Nguyễn Hoàng Phúc - EEACIU22157

## 1) VECTOR SPACE:

1. $u+v$ and $c.u$ are in $V$
2. $u+v = v+u$
3. $(u+v)+w = u+(v+w)$
4. exist a zero vector $0$ in $V$ such that $v+0 = v$
5. $u+(-u) = 0$
6. $c.(u+v) = cu + cv$
7. $(d+c)u = cu + du$
8. $c(du) = (cd)u$
9. $1u = u$

Ex: Determine whether the vectors
$v_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$
are linearly independent:

$a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$

$\Rightarrow a_1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We obtain the homogeneous system:

$\begin{cases} 3a_1 + a_2 - a_3 = 0 \\ 2a_1 + 2a_2 + 2a_3 = 0 \\ a_1 + (-a_3) = 0 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right]$

$\xrightarrow[3R_1-2R_2]{R_1-3R_3} \left[\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 0 & -4 & -8 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow[R_2-R_3]{-1/4 R_2} \left[\begin{array}{ccc|c} 3 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\xrightarrow[-1/3R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{cases} a_1 - a_3 = 0 \\ a_2 - 2a_3 = 0 \\ a_3 \text{ free} \end{cases}$

$\rightarrow \begin{cases} a_1 = h \\ a_2 = -2h \\ a_3 = h \end{cases} \rightarrow [v] = h \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ are linearly independent

Theorem: $S = \{v_1, v_2, v_3\}$ in $R^n \rightarrow S$ is linearly independent and only if $\det(A) \neq 0$

Ex: Write each vectors as linear combination of the vector in S.

$S = \{(2;0;7), (2,4,5); (2,-12,13)\}$

$u = (-1;5;-6)$

We have:

$c_1(2,0,7) + c_2(2,4,5) + c_3(2,-12,13) = (-1,5,-6)$

$\Rightarrow \begin{cases} 2c_1 + 2c_2 + 2c_3 = -1 \\ 4c_2 - 12c_3 = 5 \\ 7c_1 + 5c_2 + 13c_3 = -6 \end{cases}$

$\rightarrow \left[\begin{array}{ccc|c} 2 & 2 & 2 & -1 \\ 0 & 4 & -12 & 5 \\ 7 & 5 & 13 & -6 \end{array}\right] \xrightarrow[7R_1-R_3]{1/2 R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1/2 \\ 0 & 4 & -12 & 5 \\ 0 & 2 & -6 & 5/2 \end{array}\right]$

$\xrightarrow[1/4R_2]{R_2-2R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1/2 \\ 0 & 1 & -3 & 5/4 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 4 & -7/4 \\ 0 & 1 & -3 & 5/4 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\rightarrow \begin{cases} c_1 + 4c_3 = -7/4 \\ c_2 - 3c_3 = 5/4 \\ c_3 \text{ is free } (c_3 = c_3) \end{cases}$

$\rightarrow \begin{cases} c_1 = -7/4 - 4c_3 \\ c_2 = 5/4 + 3c_3 \\ c_3 = c_3 \end{cases}$

$\vec{u} = (-7/4 - 4c_3)(2,0,7) + (5/4 + 3c_3)(2,4,5) + c_3(2,-12,13)$

$w = (-3,13,18)$

$c_1(2,0,7) + c_2(2,4,5) + c_3(2,-12,13) = (-3,13,18)$

$\Rightarrow \begin{cases} 2c_1 + 2c_2 + 2c_3 = -3 \\ 4c_2 - 12c_3 = 13 \\ 7c_1 + 5c_2 + 13c_3 = 18 \end{cases}$

$\rightarrow \left[\begin{array}{ccc|c} 2 & 2 & 2 & -3 \\ 0 & 4 & -12 & 13 \\ 7 & 5 & 13 & 18 \end{array}\right] \xrightarrow[7R_1-R_3]{1/2 R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -3/2 \\ 0 & 4 & -12 & 15/4 \\ 0 & 0 & 0 & 7/2 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 15/4 \\ 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

$\rightarrow 0 \neq 1 \rightarrow$ Do not have solution.

$\rightarrow c_1, c_2, c_3 \in \emptyset$

$\rightarrow v$ cannot be expressed

Ex: Determine whether the set S span $R^3$

a) $S = \{(4,7,3), (-1,2,6), (2,-3,5)\}$

$\rightarrow a_1(4,7,3) + a_2(-1,2,6) + a_3(2,-3,5) = (0;0;0)$

$\Leftrightarrow \begin{cases} 4a_1 - a_2 + 2a_3 = 0 \\ 7a_1 + 2a_2 - 3a_3 = 0 \\ 3a_1 + 6a_2 + 5a_3 = 0 \end{cases} \Leftrightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \end{cases}$

$\rightarrow$ Independent $\rightarrow$ Set S span $R^3$

## LINEAR INDEPENDENT:

A set of vectors $\{v_1, v_2, v_3\}$ in a vector space V is said to be linear independent if the vector equation

$\boxed{x_1 v_1 + x_2 v_2 + \dots x_p v_p = 0}$

$(x_1 = x_2 = x_3 = x \dots = 0)$

## LINEAR DEPENDENT:

The set $\{v_1; v_2, \dots v_p\}$ is said to be linear dependent if there exist weights $c_1 \dots c_p$ not all 0, such that:

