier transform has many similarities with Fourier series.

Fourier Transform

DEVELOPMENT OF THE FOURIER TRANSFORM [APERIODIC CASE]

- The (CT) Fourier series is an extremely useful function representation.
- Unfortunately, this function representation can only be used for periodic functions, since a Fourier series is inherently periodic.
- Many functions are not periodic, however.
- Rather than abandoning Fourier series, one might wonder if we can somehow use Fourier series to develop a representation that can be applied to aperiodic functions.
- By viewing an aperiodic function as the limiting case of a T-periodic function where $T \to \infty$, we can use the Fourier series to develop a function representation that can be used for aperiodic functions, known as the Fourier transform.

DEVELOPMENT OF THE FOURIER TRANSFORM [APERIODIC CASE]

- Recall that the Fourier series representation of a T-periodic function x is given by

$$x(t) = \sum_{k=-\infty}^{\infty} \underbrace{\left(\frac{1}{T} \int_{-T/2}^{T/2} x(\tau) e^{-jk(2\pi/T)\tau} d\tau \right)}_{c_k} e^{jk(2\pi/T)t}.$$

- In the above representation, if we take the limit as $T \to \infty$, we obtain

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\left(\int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau \right)}_{X(\omega)} e^{j\omega t} d\omega$$

(i.e., as $T \to \infty$, the outer summation becomes an integral, $\frac{1}{T}$ becomes $\frac{1}{2\pi} d\omega$, and $\left(\frac{2\pi}{T}\right) k$ becomes ω).

- This representation for aperiodic functions is known as the Fourier transform representation.

CT FOURIER TRANSFORM (CTFT)

- The (CT) **Fourier transform** of the function x, denoted $\mathcal{F}x$ or X, is given by

$$\mathcal{F}x(\omega) = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt.$$

- The preceding equation is sometimes referred to as **Fourier transform analysis equation** (or **forward Fourier transform equation**).
- The **inverse Fourier transform** of X, denoted $\mathcal{F}^{-1}X$ or x, is given by

$$\mathcal{F}^{-1}X(t) = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega.$$

- The preceding equation is sometimes referred to as the **Fourier transform synthesis equation** (or **inverse Fourier transform equation**).
- As a matter of notation, to denote that a function x has the Fourier transform X, we write $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$.
- A function x and its Fourier transform X constitute what is called a **Fourier transform pair**.

EXAMPLE

Find the Fourier transform X of the function

$$x(t) = A\delta(t - t_0),$$

where A and t_0 are real constants. Then, from this result, write the Fourier transform representation of x.

Solution. From the definition of the Fourier transform, we can write

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} A\delta(t - t_0)e^{-j\omega t}\,dt \\ &= A\int_{-\infty}^{\infty} \delta(t - t_0)e^{-j\omega t}\,dt. \end{aligned}$$

Using the sifting property of the unit-impulse function, we can simplify the above result to obtain

$$\begin{aligned} X(\omega) &= A\left[e^{-j\omega t}\right]\Big|_{t=t_0} \\ &= Ae^{-j\omega t_0}. \end{aligned}$$

- **Sifting property.** For any continuous function x and any real constant t_0,

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)\,dt = x(t_0).$$

EXAMPLE

Find the inverse Fourier transform x

$$X(\omega) = 2\pi A\delta(\omega - \omega_0),$$

where A and ω_0 are real constants.

Solution. From the definition of the inverse Fourier transform, we can write

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi A\delta(\omega - \omega_0)e^{j\omega t}\,d\omega \\ &= A\int_{-\infty}^{\infty} \delta(\omega - \omega_0)e^{j\omega t}\,d\omega. \end{aligned}$$

Using the sifting property of the unit-impulse function, we can simplify the preceding equation to obtain

$$x(t) = Ae^{j\omega_0 t}.$$

Thus, we have that

$$Ae^{j\omega_0 t} \overset{\text{CTFT}}{\longleftrightarrow} 2\pi A\delta(\omega - \omega_0).$$

EXAMPLE

Find the Fourier transform X of the function $x(t) = \operatorname{rect} t$.

Solution.

$$X(\omega) = \int_{-\infty}^{\infty} \operatorname{rect}(t) e^{-j\omega t}\, dt.$$

$$\begin{aligned} X(\omega) &= \int_{-1/2}^{1/2} \operatorname{rect}(t) e^{-j\omega t}\, dt \\ &= \int_{-1/2}^{1/2} e^{-j\omega t}\, dt. \end{aligned}$$

$$\begin{aligned} X(\omega) &= \left[-\tfrac{1}{j\omega} e^{-j\omega t}\right]\Big|_{-1/2}^{1/2} \\ &= \tfrac{1}{j\omega}\left(e^{j\omega/2} - e^{-j\omega/2}\right) \\ &= \tfrac{1}{j\omega}\left[2j \sin\left(\tfrac{1}{2}\omega\right)\right] \\ &= \tfrac{2}{\omega} \sin\left(\tfrac{1}{2}\omega\right) \\ &= \left[\sin\left(\tfrac{1}{2}\omega\right)\right] / \left(\tfrac{1}{2}\omega\right) \\ &= \operatorname{sinc}\left(\tfrac{1}{2}\omega\right). \end{aligned}$$

Convergence Properties of the Fourier Transform

CONVERGENCE OF THE FOURIER TRANSFORM

- Consider an arbitrary function x.
- The function x has the Fourier transform representation $\tilde{x}$ given by

$$\tilde{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega, \quad \text{where} \quad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

- Now, we need to concern ourselves with the convergence properties of this representation.
- In other words, we want to know when $\tilde{x}$ is a valid representation of x.
- Since the Fourier transform is essentially derived from Fourier series, the convergence properties of the Fourier transform are closely related to the convergence properties of Fourier series.

CONVERGENCE OF THE FOURIER TRANSFORM: CONTINUOUS CASE

- If a function x is ***continuous*** and ***absolutely integrable*** (i.e., $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$) and the Fourier transform X of x is absolutely integrable (i.e., $\int_{-\infty}^{\infty} |X(\omega)|\, d\omega < \infty$), then the Fourier transform representation of x converges ***pointwise*** (i.e., $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right] e^{j\omega t} d\omega$ for all t).
- Since, in practice, we often encounter functions with discontinuities (e.g., a rectangular pulse), the above result is sometimes of limited value.

CONVERGENCE OF THE FOURIER TRANSFORM: FINITE ENERGY CASE

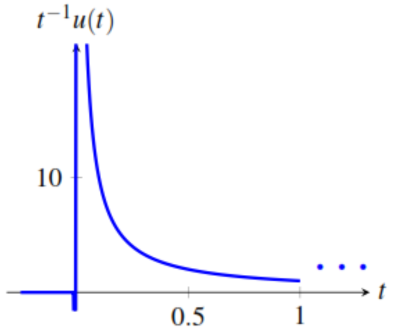
- If a function x is of ***finite energy*** (i.e., $\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty$), then its Fourier transform representation converges in the ***MSE sense***.
- In other words, if x is of finite energy, then the energy E in the difference function $\tilde{x} - x$ is zero; that is,

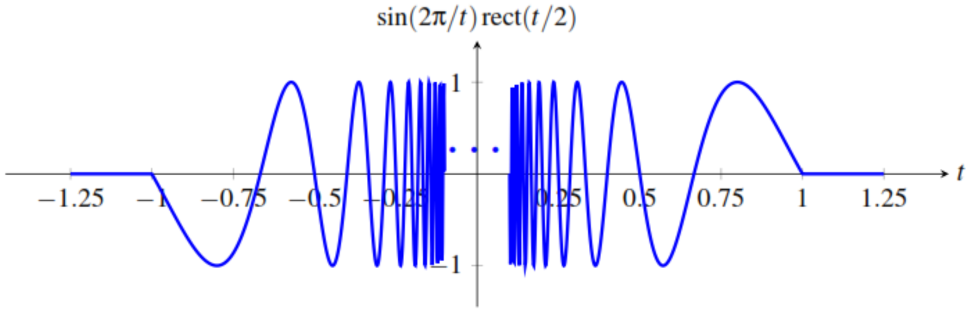
$$E = \int_{-\infty}^{\infty} |\tilde{x}(t) - x(t)|^2 \, dt = 0.$$

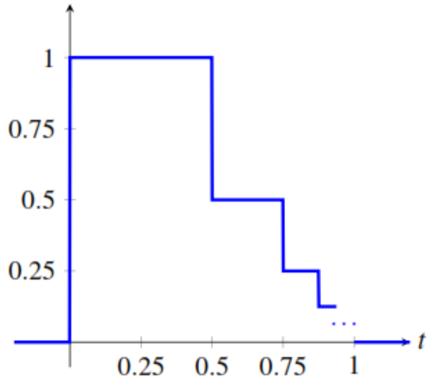
- Since, in situations of practical interest, the finite-energy condition in the above theorem is often satisfied, the theorem is frequently applicable.
- It is important to note, however, that the condition $E = 0$ does not necessarily imply $\tilde{x}(t) = x(t)$ for all t.
- Thus, the above convergence result does not provide much useful information regarding the value of $\tilde{x}(t)$ at specific values of t.
- Consequently, the above theorem is typically most useful for simply determining if the Fourier transform representation converges.

