Applied Linear Algebra
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1. Find rank, nullity (dim Null)

a)

$$A = \begin{pmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{pmatrix} \xrightarrow[R_3 - 5R_1]{R_2 + R_1} \begin{bmatrix} 1 & -4 & 9 & -7 \\ 0 & -2 & 5 & 6 \\ 0 & 14 & -35 & 42 \end{bmatrix}$$

$$\xrightarrow[R_3 - 14R_2]{\frac{-R_2}{2},\ R_1 + 4R_2} \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & 1 & -5/2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We take $x_3 = t$, $x_4 = s$, $x_1 = -5s + t$, $x_2 = -3s + \frac{5t}{2}$

$$x = \begin{pmatrix} -5s + t \\ -3s + \frac{5t}{2} \\ t \\ s \end{pmatrix} = \begin{pmatrix} 1 \\ 5/2 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -5 \\ -3 \\ 0 \\ 1 \end{pmatrix} s$$

→ the nullity of the matrix is 2 A

b)

$$B = \begin{pmatrix} 1 & 3 & -2 & 4 \\ 0 & 1 & -1 & 2 \\ -2 & -6 & 4 & -8 \end{pmatrix} \xrightarrow[R_1 - 3R_2]{R_3 + 2R_1} \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 2 \\ 0 & 12 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_2 + \frac{R_3}{12}]{R_1 - \frac{R_3}{12}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow{R_2 + R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Take $x_4 = t$, $x_1 = 0$, $x_2 = 0$, $x_3 = 2t$

$$\begin{bmatrix} 0 \\ 0 \\ 2t \\ t \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} t$$ → The nulity of the matrix B is 1 B

2.

a/

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 10 & 6 \\ 8 & -7 & 5 \end{bmatrix} \xrightarrow[R_3 - 4R_1]{R_2 - 5R_1} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 35 & 7 \\ 0 & 5 & 1 \end{bmatrix}$$

$$\xrightarrow[\frac{R_2}{7}]{7R_3 - R_2} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis of row space $\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} \right\}$

The rank of matrix = 2

b/

$$B = \begin{bmatrix} -2 & -4 & 4 & -5 \\ 3 & 6 & -6 & -4 \\ -2 & -4 & 4 & 9 \end{bmatrix} \xrightarrow[R_3 - R_1]{2R_2 + 3R_1} \begin{bmatrix} -2 & -4 & 4 & 5 \\ 0 & 0 & 0 & 7 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$$\xrightarrow[\frac{R_2}{7}]{7R_3 - 4R_2} \begin{bmatrix} -2 & -4 & 4 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis of row space $= \left\{ \begin{bmatrix} -1 \\ -4 \\ 4 \\ 5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

The rank of matrix = 2

3.

a) $S = \{(4,3,2), (0,3,2), (0,0,2)\}$

Let $u = c_1(4,3,2) + c_2(0,3,2) + c_3(0,0,2)$

$$\begin{cases} 4c_1 = 8 \\ 3c_1 + 3c_2 = 3 \\ 2c_1 + 2c_2 + 2c_3 = 8 \end{cases} \Rightarrow \begin{cases} c_1 = 2 \\ c_2 = -1 \\ c_3 = 3 \end{cases}$$

$u = (8,3,8) = 2(4,3,2) - (0,3,2) + 3(0,0,2)$

Now,

$$w = \{ (2s - t, s, t, s \mid s, t \in R \}$$

$$= \{ s(2,1,0,1) + t(-1,0,1,0) \mid s,t \in R \}$$

Basis $= \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ dim = 2

b) $S = \{(1,0,0),(1,1,0),(1,1,1)\}$

$$c_1(1,0,0) + c_2(1,1,0) + c_3(1,1,1) = (0,0,0)$$

$$\Rightarrow (c_1 + c_2 + c_3, c_2 + c_3, c_3) = (0,0,0)$$

$$\Rightarrow \left.\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_3 &= 0 \end{aligned}\right\} \Rightarrow c_1 = 0 = c_2 = 0$$

$\Rightarrow$ S is linearly independent of 3-vectors $\Rightarrow$ S is a basis of R^3

$$(8,3,8) = (c_1 + c_2 + c_3, c_2 + c_3, c_3)$$

$$\Rightarrow \begin{aligned} c_1 + c_2 + c_3 &= 8 \\ c_2 + c_3 &= 3 \\ c_3 &= 8 \end{aligned} \Rightarrow \begin{aligned} c_1 &= 5 \\ c_2 &= -5 \end{aligned}$$

$$\Rightarrow (8,3,8) = 5(1,0,0) - 5(1,1,0) + 8(1,1,1)$$

c) $S = \{(0,0,0),(1,3,4),(6,1,-2)\}$

In the set S given as S = {(0, 0, 0), (1, 3, 4), (6, 1, –2)}, the first vector is the zero vector (0, 0, 0). The presence of the zero vector automatically means that the vectors are not linearly independent because the zero vector can be represented as a scalar multiple of any other vector, which violates the definition of linear independence (no vector in the set can be written as a linear combination of others).

Since the set S does not consist of linearly independent vectors, it cannot be a basis for the vector space $\mathbb{R}^3$.

Therefore, it is not possible to express any vector, including u = (8, 3, 8), as a linear combination of the vectors in S because S is not a basis for $\mathbb{R}^3$. ~

4. Find dim and basis of the subspace:

$$H = \left\{ \begin{pmatrix} a - 4b - 2c \\ 2a + 5b - 4c \\ -a + 2c \\ -3a + 7b + 6c \end{pmatrix}, a, b, c \in R \right\}$$

$$\Rightarrow H = \left\{ a \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix} + b \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix} + c \begin{bmatrix} -2 \\ -4 \\ 2 \\ 6 \end{bmatrix} : a, b, c \in R \right\}$$

$$H = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix} \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix} \begin{bmatrix} -2 \\ -4 \\ 2 \\ 6 \end{bmatrix} \right\}$$

Now,
$$\begin{bmatrix} 1 & -4 & -2 \\ 2 & 5 & 4 \\ -1 & 0 & 2 \\ 3 & 7 & 6 \end{bmatrix} \xrightarrow[\substack{R_3 + R_1 \\ R_4 - 3R_1}]{R_2 - 2R_1} \begin{bmatrix} 1 & -4 & -2 \\ 0 & 13 & 0 \\ 0 & -4 & 0 \\ 0 & 19 & 12 \end{bmatrix} \xrightarrow{R_4 \leftrightarrow R_2} \begin{bmatrix} 1 & -4 & -2 \\ 0 & 19 & 12 \\ 0 & -4 & 0 \\ 0 & 13 & 0 \end{bmatrix}$$

$$\xrightarrow[19R_3 + 4R_2]{4R_4 + 13R_3} \begin{bmatrix} 1 & -4 & -2 \\ 0 & 19 & 12 \\ 0 & 0 & 48 \\ 0 & 0 & 0 \end{bmatrix}$$

Each column contain pivot element

$$\Rightarrow \text{Basis for } H = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix} \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix} \begin{bmatrix} -2 \\ -4 \\ 2 \\ 6 \end{bmatrix} \right\}$$

$\Rightarrow$ Dim $(H) = 3$