Applied Linear Algebra

Final

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Question 1

1. Find eigenvalues and eigenvectors of the following matrices

a)
$$A = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}$$

b)
$$A = \begin{pmatrix} 6 & -3 \\ -2 & 1 \end{pmatrix}$$

(4)

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a) $A - \lambda I = \begin{pmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{pmatrix}$

$\Rightarrow \det(A - \lambda I) = (4-\lambda)(-3-\lambda) + 10 = 0$

$\Leftrightarrow -12 - 4\lambda + 3\lambda + \lambda^2 + 10 = 0$

$\Leftrightarrow \left[\begin{array}{l} \lambda_1 = 2 \\ \lambda_2 = -1 \end{array}\right.$ (Eigen value)

Now we find eigen vectors:

$N(A - \lambda_1 I) = \left(\begin{array}{cc|c} 2 & -5 & 0 \\ 2 & -5 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & -5/2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 - \frac{5}{2}x_2 = 0 \Rightarrow x_1 = \frac{5}{2}x_2$

$\Rightarrow \left[\begin{array}{l} x_1 = 5t/2 \\ x_2 = t \text{ (free variable)} \end{array}\right. \longrightarrow$ Eigen vector: $t \begin{bmatrix} 5/2 \\ 1 \end{bmatrix}$

$N(A - \lambda_2 I) = \left(\begin{array}{cc|c} 5 & -5 & 0 \\ 2 & -2 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\quad x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

$\Rightarrow \left[\begin{array}{l} x_1 = x_2 = t \\ x_2 = x_2 = t \text{ (free variable)} \end{array}\right. \longrightarrow$ Eigen vector $t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Question 2

2. Find eigenvalues and eigenvectors of the following matrix

$$A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ 1 & -2 & 0 \end{pmatrix}$$

(2)

$A - \lambda I = \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ 1 & -2 & -\lambda \end{bmatrix} \begin{matrix} (-2-\lambda) & 2 \\ 2 & (1-\lambda) \\ 1 & -2 \end{matrix}$

$\det(A - \lambda I) = (-2-\lambda)(1-\lambda)(-\lambda) + (-12) + 12 - (1-\lambda)(-3) - (-2)(-6)(-2-\lambda)$
$- (-\lambda)(2)(2)$

$$= (-2+\lambda+\lambda^2)(-\lambda) + 3 - 3\lambda - 12(-2-\lambda) + 4\lambda$$

$$= -\lambda^3 - \lambda^2 + 2\lambda + 3 - 3\lambda + 24 + 12\lambda + 4\lambda$$

$$= -\lambda^3 - \lambda^2 + 15\lambda + 27$$

. Let $\det(A - \lambda I) = 0$

$\Leftrightarrow \quad -\lambda^3 - \lambda^2 + 15\lambda + 27 = 0$

$\Leftrightarrow \quad \left[\begin{array}{l} \lambda_1 = -3 \\ \lambda_2 = 1 + \sqrt{10} \\ \lambda_3 = 1 - \sqrt{10} \end{array}\right.$ (Eigen values)

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Now we find eigen vectors:

$$N(A - \lambda_1 I) = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ 1 & -2 & 3 & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \quad \begin{array}{l} x_1 = 0 \\ x_2 = 3/2\, x_3 \\ x_3 = x_3 = t \end{array} \longrightarrow \begin{bmatrix} 0 \\ 3t/2 \\ t \end{bmatrix} \longrightarrow t \begin{bmatrix} 0 \\ 3/2 \\ 1 \end{bmatrix}$$

Eigen space   Eigen vector

$$N(A - \lambda_2 I) = \left[\begin{array}{ccc|c} -3-\sqrt{10} & 2 & -3 & 0 \\ 2 & -\sqrt{10} & -6 & 0 \\ 1 & -2 & -1-\sqrt{10} & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1+\sqrt{10}}{3} & 0 \\ 0 & 1 & \frac{2(1+\sqrt{10})}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{l} x_1 = -\frac{1+\sqrt{10}}{3}t \\ x_2 = -\frac{2(1+\sqrt{10})}{3}t \\ x_3 = x_3 = t \text{ (free variable)} \end{array}\right. \Rightarrow t \begin{bmatrix} -\frac{1+\sqrt{10}}{3} \\ -\frac{2(1+\sqrt{10})}{3} \\ 1 \end{bmatrix} \text{ (Eigen vector)}$$

$$N(A-\lambda_3 I) = \left[\begin{array}{ccc|c} -3+\sqrt{10} & 2 & -3 & 0 \\ 2 & +\sqrt{10} & -6 & 0 \\ 1 & -2 & -1+\sqrt{10} & 0 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{-1+\sqrt{10}}{3} & 0 \\ 0 & 1 & -\frac{2(-1+\sqrt{10})}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{l} x_1 = \frac{-1+\sqrt{10}}{3}t \\ x_2 = \frac{2(-1+\sqrt{10})}{3}t \\ x_3 = x_3 = t \text{ (free variable)} \end{array}\right. \Rightarrow t \begin{bmatrix} \frac{-1+\sqrt{10}}{3} \\ \frac{2(-1+\sqrt{10})}{3} \\ 1 \end{bmatrix} \text{ (Eigen vector)}$$

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Question 1:

1. Let matrix

$$A = \begin{pmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{pmatrix}$$

Find rank(A), nullity of A (dim Nul(A)).

(4)

$$A = \begin{pmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{pmatrix} \quad \begin{matrix} R_2 \leftarrow R_1 + R_2 \\ R_3 \leftarrow 5R_2 + R_3 \end{matrix}$$

$$A = \begin{pmatrix} 1 & -4 & 9 & -7 \\ 0 & -2 & 5 & -6 \\ 0 & 4 & -10 & 12 \end{pmatrix} \quad R_3 \leftarrow 2R_2 + R_3$$

Pivot

$$A = \begin{pmatrix} 1 & -4 & 9 & -7 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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2 pivots $\Rightarrow$ Rank $(A) = 2$ (Rank là số pivot columns

dim Nul $(A) = 2$ dim Nul là số cột k° có pivot)

Question 2:

2. Let $B = \{(8, 11, 0), (7, 0, 10), (1, 4, 6)\}$ be a basis of $\mathbb{R}^3$. Find the coordinate of vector $x = (3, 19, 2)$?

