Review -
The Operational Amplifier

Textbook:

Electric Circuits

James W. Nilsson & Susan A. Riedel

8th Edition.

link: **http://twitter.com/mlinh_ee**

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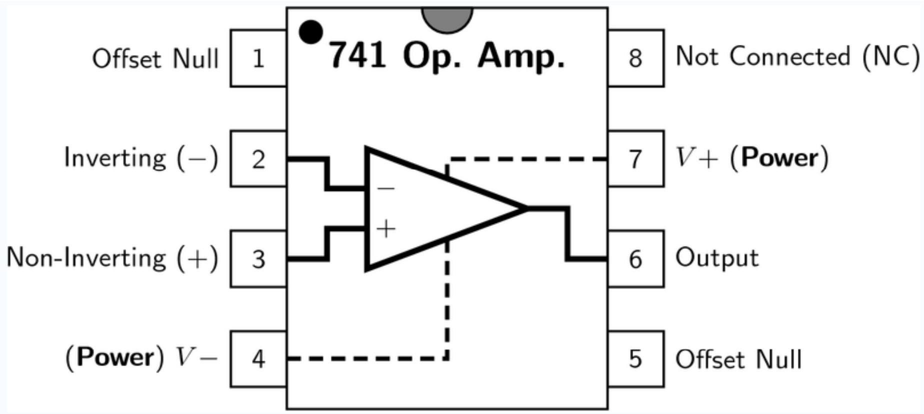
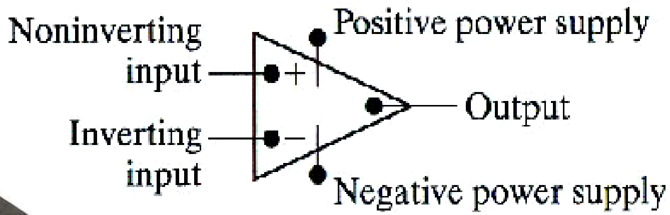
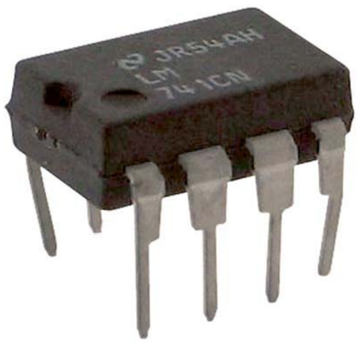
Op Amp Terminals

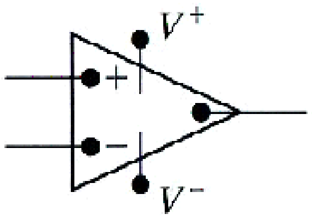


Terminals of primary interest:

- inverting input
- noninverting input
- output
- positive power supply (V^+)
- negative power supply (V^-)

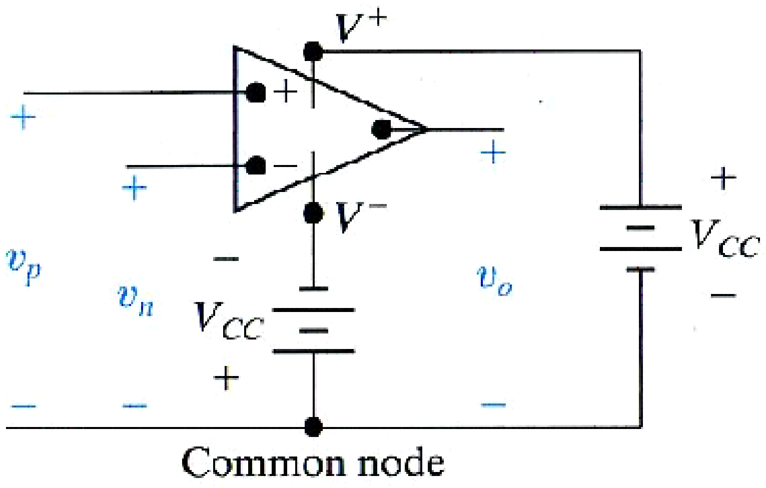






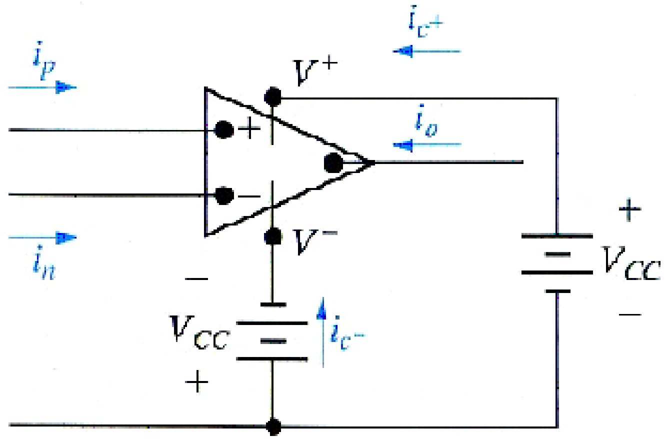
Offset null terminals may be used to compensate for a degradation in performance because of aging and imperfections.

Terminal Voltages and Currents



Terminal voltage variables

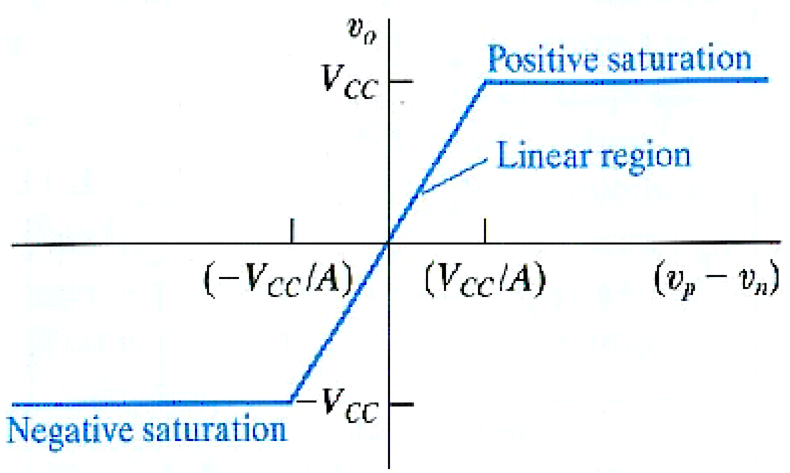
All voltages are considered as voltages rises from the common node.



Terminal current variables

All current reference directions are into the terminal of the op-amp.

Terminal Voltages and Currents



The terminal behavior of the op amp as linear circuit element is characterized by constraints on the input voltages and input currents.

Voltage transfer characteristic:

$$v_0 = \begin{cases} -V_{CC} & A(v_p - v_n) < -V_{CC} \\ A(v_p - v_n) & -V_{CC} \le A(v_p - v_n) \le +V_{CC} \\ +V_{CC} & A(v_p - v_n) > +V_{CC} \end{cases}$$

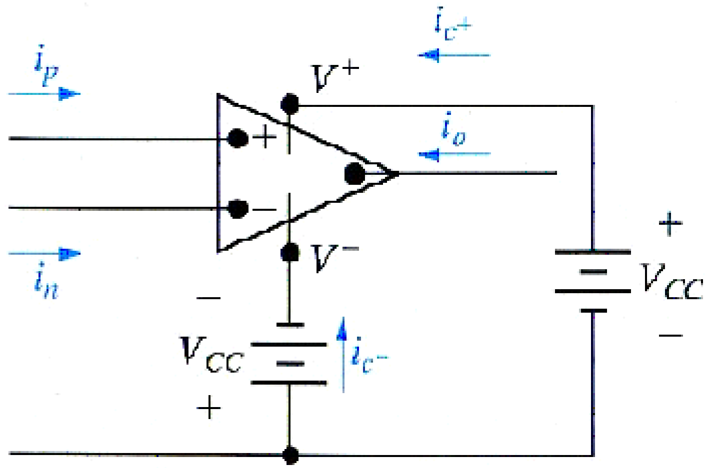
When the magnitude of the input voltage difference ($|v_p - v_n|$) is small, the op amp behaves as a linear device, as the output voltage is a linear function of the input voltages (the output voltage is equal to the difference in its input voltages times the gain, A.

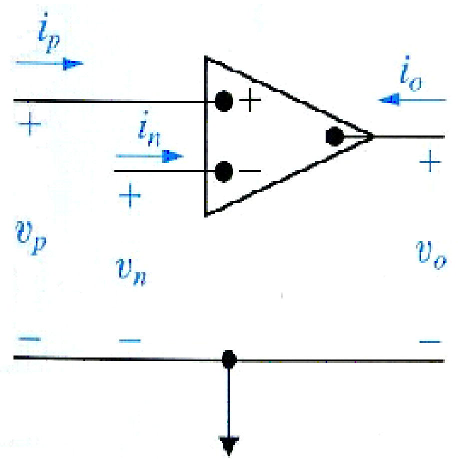
Terminal Voltages and Currents

For ideal op amp:

Input voltage constraint: $v_p = v_n$

Input current constraint: $i_p = i_n = 0$





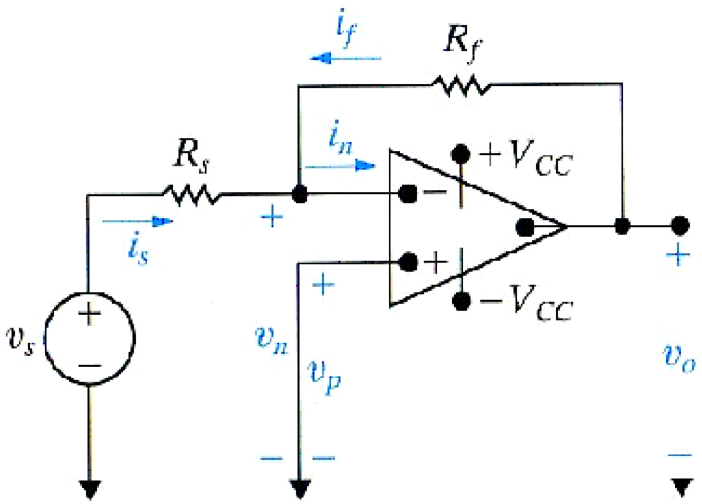
Apply Kirchhoff's current law

$$i_p + i_n + i_0 + i_{c^+} + i_{c^-} = 0$$

$$\longrightarrow \quad i_0 = -\left(i_{c^+} + i_{c^-}\right)$$

Even though the current at the input terminal is negligible, there are still appreciable current at the output terminal.

The inverting amplifier circuit



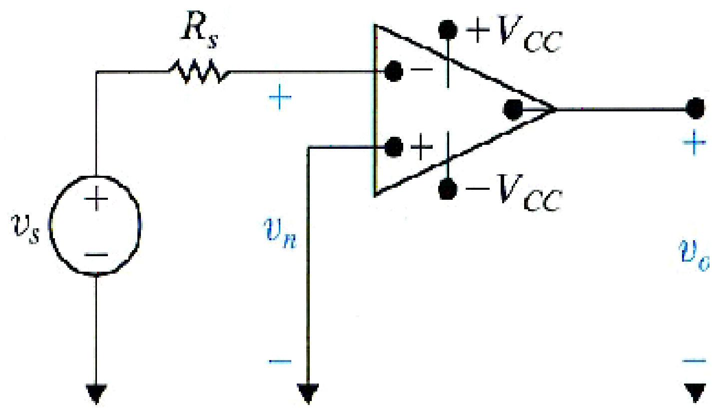
$$\boxed{v_0 = -\frac{R_f}{R_s} v_s}$$

The output voltage is an inverted, scaled replica of the input.

R_f is *negative feedback* of the circuit.

Upper limit on the gain: $\frac{R_f}{R_s} < \left| \frac{V_{CC}}{v_s} \right|$

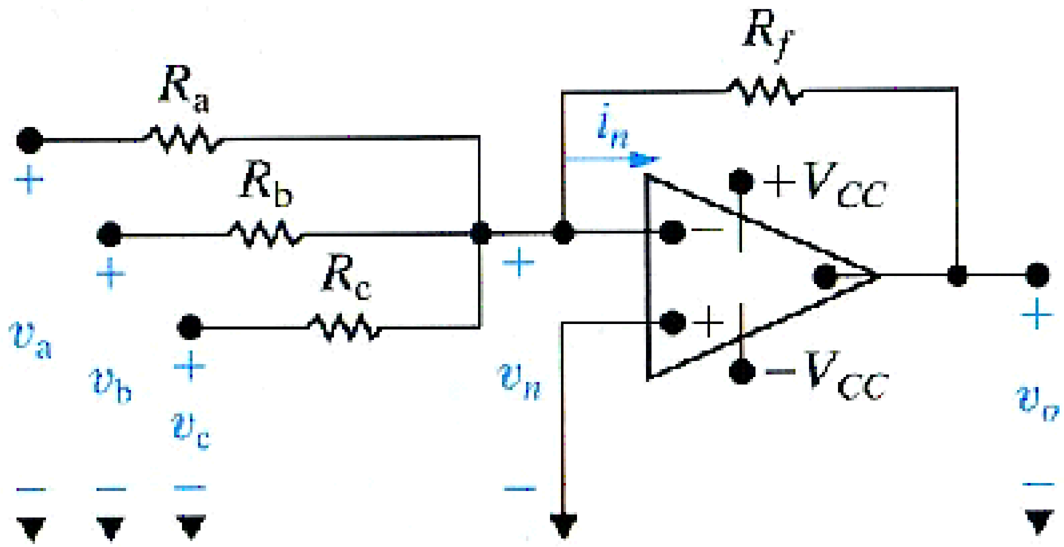
When R_f is removed, the feedback path is opened and the amplifier is called *open loop*.



$$\boxed{v_0 = -Av_n}$$

A is called the open-loop gain of the op amp

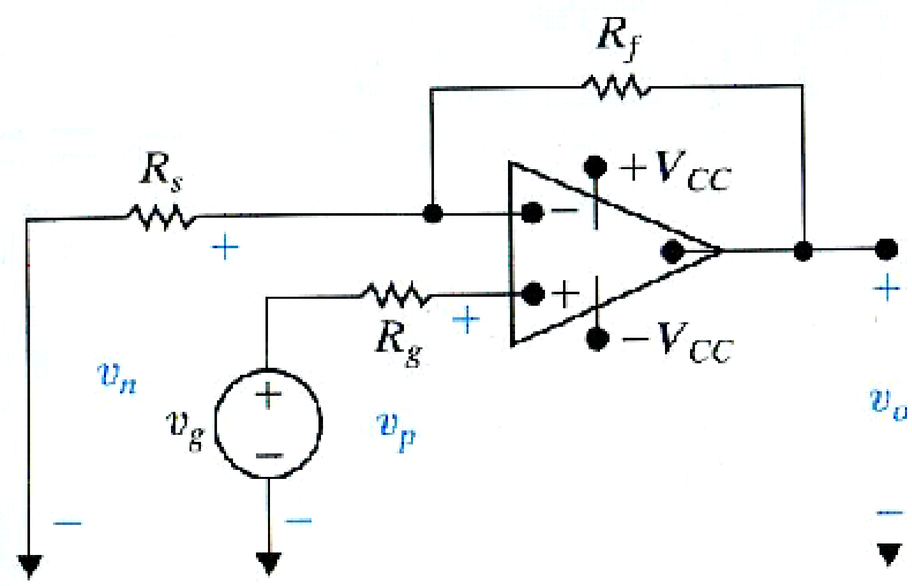
The summing amplifier circuit



$$v_0 = -\left(\frac{R_f}{R_a} v_a + \frac{R_f}{R_b} v_b + \frac{R_f}{R_c} v_c \right)$$

The output voltage of a summing amplifier is an inverted, scaled sum of the voltages applied to the input of the amplifier.

