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Introduction to Mathematical Probability and Statistics

A Calculus-based Approach

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John Travis
Mississippi College

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John Travis grew up in Mississippi and had his graduate work at the University of Tennessee and Mississippi State University. As a numerical analyst, since 1988 he has been a professor of mathematics at his undergraduate alma mater Mississippi College where he currently serves as Professor and Chair of Mathematics.

You can find him playing racquetball or guitar but not generally at the same time. He is also an active supporter and organizer for the opensouce online homework system WeBWorK.

Preface

This text is intended for a one-semester calculus-based undergraduate course in probability and statistics .

A collection of WeBWorK online homework problems are available to correlate with the material in this text. Copies of these sets of problems are available by contacting the author.

WeBWorK (webwork.maa.org) is an open-source online homework system for math and science courses. WeBWorK is supported by the MAA and the NSF and comes with a Open Problem Library (OPL) of over 35,000 homework problems. Problems in the OPL target most lower division undergraduate math courses and some advanced courses. Supported courses include college algebra, discrete mathematics, probability and statistics, single and multivariable calculus, differential equations, linear algebra and complex analysis.

Sage (sagemath.org) is a free, open source, software system for advanced mathematics, which is ideal for assisting with a study of abstract algebra. Sage can be used either on your own computer, a local server, or on SageMathCloud (https://cloud.sagemath.com).

John Travis
Clinton, Mississippi 2015

Contents

Chapter 1

Review of Calculus

This chapter is a review of power series results from Calculus.

1.1 Geometric Series

Knowledge of the use of power series is very important when dealing with both probability functions.

$$S = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

as is its extension know as the negative binomial series ($n \in \mathbb{N}$).

$$NB = \sum_{k=0}^{\infty} (-1)^k \binom{-n+k-1}{k} x^k b^{-n-k} = \frac{1}{(x+b)^n}$$

In this section, we review this series, develop its properties, and explore some of its extensions.

1.1.1 Geometric Series

Theorem 1.1.1. $S = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

Proof. Consider the partial sum

$$S_n = \sum_{k=0}^{n} x^k = 1 + x + x^2 + ... + x^n$$

$$(1-x)S_n = S_n - xS_n = 1 + x + x^2 + ... + x^n - (x + x^2 + ... + x^n + x^{n+1}) = 1 - x^{n+1}$$

$$\Rightarrow S_n = \frac{1-x^{n+1}}{1-x}$$

and so as $n \to \infty$,

$$S_n \to S = \frac{1}{1-x}$$

□

The interactive activity below shows how well the partial sums approximate $\frac{1}{1-x}$ as the number of terms increases.

```
var(’x,n,k’)
f = 1/(1-x)
@interact
def _(n = slider(2,20,1,2)):
        Sn = sum(x^k,k,0,n)
        pretty_print(html(’$S_n(x)␣=␣
            %s$’%str(latex(Sn))))
        G = plot(f,x,-1,0.9,color=’black’)
        G += plot(Sn,x,-1,0.9,color=’blue’)
        G += plot(abs(f-Sn),x,-1,0.9,color=’red’)
        G.show(title="Partial␣Sums␣(blue)␣vs␣Infinite␣
            Series␣(black)␣and␣Error␣
            (red)",figsize=(5,4))
```

1.1.2 Alternate Forms for the Geometric Series

Theorem 1.1.2 (Generalized Geometric Series)**.** *For* $k \in \mathbb{N}$*,* $\sum_{k=M}^{\infty} x^k = \frac{x^M}{1-x}$

Proof.

$$\begin{aligned}
\sum_{k=M}^{\infty} x^k &= x^M \sum_{k=0}^{\infty} x^k \\
&= x^M \frac{1}{1-x} \\
&= \frac{x^M}{1-x}
\end{aligned}$$

□

Example 1.1.3 (Integrating and Differentiating to get new Power Series)**.** The geometric power series is a nice function which is relatively easily differentiated and integrated. In doing so, one can obtain new power series which might also be very useful in their own right. Here we develop a few which are of special interest.