$$x = \begin{bmatrix} 3 \\ 19 \\ 2 \end{bmatrix} \qquad B = \left\{ \begin{bmatrix} 8 \\ 11 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 10 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} \right\}$$

$A\bar{x} = \bar{b}$ $\quad [x]_B$, $B = v_1, v_2, v_3$

$\alpha v_1 + \beta v_2 + \gamma v_3 = X$

$$\alpha \begin{bmatrix} 8 \\ 11 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 7 \\ 0 \\ 10 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 19 \\ 2 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} 8\alpha + 7\beta + \gamma = 3 \\ 11\alpha + 4\gamma = 19 \\ 10\beta + 6\gamma = 2 \end{cases}$$

$$\Rightarrow \begin{cases} \alpha = 1 \\ \beta = -1 \\ \gamma = 2 \end{cases}$$

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$$\Rightarrow [x]_B = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Question 3

3. Find the dimension and basic of the subspace

$$H = \left\{ \begin{pmatrix} a - 4b - 2c \\ 2a + 5b - 4c \\ -a + 2c \\ -3a + 7b + 6c \end{pmatrix}, a, b, c \in \mathbb{R} \right\}$$

(4)

$$H = \left\{ a \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix} + b \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix} + c \begin{bmatrix} -2 \\ -4 \\ 2 \\ 6 \end{bmatrix} \right\}$$

$$\Rightarrow H = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix} \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix} \begin{bmatrix} -2 \\ -4 \\ 2 \\ 6 \end{bmatrix} \right\}$$

Basis of H = basis of col(A) where

$$A = \begin{pmatrix} 1 & -4 & -2 \\ 2 & 5 & -4 \\ -1 & 0 & 2 \\ -3 & 7 & 6 \end{pmatrix} \quad \begin{matrix} R_2 \leftarrow (-2)R_1 + R_2 \\ R_3 \leftarrow R_1 + R_3 \\ R_4 \leftarrow 3R_1 + R_4 \end{matrix}$$

$$A = \begin{pmatrix} 1 & -4 & -2 \\ 0 & 13 & 0 \\ 0 & -4 & 0 \\ 0 & -5 & 0 \end{pmatrix} \quad \begin{matrix} R_3 \leftarrow 13/4\, R_3 + R_2 \\ R_4 \leftarrow 13/5\, R_4 + R_2 \end{matrix}$$

Pivot

$$A = \begin{pmatrix} 1 & -4 & -2 \\ 0 & 13 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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Pivot ở 2 cột đầu => basis là 2 cột đầu

$$\text{BASIS } \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix} \right\}$$

Since there are 2 vectors (2 rows)

=> DIM = 2

Question 4

4. Let $S = \{(1, -4), (3, -5)\}$, and $T = \{(-9, 1), (-5, -1)\}$ be bases in $\mathbb{R}^2$.

a. Find the change of coordinates matrix from T to S.

b. Let vector $v = (1, -2)$. Find the coordinate of v in the basis T then use the transition matrix at question a) to find the coordinate of v in the basis S.

$[v]_T$

$[v]_S$

(4)

$$S = \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \end{bmatrix} \right\} \qquad T = \left\{ \begin{bmatrix} -9 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \end{bmatrix} \right\}$$

a) Find $P_{T\to S}$

$$\begin{array}{c} \begin{array}{cccc} \quad S & & \quad T & \end{array} \\ \left(\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right) \end{array}$$

$$\Rightarrow \left(\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right)$$

$$\Rightarrow P_{T\to S} = \begin{pmatrix} 6 & 4 \\ -5 & -3 \end{pmatrix}$$

88 Quang Khưởng

b) Find $[v]_T$: $\left(\begin{array}{cc|c} -9 & -5 & 1 \\ 1 & -1 & -2 \end{array}\right)$

$$\left(\begin{array}{cc|c} 1 & 0 & -11/14 \\ 0 & 1 & 17/14 \end{array}\right)$$

$$\Rightarrow [v]_T = \begin{pmatrix} -11/14 \\ 17/14 \end{pmatrix}$$

. Find $[v]_S$: $[v_S] = P_{T\to S}\,[v_T]$

$$= \begin{pmatrix} 6 & 4 \\ -5 & -3 \end{pmatrix}\begin{pmatrix} -11/14 \\ 17/14 \end{pmatrix} = \begin{pmatrix} 1/7 \\ 2/7 \end{pmatrix}$$

Sample Test 1 (4)

Question 1

QUESTION 1 Find the inverse of matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 1 \\ 4 & -1 & 3 \end{bmatrix}$$

Nhớ c/m $\det \neq 0$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 4 & -1 & 3 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 \leftarrow (-2)R_1 + R_2 \\ R_3 \leftarrow (-4)R_1 + R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & -3 & -2 & 1 & 0 \\ 0 & -1 & -5 & -4 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_2 \leftarrow 2R_3 + R_2 \\ R_3 \leftarrow (-1)R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & -13 & -10 & 1 & 2 \\ 0 & 1 & 5 & 4 & 0 & -1 \end{array}\right) \quad R_2 \leftarrow \frac{-1}{13} R_2 \,;\; R_2 \leftrightarrow R_3$$

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$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 5 & 4 & 0 & -1 \\ 0 & 0 & 1 & \frac{10}{13} & \frac{-1}{13} & \frac{-2}{13} \end{array}\right) \quad \begin{array}{l} R_1 \leftarrow R_1 + (-2)R_3 \\ R_2 \leftarrow R_2 + (-5)R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -7/13 & 2/13 & 4/13 \\ 0 & 1 & 0 & 4/13 & 5/13 & -3/13 \\ 0 & 0 & 1 & \frac{10}{13} & \frac{-1}{13} & \frac{-2}{13} \end{array}\right) \Rightarrow A^{-1} = \begin{pmatrix} -7/13 & 2/13 & 4/13 \\ 4/13 & 5/13 & -3/13 \\ \frac{10}{13} & \frac{-1}{13} & \frac{-2}{13} \end{pmatrix}$$

