Exercises: Calculus 1

Duong Thanh PHAM

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1 Functions, Limits and Continuity

1.1 Functions, domains, compositions

Exercise 1.1. Find the domains of the functions.

(i) $f(x) = \dfrac{x}{3x-1}$

(ii) $f(x) = \dfrac{5x+4}{x^2+3x+2}$

(iii) $f(t) = \sqrt{t} + \sqrt[3]{t}$

(iv) $g(u) = \sqrt{u} + \sqrt{4-u}$

(v) $h(x) = \dfrac{1}{\sqrt{x^2-5x}}$

Exercise 1.2. Which of the following is the domain of the function $y = \sqrt{5-x} + \sqrt{x^2-2}$?

(A) $\{x \leq 5\}$

(B) $\{-\sqrt{2} \leq x \leq \sqrt{2}\}$

(C) $\{x \leq -\sqrt{2}$ or $\sqrt{2} \leq x \leq 5\}$

(D) $\{x \geq 5$ or $x \leq -\sqrt{2}\}$

Exercise 1.3. Let $f(x) = \sqrt{2014 - \cos(x+1)}$

Result

$\mathbb{R}$ (all real numbers)

(assuming a function from reals to reals)

Range of function

$\{f \in \mathbb{R} : \sqrt{2013} \leq f \leq \sqrt{2015}\}$

(i) Find the domain and the range of f.

(ii) Find functions g, h so that $f(x) = g(h(x))$, $x \in (-\infty, \infty)$.

$h(x) = \cos(x+1)$

$g(x) = \sqrt{2014 - x}$

Exercise 1.4. Find the domain of the range of the function

$$h(x) = \sqrt{4 - x^2}.$$

Domain: $4 - x^2 \geq 0$

Range: $[0;2]$

Exercise 1.5. The graph below (Figure 1) shows a function f with domain $[1,4]$ and range $[0,2]$. Let $g(x) = 2f(x+1)$.

(i) State the domain and range of $g(x)$.

(ii) Sketch the graph of $g(x)$.

Exercise 1.6. Find a formula for the inverse of the function

$$f(x) = x^2 + 2x, \quad x \geq -1.$$

$y = x^2 + 2x + 1 - 1$

$y = (x+1)^2 - 1$

$\rightarrow x = (y+1)^2 - 1$

$(\Rightarrow) y = -1 \pm \sqrt{x+1}$

$f^{-1}(x) = -1 \pm \sqrt{x+1}$

Exercise 1.7. Determine whether is even, odd, or neither.

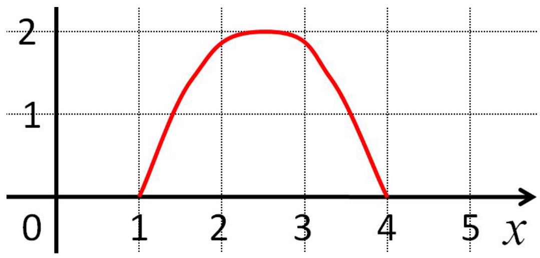


Figure 1: This figure is used in Exercise 1.5

(i) $\dfrac{x}{x^2+1}$

(ii) $\dfrac{x^2}{x^4+1}$

(iii) $\dfrac{x}{x+1}$

(iv) $x\,|x|$

(v) $1+3x^2-x^4$

(vi) $1+3x^3-x^5$

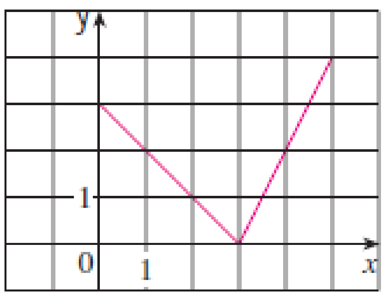
Exercise 1.8. Find an expression for the function whose graph is the given curve.

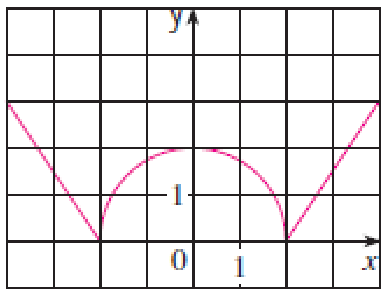
(i) The line segment joining the points $(1,-3)$ and $(5,7)$.

(ii) The bottom half of the parabola $x+(y-1)^2=0$.

(iii) The top half of the circle $x^2+(y-2)^2=4$.

(iv) The two functions whose graphs are as in the following picture.





Exercise 1.9. Let C be a circle with radius 2 centred at the point $(2,0)$.

(i) Write an equation for the circle C.

(ii) Is curve C the graph of a function of x? Explain your answer.

(iii) Write parametric equations to traverse C once, in a clockwise direction, starting from the origin.

Exercise 1.10. Let C be a circle centered at $(0,2)$ with radius 2. Which of the following is the parametric equation of C, in the counter-clockwise direction, starting from the point $(0,4)$?

(A) $x=2+2\cos\theta, \quad y=2\sin\theta \quad \text{for } \theta\in[0,2\pi]$,

(B) $x = 2\cos(\theta + \frac{\pi}{2}), \quad y = 2\sin(\theta + \frac{\pi}{2}) \quad$ for $\theta \in [0, 2\pi]$,

(C) $x = 2\cos(\theta + \frac{\pi}{2}), \quad y = 2 + 2\sin(\theta + \frac{\pi}{2}) \quad$ for $\theta \in [0, 2\pi]$,

(D) $x = 2\cos(\theta - \frac{\pi}{2}), \quad y = 2 + 2\sin(\theta - \frac{\pi}{2}) \quad$ for $\theta \in [0, 2\pi]$,

Exercise 1.11. Find the curve in the form $y = f(x)$ whose parametric curve is

$$(x(t), y(t)) = (2t - 4, 3 + t^2).$$

1.2 Limits

$-2 \quad -2^+$

Exercise 1.12. The limit $\lim_{x \to -2^+} \frac{2x+1}{x+2}$ is

- $\lim_{x \to -2^+} (2x+1) = -3$
- $\lim_{x \to -2^+} (x+2) = 0$
- $x+2 > 0 \ \forall x > -2$

$\Rightarrow \frac{2 \cdot (-1.99) + 1}{-1.99 + 2} = \frac{-}{+} = -$

(A) $-\infty$ (B) ∞ (C) 0 (D) undefined

Exercise 1.13. Evaluate the limit

$$\lim_{x \to 2} \frac{\sqrt{6-x} - 2}{\sqrt{3-x} - 1}. \quad = \frac{1}{2}$$

Exercise 1.14. What is $\lim_{h \to 0} \frac{4(0.5+h)^4 - 4(0.5)^4}{h}$?

$\Rightarrow [(2(0.5+h)^2)^2] - [(2(0.5)^2)^2]$

$\Rightarrow [(2(0.5+h)^2 - (2(0.5)^2)] \times [(2(0.5+h)^2 + (2(0.5)^2)]$

$\Rightarrow [0.5 + 2h + 2h^2 - 0.5] \times [0.5 + 2h + 2h^2 + 0.5]$

$\Rightarrow [2h + 2h^2][2h + 2h^2 + 1]$

$\Rightarrow 4h^2 + 4h^3 + 2h + 4h^3 + 4h^4 + 2h^2$

$\Rightarrow 4h + 4h^2 + 2 + 4h^2 + 4h^3 + 2h$

$\Rightarrow 4h^3 + 8h^2 + 6h + 2 = 2$

(A) 2 (C) 4

(B) $\frac{1}{2}$ (D) the

Exercise 1.15. If $\lfloor x \rfloor$ is the greatest integer that is not greater than x, then $\lim_{x \to \frac{1}{3}} \lfloor x \rfloor$ is

$\lim_{x \to \frac{1}{3}^-} \lfloor x \rfloor = 0 \quad (0.29)$

$\lim_{x \to \frac{1}{3}^+} \lfloor x \rfloor = 0 \quad (0.4)$

(A) 1 (C) $\frac{1}{3}$

(B) 0 (D) the limit does not exist

Exercise 1.16. If $\lfloor x \rfloor$ is the greatest integer that is not greater than x, then $\lim_{x \to 1} \lfloor x \rfloor$ is

$\lim_{x \to 1^-} \lfloor x \rfloor = 0 \quad (0.9)$

$\lim_{x \to 1^+} \lfloor x \rfloor = 1 \quad (1.1)$

(A) 1 (C) 2

(B) 0 (D) the limit does not exist

Exercise 1.17. If $\lceil x \rceil$ is the smallest integer that is not smaller than x, then $\lim_{x \to \frac{1}{3}} \lceil x \rceil$ is

$\lim_{x \to \frac{1}{3}^-} \lceil x \rceil = 1 \quad (0.2)$

$\lim_{x \to \frac{1}{3}^+} \lceil x \rceil = 1 \quad (0.4)$

(A) 1

(B) 0

(C) $\frac{1}{3}$

(D) the limit does not exist

Exercise 1.18. If $\lceil x \rceil$ is the smallest integer that is not smaller than x, then $\lim_{x\to 1} \lceil x \rceil$ is

$\lim_{x\to 1^-} \lceil \overset{0.9}{x} \rceil = 1$

$\lim_{x\to 1^+} \lceil \overset{1.2}{x} \rceil = 2$

(A) 1

(B) 0

(C) 2

(D) the limit does not exist

Exercise 1.19. For the function f whose graph is given, state the following.

(i) $\lim_{x\to 2} f(x) = +\infty$

(ii) $\lim_{x\to -1^-} f(x) = +\infty$

(iii) $\lim_{x\to -1^+} f(x) = -\infty$

(iv) $\lim_{x\to \infty} f(x) = 1$

(v) $\lim_{x\to -\infty} f(x) = 2$

(vi) The equations of asymptotes.

