Information

- **Course:** **Signals and Systems (EE088IU)**
- **Lecturer:** **Do Ngoc Hung**
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- **Email:** dnhung@hcmiu.edu.vn
- **Number of credits:** 3
- **TA**: **Huynh Tran Minh Tri**, email: htmtri@hcmiu.edu.vn

Reading Lists

1. B.P. Lathi, *Linear Systems and Signals*, Oxford University Press Inc., 2018

2. V. Oppenheim, A. S. Willsky with S. Hamid, *Signals and Systems*, Prentice Hall, 2nd ed., 1996.

3. A. Poularikas, *Signals and Systems with Primer with MATLAB*, CRC Press, 2016.

- **Lecture Note & Homework:** Blackboard

Course Syllabus

- Introduction of signal
- System & System Properties
- Discrete time and Continuous time Convolution methods
- Linear Time Invariant System Properties
- Fourier Series and Fourier Transforms
- Laplace Transform
- z-Transform
- Sampling

Grading

1. Midterm examination: 30%

2. Final examination : 40%

3. Attendance + Quiz + Homework: 30%

Course Policy

1. All assignments need to be submitted on the due date

2. Students are expected to do their own work at all times. Any evidence of plagiarism or cheating will be treated as grounds for failure in the class.

3. Attendance: at least 80%

COURSE CONTENTS

The concepts of signals and systems arise in a variety of fields such as, communications, bio-engineering, circuit design and others.

• Typical examples of systems include radio and television, telephone networks, radar systems, computer networks, wireless communication, military surveillance systems, and satellite communication systems.

• Although the physical attributes of the signals and systems involved in the above disciplines are different, all signals and systems have basic features in common.

• The aim of this course is to provide the fundamental and universal tools for the analysis of signals.

COURSE CONTENTS

Signal

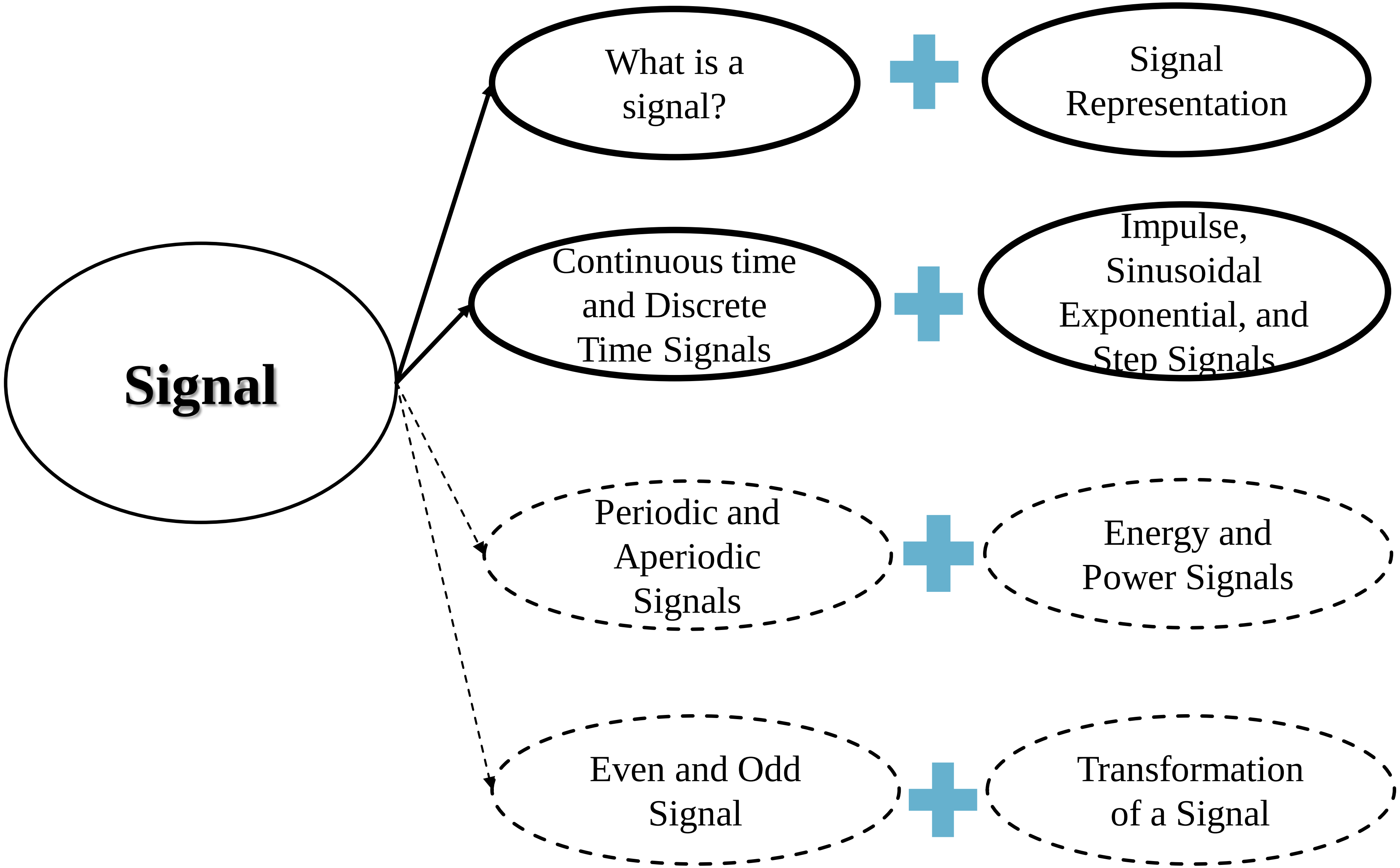
- Continuous and Discrete Time Signals
- Impulse, Exponential and Step Signals
- Periodic and Aperiodic Signals
- Power and Energy Signals
- Even and Odd Signals
- Signal representations
- Transformations of the time variable

System

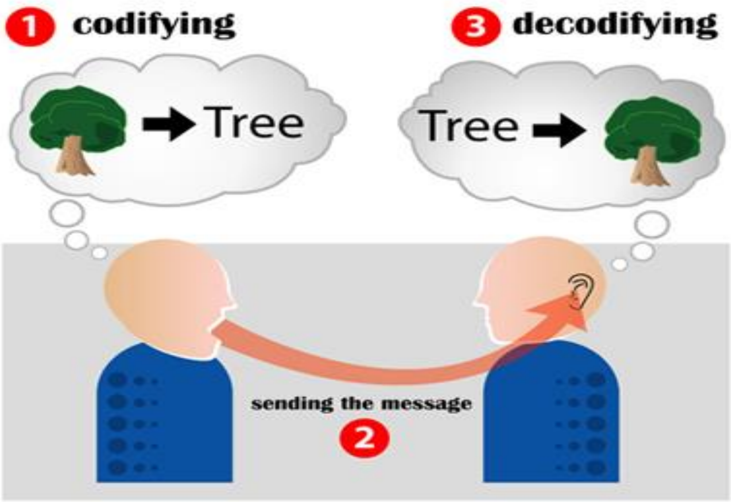
- Linearity and Superposition
- Time invariant
- Causality
- Stability
- Invertible and Non-Invertible
- Convolution

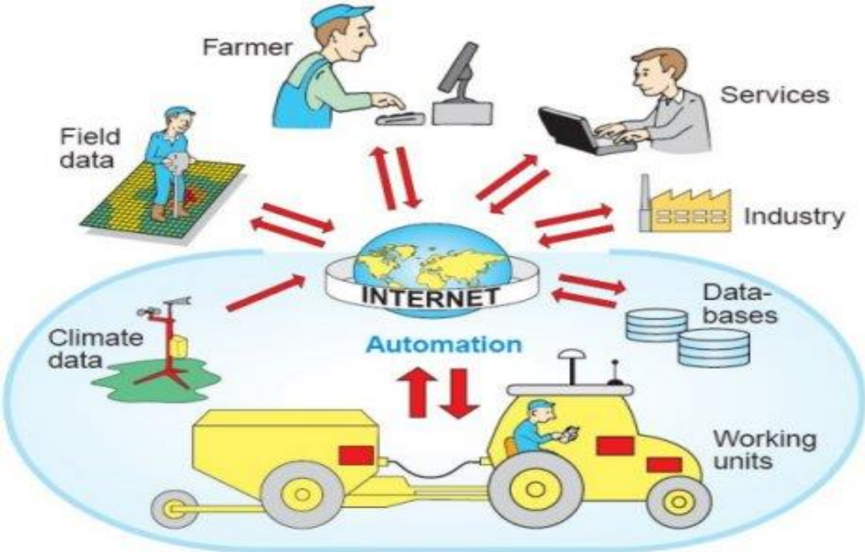
Tools

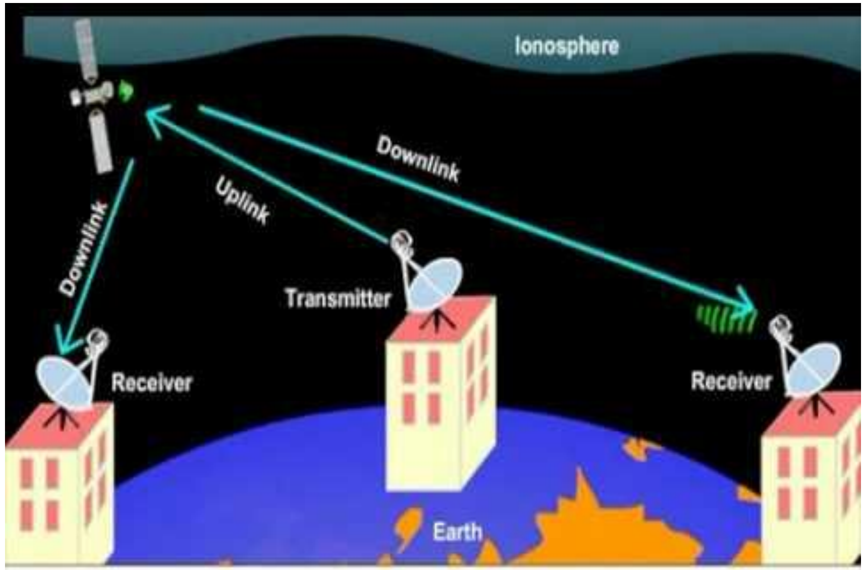
- Fourier Series, Discrete and Continuous Fourier Transforms
- Sampling method
- Laplace Transform
- Z-Transform


Signal
What is a signal?
Signal Representation
Continuous time and Discrete Time Signals
Impulse, Sinusoidal Exponential, and Step Signals
Periodic and Aperiodic Signals
Energy and Power Signals
Even and Odd Signal
Transformation of a Signal

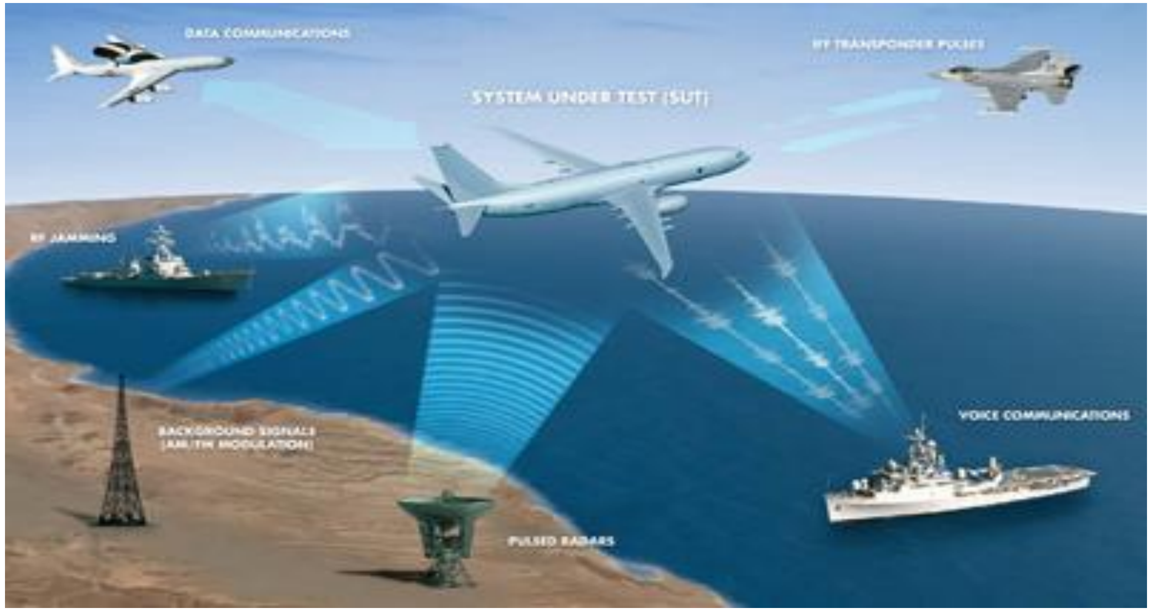
WHAT IS A SIGNAL?







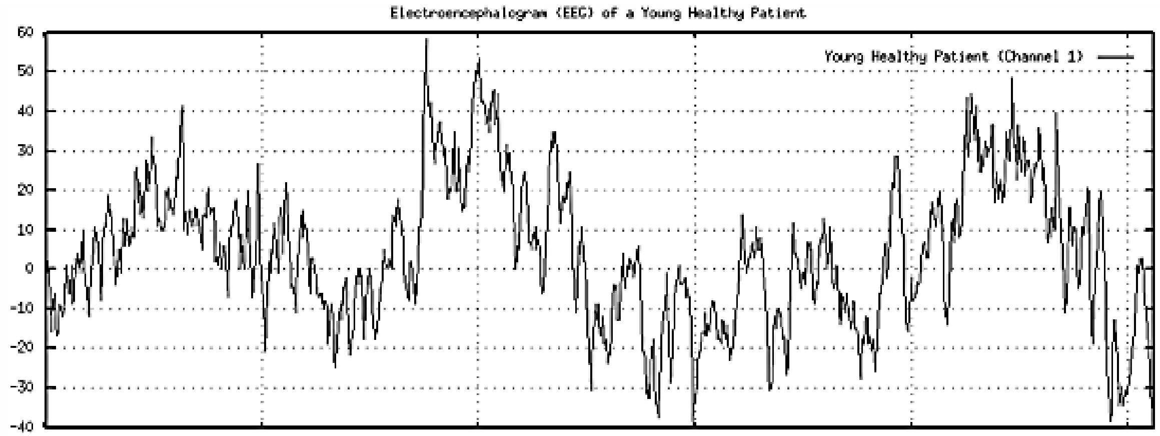
The time period for one complete orbital motion of an artificial satellite is equal to the time period of the earth's one complete rotation.



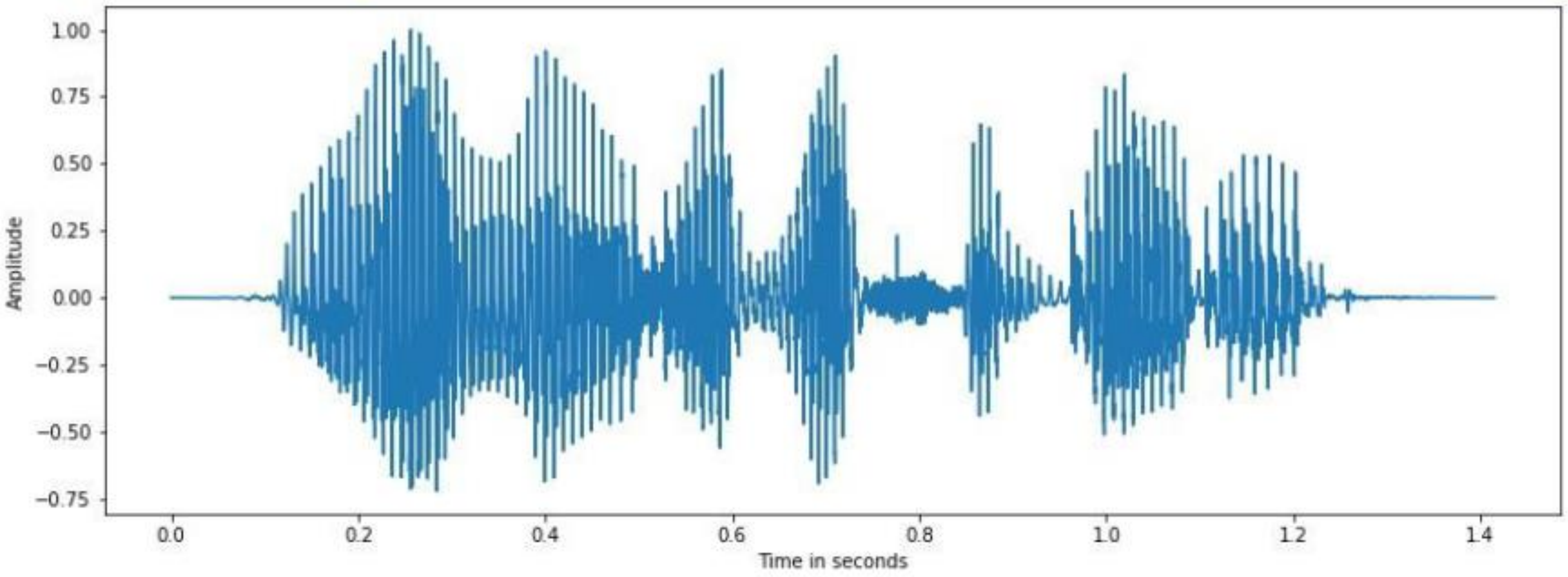
WHAT IS A SIGNAL?