DIRICHLET CONDITIONS

- The **Dirichlet conditions** for the function x are as follows:
  1. the function x is ***absolutely integrable*** (i.e., $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$);
  2. on any finite interval, x has a finite number of maxima and minima (i.e., x is of ***bounded variation***); and
  3. on any finite interval, x has a ***finite number of discontinuities*** and each discontinuity is itself ***finite***.
- Examples of functions violating the Dirichlet conditions are shown below.







CONVERGENCE OF THE FOURIER TRANSFORM: DIRICHLET CASE

- If a function x satisfies the ***Dirichlet conditions***, then:
  1. the Fourier transform representation $\tilde{x}$ converges pointwise everywhere to x, except at the points of discontinuity of x; and
  2. at each point t_a of discontinuity of x, the Fourier transform representation $\tilde{x}$ converges to

$$\tilde{x}(t_a) = \tfrac{1}{2}\left[x(t_a^+) + x(t_a^-)\right],$$

  where $x(t_a^-)$ and $x(t_a^+)$ denote the values of the function x on the left- and right-hand sides of the discontinuity, respectively.

- Since most functions tend to satisfy the Dirichlet conditions and the above convergence result specifies the value of the Fourier transform representation at every point, this result is often very useful in practice.

Properties of the Fourier Transform

PROPERTIES OF THE (CT) FOURIER TRANSFORM

Property	Time Domain	Frequency Domain
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(\omega)+a_2X_2(\omega)$
Time-Domain Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(\omega)$
Frequency-Domain Shifting	$e^{j\omega_0 t}x(t)$	$X(\omega-\omega_0)$
Time/Frequency-Domain Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Conjugation	$x^*(t)$	$X^*(-\omega)$
Duality	$X(t)$	$2\pi x(-\omega)$
Time-Domain Convolution	$x_1 * x_2(t)$	$X_1(\omega)X_2(\omega)$
Time-Domain Multiplication	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1 * X_2(\omega)$
Time-Domain Differentiation	$\frac{d}{dt}x(t)$	$j\omega X(\omega)$
Frequency-Domain Differentiation	$tx(t)$	$j\frac{d}{d\omega}X(\omega)$
Time-Domain Integration	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$

PROPERTIES OF THE (CT) FOURIER TRANSFORM

Property	
Parseval's Relation	$\int_{-\infty}^{\infty} \lvert x(t) \rvert^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(\omega) \rvert^2 d\omega$
Even Symmetry	x is even $\Leftrightarrow$ X is even
Odd Symmetry	x is odd $\Leftrightarrow$ X is odd
Real / Conjugate Symmetry	x is real $\Leftrightarrow$ X is conjugate symmetric

CTFT PAIRS

Pair	$x(t)$	$X(\omega)$
1	$\delta(t)$	1
2	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
3	1	$2\pi\delta(\omega)$
4	$\operatorname{sgn}(t)$	$\frac{2}{j\omega}$
5	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
6	$\cos(\omega_0 t)$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
7	$\sin(\omega_0 t)$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
8	$\operatorname{rect}(t/T)$	$\lvert T\rvert \operatorname{sinc}(T\omega/2)$
9	$\frac{\lvert B\rvert}{\pi}\operatorname{sinc}(Bt)$	$\operatorname{rect}\left(\frac{\omega}{2B}\right)$
10	$e^{-at}u(t)$, $\operatorname{Re}\{a\}>0$	$\frac{1}{a+j\omega}$
11	$t^{n-1}e^{-at}u(t)$, $\operatorname{Re}\{a\}>0$	$\frac{(n-1)!}{(a+j\omega)^n}$
12	$\operatorname{tri}(t/T)$	$\frac{\lvert T\rvert}{2}\operatorname{sinc}^2(T\omega/4)$

LINEARITY

- If $x_1(t) \xleftrightarrow{\text{CTFT}} X_1(\omega)$ and $x_2(t) \xleftrightarrow{\text{CTFT}} X_2(\omega)$, then

$$a_1x_1(t) + a_2x_2(t) \xleftrightarrow{\text{CTFT}} a_1X_1(\omega) + a_2X_2(\omega),$$

  where a_1 and a_2 are arbitrary complex constants.

- This is known as the **linearity property** of the Fourier transform.

TIME-DOMAIN SHIFTING (TRANSLATION)

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$x(t-t_0) \overset{\text{CTFT}}{\longleftrightarrow} e^{-j\omega t_0}X(\omega),$$

where t_0 is an arbitrary real constant.

- This is known as the **translation (or time-domain shifting) property** of the Fourier transform.

EXAMPLE

Find the Fourier transform X of the function

$$x(t) = A\cos(\omega_0 t + \theta),$$

where A, ω_0, and θ are real constants.

Solution. Let $v(t) = A\cos(\omega_0 t)$ so that $x(t) = v(t + \frac{\theta}{\omega_0})$. Also, let $V = \mathcal{F}v$.

$$\cos(\omega_0 t) \overset{\text{CTFT}}{\longleftrightarrow} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)].$$

$$\begin{aligned} V(\omega) &= \mathcal{F}\{A\cos(\omega_0 t)\}(\omega) \\ &= A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]. \end{aligned}$$

From the definition of v and the time-shifting property of the Fourier transform, we have

$$\begin{aligned} X(\omega) &= e^{j\omega\theta/\omega_0} V(\omega) \\ &= e^{j\omega\theta/\omega_0} A\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]. \end{aligned}$$

FREQUENCY-DOMAIN SHIFTING (MODULATION)

- If $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$, then

$$e^{j\omega_0 t}x(t) \xleftrightarrow{\text{CTFT}} X(\omega - \omega_0),$$

  where ω_0 is an arbitrary real constant.
- This is known as the **modulation (or frequency-domain shifting) property** of the Fourier transform.

EXAMPLE

$$x(t) = \cos(\omega_0 t)\cos(20\pi t), \quad \text{where } \omega_0 \text{ is a real constant.}$$

Find the Fourier transform X

Solution. Recall that $\cos\alpha = \frac{1}{2}[e^{j\alpha} + e^{-j\alpha}]$ for any real α.

$$\begin{aligned}
X(\omega) &= \mathcal{F}\{\cos(\omega_0 t)(\tfrac{1}{2})(e^{j20\pi t} + e^{-j20\pi t})\}(\omega) \\
&= \mathcal{F}\{\tfrac{1}{2}e^{j20\pi t}\cos(\omega_0 t) + \tfrac{1}{2}e^{-j20\pi t}\cos(\omega_0 t)\}(\omega) \\
&= \tfrac{1}{2}\mathcal{F}\{e^{j20\pi t}\cos(\omega_0 t)\}(\omega) + \tfrac{1}{2}\mathcal{F}\{e^{-j20\pi t}\cos(\omega_0 t)\}(\omega).
\end{aligned}$$

$$\cos(\omega_0 t) \overset{\text{CTFT}}{\longleftrightarrow} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)].$$

From this transform pair and the frequency-domain shifting property

$$\begin{aligned}
X(\omega) &= \tfrac{1}{2}\left(\mathcal{F}\{\cos(\omega_0 t)\}\right)(\omega - 20\pi) + \tfrac{1}{2}\left(\mathcal{F}\{\cos(\omega_0 t)\}\right)(\omega + 20\pi) \\
&= \tfrac{1}{2}\left[\pi[\delta(v - \omega_0) + \delta(v + \omega_0)]\right]\Big|_{v=\omega-20\pi} + \tfrac{1}{2}\left[\pi[\delta(v - \omega_0) + \delta(v + \omega_0)]\right]\Big|_{v=\omega+20\pi} \\
&= \tfrac{1}{2}\left(\pi[\delta(\omega + \omega_0 - 20\pi) + \delta(\omega - \omega_0 - 20\pi)]\right) + \tfrac{1}{2}\left(\pi[\delta(\omega + \omega_0 + 20\pi) + \delta(\omega - \omega_0 + 20\pi)]\right) \\
&= \tfrac{\pi}{2}\left[\delta(\omega + \omega_0 - 20\pi) + \delta(\omega - \omega_0 - 20\pi) + \delta(\omega + \omega_0 + 20\pi) + \delta(\omega - \omega_0 + 20\pi)\right].
\end{aligned}$$

TIME- AND FREQUENCY-DOMAIN SCALING (DILATION)

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$x(at) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right),$$

where a is an arbitrary nonzero real constant.

- This is known as the **dilation (or time/frequency-domain scaling) property** of the Fourier transform.

EXAMPLE

find the Fourier transform X of the function

$$x(t) = \operatorname{rect}(at),$$

where a is a nonzero real constant.

Solution. Let $v(t) = \operatorname{rect} t$ so that $x(t) = v(at)$. Also, let $V = \mathcal{F}v$.

$$\operatorname{rect} t \overset{\text{CTFT}}{\longleftrightarrow} \operatorname{sinc}\left(\frac{\omega}{2}\right),$$

From the definition of v and the time-scaling property of the Fourier transform, we have

$$X(\omega) = \frac{1}{|a|} V\left(\frac{\omega}{a}\right).$$

$$X(\omega) = \frac{1}{|a|} \operatorname{sinc}\left(\frac{\omega}{2a}\right).$$

CONJUGATION

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then
$$x^*(t) \overset{\text{CTFT}}{\longleftrightarrow} X^*(-\omega).$$
- This is known as the **conjugation property** of the Fourier transform.