H asymptotes: $y = 1$; $y = 2$

V asymptotes: $x = 2$; $x = -1$

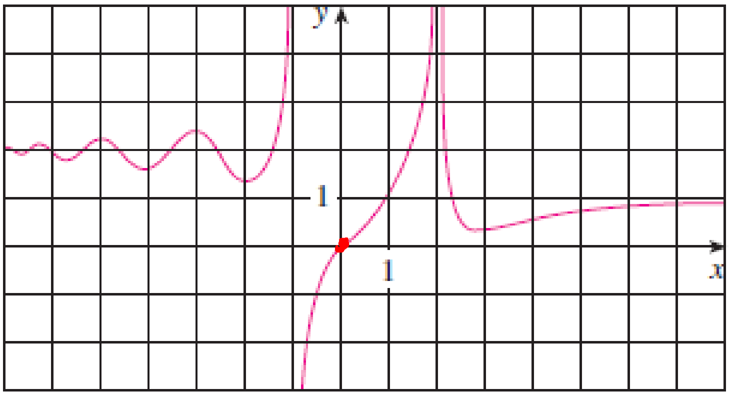


Figure 2: This figure is used in Exercise 1.19

Exercise 1.20. By using the Squeeze Theorem, or otherwise, evaluate the limit

(i)

$$\lim_{x\to \frac{\pi}{2}} (\cos x)\, \cos(\tan x).$$

(ii)

$$\lim_{x\to \pi} (x-\pi) \sin \frac{\pi}{x-\pi}.$$

(iii)

$$\lim_{x\to 0} x\cos(\ln|x|).$$

Exercise 1.21. Find the following limits:

(i) $\displaystyle\lim_{x\to 4}\frac{\sqrt{3x+4}-x}{x-4}$

(ii) $\displaystyle\lim_{x\to 0} x^2\left(\sin\frac{2017}{x}+2018\right)$

(iii) $\displaystyle\lim_{x\to 0^+}\left(e^{\sin\frac{1}{x}}\right)\sqrt{x+\sin x}.$

Exercise 1.22. Let

$$f(x)=\begin{cases}\sin x-1 & \text{if } x\neq \pi/2\\ 2 & \text{if } x=\pi/2\end{cases}.$$

Which of the following statements, I, II, and III, are true?

(I) $\displaystyle\lim_{x\to\pi/2} f(x)$ exists $= \lim_{x\to\frac{\pi}{2}}(\sin x-1) = \sin\frac{\pi}{2}-1 = 0 \to T$

(II) $f(\pi/2)$ exists $= 2 \to T$

(III) f is continuous at $x=\pi/2$. $\displaystyle\lim_{x\to\frac{\pi}{2}} f(x) \neq f\left(\frac{\pi}{2}\right) \to f$ is not continuous $\to$ False

(A) only I

(B) only II

(C) I and II

(D) all of them

Exercise 1.23. Let $f(x)=\dfrac{x^2-4x+3}{x-1}$. Choose the correct statement:

(A) $\displaystyle\lim_{x\to 1} f(x)=-2$

$f(x) = x-3$

(B) $x=1$ is a vertical asymptote.

(C) $\displaystyle\lim_{x\to 1} f(x)=-\infty$

(D) $f(x)=x-3$ for all x.

Exercise 1.24. (i) A tank contains 5000 L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min. Show that the concentration of salt after t minutes (in grams per liter) is

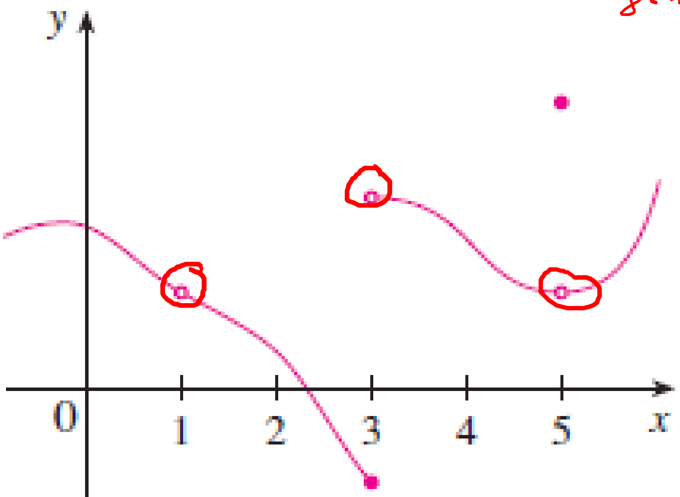
$$C(t)=\frac{30t}{200+t}.$$

(ii) What happens to the concentration as $t\to\infty$?

1.3 Continuity

Exercise 1.25. Figure 3 shows the graph of a function f. At which numbers is f discontinuous? Why?

• f is not continuous at $x = 1$ since $f(1)$ is not defined



· f is not continuous at $x = 5$ since $\lim_{x \to 5} f(x) \neq f(5)$ ⇒ DNE

Figure 3: This figure is used in Exercise 1.25

· f is not continuous at $x = 3$ since $\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$

⇒ $\lim_{x \to 3} f(x)$ DNE

Exercise 1.26. (i) From the graph of f (see Figure 4), state the numbers at which f is discontinuous and explain why.

(ii) For each of the numbers stated in part (i), determine whether is continuous from the right, or from the left, or neither.

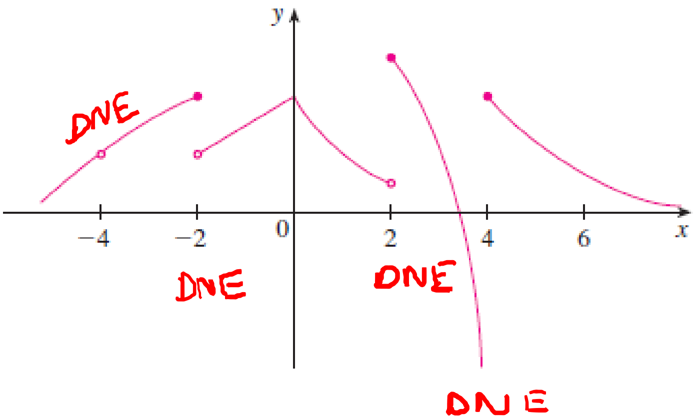


Figure 4: This figure is used in Exercise 1.26

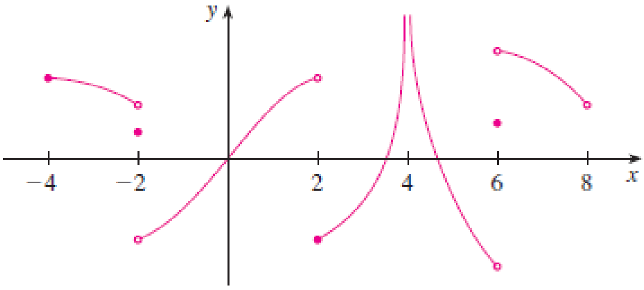


Figure 5: This figure is used in Exercise 1.27

Exercise 1.27. From the graph of g (see Figure 5), state the intervals on which g is continuous.

Exercise 1.28. Which of the following is true for the function $f(x)$ given by

$$f(x) = \begin{cases} 2x - 1 & \text{if } x < -1 \\ x^2 + 1 & \text{if } -1 \leq x \leq 1 \\ x + 1 & \text{if } x > 1. \end{cases}$$

(i) f is continuous everywhere,

(ii) f is continuous everywhere except at $x = -1$ and $x = 1$,

(iii) f is continuous everywhere except at $x = -1$,

(iv) f is continuous everywhere except at $x = 1$,

(v) None of the above.

Explain your choice in details.

$x \to 0$

(7) $f(x) = \begin{cases} 2x-1 & \text{if } x < -1 \\ x^2+1 & \text{if } -1 \leq x \leq 1 \\ x+1 & \text{if } x > 1 \end{cases}$

- When $x < -1$, then $f(x) = 2x - 1$ is continuous at every $x < -1$
- When $-1 < x < 1$, then $f(x) = x^2 + 1$ is continuous at every $-1 < x < 1$
- When $x > 1$, then $f(x) = x + 1$ is continuous at every $x > 1$

Then we have:

- $f(-1) = (-1)^2 + 1 = 2$
- $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (2x - 1) = 2.(-1) - 1 = -3$
- $\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (x^2 + 1) = (-1)^2 + 1 = 2$

$\Rightarrow f(-1) = \lim_{x \to -1^+} f(x) \neq \lim_{x \to -1^-} f(x)$

$\Rightarrow f(x)$ is not continuous at $x = -1$

- $f(1) = 1^2 + 1 = 2$
- $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 1) = 2$
- $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x + 1) = 2$

$\Rightarrow f(1) = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$

$\Rightarrow f(x)$ is continuous at every $x = 1$

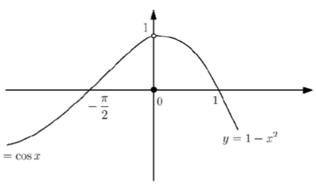
* Conclusion: iv is true.

Exercise 1.29. Let

$$g(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0. \end{cases}$$

(i) Explain why $g(x)$ is discontinuous at $x = 0$.

(ii) Sketch the graph of $g(x)$.



a) The left-hand limit of $g(x)$ at $x = 0$ is

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} \cos x = 1$$

The right-hand limit is

$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (1 - x^2) = 1$$

Therefore,

$$\lim_{x \to 0} g(x) = 1.$$

We have

$$\lim_{x \to 0} g(x) = 1 \neq 0 = g(0)$$

therefore $g(x)$ is not continuous at $x = 0$.

Exercise 1.30. Find the value of the constant k so that the function

$$f(x) = \begin{cases} kx^2 & \text{if } x \leq 2 \\ x + k & \text{if } x > 2 \end{cases}$$

is continuous on $(-\infty, \infty)$.

Exercise 1.31. The radius of the earth is roughly 4000 miles, and an object located x miles from the center of the earth weighs $w(x)$ lb, where

$$w(x) = \begin{cases} ax & \text{if} \quad 0 < x \le 4000 \\ \dfrac{b}{x^2} & \text{if} \quad x > 4000 \end{cases}$$

and a and b are positive constants.

(i) Show that $w(x)$ is continuous on $(0; \infty)$ if and only if

$$a = \frac{b}{4000^3}$$

(ii) Find any horizontal asymptotes and sketch the graph of $w(x)$.

1.4 The Intermediate Value Theorem

Exercise 1.32. Show that there are two positive real numbers c satisfying

$$\sin c = \frac{c+1}{3}.$$

Exercise 1.33. Prove that the equation

$$\ln x = e^{-x}.$$

has at least one root.

Exercise 1.34. Prove that the equation

$$x^{2016} + \frac{84}{1 + x^2 + \cos^2 x} = 119,$$

has at least two roots.

Exercise 1.35. Show that the equation

$$x^3 - 2015x^2 + 2x + 3 = 0$$

has three distinct real roots.

