7/12/2019.

$x(t) = \underset{\text{amplitude}}{A} e^{j\theta_0} e^{\underset{\text{frequency}}{j\omega_0 t}}$ : exponential signal

(1)

$$x(t) \longrightarrow \boxed{\text{LTI System } \underset{\text{unit impulse response}}{h(t)}} \longrightarrow y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$x(t) = A e^{j\omega_0 t}$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau) A e^{j\omega_0 (t-\tau)} d\tau = A e^{j\omega_0 t} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega_0 \tau} d\tau$$

Frequency response of the system: $\boxed{H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt}$

$x(t) = A e^{j\theta_0} e^{j\omega_0 t} \xrightarrow{\text{system}} y(t) = A e^{j\theta_0} e^{j\omega_0 t} H(\omega_0)$

$H(\omega_0)$ is complex value

$H(\omega_0) = |H(\omega_0)| e^{j\angle H(\omega_0)}$

$|H(\omega_0)| = |H(-\omega_0)|$ : Magnitude response is an even function

$\angle H(\omega_0) = -\angle H(-\omega_0)$ : odd function

$$\boxed{\begin{aligned} x(t) &= A e^{j\theta_0} e^{j\omega_0 t} \xrightarrow{\text{system}} y(t) = A|H(\omega_0)| e^{j(\theta_0 + \angle H(\omega_0))} \cdot e^{j\omega_0 t} \\ x(t) &= A e^{-j\theta_0} e^{-j\omega_0 t} \xrightarrow{\text{system}} y(t) = A|H(-\omega_0)| e^{j(-\theta_0 + \angle H(-\omega_0))} \cdot e^{-j\omega_0 t} \\ x(t) &= A\cos(\omega_0 t + \theta_0) \xrightarrow{\text{system}} y(t) = A|H(\omega_0)| \cos(\omega_0 t + \theta_0 + \angle H(\omega_0)) \end{aligned}}$$

$e^{-\infty} \approx 0$

$e^{+\infty} \approx \infty$

Ex 1: The LTI system has the unit impulse response: $h(t) = e^{-10t} u(t)$

a) Find the frequency response $H(\omega)$

b) Find the output $y(t)$ for the input $x(t) = 2\cos(20t + 30^\circ)$

c) ——— $y(t)$ ——— $x(t) = 3\cos(2\pi t + 30^\circ) + 4\sin(4\pi t - 60^\circ)$

a) The frequency response:

$$\boxed{H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t}\, dt} = \int_{0}^{+\infty} e^{-10t} \cdot e^{-j\omega t}\, dt$$

$$= \int_{0}^{+\infty} e^{-(j\omega+10)t} dt = \left[\frac{e^{-(j\omega+10)t}}{-(j\omega+10)}\right]_0^{\infty}$$

$$= \frac{1}{-(j\omega+10)} \times \left[\underbrace{e^{-\infty}}_{0} - \underbrace{e^{-0}}_{1}\right] = \frac{1}{j\omega+10}$$

b) $x(t) = 2\cos(20t + 30^\circ) \Rightarrow \omega_0 = 20\,\text{rad/s}$

$y(t) = 2 \cdot |H(\omega_0)| \times \cos(20t + 30^\circ + \angle H(\omega_0))$

$\omega_0 = 20\,\text{rad/s} \Rightarrow H(\omega_0) = \frac{1}{10+j20} = \frac{\sqrt{5}}{50} e^{-j63^\circ}$

$\Rightarrow y(t) = 2 \times \frac{\sqrt{5}}{50} \cos(20t + 30^\circ - 63^\circ)$

c) $x(t) = 3\cos(2\pi t + 30^\circ) + 4\sin(4\pi t - 60^\circ)$

$\omega_1 = 2\pi\ (\text{rad/s})$ ; $\omega_2 = 4\pi\ (\text{rad/s})$

$$\boxed{\begin{aligned} y(t) = {} & 3|H(\omega_1)| \cos(2\pi t + 30^\circ + \angle H(\omega_1)) \\ & + 4|H(\omega_2)| \cos(4\pi t - 60^\circ + \angle H(\omega_2)) \end{aligned}}$$

- $\omega_1 = 2\pi\ (\text{rad/s}) \rightarrow H(\omega_1) = \frac{1}{10+j2\pi} = 0.08\, e^{-j32^\circ}$
- $\omega_2 = 4\pi\ (\text{rad/s}) \rightarrow H(\omega_2) = \frac{1}{10+j4\pi} = 0.06\, e^{-j51^\circ}$

$\Rightarrow y(t) = 3 \times 0.08 \cos(2\pi t + 30^\circ - 32^\circ)$
$\quad + 4 \times 0.06 \cos(4\pi t - 60^\circ - 51^\circ)$

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## Superposition

$$x(t) = A_0 + A_1 \cos(\omega_1 t + \theta_1) + A_2 \sin(\omega_2 t + \theta_2)$$

$$\Rightarrow y(t) = A_0 H(0) + A_1 |H(\omega_1)| \cos(\omega_1 t + \theta_1 + \angle H(\omega_1)) + A_2 |H(\omega_2)| \sin(\omega_2 t + \theta_2 + \angle H(\omega_2))$$

Thm !!

| Analog Signals | Discrete time signal |
|---|---|
| $x(t) = \sum_{k=-\infty}^{\infty} X[K] \cdot e^{jK\omega_0 t}$ | $x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$ |
| $X[K] = \frac{1}{T_0} \int_{T_0} x(t) \cdot e^{-jK\omega_0 t}\, dt$ | $\int_{-\infty}^{\infty} \delta(t-t_0)\, dt = 1$ |
| $x[n]$ periodic with period $n$ | |

Ex$_2$: Find the signal $x(t)$: a) $X(\omega) = \delta(\omega)$ ; b) $X(\omega) = \delta(\omega - \omega_0)$

a) $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t}\, \delta(\omega)\, dt$

$\rightarrow X(\omega)\delta(\omega - \omega_0) = X(\omega_0)\delta(\omega - \omega_0)\Big|_{\omega_0 = 0}$