The noninverting amplifier circuit

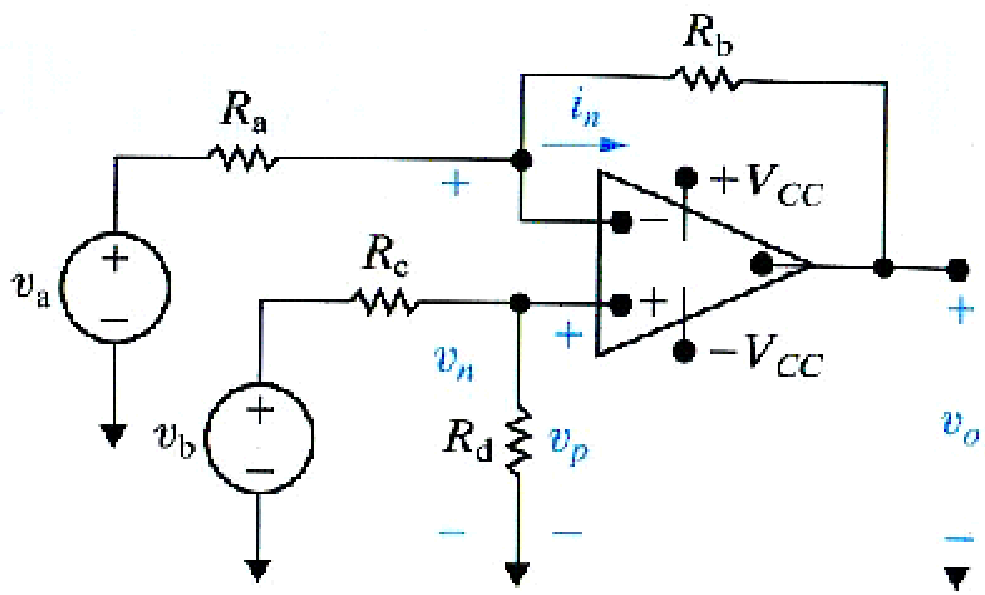


$$v_0 = \frac{R_s + R_f}{R_s} v_g$$

Requirement for operation in the linear region:

$$\frac{R_s + R_f}{R_s} < \left| \frac{V_{CC}}{V_g} \right|$$

The difference amplifier circuit



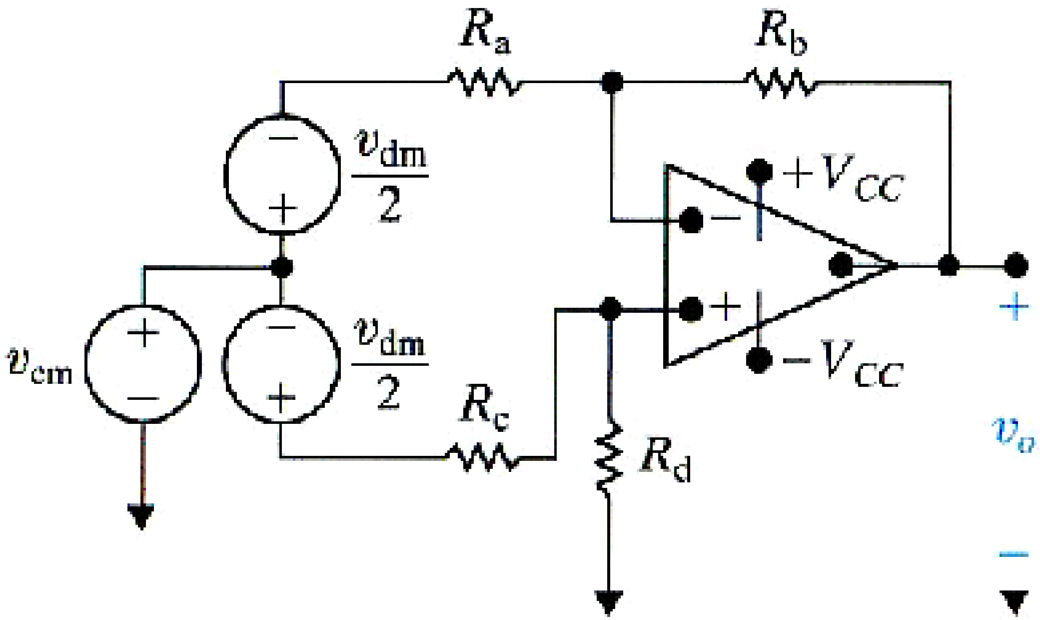
$$v_0 = \frac{R_d(R_a + R_b)}{R_a(R_c + R_d)} v_b - \frac{R_b}{R_a} v_a$$

If set $\frac{R_a}{R_b} = \frac{R_c}{R_d}$ then

$$v_0 = \frac{R_b}{R_a}(v_b - v_a)$$

The output voltage of a difference amplifier is a scaled replica of the difference between the two input voltages. The scaling is controlled by the external resistors.

The difference amplifier circuit



$$v_{dm} = v_b - v_a$$

$$v_{cm} = (v_a + v_b)/2$$

$$v_0 = A_{cm} v_{cm} + A_{dm} v_{dm}$$

An ideal differential amplifier has zero common mode gain and non-zero (usually large) differential mode gain.

In practical applications, the differential mode signal contains information of interest, and the common mode signal is the noise found in all electric signals.

Active Filter Circuits

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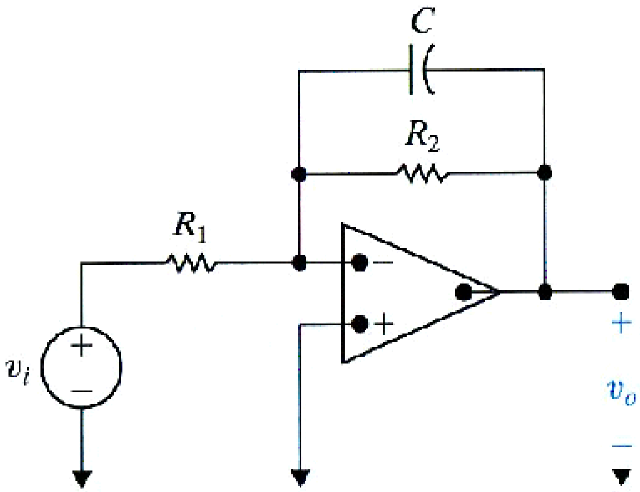
Introduction

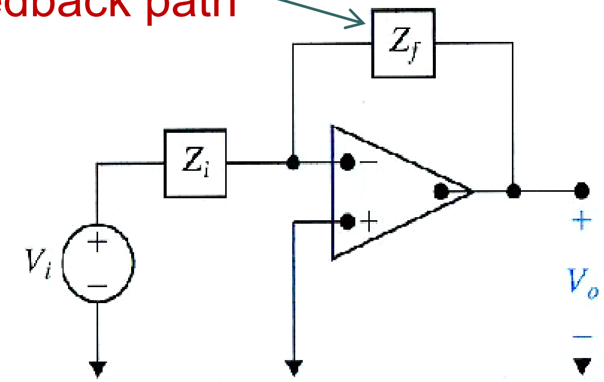
Passive filter	Active filter
Filter circuits consist of R,L,C	Filter circuits consist of op-amps
Inductors are usually large, heavy and costly, and may introduce electromagnetic field effects → disadvantage	Filter circuits without using inductor
Maximum magnitude does not exceed 1	Filter circuits provide a control over amplification.
Cut-off frequency and passband magnitude were altered with the addition of a resistive load at the output of the filter.	Does not affected → use active filter circuits to implement filter designs when gain, load variation and physical size are important parameters in the design specifications.

First-order low-pass filter



First order Low-passfilter

Feedback path



A general op amp circuit.

Transfer function of the low-pass filter:

$$H(s) = -\frac{Z_f}{Z_i} = -\frac{R_2 \parallel (1/sC)}{R_1}$$

$$\boxed{H(s) = -K\frac{\omega_c}{s+\omega_c}}$$

$$K = \frac{R_2}{R_1} \qquad \omega_c = \frac{1}{R_2 C}$$

The transfer function has the same form as for passive low-pass filter <u>except the gain K in the pass-pand</u>.

Review

Appendix D – The Decibel (textbook)

$$\text{Number of decibels} = 10\log_{10}\frac{p_{out}}{p_{in}}$$

$$\text{Number of decibels} = 20\log_{10}\frac{v_{out}}{v_{in}} = 20\log_{10}\frac{i_{out}}{i_{in}}$$

also see: Appendix E – Bode Diagrams (textbook)

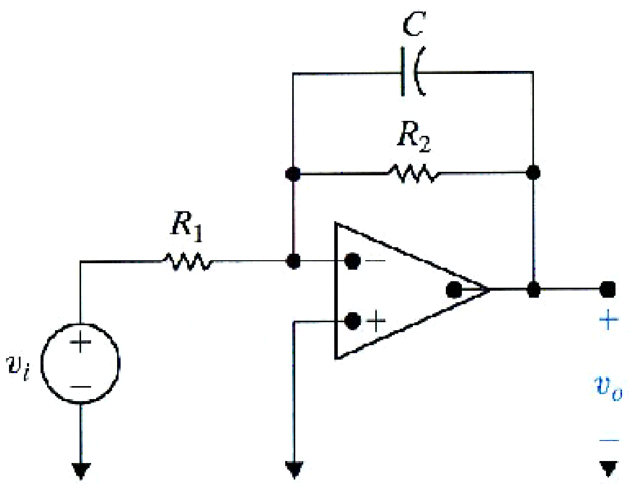
Bode plots differ from the frequency response plots in Chapter 14 in two important ways:
- First, instead of using a linear axis for the frequency values, a Bode plot uses a logarithmic axis ➔ plot a wider range of frequencies of interest.
- Second, instead of plotting the absolute magnitude of the transfer function versus frequency, the Bode magnitude is plotted in decibels (dB) versus the log of the frequency. $A_{dB} = 20\log_{10}|H(j\omega)|$

When $A_{dB} = 0$, the transfer function magnitude is 1, since $20\log_{10}(1) = 0$.
When $A_{dB} < 0$, the transfer function magnitude is between 0 and 1,
and when, $A_{dB} > 0$, the transfer function magnitude is greater than 1.

Example

Designing a low-pass Op Amp Filter

Using the circuit shown in Fig., calculate values for C and R_2 that, together with $R_1 = 1\Omega$, produce a low-pass filter having a gain of 1 in the passband and a cutoff frequency of 1 rad/s. Construct the transfer function for this filter and use it to sketch a Bode magnitude plot of the filter's frequency respone.



Solution

Calculate the required value of R_2:

$$R_2 = KR_1 = 1 \times 1 = 1\Omega$$

Calculate C to meet the specified cutoff frequency

$$C = \frac{1}{R_2\omega_c} = \frac{1}{(1)(1)} = 1\,\text{F}.$$

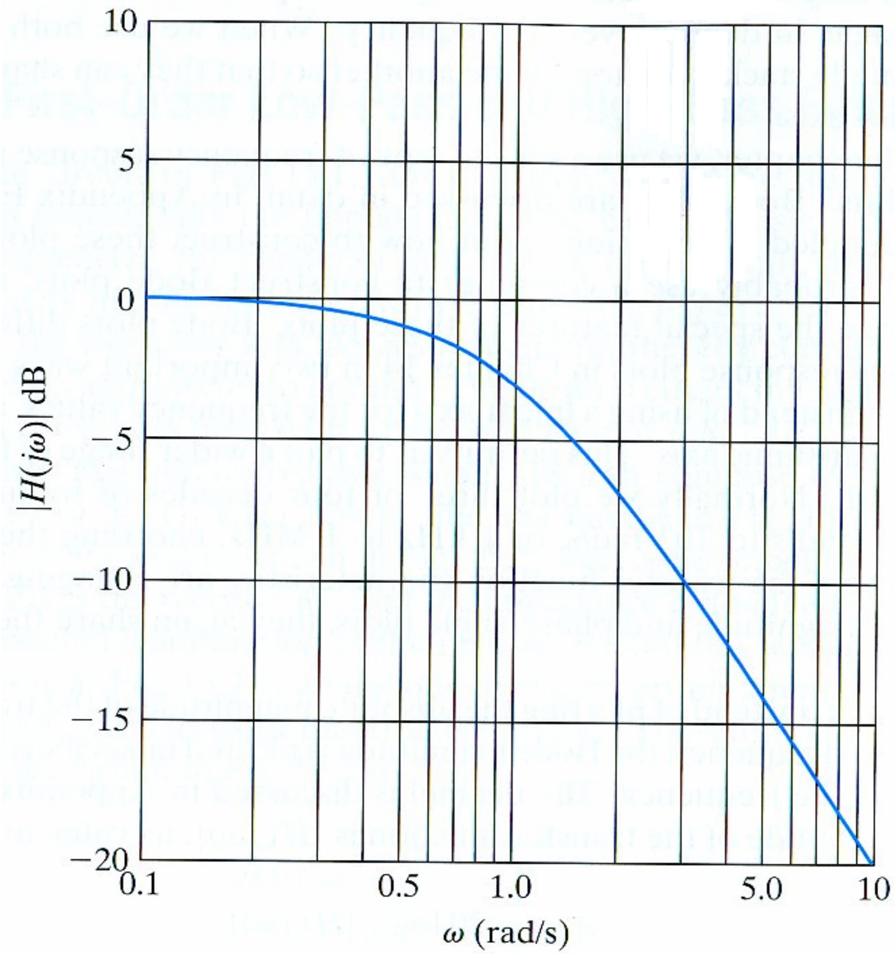
The transfer function for the low-pass filter is

$$H(s) = -K\frac{\omega_c}{s + \omega_c} = \frac{-1}{s + 1}.$$

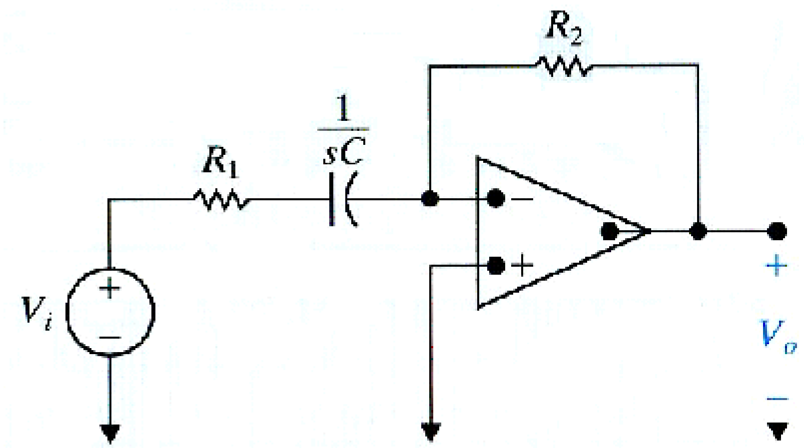
Example

Designing a low-pass Op Amp Filter

The Bode plot of $|H(j\omega)|$ is shown in Fig. This is the so-called prototype low-pass op amp filter, because it uses a resistor value of 1 Ω and a capacitor value of 1 F, and it provides a cutoff frequency of 1 rad/s.



First-order high-pass filter



It also has the same form as passive high-pass filter, except for the gain.