- A signal is a pattern of variation of some forms
- Signals are variables that carry information
- **Examples:**
- Electrical network
  - Voltages and currents in a circuit
- Acoustics
  - A song acoustic pressure (sound) over time
- Mechanics
  - Velocity of a car over time
- Videos
  - Intensity level of a pixel (camera, video) over time
- Signal exists in almost every areas of engineering, in daily life and it is essential for analysis and design of the systems. E.g. Communication

WHAT IS A SIGNAL?



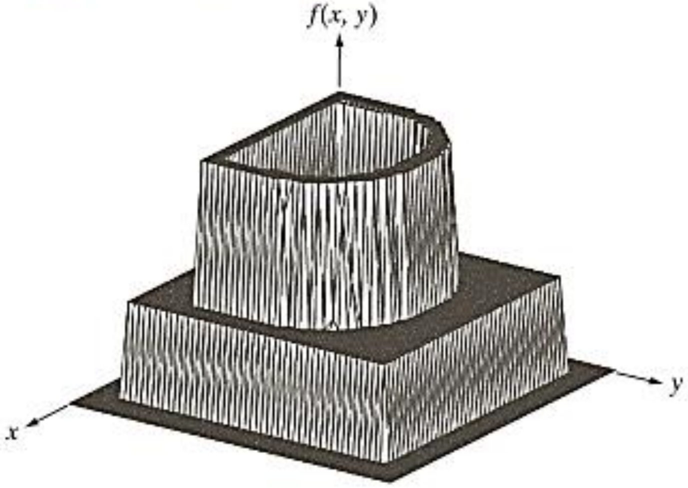
An electroencephalogram (EEG) signal

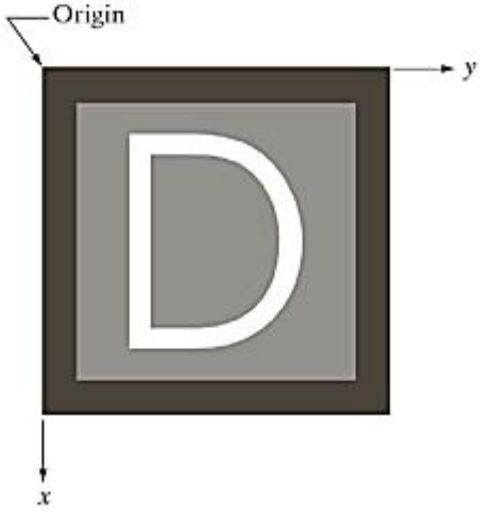


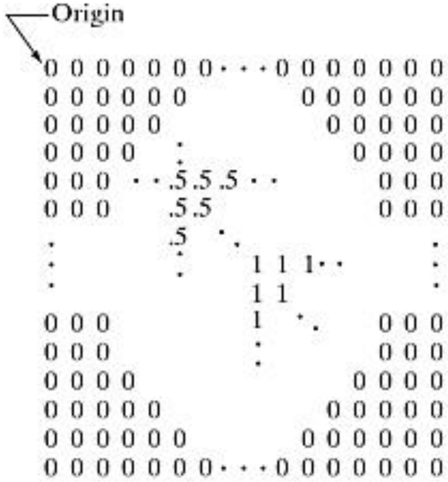
Speech signal for the utterance
“will we ever forget it”

WHAT IS A SIGNAL?

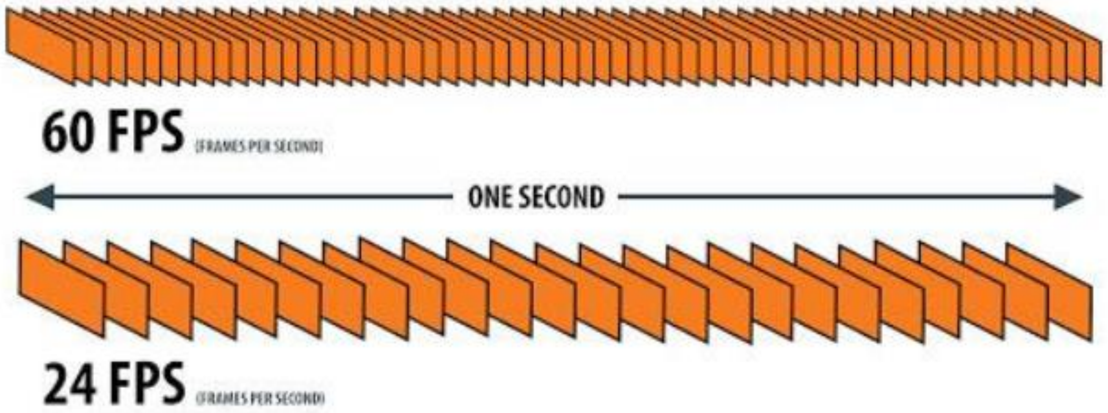
Image I(x,y)







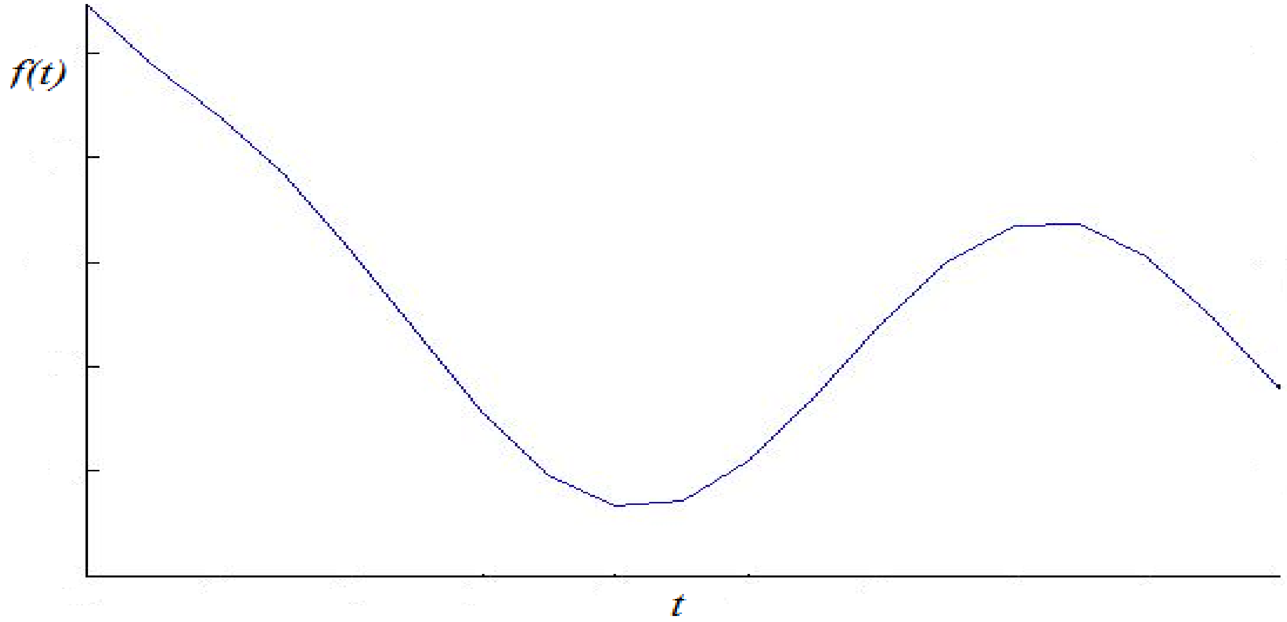
Video I(x,y,t)



- Mathematically, signals are represented as a function of **one or more independent variables**.

E.g. A picture can be represented by brightness as a function of two spatial variables, a black & white; video signal intensity is dependent on x, y coordinates and time t, $\boldsymbol{f(x, y, t)}$

- On this course, signals are considered to be a function of a single variable : **time**



❑ Continuous Time Signals

- Most signals in the real world are continuous time, as the scale is infinitesimally fine.

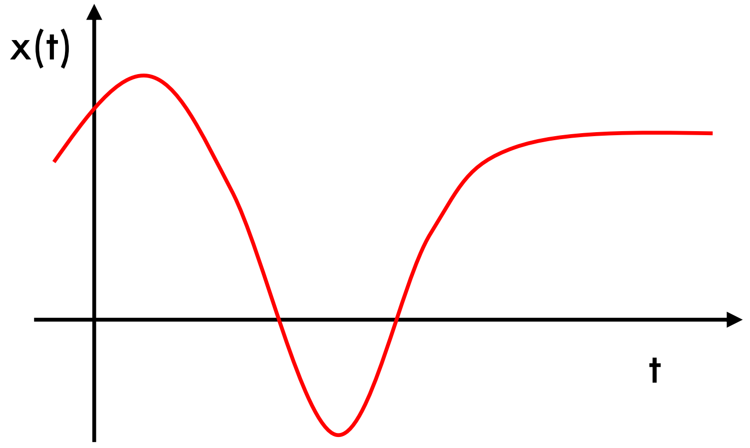
▪ **E.g**. voltage, velocity

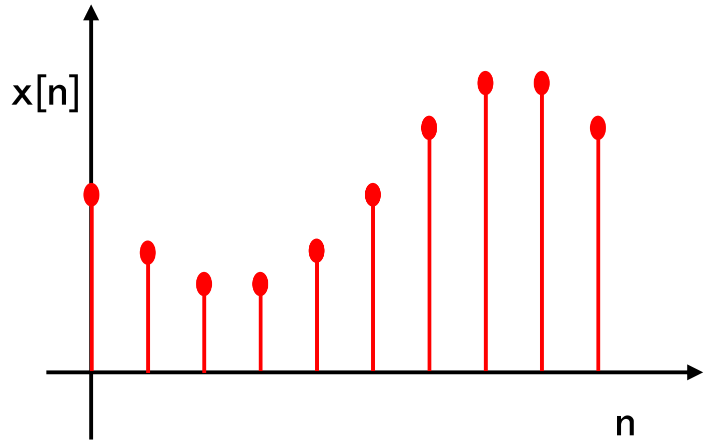
- They are functions of a continuous variable (time). Denote by **x(t)**, where the time interval may be bounded (finite) or infinite.

❑ Discrete time Signals

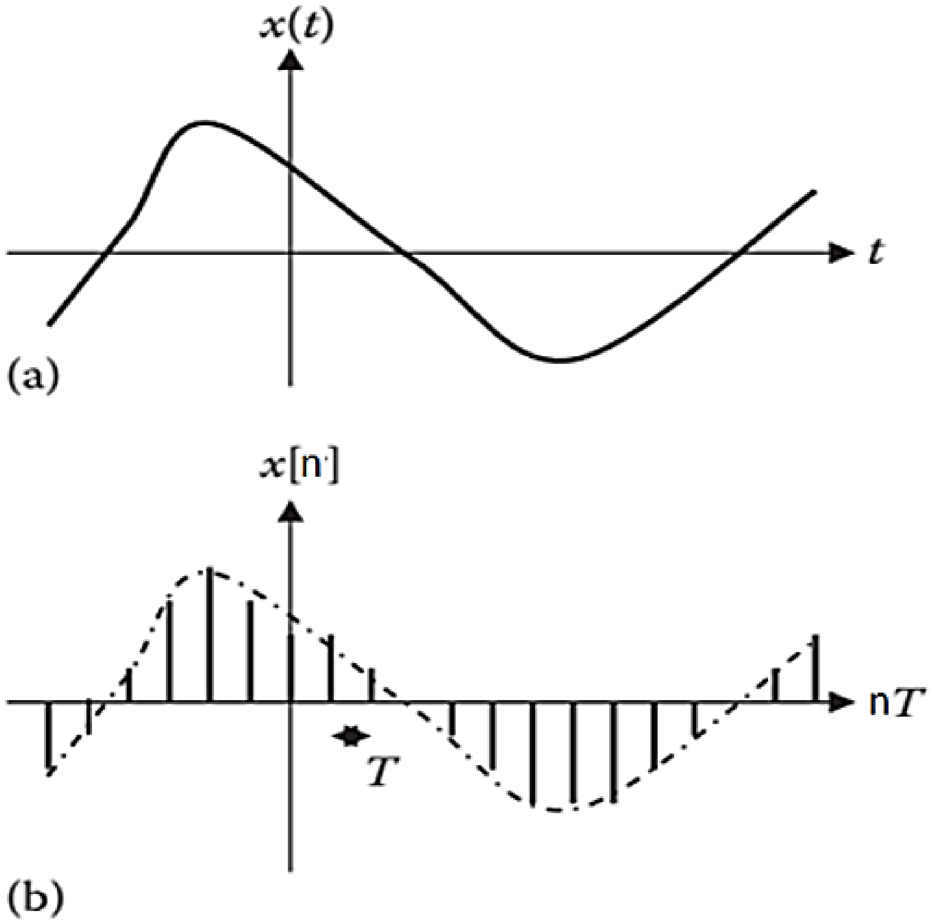
- Some real world are many digital signals are discrete time, as they are sampled. **E.g**. pixels, daily stock price (anything that a digital processes).