$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j0t}\, \delta(\omega)\, dt = \frac{1}{2\pi}$

$$\boxed{x(t) = 1 \xrightarrow[\text{Transform}]{\text{Fourier}} X(\omega) = 2\pi\,\delta(\omega)}$$

b) $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega_0 t}\, \delta(\omega - \omega_0)\, d\omega$

$= \frac{e^{j\omega_0 t}}{2\pi}$

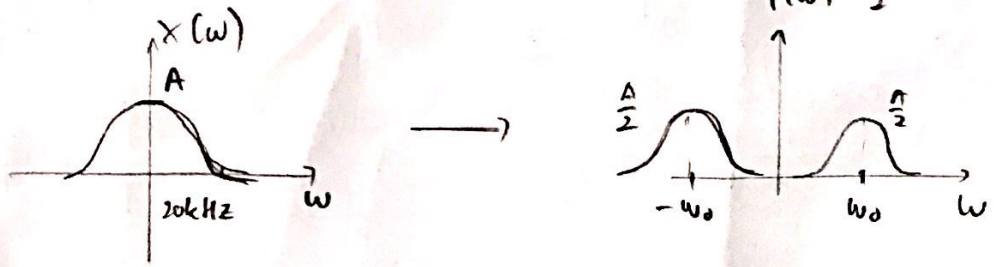
$$x(t) = e^{j\omega_0 t} \xleftrightarrow{F.T} X(\omega) = 2\pi\,\delta(\omega - \omega_0)$$

$$x(t) = e^{-j\omega_0 t} \xleftrightarrow{F.T} X(\omega) = 2\pi\,(\omega + \omega_0)$$

$$x(t) = 2\cos(\omega_0 t) \xleftrightarrow{F.T} X(\omega) = 2\pi\,\delta(\omega + \omega_0) + 2\pi\,\delta(\omega - \omega_0)$$

$$x(t) = A\cos(\omega_0 t) \xleftrightarrow{F.T} X(\omega) = \frac{A}{2} 2\pi\,\delta(\omega - \omega_0) + \frac{A}{2} 2\pi\,\delta(\omega + \omega_0)$$

$Y(\omega) = \frac{1}{2} X(\omega + \omega_0) + \frac{1}{2} X(\omega - \omega_0)$

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**Ex 3:** $x(t) \xleftrightarrow{F.T} X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

a) $y(t) = x(t) e^{j\omega_0 t} \xleftrightarrow{F.T} Y(\omega)$

b) $y(t) = x(t) \cos \omega_0 t \xleftrightarrow{F.T} Y(\omega)$

a) $Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{j\omega_0 t} e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-j(\omega-\omega_0)t} dt$

$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \; ; \; X(\omega-\omega_0) = \int_{-\infty}^{\infty} x(t) e^{-j(\omega-\omega_0)t} dt$

$\Rightarrow Y(\omega) = X(\omega - \omega_0)$

b) $x(t) \xleftrightarrow{FT} X(\omega)$ | $x(t) \cdot e^{-j\omega_0 t} \xleftrightarrow{FT} X(\omega + \omega_0)$

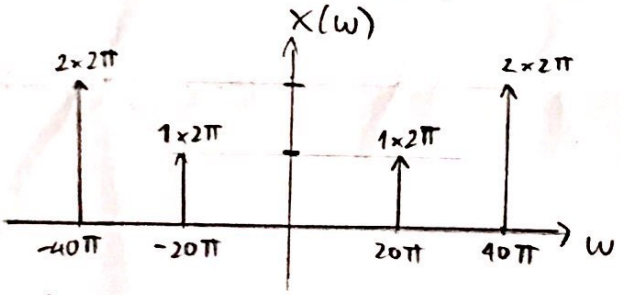
$x(t) e^{j\omega_0 t} \xleftrightarrow{FT} X(\omega - \omega_0)$

$\boxed{\text{Modulation: } x(t) \cos \omega_0 t \xleftrightarrow{FT} \frac{1}{2} X(\omega-\omega_0) + \frac{1}{2} X(\omega-\omega_0)}$ !!

**Ex 4:** $x(t) = 2\cos(2\pi 10 t) + 4\cos(2\pi 20 t)$ | a) Find & sketch $X(\omega)$ ; b) $y(t) = x(t)\cos(2\pi \cdot 1000 t)$ Find & sketch $Y(\omega)$

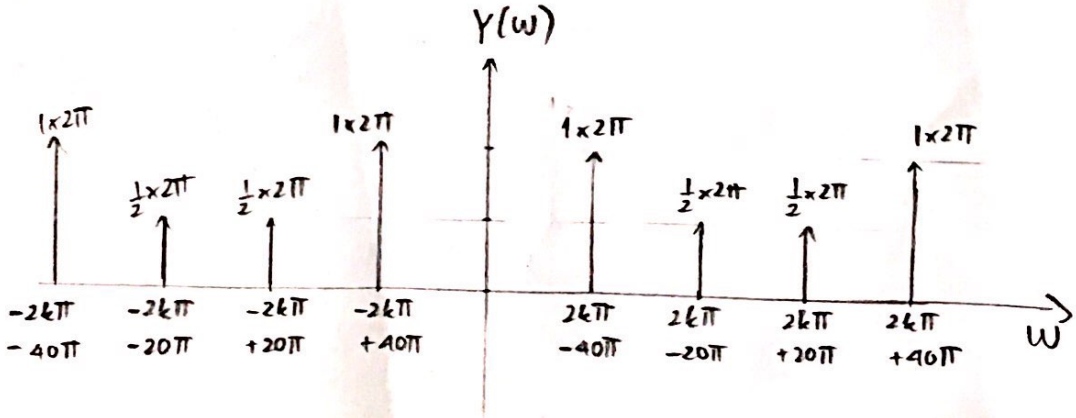
a) $x(t) = 2\cos(20\pi t) + 4\cos(40\pi t)$

$\Rightarrow X(\omega) = \left[\frac{2}{2} \times 2\pi \delta(\omega + 20\pi) + \frac{2}{2} 2\pi \delta(\omega - 20\pi)\right] + \left[\frac{4}{2} 2\pi \delta(\omega + 40\pi) + \frac{4}{2} 2\pi \delta(\omega - 40\pi)\right]$




b) $y(t) = x(t) \cos(2000\pi t)$

$\Rightarrow Y(\omega) = \frac{1}{2} X(\omega + 2000\pi) + \frac{1}{2} X(\omega - 2000\pi)$




RF Signals (Radio Frequency)

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14/12/2019

(1)

# Laplace Transform (more general)

(analyze system, Causal / X, Stable / X.)