$\boxed{c_1 v_1 + c_2 v_2 + \dots c_p v_p = 0}$

Ex: a) $S = \{(-2,1;3), (2,9,-3), (2,3,-3)\}$

$\rightarrow |D| = \begin{vmatrix} -2 & 1 & 3 \\ 2 & 9 & -3 \\ 2 & 3 & -3 \end{vmatrix} = 0 \rightarrow$ Dependent

b) $S = \{(-4,-3,4), (1,-2,-3), (6,0,0)\}$

$\rightarrow |D| = \begin{vmatrix} -4 & -3 & 4 \\ 1 & -2 & -3 \\ 6 & 0 & 0 \end{vmatrix} = -6 \neq 0 \rightarrow$ Independent

Ex: Find Rank, Nulity (dimNull) of the following matrix.

a) $A = \begin{pmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{pmatrix}$

$\xrightarrow[R_1+R_3]{5R_1-R_3} \begin{pmatrix} 1 & -4 & 9 & -7 \\ 0 & -2 & 5 & -6 \\ 0 & -14 & 35 & -42 \end{pmatrix}$

$\xrightarrow[-1/2 R_2]{-7R_2+R_3} \begin{pmatrix} 1 & -4 & 9 & -7 \\ 0 & 1 & -5/2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\xrightarrow{4R_2+R_1} \begin{pmatrix} 1 & 0 & -1 & 5 \\ 0 & 1 & -5/2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow$ Rank A = 2; Nullity A = 2

Ex: Finding a Basic for a Row Space and Rank of the following matrix:

$A = \begin{pmatrix} 2 & -3 & 1 \\ 5 & 10 & 6 \\ 8 & -7 & 5 \end{pmatrix} \xrightarrow[5R_1-2R_2]{4R_1-R_2} \begin{pmatrix} 2 & -3 & 1 \\ 0 & -35 & -7 \\ 0 & -5 & -1 \end{pmatrix}$

$\xrightarrow[-1/35R_2]{-R_2+7R_3} \begin{pmatrix} 2 & -3 & 1 \\ 0 & 1 & 1/5 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{1/2 R_1} \begin{pmatrix} 1 & -3/2 & 1/2 \\ 0 & 1 & 1/5 \\ 0 & 0 & 0 \end{pmatrix}$

Thus, Rank (A) = 2 and

$r_1 = [1 \ \ -3/2 \ \ 1/2]$ is a form of basis

$r_2 = [0 \ \ 1 \ \ 1/5]$ for a Row Space

Ex: Determine whether a set is a basis. If it is, write $u = (8,3,8)$ as a linear combination of the vectors in S.

$S = \{(4,3,2); (0,3,2); (0,0,2)\}$

$\xrightarrow{\text{Matrix A}} A = \begin{bmatrix} 4 & 0 & 0 \\ 3 & 3 & 0 \\ 2 & 2 & 2 \end{bmatrix}$

$\det(A) = 24 \neq 0 \rightarrow$ linear independent

$\rightarrow$ The vector S is a form of basis

$a(4,3,2) + b(0,3,2) + c(0,0,2) = (8,3,8)$

$\rightarrow \begin{cases} 4a + 0b + 0c = 0 \\ 3a + 3b = 0 \\ 2a + 2b + 2c = 0 \end{cases} \Rightarrow \begin{cases} a = 2 \\ b = -1 \\ c = 3 \end{cases}$

Then $u = (8,3,8)$ as a linear combination of the vector S.

$2(4,3,2) + (-1)(0,3,2) + 3(0,0,2) = (8,3,8)$

$S = \{(0,0,0); (1,3,4); (6,1,-2)\}$

$\xrightarrow{\text{Matrix B}} = \left[\begin{array}{ccc|c} 0 & 1 & 6 & 8 \\ 0 & 3 & 1 & 3 \\ 0 & 4 & -2 & 8 \end{array}\right]$

$\xrightarrow[R_3-4R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 0 & 1 & 6 & 8 \\ 0 & 0 & -17 & -21 \\ 0 & 0 & -26 & -24 \end{array}\right] \xrightarrow[R_1-6R_2]{-1/17 R_2}$

$\left[\begin{array}{ccc|c} 0 & 1 & 6 & 8 \\ 0 & 0 & 1 & 21/17 \\ 0 & 0 & -26 & -24 \end{array}\right] \xrightarrow{R_3+26R_1} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 10/17 \\ 0 & 0 & 1 & 21/17 \\ 0 & 0 & 0 & 146/17 \end{array}\right]$

The system is inconsistent with no solution
→ S is not a basis for $R^3$

**Ex:** Find the dimension and basis of the subspace:

$$H = \left\{ \begin{pmatrix} a-4b-2c \\ 2a+5b-4c \\ -a+2c \\ -3a+7b+6c \end{pmatrix}, a,b,c \in \mathbb{R} \right\}$$

$$\begin{pmatrix} a-4b-2c \\ 2a+5b-4c \\ -a+2c \\ -3a+7b+6c \end{pmatrix} = a\begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix} + b\begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix} + c\begin{bmatrix} -2 \\ -4 \\ 2 \\ 6 \end{bmatrix}$$