Question 2

QUESTION 2 Find the rank and the base of column space and null space of matrix

$$A = \begin{bmatrix} 1 & 6 & 2 & -4 \\ -3 & 2 & -2 & -8 \\ 4 & -1 & 3 & 9 \end{bmatrix} \quad \begin{array}{l} R_1 \leftarrow R_1 + 6R_3 \\ R_2 \leftarrow 3R_1 + R_2 \\ R_3 \leftarrow (-4)R_1 + R_3 \end{array}$$

$$\begin{pmatrix} 25 & 0 & 20 & 50 \\ 0 & 20 & 4 & -20 \\ 0 & -25 & -5 & 25 \end{pmatrix} \quad \begin{array}{l} R_1 \leftarrow 1/5\, R_1 \\ R_2 \leftarrow 1/4\, R_2 \\ R_3 \leftarrow -1/5\, R_3 \end{array}$$

$$\begin{pmatrix} 5 & 0 & 4 & 10 \\ 0 & 5 & 1 & -5 \\ 0 & 5 & 1 & -5 \end{pmatrix} \quad \begin{matrix} R_1 \leftarrow \frac{1}{5} R_1 \\ R_2 \leftarrow \frac{1}{5} R_2 \\ R_3 \leftarrow R_2 - R_3 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 4/5 & 2 \\ 0 & 1 & 1/5 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Pivot column

$\Rightarrow$ $\text{Rank}(A) = 2$

Basis of A: $\left\{ \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 2 \\ -1 \end{pmatrix} \right\}$

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. We solve for $A = 0 \Leftrightarrow \begin{cases} x_1 + \frac{4}{5}x_3 + 2x_4 = 0 \\ x_2 + \frac{1}{5}x_3 - x_4 = 0 \\ x_3 = t \\ x_4 = s \end{cases}$ (free variable)

$$\Leftrightarrow \begin{cases} x_1 = -\frac{4}{5}t - 2s \\ x_2 = -\frac{1}{5}t + s \\ x_3 = t \\ x_4 = s \end{cases}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4/5 \\ -1/5 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} s$$

. $\text{nul}(A) = \begin{pmatrix} -\frac{4}{5}t - 2s \\ -\frac{1}{5}t + s \\ t \\ s \end{pmatrix}$. A basis of $\text{nul}(A) = \left\{ \begin{pmatrix} -4/5 \\ -1/5 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Question 3

QUESTION 3 Let $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$

(a) Find the eigenvalues and eigenvectors of A

(b) Find nonsingular matrix P and diagonal matrix D such that $A = PDP^{-1}$

a) $A - \lambda I = \begin{pmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{pmatrix}$

$\Rightarrow \det(A - \lambda I) = (4-\lambda)(-1-\lambda) - 2(-3) = 0$

$= -4 - 4\lambda + \lambda + \lambda^2 + 6 = 0$

$= \lambda^2 - 3\lambda + 2$

Let $\det(A - \lambda I) = 0 \Rightarrow \left[\begin{array}{l} \lambda_1 = 2 \\ \lambda_2 = 1 \end{array}\right.$ (Eigen values)

. Now we find eigen vectors:

$N(A - \lambda_1 I) = \left(\begin{array}{cc|c} 2 & -3 & 0 \\ 2 & -3 & 0 \end{array}\right)$ $R_1 \leftarrow \frac{1}{2} R_1$, $R_2 \leftarrow R_1 - R_2$

$\Rightarrow \left(\begin{array}{cc|c} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \begin{cases} x_1 = 3t/2 \\ x_2 = t \text{ (free variable)} \end{cases}$

$\Rightarrow$ Eigen vector : $t \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$ with $\lambda_1 = 2$

$N(A - \lambda_2 I) = \left(\begin{array}{cc|c} 3 & -3 & 0 \\ 2 & -2 & 0 \end{array}\right)$ $R_1/3$, $R_2/2 - R_1/3$

$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \begin{cases} x_1 = x_2 = t \\ x_2 = t \text{ (free variable)} \end{cases}$

$\Rightarrow$ Eigen vector $t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with $\lambda_2 = 1$

88 Quang Khương

. Find D & P with $A = PDP^{-1}$

$\lambda_1 = 2 \quad \lambda_2 = 1$

Ans $D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad P = \begin{pmatrix} 3/2 & 1 \\ 1 & 1 \end{pmatrix}$

Eigen value

Verify:

. Since P has inverse $\Rightarrow \det(P) = 1/2 \neq 0$

$$\left(\begin{array}{cc|cc} 3/2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \quad \begin{array}{l} R_1 \leftarrow 2/3\, R_1 \\ R_3 \leftarrow (-3/2) R_2 + R_1 \end{array}$$

$$\left(\begin{array}{cc|cc} 1 & 2/3 & 2/3 & 0 \\ 0 & -1/2 & 1 & -3/2 \end{array}\right) \quad R_2 \leftarrow (-2) R_2$$

$$\left(\begin{array}{cc|cc} 1 & 2/3 & 2/3 & 0 \\ 0 & 1 & -2 & 3 \end{array}\right) \quad -2/3\, R_2 + R_1$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & 1 & -2 & 3 \end{array}\right) \Rightarrow P^{-1} = \begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix}$$

$$PDP^{-1} = \begin{pmatrix} 3/2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix} = A$$

. Find A^{1000}

. $A^{1000} = P D^{1000} P^{-1} = \begin{pmatrix} 3/2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^{1000} & 0 \\ 0 & 1^{1000} \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix}$

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Sample Test 2 (4)

Question 1:

Question 1. (25 marks) Solve the linear system of equations

$$\begin{cases} 2x_1 + x_2 + 3x_3 = 7 \\ x_1 - x_2 + x_3 = 0 \\ x_1 + 4x_2 - 2x_3 = 7 \end{cases}$$

by the following steps

(a) Write the system in matrix form $Ax = b$

(b) Find the inverse of matrix A

(c) Solve x by $x = A^{-1}b$

$$Ax = b$$

$$\Rightarrow x = b \cdot \frac{1}{A}$$

$$= b \cdot A^{-1}$$

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a) $$\begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 7 \end{pmatrix}$$

$\det \neq 0$

b) $$A = \left(\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & 4 & -2 & 0 & 0 & 1 \end{array}\right)$$

$R_1 \leftarrow R_1 + R_2$

$R_2 \leftarrow (-2)R_2 + R_1$

$R_3 \leftarrow R_2 - R_3$

$$\left(\begin{array}{ccc|ccc} 3 & 0 & 4 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 & -2 & 0 \\ 0 & -5 & 3 & 0 & 1 & -1 \end{array}\right)$$