$x(t) \xleftrightarrow[\text{Transform}]{\text{Fourier}} X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

frequency.

**Relationship** $X(\omega) = X(s)\Big|_{s=j\omega}$

$x(t) \xleftrightarrow[\text{Transform}]{\text{Laplace}} X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$ ; $s = \sigma + j\omega$

Ex: Find the laplace transform of the following system $\begin{cases} a)\ x(t) = e^{-2t}u(t) \\ b)\ x(t) = -e^{-2t}u(-t) \end{cases}$

Sol

a) Laplace Trans: $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st}\, dt$ (Step-by-Step)

$x(t) = e^{-2t}u(t)$ ← if $t < 0$, $u(t) = 0$, so we take $t = 0 \to t$ to have $u(t) = 1$.

$X(s) = \int_{0}^{+\infty} e^{-2t} \cdot 1 \cdot e^{-st}\, dt$

$X(s) = \int_{0}^{+\infty} e^{-(s+2)t}\, dt = \left[\dfrac{e^{-(s+2)t}}{-(s+2)}\right]_0^{+\infty} = \dfrac{1}{-(s+2)}\left[\underbrace{e^{-(s+2)(+\infty)}}_{\sigma + j\omega} - \underbrace{e^{-(s+2)\times 0}}_{1}\right]$

$= \dfrac{1}{-(s+2)}\left[\boxed{e^{-[\sigma+2+j\omega][+\infty]}} - 1\right]$

$= 0$ when $-(\sigma+2) < 0$ or $\sigma > -2$

$= \infty$ when $\sigma < -2$

$= \dfrac{1}{-(s+2)} \cdot (-1) = \dfrac{1}{s+2}$ when $\sigma > -2$

RoC: $\sigma > -2$ (Region of Convergence)

$z = a + jb = re^{j\theta}$ ← phase

$r$ ← amplitude / magnitude

$r = \sqrt{a^2 + b^2}$

$\theta = tg^{-1}\left(\dfrac{b}{a}\right)$

$e^{a+jb} = e^{a} e^{jb}$

ampli

$e^{(a+jb)t} = e^{at} \cdot e^{jbt}$

mag

$t \to +\infty$. depends on $a$ is (+) or (−)

$e^{-\infty} \to 0$

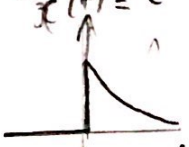
$e^{+\infty} \to \infty$

$\lim_{t \to +\infty} e^{(a+jb)t} = \begin{cases} 0 & \text{if } a < 0 \\ +\infty & \text{if } a > 0 \end{cases}$

* We can apply directly this formula next time

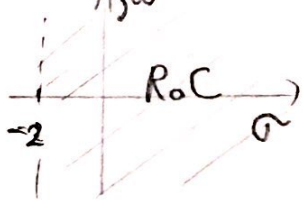
$\boxed{x(t) = e^{-at}u(t) \xleftrightarrow[\text{Transform}]{\text{Laplace}} X(s) = \dfrac{1}{s+a} \quad \text{RoC}: \sigma > -a}$

$x(t) = e^{-2t}u(t) \longrightarrow X(s) = \dfrac{1}{s+2}$ RoC: $\sigma > -2$



Causal Signal

$x(t) = 0$ for $t < 0$




when RoC is on the Right hand side

⇒ Causal Signal

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Ex: b) Find laplace transf of: $x(t) = -e^{-2t}u(-t)$

$u(-t) = \begin{cases} 1 & \text{for } t \le 0 \\ 0 & \text{for } t > 0 \end{cases}$

Def of laplace Trans: $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$.

$\Rightarrow X(s) = \int_{-\infty}^{\infty} -e^{-2t} \cdot 1 \cdot e^{-st} dt$ ; $\boxed{u = -t} \Rightarrow du = -dt$. $\sigma + j\omega$

$\lim_{t \to \infty} e^{(a+jb)t} = \begin{cases} 0, & a < 0 \\ +\infty, & a > 0 \end{cases}$

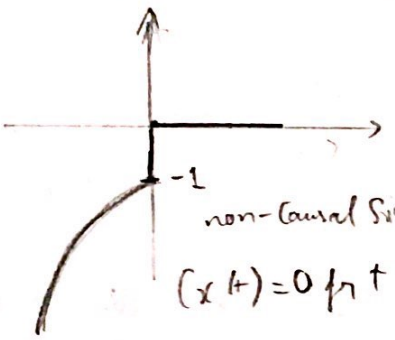
$= \int_{+\infty}^{0} e^{(s+2)u} du = \left. \frac{e^{(s+2)u}}{(s+2)} \right|_{+\infty}^{0} = \frac{e^{(s+2)\times 0}}{s+2} - \frac{e^{(s+2)(+\infty)}}{s+2}$

$= \frac{1}{s+2} + 0$ if $\sigma + 2 < 0$

$$\boxed{x(t) = -e^{-at}u(-t) \xleftrightarrow[\text{Transform}]{\text{laplace}} X(s) = \frac{1}{s+a} \quad \text{RoC}: \sigma < -a}$$

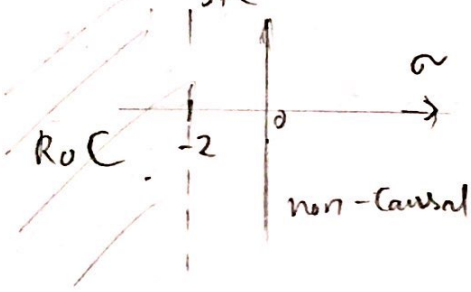
$x(t) = -e^{-2t}u(-t) \longrightarrow X(s) = \frac{1}{s+2}$, RoC: $\sigma < -2$.

