Applied Linear Algebra
Fall 2023

Name : Nguyễn Đình Ngọc Huy
ID : EEEEIU22020

Week 5:

1.

a) $A = \begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix} \xrightarrow[R_3 - 3R_1]{R_2 + 2R_1} \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 1 & -1 & 2 & -3 \end{vmatrix}$

$\xrightarrow{R_4 - R_1} \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & -4 & 2 & -5 \end{vmatrix} \xrightarrow{R_1 - 3R_2} \begin{vmatrix} 1 & 0 & -21 & -22 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & -4 & 2 & -5 \end{vmatrix}$

$\xrightarrow[R_4 + 4R_2]{R_3 + 4R_2} \begin{vmatrix} 1 & 0 & -21 & -22 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 30 & 27 \\ 0 & 0 & 30 & 27 \end{vmatrix} \xrightarrow[R_1 + 21R_3]{\frac{R_3}{30}} \begin{vmatrix} 1 & 0 & 0 & -\frac{31}{10} \\ 0 & 1 & 7 & \frac{17}{10} \\ 0 & 0 & 1 & \frac{9}{10} \\ 0 & 0 & 30 & 27 \end{vmatrix}$

$\xrightarrow[R_4 - 30R_3]{R_2 - 7R_3} \begin{vmatrix} 1 & 0 & 0 & -31/10 \\ 0 & 1 & 0 & 17/10 \\ 0 & 0 & 1 & 9/10 \\ 0 & 0 & 0 & 0 \end{vmatrix}$

Since R_4 consists all 0

Then Det (A) = 0

b) $B = \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -2 & -6 \\ -2 & -6 & 2 & 3 & 10 \\ 1 & 5 & -6 & 2 & -3 \\ 0 & 2 & -4 & 5 & 9 \end{vmatrix} \xrightarrow[R_4 - R_1]{R_3 + 2R_1} \begin{vmatrix} 1 & 3 & -1 & 0 & -2 \\ 0 & 2 & -4 & -2 & -6 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 2 & -5 & 2 & -1 \\ 0 & 2 & -4 & 5 & 9 \end{vmatrix}$

$\xrightarrow[R_1 - 3R_2]{\frac{R_2}{2}}$ $\begin{vmatrix} 1 & 0 & 5 & 3 & 7 \\ 0 & 1 & -2 & -1 & -3 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 2 & -5 & 2 & -1 \\ 0 & 2 & -4 & 5 & 9 \end{vmatrix}$ $\xrightarrow[R_5 - 2R_2]{R_4 - 2R_2}$ $\begin{vmatrix} 1 & 0 & 5 & 3 & 7 \\ 0 & 1 & -2 & -1 & -3 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & -1 & 4 & 5 \\ 0 & 0 & 0 & 7 & 15 \end{vmatrix}$

$\xrightarrow[R_3 \to -R_3]{R_3 \to R_4}$ $\begin{vmatrix} 1 & 0 & 5 & 3 & 7 \\ 0 & 1 & -2 & -1 & -3 \\ 0 & 0 & 1 & -4 & -5 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 7 & 15 \end{vmatrix}$ $\xrightarrow[R_2 + R_3]{R_1 - 5R_3}$ $\begin{vmatrix} 1 & 0 & 0 & 23 & 32 \\ 0 & 1 & 0 & -9 & -13 \\ 0 & 0 & 1 & -4 & -5 \\ 0 & 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 7 & 15 \end{vmatrix}$

$\xrightarrow[R_1 - 23R_4]{\frac{R_4}{3}}$ $\begin{vmatrix} 1 & 0 & 5 & 0 & -14 \\ 0 & 1 & -2 & -1 & -3 \\ 0 & 0 & 1 & -4 & -5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 7 & 15 \end{vmatrix}$ $\xrightarrow[R_3 + 4R_4]{R_2 + 9R_4}$