Let $f(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$. Then,

$$\begin{aligned}
f'(x) &= \sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2} \\
f''(x) &= \sum_{k=2}^{\infty} k(k-1)x^{k-1} = \frac{2}{(1-x)^3} \\
f^{(n)}(x) &= \sum_{k=n}^{\infty} k(k-1)...(k-n+1)x^{k-n} = \frac{n!}{(1-x)^{n+1}} \\
\int f(x)dx &= \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = -ln(1-x)
\end{aligned}$$

Example 1.1.4 (Playing with the base)**.**

$$\begin{aligned}
\sum_{k=0}^{\infty} a^k x^k &= \sum_{k=0}^{\infty} (ax)^k \\
&= \frac{1}{1-ax}, |x| < \frac{1}{a}
\end{aligned}$$

or perhaps

$$\sum_{k=0}^{\infty}(x-b)^k = \frac{1}{1-(x-b)}, |x-b| < 1$$

Example 1.1.5 (Application: Converting repeating decimals to fractional form)**.** Consider this example:

$$\begin{aligned} 2.48484848... &= 2 + 0.48 + 0.0048 + 0.000048 + ... \\ &= 2 + 0.48(1 + 0.01 + 0.0001 + ...) = 2 + 0.48\sum_{k=0}^{\infty}(0.01)^k \end{aligned}$$

Therefore, applying the Geometric Series

$$\begin{aligned} 2.48484848... &= 2 + 0.48\frac{1}{1-0.01} \\ &= 2 + 0.48\frac{100}{99} = 2 + \frac{48}{99} \end{aligned}$$

Example 1.1.6 (Playing around with repeating decimals)**.** Certainly most students would agree that $0.333333... = \frac{1}{3}$. So, what about 0.999999...? Simply follow the pattern above

$$\begin{aligned} 0.999999... &= 0.9 + 0.09 + 0.009 + 0.0009 + ... = 0.9(1 + 0.1 + 0.1^2 + 0.1^3 + ... \\ &= 0.9\frac{1}{1-0.1} = 0.9\frac{1}{0.9} = 1 \end{aligned}$$

1.2 Binomial Sums

The binomial series is also foundational. It is technically not a series since the sum$_iffinitebutwewon'tbotherwiththatfornow.Itisgivenby$

1.2.1

Theorem 1.2.1 (Binomial Theorem)**.** *For* $n \in \mathbb{N}$, $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$

Proof. By induction:

Basic Step: n = 1 is trivial

Inductive Step: Assume the statement is true as given for some $n \geq 1$.

Show $(a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k}$

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n \\
&= (a+b)\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k} + \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k+1} \\
&= \sum_{k=0}^{n-1} \binom{n}{k} a^{k+1} b^{n-k} + a^{n+1} + b^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^k b^{n-k+1} \\
&= \sum_{j=1}^{n} \binom{n}{j-1} a^j b^{n-(j-1)} + a^{n+1} + b^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^k b^{n+1-k} \\
&= b^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k-1} + \binom{n}{k}\right] a^k b^{n+1-k} + a^{n+1} \\
&= b^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^k b^{n+1-k} + a^{n+1} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k}
\end{aligned}$$

□

1.2.2 Binomial Series

Consider $B(a,b) = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$. This finite sum$_i sknownastheBinomialSeries.$

1.2.2.1

Show that $B(a,b) = (a+b)^n$
Show that $B(1,1) = 2^n$
Show that $B(-1,1) = 0$
Show that $B(p,1-p) = 1$
Easily, $B(x,1) = \sum_{k=0}^{n} \binom{n}{k} a^k$

1.2.3 Trinomial Series

$$(a+b+c)^n = \sum_{k_1+k_2+k_3=n} \binom{n}{k_1,k_2,k_3} a^{k_1} b^{k_2} c^{k_3}$$

where $\binom{n}{k_1,k_2,k_3} = \frac{n!}{k_1!k_2!k_3!}$. This can be generalized to any number of terms to give what is know as a multinomial series.