$R_3 \leftarrow R_3 \times \left(\frac{3}{5}\right) + R_2$

$$\left(\begin{array}{ccc|ccc} 3 & 0 & 4 & 1 & 1 & 0 \\ 0 & 3 & 1 & 1 & -2 & 0 \\ 0 & 0 & \frac{14}{5} & 1 & -\frac{7}{5} & -\frac{3}{5} \end{array}\right)$$

$1/3\, R_1$

$1/3\, R_2$

$5/14\, R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 4/3 & 1/3 & 1/3 & 0 \\ 0 & 1 & 1/3 & 1/3 & -2/3 & 0 \\ 0 & 0 & 1 & 5/14 & -1/2 & -3/14 \end{array}\right)$$

$R_1 - 4/3\, R_3$

$R_2 - 1/3\, R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/7 & 1 & 2/7 \\ 0 & 1 & 0 & 3/14 & -1/2 & 1/14 \\ 0 & 0 & 1 & 5/14 & -1/2 & -3/14 \end{array}\right)$$

$$\Rightarrow A^{-1} = \begin{pmatrix} -1/7 & 1 & 2/7 \\ 3/14 & -1/2 & 1/14 \\ 5/14 & -1/2 & -3/14 \end{pmatrix}$$

c) $$x = A^{-1} b = \begin{pmatrix} -1/7 & 1 & 2/7 \\ 3/14 & -1/2 & 1/14 \\ 5/14 & -1/2 & -3/14 \end{pmatrix} \begin{pmatrix} 7 \\ 0 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

Question 2

Question 2. (25 marks) Find the base of column space and null space of matrix

$$A = \begin{array}{c} \begin{bmatrix} 1 & 2 & 4 & 1 & 2 \\ 2 & 1 & 2 & 2 & 1 \\ 1 & 4 & 2 & 1 & 1 \\ 3 & 3 & 6 & 3 & 3 \end{bmatrix} \end{array} \begin{array}{l} R_1 + (-2) R_2 \\ R_2 + (-2) R_1 \\ R_3 + (-1) R_1 \\ R_4 + (-3) R_1 \end{array}$$

$$\begin{pmatrix} -3 & 0 & 0 & -3 & 0 \\ 0 & -3 & -6 & 0 & -3 \\ 0 & 2 & -2 & 0 & -1 \\ 0 & -3 & -6 & 0 & -3 \end{pmatrix} \begin{array}{l} -1/3\, R_1 \\ -1/3\, R_2 \\ 1/2\, R_3 \\ R_2 - R_4 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \\ \\ 2R_3 + R_2 \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \\ \\ 1/3\, R_3 \\ \\ \end{array}$$

88 Quang Khương

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \\ R_2 \leftrightarrow R_3 \\ \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \\ \\ R_2 - R_3 \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{l} \\ \\ -1/2\, R_3 \\ \\ \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ Basis $\{a_1, a_2, a_3\} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 2 \\ 6 \end{pmatrix} \right\}$

To find null space, we solve matrix A

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

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$$\begin{cases} x_1 + x_4 = 0 \\ x_2 = 0 \\ x_3 + 1/2 x_5 = 0 \\ x_4 = t \\ x_5 = s \end{cases} \quad \text{(free variable)}$$

$$\Rightarrow \begin{cases} x_1 = -t \\ x_2 = 0 \\ x_3 = 1/2\, s \\ x_4 = t \\ x_5 = s \end{cases}$$

$\Rightarrow$ nul $A = \begin{pmatrix} -t \\ 0 \\ 1/2\, s \\ t \\ s \end{pmatrix}$. Basis of nul$(A)$ = $\left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 0 \\ 1 \end{pmatrix} \right\}$

Question 3:

Question 4. (25 marks) Let $A = \begin{bmatrix} 7 & 3 \\ 0 & 4 \end{bmatrix}$

(a) Find all eigenvalues and eigenvectors of A.

(b) Find nonsingular matrix P and diagonal matrix D such that $A = PDP^{-1}$.

(c) Use the formula $A^{20} = PD^{20}P^{-1}$ to compute A^{20}.

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a) $A - \lambda I = \begin{pmatrix} 7-\lambda & 3 \\ 0 & 4-\lambda \end{pmatrix}$

$\det(A - \lambda I) = (7-\lambda)(4-\lambda) - 0 \times 3$

$= 28 - 7\lambda - 4\lambda + \lambda^2$

Let $\det(A - \lambda I) = 0$

$\Leftrightarrow \lambda^2 - 11\lambda + 28 = 0$

$\Rightarrow \left[\begin{array}{l} \lambda_1 = 7 \\ \lambda_2 = 4 \end{array}\right.$ (Eigen values)

. Now we find eigen vectors

$N(A - \lambda_1 I) = \left(\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & -3 & 0 \end{array}\right) \begin{array}{l} 1/3 R_1 \\ -1/3 R_2 \end{array}$

$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \begin{cases} x_1 = t \\ x_2 = 0 \end{cases}$

$\Rightarrow$ Eigenvector $t\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with $\lambda_1 = 7$

$N(A - \lambda_2 I) = \left(\begin{array}{cc|c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \; 1/3 R_1$

$\Rightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} x_1 = -t \\ x_2 = t \text{ (free variable)} \end{cases}$

$\Rightarrow$ Eigenvector $t\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ with $\lambda_2 = 4$

b) . Find D & P with $A = PDP^{-1}$

Ans $D = \begin{pmatrix} 7 & 0 \\ 0 & 4 \end{pmatrix}$ $\qquad$ $P = \begin{array}{cc} \lambda_1 = 7 & \lambda_2 = 4 \end{array} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

Eigen value $\qquad$ Eigen vector

Verify:

. Since P has inverse $\Rightarrow \det(P) = 1 \neq 0$

$$\left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right) \quad R_1 + R_2$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right)$$

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$$\Rightarrow P^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow PDP^{-1} = A$$

. Find $A^{20} = PD^{20}P^{-1}$

$$= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 7^{20} & 0 \\ 0 & 4^{20} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7^{20} & -4^{20} \\ 0 & 4^{20} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7^{20} & 7^{20} - 4^{20} \\ 0 & 4^{20} \end{pmatrix}$$

Question 50

Score on last try: **1 of 1 pts.** See Details for more.

Next question | Get a similar question

(4)

Find an orthogonal basis for the space spanned by $\underset{\vec{v_1}}{\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}}$, $\underset{\vec{v_2}}{\begin{bmatrix}-2\\0\\-2\\0\end{bmatrix}}$, and $\underset{\vec{v_3}}{\begin{bmatrix}-4\\2\\-6\\-4\end{bmatrix}}$.