Transfer function of the low-pass filter:

$$H(s) = -\frac{Z_f}{Z_i} = -\frac{R_2}{R_1 + (1/sC)}$$

$$\boxed{H(s) = -K\frac{s}{s+\omega_c}}$$

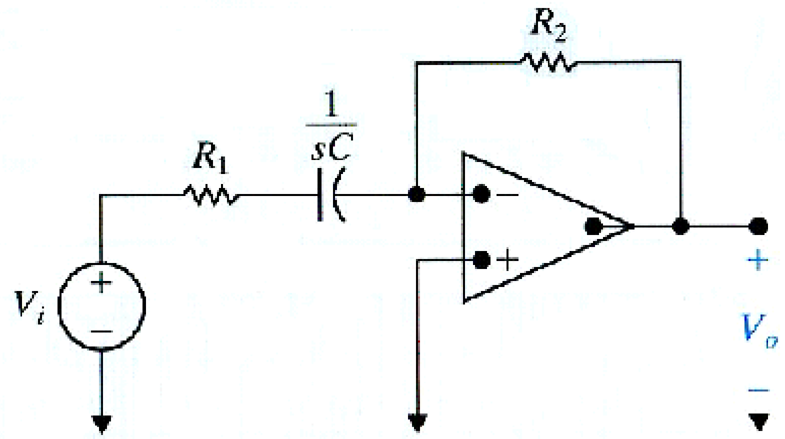
$$K = \frac{R_2}{R_1} \qquad \omega_c = \frac{1}{R_1 C}$$

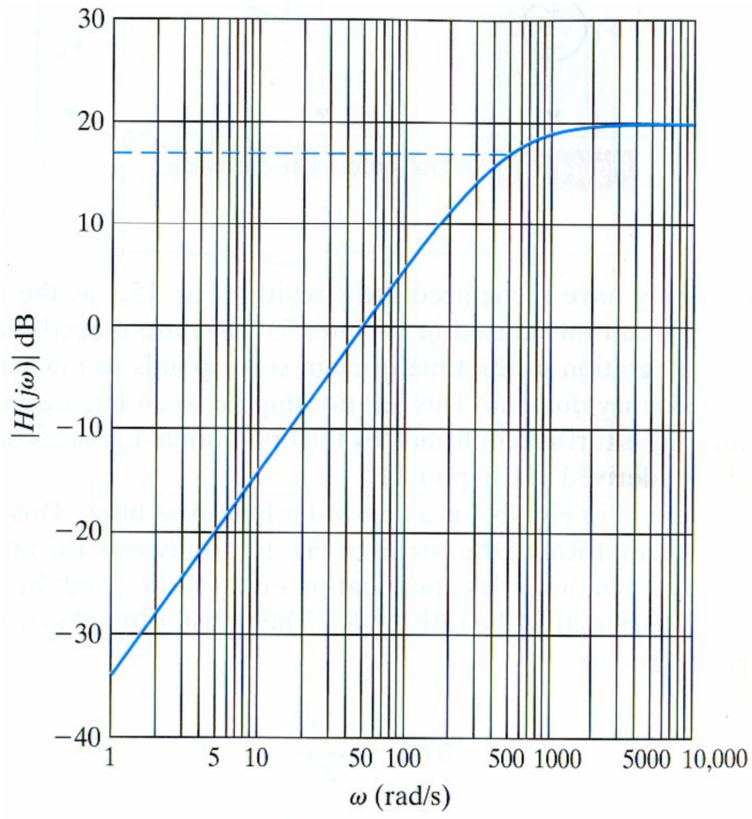
Example

Designing a high-pass Op Amp Filter

We have the Bode magnitude plot of a high-pass filter. Calculate values of R_1 and R_2 that produce the desired magnitude response.

Given C = 0.1 μF.

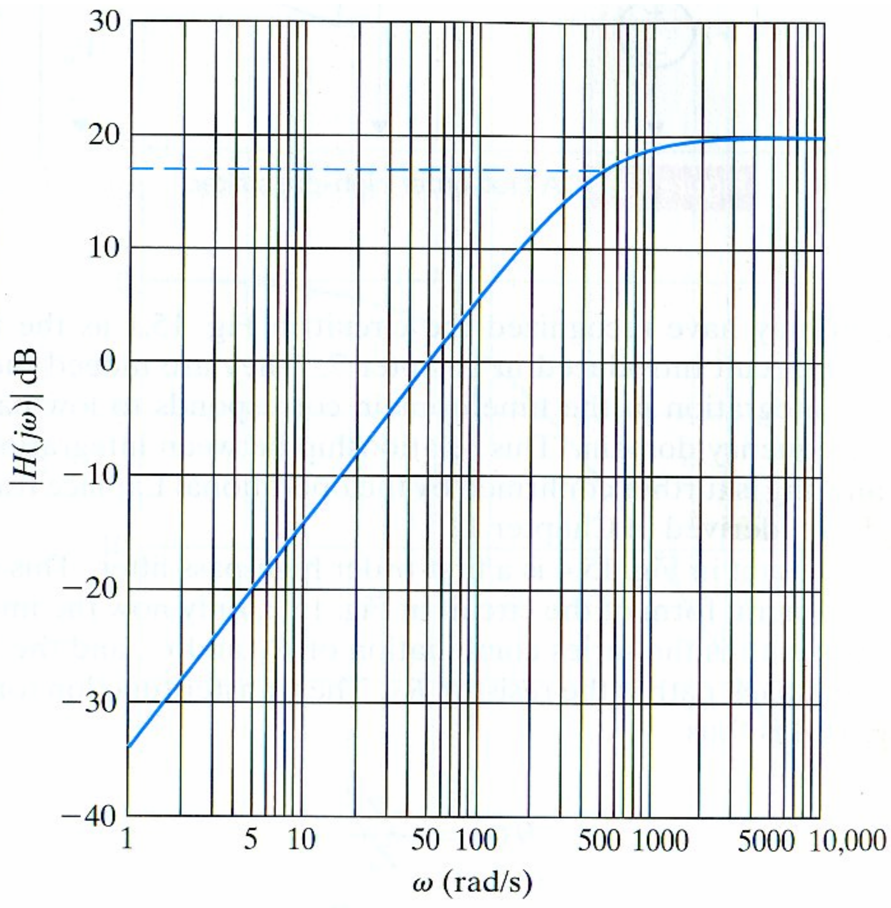




Example

Designing a high-pass Op Amp Filter

Solution:



Note that the gain in the passband is 20 dB; therefore, K = 10. Also note that the 3 dB point is 500 rad/s.

The transfer function that has the magnitude response shown in the Bode plot is given by

$$H(s) = \frac{-10s}{s + 500}$$

so

$$H(s) = \frac{-10s}{s + 500} = \frac{-(R_2/R_1)s}{s + (1/R_1C)}$$

Equating the numerators and denominators and then simplifying we get two equations:

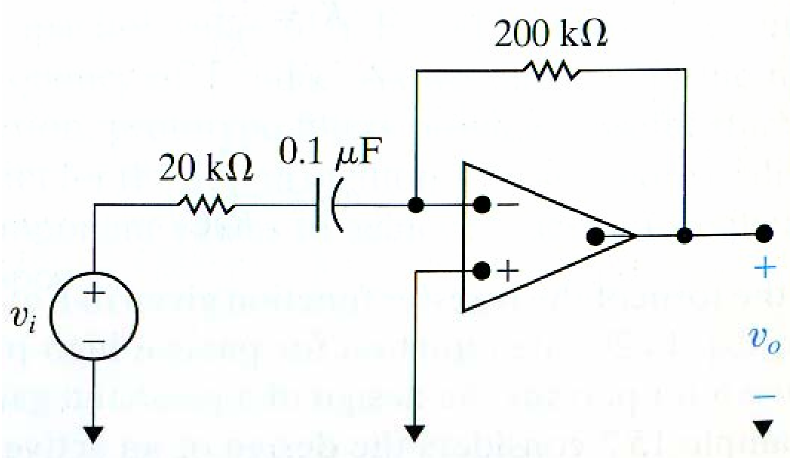
$$10 = \frac{R_2}{R_1}, \quad 500 = \frac{1}{R_1C}.$$

$$R_1 = 20\,\mathrm{k}\Omega, \quad R_2 = 200\,\mathrm{k}\Omega.$$

Example

Designing a high-pass Op Amp Filter **Solution:**

The circuit is shown in Fig.



Because we have made the assumption that the op amp in this high-pass filter circuit is ideal, the addition of any load resistor, regardless of its resistance, has no effect on the behavior of the op amp. Thus, the magnitude response of a high-pass filter with a load resistor is the same as that of a high-pass filter with no load resistor.

Problem 1

Designing a high-pass Op Amp Filter

Compute the values for R2 and C that yield a high-pass filter with a passband gain of 1 and a cutoff frequency of 1 rad/s if R1 is 1Ω. (Note: This is the prototype high-pass filter.)

Solution:

$$H(s) = \frac{-(R_2/R_1)s}{s + (1/R_1C)}$$

$$\frac{1}{R_1C} = 1\,\text{rad/s}; \qquad R_1 = 1\,\Omega, \quad \therefore \quad C = 1\,\text{F}$$

$$\frac{R_2}{R_1} = 1, \quad \therefore \quad R_2 = R_1 = 1\,\Omega$$

$$H_{\text{prototype}}(s) = \frac{-s}{s+1}$$

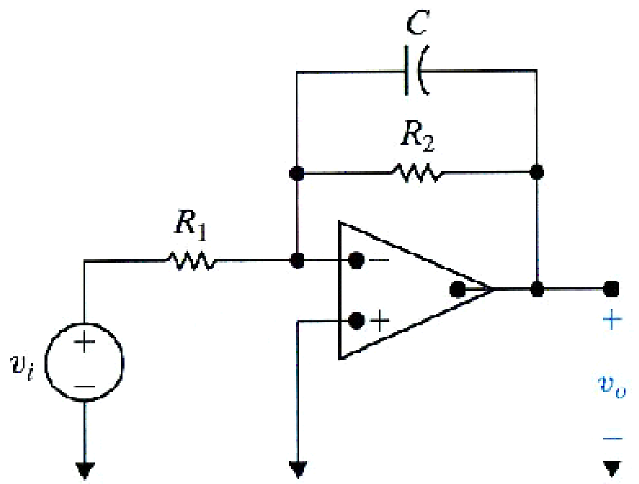
Problem 2

Designing a high-pass Op Amp Filter

Compute the resistor values needed for the low-pass filter circuit in Fig. to produce transfer function

$$H(s) = \frac{-20000}{s + 5000}$$

Assume that C = 5 μF



Solution:

$$H(s) = \frac{-(1/R_1C)}{s + (1/R_2C)} = \frac{-20{,}000}{s + 5000}$$

$$\frac{1}{R_1C} = 20{,}000; \quad C = 5\,\mu\text{F}$$

$$\Rightarrow \quad R_1 = \frac{1}{(20{,}000)(5 \times 10^{-6})} = 10\,\Omega$$

$$\frac{1}{R_2C} = 5000 \quad \Rightarrow \quad R_2 = \frac{1}{(5000)(5 \times 10^{-6})} = 40\,\Omega$$

Scaling

The designer can transform the convenient values into realistic values using the process known as **scaling**.

There are two types of scaling magnitude and frequency.

Scale a circuit in **magnitude** by multiplying the impedance at a given frequency by the scale factor k_m. Thus we multiply all resistors and inductors by k_m and all capacitors by $1/k_m$.

$$R' = k_m R; \quad L' = k_m L; \quad C' = C/k_m$$

Frequency scaling: we change the circuit parameters so that at the new frequency, the impedance of each element is the same as it was at the original frequency. If we let kf denote the frequency scale factor. Thus for frequency scaling: $R' = R; \quad L' = L/k_f; \quad C' = C/k_f$

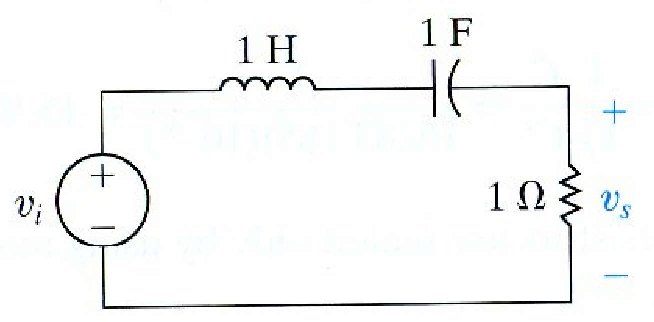
Scaling

A circuit can be scaled simultaneously in both magnitude and frequency. The scaled values (primed) in terms of (unprimed) are

$$R' = k_m R; \qquad L' = \frac{k_m}{k_f} L; \qquad C' = \frac{1}{k_m k_f} C$$

Example Scaling a Series RLC Circuit

The series RLC circuit shown in Fig. below has a center frequency of $\sqrt{1/LC} = 1\,rad/s$ a bandwidth of R/L = 1 rad/s, and thus a quality factor of 1. Use scaling to compute new values of R and L that yield a circuit with the same quality factor but with a center frequency of 500 Hz. Use a 2 μF capacitor.



Scaling

Solution:

Let computing the frequency scale factor that will shift the center frequency from 1 rad/s to 500 Hz. The unprimed variables represent values before scaling, whereas the primed variables represent values after scaling

$$k_f = \frac{\omega_o'}{\omega_o} = \frac{2\pi(500)}{1} = 3141.59.$$

the magnitude scale factor: $$k_m = \frac{1}{k_f}\frac{C}{C'} = \frac{1}{(3141.59)(2 \times 10^{-6})} = 159.155.$$

$$R' = k_m R = 159.155\ \Omega,$$

$$L' = \frac{k_m}{k_f} L = 50.66\ \text{mH}.$$

Problem 3

What magnitude and frequency scale factor will transform the prototype high-pass filter into a high pass filter with a 0.5 pF capacitor and a cutoff frequency of 10 kHz?

Solution:

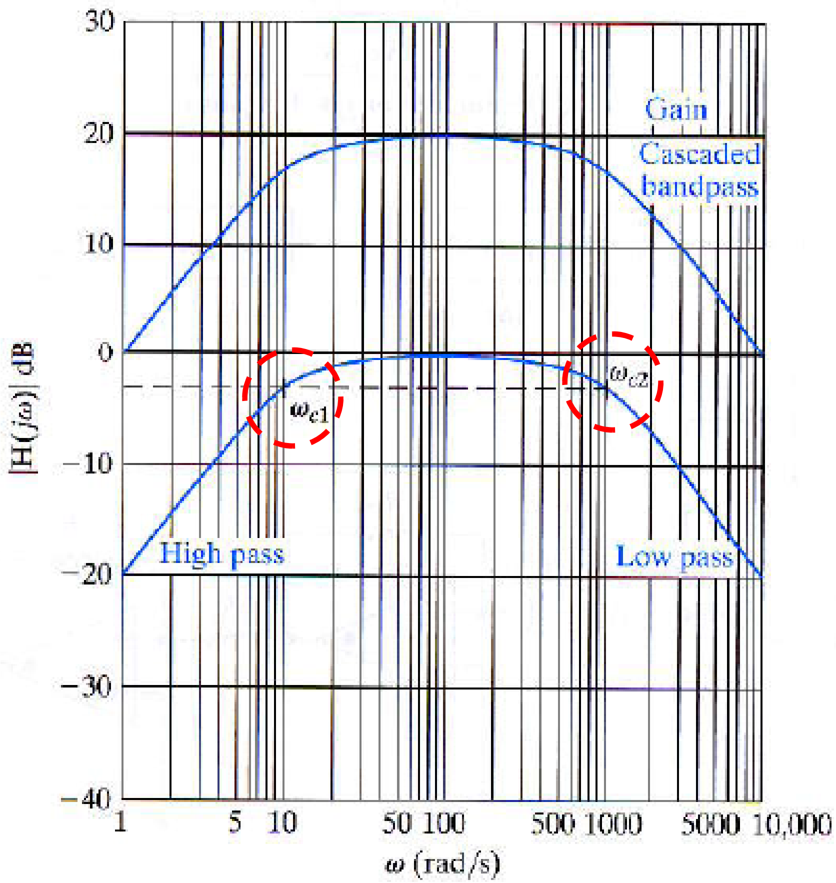
$$\omega_c = 2\pi f_c = 2\pi \times 10^4 = 20{,}000\pi \text{ rad/s}$$

$$\Rightarrow \quad k_f = 20{,}000\pi = 62{,}831.85$$

$$C' = \frac{C}{k_f k_m} \quad \Leftrightarrow \quad 0.5 \times 10^{-6} = \frac{1}{k_f k_m}$$

$$\Rightarrow \quad k_m = \frac{1}{(0.5 \times 10^{-6})(62{,}831.85)} = 31.83$$

First-order band-pass filter



Bode magnitude plot of a bandpass filter

The band-pass filter consists of 3 separate components:

1. ω_{c2} – cut-off frequency of unity-gain low-pass filter
2. ω_{c1} – cut-off frequency of unity-gain high-pass filter
3. Gain component to provide the desired level of gain in the passband.