1.3 Negative Binomial Series

$(a+b)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} a^k b^{-n-k}$

Theorem 1.3.1 (Alternate Form for Negative Binomial Series)**.** $(a+b)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} a^k b^{-n-k}$

Chapter 2

Representing Data

2.1 Measurement Scales

- Nominal - Mutually Exclusive and Exhaustive categories for which the numerical value has only identification significance. Ex: Male = 1, Female = -1
- Ordinal - Discrete values ranked from lowest to highest or vice versa. Ex: Class grades for GPA.
- Interval - Ordinal data where distance between data values is of significance. Ex: Heights and Weights.
- Ratio - Interval data where ratios of observations have meaning. Ex: Percentile rankings

2.2 Techniques for Representing Data

- Tabular Methods - based on the entire population yielding a global picture
    - frequency distributions
    - relative frequency distributions
    - cummulative frequency distributions
    - Stem-and-Leaf Displays
    - Box-and-Whisker Diagrams
- Summary Methods
    - Measures of the center
        1. Mean
        2. Median
        3. Mode
    - Measures of spread
        1. Range
        2. Variance and Standard Deviation
        3. Quantiles

- Measures of Skewness - indicates the level of symmetry of the data
  1. Pearson Coefficient
  2. Standard Skewness
  3. Bowley's Measure
- Measures of Kurtosis - indicates flatness or roundedness of the peak of the data
  1. Standard Kurtosis
  2. Coefficient of Kurtosis
- Measures of Association for Bivariate Data - indicates the likeliness of functional correlation of the data.
  1. Pearson Correlation Coefficient
  2. Spearman Rank Correlation Cooeficient
  3. Quantile-Quantile Plots
- Detection of Outliers - indicates whether abnormally large or small data distorts other techniques
  1. Z-scores
  2. Trimming
  3. Winsorizing
- Tests for Normality - indictes if the data is bell-shaped
  1. Standard Percentages relative to standard deviations from the mean
  2. Chi-square
  3. Kolmogorov-Smirnov
  4. Lilliefors
  5. Shapiro-Wilk
- Tests for Randomness - indicates whether the data has a non-systematic pattern
  1. Runs Test
  2. Mean-Square Successive Differences

Remark: Many of these measures above are relative and some are absolute.

2.3 Measures of Position

Given a collection of data, sorting the data may provide several useful descriptors. These include:

Definition 2.3.1 (Order Statistic:)**.** Given the given data set $x_1, x_2, ..., x_n$, after sorting the data label the sorted data as $y_1, y_2, ..., y_n$ where

$$y_1 \leq y_2 \leq ... \leq y_n.$$

Then, the kth order statistic is given by y_k.

For example, the age at inauguration for presidents from 1981-2016 gives the data $x_1 = 69, x_2 = 64, x_3 = 46, x_4 = 54, x_5 = 47$ (Reagan, Bush, Clinton, Bush, Obama). For this data, the order statistics are denoted $y_1 = 46, y_2 = 47, y_3 = 54, y_4 = 64, y_5 = 69$.

Definition 2.3.2 (Minimum/Maximum:)**.** The smallest and largest values in the data set. Using the notation above, minimum $= y_1$ and the maximum $= y_n$

Using the Presidential ages above, minimum $= y_1 = 46$ and maximum $= y_5 = 69$.

Definition 2.3.3 (Percentiles:)**.** A percentile is a numerical value P^p at which approximately 100p

To motivate your understanding of percentiles, consider the following data set: 2,5,8,10. The 50th percentile should be a numerical value for which approximately 50

To compute the percentile value exactly consider a percentage in the form 100p, for $0 < p < 1$, and the order statistics $y_1, y_2, ..., y_n$. Then, the 100pth percentile is given by

$$P^p = (1-r)y_m + ry_{m+1}$$

where m is the integer part of (n+1)p, namely

$$m = \lfloor (n+1)p \rfloor$$

and

$$r = (n+1)p - m,$$

the fractional part of (n+1)p. This determines a weighted average between y_m and y_{m+1} which is unique for distinct values of p provided each of the data values are distinct. Note that if some of the y-values are equal then some of these averages might be of equal numbers and will then be the common value.

Example 2.3.4 (Basic Percentiles)**.** Using the data set 2,5,8,10 with n=4 values, the 25th percentile is computed by considering

$$(n+1)p = (4+1)0.25 = 5/4 = 1.25$$

. So, m = 1 and r = 0.25. Therefore

$$P^{0.25} = 0.75 \times 2 + 0.25 \times 5 = 2.75$$

as noted above.