Exercise 1.36. Show that the equation

$$x^3 - x \sin x - 1 = x\sqrt{x+2}$$

has a real root in the interval $[0, 2]$.

Exercise 1.37. Show that the equation

$$\frac{2012}{x - 2012} + \frac{2013}{x - 2013} + \frac{2014}{x - 2014} = 0$$

has two real roots $x_1 \in (2012; 2013)$ and $x_2 \in (2013; 2014)$.

Exercise 1.38. If $f(x) = x^2 + 10 \sin x$, show that there is a number c such that $f(c) = 1000$.

Exercise 1.39. Suppose that f is continuous on $[1, 5]$ and the only solutions of the equation $f(x) = 6$ are 1 and 4. If $f(2) = 8$, explain why $f(3) > 6$.

2 Differentiation

2.1 Definition of derivative

Exercise 2.1. Let

$$f(x) = \begin{cases} 2x+1 & \text{if} \quad x < 0 \\ x^2+1 & \text{if} \quad x \geq 0. \end{cases}$$

(i) Show that $f(x)$ is continuous at $x = 0$

(ii) Is $f(x)$ differentiable at $x = 0$? Explain your answer.

(iii) Find

$$\max_{x \in [-3,5]} f(x) \quad \text{and} \quad \min_{x \in [-3,5]} f(x)$$

Exercise 2.2. Consider the function f defined by

$$f(x) = \begin{cases} -x+a & \text{if } x < 0 \\ x^2+1 & \text{if } 0 \leq x \leq 1 \\ x-b & \text{if } x > 1 \end{cases}$$

where a and b are constants.

(i) Find values of a and b such that f is continuous on $(-\infty, \infty)$.

(ii) Sketch the graph of f with a and b found in (i). Is f differentiable at $x = 1$?

Exercise 2.3. Find the values of a and b that make

$$f(x) = \begin{cases} x^2+1 & \text{if } x \geq 0, \\ a\sin x + b(x+1) & \text{if } x < 0, \end{cases}$$

differentiable everywhere.

Exercise 2.4. Let

$$f(x) = \begin{cases} \dfrac{1}{x+a} & \text{if } x < -1 \\ x & \text{if } -1 \leq x \leq 1 \\ (x-b)^2+1 & \text{if } x > 1, \end{cases}$$

(i) Find the values of a and b such that $f(x)$ is defined and continuous everywhere.

(ii) With the values of a and b found in part (i), find any horizontal asymptotes, and sketch the graph of $f(x)$.

(iii) Show that $f(x)$ is not differentiable at $x = 1$.

Exercise 2.5. Show that the function $f(x) = |x-2|$ is continuous but not differentiable at $x = 2$.

Exercise 2.6. The graph of f is given. State, with reasons, the numbers at which is not differentiable

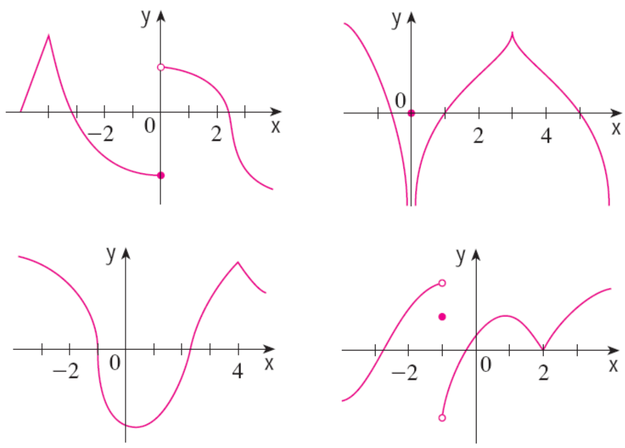


Figure 6: Exercise 2.6

2.2 Finding Derivatives

2.2.1 Direct differentiation

Exercise 2.7. Find the derivative of

$$f(x) = \sqrt{\frac{\sin x}{x}}.$$

Exercise 2.8. Let $f(x) = \sqrt{1+x^3}$; $g(x) = e^{-2x+1}$. Evaluate the derivatives $f'(x)$, $g'(x)$, and $\dfrac{d}{dx}\left(f(g(x))\right)$.

Exercise 2.9. Suppose f and g are two differentiable functions on $(-\infty, \infty)$ and $f(1) = 5$, $f'(1) = -3$, $g(1) = 1$, $g'(1) = -1$. Find $F'(1)$ where

$$F(x) = f(g(2x-1)).$$

Exercise 2.10. Differentiate the function

$$h(x) = e^{-x^2} \sin 2x.$$

Exercise 2.11. (i) Find an equation of the tangent line to the graph of $y = g(x)$ at $x = 5$ if $g(5) = -3$ and $g'(5) = 4$.

(ii) If the tangent line to $y = f(x)$ at $(4, 3)$ passes through the point $(0, 2)$, find $f(4)$ and $f'(4)$.

Exercise 2.12. Let $f(x) = \ln\left(\dfrac{x+2}{\alpha x + 3}\right)$. If the tangent to the graph of $f(x)$ at $x = 1$ has slope $-1/15$, the value of the constant α is:

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 3

Exercise 2.13. The approximate value of $y = \sqrt{4 + \sin x}$ at $x = 0.12$, obtained from the tangent to the graph at $x = 0$, is

(A) 2.00 (B) 2.03 (C) 2.06 (D) 2.12

Exercise 2.14. The radius of a sphere is measured as 3cm with possible error 0.1cm. If we use this measurement to compute the volume, then the approximation (using differentials) for the maximal error of the volume is

(A) 11.3 (B) 11.7 (C) 12.1 (D) 33.9

Exercise 2.15. Find the parabola with equation $y = ax^2 + bx$ whose tangent line at $(1,1)$ has equation $y = 3x - 2$.

Exercise 2.16. For what values of a and b is the line $2x + y = b$ tangent to the parabola $y = ax^2$ when $x = 2$?

Exercise 2.17. Find the value of c such that the line $y = \frac{3}{2}x + 6$ is tangent to the curve $y = c\sqrt{x}$.

Exercise 2.18. Suppose that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$. Find the following values

(i) $(fg)'(5)$ (ii) $(f/g)'(5)$ (iii) $(g/f)'(5)$

Exercise 2.19. Suppose that $f(2) = -3$, $g(2) = 4$, $f'(2) = -2$, and $g'(2) = 7$. Find $h'(2)$.

(i) $h(x) = 5f(x) - 4g(x)$

(ii) $h(x) = f(x)g(x)$

(iii) $h(x) = \dfrac{f(x)}{g(x)}$

(iv) $h(x) = \dfrac{g(x)}{1 + f(x)}$

Exercise 2.20. If $f(x) = e^x g(x)$, where $g(0) = 2$ and $g'(0) = 5$, find $f'(0)$.

Exercise 2.21. If $h(2) = 4$ and $h'(2) = -3$, find

$$\frac{d}{dx}\left(\frac{h(x)}{x}\right)\bigg|_{x=2}.$$

Exercise 2.22. If f and g are the functions whose graphs are shown, let $u(x) = f(x)g(x)$ and $v(x) = f(x)/g(x)$.

(i) Find $u'(1)$ (ii) Find $v'(5)$

Exercise 2.23. If F and G are the functions whose graphs are shown, let $P(x) = F(x)G(x)$ and $Q(x) = F(x)/G(x)$.

(i) Find $P'(2)$ (ii) Find $Q'(7)$

Exercise 2.24. Find $F'(1)$ where $F(x) = xf(xf(x))$, and $f(1) = 1$, $f'(1) = 2$.

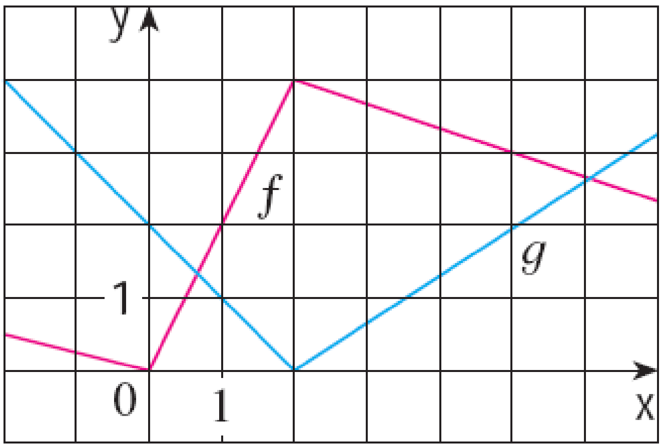


Figure 7: Exercise 2.22

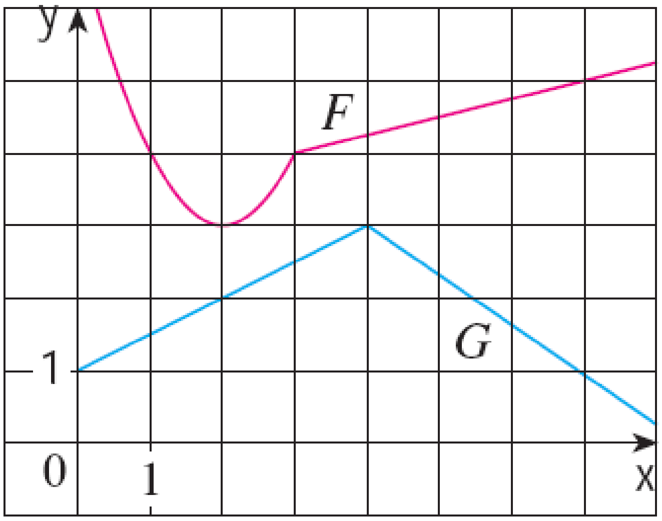


Figure 8: Exercise 2.23

2.2.2 Logarithmic differentiation

Exercise 2.25. Let $f(x) = (\tan x)^{\sin x}$, $0 < x < \frac{\pi}{2}$. Find $f'(x)$.

Exercise 2.26. Let $f(x) = x^{\cos x}$, $x > 0$. Use logarithmic differentiation to find the derivative of f.

Exercise 2.27. Use logarithmic differentiation to differentiate the function

$$y = \left(\frac{1}{x}\right)^{\ln x}$$

Exercise 2.28. Find the derivative of $y = (\sin x)^{\cos x}$, $0 < x < \pi$.