RoC on Left hand Signal




non-Causal Signal

$(x(t) = 0 \text{ for } t > 0)$.




non-Causal Signal

$x(t) = A\,\delta(t) \longleftrightarrow X(s) = A$.

$x(t) = e^{-at}u(t) \longleftrightarrow X(s) = \frac{1}{s+a}$.

$x(t) = u(t) \longleftrightarrow X(s) = \frac{1}{s}$

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(2)

Ex: An LTI System has a unit impulse response: $h(t) = e^{-2t}u(t)$

a) Find the laplace transform $H(S)$

b) Using the laplace transform to find the response $y(t)$ if $x(t) = e^{-4t}u(t)$

$$x(t) = e^{-at}u(t) \xleftrightarrow[\text{Trans}]{LL} X(S) = \frac{1}{S+a} \quad RoC: \sigma > -a$$

$a = 2$

Solution

a) $h(t) = e^{-2t}u(t)$

Transfer function

$$H(S) = \frac{1}{S+2} \quad RoC: \sigma > -2$$

b) $y(t) = x(t) * h(t)$

$Y(S) = X(S) \cdot H(S)$

$$x(t) = e^{-4t}u(t) \longleftrightarrow X(S) = \frac{1}{S+4} \quad RoC\ \sigma > -4$$

$$Y(S) = \frac{1}{S+4} \cdot \frac{1}{S+2} \quad RoC: \begin{cases} \sigma > -2\ (H_S) \\ \sigma > -4\ (X_S) \end{cases} \Rightarrow \sigma > -2$$

$$Y(S) = \frac{1}{(S+4)(S+2)} = \frac{A_1}{S+4} + \frac{A_2}{S+2} \quad \text{(partial fractions)}$$

$$\frac{1\,(S+4)}{(S+4)(S+2)} = \frac{A_1\,(S+4)}{(S+4)} + \frac{A_2\,(S+4)}{(S+2)}$$

$$A_1 = \frac{1}{S+2}\Big|_{S=-4} = \frac{1}{-4+2} = -\frac{1}{2}$$

$$A_2 = \frac{1}{S+4}\Big|_{S=-2} = \frac{1}{-2+4} = \frac{1}{2}$$

[Residue "hay dư"]

$$Y(S) = -\frac{1}{2} \cdot \frac{1}{S+4} + \frac{1}{2}\frac{1}{S+2} \quad RoC\ \sigma > -2$$

(Còn 1 phương pháp để tính hệ số)

$$\rightarrow y(t) = -\frac{1}{2}e^{-4t}u(t) + \frac{1}{2}e^{-2t}u(t)$$

Ex: A causal LTI System has the output $y(t) = e^{-5t}u(t)$ when the input $x(t) = e^{-2t}u(t)$

a) Find the unit impulse response ($h(t)$?)

b) Find the unit Step response ($x(t) = u(t) \longrightarrow y(t)$?)

Solution: a) $x(t) = A\,\delta(t) \rightarrow X(S) = A$.

$y(t) = x(t) * h(t)$

$Y(S) = X(S)\,H(S) \Rightarrow \boxed{H(S) = \frac{Y(S)}{X(S)}}$ transfer function

$y(t) = e^{-5t}u(t) \Rightarrow Y(S)$

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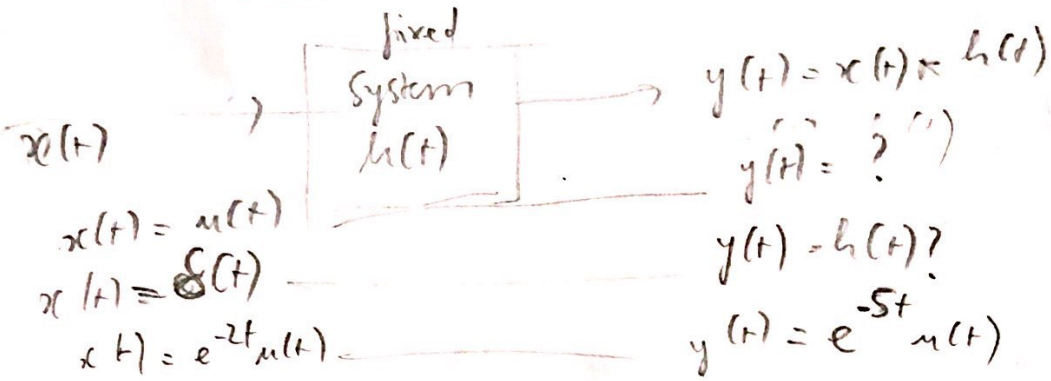


b) $H(s) = \frac{s+2}{s+5}$ $\quad RoC\ \sigma > -2$

$x(t) = u(t) \rightarrow X(s) = \frac{1}{s}$ $\quad RoC\ \sigma > 0$

$$Y(s) = X(s).H(s) = \frac{s+2}{(s+5)s} = \frac{A_1}{s} + \frac{A_2}{s+5}$$

$$A_1 = \frac{s+2}{s+5}\Big|_{s=0} = \frac{2}{5} \; ; \; A_2 = \frac{s+2}{s}\Big|_{s=-5} = \frac{3}{5}$$

$$Y(s) = \frac{2}{5}\frac{1}{s} + \frac{3}{5}\frac{1}{s+5} \rightarrow y(t) = \frac{2}{5}u(t) + \frac{3}{5}e^{-5t}u(t)$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = \left[\frac{2}{5} + \frac{3}{5}e^{-5t}\right]u(t)$$

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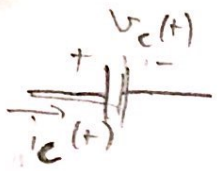
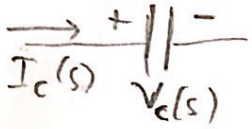
$$Z_C = \frac{1}{CS}$$