So, we have 3 vectors:

$$\vec{v_1} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} -2 \\ -4 \\ 2 \\ 6 \end{bmatrix}$$

$$\Rightarrow \text{Basis for } H = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 2 \\ 6 \end{bmatrix} \right\}$$

$\Rightarrow \dim(H) = 2$

**Ex:** Let $B = \{(8,11,0),(7,0,10),(1,4,6)\}$ be a basis for $R^3$. Find the coordinate of vector $x = (3,19,2)$?

We have the transition matrix $B' = \begin{pmatrix} 8 & 7 & 1 \\ 11 & 0 & 4 \\ 0 & 10 & 6 \end{pmatrix}$

The coordinate of the vector $x = (3,19,2)$ with respect to the basis of B are $(x_1, x_2, x_3)$. Thus,

$$\begin{pmatrix} 8 & 7 & 1 \\ 11 & 6 & 4 \\ 0 & 10 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 19 \\ 2 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 8 & 7 & 1 & 3 \\ 11 & 0 & 4 & 19 \\ 0 & 10 & 6 & 2 \end{array}\right] \xrightarrow[1/2 R_3]{11R_1-8R_2} \left[\begin{array}{ccc|c} 8 & 7 & 1 & 3 \\ 0 & 77 & -21 & -119 \\ 0 & 5 & 3 & 1 \end{array}\right]$$

$$\xrightarrow{1/7 R_2,\ 5R_2-11R_3} \left[\begin{array}{ccc|c} 8 & 7 & 1 & 3 \\ 0 & 11 & -3 & -17 \\ 0 & 0 & -48 & -96 \end{array}\right] \rightarrow \begin{cases} 8x_1 + 7x_2 + x_3 = 3 \\ 11x_2 - 3x_3 = -17 \\ -48x_3 = -96 \end{cases}$$

$$\Rightarrow \begin{cases} x_3 = 2 \\ x_2 = -1 \\ x_1 = 1 \end{cases} \Rightarrow [v_B] = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

**Ex:** $S = \{(1,-4);(3,-5)\}$ and $T = \{(-9,1),(-5,-1)\}$ be basis in $R^2$

a) Find the change of coordinates matrix from T to S (transition matrix)

$P_{T\to S} = [[T_1]_S \, [T_2]_S \, [T_3]_S]$

• $[T_1]_S$: $\begin{bmatrix} -9 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -4 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ -5 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 1 & 3 & -9 \\ -4 & -5 & 1 \end{bmatrix} \xrightarrow{4R_1+R_2} \begin{bmatrix} 1 & 3 & -9 \\ 0 & 7 & -35 \end{bmatrix}$$

$$\rightarrow \begin{cases} c_1 + 3c_2 = -9 \\ 7c_2 = -35 \end{cases} \Rightarrow \begin{cases} c_1 = 6 \\ c_2 = -5 \end{cases}$$

$$\Rightarrow [T_1]_S = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

• $[T_2]_S$: $\begin{bmatrix} -5 \\ -1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -4 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ -5 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 1 & 3 & -5 \\ -4 & -5 & -1 \end{bmatrix} \xrightarrow{4R_1+R_2} \begin{bmatrix} 1 & 3 & -5 \\ 0 & 7 & -21 \end{bmatrix}$$

$$\rightarrow \begin{cases} c_1 + 3c_2 = -5 \\ 7c_2 = -21 \end{cases} \Rightarrow \begin{cases} c_1 = 4 \\ c_2 = -3 \end{cases}$$

$$\Rightarrow [T_2]_S = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

Therefore $P_{T\to S} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$ (Transition Matrix)

b) $v = (1;-2)$. Find the coordinate of $v$ in the basis T then use the transition matrix at a) to find the coordinate of $v$ in the basis S.

The coordinate of the vector $v = (1;-2)$ with respect to the basis T are $(x_1;x_2)$

Thus, $\begin{pmatrix} -9 & -5 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$\text{Matrix} \rightarrow \left[\begin{array}{cc|c} -9 & -5 & 1 \\ 1 & -1 & -2 \end{array}\right] \xrightarrow{R_1+9R_2} \left[\begin{array}{cc|c} -9 & -5 & 1 \\ 0 & -14 & -17 \end{array}\right]$$

$$\rightarrow \begin{cases} -9x_1 - 5x_2 = 1 \\ -14x_2 = -17 \end{cases} \Rightarrow \begin{cases} x_1 = -11/14 \\ x_2 = 17/14 \end{cases}$$

$$\Rightarrow [v]_T = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -11/14 \\ 17/14 \end{pmatrix}$$

The coordinate ...