DUALITY

- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$X(t) \overset{\text{CTFT}}{\longleftrightarrow} 2\pi x(-\omega)$$

- This is known as the **duality property** of the Fourier transform.
- This property follows from the high degree of symmetry in the forward and inverse Fourier transform equations, which are respectively given by

$$X(\lambda) = \int_{-\infty}^{\infty} x(\theta) e^{-j\theta\lambda} d\theta \quad \text{and} \quad x(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\theta) e^{j\theta\lambda} d\theta.$$

- That is, the forward and inverse Fourier transform equations are identical except for a ***factor of*** 2π and ***different sign*** in the parameter for the exponential function.
- Although the relationship $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$ only directly provides us with the Fourier transform of $x(t)$, the duality property allows us to indirectly infer the Fourier transform of $X(t)$. Consequently, the duality property can be used to effectively ***double*** the number of Fourier transform pairs that we know.

EXAMPLE

find the Fourier transform X of the function

$$x(t) = \operatorname{sinc}\left(\frac{t}{2}\right).$$

Solution. From the given Fourier transform pair, we have

$$v(t) = \operatorname{rect} t \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad V(\omega) = \operatorname{sinc}\left(\frac{\omega}{2}\right).$$

By duality, we have

$$V(t) = \operatorname{sinc}\left(\frac{t}{2}\right) \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad \mathcal{F}V(\omega) = 2\pi v(-\omega) = 2\pi \operatorname{rect}(-\omega) = 2\pi \operatorname{rect} \omega.$$

Thus, we have

$$V(t) = \operatorname{sinc}\left(\frac{t}{2}\right) \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad \mathcal{F}V(\omega) = 2\pi \operatorname{rect} \omega.$$

Observing that $V = x$ and $\mathcal{F}V = X$, we can rewrite the preceding relationship as

$$x(t) = \operatorname{sinc}\left(\frac{t}{2}\right) \quad \overset{\text{CTFT}}{\longleftrightarrow} \quad X(\omega) = 2\pi \operatorname{rect} \omega.$$

TIME DOMAIN CONVOLUTION

- If $x_1(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$, then

$$x_1 * x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)X_2(\omega).$$

- This is known as the **convolution (or time-domain convolution) property** of the Fourier transform.
- In other words, a convolution in the time domain becomes a multiplication in the frequency domain.
- This suggests that the Fourier transform can be used to avoid having to deal with convolution operations.

EXAMPLE

Fourier transform X of the function

$$x(t) = x_1 * x_2(t),$$

where

$$x_1(t) = e^{-2t}u(t) \quad \text{and} \quad x_2(t) = u(t).$$

Solution. Let X_1 and X_2 denote the Fourier transforms of x_1 and x_2, respectively.

$$\begin{aligned} X_1(\omega) &= \left(\mathcal{F}\{e^{-2t}u(t)\}\right)(\omega) \\ &= \tfrac{1}{2+j\omega} \quad \text{and} \end{aligned}$$

$$\begin{aligned} X_2(\omega) &= \mathcal{F}u(\omega) \\ &= \pi\delta(\omega) + \tfrac{1}{j\omega}. \end{aligned}$$

$$\begin{aligned} X(\omega) &= (\mathcal{F}\{x_1 * x_2\})(\omega) \\ &= X_1(\omega)X_2(\omega). \end{aligned}$$

$$= \tfrac{\pi}{2}\delta(\omega) + \tfrac{1}{j2\omega - \omega^2}.$$

TIME-DOMAIN MULTIPLICATION

- If $x_1(t) \overset{\text{CTFT}}{\longleftrightarrow} X_1(\omega)$ and $x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} X_2(\omega)$, then

$$x_1(t)x_2(t) \overset{\text{CTFT}}{\longleftrightarrow} \frac{1}{2\pi} X_1 * X_2(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(\theta) X_2(\omega - \theta) d\theta.$$

- This is known as the **(time-domain) multiplication (or frequency-domain convolution) property** of the Fourier transform.
- In other words, multiplication in the time domain becomes convolution in the frequency domain (up to a scale factor of 2π).
- Do not forget the factor of $\frac{1}{2\pi}$ in the above formula!
- This property of the Fourier transform is often tedious to apply (in the forward direction) as it turns a multiplication into a convolution.

TIME-DOMAIN DIFFERENTIATION

- If $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$, then

$$\frac{dx(t)}{dt} \xleftrightarrow{\text{CTFT}} j\omega X(\omega).$$

- This is known as the **(time-domain) differentiation property** of the Fourier transform.
- Differentiation in the time domain becomes multiplication by $j\omega$ in the frequency domain.
- Of course, by repeated application of the above property, we have that $\left(\frac{d}{dt}\right)^n x(t) \xleftrightarrow{\text{CTFT}} (j\omega)^n X(\omega)$.
- The above suggests that the Fourier transform might be a useful tool when working with differential (or integro-differential) equations.

FREQUENCY-DOMAIN DIFFERENTIATION

- If $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$, then

$$tx(t) \xleftrightarrow{\text{CTFT}} j\frac{d}{d\omega}X(\omega).$$

- This is known as the **frequency-domain differentiation property** of the Fourier transform.

EXAMPLE

Find the Fourier transform X of the function $x(t) = t\cos(\omega_0 t)$, where ω_0 is a nonzero real constant.

Solution. Taking the Fourier transform of both sides of the equation for x yields

$$X(\omega) = \mathcal{F}\{t\cos(\omega_0 t)\}(\omega).$$

From the frequency-domain differentiation property of the Fourier transform, we can write

$$\begin{aligned} X(\omega) &= j\tfrac{d}{d\omega}\left[\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]\right] \\ &= j\pi\tfrac{d}{d\omega}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right] \\ &= j\pi\tfrac{d}{d\omega}\delta(\omega-\omega_0)+j\pi\tfrac{d}{d\omega}\delta(\omega+\omega_0). \end{aligned}$$

TIME-DOMAIN INTEGRATION

- If $x(t) \xleftrightarrow{\text{CTFT}} X(\omega)$, then

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\text{CTFT}} \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega).$$

- This is known as the **(time-domain) integration property** of the Fourier transform.
- Whereas differentiation in the time domain corresponds to ***multiplication*** by $j\omega$ in the frequency domain, integration in the time domain is associated with ***division*** by $j\omega$ in the frequency domain.
- Since integration in the time domain becomes division by $j\omega$ in the frequency domain, integration can be easier to handle in the frequency domain.
- The above property suggests that the Fourier transform might be a useful tool when working with integral (or integro-differential) equations.

PARSEVAL'S RELATION

- Recall that the energy of a function x is given by $\int_{-\infty}^{\infty} |x(t)|^2 \, dt$.
- If $x(t) \overset{\text{CTFT}}{\longleftrightarrow} X(\omega)$, then

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$$

  (i.e., the energy of x and energy of X are equal up to a factor of 2π).
- This relationship is known as **Parseval's relation**.
- Since energy is often a quantity of great significance in engineering applications, it is extremely helpful to know that the Fourier transform ***preserves energy*** (up to a scale factor).

EXAMPLE

Consider the function $x(t) = \operatorname{sinc}\left(\frac{1}{2}t\right)$, $X(\omega) = 2\pi \operatorname{rect} \omega$.

Compute the energy of x.

Solution. We could directly compute the energy of x as

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |x(t)|^2 \, dt \\ &= \int_{-\infty}^{\infty} \left|\operatorname{sinc}\left(\tfrac{1}{2}t\right)\right|^2 dt. \end{aligned}$$

This integral is not so easy to compute, however. Instead, we use Parseval's relation to write

$$\begin{aligned} E &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |2\pi \operatorname{rect} \omega|^2 \, d\omega \\ &= \frac{1}{2\pi} \int_{-1/2}^{1/2} (2\pi)^2 d\omega \\ &= 2\pi. \end{aligned}$$

EVEN/ODD SYMMETRY

- For a function x with Fourier transform X, the following assertions hold:

$$x \text{ is even} \Leftrightarrow X \text{ is even}; \quad \text{and}$$

$$x \text{ is odd} \Leftrightarrow X \text{ is odd}.$$

- In other words, the forward and inverse Fourier transforms preserve even/odd symmetry.

REAL FUNCTIONS

- A function x is ***real*** if and only if its Fourier transform X satisfies

$$X(\omega) = X^*(-\omega) \text{ for all } \omega$$

(i.e., X is ***conjugate symmetric***).

- Thus, for a real-valued function, the portion of the graph of $X(\omega)$ for $\omega < 0$ is ***completely redundant***, as it is determined by symmetry.
- From properties of complex numbers, one can show that $X(\omega) = X^*(-\omega)$ is equivalent to

$$|X(\omega)| = |X(-\omega)| \quad \text{and} \quad \arg X(\omega) = -\arg X(-\omega)$$

(i.e., $|X(\omega)|$ is ***even*** and $\arg X(\omega)$ is ***odd***).

- Note that x being real does ***not*** necessarily imply that X is real.