Exercise 2.29. Let $y = (ax)^{bx}$ where a and b are positive constants. The value of $y'(1)$ is:

(A) 1

(B) $b\ln a+1$

(C) $b\ln a+b$

(D) $b(\ln a+a)$

2.2.3 Implicit differentiation

Exercise 2.30. Let

$$4x^2+2xy^3-5y^2=0.$$

Find dy/dx in terms of x and y.

Exercise 2.31. Evaluate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point $(0;-2)$ on the curve $4x^2+y^2=4$.

Exercise 2.32. Let g be a differentiable function on $(-\infty,\infty)$. Assume that

$$g(x)+x\sin(g(x))=x^2,\quad x\in(-\infty,\infty).$$

Find $g'(0)$ and write the equation of the tangent line to the graph of g at the point $(0,0)$.

Exercise 2.33. Find the equation of the tangent line to the curve $x^3+y^3=x-y+6$ at the point $(2,0)$.

Exercise 2.34. Find the equation of the tangent line to the curve

$$x\sin y+x^3=(x-1)^2+1,$$

at the point $(1,0)$.

Exercise 2.35. Find the equation of the tangent line to the the graph of

$$x^2y+y^4=4+2x$$

at $(-1,1)$.

Exercise 2.36. The equation of the tangent to the hyperbola $x^2-y^2=12$ at the point $(4,2)$ on the curve is

(A) $x+y=3$

(B) $y=2x$

(C) $y=2x-6$

(D) $x+2y=6$

2.2.4 Derivative of inverse

Exercise 2.37. Let

$$f(x)=\frac{x^9}{x^8+1}$$

and let g be the inverse of f. Find $g'(1/2)$.

Exercise 2.38. Let $f(x)=x^5+4x-8$. Find $(f^{-1})'(-3)$.

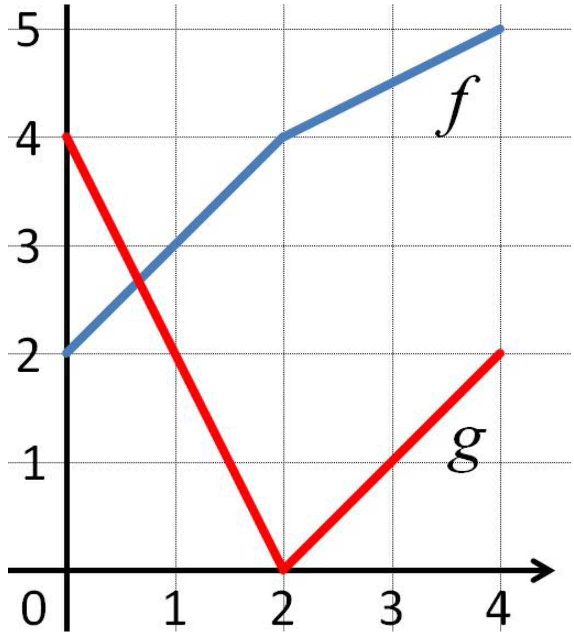


Figure 9: This figure is used in Exercise 2.39

Exercise 2.39. The figure below (Figure 9) shows the graphs of two piece-wise functions, both having domain $[0, 4]$. Evaluate the following, or explain why they do not exist: $(f \circ g)(3)$, $(f \circ g \circ g)(3)$, $g^{-1}(1)$, $f'(3)$, and $(f^{-1})'(3)$.

Exercise 2.40. This is the graph of $f'(x)$. This is the graph of $f'(x)$. This is the graph of $f'(x)$. Sorry to keep repeating myself, but you're going to be really unhappy if you misread the problem. This is the graph of $f'(x)$.

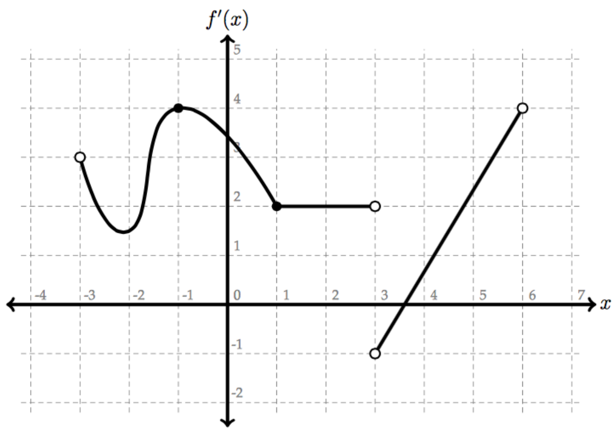


Figure 10: This figure is used in Exercise 2.40

(i) Find $f''(4)$. Explain your method.

(ii) Which is larger, $f(1)$ or $f(2)$? Explain your answer.

(iii) Suppose $f(-1) = 2$. Find $(f^{-1})'(2)$.

2.3 L'Hôpital's rule

Exercise 2.41. Evaluate the limit

$$\lim_{x\to 1} \frac{\sqrt[3]{x} - 1}{\ln(2x-1)}.$$

Exercise 2.42. Use L'Hôpital's rule to evaluate the limit

(i) $\lim\limits_{x\to 0^+} x(\ln x)^2$

(ii) $\lim\limits_{x\to 0} \dfrac{\sin x - x\cos x}{x^2 \sin x}$

(iii) $\lim\limits_{x\to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right)$

(iv) $\lim\limits_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$

Exercise 2.43. Using the L'Hopitals' rule to find the limit:

$$\lim_{x\to \frac{\pi}{2}^-} (\tan x)^{\cos x}.$$

Exercise 2.44. The limit $\lim\limits_{x\to\infty} \left(1 - \frac{\pi}{x}\right)^x$ is

(A) π (B) $-\pi$ (C) ∞ (D) Does not exist

Exercise 2.45. Evaluate the limit $\lim\limits_{x\to 0} \dfrac{\ln(2\sin x)}{\ln(3\tan x)}$.

Exercise 2.46. The limit $\lim\limits_{x\to 0^+} x(\ln x)^2$ is

(A) 0 (B) $2e$ (C) undefined (D) ∞

2.4 Related rates

Exercise 2.47. Two cars start moving at the same point. One travels south at 25 km/h and the other travels east at 60 km/h. At what rate is the distance between the cars increasing two hours later?

Exercise 2.48. A balloon is being filled with helium at the rate of 4 cm^3/min. The rate, in square cm per minute, at which the surface area is increasing when the volume is $\dfrac{32\pi}{3}$ cm^3 is

(A) 4π (B) 2 (C) 4 (D) 1

Exercise 2.49. Two carts, A and B, are connected by a rope 28 meters long that passes over a pulley P (see the figure below, Figure 11). The point Q is on the floor 12 meters directly beneath P. Cart A is being pulled away from Q at a speed of 7 m/s. How fast is cart B moving towards Q at the instant when cart A is 5 m from Q?

Exercise 2.50. A balloon is being filled with helium at the rate of 4 cm^3/min. The rate, in square cm per minute, at which the surface area is increasing when the volume is $\dfrac{32\pi}{3}$ cm^3 is

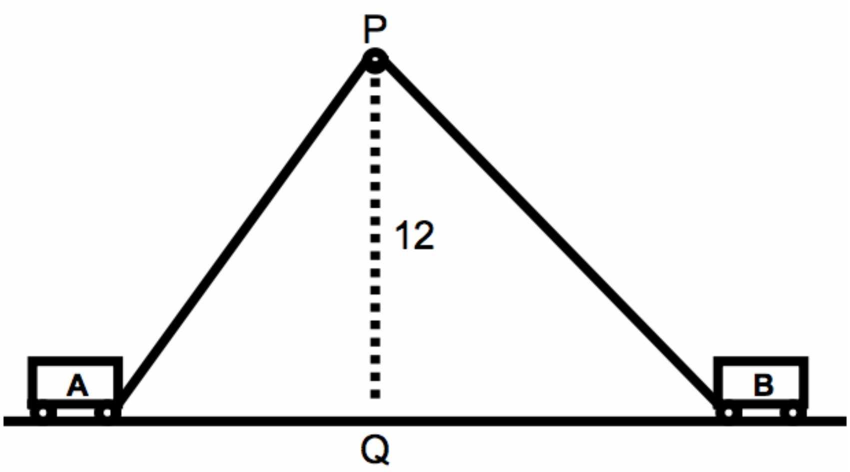


Figure 11: This figure is used in Exercise 2.49

(A) 4π (B) 2 (C) 4 (D) 1

Exercise 2.51. Two ships are sailing along straight-line courses that intersect at right angles. Ship A is approaching the intersection point at a speed of 20 knots (nautical miles per hour). Ship B is approaching the intersection at 15 knots. At what rate is the distance between the ships changing when A is 5 nautical miles from the intersection point and B is 12 nautical miles from the intersection point?

2.5 Maximum and minimum

Exercise 2.52. Find the absolute maximum and absolute minimum (if any) of $f(x) = 5 - (x+2)^{4/5}$ on $(-\infty, \infty)$.

Exercise 2.53. Find the absolute maximum and absolute minimum of the function

$$f(x) = 20 - 3x - \frac{12}{x}, \quad x \in [2, 4].$$

Exercise 2.54. Find the maximum value and the minimum value of the function

$$f(x) = x^{1/2} - x^{3/2}, \quad x \in [0, 4].$$

Exercise 2.55. Find the absolute maximum and absolute minimum values of f on the given interval

$$f(x) = \frac{x}{x^2+1}, \quad x \in [0, 2].$$

Exercise 2.56. The absolute minimum value and absolute maximum value of the function $f(x) = \dfrac{10x}{1+x^2}$ on $[0, 2]$ are, respectively

(A) 0 and 5

(B) 0 and 4

(C) -5 and 5

(D) -5 and 4

Exercise 2.57. Find the absolute maximum and absolute minimum values of the following function:

$$f(x) = \frac{x}{x^2+1}, \quad x \in [0, 2].$$

Exercise 2.58. A student wants to draw a rectangle inscribed in a semicircle of radius 8. If one side must be on the semicircle's diameter, what is the area of the largest rectangle that the student can draw?

Exercise 2.59. Find the area of the largest rectangle that can be inscribed in the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$.

Exercise 2.60. A population of animals is infected with a disease. After t days, the percentage of the population infected is modelled by the function $p(t) = 8t\, e^{-\frac{t}{12}}$ for $0 \le t \le 60$. Find the maximum value of p and the time at which it occurs.

Exercise 2.61. The concentration of a drug t hours after being injected into the arm of a patient is given by

$$C(t) = \frac{0.5t}{t^2 + 0.81}, \quad t \ge 0.$$

When does the maximum concentration occur?