$\xrightarrow[R_1 + 14R_5]{R_5 - 7R_4}$ $\xrightarrow[R_3 - 5R_5]{R_2 - 5R_5}$

$\xrightarrow{R_4 - 2R_5}$ $\longrightarrow \det(B) = 1$

c)

$C = \begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{vmatrix}$ $\xrightarrow[R_4 + 3R_1]{R_3 - 2R_1}$ $\begin{vmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -1 & -10 \end{vmatrix}$

$$\xrightarrow[R_3 - R_2]{R_2 - 3R_1} \begin{vmatrix} 1 & 0 & -4 & 11 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 10 \\ 0 & -1 & -1 & -10 \end{vmatrix} \xrightarrow[R_3 \leftrightarrow R_4]{R_4 + R_2} \begin{vmatrix} 1 & 0 & -4 & 11 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -15 \\ 0 & 0 & 0 & 10 \end{vmatrix}$$

$\rightarrow Det(C) = -10$

2.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$$

$$\rightarrow \begin{vmatrix} a & b & c \\ d & e & f \\ 3g & 3h & 3i \end{vmatrix} = 3 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 21$$

3. Find $det(B^4)$

$$B^2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 4 & 5 & 5 \\ 4 & 4 & 6 \end{pmatrix}$$

$$B^4 = \begin{pmatrix} 2 & 2 & 2 \\ 4 & 5 & 5 \\ 4 & 4 & 6 \end{pmatrix} \begin{pmatrix} 2 & 2 & 2 \\ 4 & 5 & 5 \\ 4 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 20 & 22 & 26 \\ 40 & 53 & 63 \\ 40 & 52 & 64 \end{pmatrix}$$

$Det(B^4) = 16$

1.

a)

$$A = \begin{vmatrix} 2 & 4 & -1 & 2 & 4 \\ 0 & 3 & 1 & 0 & 3 \\ 6 & -2 & 5 & 6 & -2 \end{vmatrix}$$

$$A^{-1} = \frac{1}{\text{Det}(A)} C^T$$

$$\text{Det}(A) = 30 + 24 + 0 - (-18) - (-4) - 0$$
$$= 76$$

$$A^{-1} = \frac{1}{\text{Det}(A)} \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}^T = \frac{1}{76} \begin{vmatrix} 17 & 6 & -18 \\ -18 & 16 & 28 \\ 7 & -2 & 6 \end{vmatrix}^T$$

$$\frac{1}{76} \begin{vmatrix} 17 & -18 & 7 \\ 6 & 16 & -2 \\ -18 & 28 & 6 \end{vmatrix} \Rightarrow A^{-1} = \begin{vmatrix} \frac{17}{76} & -\frac{9}{38} & \frac{7}{76} \\ \frac{3}{38} & \frac{4}{19} & -\frac{1}{38} \\ -\frac{9}{38} & \frac{7}{19} & \frac{3}{38} \end{vmatrix}$$

b)

$$B = \begin{vmatrix} 1 & 2 & 0 & 1 & 2 \\ -1 & 0 & 1 & -1 & 0 \\ 0 & 3 & 1 & 0 & 3 \end{vmatrix}$$

$$\text{Det}(B) = 0 + 0 + 0 - 0 - 3 + 2 = -1$$

$$B^{-1} = -1 \begin{vmatrix} -3 & 1 & -3 \\ -2 & 1 & -3 \\ 2 & -1 & 2 \end{vmatrix}^T \Rightarrow B^{-1} = \begin{vmatrix} 3 & 2 & -2 \\ -1 & -1 & 1 \\ 3 & 3 & -2 \end{vmatrix}$$

2.

a)