Fourier Transform of Periodic Functions

FOURIER TRANSFORM OF PERIODIC FUNCTIONS

- The Fourier transform can be generalized to also handle periodic functions.
- Consider a periodic function x with period T and frequency $\omega_0 = \frac{2\pi}{T}$.
- Define the function x_T as

$$x_T(t) = \begin{cases} x(t) & -\frac{T}{2} \leq t < \frac{T}{2} \\ 0 & \text{otherwise.} \end{cases}$$

  (i.e., $x_T(t)$ is equal to $x(t)$ over a single period and zero elsewhere).
- Let a denote the Fourier series coefficient sequence of x.
- Let X and X_T denote the Fourier transforms of x and x_T, respectively.
- The following relationships can be shown to hold:

$$X(\omega) = \sum_{k=-\infty}^{\infty} \omega_0 X_T(k\omega_0)\delta(\omega - k\omega_0),$$

$$a_k = \frac{1}{T} X_T(k\omega_0), \quad \text{and} \quad X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0).$$

FOURIER TRANSFORM OF PERIODIC FUNCTIONS

- The Fourier transform X of a periodic function is a series of impulses that occur at integer multiples of the fundamental frequency ω_0 (i.e., $X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$).
- Due to the preceding fact, the Fourier transform of a periodic function can only be nonzero at integer multiples of the fundamental frequency.
- The Fourier series coefficient sequence a is produced by sampling X_T at integer multiples of the fundamental frequency ω_0 and scaling the resulting sequence by $\frac{1}{T}$ (i.e., $a_k = \frac{1}{T} X_T(k\omega_0)$).

Fourier Transform and Frequency Spectra of Functions

THE FREQUENCY-DOMAIN PERSPECTIVE ON FUNCTIONS

- Like Fourier series, the Fourier transform also provides us with a frequency-domain perspective on functions.
- That is, instead of viewing a function as having information distributed with respect to ***time*** (i.e., a function whose domain is time), we view a function as having information distributed with respect to ***frequency*** (i.e., a function whose domain is frequency).
- The Fourier transform of a function x provides a means to ***quantify*** how much information x has at different frequencies.
- The distribution of information in a function over different frequencies is referred to as the ***frequency spectrum*** of the function.

FOURIER TRANSFORM AND FREQUENCY SPECTRA

- To gain further insight into the role played by the Fourier transform X in the context of the frequency spectrum of x, it is helpful to write the Fourier transform representation of x with $X(\omega)$ expressed in ***polar form*** as follows:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|\, e^{j[\omega t + \arg X(\omega)]}d\omega.$$

- In effect, the quantity $|X(\omega)|$ is a ***weight*** that determines how much the complex sinusoid at frequency ω contributes to the integration result x.
- The quantity $\arg X(\omega)$ determines how the complex sinusoid at frequency ω is shifted related to complex sinusoids at other frequencies.
- Perhaps, this can be more easily seen if we express the above integral as the ***limit of a sum***, derived from an approximation of the integral using the areas of rectangles, as shown on the next slide. [Recall that $\int_{-\infty}^{\infty} f(x)dx = \lim_{\Delta x \to 0} \sum_{k=-\infty}^{\infty} \Delta x f(k\Delta x)$.]

FOURIER TRANSFORM AND FREQUENCY SPECTRA

- Expressing the integral (from the previous slide) as the ***limit of a sum***, we obtain

$$x(t) = \lim_{\Delta\omega \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Delta\omega |X(\omega)| e^{j[\omega t + \arg X(\omega)]},$$

  where $\omega = k\Delta\omega$.

- In the above equation, the kth term in the summation corresponds to a complex sinusoid with fundamental frequency $\omega = k\Delta\omega$ that has had its ***amplitude scaled*** by a factor of $|X(\omega)|$ and has been ***time shifted*** by an amount that depends on $\arg X(\omega)$.

- For a given $\omega = k\Delta\omega$ (which is associated with the kth term in the summation), the ***larger*** $|X(\omega)|$ is, the larger the amplitude of its corresponding complex sinusoid $e^{j\omega t}$ will be, and therefore the ***larger the contribution*** the kth term will make to the overall summation.

- In this way, we can use $|X(\omega)|$ as a ***measure*** of how much information a function x has at the frequency ω.

FOURIER TRANSFORM AND FREQUENCY SPECTRA

- The Fourier transform X of the function x is referred to as the **frequency spectrum** of x.
- The magnitude $|X(\omega)|$ of the Fourier transform X is referred to as the **magnitude spectrum** of x.
- The argument $\arg X(\omega)$ of the Fourier transform X is referred to as the **phase spectrum** of x.
- Since the Fourier transform is a function of a real variable, a function can potentially have information at any real frequency.
- Since the Fourier transform X of a periodic function x with fundamental frequency ω_0 and the Fourier series coefficient sequence a is given by $X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$, the Fourier transform and Fourier series give consistent results for the frequency spectrum of a periodic function.
- Since the frequency spectrum is complex (in the general case), it is ***usually represented using two plots***, one showing the magnitude spectrum and one showing the phase spectrum.

FREQUENCY SPECTRA OF REAL FUNCTIONS

- Recall that, for a real function x, the Fourier transform X of x satisfies

$$X(\omega) = X^*(-\omega)$$

(i.e., X is ***conjugate symmetric***), which is equivalent to

$$|X(\omega)| = |X(-\omega)| \quad \text{and} \quad \arg X(\omega) = -\arg X(-\omega).$$

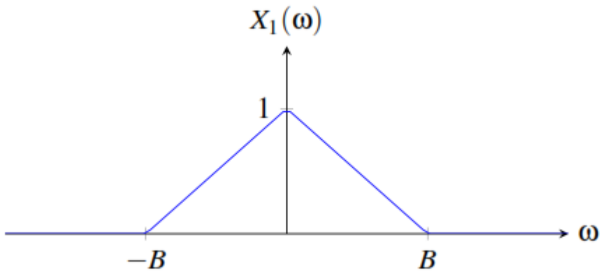
- Since $|X(\omega)| = |X(-\omega)|$, the magnitude spectrum of a real function is always ***even***.
- Similarly, since $\arg X(\omega) = -\arg X(-\omega)$, the phase spectrum of a real function is always ***odd***.
- Due to the symmetry in the frequency spectra of real functions, we typically ***ignore negative frequencies*** when dealing with such functions.
- In the case of functions that are complex but not real, frequency spectra do not possess the above symmetry, and ***negative frequencies become important***.

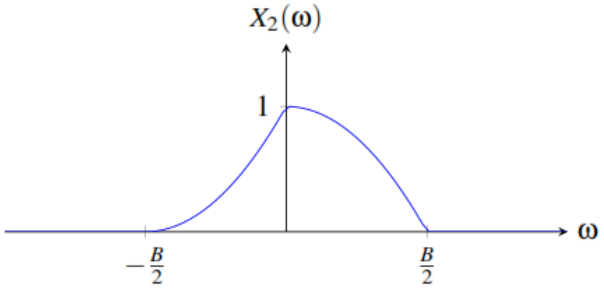
BANDWIDTH

- A function with the Fourier transform X is said to be **bandlimited** if, for some (finite) nonnegative real constant B, the following condition holds:

$$X(\omega) = 0 \text{ for all } \omega \text{ satisfying } |\omega| > B.$$

- The **bandwidth** B of a function with the Fourier transform X is defined as $B = \omega_1 - \omega_0$, where $X(\omega) = 0$ for all $\omega \notin [\omega_0, \omega_1]$.
- In the case of ***real-valued*** functions, however, this definition of bandwidth is usually amended to consider ***only nonnegative*** frequencies.
- The real-valued function x_1 and complex-valued function x_2 with the respective Fourier transforms X_1 and X_2 shown below each have bandwidth B (where only nonnegative frequencies are considered in the case of x_1).





- One can show that a function ***cannot be both time limited and bandlimited***.

ENERGY-DENSITY SPECTRA

- By Parseval's relation, the energy E in a function x with Fourier transform X is given by

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} E_x(\omega)\,d\omega,$$

where

$$E_x(\omega) = |X(\omega)|^2.$$

- We refer to E_x as the **energy-density spectrum** of the function x.
- The function E_x indicates how the energy in x is distributed with respect to frequency.
- For example, the energy contributed by frequencies in the range $[\omega_1, \omega_2]$ is given by

$$\frac{1}{2\pi}\int_{\omega_1}^{\omega_2} E_x(\omega)\,d\omega.$$

Fourier Transform and LTI Systems

FREQUENCY RESPONSE OF LTI SYSTEMS

- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the Fourier transforms of x, y, and h, respectively.
- Since $y(t) = x * h(t)$, we have that

$$Y(\omega) = X(\omega)H(\omega).$$

- The function H is called the **frequency response** of the system.
- A LTI system is ***completely characterized*** by its frequency response H.
- The above equation provides an alternative way of viewing the behavior of a LTI system. That is, we can view the system as operating in the frequency domain on the Fourier transforms of the input and output functions.
- The frequency spectrum of the output is the product of the frequency spectrum of the input and the frequency response of the system.

FREQUENCY RESPONSE OF LTI SYSTEMS

- In the general case, the frequency response H is a complex-valued function.
- Often, we represent $H(\omega)$ in terms of its magnitude $|H(\omega)|$ and argument $\arg H(\omega)$.
- The quantity $|H(\omega)|$ is called the **magnitude response** of the system.
- The quantity $\arg H(\omega)$ is called the **phase response** of the system.
- Since $Y(\omega) = X(\omega)H(\omega)$, we trivially have that

$$|Y(\omega)| = |X(\omega)|\,|H(\omega)| \quad \text{and} \quad \arg Y(\omega) = \arg X(\omega) + \arg H(\omega).$$

- The magnitude spectrum of the output equals the magnitude spectrum of the input times the magnitude response of the system.
- The phase spectrum of the output equals the phase spectrum of the input plus the phase response of the system.

