Initial value problem

Question 3. *(15 marks)* Find the solution to the initial value problem

$$xy' + (x+1)y = e^{2019x}, \qquad y(1) = 2020.$$

$y' + p$

μ

$$xy' + (x+1)y = e^{2019x}$$

$$x \cdot \frac{dy}{dx} + (x+1)y = e^{2019x}$$

$$\frac{dy}{dx} + \frac{(x+1)y}{x} = \frac{e^{2019x}}{x} \qquad (1)$$

$$P(x) = \frac{x+1}{x} = 1 + \frac{1}{x}$$

$$\mu(x) = e^{\int P(x)\,dx} = e^{\int 1 + \frac{1}{x}\,dx}$$

$$= e^{x + \ln x} = e^{x} \cdot e^{\ln x} = e^{x} \cdot x \qquad (2)$$

$$(2) \times (1) \Rightarrow e^{x} x \frac{dy}{dx} + e^{x} \cdot x \cdot \frac{(x+1)}{x} y = e^{x} \cdot x \cdot \frac{e^{2019x}}{x}$$

$$e^{x} x \frac{dy}{dx} + e^{x}(x+1)y = e^{x} \cdot e^{2019x}$$

$$= e^{2020x}$$

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$$e^x x \frac{dy}{dx} + e^x (x+1) y = e^x \cdot e^{2019x}$$

$$= e^{2020x}$$

$$\frac{d\,(xe^x \cdot y)}{dx} = e^{2020x}$$

$$\int \frac{d\,(xe^x y)}{dx}\, dx = \int e^{2020x}\, dx \qquad \int e^{ax+b} =$$

$$xe^x y = \frac{1}{2020} e^{2020x} + C \qquad (1)$$

$$e^x x \frac{dy}{dx} + e^x (x+1) y = e^x \cdot e^{2019x}$$

$$= e^{2020x}$$

$$\frac{d\,(xe^x \cdot y)}{dx} = e^{2020x}$$

$$\int \frac{d\,(xe^x y)}{dx}\, dx = \int e^{2020x}\, dx$$

$$xe^x y = \frac{1}{2020} e^{2020x} + C \qquad (1)$$

$$y(1) = 2020 \Rightarrow 1 \cdot e^1 \cdot 2020 = \frac{1}{2020} e^{2020} + C$$

$$\begin{cases} x = 1 \\ y = 2020 \end{cases} \Rightarrow C = 2020e - \frac{1}{2020} e^{2020} \qquad (2)$$

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$$xe^x y = \frac{1}{2020} e^{2020x} + C \quad (1)$$

$$y(1) = 2020 \Rightarrow 1 \cdot e^1 \cdot 2020 = \frac{1}{2020} e^{2020} + C$$

$$\begin{cases} x = 1 \\ y = 2020 \end{cases} \Rightarrow C = 2020e - \frac{1}{2020} e^{2020} \quad (2)$$

(2) into (1) $\Rightarrow xe^x y = \frac{1}{2020} e^{2020x} + 2020e - \frac{1}{2020} e^{2020}$

$$\Rightarrow y = \frac{1}{2020} \cdot \frac{e^{2020x}}{e^x \cdot x} + \frac{2020e}{xe^x} - \frac{1}{2020} \cdot \frac{e^{2020}}{xe^x}$$

Question 3. *(20 marks)* Solve the initial value problem

$$(x+1)y' + (2x+1)y = e^{-2x}, \qquad y(0) = 1.$$

$$\frac{dy}{dx} + \frac{2x+1}{x+1} y = \frac{e^{-2x}}{x+1}$$

$$\begin{array}{r|l} 2x+1 & x+1 \\ \hline -2x-2 & 2 \\ -1 & \end{array} \quad = 2 - \frac{1}{x+1}$$

$$\int 2 - \frac{1}{x+1}\, dx = 2x - \ln(x+1)$$

$$e^{2(x+1) - \ln(x+1)} = e^{2x+2} : e^{\ln(x+1)} = e^{2x+2} : (x+1) = \frac{e^{2x+2}}{x+1}$$

$$e^{2x - \ln(x+1)} = e^{2x} : e^{\ln(x+1)} = e^{2x} : (x+1) = \frac{e^{2x}}{x+1}$$