. Gaussian elimination:

$$\left[\begin{array}{ccc|c} 7 & 1 & -4 & 3 \\ -6 & -4 & 1 & 0 \\ 4 & -1 & -2 & 6 \end{array}\right] \xrightarrow[R_3 - 4R_1]{R_1 / 7} \left[\begin{array}{ccc|c} 1 & \frac{1}{7} & -\frac{4}{7} & \frac{3}{7} \\ 0 & -\frac{22}{7} & -\frac{17}{7} & \frac{18}{7} \\ 0 & -\frac{11}{7} & \frac{2}{7} & \frac{30}{7} \end{array}\right]$$

$$\xrightarrow[R_1 - \frac{R_2}{7}]{\times \frac{-7}{22} R_2} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{15}{22} & \frac{6}{11} \\ 0 & 1 & \frac{17}{22} & -\frac{9}{11} \\ 0 & 0 & \frac{3}{2} & 3 \end{array}\right] \xrightarrow[R_1 + \frac{15R_3}{22}]{\frac{2R_3}{3}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{21}{11} \\ 0 & 1 & 0 & -\frac{26}{11} \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$R_3 + \frac{11R_2}{7}$; $R_2 - \frac{17R_3}{22}$

. Cramer's Rule:

$$D = \begin{vmatrix} a & b & c \\ 7 & 1 & -4 \\ -6 & -4 & 1 \\ 4 & -1 & -2 \end{vmatrix} = -33 \;,\; D_x = \begin{vmatrix} d & b & c \\ 3 & 1 & -4 \\ 0 & -4 & 1 \\ 6 & 1 & -2 \end{vmatrix} = -63$$

$$D_y = \begin{vmatrix} a & d & c \\ 7 & 3 & -4 \\ -6 & 0 & 1 \\ 4 & 6 & -2 \end{vmatrix} = 78 \;,\; D_z = \begin{vmatrix} a & b & d \\ 7 & 1 & 3 \\ -6 & -4 & 0 \\ 4 & -1 & 6 \end{vmatrix} = -66$$

$$x = \frac{D_x}{D} = \frac{21}{11} \;,\; y = \frac{D_y}{D} = \frac{-26}{11} \;,\; z = \frac{D_z}{D} = 2$$

b)

. Gaussian elimination

$$B = \left[\begin{array}{ccc|c} 2 & 3 & -5 & 2 \\ 3 & -1 & 2 & 1 \\ 5 & 4 & -6 & 3 \end{array}\right] \xrightarrow[R_3 - 5R_1]{\frac{R_1}{2}} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{5}{2} & 1 \\ 0 & -\frac{11}{2} & \frac{19}{2} & -2 \\ 0 & -\frac{7}{2} & \frac{13}{2} & -2 \end{array}\right]$$

$R_2 - 3R_1$

$R_3 - \frac{7R_2}{11}$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{2} & -\frac{5}{2} & 1 \\ 0 & -\frac{11}{2} & \frac{19}{2} & -2 \\ 0 & 0 & \frac{5}{11} & -\frac{8}{11} \end{array}\right] \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{3}{5} \\ -\frac{12}{5} \\ -\frac{8}{5} \end{pmatrix}$$

• Cramer's Rule:

$$D = \begin{vmatrix} a & b & c \\ 2 & 3 & -5 \\ 3 & -1 & 2 \\ 5 & 4 & -6 \end{vmatrix} = -5 \;, \quad D_x = \begin{vmatrix} d & b & c \\ 2 & 3 & -5 \\ 1 & -1 & 2 \\ 3 & 4 & -6 \end{vmatrix} = -3$$

$$D_y = \begin{vmatrix} a & d & c \\ 2 & 2 & -5 \\ 3 & 1 & 2 \\ 5 & 3 & -6 \end{vmatrix} = 12 \;, \quad D_z = \begin{vmatrix} a & b & d \\ 2 & 3 & 2 \\ 3 & -1 & 1 \\ 5 & 4 & 3 \end{vmatrix} = 8$$

$$x = \frac{D_x}{D} = \frac{3}{5} \;, \quad y = \frac{D_y}{D} = \frac{-12}{5} \;, \quad z = \frac{D_z}{D} = -\frac{8}{5}$$