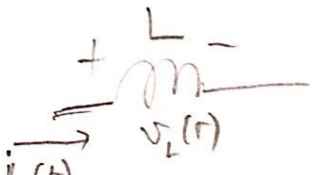



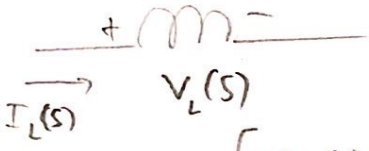

$$i_c(t) = C\frac{dv_c(t)}{dt} \longrightarrow I_c(S) = C\cdot\left[SV_c(S) - v_c(0^-)\right]$$

zero state: $v_c(0^-) = 0$

$$\boxed{\begin{aligned} &I_c(S) = CS\,V_c(S) \\ &V_c(S) = \frac{1}{CS}\cdot I_c(S) = Z_C\, I_c(S) \\ &Z_C = \frac{1}{CS} \quad \text{Impedance} \end{aligned}}$$

$$Z_L = LS$$



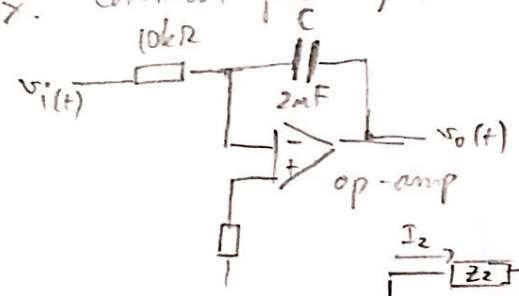




$$v_L(t) = L\frac{di_L(t)}{dt} \longrightarrow V_L(S) = L\left[SI_L(S) - v_L(0^-)\right]$$

zero-state: $v_L(0^-) = 0$

$$\boxed{\begin{aligned} &V_L(S) = LS\,I_L(S) = Z_L I_L(S) \\ &\text{Impedance: } Z_L = LS \end{aligned}}$$

Ex: Consider following circuit:



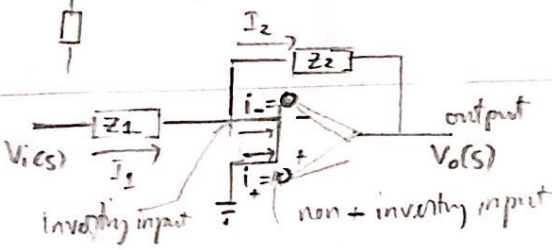

a) Find the transfer function: $H(S) = \frac{V_o(S)}{V_i(S)}$

b) Find the frequency response: $H(\omega)$ $s = j\omega$

c) Sketch the magnitude $|H(\omega)|$ and phase response $\angle H(\omega)$

d) Find $y(t)$ for $x(t) = 5\cos(10t) + 6\cos(20t)$

Review




Ideal op amp: $i_- = 0$, $i_+ = 0$

Negative feed back: $v_- = v_+$

$$I_1 = I_2$$

$$\frac{V_i - 0}{Z_1} = \frac{0 - V_o}{Z_2} \Rightarrow V_o = -\frac{Z_2}{Z_1}V_i$$

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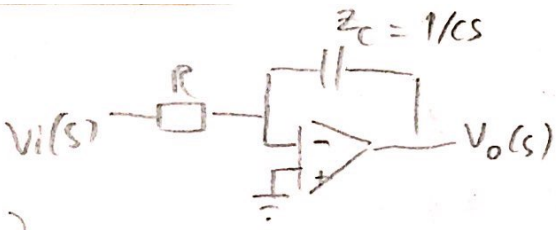
Solution a)




$$Z_2 = \frac{1}{CS} = \frac{1}{2\times10^{-6}\,s}$$
$$Z_1 = R = 10\times10^{3}\,\Omega$$

$$V_o(S) = -\frac{Z_2}{Z_1} V_i(S) \Rightarrow H(S) = \frac{V_o(S)}{V_i(S)} = -\frac{Z_2}{Z_1} = -\frac{1/CS}{R}$$

$$= \frac{-1}{CRS} = -\frac{1}{20\times10^{-3}\,s}$$

b) $x(t) = A_1 \cos(\omega_1 t) + A_2 \sin(\omega_2 t)$

$y(t) = A_1 |H(\omega)| \cos(\omega_1 t + \angle H(\omega_1) \ldots$

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21/12/2019

(1)

$x(t) \longrightarrow$ [LTI System $h(t)$ $h(t)$] $\longrightarrow y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

unit impulse response.

Laplace Trans

$X(s)$ $\quad H(s)$ (Transfer funct) $\quad Y(s) = X(s) * H(s)$

Fourier Trans: $X(\omega)$ $\quad H(\omega)$ Frequency response $\quad Y(\omega) = X(\omega) H(\omega)$

Special cases:

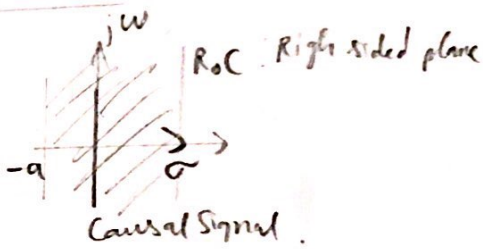
$$x(t) = A_0 + A_1 \cos(\omega_1 t + \varphi_1) + A_2 \sin(\omega_2 t + \varphi_2)$$

$$y(t) = A_0 H(0) + A_1 |H(\omega_1)| \cos(\omega_1 t + \varphi_1 + \angle H(\omega_1)) + A_2 |H(\omega_2)| \cos(\omega_2 t + \varphi_2 + \angle H(\omega_2))$$

---

$x(t) = \delta(t) \longleftrightarrow X(s) = 1$

$x(t) = e^{-at} u(t) \longleftrightarrow X(s) = \frac{1}{s+a}$ $\quad$ RoC: $\sigma > -a$



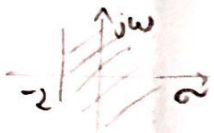

- Linearity
- Convolution
- Stability $\quad$ system is (BIBO) stable if: $|x(t)| \le A < \infty \Leftrightarrow |y(t)| \le B < \infty$

$y(t) = x(t) * h(t)$

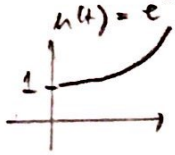
BIBO system: $y(t) = \int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

$h(t) = e^{-2t} u(t)$

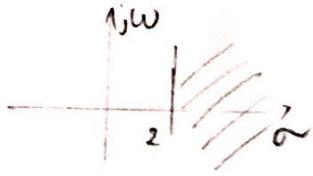
Stable $\Rightarrow H(s) = \frac{1}{s+2}$ $\quad$ RoC: $\sigma > -2$




$h(t) = e^{2t} u(t)$




non-stable $\Rightarrow H(s) = \frac{1}{s-2}$ $\quad$ RoC: $\sigma > 2$




In Laplace domain:

Stable system: $\sigma = 0$ belongs to RoC.

unstable: $\sigma = 0$ outside RoC.