These three components are cascaded in series. So:

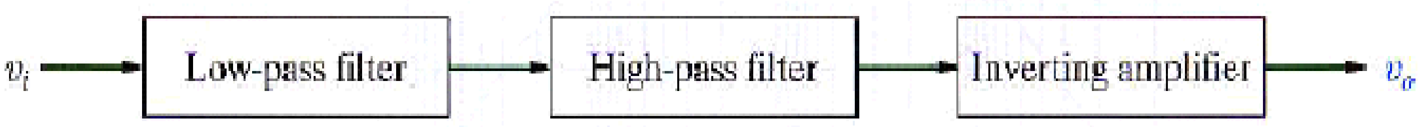
Requirement for two cut-off frequencies in broadband bandpass filter:

$$\frac{\omega_{c2}}{\omega_{c1}} \geq 2$$

First-order band-pass filter

Construct the bandpass filter

We can construct a circuit that provides each of the three components by cascading a low-pass op amp filter, a high-pass op amp filter, and an inverting amplifier, as shown in Fig. below.



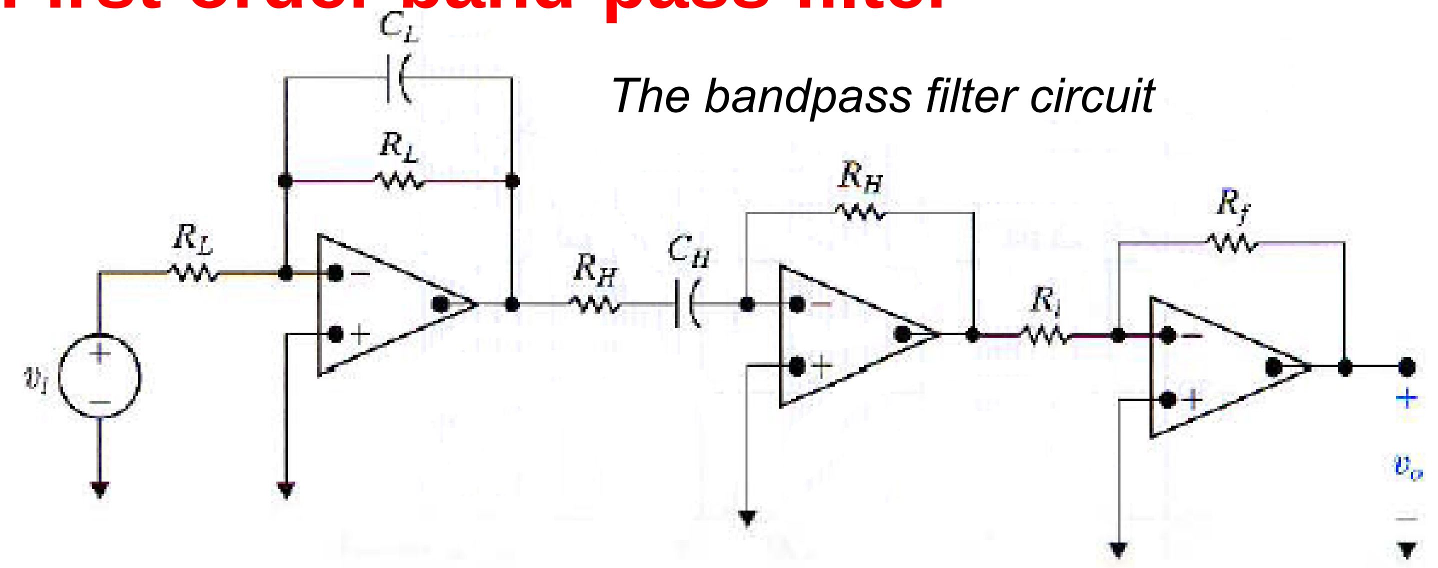
Transfer function of the band-pass filter is the product of the transfer functions of the three cascaded components:

$$H(s) = \frac{V_o}{V_i} = \left(\frac{-\omega_{c2}}{s+\omega_{c2}}\right)\left(\frac{-s}{s+\omega_{c1}}\right)\left(\frac{-R_f}{R_i}\right)$$

$$H(s) = \frac{-K\omega_{c2}s}{(s+\omega_{c1})(s+\omega_{c2})}$$

$$K = \frac{R_f}{R_i} \qquad \omega_{c2} = \frac{1}{R_L C_L} \qquad \omega_{c1} = \frac{1}{R_H C_H}$$

First-order band-pass filter



The bandpass filter circuit

Require that $\omega_{c2} >> \omega_{c1}$

➔ the transfer function for the cascaded bandpass filter: $H(s) = \dfrac{-K\omega_{c2}s}{s^2 + \omega_{c2}s + \omega_{c1}\omega_{c2}}$

Compute the values of R_L and C_L in the low-pass filter to give us the desired upper cutoff frequency, ω_{c2}:

$$\omega_{c2} = \frac{1}{R_L C_L}$$

First-order band-pass filter

Compute the values of R_H and C_H in the high-pass filter to give us the desired lower cutoff frequency, ω_{c1}:

$$\omega_{c1} = \frac{1}{R_H C_H}$$

Compute the values of R_i and R_f in the inverting amplifier to provide the desired passband gain. To do this, we consider the magnitude of the bandpass filter's transfer function, evaluated at the center frequency ω_o

$$|H(j\omega_o)| = \left| \frac{-K\omega_{c2}(j\omega_o)}{(j\omega_o)^2 + \omega_{c2}(j\omega_o) + \omega_{c1}\omega_{c2}} \right| = \frac{K\omega_{c2}}{\omega_{c2}} = K.$$

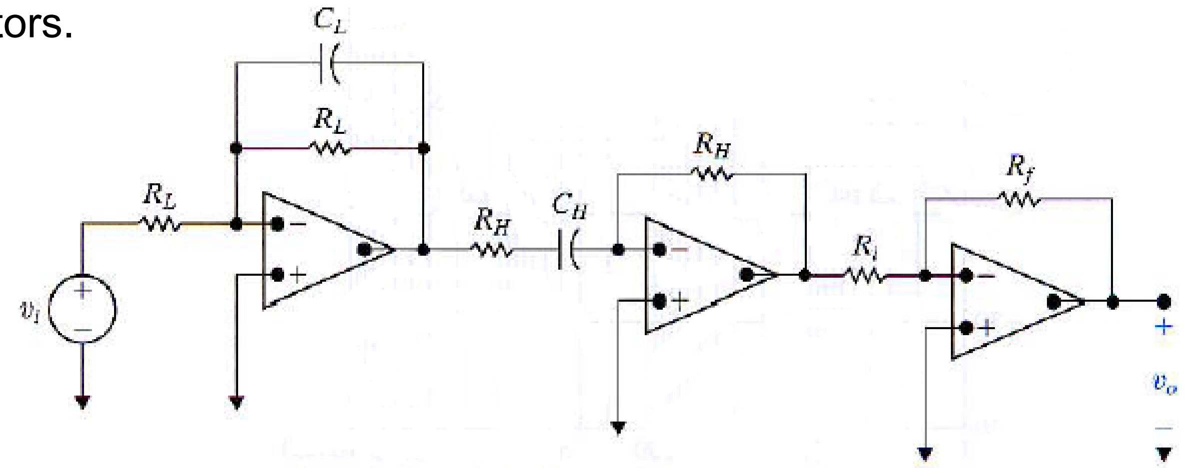
Besides, the gain of the inverting amplifier $|H(j\omega_o)| = \dfrac{R_f}{R_i}$ **(*)**

Any choice of resistors that satisfies Eq. **(*)** will produce the desired passband gain.

First-order band-pass filter

Example

Design a bandpass filter for a graphic equalizer to provide an amplification of 2 within the band of frequencies between 100 and 10,000 Hz. Use 0.2 μF capacitors.



Solution

We design each subcircuit in the cascade and meet the specified cutoff frequency values. In this case $\omega_{c2} = 100\omega_{c1}$ so we can say that $\omega_{c2} >> \omega_{c1}$

Begin with the low-pass stage. $$\omega_{c2} = \frac{1}{R_L C_L} = 2\pi(10000),$$

$$R_L = \frac{1}{[2\pi(10000)](0.2 \times 10^{-6})} \approx 80\ \Omega.$$

First-order band-pass filter

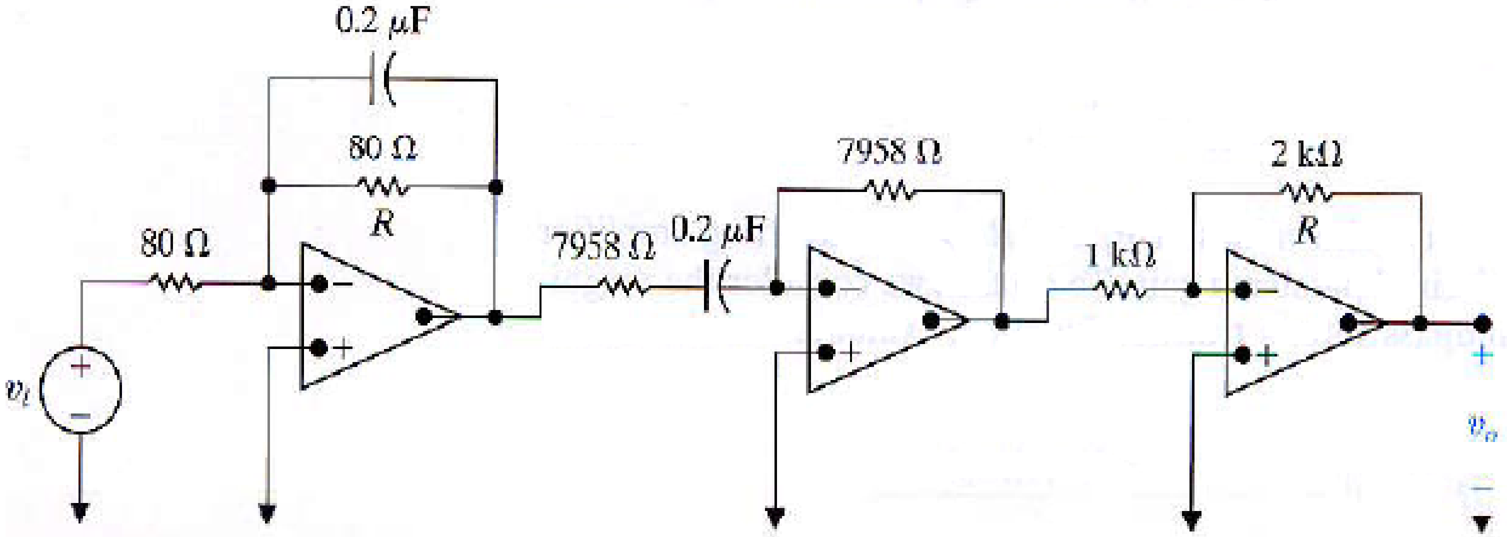
Example

Next, calculate for the high-pass stage $\omega_{c1} = \dfrac{1}{R_H C_H} = 2\pi(100),$

$$R_H = \frac{1}{[2\pi(100)](0.2 \times 10^{-6})} \approx 7958\ \Omega.$$

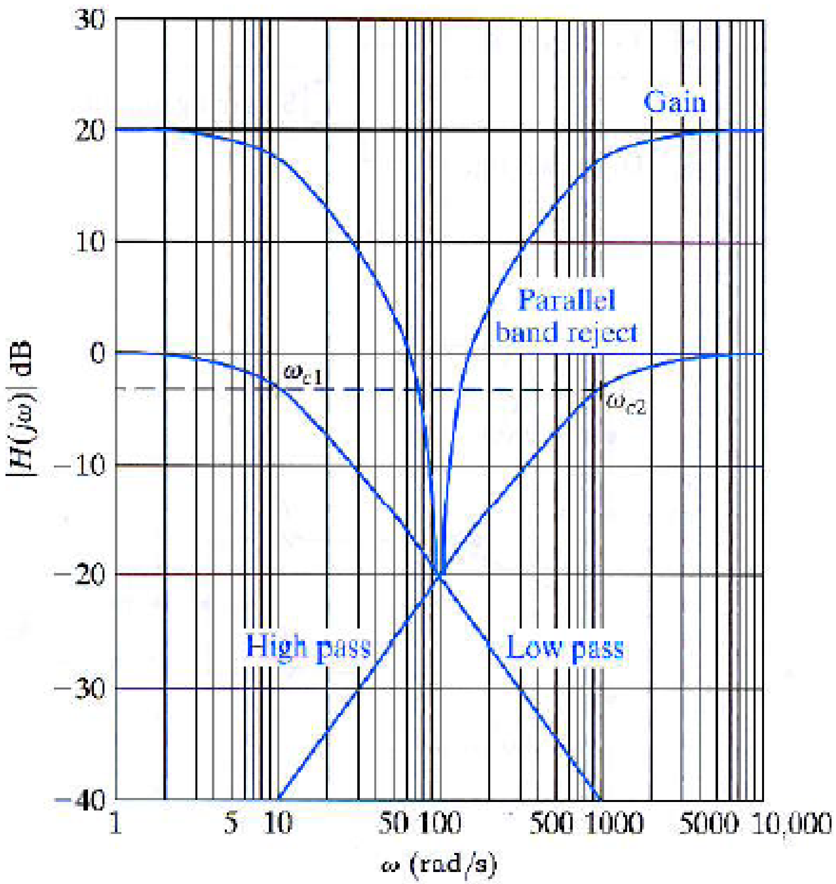
For the gain stage: one of the resistors can be selected arbitrarily. Let's select a 1 kΩ resistor for R_i . So with $H = R_f/R_i$ ➔ $R_f = HR_i = 2(1000) = 2$ kΩ

The resulting circuit is shown in Fig.



First-order band-reject filter

We can use a component approach to the design of op amp bandreject filters too



Bode magnitude plot of a band-reject filter

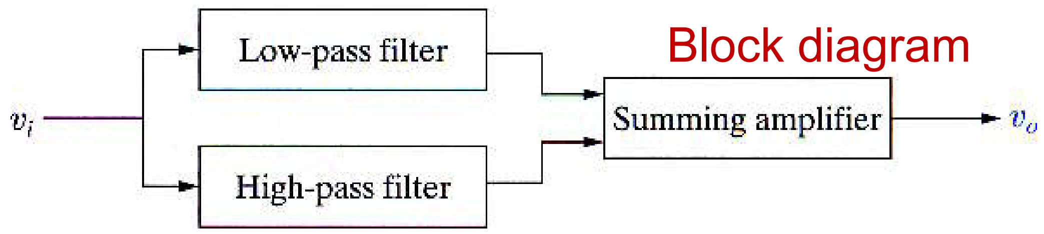
The band-reject filter consists of 3 separate components:

1. ω_{c1} – cut-off frequency of unity-gain low-pass filter
2. ω_{c2} – cut-off frequency of unity-gain high-pass filter
3. Gain component to provide the desired level of gain in the passband.

These 3 components cannot be cascaded in series, but we have to use a parallel connection and a summing amplifier to construct band-reject filter.