Thus, $\begin{pmatrix} 6 & 4 \\ -5 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$\rightarrow \left[\begin{array}{cc|c} 6 & 4 & 1 \\ -5 & -3 & -2 \end{array}\right] \xrightarrow{5R_1+6R_2} \left[\begin{array}{cc|c} 6 & 4 & 1 \\ 0 & 2 & -7 \end{array}\right]$$

$$\Rightarrow \begin{cases} 6x_1 + 4x_2 = 1 \\ 2x_2 = -7 \end{cases} \rightarrow \begin{cases} x_1 = 5/2 \\ x_2 = -7/2 \end{cases}$$

$$\Rightarrow [v]_S = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5/2 \\ -7/2 \end{pmatrix}$$

**Ex:** Find eigenvalues and eigenvectors

a) $A = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}$

• Eigenvalues: $p(\lambda) = |A - \lambda I| = \begin{vmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{vmatrix} = (4-\lambda)(-3-\lambda) + 10 = -12 - 4\lambda + 3\lambda + \lambda^2 + 10$

$\Rightarrow \lambda^2 - \lambda - 2 = 0 \Rightarrow (\lambda-2)(\lambda+1) = 0 \Rightarrow \left[\begin{array}{l} \lambda = 2 \\ \lambda = -1 \end{array}\right.$

Thus, the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = -1$

• Eigenvectors: let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

• with $\lambda_1 = 2$: $AX = 2X \Rightarrow \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\Rightarrow \begin{cases} 4x_1 - 5x_2 = 2x_1 \\ 2x_1 - 3x_2 = 2x_2 \end{cases} \Rightarrow \begin{cases} 2x_1 - 5x_2 = 0 \\ 2x_1 - 5x_2 = 0 \end{cases}$$

$\Rightarrow \frac{2}{5} x_1 = x_2$

Choose $x_2 = t \Rightarrow x_1 = \frac{5}{2} t$

Set of eigenvectors associated with $\lambda_1 = 2$ has the form $X = (\frac{5}{2}t, t)^T = t(\frac{5}{2}, 1)^T$ $(\forall t \neq 0)$

• with $\lambda_2 = -1$: $AX = (-1)X$

$$\Rightarrow \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (-1)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} 4x_1 - 5x_2 = -x_1 \\ 2x_1 - 3x_2 = -x_2 \end{cases} \Rightarrow \begin{cases} x_1 - x_2 = 0 \\ 2x_1 - 2x_2 = 0 \end{cases} \Rightarrow x_1 = x_2$$

Choose $x_2 = t \Rightarrow x_1 = t$

~~Therefo~~ Set of X: $= (t,t)^T = t(1,1)^T$ $(\forall t \neq 0)$

**Ex:** Find eigenvalues and eigenvectors

$A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ 1 & -2 & 0 \end{pmatrix}$

• Eigenvalues: $p(\lambda) = |A - \lambda I| = \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ 1 & -2 & -\lambda \end{vmatrix}$

$$\Rightarrow (-2-\lambda)\begin{vmatrix} 1-\lambda & -6 \\ -2 & -\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & -6 \\ 1 & -\lambda \end{vmatrix} - 3\begin{vmatrix} 2 & 1-\lambda \\ 1 & -2 \end{vmatrix}$$

$\Rightarrow -\lambda^3 - \lambda^2 + 15\lambda + 27 = 0$

$\Rightarrow \left[\begin{array}{l} \lambda_1 = \sqrt{10} + 1 \\ \lambda_2 = \sqrt{10} + 1 \\ \lambda_3 = -3 \end{array}\right.$ are eigenvalues

• Eigenvector: Let $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

- With $\lambda_1 = \sqrt{10} + 1$

$$(A - \lambda_1 I)X = 0 \Rightarrow \left[\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ 1 & -2 & 0 \end{pmatrix} - \begin{pmatrix} \sqrt{10}+1 & 0 & 0 \\ 0 & \sqrt{10}+1 & 0 \\ 0 & 0 & \sqrt{10}+1 \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -\sqrt{10}-3 & 2 & -3 \\ 2 & -\sqrt{10} & -6 \\ 1 & -2 & -\sqrt{10}-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\text{Matrix} \rightarrow \left[\begin{array}{ccc|c} -\sqrt{10}-3 & 2 & -3 & 0 \\ 2 & -\sqrt{10} & -6 & 0 \\ 1 & -2 & -\sqrt{10}-1 & 0 \end{array}\right]$$

$$\xrightarrow{RREF} \begin{cases} x_1 + \left(\frac{1+\sqrt{10}}{3}\right)x_3 = 0 \\ x_2 + \left(\frac{2+2\sqrt{10}}{3}\right)x_3 = 0 \\ x_3 = x_3 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{-\sqrt{10}-1}{3} x_3 \\ x_2 = \frac{-2-2\sqrt{10}}{3} x_3 \\ x_3 = x_3 \end{cases}$$

$$\Rightarrow X_1 = \begin{bmatrix} \frac{-\sqrt{10}-1}{3} \\ \frac{-2-2\sqrt{10}}{3} \\ 1 \end{bmatrix} x_3 = v_1$$

Same for $\lambda_2, \lambda_3$

Thus, the eigenvalues are $\lambda_1 = \sqrt{10} + 1$; $\lambda_2 = \sqrt{10} + 1$, $\lambda_3 = -3$

• Eigenvectors: $v_1, v_2, v_3$.