- They are functions of a discrete variable (i.e. time steps). Denote by **x[n]**, where n is an integer value 0,1,2,3,…



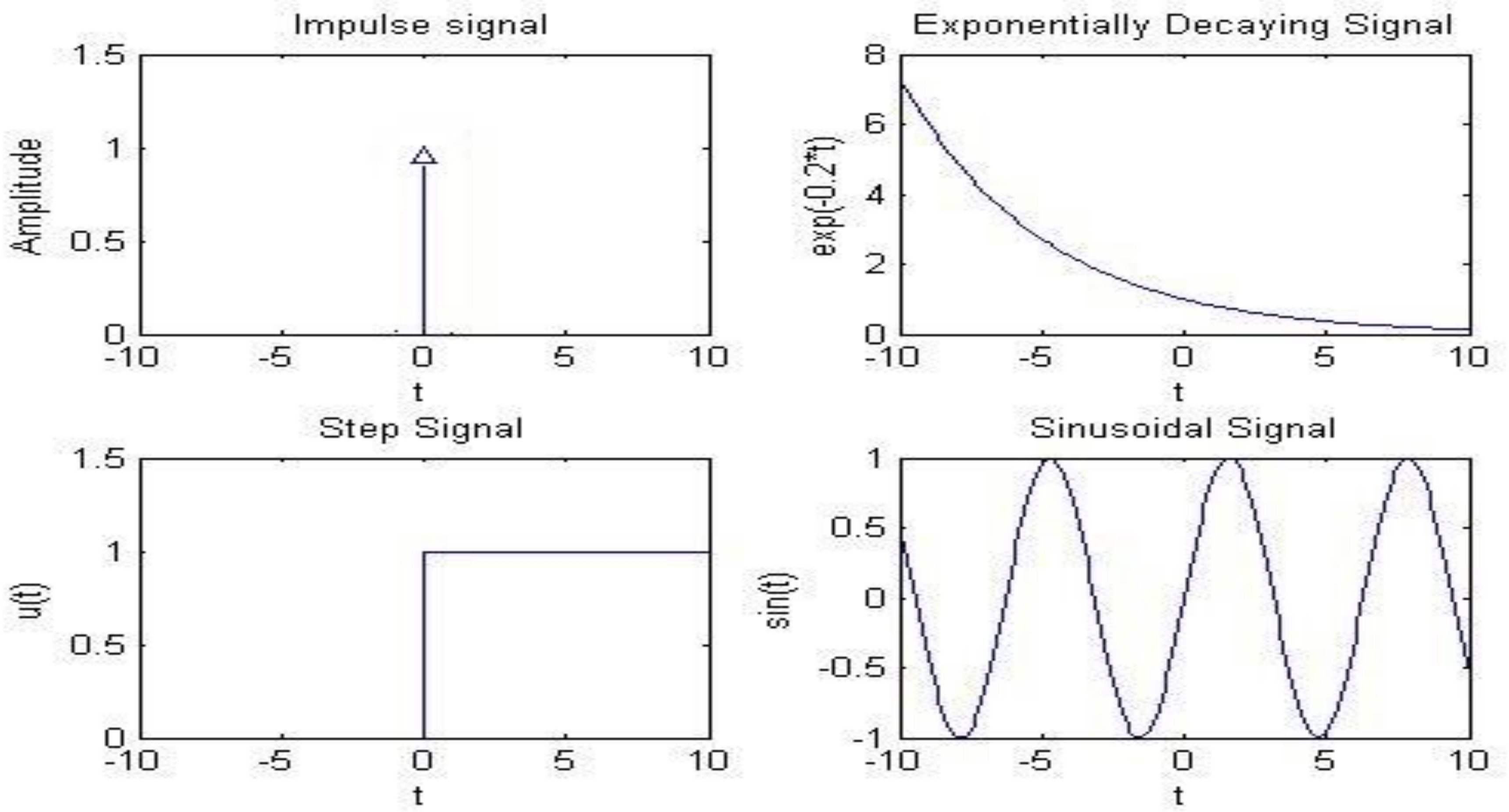


Continuous-time signal x(t) is sampled uniformly with sampling period T to produce the discrete-time signal x[n]



We simplify notation by letting $x(nT) \triangleq x[n]$

COMMON CONTINUOUS TIME SIGNALS



Note: Many other signals can be created from this simple set of basis signals

- The discrete **unit impulse signal** is defined

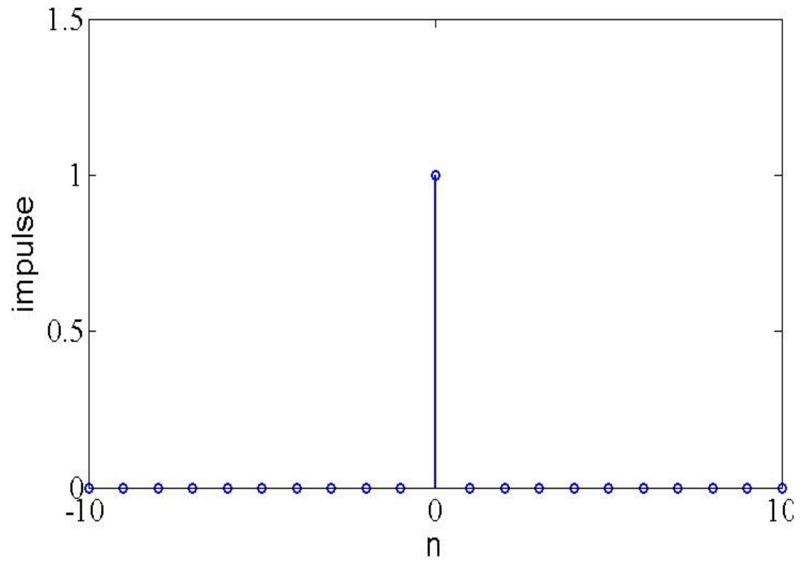
$$x[n] = \delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

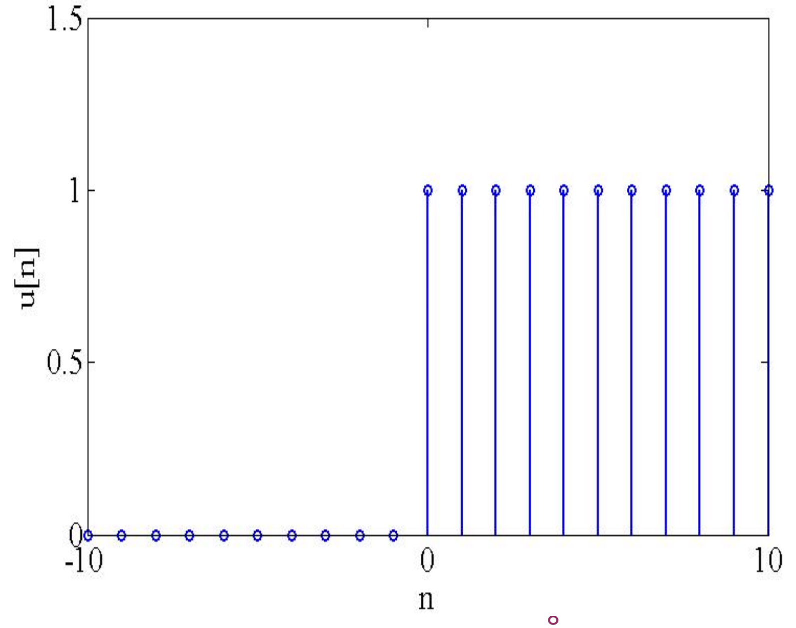
- The discrete **unit step signal** is defined:
- $$x[n] = u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$
- **Note**:

1. The unit impulse is the first difference (derivative) of the step signal

- $\delta[n] = u[n] - u[n-1]$

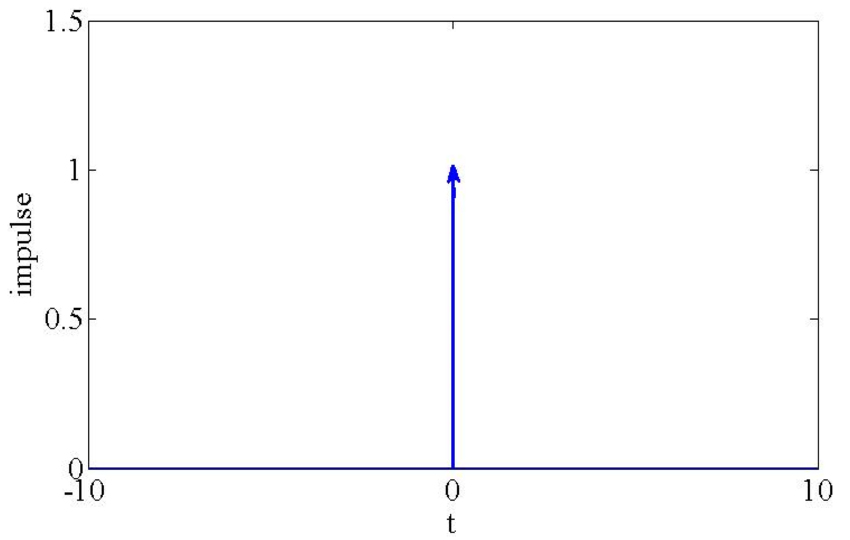
2. The unit step is the running sum (integral) of the unit impulse.





- The **continuous unit impulse signal** is defined

$$x(t) = \delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases}$$



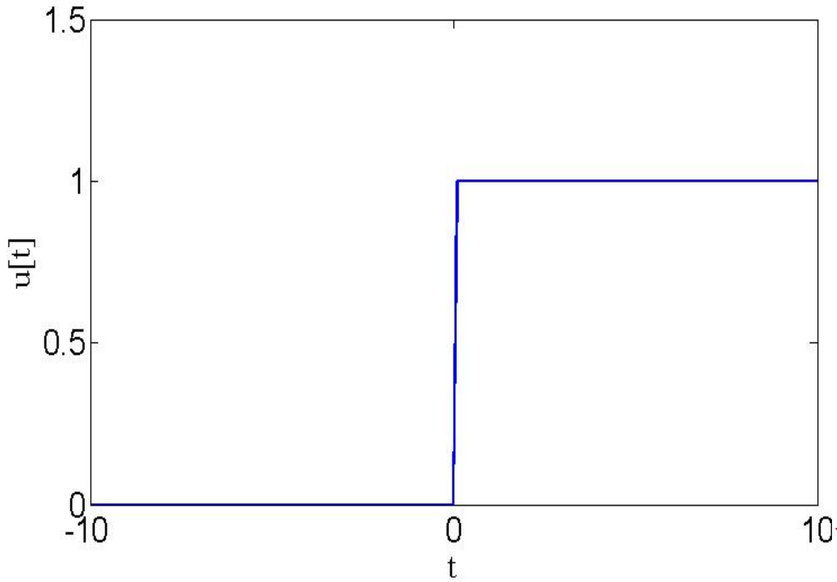
Note that it is discontinuous at $t = 0$

The arrow is used to denote area (1), rather than actual value (∞). Again, useful for an infinite basis.

- The **continuous unit step signal** is defined as

$$x(t) = u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

$$x(t) = u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

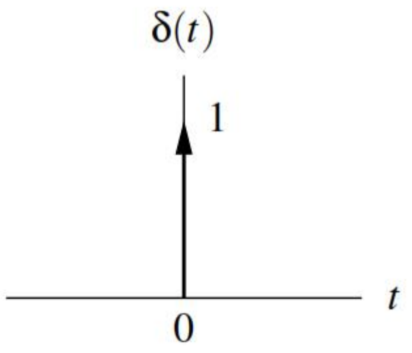


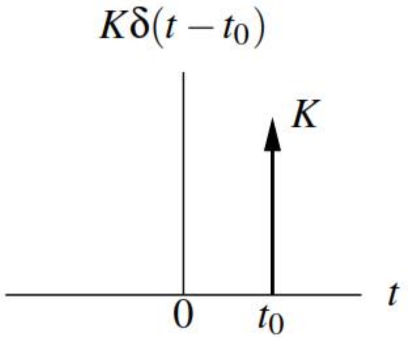
- The **unit-impulse function** (also known as the **Dirac delta function** or **delta function**), denoted δ, is defined by the following two properties:

$$\delta(t) = 0 \quad \text{for } t \neq 0 \quad \text{and}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1.$$

- Technically, δ is not a function in the ordinary sense. Rather, it is what is known as a *generalized function*. Consequently, the δ function sometimes behaves in unusual ways.
- Graphically, the delta function is represented as shown below.



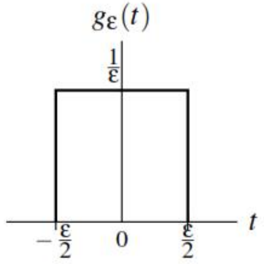


UNIT IMPULSE FUNCTION AS A LIMIT

- Define

$$g_\varepsilon(t) = \begin{cases} 1/\varepsilon & |t| < \varepsilon/2 \\ 0 & \text{otherwise.} \end{cases}$$

- The function g_ε has a plot of the form shown below.



- Clearly, for any choice of ε, $\int_{-\infty}^{\infty} g_\varepsilon(t)\,dt = 1$.
- The function δ can be obtained as the following limit:

$$\delta(t) = \lim_{\varepsilon \to 0} g_\varepsilon(t).$$

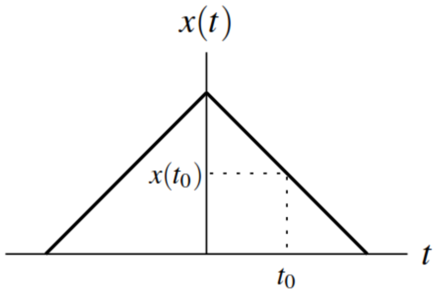
- That is, δ can be viewed as a ***limiting case of a rectangular pulse*** where the pulse width becomes infinitesimally small and the pulse height becomes infinitely large in such a way that the integral of the resulting function remains unity.

- **Equivalence property.** For any continuous function x and any real constant t_0,

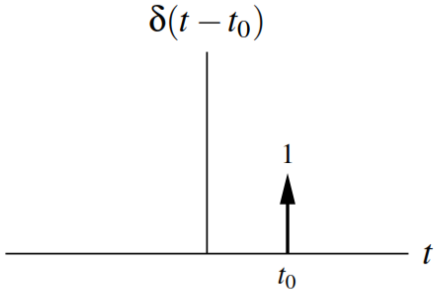
$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0).$$

- **Sifting property.** For any continuous function x and any real constant t_0,

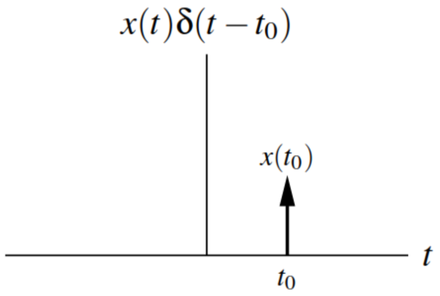
$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0).$$



Function x



Time-Shifted Unit-Impulse Function

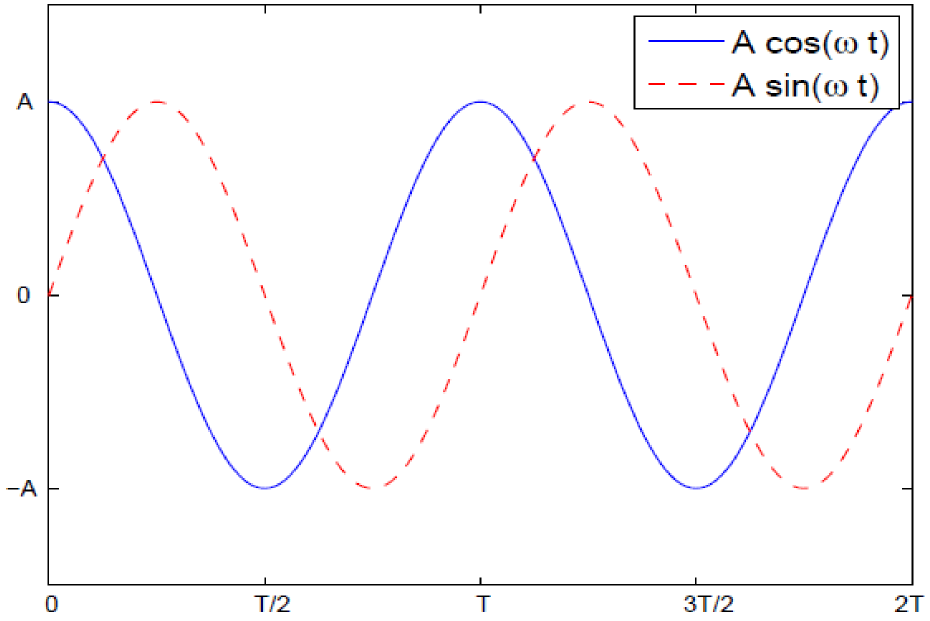


Product

- The signal is generally described as a function of time by:

$$y = Asin(2\pi ft + \varphi) = Asin(\omega t + \varphi)$$

where A is the amplitude (the peak deviation of the function from zero), f is the function (the number of oscillation that occur each second of time), ω is the angular frequency with unit as radians/second, and φ is the phase with unit as radians.



$$x(t) = Asin(2\pi ft)$$
$$= Acos\left(2\pi ft - \frac{\pi}{2}\right)$$

GENERAL COMPLEX EXPONENTIAL SIGNALS

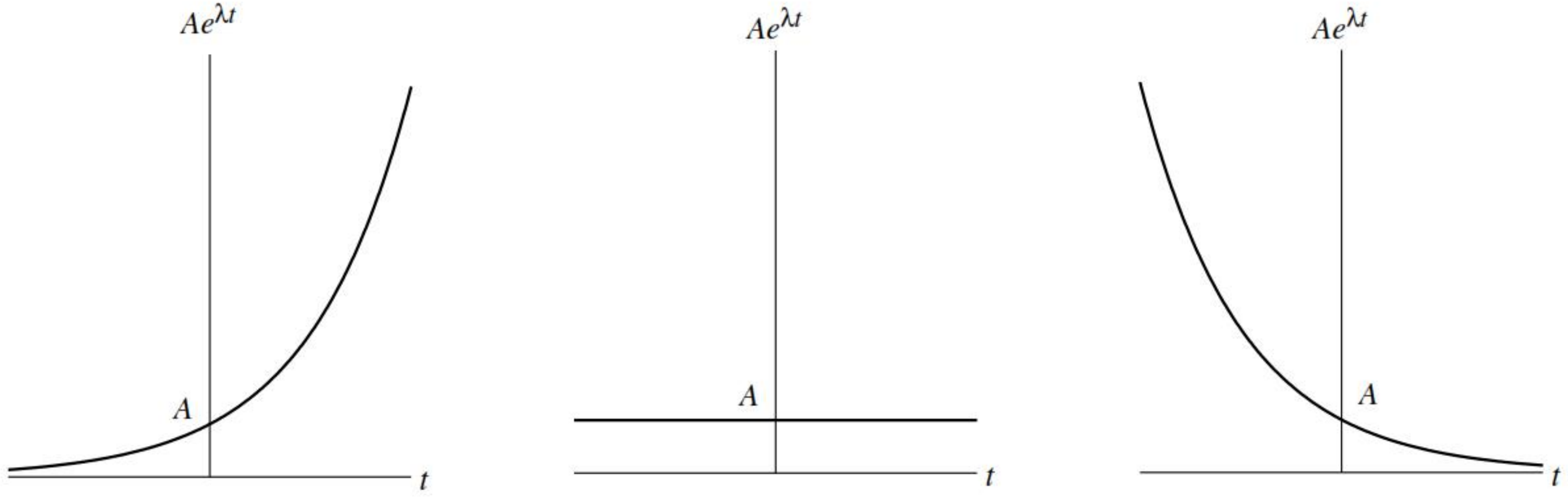
- A **complex exponential function** is a function of the form

$$x(t) = Ae^{\lambda t},$$

  where A and λ are ***complex*** constants.