Similarly, the 75th percentile is given by

$$(n+1)p = (4+1)0.75 = 15/4 = 3.75$$

. So, m = 3 and r = 0.75. Therefore

$$P^{0.75} = 0.25 \times 8 + 0.75 \times 10 = 9.5$$

It is interesting to note that 3 also lies between 2 and 5 as does 2.75 and has the same percentages above (75 percent) and below (25 percent). However, it should designate a slightly larger percentile location. Indeed, going backward:

$$\begin{aligned} 3 &= (1-r) \times 2 + r \times 5 \\ \Rightarrow r &= \frac{1}{3} \\ \Rightarrow (n+1)p &= 1 + \frac{1}{3} = \frac{4}{3} \\ \Rightarrow p &= \frac{4}{15} \approx 0.267 \end{aligned}$$

and so 3 would actually be at approximately the 26.7th percentile.

Definition 2.3.5 (Quartiles:)**.** Given a sorted data set, the first, second, and third quartiles are the values of $Q_1 = P^{0.25}, Q_2 = P^{0.5}$ and $Q_3 = P^{0.75}$.

Definition 2.3.6 (Deciles:)**.** Given a sorted data set, the first, second, ..., ninth deciles are the value of $D_1 = P^{0.1}, D_2 = P^{0.2}, ..., D_9 = P^{0.9}$

For your data set 2,5,8,10, $Q_1 = 2.75, Q_2 = 6.5$, and $Q_3 = 9.5$.

Definition 2.3.7 (5-number summary)**.** Given a set of data, the 5-number summary is a vector of the order statistics given by < minimum, Q_1, Q_2, Q_3, maximum >.

Returning to our previous example, the five number summary would be $< 2, 2.75, 6.5, 9.5, 10 >$

2.4 Measures of the Middle

Definition 2.4.1 (Arithmetic Mean)**.** Suppose X is a discrete random variable with range $R = x_1, x_2, ..., x_n$. The arithmetic mean is given by

$$AM = \frac{x_1 + ... + x_n}{n} = \frac{\sum_{k=1}^{n} x_k}{n}.$$

If this data comes from sample data then we call it a sample mean and denote this value by $\overline{x}$. If this data comes from the entire universe of possibilities then we call it a population mean and denote this value by μ.

To illustrate, consider the previous data set: 2,5,8,10. The arithmetic mean is given by

$$\frac{2+5+8+10}{4} = \frac{25}{4} = 6.25.$$

The mean is often called the centroid in the sense that if the x values were locations of objects of equal weight, then the centroid would be the point where this system of n masses would balance.

The values can all be provided with varying weights if desired and the result is called the weighted arithmetic mean and is given by

$$\frac{m_1 x_1 + ... + m_n x_n}{m_1 + ... + m_n} = \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k}.$$

Other Means:

Definition 2.4.2 (Geometric Mean)**.**

$$GM = (x_1 x_2 ... x_n)^{1/n}$$

Again, consider 2,5,8,10. The geometric mean is given by

$$(2 \times 5 \times 8 \times 10)^{1/4} = 800^{1/4} \approx= 5.318$$

Definition 2.4.3 (Harmonic Mean)**.**

$$\frac{1}{HM} = \frac{1}{n}\sum_{k=1}^{n}\frac{1}{x_k}$$

Once again, consider 2,5,8,10. The harmonic mean is given by first computing

$$\frac{1}{4}(\frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \frac{1}{10}) = 800^{1/4} \approx= 0.23125$$

and so HM $= \frac{1}{0.23125} approx 4.32$

Theorem 2.4.4 (Relative sizes of Means)**.** $HM \leq GM \leq AM$.

Theorem 2.4.5 (Mean Formula)**.** $AMHM = GM^2$

Definition 2.4.6 (Median:)**.** A positional measure of the middle is often utilized by finding the location of the 50th percentile. This value is also called the median and indicates the value at which approximately half the sorted data lies below and half lies above.

For data sets with an odd number of values, this is the "middle" data value if one were to successively cross off pairs from the two ends of the sorted date. For data sets with an even number of values, this is a average of the two data values left after crossing off these pairs. Using the order statistics, the median equals

$$y_{\frac{n+1}{2}}$$

if n is odd and

$$\frac{y_{\frac{n}{2}} + y_{\frac{n}{2}+1}}{2}$$

if n is even.

From the Presidential data, note that you are considering an odd number of data values and so the median is given by 54.