**What is $R^n$ = ?**

$R$ denotes the set of real numbers

$R^2$ denotes the set of all column vectors with 2 entries

$R^3$ denotes the set of all column vectors with 3 entries

length of vector $R^2$: $d(O,X) = ||\vec{x}|| = \sqrt{x_1^2 + x_2^2}$

$R^3$: $d(O,X) = \sqrt{x_1^2 + x_2^2 + x_3^2}$

$\Rightarrow d(A,B) = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

Ex: let $T: \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation such that:

$$T\begin{bmatrix}1\\3\\1\end{bmatrix} = \begin{bmatrix}4\\4\\0\\-2\end{bmatrix} \text{ and } T\begin{bmatrix}4\\0\\5\end{bmatrix} = \begin{bmatrix}4\\5\\-1\\5\end{bmatrix}$$

Find $T\begin{bmatrix}-7\\3\\-9\end{bmatrix}$

$$\begin{bmatrix}-7\\3\\-9\end{bmatrix} = a\begin{bmatrix}1\\3\\1\end{bmatrix} + b\begin{bmatrix}4\\0\\5\end{bmatrix}$$

$$\to \left[\begin{array}{cc|c}1 & 4 & -7\\3 & 0 & 3\\1 & 5 & -9\end{array}\right] \to \left[\begin{array}{cc|c}1 & 0 & 1\\0 & 1 & -2\\0 & 0 & 0\end{array}\right]$$

Thus $a = 1, b = -2$

$$\begin{bmatrix}-7\\3\\-9\end{bmatrix} = 1\begin{bmatrix}1\\3\\1\end{bmatrix} - 2\begin{bmatrix}4\\0\\5\end{bmatrix}$$

$$T\begin{bmatrix}-7\\3\\-9\end{bmatrix} = T\left(\begin{bmatrix}1\\3\\1\end{bmatrix} - 2\begin{bmatrix}4\\0\\5\end{bmatrix}\right)$$

$$= T\begin{bmatrix}1\\3\\1\end{bmatrix} - 2T\begin{bmatrix}4\\0\\5\end{bmatrix}$$

$$= \begin{bmatrix}4\\4\\0\\-2\end{bmatrix} - 2\begin{bmatrix}4\\5\\-1\\5\end{bmatrix} = \begin{bmatrix}-4\\-6\\2\\-12\end{bmatrix}$$

<< Not all transformations are matrix transformation >>

We have $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{x}) = \vec{x} + \begin{bmatrix}1\\-1\end{bmatrix}$ for all $\vec{x} \in \mathbb{R}^2$

Since every matrix transformation is a linear transformation, we consider $T(\vec{0})$ where $\vec{0}$ is the zero vector of $\mathbb{R}^2$

$$T = \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix} \neq \begin{bmatrix}0\\0\end{bmatrix}$$

(*) Find the matrix of T:

$$T\begin{bmatrix}1\\5\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix},\ T\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$$

AnD: $\left[\begin{array}{cc|cc}1 & 1 & \vec{e}_1 & \vec{e}_2\\5 & 4 & & \end{array}\right] = \left[\begin{array}{cc|cc}1 & 1 & 1 & 0\\5 & 4 & 0 & 1\end{array}\right]$

$$\to \left[\begin{array}{cc|cc}1 & 0 & -4 & 1\\0 & 1 & 5 & -1\end{array}\right]$$

$$\vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix} = -4\begin{bmatrix}1\\5\end{bmatrix} + 5\begin{bmatrix}1\\4\end{bmatrix}$$

$$\vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix} = 1\begin{bmatrix}1\\5\end{bmatrix} - 1\begin{bmatrix}1\\4\end{bmatrix}$$

$$T(\vec{e}_1) = T\left(-4\begin{bmatrix}1\\5\end{bmatrix} + 5\begin{bmatrix}1\\4\end{bmatrix}\right)$$

$$= -4T\begin{bmatrix}1\\5\end{bmatrix} + 5T\begin{bmatrix}1\\4\end{bmatrix}$$

$$= -4\begin{bmatrix}1\\2\end{bmatrix} + 5\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}11\\2\end{bmatrix}$$

Doing the we have $T(\vec{e}_2) = \begin{bmatrix}-2\\0\end{bmatrix}$

$$\Rightarrow A = \boxed{\begin{bmatrix}T(\vec{e}_1) & T(\vec{e}_2)\end{bmatrix}} = \begin{bmatrix}11 & -2\\2 & 0\end{bmatrix}$$

$\mathbb{R}^2 \to \mathbb{R}^2$: $T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x+2y\\x-y\end{bmatrix}$

$\mathbb{R}^2 \to \mathbb{R}^3$: $T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2x\\y\\-x+2y\end{bmatrix}$

(*) Rotation in $\mathbb{R}^2$: $R_\theta = \mathbb{R}^2 \to \mathbb{R}^2$

$E_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $E_2 = \begin{bmatrix}0\\1\end{bmatrix}$

$$R_\theta(E_1) = R_\theta\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\theta\\\sin\theta\end{bmatrix}$$

$$R_\theta(E_2) = R_\theta\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin\theta\\\cos\theta\end{bmatrix}$$

$$A = \begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix} \text{ (matrix A)}$$

(*) Surjection: Let $\mathbb{R}^n \to \mathbb{R}^m$

$$T\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}a+b\\0\end{bmatrix} \forall \begin{bmatrix}a\\b\end{bmatrix} \in \mathbb{R}^2$$

$\ldots = -1,\ x_2 = 5/2$

Ex: $T = \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1 & 2\\3 & 5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

$T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix}1 & 2\\3 & 5\end{bmatrix}$

Since $\det(A) = -1 \neq 0 \to A$ is invertible for every choice of $\vec{b}$, $A\vec{x} = \vec{b}$.