Exercise 2.62. A university campus suffers an outbreak of an infectious disease. The percentage of students infected by the disease after t days can be modelled by the function $p(t) = 5te^{-0.1t}$ for $0 \le t \le 30$. After how many days is the percentage of students infected a maximum?

(A) After 30 days

(B) After 5 days

(C) After 20 days

(D) After 10 days

Exercise 2.63. The population of a bacterial colony t hours after the introduction of a toxin is estimated by

$$P(t) = \frac{15t^2 + 10}{t^3 + 6} \text{(millions)}.$$

When does the largest population occur? What is the largest population?

Exercise 2.64. A lake polluted by bacteria is treated with an antibacterial chemical. After t days, the number of bacteria per milliliter of water can be modelled by the function

$$N(t) = 32\left(\frac{t}{4} - 2\ln\frac{t}{5}\right) \quad \text{for } 1 \le t \le 15.$$

On this time period, find the maximum and minimum values of N and the times at which they occur.

Exercise 2.65. The isosceles triangle shown above (Figure 12) has height AQ of length 3 and base BC of length 2. A point P is placed along the line segment AQ. What is the minimum value of the sum of the distances from P to A, P to B, and P to C?

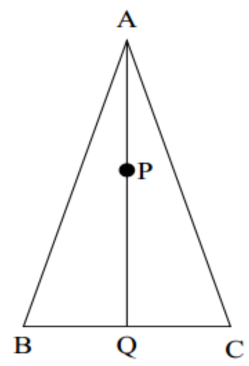


Figure 12: This figure is used in Exercise 2.65

Exercise 2.66. On the curve $y = x + \frac{4}{x}$, the point $(2, 4)$ is

(A) both a local minimum and an absolute minimum

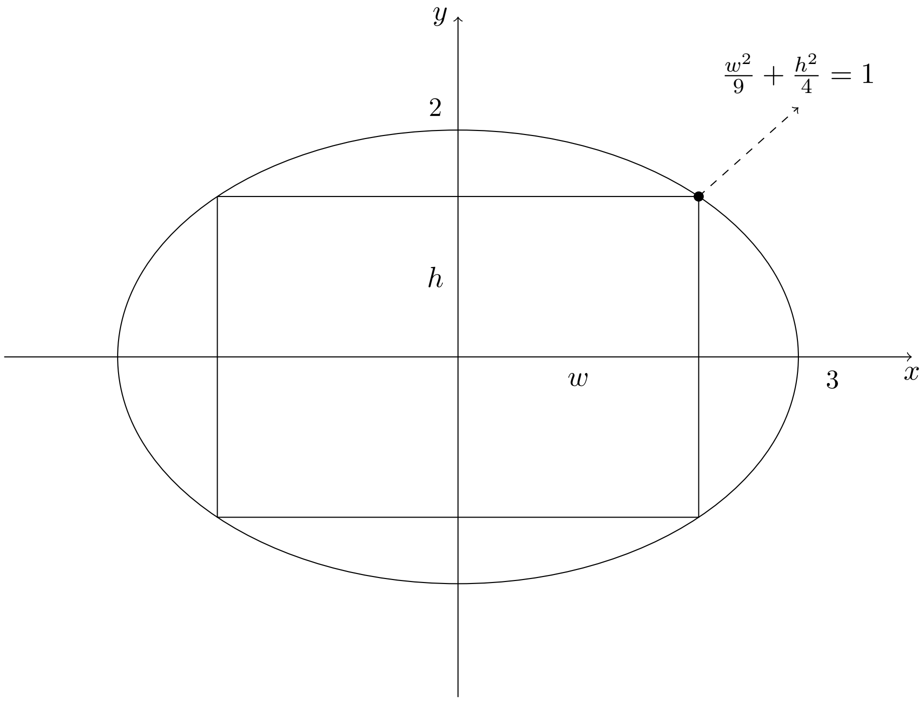
(B) an absolute minimum but not a local minimum

(C) a local minimum but not an absolute minimum

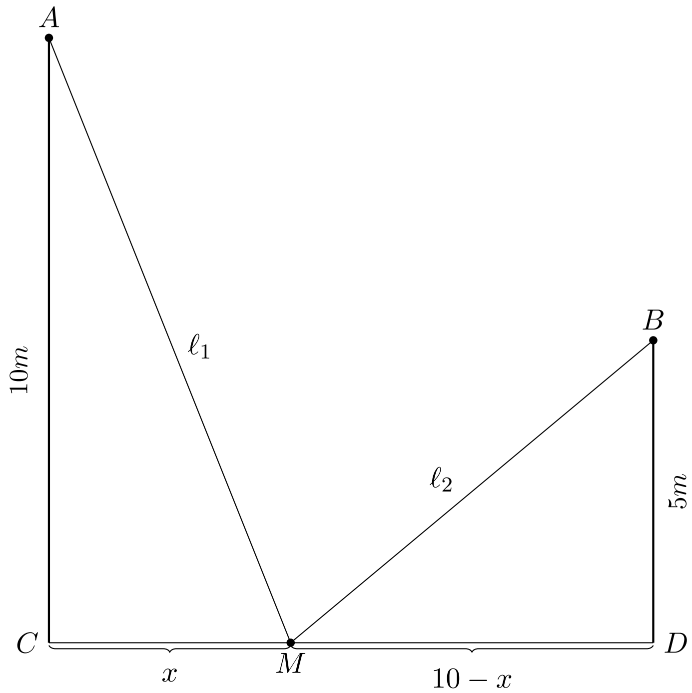
(D) neither a local minimum nor an absolute minimum

Exercise 2.67. A cylinder is generated by rotating a rectangle about one of its edges. The perimeter of the rectangle is 30 cm. To obtain a cylinder of maximum volume, what dimensions should the rectangle have?

Exercise 2.68. Find the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

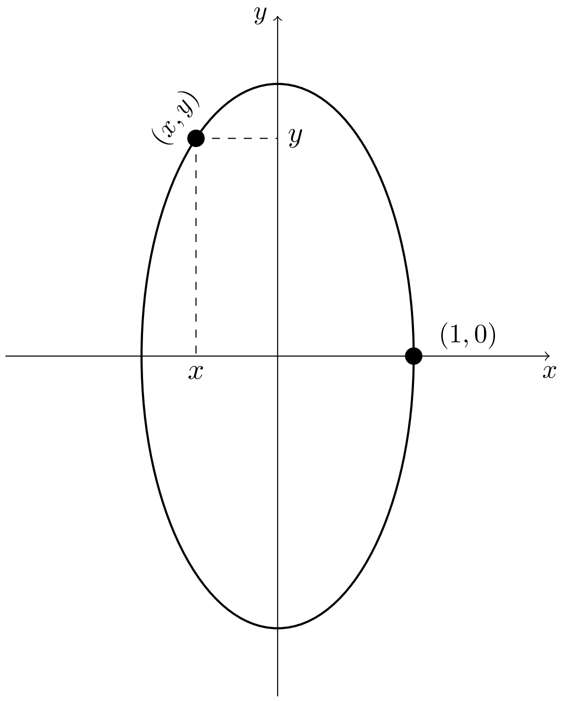


Exercise 2.69. Two vertical poles (one is of 10m length and the other is of 5m) and stay 10m away from each other (see the figure). Find the point M on the ground between the two poles so that the sum of distances from M to the two high endpoints of the two poles attains its minimum value.



Hint: We want to find absolute minimum of the function $\ell(x)=\ell_1(x)+\ell_2(x)$*, where* $\ell_1(x)=\sqrt{x^2+100}$ *and* $\ell_2(x)=\sqrt{(10-x)^2+25}$ *due to the Pythagorean Formula.*

Exercise 2.70. Find the points on the ellipse $4x^2+y^2=4$ that are farthest away from the point $(1,0)$.



Hint: If the point (x, y) lies on the ellipse then we have $4x^2 + y^2 = 4$. The distance between the two points (x, y) and $(1, 0)$ is given by

$$d = \sqrt{(x-1)^2 + y^2}.$$

2.6 Proving inequality by the Mean value theorem

Exercise 2.71. Show that

$$\ln x < x \quad \forall x > 0.$$

Exercise 2.72. Show that

$$\sqrt{1+h} < 1 + \frac{1}{2}h \quad \forall h > 0.$$

Exercise 2.73. Show that $e^u \geq 1 + u$ for all $u \geq 0$.

Exercise 2.74. Show that

$$e^x < 1 + xe^x \quad \forall x > 0.$$

Exercise 2.75. Use the mean value theorem to prove that

$$\frac{1}{2\sqrt{1+x}} + \sqrt{x} < \sqrt{1+x} < \sqrt{x} + \frac{1}{2\sqrt{x}}, \quad x > 0.$$

2.7 Others

Exercise 2.76. The figure shows the graphs of f, f', and f''. Identify each curve, and explain your choices.

Exercise 2.77. The figure shows the graphs of three functions. One is the position function of a car, one is the velocity of the car, and one is its acceleration. Identify each curve, and explain your choices.