Definition 2.4.7 (Midrange:)**.** A mixture of the mean and median where one takes the simple average of the maximum and minimum values in the data set. Using the order statistics, this equals

$$\frac{y_1 + y_n}{2}$$

From the Presidential data, the maximum is 69 and the minimum is 46 so the midrange is 57.5, the average of these two.

Mean utilizes all of the data values so each term is important. Utilizes them all even if some of the data values might suffer from collection errors. Median ignores outliers (which might be a result of collection errors) but does not account for the relative differences between terms. Midrange is very easy to compute but ignores the relative differences for all terms but the two extremes.

Example 2.4.8 (Numerical Example of these Quantitative Measures)**.** The US Census Bureau reported the following state populations (in millions) for 2013: Spreadsheet

Determine the minimum, maximim, midrange, and mean for this data. Notice that these are already in order so you can presume $y_1 = 0.6$ million is the minimum and $y_{50} = 38.3$ million is the maximum. Therefore, the midrange is given by

$$\frac{0.6 + 38.3}{2} = \frac{38.9}{2} = 19.45\text{million}.$$

Note, in this collection of "states" data the District of Columbia is included so that the number of data items is n=51. The mean of this data takes a bit of arithmetic but gives

$$\frac{\sum_{k=1}^{51} y_k}{51} = \frac{316.1}{51} \approx 6.20$$

million residents.

Since the number of states is odd, the median is found by looking at the 26th order statistics. In this case, that is the 4.6 million residents of Louisiana.

State	Population
Wyoming	0.6
Vermont	0.6
District of Columbia	0.6
North Dakota	0.7
Alaska	0.7
South Dakota	0.8
Delaware	0.9
Montana	1
Rhode Island	1.1
New Hampshire	1.3
Maine	1.3
Hawaii	1.4
Idaho	1.6
West Virginia	1.9
Nebraska	1.9
New Mexico	2.1
Nevada	2.8
Kansas	2.9
Utah	2.9
Arkansas	3
Mississippi	3
Iowa	3.1
Connecticut	3.6
Oklahoma	3.9
Oregon	3.9
Kentucky	4.4
Louisiana	4.6
South Carolina	4.8
Alabama	4.8
Colorado	5.3
Minnesota	5.4
Wisconsin	5.7
Maryland	5.9
Missouri	6
Tennessee	6.5
Indiana	6.6
Arizona	6.6
Massachusetts	6.7
Washington	7
Virginia	8.3
New Jersey	8.9
North Carolina	9.8
Michigan	9.9
Georgia	10
Ohio	11.6
Pennsylvania	12.8
Illinois	12.9
Florida	19.6
New York	19.7
Texas	26.4
California	38.3

2.5 Measures of Spread

Definition 2.5.1 (Range:)**.** Using the order statistics,

$$y_n - y_1.$$

Easy to compute. Ignores the spread of all the data in between.

From the Presidential data, the maximum is 69 and the minimum is 46 so the range is 23, the difference of these two.

Definition 2.5.2 (Interquartile Range (IQR):)**.** $P^{0.75} - P^{0.25}$.

For the data set 2, 5, 8, 10, you have found that $Q_1 = 2.75$ and $Q_3 = 9.5$. Therefore,

$$IQR = 9.5 - 2.75 = 6.75.$$

Average Deviation from the Mean: Given a data set $x_1, x_2, ..., x_n$ with mean μ each term deviates from the mean by the value $x_k - \mu$. So, averaging these gives

$$\frac{\sum_{k=1}^{n}(x_k - \mu)}{n} = \frac{\sum_{k=1}^{n} x_k}{n} - \frac{\sum_{k=1}^{n} \mu}{n} = \mu - \mu = 0$$

which is always zero for any provided set of data. This cancellation makes this measure not useful. To avoid cancellation, perhaps removing negatives would help.

Average Absolute Deviation from the Mean:

$$\frac{\sum_{k=1}^{n} |x_k - \mu|}{n}$$

which, although nicely stated, is difficult to deal with algebraically since the absolute values do not simplify well algebraically. To avoid this algebraic roadblock, we can look for another way to nearly accomplish the same goal by squaring and then square rooting.

Average Squared Deviation from the Mean:

$$\frac{\sum_{k=1}^{n}(x_k - \mu)^2}{n}$$

which will always be non-negative but can be easily expanded using algebra. Since this is a mouthful, this measure is generally called the variance.