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Ex: A causal LTI system has the transfer function:

$$H(s) = \frac{(s+2)}{(s+1)(s+4)}$$

a) Find the unit impulse response $h(t)$

b) Discuss the stability of the system

c) Find the output $y(t)$ for input $x(t) = e^{-5t}u(t)$

Solution

a) Given: $H(s) = \frac{s+2}{(s+1)(s+4)}$, causal $\longrightarrow$ Find $h(t)$?

$x(t) = \delta(t) \longleftrightarrow H(s) = 1$

$x(t) = e^{-at}u(t) \longleftrightarrow H(s) = \frac{1}{s+a}$ RoC $\sigma > -a$

we have partial fraction: $H(s) = \frac{s+2}{(s+1)(s+4)} = \frac{A_1}{s+1} + \frac{A_2}{s+4}$

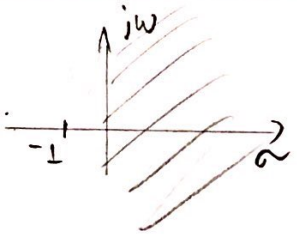
$A_1 = \left.\frac{s+2}{s+4}\right|_{s=-1} = \frac{(-1)+2}{(-1)+4} = \frac{1}{3}$ ; $A_2 = \left.\frac{s+2}{(s+1)}\right|_{s=-4} = \frac{2}{3} \Rightarrow H(s) = \frac{1}{3} \times \frac{1}{s+1} + \frac{2}{3} \times \frac{1}{s+4}$

we have: $H(s) = \frac{1}{3} \times \frac{1}{s+1} + \frac{2}{3} \times \frac{1}{s+4} \longrightarrow h(t) = \frac{1}{3} e^{-t} u(t) + \frac{2}{3} e^{-4t} u(t)$

b) $H(s) = \frac{1}{3} \frac{1}{s+1} + \frac{2}{3} \cdot \frac{1}{s+4}$

Causal $\quad$ RoC $\begin{cases} \sigma > -1 \\ \sigma > -4 \end{cases}$

$\rightarrow$ RoC $\sigma > -1$.




$\sigma = 0$ belongs to RoC

(OR: RoC includes $\sigma = 0$)

$\Rightarrow$ System is stable.

c) input $x(t) = e^{-5t}u(t) \longrightarrow$ output $y(t) = ?$

we can find by using convolution, but it's time consuming so we can use direct formula:

$\boxed{Y(s) = X(s)H(s)}$ ; $X(s) = \frac{1}{s+5} \rightarrow Y(s) = \frac{1}{(s+5)} \times \frac{(s+2)}{(s+1)(s+4)}$

$A_1 = \left.\frac{(s+2)}{(s+1)(s+4)}\right|_{s=-5} = -\frac{3}{4}$

$A_2 = \left.\frac{(s+2)}{(s+5)(s+4)}\right|_{s=-1} = \frac{1}{12}$

$A_3 = \left.\frac{(s+2)}{(s+5)(s+1)}\right|_{s=-4} = \frac{2}{3}$

$Y(s) = -\frac{3}{4} \cdot \frac{1}{s+5} + \frac{1}{12} \cdot \frac{1}{s+1} + \frac{2}{3} \frac{1}{s+4}$

$y(t) = -\frac{3}{4} e^{-5t} u(t) + \frac{1}{2} e^{-t} u(t) + \frac{2}{3} e^{-4t} u(t)$

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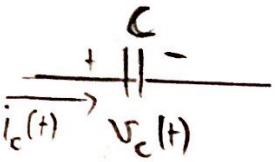
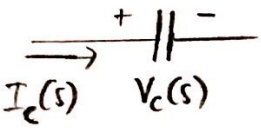
$$x(t) \longleftrightarrow X(s)$$

$$y(t) = \frac{dx(t)}{dt} \longleftrightarrow Y(s) = sX(s) - x(0^-)$$

Impedance $\longrightarrow Z_c = \frac{1}{Cs}$



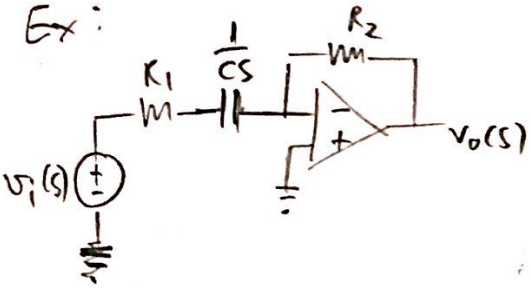

$$i_c(t) = C\frac{dv_c(t)}{dt} \longrightarrow I_c(s) = C[sV_c(s) - v_c(0^-)]$$

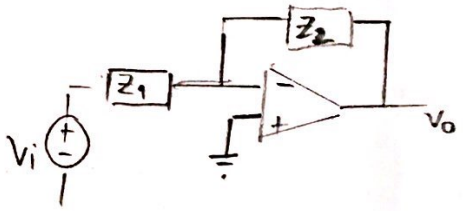
Zero_State : $v_c(0^-) = 0$

$I_c(s) = Cs\,V_c(s) \Rightarrow$ Ohm Law : $V_c(s) = \frac{1}{Cs} I_c(s)$

---

Ex :




Inverting Amplifier

$$V_o = -\frac{Z_2}{Z_1} V_i$$

a)

$$Z_1 = R_1 + \frac{1}{Cs} = \frac{R_1Cs + 1}{Cs}$$

$$V_o(s) = -\frac{Z_2}{Z_1} V_i(s) = -\frac{R_2Cs}{R_1(s+1)} V_c(s)$$

Transfer function:

$$H(s) = \frac{V_o(s)}{V_i(s)} = -\frac{R_2Cs}{R_1Cs+1}$$

$$= \frac{-10^5 10^{-7} s}{10^4 \cdot 10^{-7} s + 1} = -\frac{0.01s}{0.001s+1} = -\frac{10s}{s+1000}.$$

b) $H(s) \longrightarrow H(\omega)$ ?