First-order band-reject filter

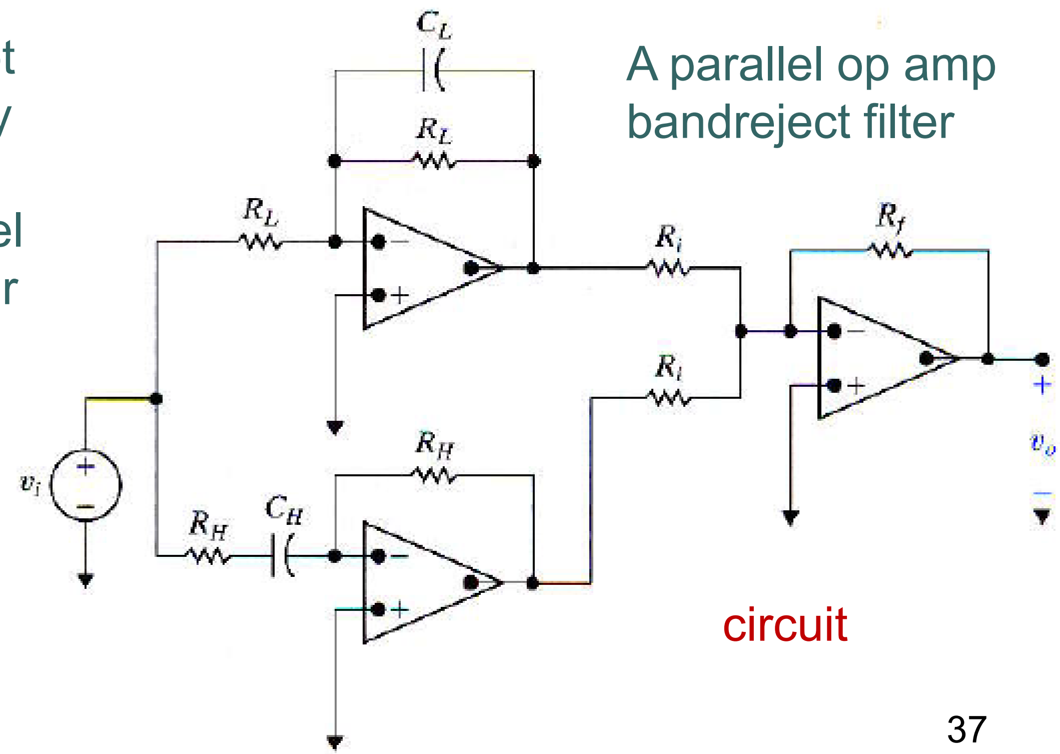
Construct the band-reject filter



Block diagram

The most important difference is that these three components cannot be cascaded in series because they do not combine additively on the Bode plot. Instead, we use a parallel connection and a summing amplifier

A parallel op amp bandreject filter



circuit

First-order band-reject filter

Assume that the two cutoff frequencies are widely separated → the resulting design is a broadband bandreject filter, and $\omega_{c2} >> \omega_{c1}$. Then each component of the parallel design car be created independently, and the cutoff frequency specifications will be satisfied. The transfer function of the resulting circuit is the sum of the low-pass and high-pass filter transfer functions

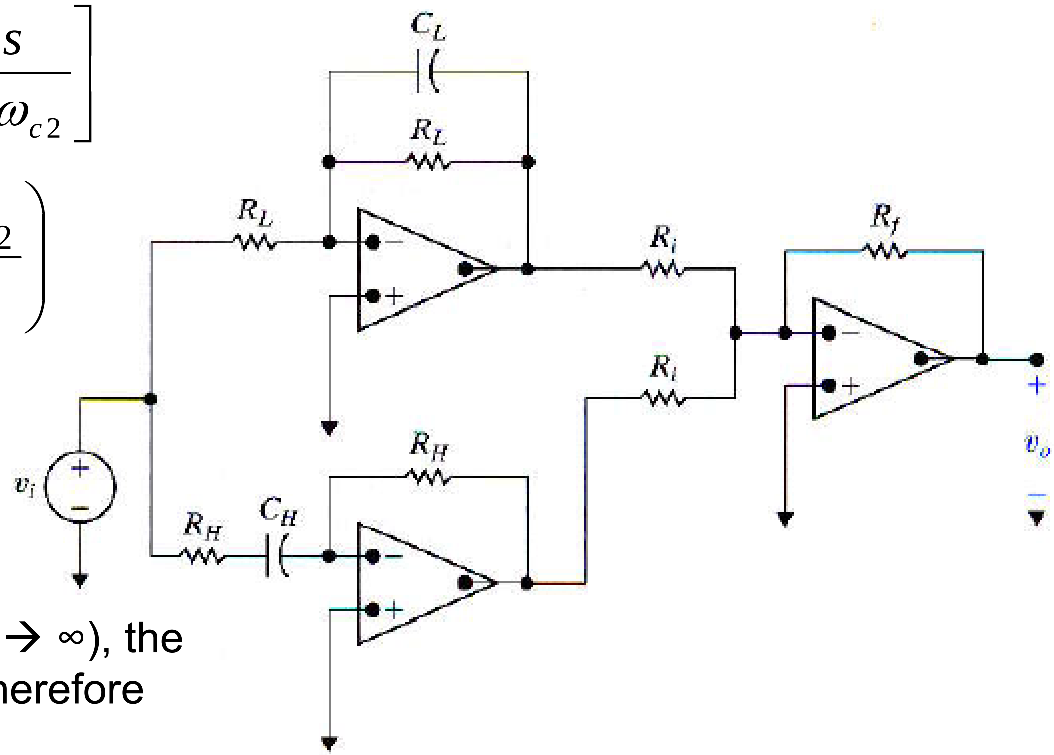
$$H(s) = \left(-\frac{R_f}{R_i}\right)\left[\frac{-\omega_{c1}}{s+\omega_{c1}} + \frac{-s}{s+\omega_{c2}}\right]$$

$$H(s) = \frac{R_f}{R_i}\left(\frac{s^2 + 2\omega_{c1}s + \omega_{c1}\omega_{c2}}{(s+\omega_{c1})(s+\omega_{c2})}\right)$$

the cutoff frequencies are

$$\omega_{c1} = \frac{1}{R_L C_L} \qquad \omega_{c2} = \frac{1}{R_H C_H}$$

In the two passbands (as s → 0 and s → ∞), the gain of the transfer function is R_f/R_i. Therefore $K = R_f/R_i$



First-order band-reject filter

Example

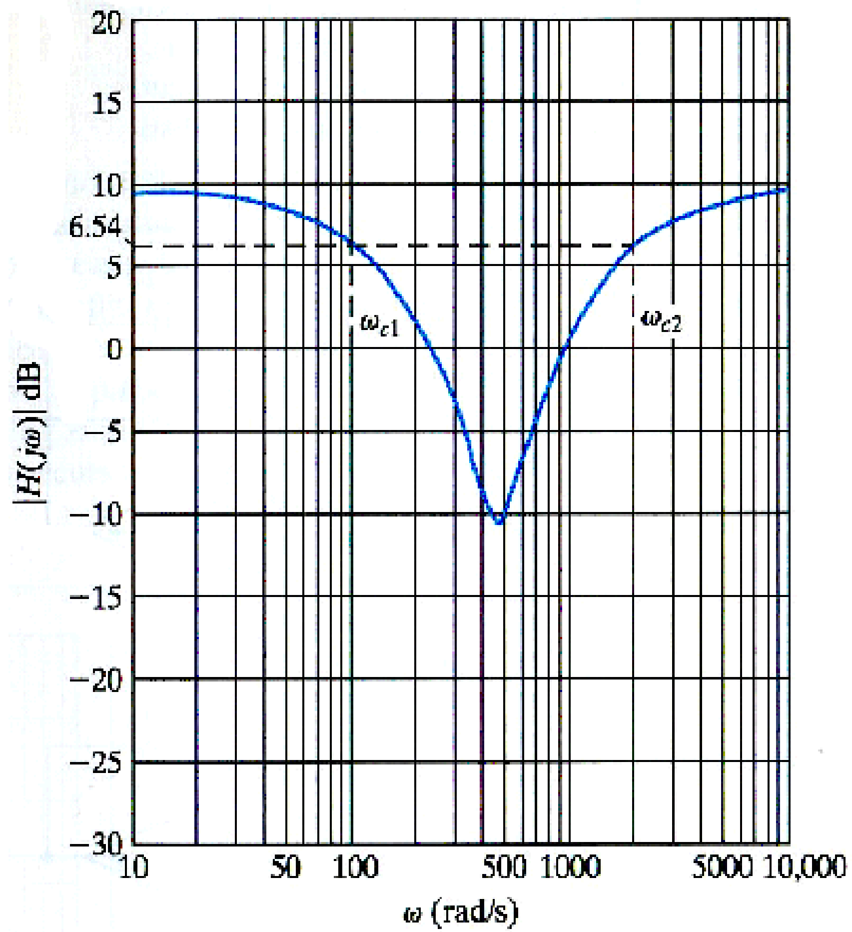
Design a circuit based on the parallel band-reject op-amp filter as in the figure. Use 0.5 μF capacitors for the design.

Solution

From the Bode figure, we see that the bandreject filter has cutoff frequencies of 100 rad/s and 2000 rad/s and a gain of 3 in the pass bands → $\omega_{c2} = 20\omega_{c1}$

So, assume that: $\omega_{c2} >> \omega_{c1}$

From low-pass filter model and use scaling to meet the specifications for cutoff frequency and capacitor value. The frequency scale factor $k_f = 100$ → shifts the cutoff frequency from 1 rad/s to 100 rad/s. The magnitude scale factor f_m is 20,000, which permits the use of a 0.5 μF capacitor



First-order band-reject filter

Example

$$R_L = 20\ \mathrm{k\Omega},$$

$$C_L = 0.5\ \mu\mathrm{F}.$$

Cutoff frequency of the low pass filter

$$\omega_{c1} = \frac{1}{R_L C_L} = \frac{1}{(20 \times 10^3)(0.5 \times 10^{-6})} = 100\ \mathrm{rad/s}.$$

Similarly, to design high pass filter, from the high-pass op amp filter model. We have k_f = 2000, and k_m = 1000, resulting in the following scaled component values

$$R_H = 1\ \mathrm{k\Omega},$$

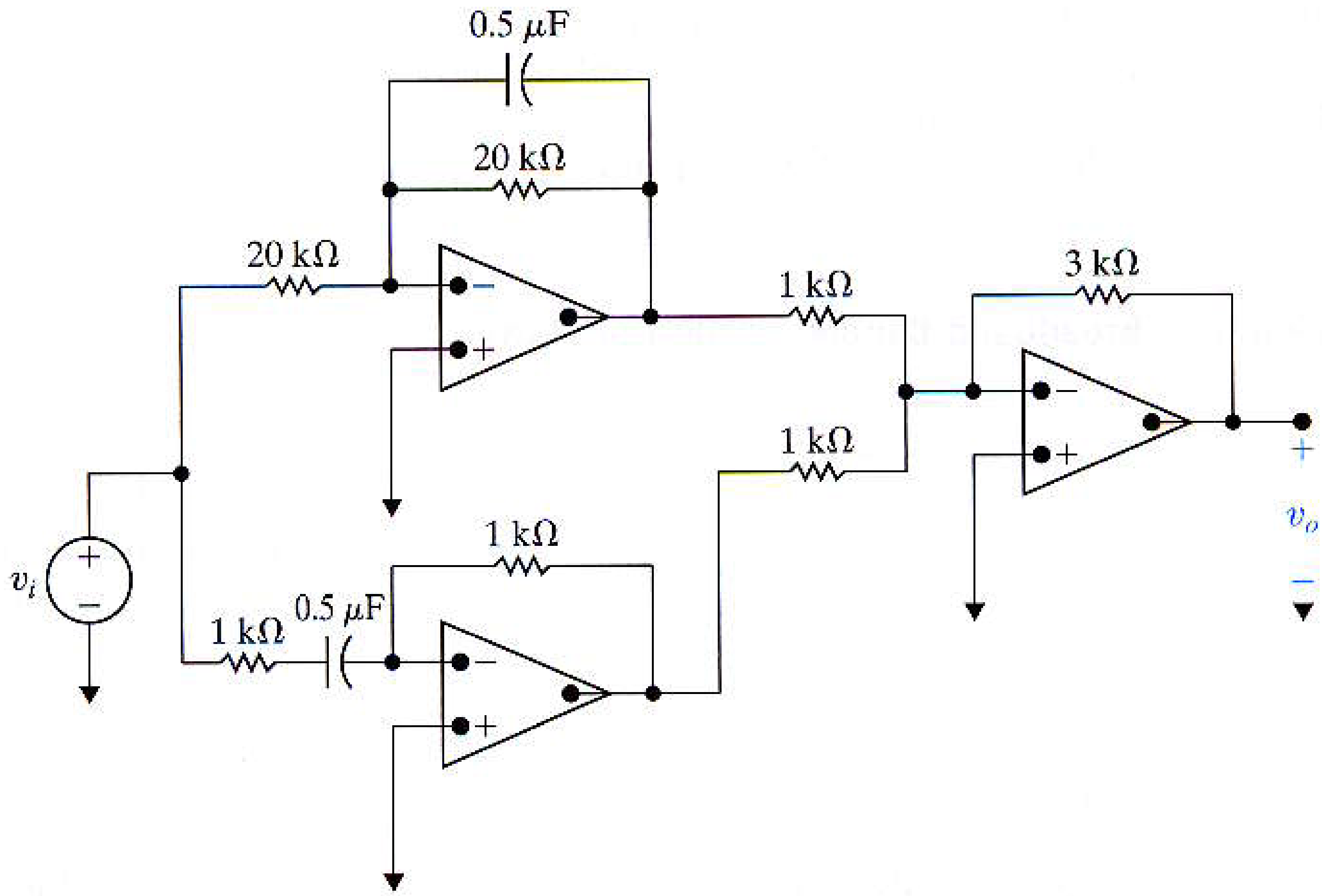
$$C_H = 0.5\ \mu\mathrm{F}.$$

Finally, because the cutoff frequencies are widely separated, we can use the ratio R_f/R_i to establish the desired passband gain of 3.

Choose $R_i = 1\mathrm{k\Omega}$, (as for R_H). ➔ $R_f = 3\mathrm{k\Omega}$, and $K = R_f/R_i = 3000/1000 = 3$

The resulting parallel op amp bandreject filter circuit is shown in Fig.

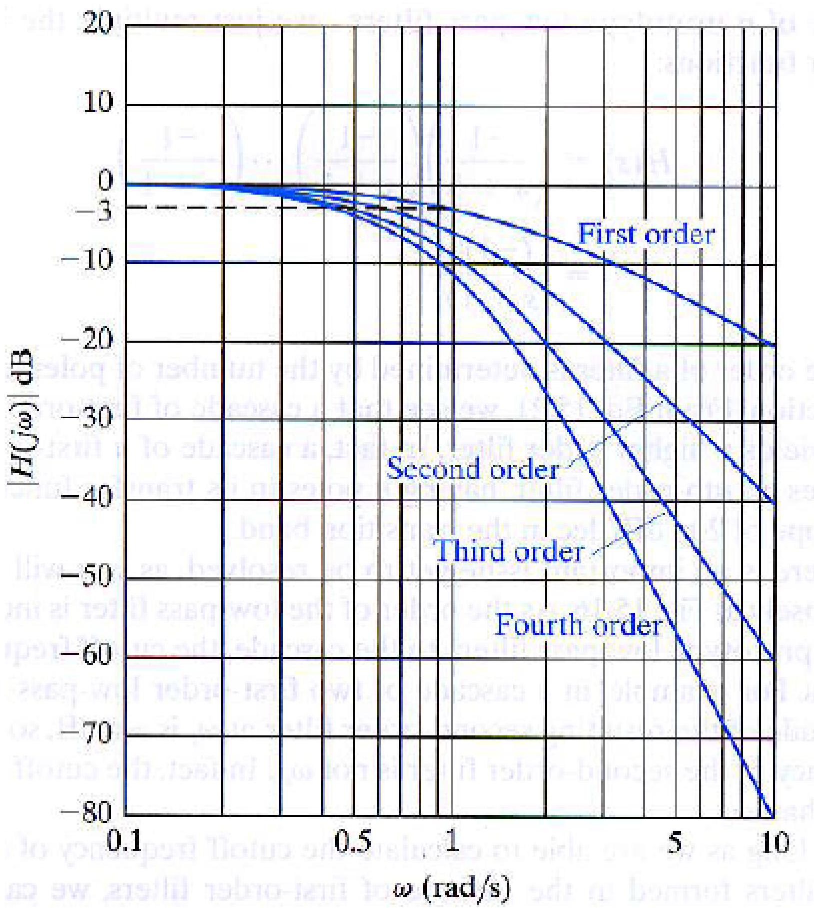
Example



The resulting bandreject filter circuit design example

High Order Op Amp Filters

This figure shows the Bode magnitude plots of a cascade of identical prototype low pass filters and includes plots of just one filter, two in cascade, three in cascade, and four in cascade.