- A complex exponential can exhibit one of a number of ***distinct modes of behavior***, depending on the values of its parameters A and λ.
- For example, as special cases, complex exponentials include real exponentials and complex sinusoids.

- A **real exponential function** is a special case of a complex exponential $x(t) = Ae^{\lambda t}$, where A and λ are restricted to be ***real*** numbers.
- A real exponential can exhibit one of ***three distinct modes*** of behavior, depending on the value of λ, as illustrated below.
- If $\lambda > 0$, $x(t)$ ***increases*** exponentially as t increases (i.e., a growing exponential).
- If $\lambda < 0$, $x(t)$ ***decreases*** exponentially as t increases (i.e., a decaying exponential).
- If $\lambda = 0$, $x(t)$ simply equals the ***constant*** A.



$\lambda > 0$ $\lambda = 0$ $\lambda < 0$

COMPLEX SINUSOIDAL FUNCTIONS

- A complex sinusoidal function is a special case of a complex exponential $x(t) = Ae^{\lambda t}$, where A is ***complex*** and λ is ***purely imaginary*** (i.e., $\mathrm{Re}\{\lambda\} = 0$).
- That is, a **complex sinusoidal function** is a function of the form

$$x(t) = Ae^{j\omega t},$$

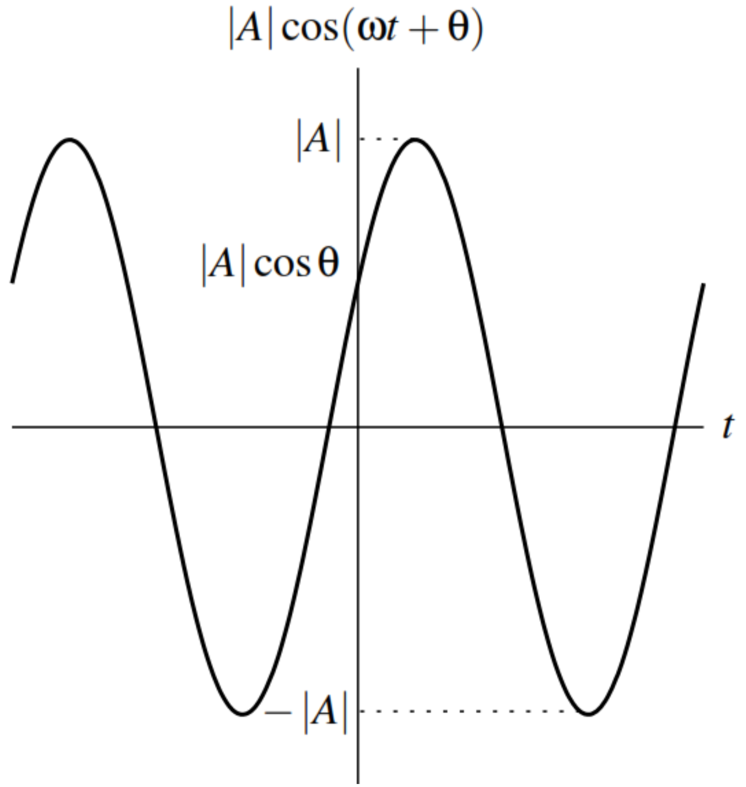
  where A is ***complex*** and ω is ***real***.
- By expressing A in polar form as $A = |A|\, e^{j\theta}$ (where θ is real) and using Euler's relation, we can rewrite $x(t)$ as

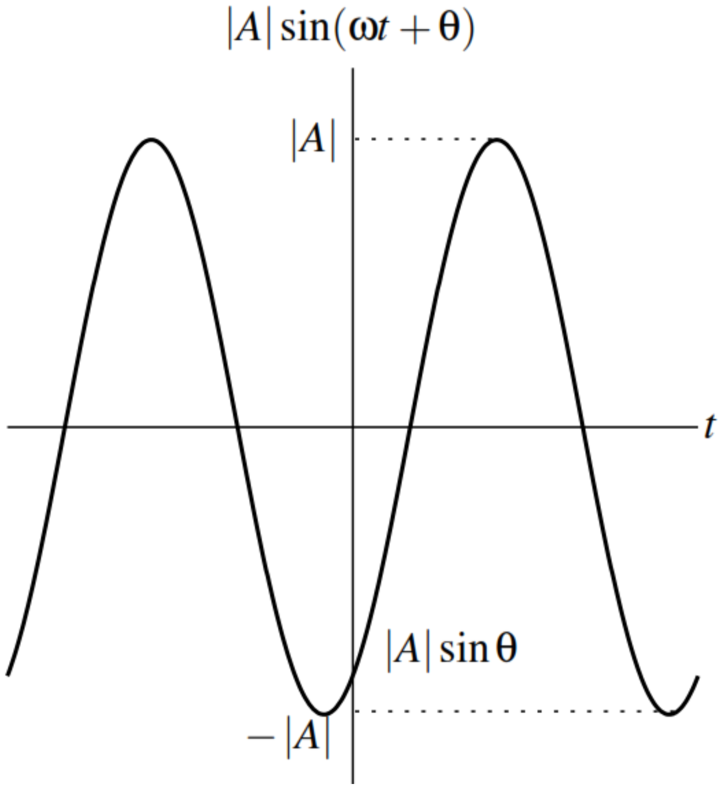
$$x(t) = \underbrace{|A|\cos(\omega t + \theta)}_{\mathrm{Re}\{x(t)\}} + j\underbrace{|A|\sin(\omega t + \theta)}_{\mathrm{Im}\{x(t)\}}.$$

- Thus, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are the same except for a time shift.
- Also, x is periodic with ***fundamental period*** $T = \frac{2\pi}{|\omega|}$ and ***fundamental frequency*** $|\omega|$.

COMPLEX SINUSOIDAL FUNCTIONS

- The graphs of $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ have the forms shown below.





GENERAL COMPLEX EXPONENTIAL SIGNALS

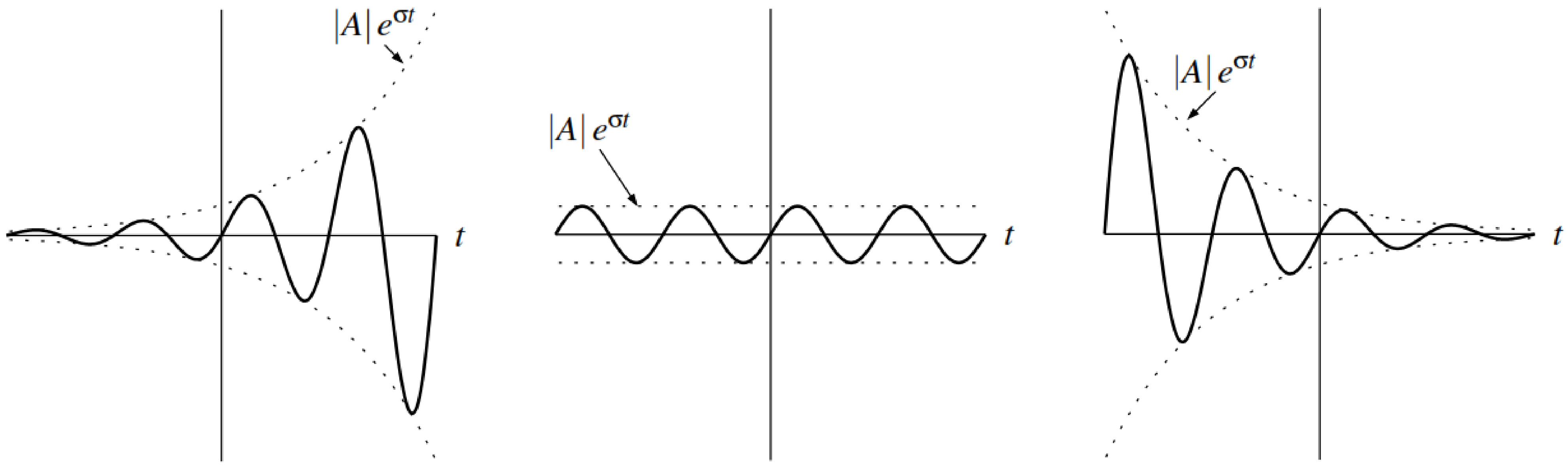
- In the most general case of a complex exponential function $x(t) = Ae^{\lambda t}$, A and λ are both ***complex***.
- Letting $A = |A| e^{j\theta}$ and $\lambda = \sigma + j\omega$ (where θ, σ, and ω are real), and using Euler's relation, we can rewrite $x(t)$ as

$$x(t) = \underbrace{|A| e^{\sigma t} \cos(\omega t + \theta)}_{\mathrm{Re}\{x(t)\}} + j \underbrace{|A| e^{\sigma t} \sin(\omega t + \theta)}_{\mathrm{Im}\{x(t)\}}.$$

- Thus, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the product of a real exponential and real sinusoid.
- One of ***three distinct modes*** of behavior is exhibited by $x(t)$, depending on the value of σ.
- If $\sigma = 0$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are ***real sinusoids***.
- If $\sigma > 0$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the ***product of a real sinusoid and a growing real exponential***.
- If $\sigma < 0$, $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are each the ***product of a real sinusoid and a decaying real exponential***.

GENERAL COMPLEX EXPONENTIAL SIGNALS

- The ***three modes of behavior*** for $\mathrm{Re}\{x\}$ and $\mathrm{Im}\{x\}$ are illustrated below.



$\sigma > 0$ $\quad\quad$ $\sigma = 0$ $\quad\quad$ $\sigma < 0$

- From Euler's relation, a complex sinusoid can be expressed as the sum of two real sinusoids as

$$Ae^{j\omega t} = A\cos(\omega t) + jA\sin(\omega t).$$

- Moreover, a real sinusoid can be expressed as the sum of two complex sinusoids using the identities

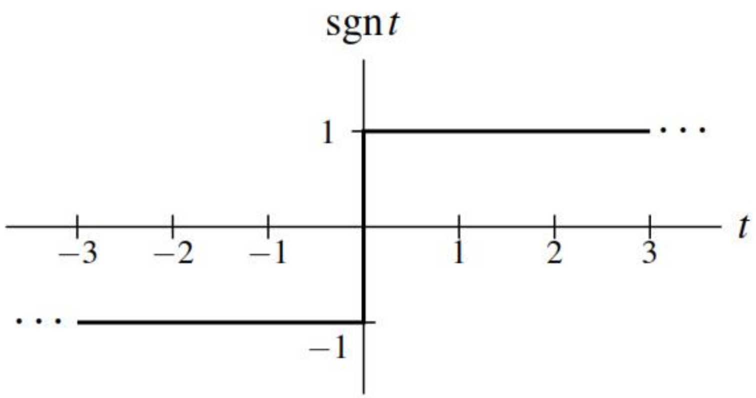
$$A\cos(\omega t + \theta) = \frac{A}{2}\left[e^{j(\omega t+\theta)} + e^{-j(\omega t+\theta)}\right] \quad \text{and}$$
$$A\sin(\omega t + \theta) = \frac{A}{2j}\left[e^{j(\omega t+\theta)} - e^{-j(\omega t+\theta)}\right].$$

SIGNUM FUNCTION

- The **signum function**, denoted sgn, is defined as

$$\operatorname{sgn} t = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0. \end{cases}$$

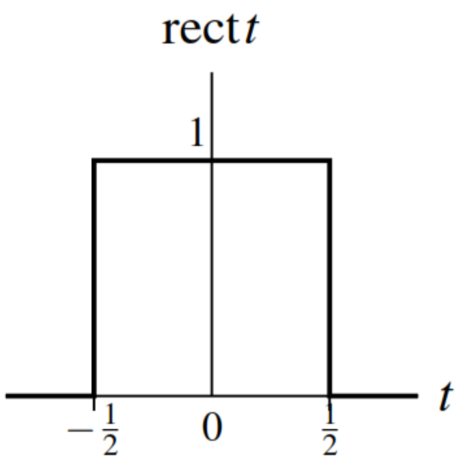
- From its definition, one can see that the signum function simply computes the ***sign*** of a number.
- A plot of this function is shown below.



- The **rectangular function** (also called the unit-rectangular pulse function), denoted rect, is given by

$$\operatorname{rect} t = \begin{cases} 1 & -\frac{1}{2} \leq t < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

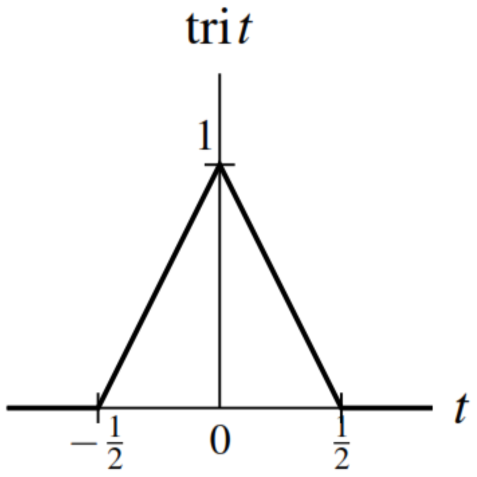
- Due to the manner in which the rect function is used in practice, the actual ***value of*** $\operatorname{rect} t$ ***at*** $t = \pm\frac{1}{2}$ is unimportant. Sometimes different values are used from those specified above.
- A plot of this function is shown below.



- The **triangular function** (also called the unit-triangular pulse function), denoted tri, is defined as

$$\operatorname{tri} t = \begin{cases} 1 - 2|t| & |t| \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

- A plot of this function is shown below.

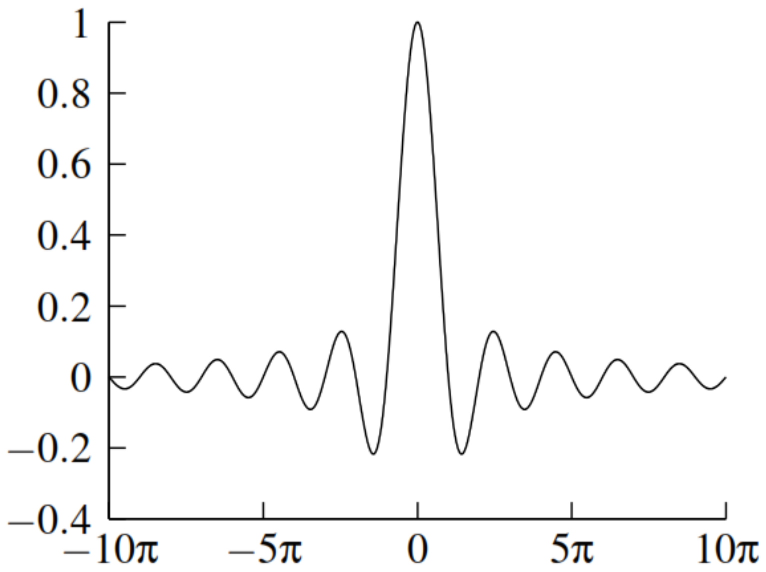


CARDINAL SINE FUNCTION

- The **cardinal sine** function, denoted sinc, is given by

$$\operatorname{sinc} t = \frac{\sin t}{t}.$$

- By l'Hopital's rule, $\operatorname{sinc} 0 = 1$.
- A plot of this function for part of the real line is shown below. [Note that the oscillations in $\operatorname{sinc} t$ do not die out for finite t.]