$$H(\omega) = H(s)\Big|_{s=j\omega} = -\frac{10j\omega}{j\omega + 1000}$$

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## Z - transform (discrete - time Signal & System)

$$x[n] \xleftrightarrow{Z\text{-transform}} X(Z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Ex: Find the Z - transform of the following Signals:

a) $x[n] = \{2, -1, 3, -2\}$; b) $x[n] = \delta[n]$, c) $x[n] = 0.5^{n} u[n]$

a) $x[n] = \{\underset{\uparrow}{2}; \underset{\uparrow}{-1}; \underset{\uparrow}{3}; \underset{\uparrow}{-2}\}$

$x[0]$ $x[1]$ $x[2]$ $x[3]$

$$X(Z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = x[0] z^{-0} + x[1] z^{-1} + x[2] z^{-2} + x[3] z^{-3}$$

$X(Z) = 2 - 1Z^{-1} + 3Z^{-2} - 2Z^{-3}$ ; RoC: $Z \neq 0$ ($0^{-1}, 0^{-2}$, ERROR)

RoC = Z such that $X(Z)$ is finite.

b) $x[n] = \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$ $\qquad x[n] = \delta[n] \longleftrightarrow X[Z] = 1$

$X[Z] = 1 . Z^{0} = 1$

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28/12/2019 —

# Z - Transform

$$\boxed{x[n] \xleftrightarrow{\text{z-Transform}} X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}}$$

**Ex 1:** Find the z-transform of the following signal: a) $x[n] = \{2, -1, 3, -2\}$

b) $x[n] = \delta[n]$

c) $x[n] = 0.5^n u[n]$

a) $x[n] = \{\underset{x[0]}{2}, \underset{x[1]}{-1}, \underset{x[2]}{3}, \underset{x[3]}{-2}\} \Rightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n] . z^{-n} = x[0] z^{-0} + x[1] z^{-1} + x[2] z^{-2} + x[3] z^{-3}$

(RoC = z such that X(z) is finite)

Therefore $X(z) = 2 - 1z^{-1} + 3z^{-2} - 2z^{-3}$

RoC: $z \neq 0$

b) $x[n] = \delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$ | $X(z) = 1 . z^0 = 1$.

So: $\boxed{x[n] = \delta[n] \xleftrightarrow{\text{z-transform}} X(z) = 1}$

* Find the z-transform of following signal: a) $x[n] = \delta[n]$ ; b) $x[n] = \delta[n-2]$

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}}$$

a) $x[n] = \delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$

z-transform: $X(z) = 1 z^{-0} = 1$.

$\Rightarrow x[n] = \delta[n] \xleftrightarrow{\text{z-trans}} X(z) = 1$.

b) $x[n] = \delta[n-2] = \begin{cases} 1, & n = 2 \\ 0, & n \neq 2 \end{cases}$ | $X(z) = 1 . z^{-2}$, RoC $z \neq 0$

So: $\boxed{x[n] = \delta[n-n_0] \xleftrightarrow{\text{z-transform}} X(z) = z^{-n_0} \quad \text{RoC: } z \neq 0, n_0 > 0}$

c) $x[n] = 0.5^n u[n]$

Unit Step Signal

$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$ | $x[n] = \begin{cases} 0.5^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$ | $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{+\infty} 0.5^n z^{-n} = \sum_{n=0}^{+\infty} [0.5 z^{-1}]^n$

$= \sum_{n=0}^{+\infty} [0.5 z^{-1}]^0 = \frac{1}{1 - 0.5 z^{-1}}$, $|z| > 0.5$

$|0.5 z^{-1}| < 1$

$|z| > 0.5$

$$\boxed{x[n] = a^n u[n] \xleftrightarrow{\text{Z-Trans}} X(z) = \frac{1}{1 - a z^{-1}}, \text{ RoC } |z| > |a|}$$

$\boxed{x[n] = 0 \text{ for } n < 0 \text{ (causal Signal)}}$

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Ex: The LTI discrete system has the unit impulse response: $h[n] = [2, 1, -1]$

a) Find the transfer function $H(z)$

b) Using the Z - Transform to find the output $y[n]$ for the input $x[n] = [3, -1, 2]$

a) $h[n] = [\underset{h_0}{2}, \underset{h_1}{1}, \underset{h_2}{-1}]$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = h[0] z^{-0} + h[1] z^{-1} + h[2] z^{-2}$$

transfer function: $H(z) = 2 + 1 \cdot z^{-1} - 1 \cdot z^{-2}$, ROC $z \neq 0$.

b) $y[n] = x[n] * h[n]$

$Y(z) = X(z) H(z)$ ; $x[n] = [\overset{x[0]}{3}, \overset{x[1]}{-1}, \overset{x[2]}{2}]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = x[0] z^{-0} + x[1] z^{-1} + x[2] z^{-2} = 3 - 1 \cdot z^{-1} + 2 z^{-2}$$

$$Y(z) = X(z) H(z) = (3 - 1z^{-1} + 2z^{-2})(2 + 1z^{-1} - 1z^{-2}) = 6 + z^{-1} + 0z^{-2} + 3z^{-3} - 2z^{-4}$$

$\Rightarrow y[n] = [6, 1, 0, 3, -2]$

(Infinite Impulse Response, IIR)

Ex: A Causal LTI system has the unit impulse response $h[n] = 0.5^n u[n]$

a) Find the transfer function $H(z)$

b) Find the output $y[n]$ for the input $x[n] = 0.8^n u[n]$

normally $< 1$, so that $n \to \infty$, $x[n] \to 0$

$$\boxed{x[n] = a^n u[n] \xleftrightarrow[\text{Transform}]{Z} X(z) = \frac{1}{1 - a z^{-1}}, \quad \text{ROC } |z| > |a|}$$

a) $h[n] = 0.5^n u[n]$

Transfer function: $H(z) = \dfrac{1}{1 - 0.5 z^{-1}}$ ROC $|z| > 0.5$.