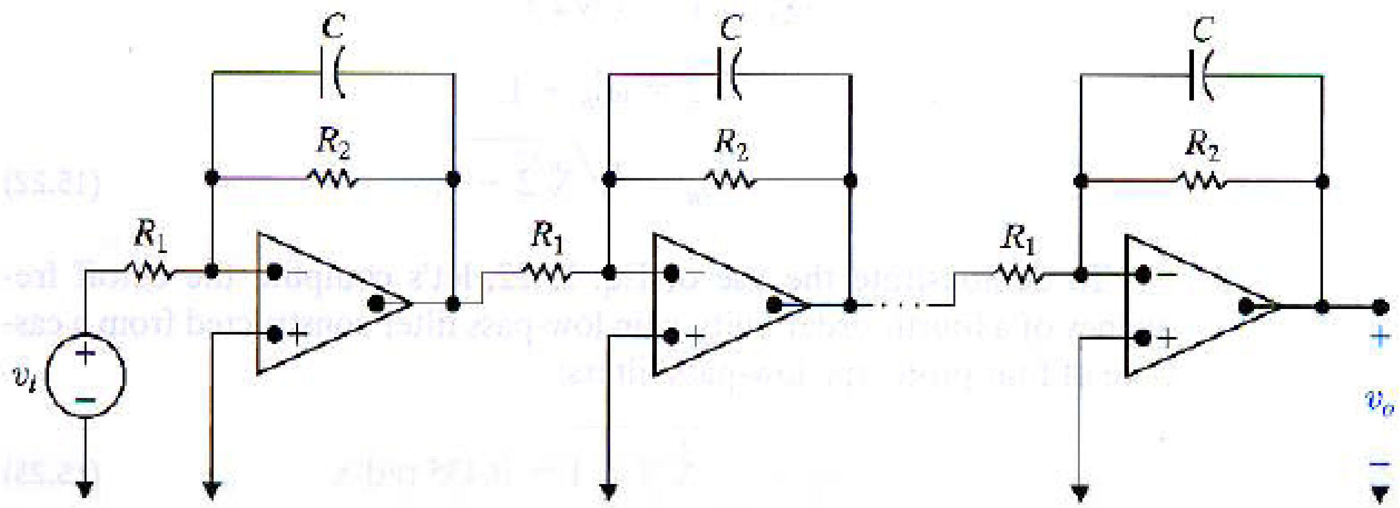


As the order of the low-pass filter is increased by adding prototype low-pass filters to the cascade, the cutoff frequency also change

High Order Op Amp Filters

In general, an n-element cascade of identical low-pass filter will transition from the passband to the stopbard with a slope of 20n dB/dec. Both the block diagram and the circuit diagram for such a cascade are shown in Fig.

(a)



(b)

High Order Op Amp Filters

The transfer function for a cascade of n prototype low-pass filters

$$H(s) = \left(\frac{-1}{s+1}\right)\left(\frac{-1}{s+1}\right)\cdots\left(\frac{-1}{s+1}\right) = \frac{(-1)^n}{(s+1)^n}$$

The order of a filter is determined by the number of poles in its transfer function

To calculate the cutoff frequency (ω_c) of the higher order filter formed in the cascade of first-order filter, we can use frequency scaling to calculate component values that move the ω_c to its specified location. By solving for the value of ω_c that results in $|H(j\omega)| = 1/\sqrt{2}$

$$H(s) = \frac{(-1)^n}{(s+1)^n}, \iff |H(j\omega_{cn})| = \left|\frac{1}{(j\omega_{cn}+1)^n}\right| = \frac{1}{\sqrt{2}},$$

$$\Rightarrow \frac{1}{\omega_{cn}^2+1} = \left(\frac{1}{\sqrt{2}}\right)^{2/n} \Rightarrow \sqrt[n]{2} = \omega_{cn}^2 + 1 \Rightarrow \omega_{cn} = \sqrt{\sqrt[n]{2}-1}.$$

High Order Op Amp Filters

For example, n = 4:

$$\omega_{c4} = \sqrt{\sqrt[4]{2} - 1} = 0.435 \text{ rad/s.}$$

Thus we can design a fourth order low-pass filter with any arbitrary cutoff frequency by starting with a fourth-order cascade consisting of prototype low-pass filters and then scaling the components by $k_f = \omega_c/0.435$ to place the cutoff frequency at any value of ω_c desired.

Example: Designing a fouth-order low-pass op amp fitter with a cutoff frequency of 500Hz and a passband gain of 10. Use 1 μF capacitors. Sketch the Bode magnitude plot for this filter.

High Order Op Amp Filters

Example Solution

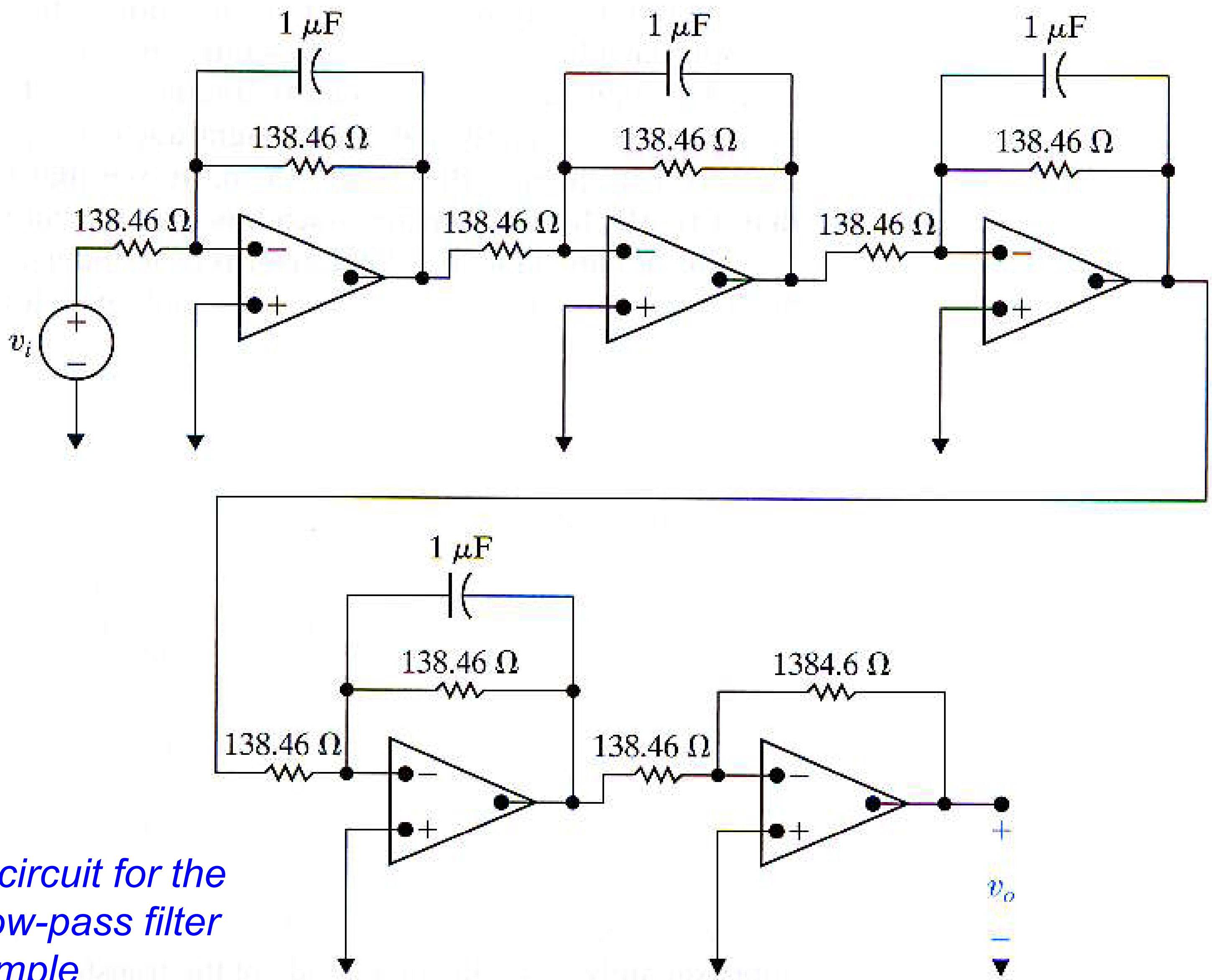
We have already used Eq. $\omega_{c4} = \sqrt{\sqrt[4]{2} - 1} = 0.435 \text{ rad/s}$. calculate the cutoff frequency for the resulting fourth-order low-passfilter as 0.435 rad/s.
A frequency scale factor of k_f = 7222.39 will scale the component values to give a 500 Hz cutoff frequency.
A magnitude scale factor of k_m = 138.46 permits the use of 1 μF capacitors.
The scaled component values are thus $R = 138.46\ \Omega; \quad C = 1\ \mu\text{F}$

Finally, add an inverting amplifier stage with a gain of R_f/R_i = 10. As usual, we can arbitrarily select one of the two resistor values. Because we are already using 138.46 Ω resistors, let R_i = 138.46 Ω; then,

$$R_f = 10R_i = 1384.6\ \Omega.$$

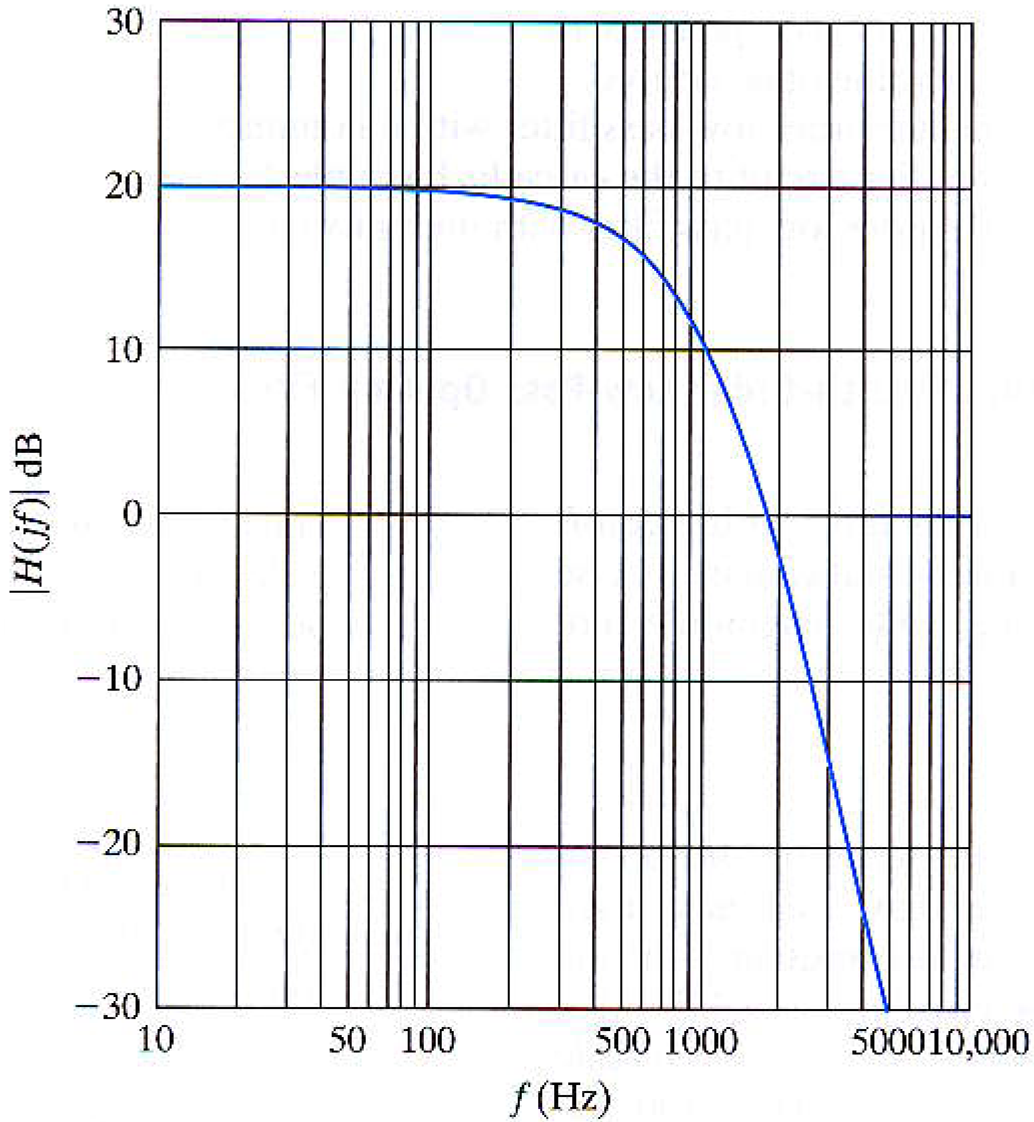
The circuit for this cascaded the fourth order low-pass filter is shown in Fig. It has the transfer function

$$H(s) = -10\left[\frac{7222.39}{s + 7222.39}\right]^4$$



The cascade circuit for the fourth order low-pass filter design in example

The Bode magnitude plot for this transfer function



Butterworth Filters

A unity-gain Butterworlh low-pass filter has a transfer function whose magnitude is given by

$$|H(j\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_c)^{2n}}} \qquad (*)$$

where n is an integer that denotes the order of the filter

1. The cutoff frequency is ω_c rad/s for all values of n.
2. If n is large enough, the denominator is always close to unity when $\omega < \omega_c$.
3. In the expression for $|H(j\omega)|$, the exponent of ω/ω_c is always even.

Given an equation for the magnitude of the transfer function, how do we find H(s)?

Set ω_c equal to 1 rad/s in Eq. (*)
We use scaling to transform the prototype filter to a filter that meets the given filtering specifications. Note that s = jω

$$|H(j\omega)|^2 = H(j\omega)H(-j\omega) = H(s)H(-s)$$

Butterworth Filters

With $s^2 = \omega^2$. Thus,

$$|H(j\omega)|^2 = \frac{1}{1+\omega^{2n}} = \frac{1}{1+(\omega^2)^n} = \frac{1}{1+(-s^2)^n} = \frac{1}{1+(-1)^n s^{2n}}$$

$$\Rightarrow \quad H(s)H(-s) = \frac{1}{1+(-1)^n s^{2n}}$$

The procedure for finding $H(s)$ for a given value of n is as follows:

1. Find the roots of the polynomial

$$1 + (-1)^n s^{2n} = 0.$$

2. Assign the left-half plane roots to $H(s)$ and the right-half plane roots to $H(-s)$.
3. Combine terms in the denominator of $H(s)$ to form first- and second-order factors.

Butterworth Filters

Example: Find the Butterworth transfer functions for n = 2 and n = 3.

Solution

For n = 2, we find the roots of the polynomial: $1 + (-1)^2 s^4 = 0$

Rearranging terms we find $s^4 = -1 = 1\angle 180^\circ$

Therefore, the four roots are:

$$s_1 = 1\angle 45^\circ = 1/\sqrt{2} + j/\sqrt{2},$$

$$s_2 = 1\angle 135^\circ = -1/\sqrt{2} + j/\sqrt{2},$$

$$s_3 = 1\angle 225^\circ = -1/\sqrt{2} + -j/\sqrt{2},$$

$$s_4 = 1\angle 315^\circ = 1/\sqrt{2} + -j/\sqrt{2}.$$

Roots s_2 and s_3 are in the left-half plane. So:

$$H(s) = \frac{1}{(s + 1/\sqrt{2} - j/\sqrt{2})(s + 1/\sqrt{2} + j/\sqrt{2})}$$

$$= \frac{1}{(s^2 + \sqrt{2}s + 1)}$$

Butterworth Filters

Solution

For n = 3, we find the roots of the polynomial $1 + (-1)^3 s^6 = 0$

$$s^6 = 1\angle 0^\circ = 1\angle 360^\circ$$

Therefore, the six roots are

$$s_1 = 1\angle 0^\circ = 1, \qquad s_4 = 1\angle 180^\circ = -1 + j0,$$

$$s_2 = 1\angle 60^\circ = 1/2 + j\sqrt{3}/2, \qquad s_5 = 1\angle 240^\circ = -1/2 + -j\sqrt{3}/2,$$

$$s_3 = 1\angle 120^\circ = -1/2 + j\sqrt{3}/2, \qquad s_6 = 1\angle 300^\circ = 1/2 + -j\sqrt{3}/2.$$

Roots s3, s4, and s5 are in the leff half plane. Thus

$$H(s) = \frac{1}{(s+1)(s+1/2-j\sqrt{3}/2)(s+1/2+j\sqrt{3}/2)} = \frac{1}{(s+1)(s^2+s+1)}$$

Note that in passing that the roots of the Butterworth polynomial are always equally spaced around the unit circle in the s plane.