Exercise 2.78. At $x = 1$ the curve $y = x^5 - 3x^3 + 1$ is

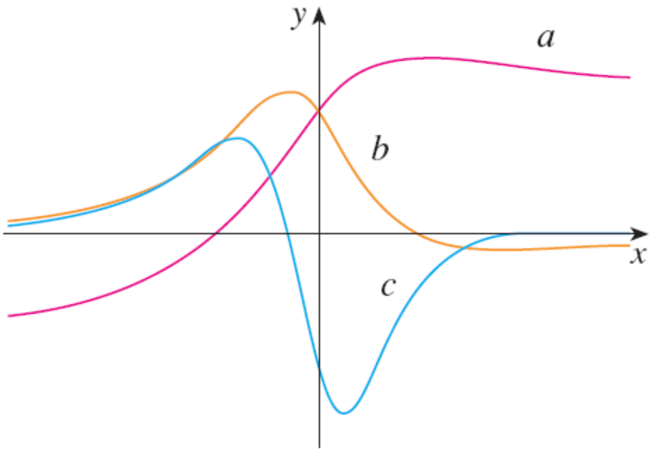


Figure 13: Exercise 2.76

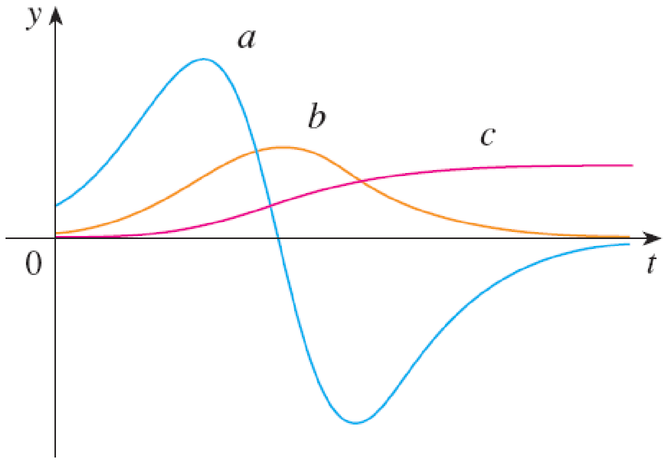


Figure 14: Exercise 2.77

(A) increasing and concave down

(B) decreasing and concave down

(C) increasing and concave up

(D) decreasing and concave up

Exercise 2.79. Let $f(x) = x^{2/3}(6-x)^{1/3}$. Then

(A) $x = 0$ is a critical point of f but f does not attain any extreme value at this point,

(B) $x = 0$ is not a critical point of f but f attains a local maximum at this point,

(C) $x = 0$ is a critical point of f and f attains a local minimum at this point

(D) $x = 0$ is not a critical point of f but f attains a local minimum at this point

Exercise 2.80. Assume f is twice differentiable on $[0, 1]$ such that $f(0) \leq 0$, $f(1) \leq 0$, and $f''(x) \geq 0$ for all $x \in (0, 1)$. Show that $f(x) \leq 0$ for all $x \in (0, 1)$.

Exercise 2.81. The formula below gives the value V of a truck (in dollars) at time t years after the truck was purchased. Find $V'(2)$ and interpret its practical meaning.

$$V(t) = \frac{32000}{1 + 0.3t + 0.1t^2}.$$

Exercise 2.82. A particle is moving along an x-axis. Its x-coordinate (in meters) at time t seconds is given by

$$x(t) = t^3 - 3t + 1, \quad t \geq 0:$$

(i) Find the velocity of the particle at time t.

(ii) When is the particle at rest?

(iii) Find the total distance traveled by the particle during the first 3 seconds.

Exercise 2.83. The equation of motion of a particle is $s = t^3 - 3t$, where s is in meters and t is in seconds. Find

(i) the velocity and acceleration as functions of t,

(ii) the acceleration after 2s, and

(iii) the acceleration when the velocity is 0.

Exercise 2.84. A particle is moving such that its height h at time t is given by $h(t) = \frac{1}{3}t^3 - 4t^2 + 15t + 3$. At which time is the particle moving downwards?

(A) $t = 1$ (B) $t = 4$ (C) $t = 6$ (D) $t = 10$

Exercise 2.85. The median weight of a boy whose age is between 0 and 36 months can be approximated by the function

$$w(t) = 8.15 + 1.82t - 0.0596t^2 + 0.000758t^3,$$

where t is meansured in months and w is meansured in pounds.

Use this approximation to find the following for a boy with median weight:

(a) The rate of change of weight with respect to time.

(b) The weight of the baby at age 10 months.

(c) The rate of change of the baby's weight with respect to time at age 10 months.

Exercise 2.86. Suppose f is continuous on $[1, 2]$ and differentiable on $(1, 2)$. Show that if

$$f(1) - \frac{f(2)}{2} = 0,$$

then there is a $c \in (1, 2)$ such that

$$cf'(c) = f(c).$$

Exercise 2.87. Suppose f is continuous on $[0,1]$ and differentiable on $(0,1)$. Assume that

$$f(0) < f(1) \quad \text{and} \quad f(x) + f'(x) < 2017 \quad \forall x \in (0,1).$$

Prove that

$$f(x) < 2017 \quad \forall x \in (0,1).$$

Exercise 2.88. The acceleration due to gravity, g, at a height of x kilometers above the surface of the earth can be modeled by the function

$$g(x) = g_0 \left(\frac{R}{R+x} \right)^2$$

where g_0 and R are constants, g_0 being the acceleration at the earth's surface and R the radius of the earth.

(i) Find the linearization of $g(x)$ about $x = 0$.

(ii) Use the linearization to approximate g at a height of 120 km, taking $g_0 = 9 : 81$ m/s2 and $R = 6370$ km.

Exercise 2.89. The graph shows the velocity function v of a moving particle. If at time $t = 0$ the particle is at the origin, at which time is it *furthest* from the origin?

(A) A (B) B (C) C (D) D

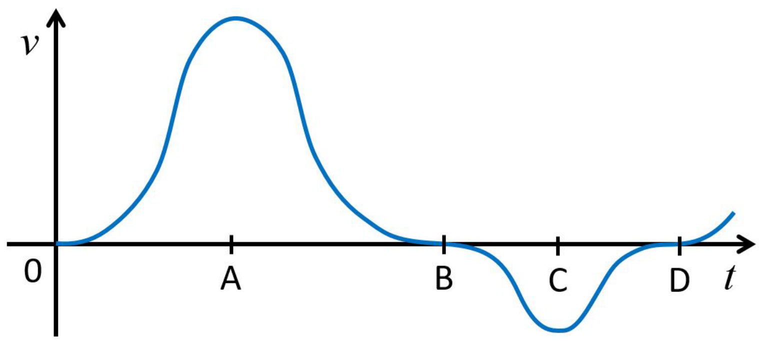


Figure 15: This figure is used in Exercise 2.89

Exercise 2.90. Suppose that a car moves on a straight line, with its position from a starting point (in feet) at time t (in second) given by:

$$s(t) = t^3 - 2t^2 - 7t + 9.$$

When does the car move forward? When does it move backward?

(A) backward on $[0,7]$ and forward on $[7,+\infty)$

(B) forward on $(0,7/3)$ and then backward

(C) backward on $(-1, 7/3)$ and forward on $(7/3, +\infty)$

(D) backward during the first 7/3 seconds and then forward

Exercise 2.91. A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible. He could proceed in any of three ways (see Figure 16):

(a) Row his boat directly across the river to point C and then run to B

(b) Row directly to B

(c) Row to some point D between C and B and then run to B.

If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible?

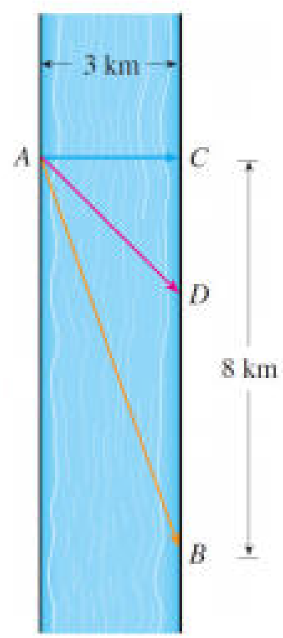


Figure 16: This figure is used in Exercise 2.91

Exercise 2.92. The height (in feet) of an object projected up into the sky is

$$h(t) = -16t^2 + at,$$

where t is the time after projection measured in seconds and a is a constant. Suppose an object was projected at $t = 0$ and dropped down on the ground at $t = 3$ (three seconds after projection).

(i) Find the number a.

(ii) Find the greatest height that the object has achieved.

(iii) What is the falling speed of the object when it hit the ground?

Exercise 2.93. Patrick throws a tomato straight up into the air. The height y of the tomato at time t is given by

$$y(t) = -16t^2 + 20t + 6 \quad \text{(Units are meters and seconds).}$$

(i) Compute the average velocity of the tomato between $t = 0.9$ and $t = 1$.

(ii) Find the maximum height reached by the tomato.

Exercise 2.94. A sphere of radius r has volume $V = \frac{4}{3}\pi r^3$ and circumference $C = 2\pi r$.

(i) Find an equation for the volume V as a function of the circumference C

(ii) A sphere is measured to have circumference 56 cm with a possible error of 0.5 cm. Use differentials to estimate the maximum possible error in the calculated volume of the sphere. Also find the percentage error.

Exercise 2.95. The cost (in dollars) of producing units of a certain commodity is $C(x) = 5000 + 10x + 0.05^2$.

(i) Find the average rate of change of C with respect to x when the production level is changed

- from $x = 100$ to $x = 105$
- from $x = 100$ to $x = 101$

(ii) Find the instantaneous rate of change of C with respect to x when $x = 100$. (This is called the marginal cost).

Exercise 2.96. Use Newton's method to find, correct to three decimal places, the root of the equation $\cos x = x$.

Hint: You may need to use the following iterative formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Exercise 2.97. Plant scientists observe that for a certain type of tree, the growth rate $r(t)$, in centimeters per year, can be modeled by the function

$$r(t) = 15 + \frac{120}{t^4}, \quad 1 \leq t \leq 5$$

where t is the age of the tree in years. During the second year of its life the tree will grow by

(A) $\frac{500}{27} \approx 18.5$ cm

(B) 25 cm

(C) 40 cm

(D) 50 cm

Exercise 2.98. A company sells a new model of computer graphic card at the price of $\$1300 - 0.5x$ where x is the number of graphic cards manufactured per day. The parts for each card cost \$360. The labor and overhead for running the plant cost \$2900 per day. How many graphic cards should the company manufacture and sell per day to maximize their profit?
Knowing that, Profit = Revenue Cost , and Revenue = Number of products × Product price.

Fermat's, Rolle's and Mean Value Theorems

Exercise 2.99. Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be?

Exercise 2.100. Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

(i) $f(x) = 3x^2 + 2x + 5, \quad [-1, 1]$

(ii) $f(x) = x^3 + x - 1, \quad [0, 2]$

(iii) $f(x) = e^{-2x}, \quad [0, 3]$

(iv) $f(x) = \frac{x}{x+2}, \quad [1, 4]$

Exercise 2.101. Let $f(x) = (x-3)^{-2}$. Show that there is no value of c in $(1, 4)$ such that $f(4) - f(1) = f'(c)(4-1)$. Why does this not contradict the Mean Value Theorem?

Exercise 2.102. Let $f(x) = 2 - |2x - 1|$. Show that there is no value of c such that $f(3) - f(1) = f'(c)(3-0)$. Why does this not contradict the Mean Value Theorem?

Exercise 2.103. Show that the equation $1 + 2x + x^3 + 4x^5 = 0$ has exactly one real root.