FREQUENCY RESPONSE OF LTI SYSTEMS

- Since the frequency response H is simply the frequency spectrum of the impulse response h, if h is ***real***, then

$$|H(\omega)| = |H(-\omega)| \quad \text{and} \quad \arg H(\omega) = -\arg H(-\omega)$$

(i.e., the magnitude response $|H(\omega)|$ is ***even*** and the phase response $\arg H(\omega)$ is ***odd***).

INTERPRETATION OF MAGNITUDE AND PHASE RESPONSE

- Recall that a LTI system $\mathcal{H}$ with frequency response H is such that

$$\mathcal{H}\left\{e^{j\omega t}\right\}(t) = H(\omega)e^{j\omega t}.$$

- Expressing $H(\omega)$ in polar form, we have

$$\begin{aligned}\mathcal{H}\{e^{j\omega t}\}(t) &= |H(\omega)|\, e^{j\arg H(\omega)} e^{j\omega t} \\ &= |H(\omega)|\, e^{j[\omega t + \arg H(\omega)]} \\ &= |H(\omega)|\, e^{j\omega(t + \arg[H(\omega)]/\omega)}.\end{aligned}$$

- Thus, the response of the system to the function $e^{j\omega t}$ is produced by applying two transformations to this function:
  - (amplitude) scaling by $|H(\omega)|$; and
  - translating by $-\frac{\arg H(\omega)}{\omega}$.
- Therefore, the magnitude response determines how different complex sinusoids are ***scaled*** (in amplitude) by the system.
- Similarly, the phase response determines how different complex sinusoids are ***translated*** (i.e., delayed/advanced) by the system.

MAGNITUDE DISTORTION

- Recall that a LTI system $\mathcal{H}$ with frequency response H is such that

$$\mathcal{H}\{e^{j\omega t}\}(t) = |H(\omega)|\, e^{j\omega(t+\arg[H(\omega)]/\omega)}.$$

- If $|H(\omega)|$ is a constant (for all ω), every complex sinusoid is scaled by the same amount when passing through the system.
- A system for which $|H(\omega)| = 1$ (for all ω) is said to be **allpass**.
- In the case of an allpass system, the magnitude spectra of the system's input and output are identical.
- If $|H(\omega)|$ is not a constant, different complex sinusoids are scaled by different amounts, resulting in what is known as **magnitude distortion**.

PHASE DISTORTION

- Recall that a LTI system $\mathcal{H}$ with frequency response H is such that

$$\mathcal{H}\{e^{j\omega t}\}(t) = |H(\omega)|\, e^{j\omega(t+\arg[H(\omega)]/\omega)}.$$

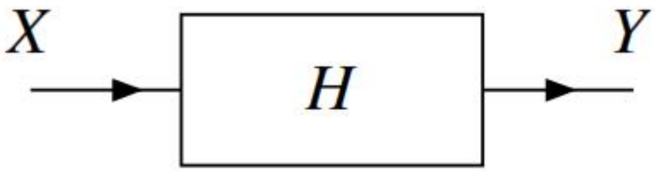
- The preceding equation can be rewritten as

$$\mathcal{H}\{e^{j\omega t}\}(t) = |H(\omega)|\, e^{j\omega[t-\tau_p(\omega)]} \quad \text{where} \quad \tau_p(\omega) = -\frac{\arg H(\omega)}{\omega}.$$

- The function τ_p is known as the **phase delay** of the system.
- If $\tau_p(\omega) = t_d$ (where t_d is a constant), the system shifts all complex sinusoids by the same amount t_d.
- Since $\tau_p(\omega) = t_d$ is equivalent to the (unwrapped) phase response being of the form $\arg H(\omega) = -t_d\omega$ (which is a linear function with a zero constant term), a system with a constant phase delay is said to have **linear phase**.
- In the case that $\tau_p(\omega) = 0$, the system is said to have **zero phase**.
- If $\tau_p(\omega)$ is not a constant, different complex sinusoids are shifted by different amounts, resulting in what is known as **phase distortion**.

BLOCK DIAGRAM REPRESENTATIONS OF LTI SYSTEMS

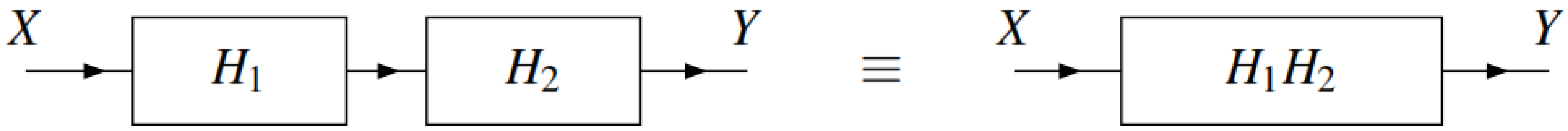
- Consider a LTI system with input x, output y, and impulse response h, and let X, Y, and H denote the Fourier transforms of x, y, and h, respectively.
- Often, it is convenient to represent such a system in block diagram form in the frequency domain as shown below.



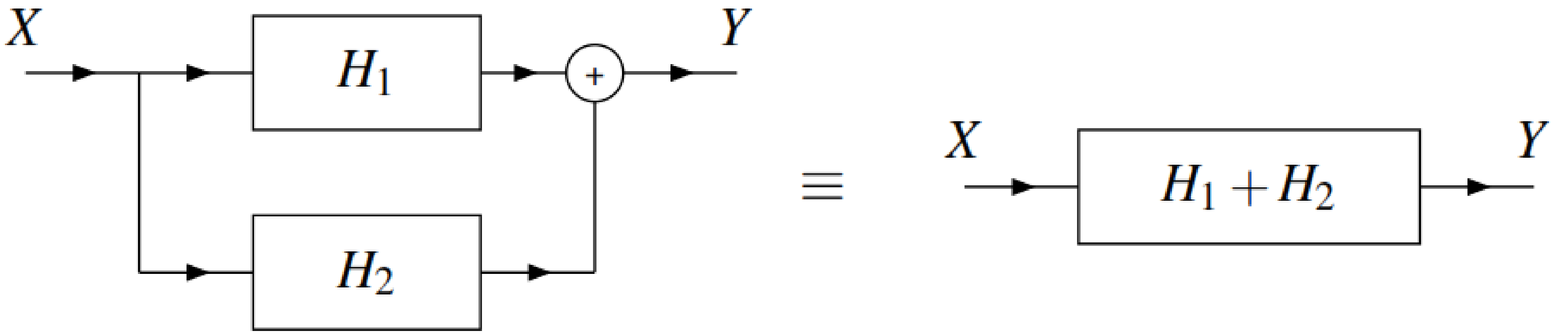
- Since a LTI system is completely characterized by its frequency response, we typically label the system with this quantity.

INTERCONNECTION OF LTI SYSTEMS

- The *series* interconnection of the LTI systems with frequency responses H_1 and H_2 is the LTI system with frequency response $H_1 H_2$. That is, we have the equivalence shown below.



- The *parallel* interconnection of the LTI systems with frequency responses H_1 and H_2 is the LTI system with the frequency response $H_1 + H_2$. That is, we have the equivalence shown below.



Application: Filtering

FILTERING

- In many applications, we want to ***modify the spectrum*** of a function by either amplifying or attenuating certain frequency components.
- This process of modifying the frequency spectrum of a function is called **filtering**.
- A system that performs a filtering operation is called a **filter**.
- Many types of filters exist.
- **Frequency selective filters** pass some frequencies with little or no distortion, while significantly attenuating other frequencies.
- Several basic types of frequency-selective filters include: lowpass, highpass, and bandpass.

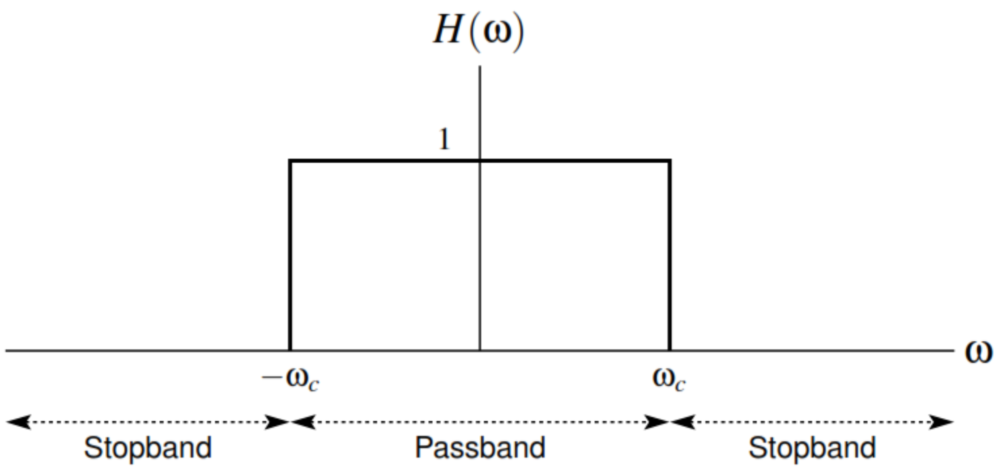
IDEAL LOWPASS FILTER

- An **ideal lowpass filter** eliminates all frequency components with a frequency whose magnitude is greater than some cutoff frequency, while leaving the remaining frequency components unaffected.
- Such a filter has a ***frequency response*** H of the form

$$H(\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where ω_c is the **cutoff frequency**.