Butterworth Filters

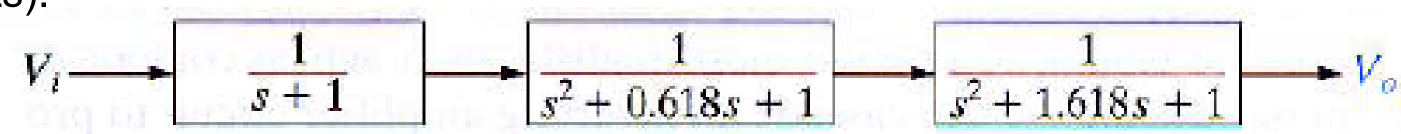
TABLE 15.1 Normalized (so that $\omega_c = 1$ rad/s) Butterworth Polynomials up to the Eighth Order

n	nth-Order Butterworth Polynomial
1	$(s + 1)$
2	$(s^2 + \sqrt{2}s + 1)$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.765s + 1)(s^2 + 1.848s + 1)$
5	$(s + 1)(s^2 + 0.618s + 1)(s^2 + 1.618s + 1)$
6	$(s^2 + 0.518s + 1)(s^2 + \sqrt{2} + 1)(s^2 + 1.932s + 1)$
7	$(s + 1)(s^2 + 0.445s + 1)(s^2 + 1.247s + 1)(s^2 + 1.802s + 1)$
8	$(s^2 + 0.390s + 1)(s^2 + 1.111s + 1)(s^2 + 1.6663s + 1)(s^2 + 1.962s + 1)$

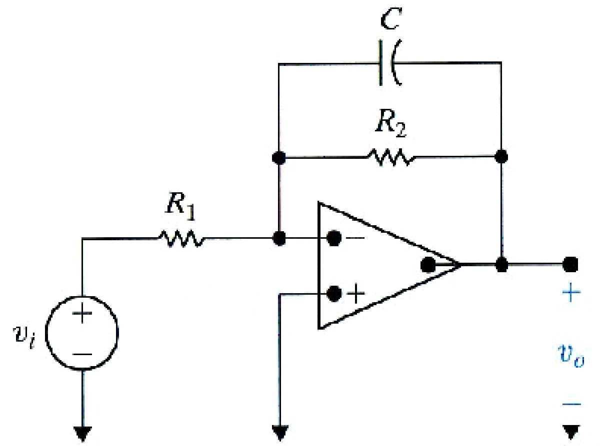
To assist in the design of Butterworth filters, Table 15.1 lists the Butterworth polynomials up to n = 8.

Butterworth Filter Circuits

The Butterworth polynomials in Table15.1 are the product of first- and second-order factors ➔ we can construct a circuit whose transfer function has a Butterworth polynomial in its denominator by cascading op amp circuits, each of which provides one of the needed factors. A block diagram of such a cascade is shown in Fig. below, using a fifth-order Butterworth polynomial as an example (a cascade of first-and second-order circuits).

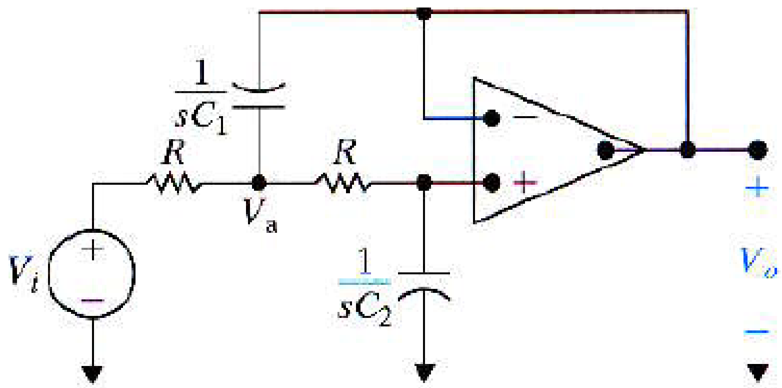


All odd-order Butterworth polynomials include the factor (s + 1) → all odd-order Butterworth filter circuits must have a subcircuit that provides the transfer function H(s) = 1/(s + 1). This is the transfer function of the prototype low-pass op amp filter



Butterworth Filter Circuits

Now, to find a circuit that provides a transfer function of the form
H(s)= 1/(s^2+ b$_1$s + 1).



The s-domain nodal equations at the noninverting terminal of the op amp and at the node labeled V$_a$:

$$\frac{V_a - V_i}{R} + (V_a - V_o)sC_1 + \frac{V_a - V_o}{R} = 0$$

$$V_o sC_2 + \frac{V_o - V_a}{R} = 0.$$

$$\left.\begin{aligned}(2 + RC_1s)V_a - (1 + RC_1s)V_o &= V_i,\\ -V_a + (1 + RC_2s)V_o &= 0.\end{aligned}\right\}$$

Using Cramer's rule

$$V_o = \frac{\begin{vmatrix} 2+RC_1s & V_i \\ -1 & 0 \end{vmatrix}}{\begin{vmatrix} 2+RC_1s & -(1+RC_1s) \\ -1 & 1+RC_2s \end{vmatrix}}$$

$$= \frac{V_i}{R^2C_1C_2s^2 + 2RC_2s + 1}$$

Butterworth Filter Circuits

The transfer function for the circuit
$$H(s) = \frac{V_o}{V_i} = \frac{\frac{1}{R^2C_1C_2}}{s^2 + \frac{2}{RC_1}s + \frac{1}{R^2C_1C_2}}$$

Finally, set R = 1Ω ⇨ $H(s) = \dfrac{\frac{1}{C_1C_2}}{s^2 + \frac{2}{C_1}s + \frac{1}{C_1C_2}}$

This Eq. required for the 2nd-order circuit in the Butterworth cascade or the form

$$H(s) = \frac{1}{s^2 + b_1 s + 1}$$

Choose capacitor values so that $b_1 = \dfrac{2}{C_1}$ and $1 = \dfrac{1}{C_1C_2}$.

The procedure for designing an nth-order Butterworth low-pass filter circuit with a cutoff frequency of ω_c = 1 rad/s and a gain of 1 in the passband: Use frequency scaling to calculate revised capacitor values that yield any other cutoff frequency; and use magnitude scaling to provide more realistic or practical component values in our design.

➔ We can cascade an inverting amplifier circuit to provide a gain other than 1 in the passband.

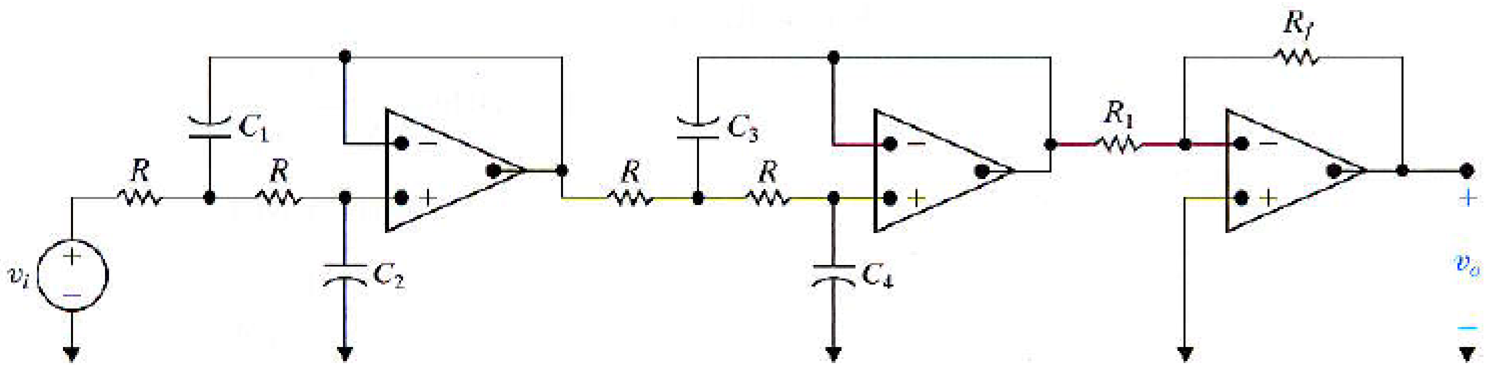
Butterworth Filter Circuits

Example: Design a 4th-order Butterworth low-pass filter with a cutoff frequency of 500 Hz and a passband gain of 10. Use as many 1 kΩ resistors as possible. Compare the Bode magnitude plot for this Butterworth filter with that of the identical cascade filter in previous example

Solution From Table 15.1, we find that the 4th-order Butterworth polynomial is

$$(s^2 + 0.765s + 1)(s^2 + 1.848s + 1).$$

Need a cascade of two 2nd-order filters to yield the 4th-order transfer function plus an inverting amplifier circuit for the passband gain of 10. The circuit is shown in Fig. below



Butterworth Filter Circuits

Solution

Let the first stage of the cascade implement the transfer function for the polynomial ($s^2 + 0.765s + 1$).

Use Eq. $b_1 = \frac{2}{C_1}$ and $1 = \frac{1}{C_1 C_2}$. ⇨ $C_1 = 2.61$ F, $C_2 = 0.38$ F.

Let the second stage of the cascade implement the transfer function for the polynomial ($s^2 + 1.848s + 1$). ⇨ $C_3 = 1.08$ F, $C_4 = 0.924$ F.

The preceding values for C_1, C_2, C_3, & C_4 yield a **4th-order** Butterworth filter with a $\mathbf{f_c}$ of 1 rad/s. $\mathbf{K_f = 3141.6}$ will move $\mathbf{f_c}$ to 500 Hz. A magnitude scale factor of $\mathbf{k_m = 1000}$ will permit the use of 1kΩ resistor in place of 1Ω resistors.

The resulting scaled component values:
$R = 1k\Omega$; $C_1 = 831nF$; $C_2 = 121nF$; $C_3 = 344nF$; $C_4 = 294nF$

Finally, we need to specify the resistor values in the inverting amplifier stage to yield a passband gain of 10. Let $R_1 = 1k\ \Omega$; then $R_f = 10R_1 = 10k\Omega$

Figure compares the magnitude responses of the 4th-order identical cascade filter from previous Example and the Butterworth filter we just designed. Note that both filters provide a passband gain of 10 (20 dB) and a cutoff frequency of 500 Hz, but the Butterworth filter is closer to an ideal low-pass filter due to its flatter passband and steeper rolloff at the cutoff frequency. Thus, the Butterworth design is preferred over the identical cascade design.

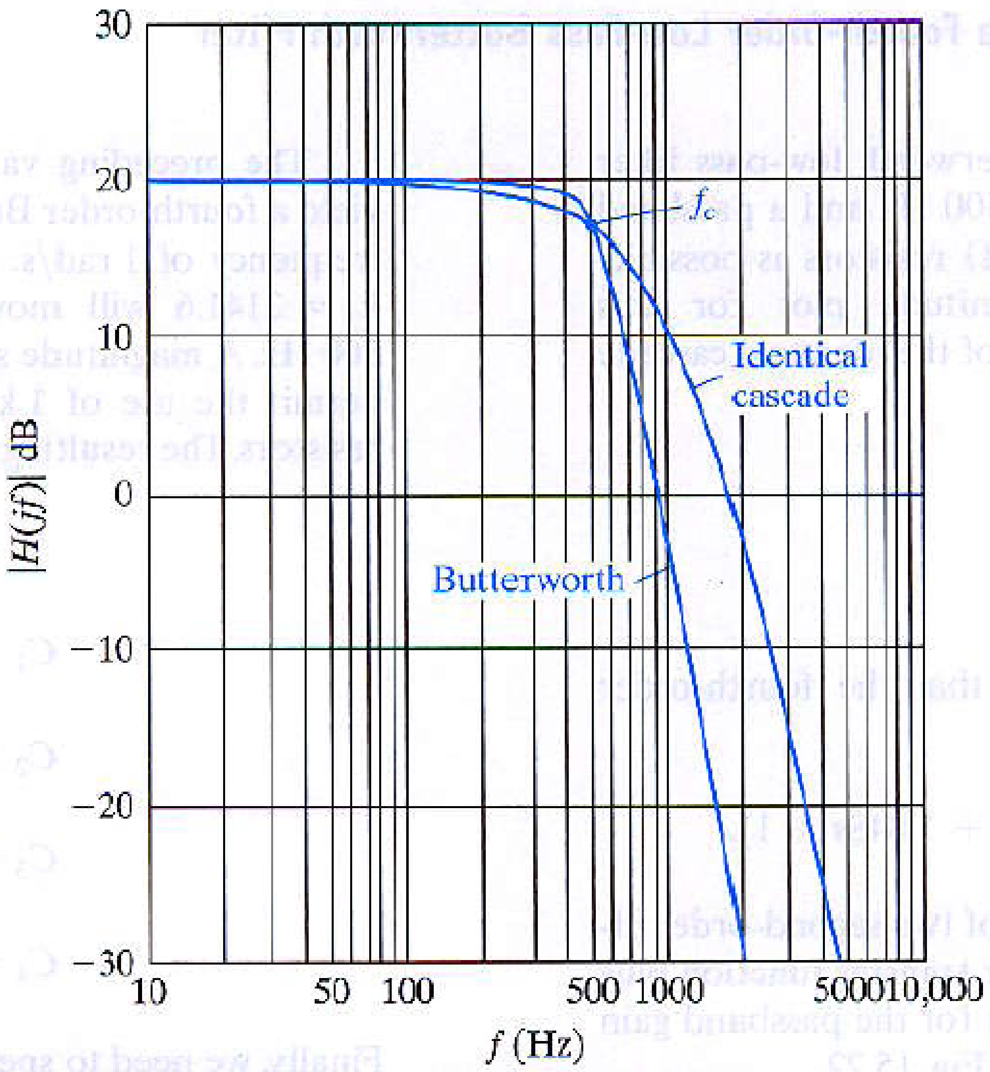


Figure 15.23 ▲ A comparison of the magnitude responses for a fourth-order low-pass filter using the identical cascade and Butterworth designs.

The Order of a Butterworth Filter

Read @ home!

Narrow Bandpass & Bandreject Filters

So far, we mentioned only about **broadband filters** or **low Q** filters that constructed from simple high-pass & low-pass filters.

This sort of broadband filter with the transfer functions for cascaded bandpass & parallel bandreject fillers that have discrete real poles. The synthesis techniques work best for cutoff frequencies that are widely separated and therefore yield the lowest quality factors.
Consider the transfer function that results

$$H(s) = \left(\frac{-\omega_c}{s+\omega_c}\right)\left(\frac{-s}{s+\omega_c}\right) = \frac{s\omega_c}{s^2 + 2\omega_c s + \omega_c^2} = \frac{0.5\beta s}{s^2 + \beta s + \omega_c^2}.$$

This is standard form of the transfer function of a bandpass filter

The bandwidth and center frequency:

$$\beta = 2\omega_c, \qquad \omega_o^2 = \omega_c^2. \quad \Rightarrow \quad Q = \frac{\omega_o}{\beta} = \frac{\omega_c}{2\omega_c} = \frac{1}{2}$$

Thus with discrete real poles, the highest quality bandpass filter (or bandreject filter) we can achieve has $Q = 1/2$.

Narrow Bandpass & Bandreject Filters

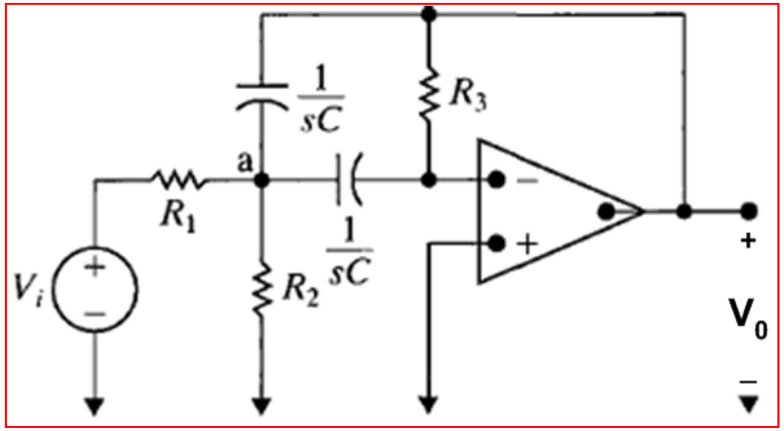
To build active filters with high quality factor values, we need an op amp circuit that can produce a transfer function with complex conjugate poles.