Orthogonal basis : u

Orthonormal basis : e

Vector projection of $\vec{v}$ onto $\vec{u}$: $proj_{\vec{u}}(\vec{v})$

$$\frac{\vec{v}\cdot\vec{u}}{|\vec{u}|^2}\cdot\vec{u}$$

. Using Gram - Schmidt process

$$\begin{cases} \vec{u_1} = \vec{v_1} = (0; 1; 0; -1) \\ \vec{e_1} = \dfrac{\vec{u_1}}{|\vec{u_1}|} = \left(0; \dfrac{\sqrt{2}}{2}; 0; \dfrac{-\sqrt{2}}{2}\right) \end{cases}$$

$$\begin{cases} \vec{u_2} = \vec{v_2} - proj_{\vec{u_1}}(\vec{v_2}) \\ \Rightarrow \vec{u_2} = (-2; 0; -2; 0) \\ \vec{e_2} = \dfrac{\vec{u_2}}{|\vec{u_2}|} = \left(\dfrac{-\sqrt{2}}{2}; 0; \dfrac{-\sqrt{2}}{2}; 0\right) \end{cases}$$

$$\begin{cases} \vec{u_3} = \vec{v_3} - proj_{\vec{u_1}}(\vec{v_3}) - proj_{\vec{u_2}}(\vec{v_3}) = (1; -1; -1; -1) \\ \vec{e_3} = \dfrac{\vec{u_3}}{|\vec{u_3}|} = \left(\dfrac{1}{2}; \dfrac{-1}{2}; \dfrac{-1}{2}; \dfrac{-1}{2}\right) \end{cases}$$

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$\Rightarrow$ Orthogonal basis / vector : $\left\{ \underset{u_1}{\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}}, \underset{u_2}{\begin{pmatrix}-2\\0\\-2\\0\end{pmatrix}}, \underset{u_3}{\begin{pmatrix}1\\-1\\-1\\-1\end{pmatrix}} \right\}$

$\Rightarrow$ Orthonormal basis / vector $\{(e_1), (e_2); (e_3)\}$

Các dạng phụ (1)

1) Determine whether $W = \{(x, 2x) : x \text{ is a real number}\}$ with standard operations in $\mathbb{R}^2$ is a vector space

1. $u + v$ is in V (closure under addition)

$$(r, 2r) + (s, 2s) = (r + s, 2r + 2s)$$
$$= [\underbrace{r+s}_{x}, \underbrace{2(r+s)}_{2x}] \checkmark$$

2. $u + v = v + u$ (commutativity of addition)

$$u + v = (r, 2r) + (s, 2s) = (r + s, 2r + 2s)$$
$$= (s + r, 2s + 2r)$$
$$= (s, 2s) + (r, 2r) = v + u \checkmark$$

3. $u + (v + w) = (u + v) + w$ (Associativity of addition)

$$(r, 2r) + \big((s, 2s) + (t, 2t)\big) = (r, 2r) + (s + t, 2s + 2t)$$
$$= (r + s + t, 2r + 2s + 2t)$$
$$= \big((r+s) + t, (2r + 2s) + 2t\big)$$
$$= (r + s, 2r + 2s) + (t, 2t)$$
$$= \big((r, 2r) + (s, 2s)\big) + (t, 2t) \checkmark$$

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4. V has a vector 0 such that : $u + 0 = 0 + u = u$ (additive identity)

$(0, 0) = (0, 2(0))$ is in W

$$(0, 0) + (r, 2r) = (r, 2r) = (r, 2r) + (0, 0) \checkmark$$

5. For every vector u in V, there is a vector $-u$ in V such that :

$u + (-u) = 0$ (additive inverse)

$$(r, 2r) + (-r, -2r) = (r - r, 2r - 2r)$$
$$= (0, 0)$$

So $(-r, -2r) = -(r, 2r) \checkmark$

6. cu in in V (closure under scalar multiplication)

$$c(r, 2r) = (cr, c2r)$$
$$= (\underbrace{cr}_{x}, \underbrace{2(cr)}_{2x})$$

$\Rightarrow$ Axiom hold

7. $c(u+v) = cu + cv$ (distributive properties)

$$c\left[(r, 2r) + (s, 2s)\right] = c(r+s, 2r+2s)$$
$$= (c(r+s), c(2r+2s))$$
$$= (cr + cs, 2cr + 2cs)$$
$$= (cr, 2cr) + (cs, 2cs)$$
$$= c(r, 2r) + c(s, 2s) \checkmark$$

$\Rightarrow$ Seventh axiom hold

8. $(c+d)u = cu + du$ (distributive properties)

$$(c+d)(r, 2r) = ((c+d)r, (c+d)2r)$$
$$= (cr + dr, c2r + d2r)$$
$$= (cr, 2cr) + (dr, 2dr)$$
$$= c(r+2r) + d(r+2r)$$

$\Rightarrow$ Eighth axiom hold $\checkmark$

9. $c(du) = (cd)u$ (associative property)

$$c(d(r, 2r)) = c(dr, 2dr)$$
$$= (cdr, 2cdr)$$
$$= cd(r, 2r) \checkmark$$

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10) $1(u) = u$ (multiplication - identity)

$$1(r, 2r) = (1r, 1\,2r)$$
$$= (r, 2r) \checkmark$$

$\Rightarrow$ W is a vector space

'a" to make the matrix orthogonal

(2)

$$A = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \\ \sqrt{2}/2 & a & 0 \end{bmatrix}$$

To let it be, $A \times A^T = I$

$$\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 1 \\ \sqrt{2}/2 & a & 0 \end{bmatrix} \times \begin{bmatrix} \sqrt{2}/2 & 0 & \sqrt{2}/2 \\ -\sqrt{2}/2 & 0 & a \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Consider (1, 3)

$$\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{\sqrt{2}}{2} a\right) + 0 = 0$$

$$\Rightarrow a = \frac{\sqrt{2}}{2}$$

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Determine whether the set $\mathbb{R}^2$, along with standard operations

. $(x_1, y_1) + (x_2, y_2) = (x_1 x_2, y_1 y_2)$

. $c(x_1, y_1) = (cx_1, cy_1)$

(1)