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Second order Linear Differential equation

$$P(x).y'' + Q(x)y' + R(x)y = G(x) \begin{cases} G(x) = 0 : \text{homogeneous} \\ G(x) \neq 0 : \text{non-homogeneous} \end{cases}$$

$ar^2 + br + c$

$\Delta = b^2 - 4ac > 0 \quad r_1 = 4 \; ; \; r_2 = -3$

$\Rightarrow Y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$

$\Delta = b^2 - 4ac = 0 \quad r = 5$

$\Rightarrow Y = C_1 e^{rx} + C_2 . x e^{rx}$

$\Delta = b^2 - 4ac < 0 \quad r_1 = \alpha + \beta i \quad r_2 = \alpha - \beta i$

$\Rightarrow Y = e^{\alpha x} [C_1 \cos(\beta x) + C_2 \sin(\beta x)]$

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$Y'' + 5Y' + 6Y = x^2$

$Y_p(x) = Ax^2 + Bx + C$

$Y'' + 9Y = e^{2x}$

$Y_p(x) = Ae^{2x}$

$Y'' \to r^2$

$Y' \to r$

$Y \to$ number

$Y'' + 3Y' + 2Y = \cos x$

$Y_p(x) = A\cos x + B\sin x$

$Y'' + Y = \cos x$

$r^2 + 1 = 0 \Rightarrow r = \sqrt{\pm 1}$

$\Rightarrow \left[\begin{array}{l} r_1 = 0 - 1i \\ r_2 = 0 + 1i \end{array} \right.$

$Y_c = e^{\alpha x} [C_1 \cos \beta x + C_2 \sin \beta x]$

$= e^0 [C_1 \cos x + C_2 \sin x]$

$= C_1 \cos x + C_2 \sin x$

$Y_p = A x\cos x + B x \sin x$

$$Y'' - 9Y = xe^{x} + \sin 2x$$

$$r^2 - 9 = 0 \Rightarrow r = \pm 3 \Rightarrow Y_c = C_1 e^{3x} + C_2 e^{-3x}$$

$$Y_{P_1} = (Ax + B)e^{x} \qquad Y_{P_2} = C\cos 2x + D\sin 2x$$

Question 4. *(20 marks) Find the general solution of the differential equation*

$$y'' - 6y' + 9y = 2018e^{3x} + e^{x}(x+1).$$

2017 – 18 : S2 $\qquad Y_G = Y_c + Y_p$

4) $$y'' - 6y' + 9y = 2018e^{3x} + e^{x}(x+1)$$

$$r^2 - 6r + 9 = 0 \qquad \left[\begin{array}{l} r_1 = 3 \\ r_2 = 3 \end{array}\right.$$

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$$Y_c = C_1 e^{3x} + C_2 . x e^{3x}$$

Question 4. (25 marks) Find the general solution of the differential equation

$$y'' - 6y' + 9y = 2018e^{3x} + e^{x}(x+1).$$

2017 – 18: S2 $\quad Y_G = Y_c + Y_p$

4) $\quad y'' - 6y' + 9y = 2018e^{3x} + e^{x}(x+1)$

$r^2 - 6r + 9 = 0 \quad \left[\begin{array}{l} r_1 = 3 \\ r_2 = 3 \end{array}\right.$