(*) Not one-to-one:

$$A = \begin{bmatrix}1 & 2 & -1\\3 & 5 & 0\end{bmatrix} \to \begin{bmatrix}1 & 2 & -1\\0 & 1 & -3\end{bmatrix} = R$$

A has 2 rank, $A\vec{x} = 0$ (many columns) so $\vec{x} = 0$ is not the only solution, $T_A$ is one-to-one. (Has unique solution → one-to-one).

(*) One-to-one and onto: $A\vec{x} = \vec{b}$ has unique solution $\vec{x}$ for every $\vec{b}$ in $\mathbb{R}^2$.

(*) Kernel and Image:

Ex: Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$T\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = \begin{bmatrix}a+b+c\\c-a\end{bmatrix}$$

Find a basis for ker(T) and im(T)

$V = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow a+b+c = 0,\ c-a = 0$ kernel

$$\Rightarrow \left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = \left\{\begin{bmatrix}t\\-2t\\t\end{bmatrix} : t \in \mathbb{R}\right\} = \left\{t\begin{bmatrix}1\\-2\\1\end{bmatrix} : t \in \mathbb{R}\right\}$$

$$\operatorname{im}(T) = \left\{\begin{bmatrix}a+b+c\\c-a\end{bmatrix} : a,b,c \in \mathbb{R}\right\}$$

$$= \left\{\begin{bmatrix}a\\-a\end{bmatrix} + \begin{bmatrix}b\\0\end{bmatrix} + \begin{bmatrix}c\\c\end{bmatrix} : a,b,c \in \mathbb{R}\right\}$$

$$= \left\{a\begin{bmatrix}1\\-1\end{bmatrix} + b\begin{bmatrix}1\\0\end{bmatrix} + c\begin{bmatrix}1\\1\end{bmatrix} : a,b,c \in \mathbb{R}\right\}$$

thus $\operatorname{im}(T) = \operatorname{span}\left\{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$

Then vectors are not linear independent but ~~the list~~ are the basis for the image of T: $\left\{\begin{bmatrix}1\\-1\end{bmatrix}; \begin{bmatrix}1\\0\end{bmatrix}\right\}$

(*) Dimension of Kernel & Image:

$$\boxed{k = \dim(\ker(T)) + \dim(\operatorname{im}(T))}$$

$$P^{-1}AP = D \Leftrightarrow P(P^{-1}AP) = PD$$

• Eigenvector: Let $X = \begin{pmatrix}x_1\\x_2\end{pmatrix}$

$\Leftrightarrow P(P^{-1}AP)P^{-1} = PDP^{-1}$

$\Leftrightarrow (PP^{-1})A(PP^{-1}) = PDP^{-1}$

$\Leftrightarrow |A| = PDP^{-1} \Leftrightarrow A = PDP^{-1}$

Ex: let $A = \begin{bmatrix}4 & -2\\-1 & 3\end{bmatrix}$. Find $A^{100}$

$$P = \begin{bmatrix}1 & -2\\1 & 1\end{bmatrix} \to P^{-1} = \frac{1}{3}\begin{bmatrix}1 & 2\\-1 & 1\end{bmatrix}$$

$$P^{-1}AP = \frac{1}{3}\begin{bmatrix}1 & 2\\-1 & 1\end{bmatrix}\begin{bmatrix}4 & -2\\-1 & 3\end{bmatrix}\begin{bmatrix}1 & -2\\1 & 1\end{bmatrix}$$

$$= \begin{bmatrix}2 & 0\\0 & 5\end{bmatrix} = D$$

$$A^{100} = (PDP^{-1})^{100} = PD^{100}P^{-1}$$

$$D^{100} = \begin{bmatrix}2^{100} & 0\\0 & 5^{100}\end{bmatrix} \quad \boxed{P^{-1} = \frac{1}{ad-bc}\begin{bmatrix}d & -b\\-c & a\end{bmatrix}}$$

$$A^{100} = \begin{bmatrix}1 & -2\\1 & 1\end{bmatrix}\begin{bmatrix}2^{100} & 0\\0 & 5^{100}\end{bmatrix}\left(\frac{1}{3}\right)\begin{bmatrix}1 & 2\\-1 & 1\end{bmatrix}$$

(*) Diagonalize the matrix 2×2:

Ex: $\begin{bmatrix}6 & 10\\-2 & -3\end{bmatrix}$ $\det(A - \lambda I) = 0 \Rightarrow \lambda_1 = 1$ or $\lambda_2 = 2$

$\Rightarrow D = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$ $\quad A = PDP^{-1}$

for $\boxed{\lambda_1 = 1}$: $(A - \lambda I)\vec{x} = 0$

$$\Leftrightarrow \left(\begin{bmatrix}6 & 10\\-2 & -3\end{bmatrix} - (1)\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\right)\vec{x} = 0$$

$$\Rightarrow \begin{bmatrix}5 & 10\\-2 & -5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \to \begin{bmatrix}5 & 10 & 0\\-2 & -5 & 0\end{bmatrix}$$

$$\to \begin{bmatrix}1 & 2 & 0\\0 & 0 & 0\end{bmatrix} \to \begin{bmatrix}x_1\\x_2\end{bmatrix} = t\begin{bmatrix}-2\\1\end{bmatrix}$$