b) $y[n] = x[n] * h[n]$

$Y(z) = X(z) H(z)$

$x[n] = 0.8^n u[n] \longrightarrow X(z) = \dfrac{1}{1 - 0.8 z^{-1}}$, ROC $|z| > 0.8$

$Y(z) = X(z) H(z) = \dfrac{1}{1 - 0.8 z^{-1}} \times \dfrac{1}{1 - 0.5 z^{-1}}$, ROC $|z| > 0.8$

partial fraction:

$$Y(z) = \frac{1}{(1 - 0.8 z^{-1})(1 - 0.5 z^{-1})} = \frac{A_1}{1 - 0.8 z^{-1}} + \frac{A_2}{1 - 0.5 z^{-1}}$$

$$A_1 = \left.\frac{1}{1 - 0.5 z^{-1}}\right|_{z = 0.8} = \frac{1}{1 - 0.5 \times 0.8^{-1}} = \frac{8}{3}$$

$$A_2 = \left.\frac{1}{1 - 0.8 z^{-1}}\right|_{z = 0.5} = \frac{1}{1 - 0.8 \times 0.5^{-1}} = -\frac{5}{3}$$

$$Y(z) = \frac{8}{3} \frac{1}{(1 - 0.8 z^{-1})} - \frac{5}{3} \frac{1}{(1 - 0.5 z^{-1})}$$

transfer back in time domain

Causal:

$$y[n] = \frac{8}{3} \times 0.8^n u[n] - \frac{5}{3} \times 0.5^n u[n]$$

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Ex: A causal LTI discrete time system has a transfer function $H(z) = \dfrac{1+0.5z^{-1}}{1+0.1z^{-1}-0.06z^{-2}}$

a) Derive the equation to express the relationship between the input $x[n]$ and the input $y[n]$. (i.e I/O difference equation)

b) Given the input $x[n] = 0.4^n u[n]$. Find $y[n]$?

$$\boxed{\begin{array}{l} x[n] \longleftrightarrow X(z) \\ x[n-D] \longleftrightarrow z^{-D}X(z) \end{array}}$$

a) $Y(z) = X(z)H(z)$

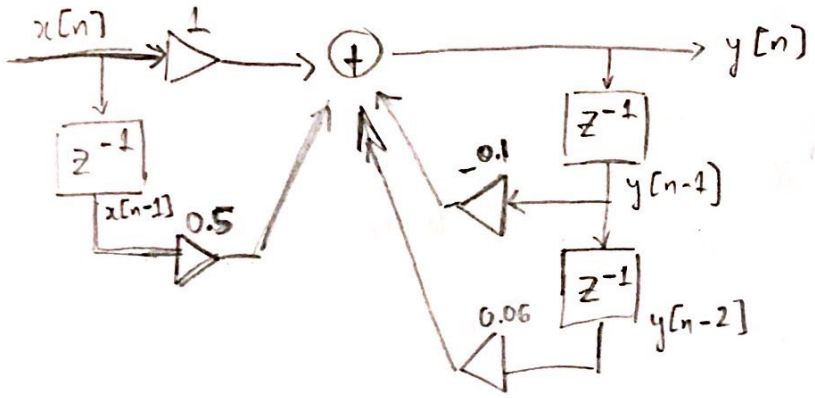
$$Y(z) = X(z)\,\frac{1+0.5z^{-1}}{1+0.1z^{-1}-0.06z^{-2}} \Rightarrow Y(z)\left[1+0.1z^{-1}-0.06z^{-2}\right] = X(z)\left[1+0.5z^{-1}\right]$$

$$\Rightarrow Y(z) + 0.1z^{-1}Y(z) - 0.06z^{-2}Y(z) = X(z) + 0.5z^{-1}X(z)$$

$$y[n] + 0.1y[n-1] - 0.06\,y[n-2] = x[n] + 0.5\,x[n-1]$$

$$y[n] = -0.1y[n-1] + 0.06\,y[n-2] + x[n] + 0.5\,x[n-1]$$

Block diagram realization of system (basic blocks: adder, amplifier, delay)




b) $Y(z) = X(z)H(z)$

$$Y(z) = X(z)\,\frac{1+0.5z^{-1}}{1+0.1z^{-1}-0.06z^{-2}}$$

$x[n] = 0.4^n u[n]$

$$X(z) = \frac{1}{1-0.4z^{-1}}\ ;\ Y(z) = \frac{1}{(1-0.4z^{-1})}\,\frac{1+0.5z^{-1}}{(1+0.1z^{-1}-0.06z^{-2})}$$

$$= \frac{1+0.5z^{-1}}{(1-0.4z^{-1})(1-0.2z^{-1})(1+0.3z^{-1})}$$

$$= \frac{A_1}{1-0.4z^{-1}} + \frac{A_2}{1-0.2z^{-1}} + \frac{A_3}{1+0.3z^{-1}}$$

Causal: $y[n] = A_1\,0.4^n u[n] + A_2\,0.2^n u[n] + A_3(-0.3)^n u[n]$

$(1+0.1z^{-1}-0.06z^{-2})$
$= (1-a_1z^{-1})(1-a_2z^{-1})$
$= (1-0.2z^{-1})(1+0.3z^{-1})$

$1+0.1z^{-1}-0.06z^{-2} = 0$
$\Leftrightarrow z^2+0.1z-0.06 = 0 \Rightarrow \begin{cases} a_1 = 0.2 \\ a_2 = -0.3 \end{cases}$

$$A_1 = \left.\frac{1+0.5z^{-1}}{(1-0.2z^{-1})(1+0.3z^{-1})}\right|_{z=0.4}$$

$$A_2 = \left.\frac{1+0.5z^{-1}}{(1-0.4z^{-1})(1+0.3z^{-1})}\right|_{z=0.2}$$

$$A_3 = \left.\frac{1+0.5z^{-1}}{(1-0.4z^{-1})(1-0.2z^{-1})}\right|_{z=-0.3}$$

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