1. $u + v$ is in V

$$(x_1, y_1) + (x_2, y_2) = (x_1 x_2, y_1 y_2)$$

is in $\mathbb{R}^2$ ✓

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2. $u + v = v + u$

$$(x_1, y_1) + (x_2, y_2) = (x_1 x_2, y_1 y_2) = (x_2 x_1, y_2 y_1) = (x_2, y_2) + (x_1, y_1)$$ ✓

3. $u + (v + w) = (u + v) + w$

$$(x_1, y_1) + ((x_2, y_2) + (x_3, y_3)) = (x_1, y_1) + (x_2 x_3, y_2 y_3) = (x_1 x_2 x_3, y_1 y_2 y_3) = (x_1 x_2, y_1 y_2) + (x_3, y_3) = ((x_1, y_1) + (x_2, y_2)) + (x_3, y_3)$$

$\Rightarrow$ Third axiom hold

4. V has a vector 0 such that

$u + 0 = 0 + u = u$ $\quad\quad \vec{0} = (1, 1)$

$$(x_1, y_1) + (1, 1) = (x_1 . 1, y_1 1) = (x_1, y_1)$$

$$(1, 1) + (x_1, y_1) = (1x_1, 1y_1) = (x_1, y_1)$$ ✓

5. For every vector u in V, there is a vector $-u$ in V such that :

$u + (-u) = 0$ (additive inverse)

$x_1 \neq 0$, $y_1 \neq 0$

$$(x_1, y_1) + \left(\frac{1}{x_1}, \frac{1}{y_1}\right) = \left(x_1 . \frac{1}{x_1}, y_1 . \frac{1}{y_1}\right) = (1, 1) = \vec{0}$$

$$-(x_1, y_1) = \left(\frac{1}{x_1}, \frac{1}{y_1}\right)$$

But $(0,0)$ is in $\mathbb{R}^2$ and has no addtive inverse $\Rightarrow$ Fifth axiom failed

$\Rightarrow$ The set is not vector space

Prove $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x_3 = x_1 + x_2 \right\}$ is a subspace of $\mathbb{R}^3$

Need to show

(1) $V \neq 0$

(2) $\vec{x}, \vec{y} \in V \Rightarrow \vec{x} + \vec{y} \in V$

(3) $\alpha \in \mathbb{R}, \vec{x} \in V \Rightarrow \alpha \vec{x} \in V$

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Proof: (1) $0 = 0 + 0$, $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in V$, so $V \neq 0$

(2) Suppose $\vec{x}, \vec{y} \in V$

$$\vec{x} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \vec{y} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$ and $\begin{cases} x_3 = x_1 + x_2 \\ y_3 = y_1 + y_2 \end{cases}$

$$\vec{x} + \vec{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix} \quad \text{and} \quad \begin{aligned} x_3 + y_3 &= (x_1 + x_2) + (y_1 + y_2) \\ &= (x_1 + y_1) + (x_2 + y_2) \end{aligned}$$

so $\vec{x} + \vec{y} \in V$

(3) Take any $\vec{x} \in V$ and $\alpha \in \mathbb{R}$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$x_1, x_2, x_3 \in \mathbb{R}$ and $x_3 = x_1 + x_2$

$$\alpha \vec{x} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{pmatrix} \quad \text{and} \quad \begin{aligned} \alpha x_3 &= \alpha (x_1 + x_2) \\ &= \alpha x_1 + \alpha x_2 \end{aligned}$$

, so $\alpha \vec{x} \in V$

$\therefore$ V is a subspace

Determine whether $W = \{(a,b,c) \mid a = b^2\}$ is a subspace of $\mathbb{R}^3$

Need to show

(1) $V \neq 0$

(2) $\vec{x}, \vec{y} \in V \Rightarrow \vec{x} + \vec{y} \in V$

(3) $\alpha \in \mathbb{R}$, $\vec{x} \in V \Rightarrow \alpha \vec{x} \in V$

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(1) $(0,0,0) \in W$ because $0 = 0^2$ $(a = b^2)$

(2) $\underset{a}{(4}, \underset{b}{2}, \underset{c}{1}) \in W$ $\quad 4 = 2^2$

$\underset{a}{(9}, \underset{b}{3}, \underset{c}{1}) \in W$ $\quad 9 = 3^2$

Then $(4,2,1) + (9,3,1) = (13,5,2)$

Since $13 \neq 5^2 \Rightarrow$ Not closed under vector addition

$\therefore$ Not a subspace

Determine whether $W = \{(x,y,z) \mid x^2 + y^2 + z^2 \leq 1\}$ is a subspace of $\mathbb{R}^3$

(1) $0^2 + 0^2 + 0^2 \leq 1$

$(0,0,0) \in W$, $W \neq \emptyset$

(2) $(1,0,0) \in W$ b/c $1^2 + 0^2 + 0^2 \leq 1$

$(0,1,0) \in W$ b/c $0^2 + 1^2 + 0^2 \leq 1$

$(1,0,0) + (0,1,0) = (1,1,0)$

But $1^2 + 1^2 + 0^2 = 2 > 1$, so $(1,1,0) \notin W \Rightarrow$ Not closed under vector addition

$\Rightarrow$ Not subspace

Prove $W = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; a = -b \right\}$ is a subspace of $\mathbb{R}^2$

(1)

Need to show

(1) $V \neq 0$

(2) $\vec{x}, \vec{y} \in V \Rightarrow \vec{x} + \vec{y} \in V$

(3) $\alpha \in \mathbb{R}, \vec{x} \in V \Rightarrow \alpha \vec{x} \in V$

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(1) $0 = -0$, so $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in W$, so $W \neq \emptyset$

(2) Spse, $x, y \in W$

$$X = \begin{pmatrix} a \\ b \end{pmatrix}, \quad Y = \begin{pmatrix} c \\ d \end{pmatrix} \text{ with } a = -b \text{ and } c = -d$$

$$X + Y = \begin{pmatrix} a + c \\ b + d \end{pmatrix} = \begin{pmatrix} -b - d \\ b + d \end{pmatrix} = \begin{pmatrix} -(b+d) \\ b + d \end{pmatrix}$$

So $x + y \in W$

(3) Spse $\alpha \in \mathbb{R}$, $x \in W$

$$\alpha x = \alpha \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix} \text{ with } \alpha a = -\alpha b$$