Using the average squared deviation from the mean, differences have been squared. Thus all values added are non-negative but very small ones have been made even smaller and larger ones have possibly been made much larger. To undo this scaling issue, one must take a square root to get things back into the right ball park.

Definition 2.5.3 (Variance and Standard Deviation)**.** The variance is the average squared deviation from the mean. If this data comes from the entire universe of possibilities then we call it a population variance and denote this value by σ^2. Therefore

$$\sigma^2 = \frac{\sum_{k=1}^{n}(x_k - \mu)^2}{n}$$

The standard deviation is the square root of the variance. If this data comes from the entire universe of possibilities then we call it a population standard deviation and denote this value by σ. Therefore

$$\sigma = \sqrt{\frac{\sum_{k=1}^{n}(x_k - \mu)^2}{n}}.$$

From the data 2,5,8,10, you have found that the mean is 6.25. Computing the variance then involves accumulating and averaging the squared differences of each data value and this mean. Then

$$\begin{aligned}&\frac{1}{4}\left((2-6.25)^2+(5-6.25)^2+(8-6.25)^2+(10-6.25)^2\right)\\&=\frac{18.0625+1.5625+3.0625+14.0625}{4}\\&=\frac{36.75}{4}\\&=9.1875.\end{aligned}$$

If data comes from a sample of the population then we call it a sample variance and denote this value by v. Since sample data tends to reflect certain "biases" then we increase this value slightly by $\frac{n}{n-1}$ to give the sample variance

$$s^2=\frac{n}{n-1}\frac{\sum_{k=1}^{n}(x_k-\overline{x})^2}{n}=\frac{\sum_{k=1}^{n}(x_k-\overline{x})^2}{n-1}.$$

and the sample standard deviation similarly as the square root of the sample variance.

Theorem 2.5.4 (Alternate Forms for Variance)**.**

$$\begin{aligned}\sigma^2&=\left(\frac{\sum_{k=1}^{n}x_k^2}{n}\right)-\mu^2\\&=\left[\frac{\sum_{k=1}^{n}x_k(x_k-1)}{n}\right]+\mu-\mu^2\end{aligned}$$

Proof. □

The Population of the individual USA states according to the 2013 Census Consider the data set

Exercise 2.5.5 (Numerical Example of these Quantitative Measures)**.**

2.6 Grouped Data

As you considered the measures of the center and spread before, each data point was considered individually. Often, data may however be grouped into categories and perhaps expressed as a frequency distribution. In this case, rather than considering x_k to be the kth data value can take advantage of the grouping to perhaps save a bit on arithmetic.

Indeed, let's assume that data is grouped into m categories $x_1, x_2, ..., x_m$ with corresponding frequencies $f_1, f_2, ..., f_m$. Then, for example, when computing the mean rather than adding x_1 with itself f_1 times just compute $x_1 \times f_1$ for the first category and continuing through the remaining categories. This gives the following grouped data formula for the mean

$$\mu=\frac{x_1f_1+...+x_mf_m}{f_1+...+f_m}=\frac{\sum_{k=1}^{m}x_kf_k}{\sum_{k=1}^{m}f_k}.$$

and the following grouped data formula for the variance

$$\sigma^2=\frac{\sum_{k=1}^{m}(x_k-\mu)^2f_k}{\sum_{k=1}^{m}f_k}=\frac{\sum_{k=1}^{m}x_k^2f_k}{\sum_{k=1}^{m}f_k}-\mu^2$$

Exercise 2.6.1.

2.7 Other Point Measures

Beyond measures of the middle and of spread includes a way you can determine if data is heaped up to one side or the other of the mean. One such measure is the skewness...

Definition 2.7.1 (Skewness)**.** The Skewness of $x_1, x_2, ..., x_n$ is given by

$$\frac{1}{\sigma^3}\frac{\sum_{k=1}^n (x_k - \overline{x})^3}{n}.$$

A positive skewness indicates that the positive $(x_k - \overline{x})^3$ terms overwhelm the negative terms. Therefore, this indicates data which is strung out to the right. Likewise, a negative skewness indicates data which is strung out to the left.