- The **floor function**, denoted $\lfloor \cdot \rfloor$, is a function that maps a real number x to the largest integer not more than x.
- In other words, the floor function rounds a real number to the nearest integer in the direction of negative infinity.
- For example,

$$\left\lfloor -\tfrac{1}{2} \right\rfloor = -1, \quad \left\lfloor \tfrac{1}{2} \right\rfloor = 0, \quad \text{and} \quad \lfloor 1 \rfloor = 1.$$

- The **ceiling function**, denoted $\lceil \cdot \rceil$, is a function that maps a real number x to the smallest integer not less than x.
- In other words, the ceiling function rounds a real number to the nearest integer in the direction of positive infinity.
- For example,

$$\left\lceil -\tfrac{1}{2} \right\rceil = 0, \quad \left\lceil \tfrac{1}{2} \right\rceil = 1, \quad \text{and} \quad \lceil 1 \rceil = 1.$$

- Several useful properties of the floor and ceiling functions include:

$$\lfloor x+n \rfloor = \lfloor x \rfloor + n \quad \text{for } x \in \mathbb{R} \text{ and } n \in \mathbb{Z};$$

$$\lceil x+n \rceil = \lceil x \rceil + n \quad \text{for } x \in \mathbb{R} \text{ and } n \in \mathbb{Z};$$

$$\lceil x \rceil = -\lfloor -x \rfloor \quad \text{for } x \in \mathbb{R};$$

$$\lfloor x \rfloor = -\lceil -x \rceil \quad \text{for } x \in \mathbb{R};$$

$$\left\lceil \frac{m}{n} \right\rceil = \left\lfloor \frac{m+n-1}{n} \right\rfloor = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 \quad \text{for } m,n \in \mathbb{Z} \text{ and } n > 0; \quad \text{and}$$

$$\left\lfloor \frac{m}{n} \right\rfloor = \left\lceil \frac{m-n+1}{n} \right\rceil = \left\lceil \frac{m+1}{n} \right\rceil - 1 \quad \text{for } m,n \in \mathbb{Z} \text{ and } n > 0.$$

PERIODIC SIGNALS

- An important class of signals is the class of periodic signals.
- A periodic CT signal *x(t)* has the property:

$$x(t) = x(t + T)$$

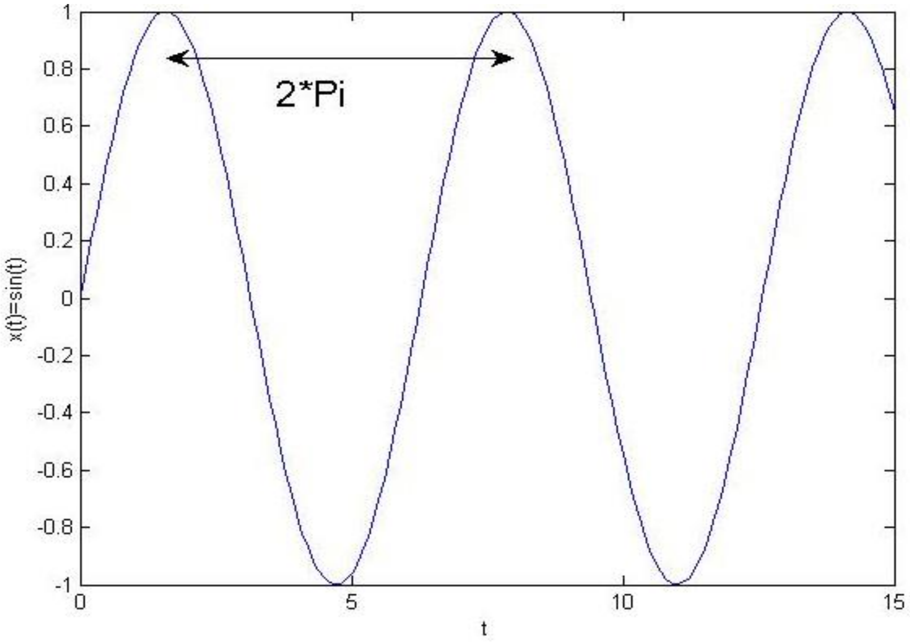
where *T>0*, for all *t*.

Smallest value of *T* is the fundamental period.

Examples:

$$\cos(t + 2\pi) = \cos(t)$$
$$\sin(t + 2\pi) = \sin(t)$$

Are both periodic with period 2π



- A **DT signal** is **periodic** if:

$$\boldsymbol{x[n] = x[n + N]}$$

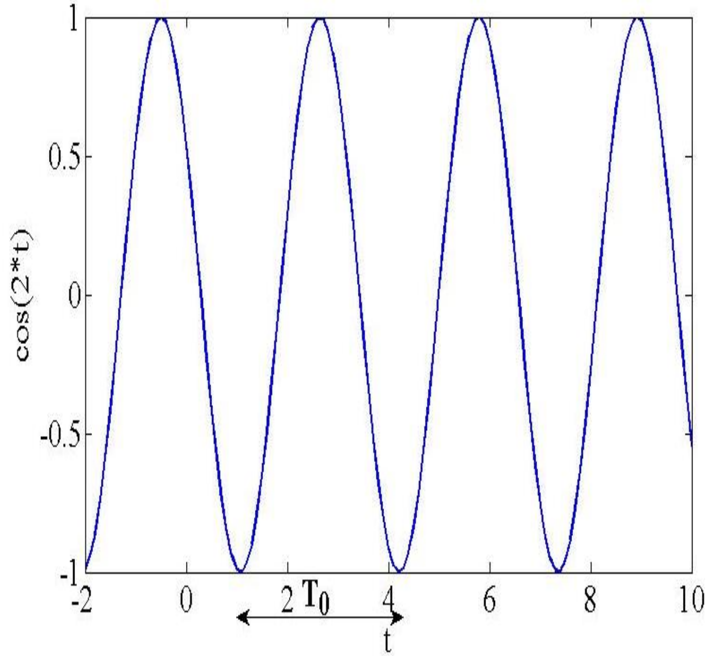
for an integer $N > 0, \forall n$. Smallest value of *N* is the fundamental period.

- Consider when $a = j\omega_0$ is purely imaginary (complex signal):

$$x(t) = Ce^{j\omega_0 t}$$

- By **Euler's relationship**, this is:

$$e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$$



This is a periodic signals because:

$$e^{j\omega_0(t+T)} = \cos\omega_0(t+T) + j\sin\omega_0(t+T)$$

when $T = \dfrac{2\pi}{\omega_0}$.

- This is related to real valued sinusoidal signal:

$x(t) = \cos(\omega_0 t)\ , \omega_0 = 2\pi f_0$ $\qquad$ $T_0 = \dfrac{2\pi}{\omega_0}$

T_0 is the **fundamental period** (s)
ω_0 is the **fundamental frequency** (rad/s)

because:

$\cos(\omega_0 t) = (e^{j\omega_0 t} + e^{-j\omega_0 t})/2;$ $\qquad$ $\sin(\omega_0 t) = (e^{j\omega_0 t} - e^{-j\omega_0 t})/2j$

Ex1: $x(t) = je^{j5t}$

$$je^{j5t} = jcos5t - sin5t$$

This is periodic if there exists *T>0*:

$$x(t) = x(t+T) = jcos5(t+T) - sin5(t+T)$$

This is true for sinusoidal signals:

$5T = 2k\pi$ *k*=…,-2,-1,1,2..

T= 2π/5

Signal is periodic and fundamental period is *T*= 2π/5.

Ex2: $x[n] = e^{j7\pi n}$

$$e^{j7\pi n} = cos7\pi n + jsin7\pi n$$

This is periodic if there exists *T>0*:

$$x[n] = x(n+N) = cos7\pi(n+N) + jsin7\pi(n+N)$$

This is true for sinusoidal signals:

$7\pi N = 2k\pi$ *k*=…,-2,-1,1,2..

N= $2k$/7; *N*=2 when *k*=7

Signal is periodic and fundamental period is *N*=2.

- **Total energy** (similar to electrical) of a continuous time signal $x(t)$ over $[t_1, t_2]$ is:

$$E = \int_{t_1}^{t_2} |x(t)|^2 dt$$

where $|.|$ denotes the magnitude of the (complex) number/signal. Similarly for a DT (discrete time) signal over $[n_1, n_2]$:

$$E = \sum_{n_1}^{n_2} |x[n]|^2$$

By dividing the energy by $(t_2 - t_1)$ and $(n_2 - n_1 + 1)$, respectively, gives the **average power**, P

$$P = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |x(t)|^2 dt$$

$$P = \frac{1}{n_2 - n_1 + 1} \sum_{n_1}^{n_2} |x[n]|^2$$

ENERGY AND POWER OVER INFINITE TIME

- Examining the energy over an infinite time interval $(-\infty, \infty)$:

$$E_\infty = lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$E_\infty = lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

- If the sums or integrals do not converge (*x[t]* or *x[n]* equals a nonzero constant value for all time), the signal energy is infinite. The corresponding power is

$$P_\infty = lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$P_\infty = lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

- Two important (sub) classes of signals
  1. Finite total energy (zero average power)
  2. Finite average power (infinite total energy)

POWER & ENERGY OF PERIODIC SIGNALS

- Periodic signals, in particular complex periodic and sinusoidal signals, have infinite total energy but finite average power.

Consider energy over one period:

$$E_{period} = \int_0^{T_0} \left|e^{j\omega_0 t}\right|^2 dt$$

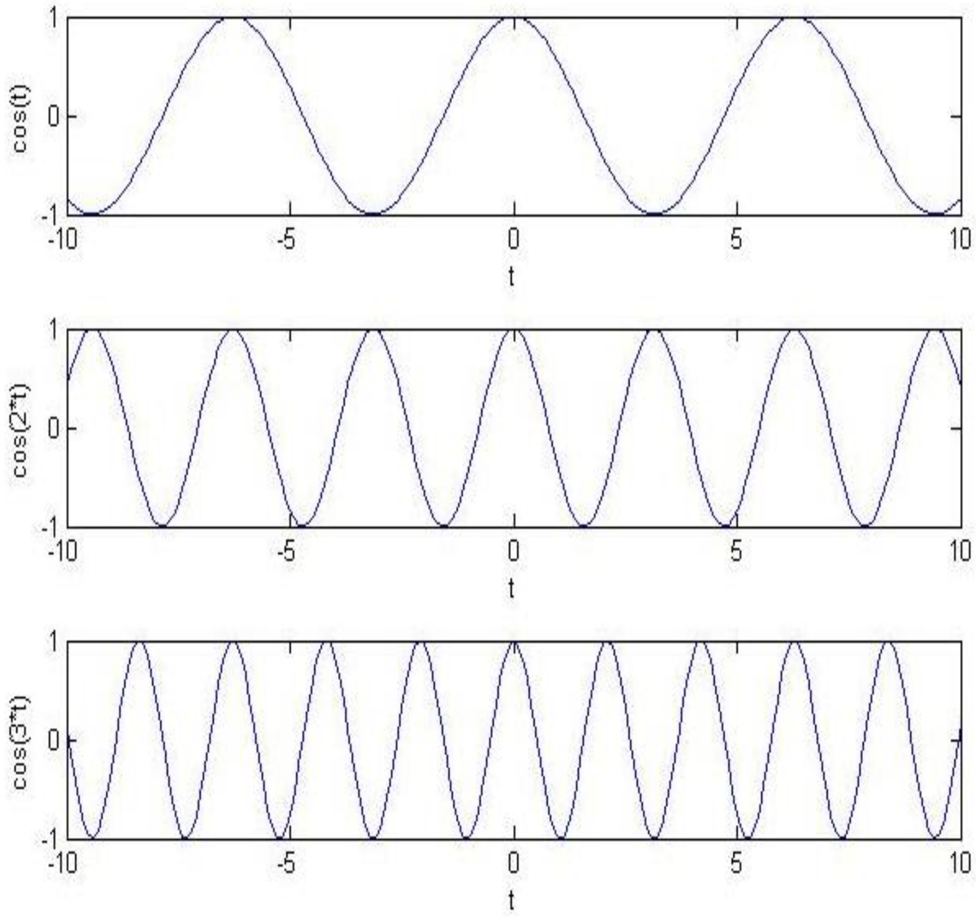
$$= \int_0^{T_0} 1dt = T_0$$

Therefore:

$$E_\infty = \infty$$

Average power:

$$P_{period} = \frac{1}{T_0} E_{period} = 1$$



EXAMPLES: SIGNAL POWER AND ENERGY

$x(t) = e^{-2t}u(t)$

$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt$

$= \int_{-\infty}^{\infty} |e^{-2t}u(t)|^2 dt$

$= \int_{0}^{\infty} e^{-4t} dt$

$= \frac{1}{4}$

$x[n] = (\frac{1}{2})^n u[n]$

$E_\infty = \sum_{n=-\infty}^{\infty} |x[n]|^2$

$= \sum_{n=-\infty}^{\infty} \left|(\frac{1}{2})^n u[n]\right|^2$

$= \sum_{n=0}^{\infty} \frac{1}{4}^n$

$= \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$

$$P_\infty = lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$= lim_{T\to\infty} \frac{1}{2T} * \frac{1}{4}$

$= 0$

$$P_\infty = lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

$= lim_{N\to\infty} \frac{1}{2N+1} * \frac{4}{3}$

$= 0$

ODD AND EVEN SIGNALS

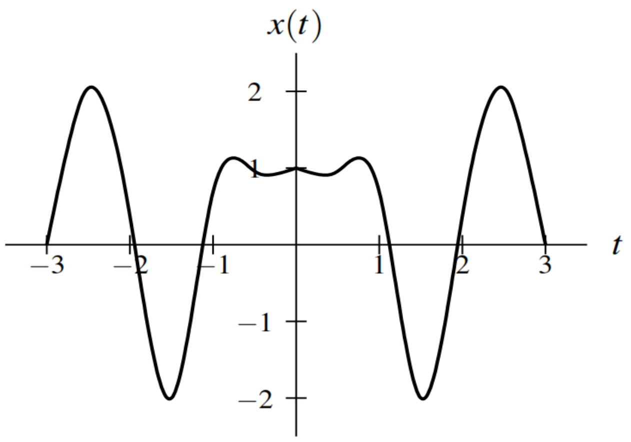
- A function x is said to be **even** if it satisfies

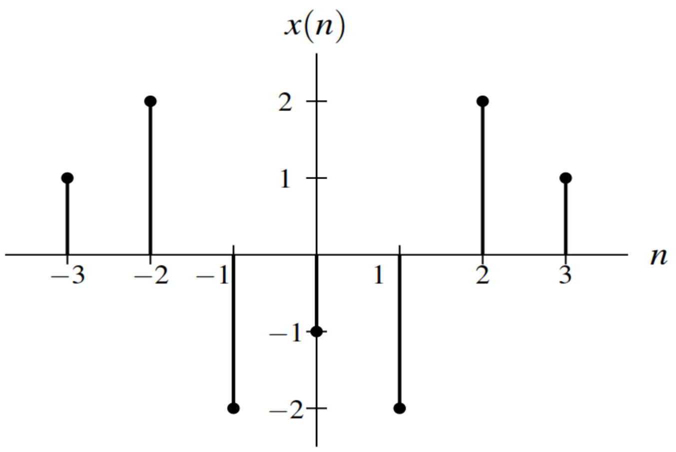
$$x(t) = x(-t) \quad \text{for all } t \text{ (where } t \text{ is a real number).}$$

- A sequence x is said to be **even** if it satisfies

$$x(n) = x(-n) \quad \text{for all } n \text{ (where } n \text{ is an integer).}$$

- Geometrically, the graph of an even signal is ***symmetric*** with respect to the vertical axis.
- Some examples of even signals are shown below.