- A plot of this frequency response is given below.



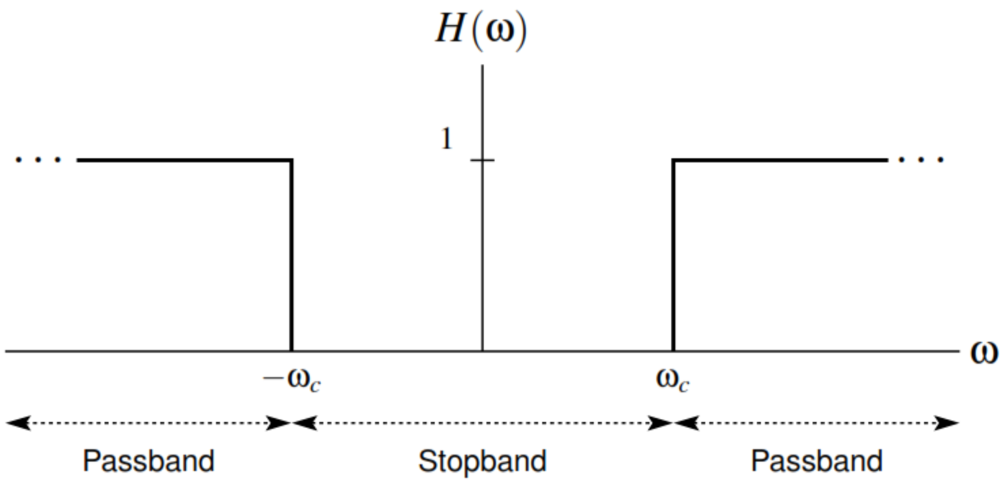
IDEAL HIGHPASS FILTER

- An **ideal highpass filter** eliminates all frequency components with a frequency whose magnitude is less than some cutoff frequency, while leaving the remaining frequency components unaffected.
- Such a filter has a ***frequency response*** H of the form

$$H(\omega) = \begin{cases} 1 & |\omega| \geq \omega_c \\ 0 & \text{otherwise,} \end{cases}$$

where ω_c is the **cutoff frequency**.

- A plot of this frequency response is given below.

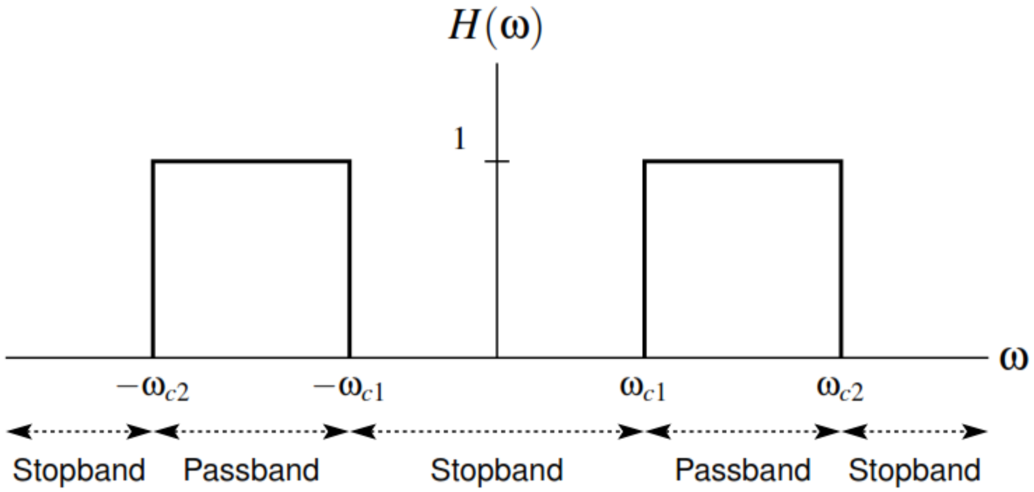


IDEAL BANDPASS FILTER

- An **ideal bandpass filter** eliminates all frequency components with a frequency whose magnitude does not lie in a particular range, while leaving the remaining frequency components unaffected.
- Such a filter has a ***frequency response*** H of the form

$$H(\omega) = \begin{cases} 1 & \omega_{c1} \leq |\omega| \leq \omega_{c2} \\ 0 & \text{otherwise,} \end{cases}$$

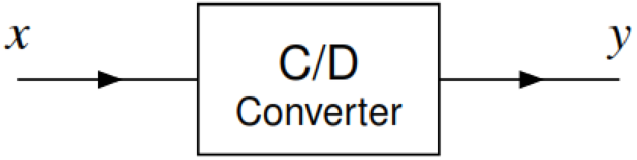
  where the limits of the passband are ω_{c1} and ω_{c2}.
- A plot of this frequency response is given below.



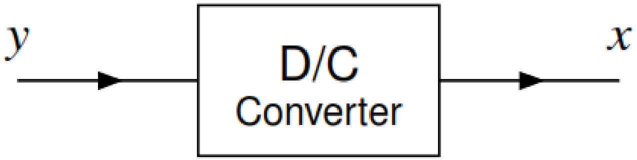
Application: Sampling and Interpolation

SAMPLING AND INTERPOLATION

- Often, we want to be able to transform a continuous-time signal (i.e., a function) into a discrete-time signal (i.e., a sequence) and vice versa.
- This is accomplished through processes known as ***sampling*** and ***interpolation***.
- **Sampling**, which is performed by a **continuous-time to discrete-time (C/D) converter** shown below, transforms a function x to a sequence y.



- **Interpolation**, which is performed by a **discrete-time to continuous-time (D/C) converter** shown below, transforms a sequence y to a function x.



- Note that, unless very special conditions are met, the sampling process loses information (i.e., is ***not invertible***).

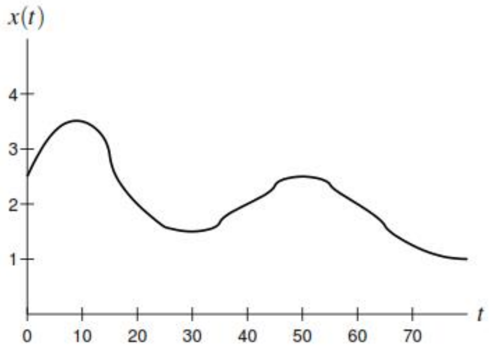
PERIODIC SAMPLING

- Although sampling can be performed in many different ways, the most commonly used scheme is **periodic sampling**.
- With this scheme, a sequence y of samples is obtained from a function x according to the relation

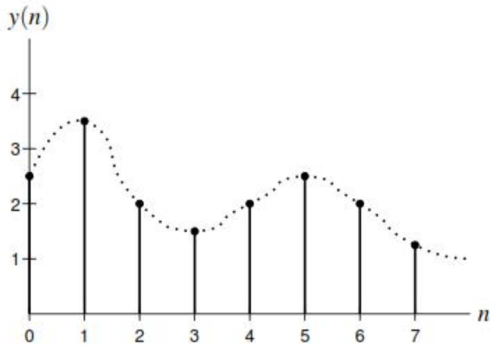
$$y(n) = x(Tn) \quad \text{for all integer } n,$$

where T is a (strictly) positive real constant.

- As a matter of terminology, we refer to T as the **sampling period**, and $\omega_s = \frac{2\pi}{T}$ as the (angular) **sampling frequency**.
- An example of periodic sampling is shown below, where the function x has been sampled with ***sampling period*** $T = 10$, yielding the sequence y.



Function to Be Sampled



Sequence Produced by Sampling

INVERTIBILITY OF SAMPLING

- Unless constraints are placed on the functions being sampled, the sampling process is ***not invertible***.
- In other words, in the absence of any constraints, a function cannot be uniquely determined from a sequence of its equally-spaced samples.
- Consider, for example, the functions x_1 and x_2 given by

$$x_1(t) = 0 \quad \text{and} \quad x_2(t) = \sin(2\pi t).$$

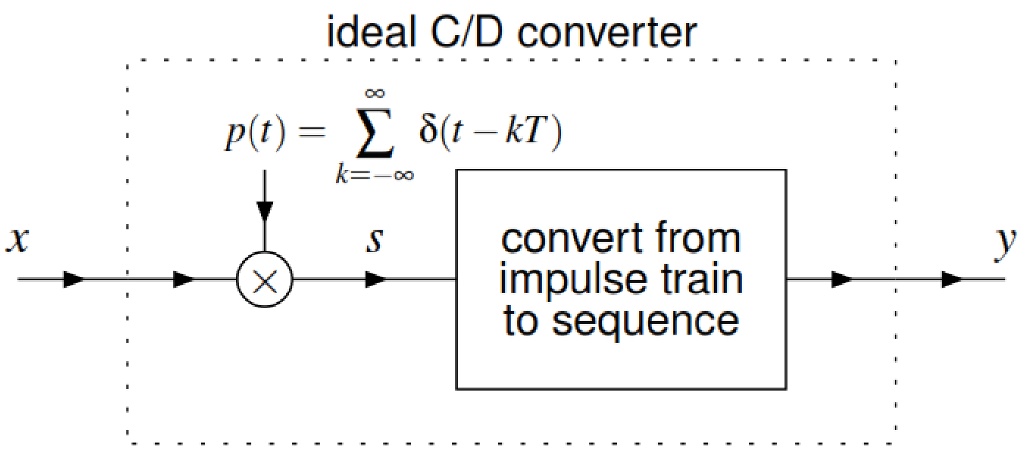
- Sampling x_1 and x_2 with the sampling period $T = 1$ yields the respective sequences

$$y_1(n) = x_1(Tn) = x_1(n) = 0 \quad \text{and}$$
$$y_2(n) = x_2(Tn) = \sin(2\pi n) = 0.$$

- So, although x_1 and x_2 are ***distinct***, y_1 and y_2 are ***identical***.
- Given the sequence y where $y = y_1 = y_2$, it is impossible to determine which function was sampled to produce y.
- Only by imposing a carefully chosen set of constraints on the functions being sampled can we ensure that a function can be exactly recovered from only its samples.