So $\alpha \vec{x} \in W$

$\therefore$ W is a subspace of $\mathbb{R}^2$

Determine $W = \{ (a, b, c) \mid a \leq b \leq c \}$ is a subspace of $\mathbb{R}^3$

(1) $(0, 0, 0) \in W$ b/c $0 \leq 0 \leq 0$

(2) $(1, 2, 3) \in W$ b/c $1 \leq 2 \leq 3$

and $-1 \in \mathbb{R}$

Then $-1(1, 2, 3) = (-1, -2, -3)$

but $-1 < -2 < -3$ is false not closed under vector scalar multiplication

$\Rightarrow$ Not subspace

Linear Combination

(2)

Write w as a linear combination of u & v

$$u = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix} \qquad V = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \end{bmatrix} \qquad w = \begin{bmatrix} -11 \\ 4 \\ -2 \\ -9 \end{bmatrix}$$

$$\begin{array}{c} \begin{array}{ccc} u & v & w \end{array} \\ \left[\begin{array}{cc|c} 1 & 3 & -11 \\ 1 & 0 & 4 \\ 2 & 2 & -2 \\ -1 & 1 & -9 \end{array}\right] \end{array} \xrightarrow[R_4 + R_2]{R_3 - 2R_2} \left[\begin{array}{cc|c} 1 & 3 & -11 \\ 1 & 0 & 4 \\ 0 & 2 & -10 \\ 0 & 1 & -5 \end{array}\right] \xrightarrow[R_3/2]{R_1 - R_2} \left[\begin{array}{cc|c} 0 & 3 & -15 \\ 1 & 0 & 4 \\ 0 & 1 & -5 \\ 0 & 1 & -5 \end{array}\right]$$

$$\xrightarrow[R_1/3]{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -5 \\ 0 & 1 & -5 \\ 0 & 1 & -5 \end{array}\right] \xrightarrow[R_4 - R_2]{R_3 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow \quad x = 4$

$\quad\quad y = -5$

$\Rightarrow \quad 4u + (-5)\,v = w$

$$\Rightarrow \quad 4\begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix} + (-5)\begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -11 \\ 4 \\ -2 \\ -9 \end{bmatrix}$$

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If possible, write $\vec{v} = (-1, 7, 2)$ as a linear combination of $\vec{u_1} = (1,3,5)$
$u_2 = (2,-1,3)$ and $\vec{u_3} = (-3, 2, -4)$

$$c_1 . \vec{u_1} + c_2 . \vec{u_2} + c_3 . \vec{u_3} = \vec{v}$$

$$c_1(1,3,5) + c_2(2,-1,3) + c_3(-3,2,-4) = (-1,7,2)$$

$$c_1 + 2c_2 - 3c_3 = -1$$

$$3c_1 - c_2 + 2c_3 = 7$$

$$5c_1 + 3c_2 - 4c_3 = 2$$

$$\begin{array}{c} \begin{array}{cccc} \vec{u_1} & \vec{u_2} & \vec{u_3} & \vec{v} \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & -3 & -1 \\ 3 & -1 & 2 & 7 \\ 5 & 3 & -4 & 2 \end{array}\right] \\ \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \end{array} \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 0 & 1/7 & 0 \\ 0 & 1 & -11/7 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad 0 = 1$$

$\therefore$ $\vec{v}$ can not be expressed as a linear comb

Spanning Vector

Let $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, does $\{v_1, v_2\}$ span $\mathbb{R}^2$

(Let $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ Can we write $x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$?)

$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \overset{Aug}{\Rightarrow}$

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & -2 & b \end{array}\right] \quad R_2 \leftarrow R_2 - R_1$$

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & -3 & b-a \end{array}\right] \quad R_2 \leftarrow -\frac{1}{3} R_2$$

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & 1 & \frac{a-b}{3} \end{array}\right] \quad R_1 \leftarrow R_1 - R_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & a - \frac{a-b}{3} \\ 0 & 1 & \frac{a-b}{3} \end{array}\right]$$

$$\Rightarrow \boxed{\begin{aligned} x_1 &= a - \frac{a-b}{3} \\ x_2 &= \frac{a-b}{3} \end{aligned}}$$

Yes, these span $\mathbb{R}^2$

(2)

$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

$\det = (1 \times (-2)) - 1 \times 1$

$= -2 - 1 = -3 \neq 0$

$\Rightarrow$ Span

Let $v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

(2)

Does v_1, v_2 span $\mathbb{R}^2$?

$$\left[\begin{array}{cc|c} 1 & -2 & a \\ -2 & 4 & b \end{array}\right] \quad R_2 \leftarrow 2R_1 + R_2$$

$$\left[\begin{array}{cc|c} 1 & -2 & a \\ 0 & 0 & 2a+b \end{array}\right]$$

$\Rightarrow \quad 0 = 2a + b$

Not true $\forall\, a, b$

$\Rightarrow$ No solution

$\Rightarrow$ Vectors do not span $\mathbb{R}^2$

Can $(-1, 0, 0)$ $(2, 0, 0)$ $(0, 0, 3)$ span $\mathbb{R}^3$

$$x_1 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 2 & 0 & a \\ 0 & 0 & 0 & b \\ 0 & 0 & 3 & c \end{array}\right]$$

$0 = b$ Not true $\forall\, b$

$\Rightarrow$ No sol

$\Rightarrow$ Vectors do not span $\mathbb{R}^3$

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Mẹo : det $= 0 \Rightarrow$ k° Span

det $\neq 0 \Rightarrow$ Span

Vector independence

$$\begin{array}{ccc} v_1 & v_2 & v_3 \\ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} & \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \end{array}$$

$$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right]$$

(2)

$$\xrightarrow{R_2/2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 + R_2 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

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$$\xrightarrow{R_3/3} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\begin{cases} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{cases}$$

1 2

$$\begin{cases} x_1 = 1 \\ x_2 = \underline{s} \\ x_3 = \underline{3s} \end{cases}$$

3 6

=> Linearly Independent

Bậc thang → det ≠ 0 → L. In

→ det = 0 → L. De

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & 4 & 5 & 8 \\ 0 & 0 & 6 & 9 \end{array}\right] \Rightarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 7 \\ 4x_2 + 5x_3 = 8 \\ 6x_3 = 9 \end{cases}$$

$$\det \neq 0$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & 4 & 5 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\det = 0$$