ODD AND EVEN SIGNALS

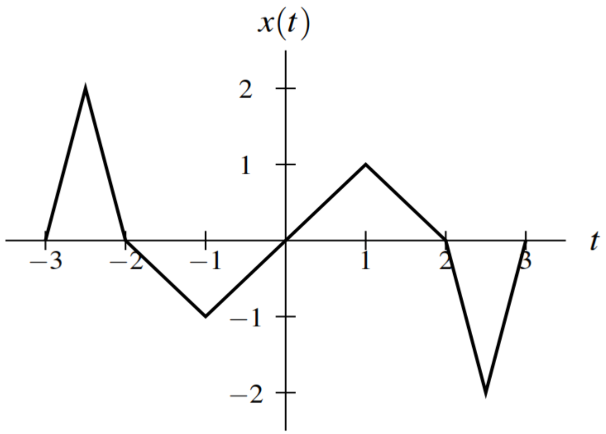
- A function x is said to be **odd** if it satisfies

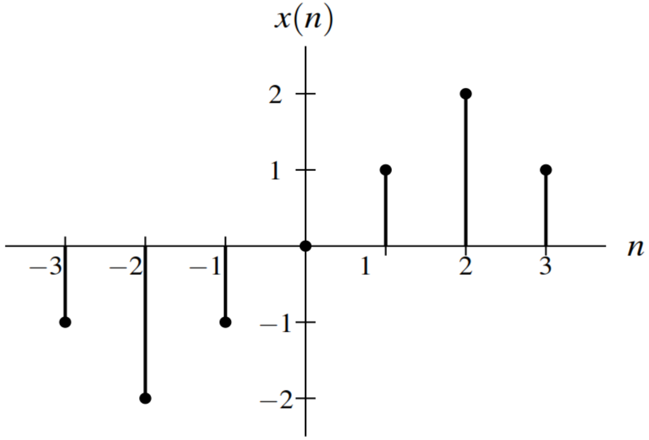
$$x(t) = -x(-t) \quad \text{for all } t \text{ (where } t \text{ is a real number).}$$

- A sequence x is said to be **odd** if it satisfies

$$x(n) = -x(-n) \quad \text{for all } n \text{ (where } n \text{ is an integer).}$$

- An odd signal x must be such that $x(0) = 0$.
- Geometrically, the graph of an odd signal is ***symmetric*** with respect to the origin.
- Some examples of odd signals are shown below.





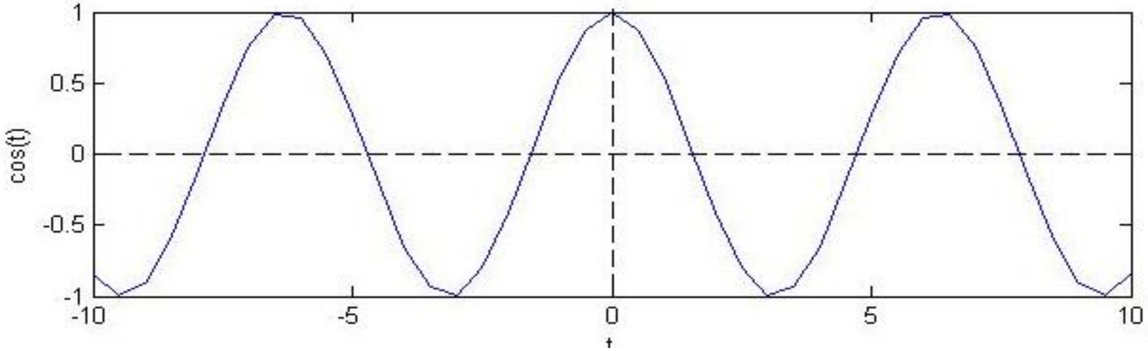
ODD AND EVEN SIGNALS

- An **even** signal is identical to its time reversed signal, i.e. It can be reflected in the origin and is equal to the original.

$$x(t) = x(-t)$$

Examples:

$$x(t) = \cos(t)$$



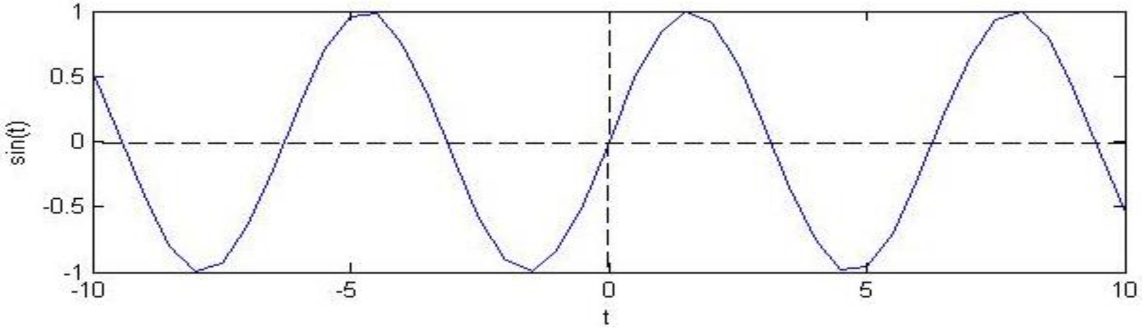
- An **odd** signal is identical to its negated, time reversed signal, i.e. it is equal to the negative reflected signal

$$x(t) = -x(t)$$

Examples:

$$x(t) = \sin(t)$$

$$x(t) = \mathrm{t}$$



<u>**Note:**</u> Even/Odd property is important because any signal can be expressed as the sum of an odd signal and an even signal.

CONJUGATE SYMMETRY

- A function x is said to be **conjugate symmetric** if it satisfies

$$x(t) = x^*(-t) \quad \text{for all } t \text{ (where } t \text{ is a real number).}$$

- A sequence x is said to be **conjugate symmetric** if it satisfies

$$x(n) = x^*(-n) \quad \text{for all } n \text{ (where } n \text{ is an integer).}$$

- The real part of a conjugate symmetric function or sequence is even.
- The imaginary part of a conjugate symmetric function or sequence is odd.
- An example of a conjugate symmetric function is a complex sinusoid $x(t) = \cos \omega t + j \sin \omega t$, where ω is a real constant.

- A central concept is signal analysis is the transformation of one signal into another signal. Only need to concentrate in simple transformations that involve a transformation of the time axis only.

- A linear **time shift** signal transformation is given by:

$$y(t) = x(at + b)$$

where b represents a signal offset from 0, and the a parameter represents a signal **compression if $|a| > 1$, stretching if $0 < |a| < 1$** and a **reflection if $a < 0$**.

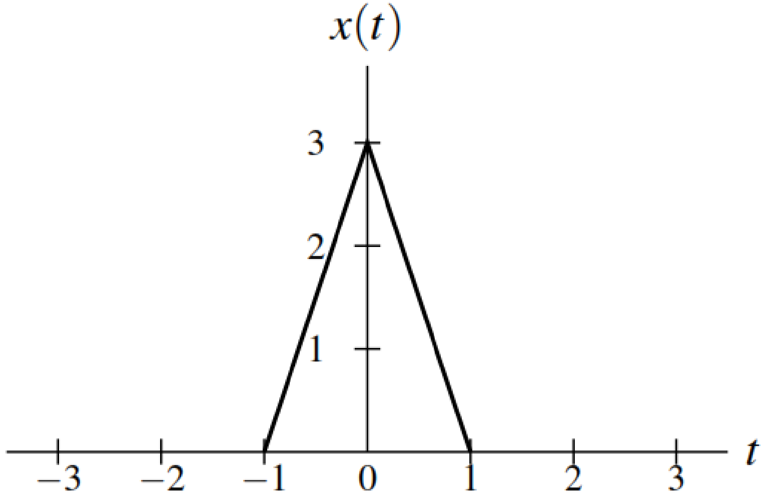
- **Time shifting** (also called **translation**) maps the input function x to the output function y as given by

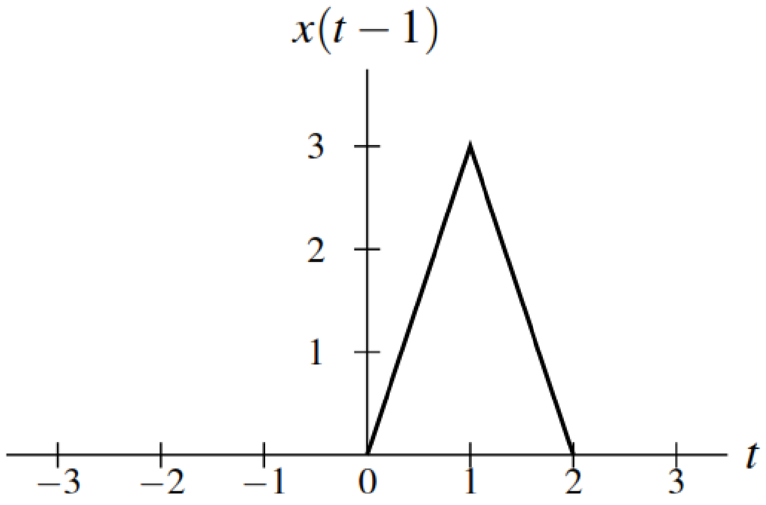
$$y(t) = x(t-b),$$

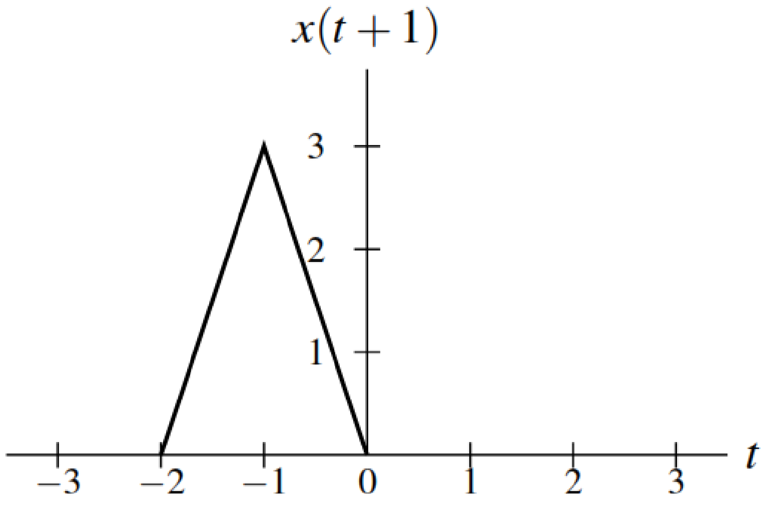
where b is a real number.

- Such a transformation shifts the function (to the left or right) along the time axis.
- If $b > 0$, y is ***shifted to the right*** by $|b|$, relative to x (i.e., delayed in time).
- If $b < 0$, y is ***shifted to the left*** by $|b|$, relative to x (i.e., advanced in time).

TRANSFORMATION OF SIGNAL



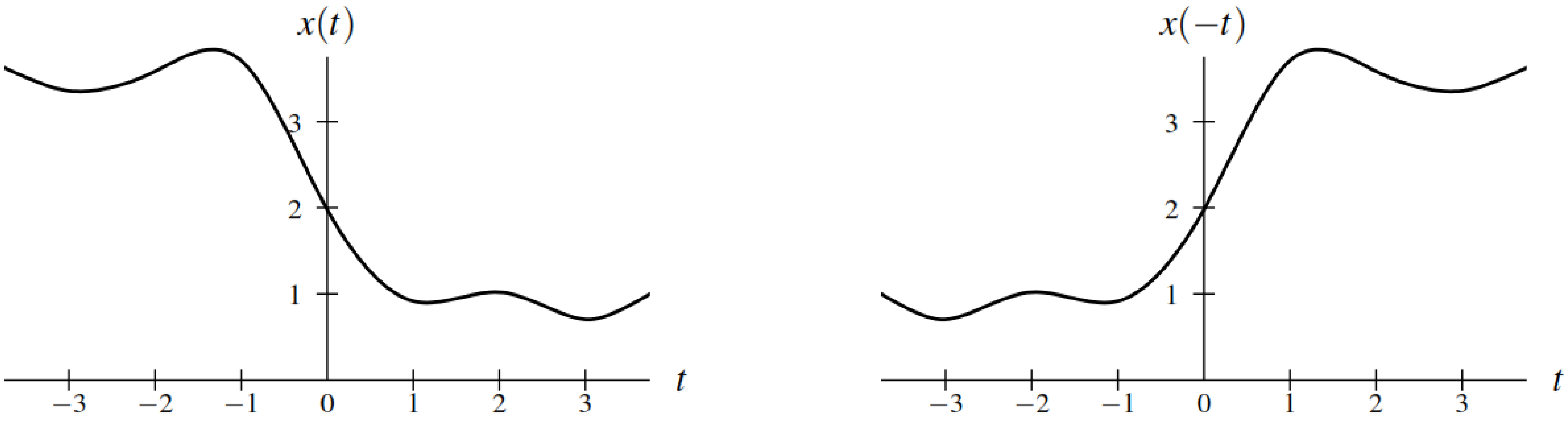




- **Time reversal** (also known as **reflection**) maps the input function x to the output function y as given by

$$y(t) = x(-t).$$

- Geometrically, the output function y is a reflection of the input function x about the (vertical) line $t = 0$.



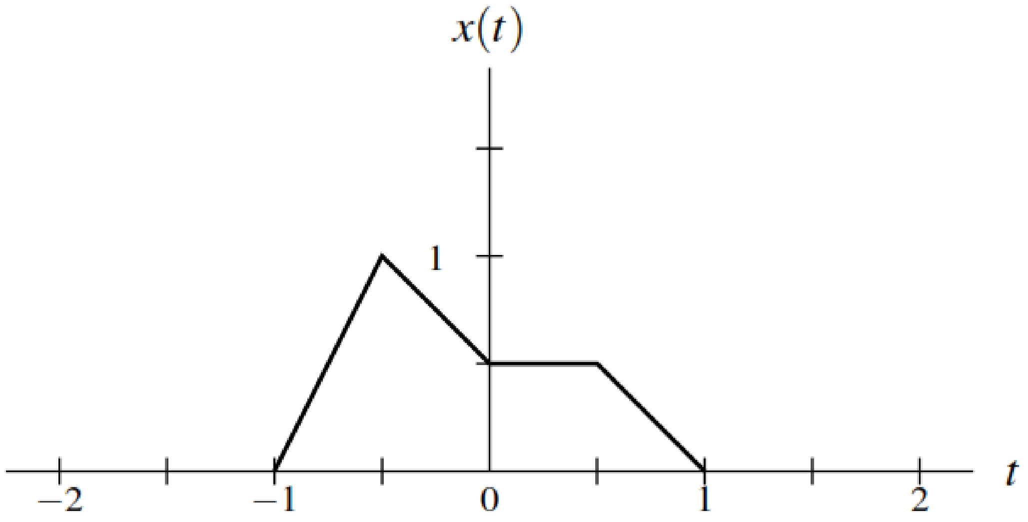
- **Time compression/expansion** (also called **dilation**) maps the input function x to the output function y as given by

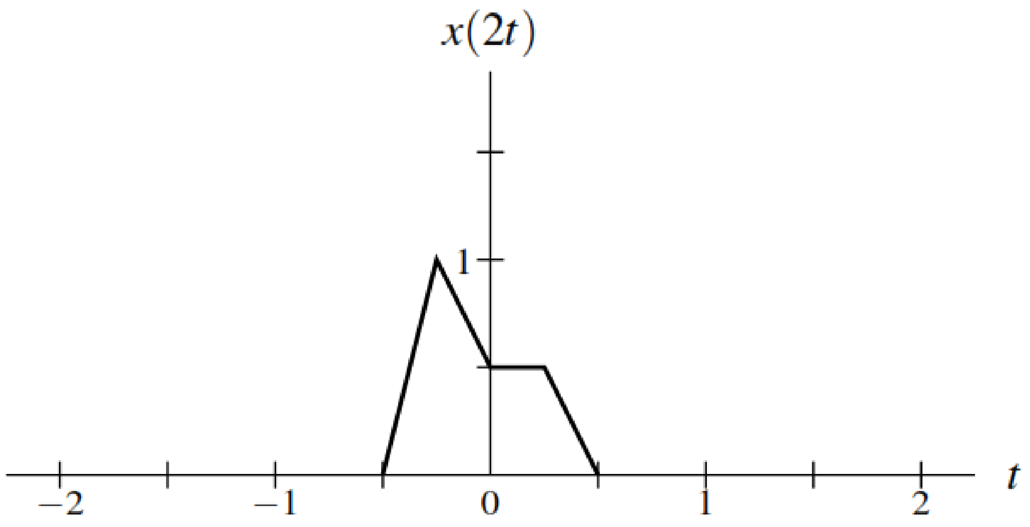
$$y(t) = x(at),$$

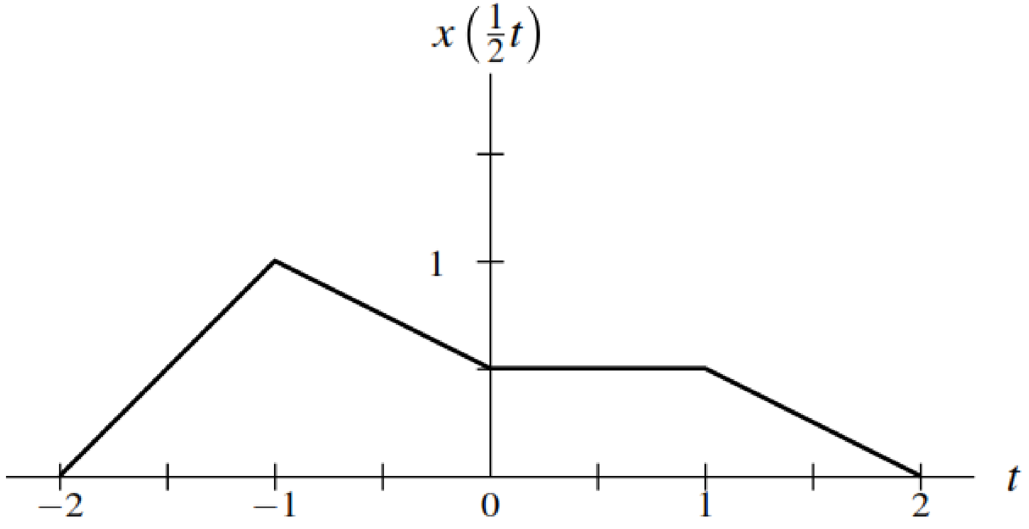
where a is a ***strictly positive*** real number.