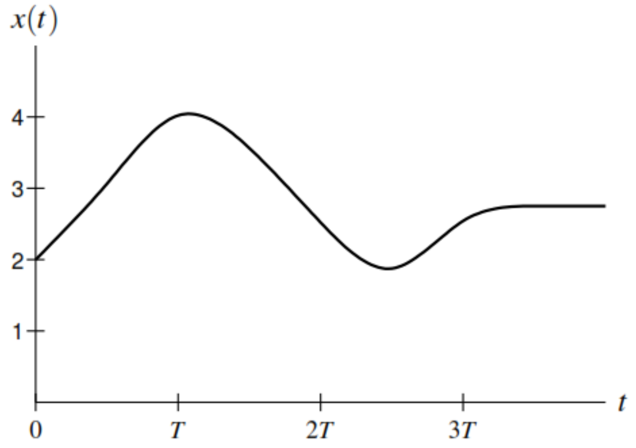
MODEL OF SAMPLING

- An **impulse train** is a function of the form $v(t) = \sum_{k=-\infty}^{\infty} c_k \delta(t - kT)$, where c_k and T are real constants.
- For the purposes of analysis, sampling with sampling period T and frequency $\omega_s = \frac{2\pi}{T}$ can be modelled as shown below.

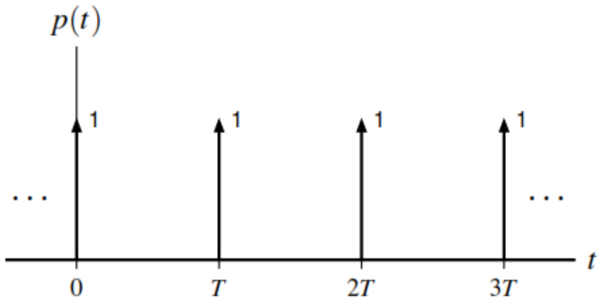


- The sampling of a function x to produce a sequence y consists of the following two steps (in order):
  1. Multiply the function x to be sampled by a periodic impulse train p, yielding the impulse train $s(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$.
  2. Convert the impulse train s to a sequence y by forming y from the weights of successive impulses in s so that $y(n) = x(nT)$.

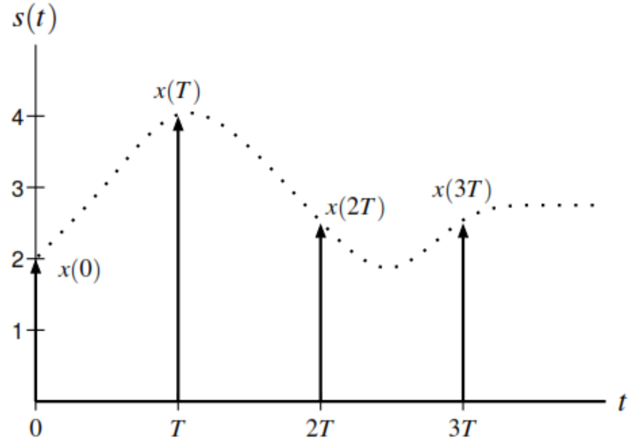
MODEL OF SAMPLING: VARIOUS SIGNALS



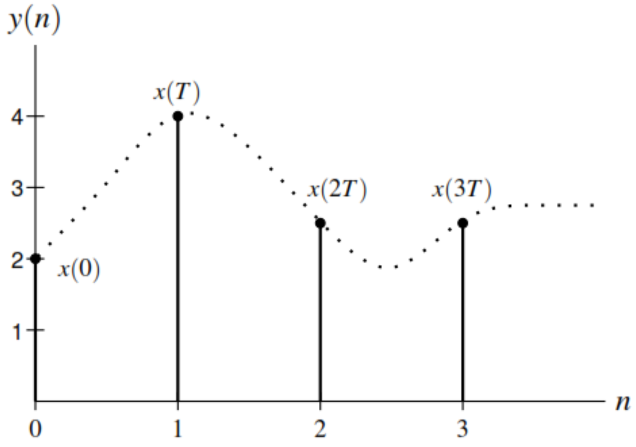
Input Function



Periodic Impulse Train

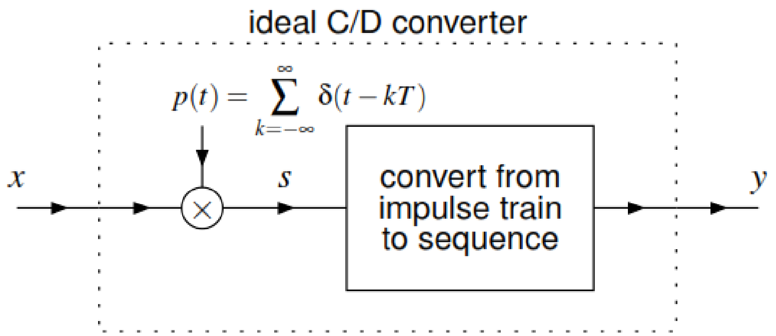


Impulse-Sampled Function
(Continuous-Time)



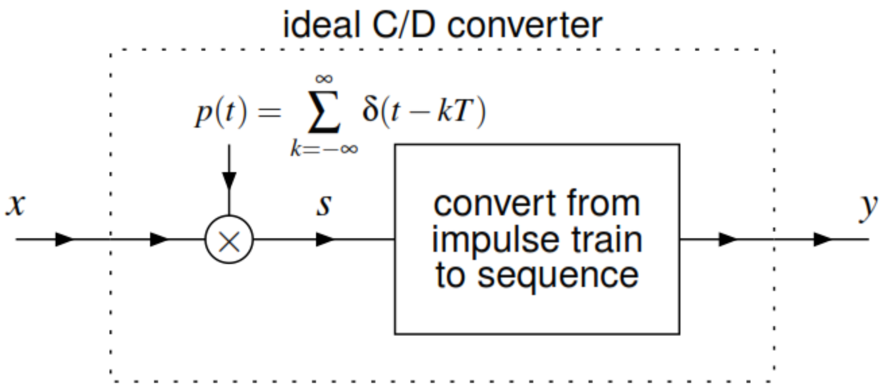
Output Sequence ***(Discrete-Time)***

MODEL OF SAMPLING: INVERTIBILITY OF SAMPLING REVISITED



- Since sampling is not invertible and our model of sampling consists of only two steps, at least one of these two steps must not be invertible.
- Recall the two steps in our model of sampling are as follows (in order):
  1. $x \longrightarrow s(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$; and
  2. $s(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \longrightarrow y(n) = x(nT)$.
- Step 1 cannot be undone (unless we somehow restrict which functions x can be sampled).
- Step 2 is always invertible.
- Therefore, the fact that sampling is not invertible is entirely due to step 1.

MODEL OF SAMPLING: CHARACTERIZATION



- In the time domain, the impulse-sampled function s is given by

$$s(t) = x(t)p(t) \quad \text{where} \quad p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

- In the Fourier domain, the preceding equation becomes

$$S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s) \quad (\text{where } \omega_s = \tfrac{2\pi}{T}).$$

- Thus, the spectrum of the impulse-sampled function s is a scaled sum of an infinite number of ***shifted copies*** of the spectrum of the original function x.