$Y_c = C_1 e^{3x} + C_2 . x e^{3x}$

$(uv)' = u'v + uv'$

$(e^u)' = u'e^u$

$g_1(x) = 2018e^{3x} \Rightarrow Y_{P_1} = Ae^{3x} x^2$

$Y_{P_1}' = 3Ae^{3x}x^2 + 2Ae^{3x}x = Ae^{3x}(3x^2 + 2x)$

$Y_{P_1}'' = 3Ae^{3x}(3x^2 + 2x) + Ae^{3x}(6x + 2)$

$= Ae^{3x}(9x^2 + 6x + 6x + 2)$

$= Ae^{3x}(9x^2 + 12x + 2)$

$Ae^{3x}(9x^2 + 12x + 2) - 6Ae^{3x}(3x^2 + 2x) + 9Ae^{3x}x^2 = 2018e^{3x}$

$(uv)' = u'v + uv'$

$g_1(x) = 2018e^{3x} \Rightarrow Y_{P_1} = Ae^{3x} \cdot x^2$

$(e^u)' = u'e^u$

$Y_{P_1}' = 3Ae^{3x}x^2 + 2Ae^{3x}x = Ae^{3x}(3x^2 + 2x)$

$$\begin{aligned} Y_{P_1}'' &= 3Ae^{3x}(3x^2+2x) + Ae^{3x}(6x+2) \\ &= Ae^{3x}(9x^2 + 6x + 6x + 2) \\ &= Ae^{3x}(9x^2 + 12x + 2) \end{aligned}$$

$Ae^{3x}(9x^2 + 12x + 2) - 6Ae^{3x}(3x^2 + 2x) + 9Ae^{3x}x^2 = 2018e^{3x}$

$$\begin{aligned} & A(9x^2 + 12x + 2) \\ + & A(-18x^2 - 12x) \\ + & A(9x^2) = 2018 \end{aligned}$$

$\Rightarrow 2A = 2018$

$\Rightarrow A = 1009$

$Y_{P_1} = 1009e^{3x} \cdot x^2$

$$\begin{aligned} Y_{P_1}'' &= 3Ae^{3x}(3x^2+2x) + Ae^{3x}(6x+2) \\ &= Ae^{3x}(9x^2 + 6x + 6x + 2) \\ &= Ae^{3x}(9x^2 + 12x + 2) \end{aligned}$$

$Ae^{3x}(9x^2 + 12x + 2) - 6Ae^{3x}(3x^2 + 2x) + 9Ae^{3x}x^2 = 2018e^{3x}$

$$\begin{aligned} & A(9x^2 + 12x + 2) \\ + & A(-18x^2 - 12x) \\ + & A(9x^2) = 2018 \end{aligned}$$

$\Rightarrow 2A = 2018$

$\Rightarrow A = 1009$

$Y_{P_1} = 1009e^{3x} \cdot x^2$

$g_2(x) = e^x(x+1) \Rightarrow Y_{P_2} = (Bx + C)e^x$

$Y_{P_2}' = Be^x + (Bx + C)e^x = e^x(Bx + B + C)$

$$\begin{aligned} Y_{P_2}'' &= e^x(Bx + B + C) + e^x B \\ &= e^x(Bx + 2B + C) \end{aligned}$$

$$
\begin{aligned}
& A\,(9x^2 + 12x + 2) \\
+\ & A\,(-18x^2 - 12x) \\
+\ & A\,(9x^2) = 2018
\end{aligned}
\qquad \Rightarrow 2A = 2018 \quad \Rightarrow A = 1009
$$

$$Y_{p_1} = 1009\, e^{3x} \cdot x^2$$

$$g_2(x) = e^x (x+1) \quad \Rightarrow \quad Y_{p_2} = (Bx + C)e^x$$

$$Y'_{p_2} = Be^x + (Bx + C)e^x = e^x (Bx + B + C)$$

$$Y''_{p_2} = e^x (Bx + B + C) + e^x B = e^x (Bx + 2B + C)$$

$$e^x (Bx + 2B + C) - 6 e^x (Bx + B + C) + 9(Bx + C)e^x = e^x (x+1)$$

$$e^x \begin{pmatrix} Bx + 2B + C \\ -6Bx - 6B - 6C \\ 9Bx + 9C \end{pmatrix} = e^x (x+1) \Rightarrow \begin{cases} 4B = 1 \\ -4B + 4C = 1 \end{cases} \Rightarrow \begin{cases} B = 1/4 \\ C = 1/2 \end{cases}$$

$$A\,(9x^2) = 2018$$

$$Y_{p_1} = 1009\, e^{3x} \cdot x^2$$

$$g_2(x) = e^x (x+1) \quad \Rightarrow \quad Y_{p_2} = (Bx + C)e^x$$

$$Y'_{p_2} = Be^x + (Bx + C)e^x = e^x (Bx + B + C)$$

$$Y''_{p_2} = e^x (Bx + B + C) + e^x B = e^x (Bx + 2B + C)$$

$$e^x (Bx + 2B + C) - 6 e^x (Bx + B + C) + 9(Bx + C)e^x = e^x (x+1)$$

$$e^x \begin{pmatrix} Bx + 2B + C \\ -6Bx - 6B - 6C \\ 9Bx + 9C \end{pmatrix} = e^x (x+1) \Rightarrow \begin{cases} 4B = 1 \\ -4B + 4C = 1 \end{cases} \Rightarrow \begin{cases} B = 1/4 \\ C = 1/2 \end{cases}$$

$$Y_{p_2} = \left(\frac{1}{4}x + \frac{1}{2}\right) e^x$$

$$Y_G = y_c + y_p = C_1 e^{3x} + C_2 x e^{3x} + 1009 e^{3x} x^2 + \left(\frac{1}{4}x + \frac{1}{2}\right) e^x$$

18-19 . S2

4)

Question 4. a) *(15 marks)* Determine the form of a particular solution of the differential equation

$$y'' - 4y' + 4y = e^{2x}(x^3+1) + e^x(x+1)\sin x.$$

b) *(15 marks)* Find the general solution of the differential equation

$$y'' - 4y' + 3y = xe^x.$$

$$y'' - 4y' + 4y = e^{2x}(x^3+1) + e^x(x+1)\sin x$$

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

$$r_1 = r_2 = 2$$

$$\alpha = 2$$

$$Y_c = C_1 e^{2x} + C_2 x e^{2x}$$

$$g_1(x) = e^{2x}(x^3+1) \Rightarrow Y_{p_1} = e^{2x}\boxed{Ax^3 + Bx^2 + Cx + D}\,x^2$$

Particular Solution

18-19 . S2

4)

Question 4. a) *(15 marks)* Determine the form of a particular solution of the differential equation

$$y'' - 4y' + 4y = e^{2x}(x^3+1) + e^x(x+1)\sin x.$$

b) *(15 marks)* Find the general solution of the differential equation

$$y'' - 4y' + 3y = xe^x.$$

$$y'' - 4y' + 4y = e^{2x}(x^3+1) + e^x(x+1)\sin x$$

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

$$r_1 = r_2 = 2$$

$$Y_c = C_1 e^{2x} + C_2 x e^{2x} \qquad A\sin x + B\cos x$$

$$g_1(x) = e^{2x}(x^3+1) \Rightarrow Y_{p_1} = e^{2x} \quad (Ax+B)\sin x + (Dx+C)\cos x$$

$$g_2(x) = e^x(x+1)\sin x \Rightarrow Y_{p_2} = e^x[(Ex+F)\sin x + (Gx+H)\cos x]$$

$$\alpha = 1$$

18-19 . S2

4)

Question 4. a) *(15 marks)* Determine the form of a particular solution of the differential equation

$$y'' - 4y' + 4y = e^{2x}(x^3+1) + e^x(x+1)\sin x.$$

b) *(15 marks)* Find the general solution of the differential equation

$$y'' - 4y' + 3y = xe^x.$$

$$y'' - 4y' + 4y = e^{2x}(x^3+1) + e^x(x+1)\sin x$$

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

$$r_1 = r_2 = 2$$

$$Y_c = C_1 e^{2x} + C_2 x e^{2x}$$

$$g_1(x) = e^{2x}(x^3+1) \Rightarrow Y_{p_1} = e^{2x}(Ax^3 + Bx^2 + Cx + D).\,x^2$$

CBS

$$g_2(x) = e^x(x+1)(\sin x) \Rightarrow Y_{p_2} = e^x[(Ex+F)\sin x + (Gx+H)\cos x]$$

$$Y_p = Y_{p_1} + Y_{p_2}$$

b) $y'' - 4y' + 3y = x e^{x}$

$r^2 - 4r + 3 = 0 \Rightarrow \begin{cases} r = 3 \\ r = 1 \end{cases}$

$Y_c = C_1 e^{3x} + C_2 e^{x}$ $\qquad \alpha = 1$

$g_1(x) = x e^{x} \Rightarrow Y_{p_1} = (Ax + B) e^{x} \cdot x$

$= (Ax^2 + Bx) e^{x}$

$r^2 - 4r + 3 = 0 \Rightarrow \begin{cases} r = 3 \\ r = 1 \end{cases}$

$Y_c = C_1 e^{3x} + C_2 e^{x}$

$g_1(x) = x e^{x} \Rightarrow Y_{p_1} = (Ax + B) e^{x} \cdot x$

$= (Ax^2 + Bx) e^{x}$

$Y'_{p_1} = (2Ax + B) e^{x} + (Ax^2 + Bx) e^{x}$

$= e^{x} \left(Ax^2 + (B + 2A) x + B \right)$

$Y''_{p_1} = e^{x} (2Ax + B + 2A + Ax^2 + (B + 2A) x + B$

b) $y'' - 4y' + 3y = xe^x$

$r^2 - 4r + 3 = 0 \Rightarrow \begin{cases} r = 3 \\ r = 1 \end{cases}$

$y_c = C_1 e^{3x} + C_2 e^x$

$g_1(x) = xe^x \Rightarrow y_{p_1} = (Ax + B)e^x \cdot x$
$= (Ax^2 + Bx)e^x$

$y'_{p_1} = (2Ax + B)e^x + (Ax^2 + Bx)e^x$
$= e^x(Ax^2 + (B+2A)x + B)$

$y''_{p_1} = e^x(2Ax + B + 2A + Ax^2 + (B+2A)x + B)$
$= e^x(Ax^2 + x(4A+B) + 2B + 2A)$

$y_c = C_1 e + C_2 e$

$g_1(x) = xe^x \Rightarrow y_{p_1} = (Ax + B)e^x \cdot x$
$= (Ax^2 + Bx)e^x$

$y'_{p_1} = (2Ax + B)e^x + (Ax^2 + Bx)e^x$
$= e^x(Ax^2 + (B+2A)x + B)$

$y''_{p_1} = e^x(2Ax + B + 2A + Ax^2 + (B+2A)x + B)$
$= e^x(Ax^2 + x(4A+B) + 2B + 2A)$

$y'' - 4y' + 3y = xe^x.$

$$e^x \left(\begin{aligned} &Ax^2 + x(4A+B) + 2B + 2A \\ &-4Ax^2 + x(-8A-4B) - 4B \\ &3Ax^2 + x \quad (3B) \end{aligned} \right) = xe^x$$

$$= (Ax^2 + Bx)\, e^x$$

$$Y'_{P_1} = (2Ax + B)\,e^x + (Ax^2 + Bx)\,e^x$$
$$= e^x\,(Ax^2 + (B+2A)\,x + B)$$

$$Y''_{P_1} = e^x\,(2Ax + B + 2A + Ax^2 + (B+2A)x + B)$$
$$= e^x\,(Ax^2 + x(4A+B) + 2B + 2A)$$

$y'' - 4y' + 3y = xe^x.$

$$e^x \left(\begin{array}{l} Ax^2 + x(4A+B) + 2B + 2A \\ -4Ax^2 + x(-8A-4B) - 4B \\ 3Ax^2 + x \quad (3B) \end{array} \right) = x e^x$$

$$\begin{cases} -4A = 1 \\ -2B + 2A = 0 \end{cases} \Rightarrow \begin{cases} A = -1/4 \\ B = -1/4 \end{cases}$$

$$Y_{P_1} = \left(-\frac{1}{4}x^2 - \frac{1}{4}x\right) e^x$$

$$Y_G = C_1 e^{3x} + C_2 e^x + e^x\left(-\frac{1}{4}x^2 - \frac{1}{4}x\right)$$

1718 - S1

3) $$y'' - 4y' + 20y = (x+2)\,e^x + x\,e^{2x}$$

$$r^2 - 4r + 20 = 0$$

$$\left[\begin{array}{l} r_1 = 2 + 4i \\ r_2 = 2 - 4i \end{array}\right.$$

$$r_1 = \alpha + \beta i$$
$$r_2 = \alpha - \beta i$$

$$\begin{array}{l} \alpha = 2 \\ \beta = 4 \end{array} \rightarrow Y_c = C_1 e^{2x} \cos 4x + C_2 e^{2x} \sin 4x$$