- Such a transformation is associated with a compression/expansion along the time axis.
- If $a > 1$, y is ***compressed*** along the horizontal axis by a factor of a, relative to x.
- If $a < 1$, y is ***expanded*** (i.e., stretched) along the horizontal axis by a factor of $\frac{1}{a}$, relative to x.

TRANSFORMATION OF SIGNAL







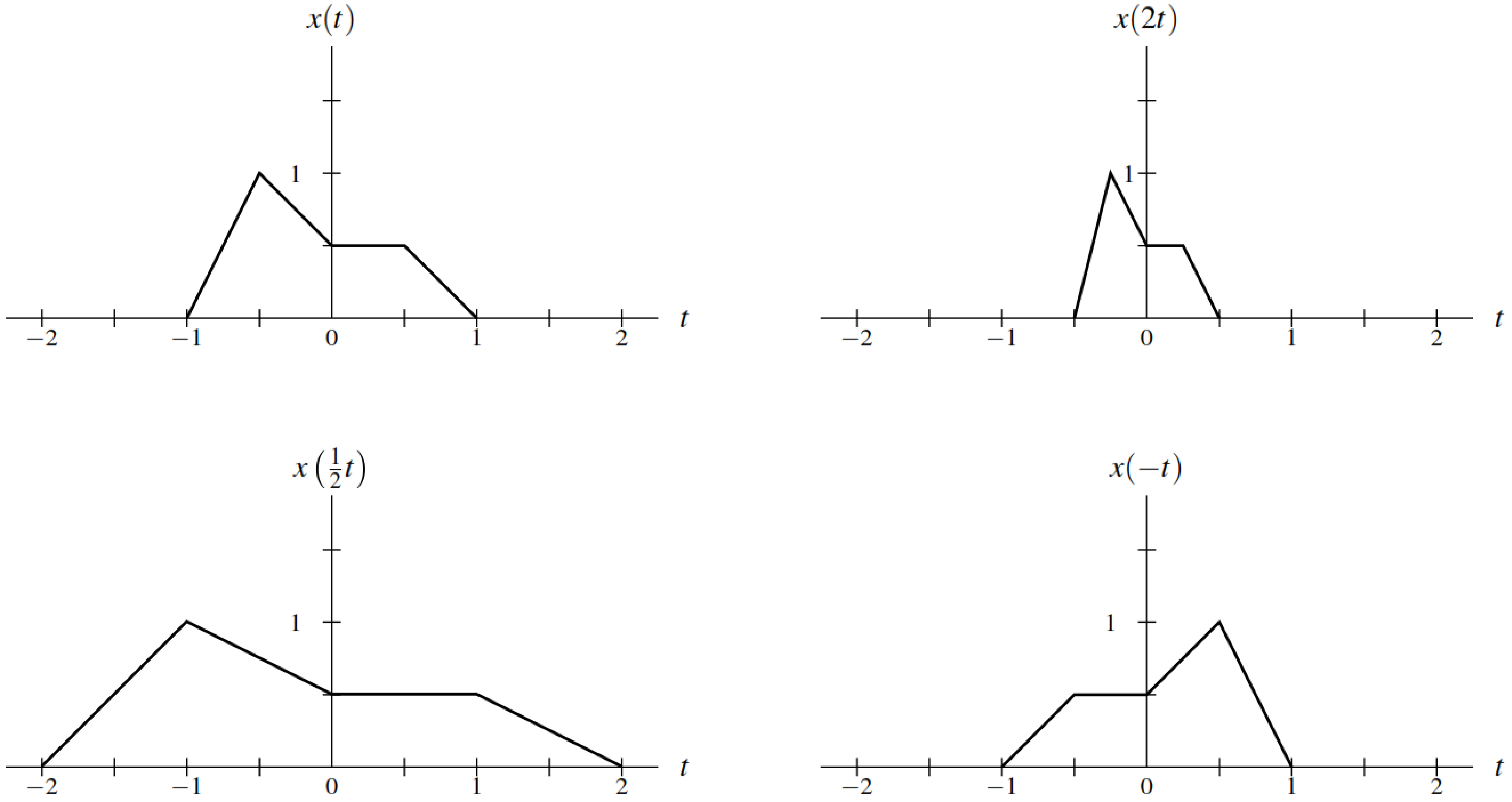
- **Time scaling** maps the input function x to the output function y as given by

$$y(t) = x(at),$$

where a is a ***nonzero*** real number.

- Such a transformation is associated with a dilation (i.e., compression/expansion along the time axis) and/or time reversal.
- If $|a| > 1$, the function is ***compressed*** along the time axis by a factor of $|a|$.
- If $|a| < 1$, the function is ***expanded*** (i.e., stretched) along the time axis by a factor of $\left|\frac{1}{a}\right|$.
- If $|a| = 1$, the function is neither expanded nor compressed.
- If $a < 0$, the function is also time reversed.
- Dilation (i.e., expansion/compression) and time reversal ***commute***.
- Time reversal is a special case of time scaling with $a = -1$; and time compression/expansion is a special case of time scaling with $a > 0$.

TRANSFORMATION OF SIGNAL

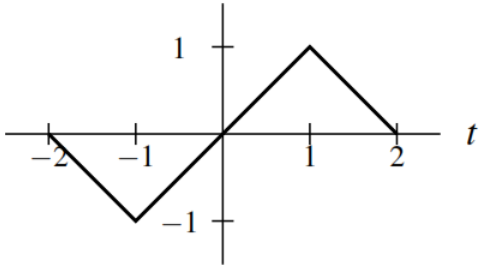


- Consider a transformation that maps the input function x to the output function y as given by
$$y(t) = x(at - b),$$
where a and b are real numbers and $a \neq 0$.
- The above transformation can be shown to be the combination of a time-scaling operation and time-shifting operation.
- Since time scaling and time shifting ***do not commute***, we must be particularly careful about the order in which these transformations are applied.
- The above transformation has two distinct but equivalent interpretations:
  1. first, time shifting x by b, and then time scaling the result by a;
  2. first, time scaling x by a, and then time shifting the result by b/a.
- Note that the time shift is not by the same amount in both cases.
- In particular, note that when time scaling is applied first followed by time shifting, the time shift is by b/a, not b.

Given x as shown below, find

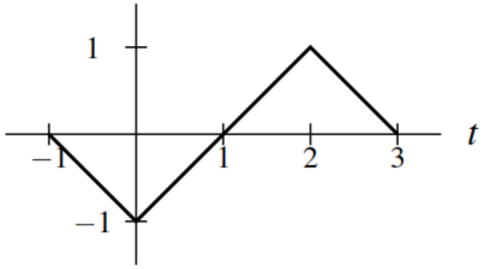
$y(t) = x(2t-1)$.

$x(t)$

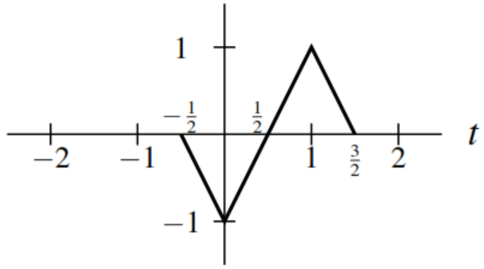


time shift by 1 and then time scale by 2

$p(t) = x(t-1)$

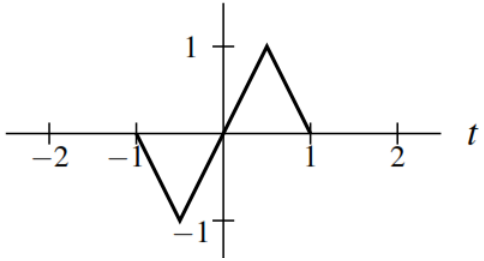


$y(t) = p(2t)$

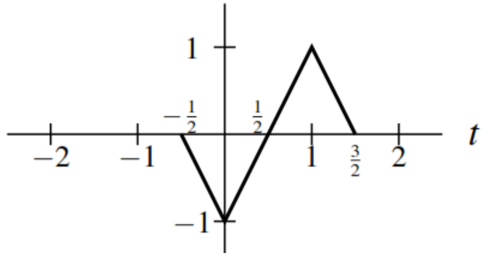


time scale by 2 and then time shift by $\frac{1}{2}$

$q(t) = x(2t)$



$y(t) = q\left(t - \frac{1}{2}\right)$

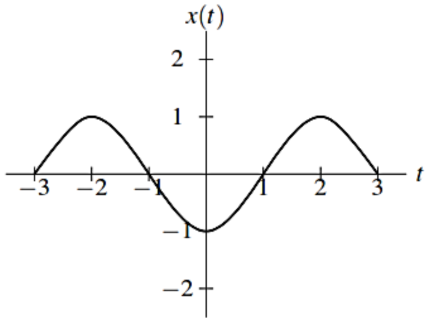


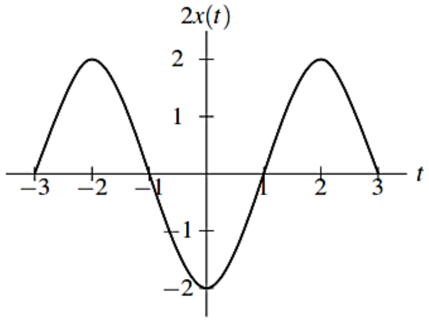
- **Amplitude scaling** maps the input function x to the output function y as given by

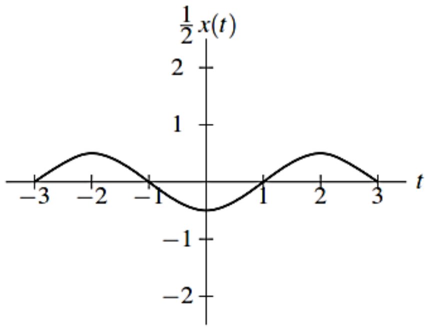
$$y(t) = ax(t),$$

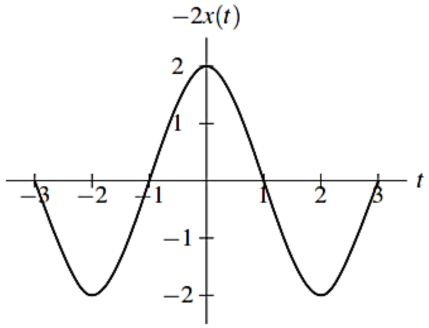
where a is a real number.

- Geometrically, the output function y is ***expanded/compressed*** in amplitude and/or ***reflected*** about the horizontal axis.









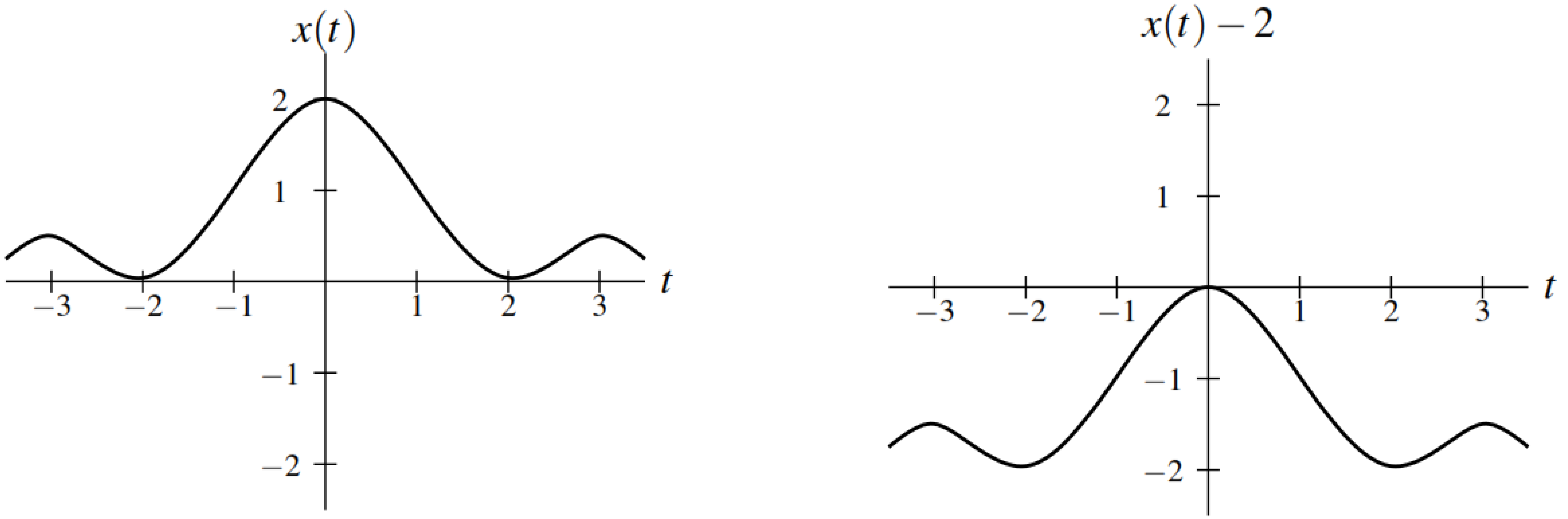
TRANSFORMATION OF SIGNAL

- **Amplitude shifting** maps the input function x to the output function y as given by

$$y(t) = x(t) + b,$$

where b is a real number.

- Geometrically, amplitude shifting adds a ***vertical displacement*** to x.



- We can also combine amplitude scaling and amplitude shifting transformations.
- Consider a transformation that maps the input function x to the output function y, as given by

$$y(t) = ax(t) + b,$$

where a and b are real numbers.

- Equivalently, the above transformation can be expressed as

$$y(t) = a\left[x(t) + \frac{b}{a}\right].$$

- The above transformation is equivalent to:
  1. first amplitude scaling x by a, and then amplitude shifting the resulting function by b; or
  2. first amplitude shifting x by b/a, and then amplitude scaling the resulting function by a.

PRESENTING A RECTANGULAR PULSE USING UNIT STEP FUNCTION

- For real constants a and b where $a \leq b$, consider a function x of the form

$$x(t) = \begin{cases} 1 & a \leq t < b \\ 0 & \text{otherwise} \end{cases}$$

(i.e., x is a ***rectangular pulse*** of height one, with a ***rising edge at*** a and ***falling edge at*** b).

- The function x can be equivalently written as

$$x(t) = u(t-a) - u(t-b)$$

(i.e., the difference of two time-shifted unit-step functions).

- Unlike the original expression for x, this latter expression for x ***does not involve multiple cases***.
- In effect, by using unit-step functions, we have collapsed a formula involving multiple cases into a single expression.