Exercise 2.104. Show that the equation $2x - 1 - \sin x = 0$ has exactly one real root.

Exercise 2.105. Show that the equation $x^3 - 15x + c = 0$ has at most one root in the interval $[-2, 2]$.

Exercise 2.106. Show that the equation $x^4 + 4x + c = 0$ has at most two real roots.

Exercise 2.107. (i) Show that a polynomial of degree 3 has at most three real roots.

(ii) Show that a polynomial of degree n has at most n real roots.

Exercise 2.108. (i) Suppose that f is differentiable on $\mathbb{R}$ and has two roots. Show that f' has at least one root.

(ii) Suppose f is twice differentiable on $\mathbb{R}$ and has three roots. Show that f'' has at least one real root.

(iii) Can you generalize parts (i) and (ii)?

Exercise 2.109. If $f(1) = 10$ and $f'(x) \geq 2$ for $1 \leq x \leq 4$, how small can $f(4)$ possibly be?

Exercise 2.110. Suppose that $3 \leq f'(x) \leq 5$ for all values of x. Show that

$$18 \leq f(8) - f(2) \leq 30.$$

Exercise 2.111. Does there exist a function f such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2$ for all x?

Exercise 2.112. Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b). Suppose also that $f(a) = g(a)$ and $f'(x) < g'(x)$ for $a < x < b$. Prove that $f(b) < g(b)$. [Hint: Apply the Mean Value Theorem to the function $h = f - g$.]

Exercise 2.113. Suppose is an odd function and is differentiable everywhere. Prove that for every positive number b, there exists a number c in $(-b, b)$ such that $f'(c) = f(b)/b$.

Exercise 2.114. Use the Mean Value Theorem to prove the inequality

$$|\sin a - \sin b| \leq |a - b| \quad \forall a, b.$$

Exercise 2.115. A number a is called a fixed point of a function f if $f(a) = a$. Prove that if $f'(x) \neq 1$ for all real numbers x, then f has at most one fixed point.

3 Integration

3.1 Integration techniques

Exercise 3.1. Evaluate the integral $\displaystyle\int \frac{x}{\sqrt{1+x^2}} dx$

Exercise 3.2. The value of $\displaystyle\int_0^1 e^x \sin(e^x)\, dx$ is

(A) $\sin e - \sin 1$

(B) $\sin 1$

(C) $\sin e$

(D) 0

Exercise 3.3. (i) $\displaystyle\int_1^2 x \ln x\, dx$ (ii) $\displaystyle\int_0^1 x^2 e^x dx$

Exercise 3.4. The value of $\displaystyle\int_1^2 \ln(3x)\, dx$ is

(A) -1

(B) $\ln(12) - 1$

(C) $\ln 3 - 1$

(D) 0

Exercise 3.5. The value of $\displaystyle\int_1^2 \frac{\ln x}{x^2}\, dx$

(A) $\dfrac{1}{2}$ (B) $\ln \dfrac{1}{2}$ (C) $\dfrac{1}{2} - \dfrac{\ln 2}{2}$ (D) $\ln \dfrac{\sqrt{2}}{2}$

Exercise 3.6. Given the the table that shows the values of f, f', g, g' at $x = 0$ and $x = 1$:

x	0	1
f(x)	2	4
f'(x)	6	-3
g(x)	-4	3
g'(x)	2	-1

If $\displaystyle\int_0^1 f'(x)g(x) = 5$, then $\int_0^1 f(x)g'(x)dx$ equals

(A) -14 (B) -2 (C) 7 (D) 15

Exercise 3.7. Evaluate the integrals

(i) $\displaystyle\int \frac{\sin x\sqrt{5-\cos x}}{2}dx$

(ii) $\displaystyle\int \frac{(\sin x)^3}{\cos x + 2}\,dx$

(iii) $\displaystyle\int (\sin x + 8)\cos^3 x\,dx$

Exercise 3.8. The value of $\displaystyle\int_0^{\pi/2} \sin x\,\cos x\sqrt{1+\sin^2 x}\,dx$ is

(A) $\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}$

(B) $\dfrac{\sqrt{2}}{2}$

(C) $\dfrac{\sqrt{3}}{2} - 1$

(D) $\dfrac{1}{3}(2\sqrt{2}-1)$

Exercise 3.9. The indefinite integral $\displaystyle\int (\sin(2x) + \cos(2x))dx$ equals

(A) $\dfrac{1}{2}\cos(2x) + \dfrac{1}{2}\sin(2x)$

(B) $-\dfrac{1}{2}\cos(2x) + \dfrac{1}{2}\sin(2x)$

(C) $2\cos(2x) - 2\sin(2x) + C$

(D) $-\dfrac{1}{2}\cos(2x) + \dfrac{1}{2}\sin(2x) + C$

Exercise 3.10. Evaluate the integral:

$$\int_2^3 \frac{x-8}{3x^2+2x-8}\,dx.$$

Exercise 3.11. In the partial fraction decomposition

$$\frac{1}{x^2+2x-3} = \frac{A}{x-1} + \frac{B}{x+3},$$

what is the value of B?

(A) $\frac{1}{4}$ (B) $-\frac{1}{4}$ (C) 4 (D) $\frac{1}{2}$

Exercise 3.12. The value of $\int_3^4 \frac{-x-12}{(x+5)(x-2)}\,dx$ is

(A) $\ln\frac{9}{32}$ (B) $\ln 9$ (C) $\ln\frac{3}{32}$ (D) $\ln 32$

Exercise 3.13. Evaluate the following integrals

$$\int_0^{\ln 2} \frac{e^{2t}}{e^{2t}+3e^t+2}\,dt.$$

abc

Exercise 3.14. Evaluate the integral $\int \frac{dx}{x^2\sqrt{9-x^2}}$.

Exercise 3.15. The value of $\int_0^1 \frac{dx}{(x^2+1)\sqrt{x^2+1}}\,dx$ is

(A) $\frac{1}{2\sqrt{2}}-1$ (B) $\frac{\sqrt{\pi}}{2}$ (C) $\frac{1}{\sqrt{2}}$ (D) $\frac{2}{3\pi}$

Exercise 3.16. Evaluate

$$\int_0^1 \sqrt{t^4+t^6}\,dt.$$

Exercise 3.17. Evaluate the integrals $\int_{-1}^1 (1+|t^3|)dt$.

3.2 Improper integrals

Exercise 3.18.

$$\int_0^\infty (x+1)e^{-2x}\,dx.$$

Exercise 3.19. Evaluate

$$\int_0^\infty xe^{-x^2+1}dx.$$

Exercise 3.20. Show that the following improper integral converges, determine its value.

$$\int_0^\infty (2x-1)e^{-x}.$$

Exercise 3.21. Find the area under the curve $y=\frac{1}{(x+2)^2}$ from $x=0$ to $x=\infty$.

Exercise 3.22. Evaluate

$$\int_e^{2e} \frac{1}{x\sqrt{\ln x-1}}dx$$

Exercise 3.23. Show that the following improper integral converges

$$\int_0^\infty \frac{\arctan x}{(1+x^2)^{2017}}\,dx.$$

Exercise 3.24. The integral $\int_{-1}^2 \frac{8}{x^3}\,dx$ is

(A) 3 (B) -5 (C) 20 (D) divergent

Exercise 3.25. In probability theory, the mean of the standard normal distribution is

$$\mu = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x\, e^{-x^2/2}\, dx.$$

Evaluate μ.

Exercise 3.26. Let R be the region between the graph of $f(x) = e^{-2x}$ and the x-axis for $x \geq 3$. The area of R is

(A) $\frac{1}{2e^6}$ (B) $\frac{1}{e^6}$ (C) $\frac{2}{e^6}$ (D) $-\frac{1}{2e^6}$

Exercise 3.27. The value of $\int_1^{\infty} \frac{dx}{x^5}$ is

(A) $\frac{1}{5}$ (B) undefined (C) $\frac{1}{4}$ (D) ∞

Answer: C

Exercise 3.28. The value of $\int_1^2 \frac{dx}{\sqrt{x-1}}$ is

(A) ∞ (B) undefined (C) $\frac{1}{2}$ (D) 2

Answer: D

3.3 Fundamental theorems of Calculus

Exercise 3.29. Evaluate $\lim_{x\to 0} \frac{\int_0^x \cos(t^2)dt}{x}$

Exercise 3.30. The limit $\lim_{x\to 0} \frac{\int_0^x \cos(\pi t^2)dt}{x}$ is

(A) 1 (B) 0 (C) -1 (D) ∞

Exercise 3.31. The limit $\lim_{x\to 0} \frac{\int_{x^2}^0 \sqrt{1+t^3}dt}{x^2}$. is

(A) -1 (B) 1 (C) ∞ (D) Does not exist

Exercise 3.32. Find a function $y = f(x)$ such that

$$\int_0^x f(t)dt = \frac{\sin x}{x^2+1} \quad \text{for all } x.$$

Exercise 3.33. Find a continuous function $f(x)$ and a number $a > 0$ such that

$$16 + \int_a^x t^2 f(t)\, dt = x^4$$

for all values of x.

(Hint: Differentiate both sides with respect to x.)

Exercise 3.34. Find a continuous function $f(x)$ on $(-\infty, \infty)$ such that

$$\int_0^{2017x} f(t)dt = \frac{e^x}{x^2+1} - 1, \quad \forall x \in (-\infty, \infty).$$

Exercise 3.35. Let $F(x) = \displaystyle\int_x^{2x+1} \sqrt{t^2+2t+5}dt$. Find $F'(x)$.

Exercise 3.36. Given $F(x) = \displaystyle\int_1^{\pi x} \sqrt{t + \sin t}\; dt$, the value of $F'(1)$ is

(A) $\sqrt{\pi}$

(B) $\sqrt{\pi} - 1$

(C) $\pi\sqrt{\pi}$

(D) $\pi(\sqrt{\pi} - 1)$

Exercise 3.37. Let $F(x) = \displaystyle\int_{x^2}^{x^3} \sqrt{t^4+1}dt$. Find $F'(x)$.

Exercise 3.38. If $f'(x) = 2x - 3\sqrt{x}$ and $f(0) = 3$, the value of $f(1)$ is

(A) 6 (B) 2 (C) 0 (D) $-\frac{1}{2}$

Exercise 3.39. Given $f''(x) = 12x^2 + e^x$, $f'(0) = 4$ and $f(0) = 1$, find $f(x)$.