Same for $\lambda_2 = 2 \Rightarrow \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-5/2\\1\end{bmatrix}$

$$\Rightarrow P = \begin{bmatrix}-2 & -5/2\\1 & 1\end{bmatrix} \to P^{-1} = \begin{bmatrix}2 & 5\\-1 & -2\end{bmatrix}$$

$$A^n = (PDP^{-1})^n = PD^nP^{-1}$$

(*) Diagonalize the matrix 3×3:

Ex: $\begin{bmatrix}-1 & -19 & -4\\0 & -2 & 0\\0 & 15 & 3\end{bmatrix}$ $\det(A - \lambda I) = 0$; $\text{cofactor}_{ij} = (-1)^{i+j}M_{ij}$

$$\det\left(\begin{bmatrix}1-\lambda & -19 & -4\\0 & -2-\lambda & 0\\0 & 15 & 3-\lambda\end{bmatrix}\right) = 0\cdot(-1)^{2+1}M_{2,1} + (-2-\lambda)\cdot(-1)^{2+2}\begin{vmatrix}1-\lambda & -4\\0 & 3-\lambda\end{vmatrix} + 0\cdot(-1)^{2+3}M_{2,3} \Rightarrow D = \begin{bmatrix}-2 & 0 & 0\\0 & -1 & 0\\0 & 0 & 3\end{bmatrix}$$

$$= (-2-\lambda)(-1-\lambda)(3-\lambda) = 0$$

For $\lambda_1 = -2$: $(A - \lambda I)\vec{x} = 0 \Rightarrow \vec{x} = t\begin{bmatrix}-7/3\\-4/3\\1\end{bmatrix}$

$P \to P^{-1}$

**Ex:** Find the length of $\vec{u} = \begin{bmatrix}1\\3\\5\\2\end{bmatrix}$

By the properties of dot product $||\vec{u}||^2 = \vec{u}\cdot\vec{u}$

$\vec{u}\cdot\vec{u} = (1)(1)+(3)(3)+(5)(5)+(2)(2) = 39$
$= 1 + 9 + 25 + 4 = 39$

$||\vec{u}|| = \sqrt{39}$

$$\boxed{\cos\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\cdot||\vec{v}||}}$$

**Ex:** let $\vec{u} = [1, 1, 0]^T$ and $\vec{v} = [3\ 2\ 0]^T \in R^3$. Show that $\vec{w} = [4\ 5\ 0]^T$ B in span$\{\vec{u}, \vec{v}\}$

$\vec{w} = a\vec{u} + b\vec{v}$

$$\begin{bmatrix}4\\5\\0\end{bmatrix} = a\begin{bmatrix}1\\1\\0\end{bmatrix} + b\begin{bmatrix}3\\2\\0\end{bmatrix}$$

This is equivalent to the following system of equation.

$a + 3b = 4$
$a + 2b = 5$

$$\left[\begin{array}{cc|c}1 & 3 & 4\\1 & 2 & 5\end{array}\right] \to \left[\begin{array}{cc|c}1 & 0 & 7\\0 & 1 & -1\end{array}\right]$$

The solution is $a = 7$; $b = -1$. This means that $\vec{w} = 7\vec{u} - \vec{v}$. Therefore we can say $\vec{w}$ is in span $\{\vec{u}, \vec{v}\}$.

**Ex:** Find ker(A) for the matrix

$$A = \begin{bmatrix}1 & 2 & 1\\0 & -1 & 1\\2 & 3 & 3\end{bmatrix} \to \left[\begin{array}{ccc|c}1 & 2 & 1 & 0\\0 & -1 & 1 & 0\\2 & 3 & 3 & 0\end{array}\right]$$

$$\to \left[\begin{array}{ccc|c}1 & 0 & 3 & 0\\0 & 1 & -1 & 0\\0 & 0 & 0 & 0\end{array}\right] \Rightarrow \left\{\begin{bmatrix}-3t\\t\\t\end{bmatrix} t \in R\right\}$$

Therefore the null space of A is the span of this vector

$$\ker(A) = \text{span}\left\{\begin{bmatrix}-3\\1\\1\end{bmatrix}\right\}$$

$R^2$: $\vec{i} = \begin{bmatrix}1\\0\end{bmatrix}$, $\vec{j} = \begin{bmatrix}0\\1\end{bmatrix} \to \begin{bmatrix}a\\b\end{bmatrix} = a\vec{i} + b\vec{j}$

$R^3$: $\vec{i} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\vec{j} = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\vec{k} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$\to \begin{bmatrix}a\\b\\c\end{bmatrix} = a\vec{i} + b\vec{j} + c\vec{k}$

**Ex:** $\vec{v} = \begin{bmatrix}-1\\3\\2\end{bmatrix}$ is not a unit vector since $||\vec{v}|| = \sqrt{14}$

$\vec{u} = \frac{1}{\sqrt{14}}\vec{v}\begin{bmatrix}-1/\sqrt{14}\\3/\sqrt{14}\\2/\sqrt{14}\end{bmatrix}$

$||\vec{u}|| = \frac{1}{\sqrt{14}}||\vec{v}|| = \frac{1}{\sqrt{14}}\sqrt{14} = 1$

If $\vec{v}$ and $\vec{w}$ are nonzero that have the same direction: $\vec{v} = \frac{||\vec{v}||}{||\vec{w}||}\vec{w}$