At the inverting input $\dfrac{V_a}{1/sC} = \dfrac{-V_o}{R_3}$

Solving for V_a $\boxed{V_a = \dfrac{-V_o}{sR_3C}}$

@ node a: $\dfrac{V_i - V_a}{R_1} = \dfrac{V_a - V_o}{1/sC} + \dfrac{V_a}{1/sC} + \dfrac{V_a}{R_2}$,

$$\Rightarrow V_i = (1 + 2sR_1C + R_1/R_2)V_a - sR_1CV_o$$



$$\Rightarrow H(s) = \frac{\dfrac{-s}{R_1C}}{s^2 + \dfrac{2}{R_3C}s + \dfrac{1}{R_{eq}R_3C^2}}$$

(transfer function V_0/V_i)

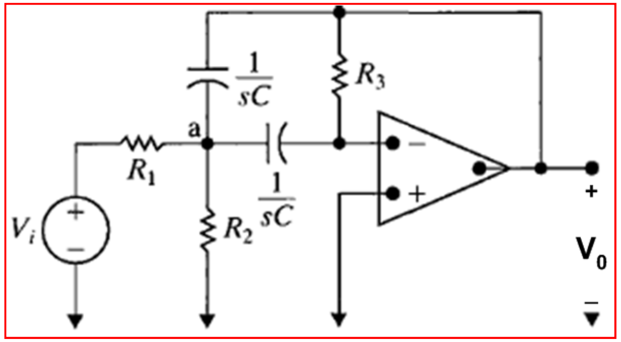
$$R_{eq} = R_1 \| R_2 = \frac{R_1R_2}{R_1 + R_2}$$

The standard form of the transfer function for a bandpass filter, $\boxed{H(s) = \dfrac{-K\beta s}{s^2 + \beta s + \omega_o^2}}$

equate terms

$$\beta = \frac{2}{R_3C}; \quad K\beta = \frac{1}{R_1C}; \quad \omega_o^2 = \frac{1}{R_{eq}R_3C^2}.$$

Narrow Bandpass & Bandreject Filters



Consider the prototype version of the circuit

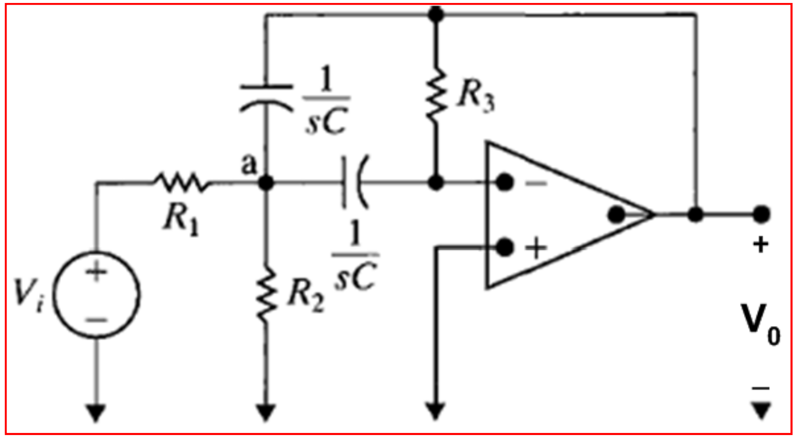
$$\boxed{\omega_o = 1 \text{ rad/s and } C = 1 \text{ F}}$$

$$R_1 = Q/K,$$
$$R_2 = Q/(2Q^2 - K),$$
$$R_3 = 2Q.$$

Finally, use the scaling is used to specify practical values for the circuit components.

Narrow Bandpass & Bandreject Filters

Example: design a **bandpass filter**, using the circuit in Fig. below which has a center frequency of **3000 Hz**, a **quality factor** of **10**, and a passband **gain** of **2**. Use **0.01 μF capacitors** in your design. Compute the transfer function of your circuit, and sketch a Bode plot of its magnitude response.



Narrow Bandpass & Bandreject Filters

[Solution]

Narrow Bandpass & Bandreject Filters

The parallel implementation of a **bandreject filter** that combines low-pass and high-pass filter components with a summing amplifier

@ node a

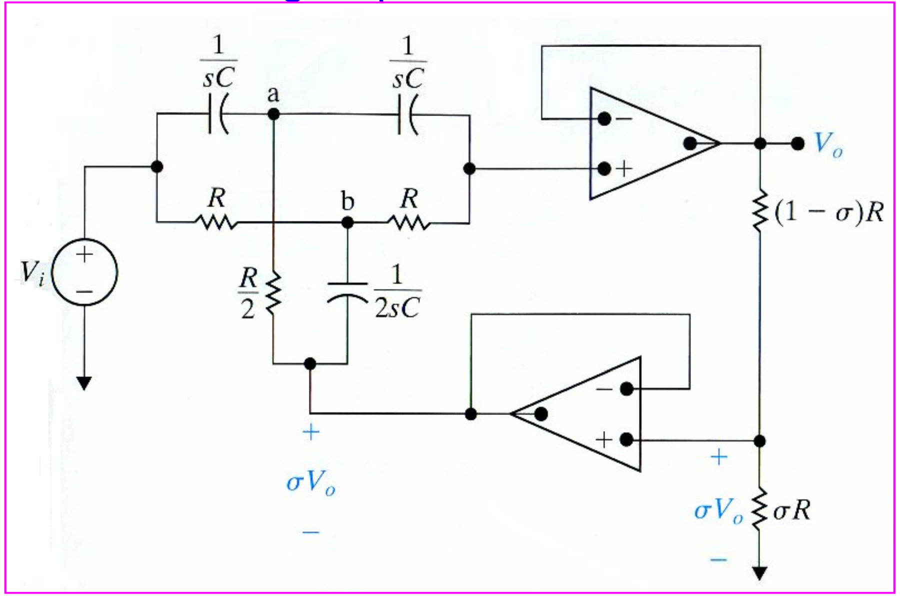
$$(V_a - V_i)sC + (V_a - V_o)sC + \frac{2(V_a - \sigma V_o)}{R} = 0$$

$$\Leftrightarrow \boxed{V_a[2sCR + 2] - V_o[sCR + 2\sigma] = sCRV_i.}$$

@ node b

$$\frac{V_b - V_i}{R} + \frac{V_b - V_o}{R} + (V_b - \sigma V_o)2sC = 0$$

$$\Leftrightarrow \boxed{V_b[2 + 2RCs] - V_o[1 + 2\sigma RCs] = V_i.}$$



A high-Q active bandreject filter aka. twin-T notch filter

Summing the currents away from the noninverting input terminal of the top op amp gives

$$(V_o - V_a)sC + \frac{V_o - V_b}{R} = 0 \Rightarrow \boxed{-sRCV_a - V_b + (sRC + 1)V_o = 0.}$$

Use Cramer's rule to solve for V_0:

Narrow Bandpass & Bandreject Filters

$$V_o = \frac{\begin{vmatrix} 2(RCs+1) & 0 & sCRV_i \\ 0 & 2(RCs+1) & V_i \\ -RCs & -1 & 0 \end{vmatrix}}{\begin{vmatrix} 2(RCs+1) & 0 & -(RCs+2\sigma) \\ 0 & 2(RCs+1) & -(2\sigma RCs+1) \\ -RCs & -1 & RCs+1 \end{vmatrix}} = \frac{(R^2C^2s^2+1)V_i}{R^2C^2s^2+4RC(1-\sigma)s+1}$$

The transfer function:

$$H(s) = \frac{V_o}{V_i} = \frac{\left(s^2 + \frac{1}{R^2C^2}\right)}{\left[s^2 + \frac{4(1-\sigma)}{RC}s + \frac{1}{R^2C^2}\right]},$$

The standard form for the transfer function of a bandreject filter

$$H(s) = \frac{s^2+\omega_0^2}{s^2+\beta s+\omega_0^2}.$$

Equating both Equ.

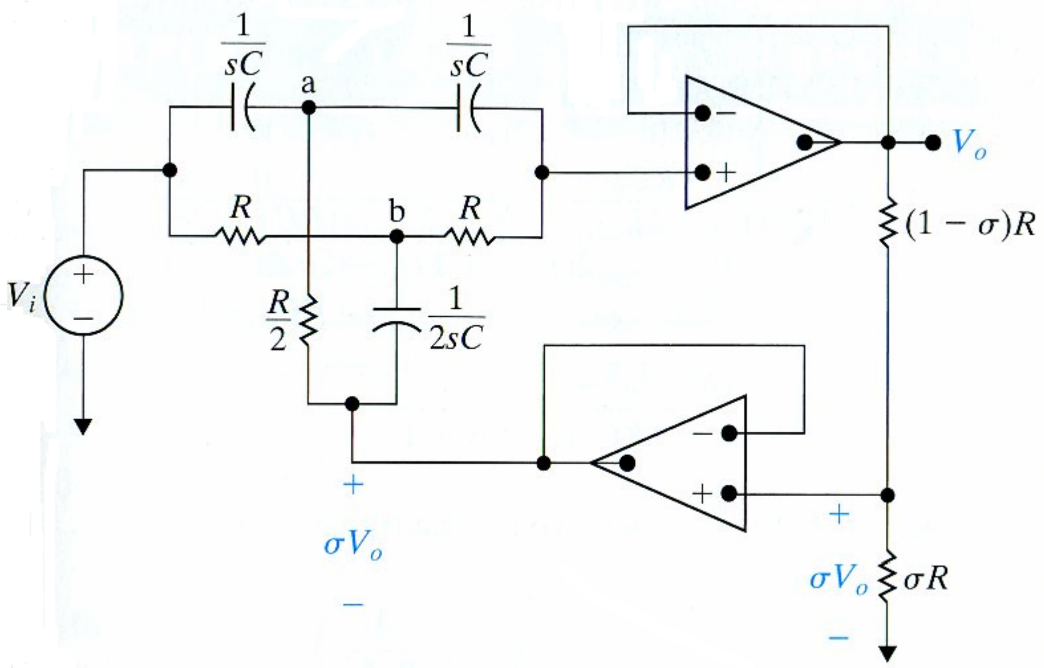
$$\omega_o^2 = \frac{1}{R^2C^2}, \quad \beta = \frac{4(1-\sigma)}{RC}.$$

In this circuit, we have three parameters *(R, C, and σ) and two design* constraints *(ω_0 and β). Thus one parameter is chosen arbitrarily. Usually, we select capacitor value.*

Narrow Bandpass & Bandreject Filters

$$\Rightarrow \quad \boxed{R = \frac{1}{\omega_o C},} \quad \boxed{\sigma = 1 - \frac{\beta}{4\omega_o} = 1 - \frac{1}{4Q}.}$$

Example: Design a high-Q active bandreject filter (based on the circuit in Fig. below) with a center frequency of 5000 rad/s and a bandwidth (β) of 1000 rad/s. Use 1μF capacitors in your design.



Narrow Bandpass & Bandreject Filters

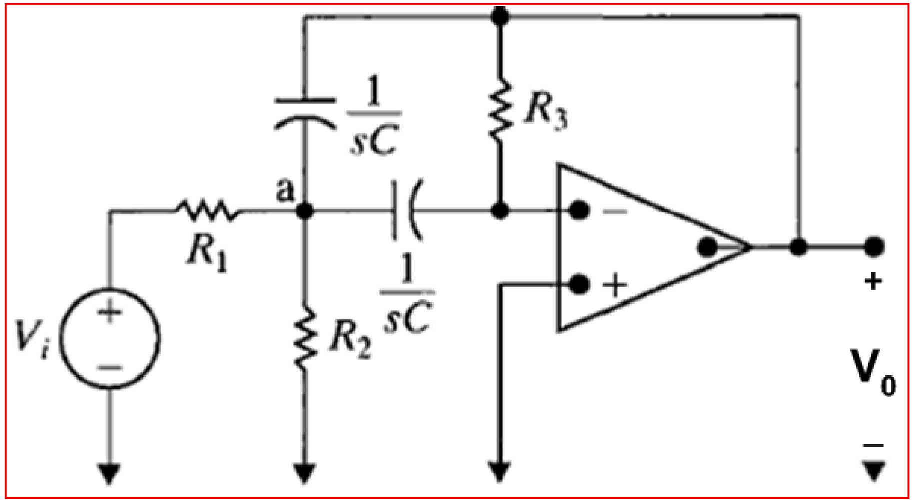
Solution

Narrow Bandpass & Bandreject Filters

Solution

Narrow Bandpass & Bandreject Filters

Problem4: Design an active bandpass filter (as shown the the figure below) with *Q = 8, K = 5, and* ω_0 *= 1000 rad/s. Use 1*μ*F capacitors and specify the values of all resistors.*



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Solution:

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Solution:

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Problem 5:

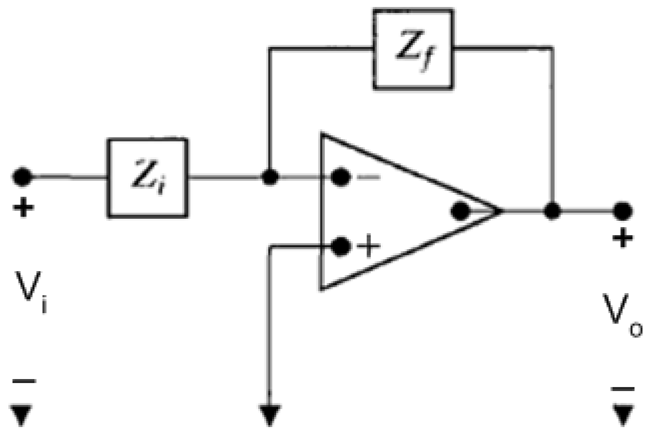
Design an active unity-gain bandreject filter with $\omega_0 = 1000$ rad/s and Q = 4. Use 2μF capacitors, and specify the values of R and σ

Narrow Bandpass & Bandreject Filters

Solution:

Exercise 1

Find the transfer function V_0/V_i *for the circuit* shown in Fig. if Z_f *is the equivalent impedance* of the feedback circuit, Z_i is the equivalent impedance of the input circuit, and the operational amplifier is ideal.

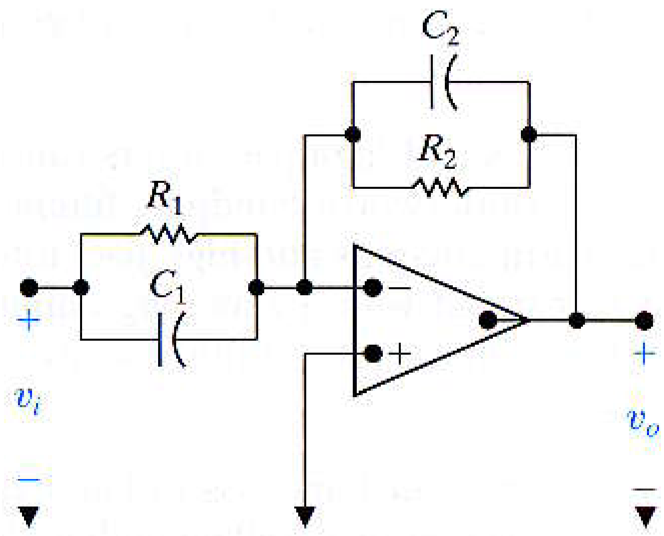


Exercise 1

Sol.:

Exercise 2

a) Use the results of exercise1 to find the transfer function of the circuit showning fig.
b) What is the gain of the circuit as $\omega \rightarrow 0$?
c) What is the gain of the circuit as $\omega \rightarrow \infty$?
d) Do your answers to (b) and (c) make sense in terms of known circuit behavior?



Exercise 2

Sol.:

Exercise 2

Sol.: