Definition : $V \neq \emptyset$ ( non empty ) is a vector space if :

(i) V is equipped by :

- Addition : $\forall x, y \in V \to x + y \in V$

- Multiplication (Scalar) : $\forall x \in V$ thì $kx \in V$ $(k \in R)$

(ii) Thoả các tính chất :

- $x + y = y + x$ (giao hoán)
- $(x+y) + z = x + (y+z)$ (kết hợp)
- $\exists$ ptử $\theta$ : $x + \theta = \theta + x = x$
- $\exists$ ptử $-x$ : $x + (-x) = \theta$
- $k(x+y) = kx + ky$ (Phân phối)
- $(h+k)x = hx + kx$
- $h(kx) = (hk)x$ (kết hợp)
- $1 \cdot x = x$

Some special vector spaces:

a) Vector space : $\boxed{R^n = \{(x_1, x_2, \dots, x_n)\}}$

$(x_1, x_2, \dots x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$

$k(x_1, x_2, \dots x_n) = (kx_1, kx_2, \dots kx_n)$

b) $\boxed{\text{Space vector } M_{m\times n} = \text{tập các ma trận có } m\times n}$

$$(a_{ij})_{m\times n} + (b_{ij})_{m\times n} = (a_{ij} + b_{ij})_{m\times n}$$

$$k.(a_{ij})_{m\times n} = (k a_{ij})_{m\times n}$$

c) $\boxed{\text{Space vector : } P_n = \text{tập hợp các đa thức có bậc} \leq n}$

$$(a_0 + a_1 x + \dots a_n x^n) + (b_0 + b_1 x + \dots + b_m x^n)$$
$$= (a_0 + b_0) + (a_1 + b_1)x + \dots + (a_n + b_n) x^n$$

$$k(a_0 + a_1 x + \dots + a_n x^n) = k a_0 + k a_1 x + \dots + k a_n x^n$$

## Linear Independence and Dependence of Vectors :

ĐN: Cho $S = \{a_1, a_2, \dots a_n\} \subset$ vector space $V$

Linear combination: $c_1 a_1 + c_2 a_2 + \dots + c_n a_n$

where $c_1, c_2, c_n$ are any scalar. Now consider the equation

(*) $c_1 a_1 + c_2 a_2 + \dots + c_n a_n = \theta$ (vector 0)

(*) có $n_0$ duy nhất ($c_1 = c_2 = \dots = c_n = 0$)

$\Rightarrow$ S is linearly independent

(*) có vô số $n_0$ ($\exists c \neq 0$) $\Rightarrow$ Linearly dependent

Note:

$$(*) \Leftrightarrow \underbrace{(a_1, a_2, \dots a_n)}_{\text{Ma trận toạ độ S là A}} \begin{pmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{pmatrix} = 0$$

Nếu A vuông $\begin{cases} \det A \neq 0 \rightarrow \text{linearly independent} \\ \det A = 0 \rightarrow \text{linearly dependent} \end{cases}$

Nếu A ko vuông $\Rightarrow$ Reduce echelon form $(A \mid 0)$
(biến đổi sơ cấp)

Ex: check L/I or L/D

$S = \{2-x, 2x-x^2, 6-5x+x^2\} \subset P_2$

Consider the linear combination:

$$c_1(2-x) + c_2(2x-x^2) + c_3(6-5x+x^2) = 0$$

$$\begin{cases} 2c_1 + \quad\quad + 6c_3 = 0 \\ -c_1 + 2c_2 - 5c_3 = 0 \\ \quad\quad -c_2 + c_3 = 0 \end{cases}$$

$$\cdot\ \mathrm{Det}(A) = \begin{vmatrix} 2 & 0 & 6 \\ -1 & 2 & -5 \\ 0 & -1 & 1 \end{vmatrix} \begin{matrix} 2 & 0 \\ -1 & 2 \\ 0 & 1 \end{matrix}$$

$= 0 \rightarrow$ Linearly dependent

$$S = \left\{ \begin{pmatrix} 6 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \right\} \subset M_{2\times 2}$$

$$\cdot\ \mathrm{Det}\ A = \begin{vmatrix} 6 & 1 & 1 & 4 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0 \Rightarrow S \text{ linearly dependent}$$

$S = \{(1,0,1,0),(2,0,1,2),(2,0,2,4)\} \subset R^4$

* A/0
$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right)$$ luôn đúng => remove

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right)$$ Mt vuông → tính det

$$\text{Det}\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} = \boxed{-4 \neq 0}$$

=> S is linearly $\boxed{\text{independent}}$

$S = \left\{ \begin{pmatrix} 1 & -1 \\ 4 & 5 \end{pmatrix}, \begin{pmatrix} 4 & 3 \\ -2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & -8 \\ 22 & 23 \end{pmatrix} \right\} \subset M_{2\times 2}$

A/0
$$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 4 & 1 & 0 \\ -1 & -2 & -8 & 0 \\ 4 & 3 & 22 & 0 \\ 5 & 3 & 23 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 4 & 1 & 0 \\ 0 & 7 & -7 & 0 \\ 0 & -18 & 18 & 0 \\ 0 & -17 & 18 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 4 & 1 & 0 \\ 0 & 7 & -7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \Leftrightarrow \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases}$$

=> S is Lineasly dependent

Find $m$ such that $S$ is linearly dependent

$S=\{(1,2,m),(-1,m,-2),(1,4+m,2m-2)\} \subset R^3$

* $S$ is linearly dependent

$$\Leftrightarrow \left|\begin{matrix} 1 & -1 & 1 \\ 2 & m & 4+m \\ m & -2 & 2m-2 \end{matrix}\right| \begin{matrix} 1 & -1 \\ 2 & m \\ m & -2 \end{matrix} = 0$$

$$\Leftrightarrow m(2m-2) - m(4+m) + 1(2\times -2) - m^2 + 2(4+m) + 2(2m-2) = 0$$

$\Leftrightarrow 0 = 0$ (lđ)

$\Leftrightarrow \forall m$ statisfied