(A) $f(x) = x^4 + e^x + 3x$

(B) $f(x) = 4x^3 + e^x + 3$

(C) $f(x) = x^4 + e^x - 3$

(D) $f(x) = x^4 + e^x + 4x + 1$

Exercise 3.40. An electronics factory uses a certain rare metal. The metal is used up at a rate $r(t) = 2e^{0.04t}$ tons per year, where t is time in years. At time $t = 0$ the factory has a reserve of 40 tons. How long will it take for the factory to use up this reserve?

Exercise 3.41. A particle moves along a straight line and its velocity at time t is given by $v(t) = 2t + \cos 2t$. If its initial position is $s(0) = 1$, then its position at $t = T$ is

(A) $s(T) = T^2 - \dfrac{1}{2}\sin(2T) + 1$

(B) $s(T) = T^2 + \dfrac{1}{2}\sin(2T) + 1$

(C) $s(T) = T^2 - 2\sin(2T) + 1$

(D) $s(T) = T^2 + 2\sin(2T) + 1$

Exercise 3.42. An oil storage tank cracks at time $t = 0$ and oil leaks from the tank at a rate of $r(t) = e^{-0.1t}$ liters per minute. How much oil leaks out during the first ten minutes?

(A) 10 liters

(B) $10 - \dfrac{10}{e}$ liters

(C) $10 + \dfrac{10}{e}$ liters

(D) $1 - \dfrac{1}{e}$ liters

Exercise 3.43. Assume that $f(1) = 10$ and $f'(x) \geq 2$ for all values of $x \in [1, 4]$. Which of the following statements must be true?

(A) $f(4) < 25$

(B) $f(4) \geq 16$

(C) $f(4) \leq 18$

(D) $f(4) \leq 12$

3.4 Average value of a function, Areas, volumes, arc lengths

Exercise 3.44. The average value of the function $f(x) = \dfrac{1}{1+x^2}$ on $[0, 1]$ is

(A) $\dfrac{1}{2}$ (B) $\dfrac{3}{4}$ (C) $\dfrac{1}{2}\ln 2$ (D) $\dfrac{\pi}{4}$

Exercise 3.45. A cup of hot tea with initial temperature 90^o C is placed in a room where the temperature is 25^o C. According to Newton's Law of Cooling, the temperature of the tea after t minutes is

$$T(t) = 25 + 65e^{-\frac{t}{50}}.$$

Evaluate the average temperature of the tea during the first hour.

(Hint: The average value of a continuous function f on $[a, b]$ is defined to be the number $\frac{1}{b-a}\int_a^b f(x)dx$.)

Exercise 3.46. Consider the region A bounded by the curves $y = x^2 - 1$ and the line $y = 3x - 3$.

(a) Find the area of A.

(b) Find volume of the solid obtained by rotating A about the x-axis.

Exercise 3.47. Let G be the region enclosed by the the curves $y = x^{3/2}$, $x = 0$, $x = 4$ and $y = 0$.

(a) Find the area of of the region G.

(b) Find the volume of the solid generated by rotating the region G about the x-axis.

(c) Find the circumference of the region G.

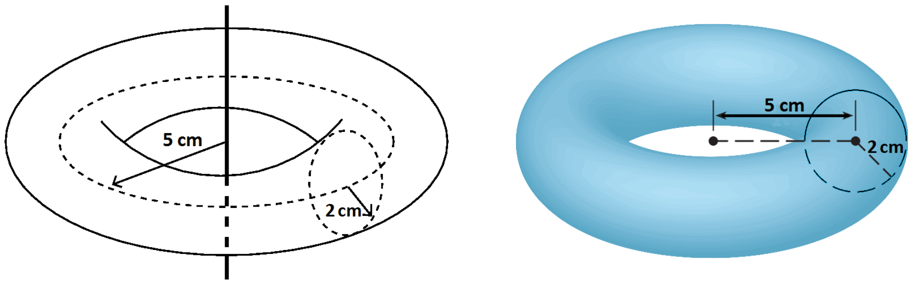


Figure 17: This figure is used in Exercise 3.48

Exercise 3.48. A torus (a donut-shaped solid) is generated by rotating a circle of radius 2 cm about an axis 5 cm from the center of the circle, as shown in the figures below (Figure 17). Find the volume of the torus.
[Hint: Use the method of cylindrical shells, $V = \int_a^b 2\pi x f(x)\,dx$*].*

Exercise 3.49. Find the area of the region enclosed by $y = x^2 - 2x$ and $y = x + 4$.

Exercise 3.50. Find the area of the region enclosed by $y = \sqrt{x^3}$ and $y = x$.

Exercise 3.51. Determine the area of the region bounded by $y = 4x + 3$, $y = 6 - x - 2x^2$, and two vertical lines $x = -4$ and $x = 2$.

Exercise 3.52. Consider the region R bounded by the curves $y = 5 - x^2$ and $y = 4x^2$.

(i) Find the area of R.

(ii) Find volume of the solid obtained by rotating R about the x-axis.

Exercise 3.53. Let R be the region enclosed by $y = 12 - x^2$ and $y = |x|$.

(i) Sketch the region R.

(ii) Find the area of R.

Exercise 3.54. Let G be the region enclosed by the curves $y = \frac{4}{x}$ and $y = 5 - x$

(i) Find the area of the region G.

(ii) Find the volume of the solid generated by rotating the region G about the x-axis.

Exercise 3.55. Find the area of the region S enclosed by $y = x$ and $y = 5x - x^2$.

Exercise 3.56. A city's water storage tank is in the shape of the solid of revolution generated by rotating the region under the graph of

$$f(x) = 20\sqrt{1 - \frac{x^2}{30^2}}$$

from $x = -30$ to $x = 30$m about the x-axis. What is the volume of this tank?

Exercise 3.57. The region bounded by the curves $y = \sqrt{x}$, $x = 0$ and $y = 1$ is revolved around the x-axis. The volume of the solid generated is given by which integral?

(A) $\pi \int_0^1 x\, dx$

(B) $\pi \int_0^1 (1-x)\, dx$

(C) $\pi \int_0^1 x(1-\sqrt{x})\, dx$

(D) $\pi \int_0^1 (1-\sqrt{x})^2\, dx$

Exercise 3.58. Which integral represents the volume of the solid generated when the region under the curve $y = \cos x$ on $[0, \pi/2]$ is rotated about the y-axis?

(A) $\pi \int_0^{\pi/2} \cos x\, dx$

(B) $2\pi \int_0^1 \cos x\, dx$

(C) $2\pi \int_0^{\pi/2} x \cos x\, dx$

(D) $\pi \int_0^{\pi} \sin^2 x\, dx$

3.5 Others

Exercise 3.59. Let f be a continuous and decreasing function on $[0, 1]$. Show that

$$\int_0^{1/2} f(x)dx \geq \frac{1}{2} \int_0^1 f(x)dx.$$

Exercise 3.60. The demand function for a certain type of seed is

$$p(x) = \frac{500}{(3x+4)^2}.$$

Find the consumer surplus when the production level x is 2 units.
(Hint: The demand function is $p(x)$, where p is price and x is production level. The consumer surplus at production level X is defined as $\int_0^X [p(x) - P]dx$ where $P = p(X)$.)

Exercise 3.61. (a) Show that

$$1 \leq \sqrt{1+x^3} \leq 1 + x^3 \quad \text{for all } 0 \leq x \leq 1.$$

(b) Show that

$$1 \leq \int_0^1 \sqrt{1+x^3}\, dx \leq \frac{5}{4}.$$

Exercise 3.62. Use the Midpoint Rule by dividing the interval $[0, 1]$ into 5 equal subintervals to evaluate the integral

$$\int_0^1 \sqrt{1+x^3}\, dx.$$

Exercise 3.63. By using the Simpson rule and dividing the interval $[0, 1]$ into 6 equal subintervals, approximate the integral

$$\int_0^3 \sqrt{1+x}\, dx.$$

(Give your answer correct to at least 3 decimal places.)
How many subintervals should we use if we want to guarantee that the error will be less than 10^{-4}?

Exercise 3.64. The expression

$$\frac{1}{50}\left(\sqrt{\frac{1}{50}} + \sqrt{\frac{2}{50}} + \cdots + \sqrt{\frac{50}{50}}\right)$$

is the Riemann sum approximation for

(A) $\int_0^1 \sqrt{\frac{x}{50}}\, dx$

(B) $\int_0^1 \sqrt{x}\, dx$

(C) $\frac{1}{50} \int_0^1 \sqrt{\frac{x}{50}}\, dx$

(D) $\frac{1}{50} \int_0^1 \sqrt{x}\, dx$

Exercise 3.65. For the function f shown in the figure, the approximate value of the integral $\displaystyle\int_0^6 f(x)\, dx$ found using six rectangles of equal width and right hand endpoints is

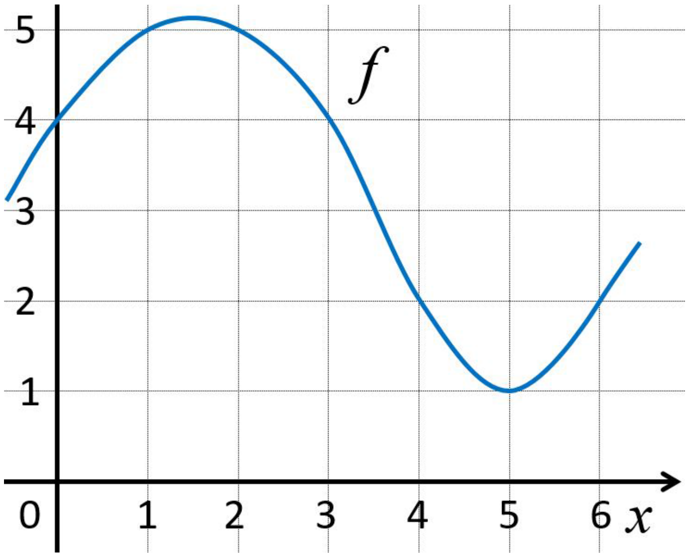


Figure 18: This figure is used in Exercise 3.65

(A) 19

(B) 20

(C) 21

(D) 23.

Exercise 3.66. If $f(1) = 3$ and $\displaystyle\int_0^1 f(x)dx = 2$, then the value of $\displaystyle\int_0^1 xf'(x)dx$ is

(A) 3

(B) 2

(C) 5

(D) 1