In addition to skewness, data might tend to be clustered around the mean and often in a "bell-shaped" manner. The kurtosis can be used o measure how closely data resembles a bell-shaped collection.

Definition 2.7.2 (Kurtosis)**.** The Kurtosis of $x_1, x_2, ..., x_n$ is given by

$$\frac{1}{\sigma^4}\frac{\sum_{k=1}^n (x_k - \overline{x})^4}{n}.$$

A kurtosis of 3 indicates that the data is perfectly bell shaped (a "normal" distribution) whereas data further away from 3 indicates data that is less bell shaped.

Theorem 2.7.3 (Alternate Formulas for Skewness and Kurtosis)**.** *Skewness* $=$

$$\frac{1}{s^3}\left[\frac{\sum_{k=1}^n x_k^3}{n} - 3v\overline{x} - \overline{x}^3\right]$$

and Kurtosis $=$

$$\frac{1}{s^4}\left[\frac{\sum_{k=1}^n x_k^4}{n} - 4\overline{x}\frac{\sum_{k=1}^n x_k^3}{n} + 6\overline{x}^2 v - 3\overline{x}^4\right]$$

Proof. For skewness, expand the cubic and break up the sum. Factoring out constants (such as $\overline{x}$) gives

$$\begin{aligned}
&\frac{\sum_{k=1}^n (x_k - \overline{x})^3}{n} \\
&= \frac{\sum_{k=1}^n x_k^3}{n} - 3\overline{x}\frac{\sum_{k=1}^n x_k^2}{n} + 3\overline{x}^2\frac{\sum_{k=1}^n x_k}{n} - \frac{\sum_{k=1}^n \overline{x}^3}{n} \\
&= \frac{\sum_{k=1}^n x_k^3}{n} - 3\overline{x}(v + \overline{x}^2) + 3\overline{x}^3 - \overline{x}^3 \\
&= \frac{\sum_{k=1}^n x_k^3}{n} - 3\overline{x}v - \overline{x}^3
\end{aligned}$$

and divide by the cube of the standard deviation to finish. Note that the first expansion in the derivation above can be used quickly if the data is collected in a table and powers easily computed.

For kurtosis, similarly expand the quartic and break up the sum as before. Note that you can extract the value of the cubic term by solving for that term

k	x_k
1	8
2	12
3	6
4	3
5	1
6	2

in the skewness formula above. Then,

$$\begin{aligned}
&\frac{\sum_{k=1}^{n}(x_k-\overline{x})^4}{n} \\
&= \frac{\sum_{k=1}^{n} x_k^4}{n} - 4\overline{x}\frac{\sum_{k=1}^{n} x_k^3}{n} + 6\overline{x}^2\frac{\sum_{k=1}^{n} x_k^2}{n} - 4\overline{x}^3\frac{\sum_{k=1}^{n} x_k}{n} + \frac{\sum_{k=1}^{n} \overline{x}^4}{n} \\
&= \frac{\sum_{k=1}^{n} x_k^4}{n} - 4\overline{x}\frac{\sum_{k=1}^{n} x_k^3}{n} + 6\overline{x}^2(v+\overline{x}^2) - 4\overline{x}^4 + \overline{x}^4 \\
&= \frac{\sum_{k=1}^{n} x_k^4}{n} - 4\overline{x}\frac{\sum_{k=1}^{n} x_k^3}{n} + 6\overline{x}^2 v - 3\overline{x}^4
\end{aligned}$$

and then divide by the fourth power of the standard deviation. Note again that the first expansion in the derivation above might also be a useful shortcut. □

2.8 Graphical Representation of Data

Data sets can range from small to very large. Visual representations of these data sets often allow you to see trends and reveal a lot about the distribution of the data values.