7. For which values of t is each set linearly independent?

$$S = \{(t,1,1),(1,t,1),(1,1,t)\}$$

$$\begin{bmatrix} t & 1 & 1 \\ 1 & t & 1 \\ 1 & 1 & t \end{bmatrix} \neq 0$$

88 Quang Khương

(2)

$\Rightarrow$ $\det \neq 0$

$\Leftrightarrow$ $t(t^2-1) - (t-1) + (1-t) \neq 0$

$\Leftrightarrow$ $t^3 - t - t + 1 - t + 1 \neq 0$

$\Leftrightarrow$ $t^3 - 3t + 2 \neq 0$

$\Rightarrow$ $\begin{cases} t \neq -2 \\ t \neq 1 \end{cases}$

$\{1, 2, 3, \ldots\}$

$0x = 0$

Vô số nghiệm $\quad 0 \; 0 \; 0 \mid 0$

Vô nghiệm $\quad 0 \; 0 \; 0 \mid \neq 0$

$0x = 3$

$\Rightarrow x = \dfrac{3}{0}$

THE INTERNATIONAL UNIVERSITY - VIETNAM NATIONAL UNIVERSITY - HCMC

FINAL EXAMINATION

Semester 2, 2022-2023 • Date: June 07, 2023 • Duration: 75 minutes

SUBJECT: Applied Linear Algebra

Department of Mathematics Vice Chair:	Lecturer
Dr. Nguyen Minh Quan	Dr. Le Minh Tuan

INSTRUCTIONS: Each student is allowed a scientific calculator and a maximum of TWO double-sided sheets of reference material (size A4 or similar) marked with their name and ID. ***All other documents and electronic devices are forbidden.***

Solve each of the following problems. Show your work clearly. **You must write out all relevant steps. Simply having the correct answer does not give you credit.**

Problem 1 (20 points).
Evaluate the following determinant:
$$\begin{vmatrix} 1 & -3 & 2 & 0 \\ -3 & -1 & 0 & -2 \\ 2 & 1 & 3 & 1 \\ 2 & 0 & -3 & 0 \end{vmatrix}.$$

Mid: Thay đổi Row $\times (-1)$

Problem 2 (20 points).
Find all real numbers λ such that the homogeneous system:
$$\underbrace{\left(\lambda\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}2 & 3\\2 & -3\end{bmatrix}\right)}_{\det} x = 0$$
has a nontrivial solution. vô số nghiệm

$0x = 0$

det $\left\{\begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix} - \begin{bmatrix}2 & 3\\2 & -3\end{bmatrix}\right\} = 0$

det $= 0$

$\det \left\{\begin{bmatrix}\lambda - 2 & -3\\ -2 & \lambda + 3\end{bmatrix}\right\} = 0$

$\Rightarrow (\lambda - 2)(\lambda + 3) - (-2)(-3) = 0$

Problem 3 (30 points).
Let $S = \left\{\begin{bmatrix}3\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\3\end{bmatrix}, \begin{bmatrix}2\\2\\2\end{bmatrix}\right\}$ *and* $\mathbf{v} = \begin{bmatrix}31\\19\\25\end{bmatrix}$.

a) *Show that S is an ordered basis of $\mathbb{R}^3$.* span R^3

b) *Find $[\mathbf{v}]_S$, the coordinates of $\mathbf{v}$ with respect to S.*

$T(u+v) = T(u) + T(v)$ and $T(cu) = cT(u)$

Problem 4 (30 points).
Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a mapping defined by $\boxed{T\left(\begin{bmatrix}w\\x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}2w - 3x + 4y + z\\ w - x + 3y - 5z\\ 3w + x - 2y - 2z\end{bmatrix}}$, *for all* $\begin{bmatrix}w\\x\\y\\z\end{bmatrix} \in \mathbb{R}^4$.

a) *Show that T is linear transformation.*

b) *Does T map $\mathbb{R}^4$ onto $\mathbb{R}^3$?*

RREF

.... THE END! ....

| Nếu có 1 hàng vô số nghiệm, toàn là 0 (det = 0)

$\rightarrow$ T doesn't map R^4 onto R^3

Problem 4:

(i) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \quad c = 2$

$$T(\vec{u}+\vec{v}) = T\begin{pmatrix} 3 & w \\ 5 & x \\ 7 & y \\ 9 & z \end{pmatrix} = \begin{bmatrix} 2w-3x+4y+z \\ w-x+3y-5z \\ 3w+x-2y-2z \end{bmatrix} = \begin{pmatrix} 28 \\ -26 \\ -18 \end{pmatrix}$$

$$T\begin{pmatrix} 1 & w \\ 2 & x \\ 3 & y \\ 4 & z \end{pmatrix} + T\begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix} = \begin{bmatrix} 2w-3x+4y+z \\ w-x+3y-5z \\ 3w+x-2y-2z \end{bmatrix} + \begin{bmatrix} 2w-3x+4y+z \\ w-x+3y-5z \\ 3w+x-2y-2z \end{bmatrix}$$

$$\begin{pmatrix} 12 \\ -12 \\ -9 \end{pmatrix} + \begin{pmatrix} 16 \\ -14 \\ -9 \end{pmatrix} = \begin{pmatrix} 28 \\ -26 \\ -18 \end{pmatrix}$$

(ii) $T(c\vec{u}) = c \cdot T\vec{u}$

$$T\begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \end{pmatrix} = \begin{bmatrix} 2w-3x+4y+z \\ w-x+3y-5z \\ 3w+x-2y-2z \end{bmatrix} = \begin{pmatrix} 24 \\ -24 \\ -18 \end{pmatrix}$$

$$2 \cdot T\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = 2 \cdot \begin{bmatrix} 2w-3x+4y+z \\ w-x+3y-5z \\ 3w+x-2y-2z \end{bmatrix} = 2\begin{pmatrix} 12 \\ -12 \\ -9 \end{pmatrix}$$

$$= \begin{pmatrix} 24 \\ -24 \\ -18 \end{pmatrix}$$

$\Rightarrow$ Linear Trans

Đặc biệt cảm ơn :

Nguyễn Thành Luân

Mai Khánh Thuận

Mến chúc tất cả các bạn thi tốt

Quang Khương

Aladin

iem-buddy

College Tutoring

Quang Khuong via Aladin / Tớ review:

- Deterministic model in OR
- Engineering Economy
- Probability - Statistics
- Mechanics - Dynamics
- Linear Algebra
- Cal / Phy các kiểu