$\to 1$

$$y'' - 4y' + 20y = \underline{(x+2)}e^{x} + \underline{x}e^{2x}$$

$$r^2 - 4r + 20 = 0$$

$$\left[\begin{array}{l} r_1 = 2+4i \\ r_2 = 2-4i \end{array}\right.$$

$r_1 = \alpha + \beta i$
$r_2 = \alpha - \beta i$

$\alpha = 2$
$\beta = 4$
$$\Rightarrow Y_c = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x$$

$$r^2 - 4r + 20 = 0$$

$$\left[\begin{array}{l} r_1 = 2+4i \\ r_2 = 2-4i \end{array}\right.$$

$r_1 = \alpha + \beta i$
$r_2 = \alpha - \beta i$

$\alpha = 2$
$\beta = 4$
$$\Rightarrow Y_c = C_1 e^{2x} \cos 4x + C_2 e^{2x} \sin 4x$$

$\alpha = 1$

$$g_1(x) = (x+2)e^{x} \Rightarrow Y_{p_1} = (Ax+B)e^{x}$$

$$Y'_{p_1} = e^{x}(A + Ax + B)$$

$$Y''_{p_1} = e^{x}(A + A + Ax + B) = e^{x}(Ax + 2A + B)$$

$$e^{x}\begin{pmatrix} Ax + 2A + B \\ -4Ax - 4A - 4B \\ 20Ax \quad + 20B \end{pmatrix} = (x+2)e^{x}$$

$$\begin{cases} 17A = 1 \\ -2A + 17B = 2 \end{cases} \Rightarrow \begin{cases} A = 1/17 \\ B = 36/289 \end{cases}$$

$$\Rightarrow Y_p = \left(\frac{1}{17}x + \frac{36}{289}\right)e^{x}$$

$$g_2(x) = x e^{2x} \Rightarrow Y_{p_2} = (Cx + D)\cdot e^{2x}$$

$$Y'_{p_2} = e^{2x}(C + 2Cx + 2D)$$

$$Y''_{p_2} = e^{2x}(2C + 2C + 4Cx + 4D)$$
$$= e^{2x}(4Cx + 4C + 4D)$$

$$e^{x}\left(\begin{array}{lll} Ax & +2A & +B \\ -4Ax & -4A & -4B \\ 20Ax & & +20B \end{array}\right) = (x+2)e^{x}$$

$$\begin{cases} 17A = 1 \\ -2A + 17B = 2 \end{cases} \Rightarrow \begin{cases} A = 1/17 \\ B = 36/289 \end{cases}$$

$$\Rightarrow Y_p = \left(\frac{1}{17}x + \frac{36}{289}\right)e^{x}$$

$$g_2(x) = x e^{2x} \Rightarrow Y_{p_2} = (Cx + D)\cdot e^{2x}$$

$$Y'_{p_2} = e^{2x}(C + 2Cx + 2D)$$

$$Y''_{p_2} = e^{2x}(2C + 2C + 4Cx + 4D)$$
$$= e^{2x}(4Cx + 4C + 4D)$$

$$e^{2x}\left(\begin{array}{lll} 4Cx & +4C & +4D \\ -8Cx & -4C & -8D \\ 20Cx & & +20D \end{array}\right) = x e^{2x}$$

$$-2A+17B=2 \Rightarrow B=36/289$$

$$\Rightarrow Y_p=\left(\frac{1}{17}x+\frac{36}{289}\right)e^{x}$$

$$g_2(x)=xe^{2x} \Rightarrow Y_{p_2}=(Cx+D)\cdot e^{2x}$$

$$Y'_{p_2}=e^{2x}(C+2Cx+2D)$$

$$Y''_{p_2}=e^{2x}(2C+2C+4Cx+4D)$$
$$=e^{2x}(4Cx+4C+4D)$$

$$e^{2x}\left(\begin{array}{lll} 4Cx & +4C & +4D \\ -8Cx & -4C & -8D \\ 20Cx & & +20D \end{array}\right)=xe^{2x}$$

$$\begin{cases} 16C=1 \\ 16D=0 \end{cases} \Rightarrow \begin{cases} C=1/16 \\ D=0 \end{cases}$$

$$Y_{p_2}=\frac{1}{16}x\,e^{2x}$$

$$Y_G=Y_C+Y_{p_1}+Y_{p_2}$$

/ 2:28:16