. opposite direction: $\vec{v} = -\frac{||\vec{v}||}{||\vec{w}||}\vec{w}$

⊕ **A formula for $proj_{\vec{u}}\vec{v}$**

The defining properties of $\vec{v}_{\parallel}$ and $\vec{v}_{\perp}$

1. $\vec{v}_{\parallel}$ is parallel to $\vec{u}$
2. $\vec{v}_{\perp}$ is orthogonal to $\vec{u}$
3. $\vec{v}_{\parallel} + \vec{v}_{\perp} = \vec{v}$

$0 = \vec{v}_{\perp}\cdot\vec{u} = (\vec{v} - \vec{v}_{\parallel})\cdot\vec{u}$
$= (\vec{v} - t\vec{u})\cdot\vec{u} = \vec{v}\cdot\vec{u} - t(\vec{u}\cdot\vec{u})$

$\vec{v}_{\parallel} = \left(\frac{\vec{v}\cdot\vec{u}}{||\vec{u}||^2}\right)\vec{u}$ and $\vec{v}_{\perp} = \vec{v} - \vec{v}_{\parallel}$

Theorem:

1. $proj_{\vec{u}}\vec{v} = \left(\frac{\vec{v}\cdot\vec{u}}{||\vec{u}||^2}\right)\vec{u}$
2. $\vec{v} - \left(\frac{\vec{v}\cdot\vec{u}}{||\vec{u}||^2}\right)\vec{u}$ is orthogonal to $\vec{u}$

⊕ **Properties of the Dot Product:**

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $R^n$ and let $h \in R$.

1. $\vec{u}\cdot\vec{v}$ ±Z real number
2. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
3. $\vec{u}\cdot\vec{0} = 0$
4. $\vec{u}\cdot\vec{u} = ||\vec{u}||^2$
5. $(h\vec{u})\cdot\vec{v} = h(\vec{u}\cdot\vec{v}) = \vec{u}\cdot h\vec{v}$
6. $\vec{u}\cdot(\vec{v} + \vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\vec{w}$
   $\vec{u}\cdot(\vec{v} - \vec{w}) = \vec{u}\vec{v} - \vec{u}\vec{w}$

⊕ **Box Product:**

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \det\begin{bmatrix}u_1 & v_1 & w_1\\u_2 & v_2 & w_2\\u_3 & v_3 & w_3\end{bmatrix}$$

⊕ **Row & Column Space:**

- Column space: a space of columns.
- Row Space: a space of rows.

**Ex:** Find rank of matrix A, describe the column and row spaces.

$$A = \begin{bmatrix}1 & 2 & 1 & 3 & 2\\1 & 3 & 6 & 0 & 2\\3 & 7 & 8 & 6 & 6\end{bmatrix}$$

$$\xrightarrow{RREF}\begin{bmatrix}1 & 0 & -9 & 9 & 2\\0 & 1 & 5 & -3 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$\Rightarrow$ Rank (A) = 2

**Note:** The 1st two columns are the privot columns. All columns of the above RREF matrix are in span$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$

To construct the column space, we use the privot columns of the original matrix — in this case, the 1st and 2nd columns. Therefore the column space A is span$\left\{\begin{bmatrix}1\\1\\3\end{bmatrix}, \begin{bmatrix}2\\3\\7\end{bmatrix}\right\}$

The row space of A
span $\{[1\ 0\ -9\ 9\ 2], [0\ 1\ 5\ -3\ 0]\}$

**Notice:** - Column space are from the original matrix.
- Row space are from the RREF matrix

⊛ **Null Space: (Kernel of A)**

$$\boxed{\ker(A) = \{X : AX = 0\}}$$

**Ex:** Find ker(A) for matrix A

$$A = \begin{bmatrix}1 & 2 & 1\\0 & -1 & 1\\2 & 3 & 3\end{bmatrix} \to \left[\begin{array}{ccc|c}1 & 2 & 1 & 0\\0 & -1 & 1 & 0\\2 & 3 & 3 & 0\end{array}\right]$$

$$\xrightarrow{PREF}\left[\begin{array}{ccc|c}1 & 0 & -3 & 0\\0 & 1 & -1 & 0\\0 & 0 & 0 & 0\end{array}\right] \to \left\{\begin{bmatrix}3t\\t\\t\end{bmatrix} : t \in R\right\}$$

Therefore the null space of A is the span of this vector:

$$\ker(A) = \text{span}\left\{\begin{bmatrix}3\\1\\1\end{bmatrix}\right\}$$

⊛ **Nullity, the dimension of the null space of a matrix**

$$\boxed{\text{rank}(A) + \text{null}(A) = n}$$

⊕ **Tranformation:**

**Ex:** $T: R^3 \to R^4$ defined by

$$T = \begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}a+b\\b+c\\a-c\\c-b\end{bmatrix}$$ is a tranformation that transform the vector $\begin{bmatrix}1\\4\\7\end{bmatrix}$

in $R^3$ into the vector

$$T\begin{bmatrix}1\\4\\7\end{bmatrix} = \begin{bmatrix}1+4\\4+7\\1-7\\7-4\end{bmatrix} = \begin{bmatrix}5\\11\\-6\\3\end{bmatrix}$$

⊕ **Linear Tranformations:**

1. $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$
2. $T(a\vec{x}) = aT(\vec{x})$