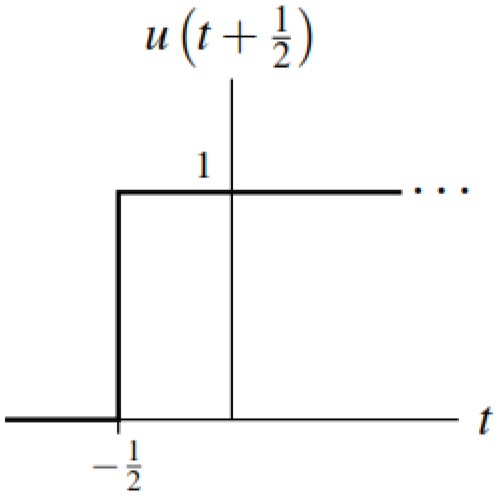
EXAMPLE

(Rectangular function). Show that the rect function can be expressed in terms of u as

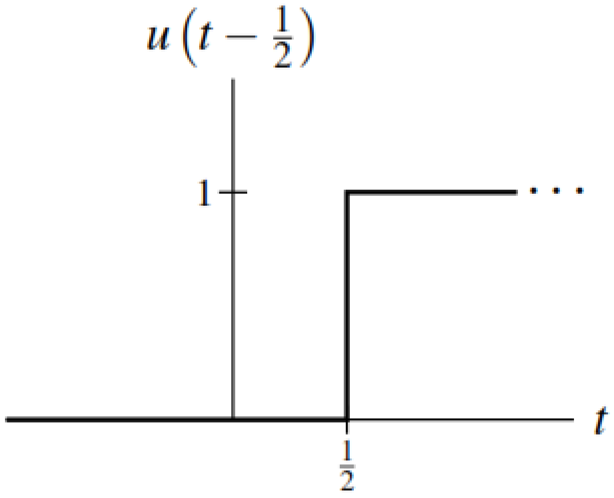
$$\operatorname{rect} t = u\left(t+\tfrac{1}{2}\right) - u\left(t-\tfrac{1}{2}\right).$$

Solution. Using the definition of u and time-shift transformations, we have

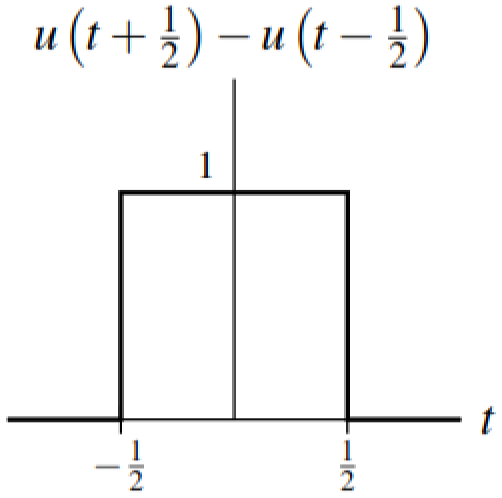
$$u\left(t+\tfrac{1}{2}\right) = \begin{cases} 1 & t \geq -\frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad u\left(t-\tfrac{1}{2}\right) = \begin{cases} 1 & t \geq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$



(a)



(b)



(c)

Thus, we have

$$\begin{aligned}
u\left(t+\tfrac{1}{2}\right)-u\left(t-\tfrac{1}{2}\right) &= \begin{cases} 0-0 & t<-\frac{1}{2} \\ 1-0 & -\frac{1}{2}\leq t<\frac{1}{2} \\ 1-1 & t\geq\frac{1}{2} \end{cases} \\
&= \begin{cases} 0 & t<-\frac{1}{2} \\ 1 & -\frac{1}{2}\leq t<\frac{1}{2} \\ 0 & t\geq\frac{1}{2} \end{cases} \\
&= \begin{cases} 1 & -\frac{1}{2}\leq t<\frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \\
&= \operatorname{rect} t.
\end{aligned}$$

EXAMPLE

(Piecewise-linear function). Consider the piecewise-linear function x given by

$$x(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ 3-t & 2 \le t < 3 \\ 0 & \text{otherwise.} \end{cases}$$

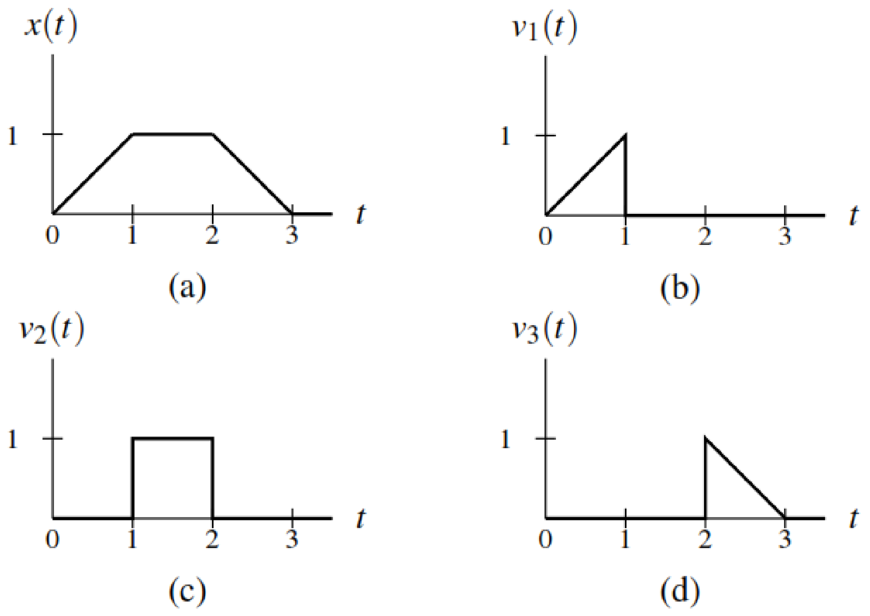
Find a single expression for $x(t)$ (involving unit-step functions) that is valid for all t.

Solution.

$$v_1(t) = t[u(t) - u(t-1)].$$

$$v_2(t) = u(t-1) - u(t-2).$$

$$v_3(t) = (3-t)[u(t-2) - u(t-3)].$$



EXAMPLE

$$
\begin{aligned}
x(t) &= v_1(t) + v_2(t) + v_3(t) \\
&= t[u(t) - u(t-1)] + [u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)] \\
&= tu(t) + (1-t)u(t-1) + (3-t-1)u(t-2) + (t-3)u(t-3) \\
&= tu(t) + (1-t)u(t-1) + (2-t)u(t-2) + (t-3)u(t-3).
\end{aligned}
$$

Thus, we have found a single expression for $x(t)$ that is valid for all t.

Determine whether the following signals are energy signals, power signals, or neither.

(a) $x(t) = \begin{cases} e^{-at}, & 0 < t < \infty, \quad a > 0 \\ 0, & \text{otherwise} \end{cases}$

(b) $x(t) = A\cos(\omega t + \theta)$

(c) $x[n] = 10e^{j2n}$

(a) The normalized energy of the signal is

$$E = \int_{-\infty}^{\infty} x(t)^2 dt = \int_{0}^{\infty} e^{-2at} dt = \left.\frac{e^{-2at}}{-2a}\right|_{0}^{\infty} = \frac{1}{2a} < \infty$$

confirming that $x(t)$ is an energy signal.

(b) The sinusoidal signal has period $T = 2\pi/\omega$.

The normalized average power is

$$p = \frac{1}{T}\int_{0}^{T} x(t)^2 dt = \frac{\omega}{2\pi}\int_{0}^{T} A^2 \cos^2(\omega t + \theta) dt$$

$$= \frac{A^2\omega}{2\pi}\int_{0}^{2\pi/\omega} \frac{1}{2}\left[1 + \cos(2\omega t + 2\theta)\right] dt$$

$$= \frac{A^2\omega}{2\pi}\frac{1}{2}\frac{2\pi}{\omega} = \frac{A^2}{2} < \infty$$

showing that $x(t)$ is a power signal. All periodic signals are generally power signals.

(c) $|x[n]| = |10e^{j2n}| = 10|e^{j2n}| = 10\,|\cos(2n) + j\sin(2n)|$

$$= 10\sqrt{\left[\cos^2(2n) + \sin^2(2n)\right]} = 10$$

The normalized average power is

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 10^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} 100(2N+1) = 100 < \infty$$

that is, $x[n]$ is a power signal.

SYMMETRY AND ADDITION/MULTIPLICATION

- Sums involving even and odd functions have the following properties:
  - The sum of two even functions is even.
  - The sum of two odd functions is odd.
  - The sum of an even function and odd function is neither even nor odd, provided that neither of the functions is identically zero.
- That is, the *sum* of functions with the *same type of symmetry* also has the *same type of symmetry*.
- Products involving even and odd functions have the following properties:
  - The product of two even functions is even.
  - The product of two odd functions is even.
  - The product of an even function and an odd function is odd.
- That is, the *product* of functions with the *same type of symmetry* is *even*, while the *product* of functions with *opposite types of symmetry* is *odd*.

DECOMPOSITION OF A FUNCTION INTO EVEN AND ODD PARTS

- Every function x has a ***unique*** representation of the form

$$x(t) = x_e(t) + x_o(t),$$

  where the functions x_e and x_o are ***even*** and ***odd***, respectively.

- In particular, the functions x_e and x_o are given by

$$x_e(t) = \tfrac{1}{2}[x(t) + x(-t)] \quad \text{and} \quad x_o(t) = \tfrac{1}{2}[x(t) - x(-t)].$$

- The functions x_e and x_o are called the **even part** and **odd part** of x, respectively.
- For convenience, the even and odd parts of x are often denoted as $\mathrm{Even}\{x\}$ and $\mathrm{Odd}\{x\}$, respectively.

SUM OF PERIODIC FUNCTIONS

- **Sum of periodic functions.** For two periodic functions x_1 and x_2 with fundamental periods T_1 and T_2, respectively, and the sum $y = x_1 + x_2$:
  1. The sum y is periodic if and only if the ratio T_1/T_2 is a ***rational number*** (i.e., the quotient of two integers).
  2. If y is periodic, its fundamental period is rT_1 (or equivalently, qT_2, since $rT_1 = qT_2$), where $T_1/T_2 = q/r$ and q and r are integers and ***coprime*** (i.e., have no common factors). (Note that rT_1 is simply the least common multiple of T_1 and T_2.)
- Although the above theorem only directly addresses the case of the sum of two functions, the case of N functions (where $N > 2$) can be handled by applying the theorem repeatedly $N - 1$ times.

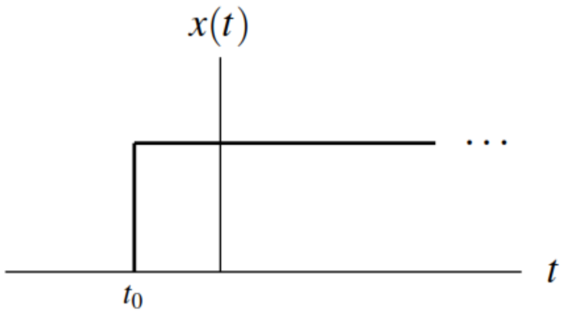
RIGHT-SIDED FUNCTIONS

- A function x is said to be **right sided** if, for some (finite) real constant t_0, the following condition holds:

$$x(t) = 0 \quad \text{for all } t < t_0$$

(i.e., x is ***only potentially nonzero to the right of*** t_0).

- An example of a right-sided function is shown below.



- A function x is said to be **causal** if

$$x(t) = 0 \quad \text{for all } t < 0.$$

- A causal function is a ***special case*** of a right-sided function.
- A causal function is not to be confused with a causal system. In these two contexts, the word "causal" has very different meanings.

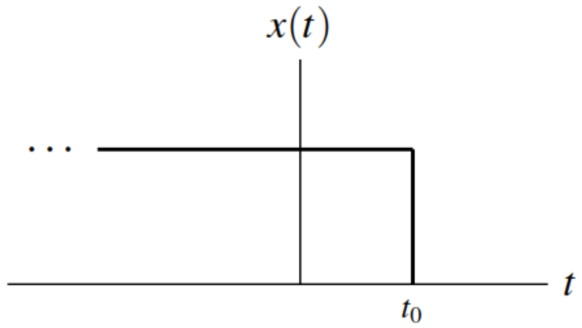
LEFT-SIDED FUNCTIONS

- A function x is said to be **left sided** if, for some (finite) real constant t_0, the following condition holds:

$$x(t) = 0 \quad \text{for all } t > t_0$$

(i.e., x is ***only potentially nonzero to the left of*** t_0).

- An example of a left-sided function is shown below.



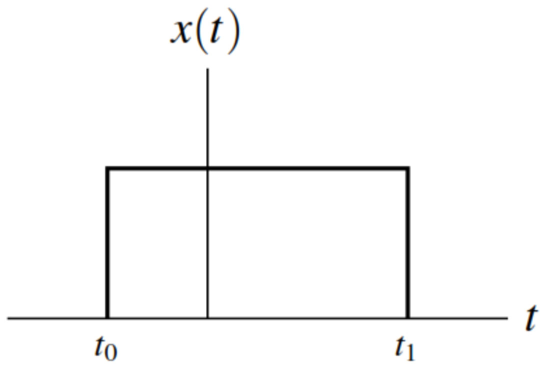
- Similarly, a function x is said to be **anticausal** if

$$x(t) = 0 \quad \text{for all } t > 0.$$

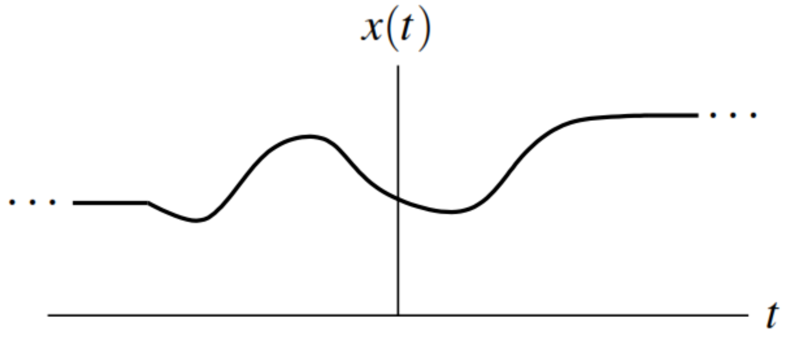
- An anticausal function is a ***special case*** of a left-sided function.
- An anticausal function is not to be confused with an anticausal system. In these two contexts, the word “anticausal” has very different meanings.

FINITE-DURATION AND TWO-SIDED FUNCTIONS

- A function that is both left sided and right sided is said to be **finite duration** (or **time limited**).
- An example of a finite duration function is shown below.



- A function that is neither left sided nor right sided is said to be **two sided**.
- An example of a two-sided function is shown below.



BOUNDED FUNCTIONS

- A function x is said to be **bounded** if there exists some (*finite*) positive real constant A such that

$$|x(t)| \leq A \quad \text{for all } t$$

(i.e., $x(t)$ is *finite* for all t).

- For example, the sine and cosine functions are bounded, since

$$|\sin t| \leq 1 \text{ for all } t \quad \text{and} \quad |\cos t| \leq 1 \text{ for all } t.$$

- In contrast, the tangent function and any nonconstant polynomial function p (e.g., $p(t) = t^2$) are unbounded, since

$$\lim_{t \to \pi/2} |\tan t| = \infty \quad \text{and} \quad \lim_{|t| \to \infty} |p(t)| = \infty.$$

EXAMPLE

Let $x_1(t) = \sin(\pi t)$ and $x_2(t) = \sin t$. Determine whether the function $y = x_1 + x_2$ is periodic.

Solution. Denote the fundamental periods of x_1 and x_2 as T_1 and T_2, respectively. We then have

$$T_1 = \tfrac{2\pi}{\pi} = 2 \quad \text{and} \quad T_2 = \tfrac{2\pi}{1} = 2\pi.$$

Here, we used the fact that the fundamental period of $\sin(\alpha t)$ is $\frac{2\pi}{|\alpha|}$. Thus, we have

$$\frac{T_1}{T_2} = \frac{2}{2\pi} = \frac{1}{\pi}.$$

Since π is an irrational number, $\frac{T_1}{T_2}$ is not rational. Therefore, y is not periodic.

EXAMPLE

Let $x_1(t) = \cos(2\pi t + \frac{\pi}{4})$ and $x_2(t) = \sin(7\pi t)$. Determine if the function $y = x_1 + x_2$ is periodic, and if it is, find its fundamental period.

Solution. Let T_1 and T_2 denote the fundamental periods of x_1 and x_2, respectively. Thus, we have

$$T_1 = \frac{2\pi}{2\pi} = 1 \quad \text{and} \quad T_2 = \frac{2\pi}{7\pi} = \frac{2}{7}.$$

Taking the ratio of T_1 to T_2, we have

$$\frac{T_1}{T_2} = \frac{7}{2}.$$

Since $\frac{T_1}{T_2}$ is a rational number, y is periodic. Let T denote the fundamental period of y. Since 7 and 2 are coprime,

$$T = 2T_1 = 7T_2 = 2.$$