SAMPLING: FOURIER SERIES FOR A PERIODIC IMPULSE TRAIN

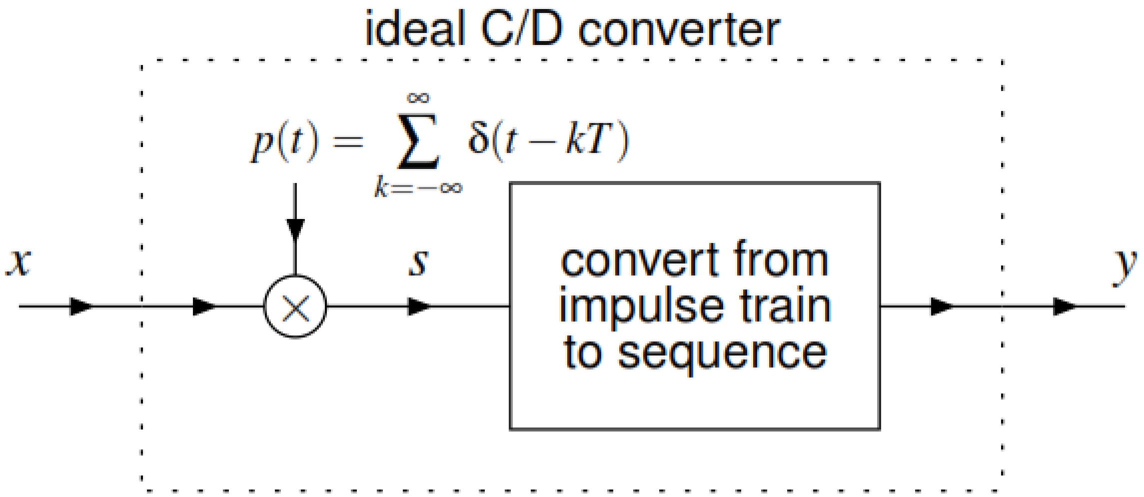
$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT), \quad \omega_s = \frac{2\pi}{T}$$

$$p(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}$$

$$\begin{aligned} c_k &= \frac{1}{T} \int_{-T/2}^{T/2} p(t) e^{-jk\omega_s t} dt \\ &= \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_s t} dt \\ &= \frac{1}{T} \int_{-\infty}^{\infty} \delta(t) e^{-jk\omega_s t} dt \\ &= \frac{1}{T} \\ &= \frac{\omega_s}{2\pi} \end{aligned}$$

$$p(t) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t}$$

SAMPLING: MULTIPLICATION BY A PERIODIC IMPULSE TRAIN



$$s(t) = p(t)x(t), \quad p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT), \quad \omega_s = \frac{2\pi}{T}$$

$$p(t) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t}$$

$$s(t) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t} x(t)$$

$$X = \mathfrak{F}x, \quad S = \mathfrak{F}s$$

$$S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$$

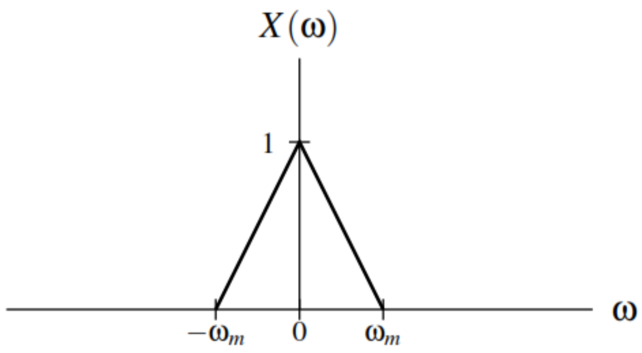
MODEL OF SAMPLING: ALIASING

- Consider frequency spectrum S of the impulse-sampled function s given by

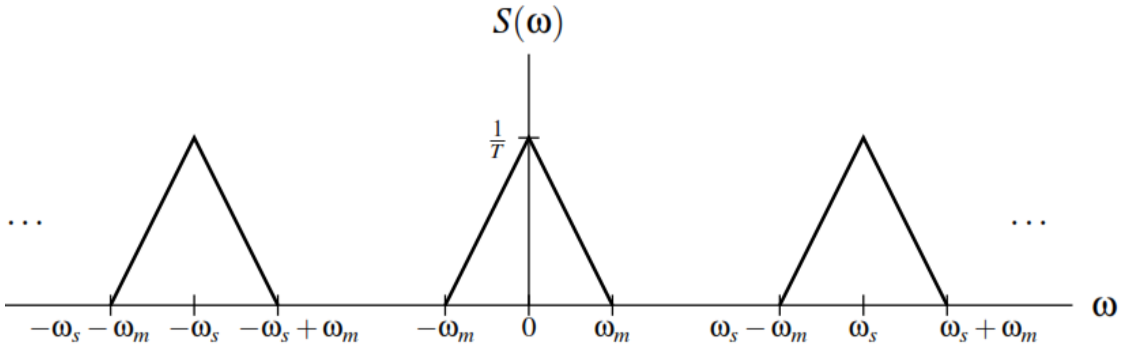
$$S(\omega) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{\infty} X(\omega - k\omega_s).$$

- The function S is a scaled sum of an infinite number of ***shifted copies*** of X.
- Two distinct behaviors can result in this summation, depending on ω_s and the bandwidth of x.
- In particular, the nonzero portions of the different shifted copies of X can either:
  1. overlap; or
  2. not overlap.
- In the case where overlap occurs, the various shifted copies of X add together in such a way that the original shape of X is lost. This phenomenon is known as **aliasing**.
- When aliasing occurs, the original function x cannot be recovered from its samples in y.

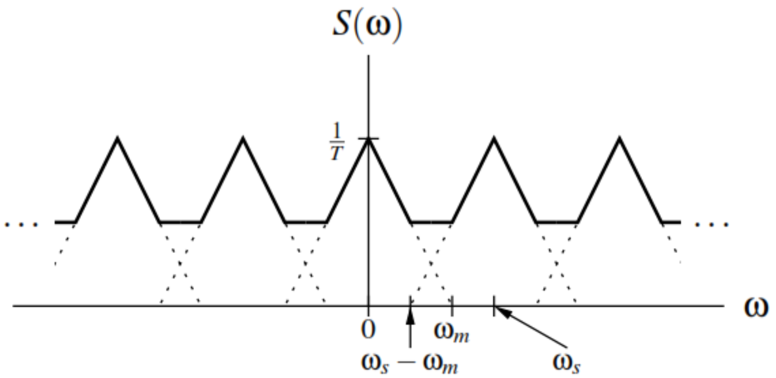
MODEL OF SAMPLING: ALIASING



Spectrum of Input Function
(Bandwidth ω_m)



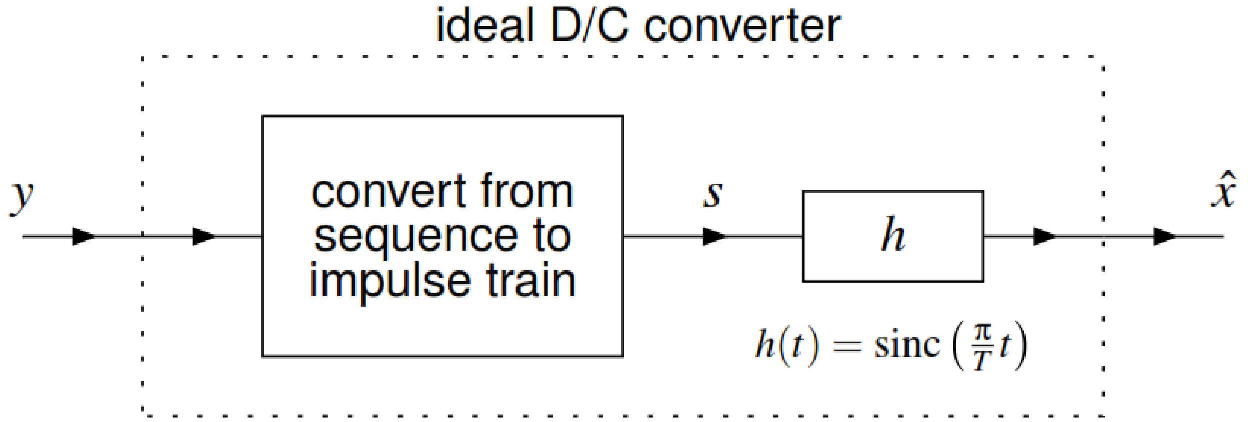
Spectrum of Impulse-Sampled Function:
No Aliasing Case
($\omega_s > 2\omega_m$)



Spectrum of Impulse-Sampled Function:
Aliasing Case
($\omega_s \leq 2\omega_m$)

MODEL OF INTERPOLATION

- For the purposes of analysis, interpolation can be modelled as shown below.



- The reconstruction of a function x from its sequence y of samples (i.e., bandlimited interpolation) consists of the following two steps (in order):
  1. Convert the sequence y to the impulse train s by using the samples in y as the weights of successive impulses in s so that $s(t) = \sum_{n=-\infty}^{\infty} y(n)\delta(t - Tn)$.
  2. Apply the lowpass filter with impulse response h to s to produce $\hat{x}$ so that $\hat{x}(t) = s * h(t) = \sum_{n=-\infty}^{\infty} y(n) \operatorname{sinc}\left[\frac{\pi}{T}(t - Tn)\right]$.
- The lowpass filter is used to eliminate the extra copies of the originally-sampled function's spectrum present in the spectrum of s.

SAMPLING THEOREM

- **Sampling Theorem.** Let x be a function with Fourier transform X, and suppose that $|X(\omega)| = 0$ for all ω satisfying $|\omega| > \omega_M$ (i.e., x is bandlimited to frequencies $[-\omega_M, \omega_M]$). Then, x is uniquely determined by its samples $y(n) = x(Tn)$ for all integer n, if

  $$\omega_s > 2\omega_M,$$

  where $\omega_s = \frac{2\pi}{T}$. The preceding inequality is known as the **Nyquist condition**. If this condition is satisfied, we have that

  $$x(t) = \sum_{n=-\infty}^{\infty} y(n) \operatorname{sinc}\left[\tfrac{\pi}{T}(t - Tn)\right],$$

  or equivalently (i.e., rewritten in terms of ω_s instead of T),

  $$x(t) = \sum_{n=-\infty}^{\infty} y(n) \operatorname{sinc}\left(\tfrac{\omega_s}{2}t - \pi n\right).$$

- We call $\frac{\omega_s}{2}$ the **Nyquist frequency** and $2\omega_M$ the **Nyquist rate**.