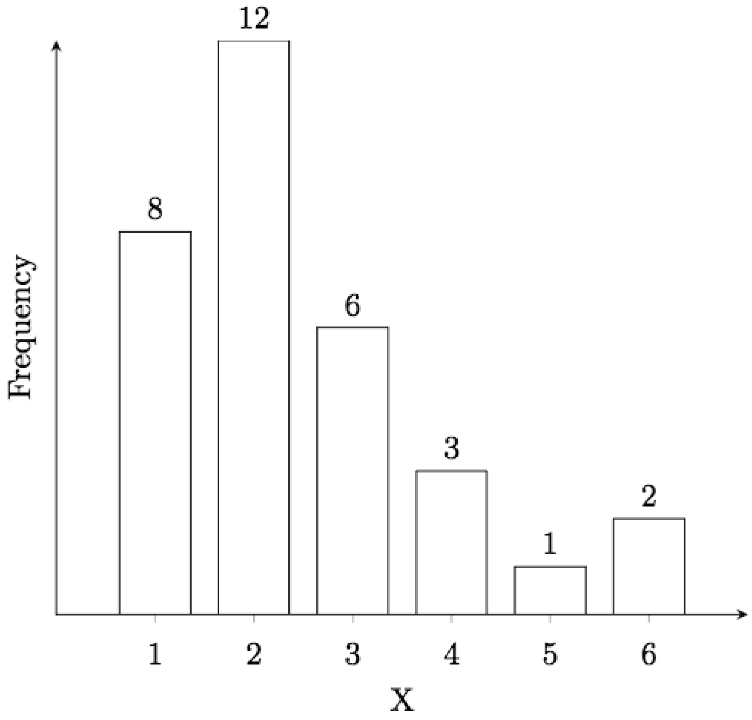
Also, probability mass functions for discrete variables can be graphed as a set of points but sometimes these points do not convey size very well. A visual representation of these functions needs to be addressed.

2.8.1 Histograms

Frequency Histograms - height matters

Consider the data set given by

A frequency histogram representing this data can be given by



Experiment with creating your own histogram by inputting your data into the interactive cell below.

```
#  This function is used to convert an input string into
    separate entries
def g(s): return str(s).replace(’,’,’␣’).replace(’(’,’␣
    ’).replace(’)’,’␣’).split()

@interact
def _(freq =
    input_box("1,1,1,1,2,2,2,3,3,3,3,1,5",label="Enter␣
    data␣separated␣by␣commas")):
     freq = g(freq)
     freq = [int(k) for k in freq]
     m = min(freq)
     M = max(freq)
     bn = M-m+1
     histogram( freq, range=[m-1/2,M+1/2], bins = bn,
         align="mid", linewidth=2, edgecolor="blue",
         color="yellow").show()
```

Relative Frequency Histograms - In this case, area describes your data. Notice in the interactive cell above that each bar is of width one. Therefore, frequency = area. In some instances where data may be grouped the total width of the interval may be different and so the height will need to be adjusted so that the total area of each bar corresponds to the relative frequency of that category.

Cummulative Histograms. In these a running total is presented using all values from the given point and below.

```
#  This function is used to convert an input string into
    separate entries
def g(s): return str(s).replace(’,’,’␣’).replace(’(’,’␣
    ’).replace(’)’,’␣’).split()

@interact
def _(freq =
    input_box("1,1,1,1,2,2,2,3,3,3,3,1,5",label="Enter␣
    data␣separated␣by␣commas")):
     freq = g(freq)
     freq = [int(k) for k in freq]
     top = len(freq)
     m = min(freq)
     M = max(freq)
     bn = M-m+1
     histogram( freq, range=[m-1/2,M+1/2], cumulative =
         "true", bins = bn, align="mid", linewidth=2,
         edgecolor="blue", color="yellow").show(ymax=top)
```

Stem-and-Leaf Plot - Histogram with data. Using the state population data above, consider organizing the data but using a "two-pass sort" where you first roughly break data up into groups based upon ranges which relate to their first digit(s). In this case, let’s break up into groups according to populations corresponding to 0-4 million, 5-9 million, 10-14 million, 15-19, million, 20-24 million, 25-29 million, 30-35 million, and 35-39 million. We can represent these classes by using the stems 0L, 0H, 1L, 1H, 2L, 2H, 3L, and 3H where the L and H represent the one’s digits L in 0, 1, 2, 3, 4 and H in 5, 6, 7, 8, 9. Once we group the data into these smaller groups then we can write the remaining portion of the number horizontally as leaves (in this case with one decimal place for all values.) This gives a step-and-leaf plot. If we additionally sort the data in the leaves then this gives you an ordered stem-and-leaf plot. For the state population data, the ordered stem-and-leaf plot is given by

Table 1: Stem Plot for State Populations

Stem	Leaf
0L	06 06 07 07 08 09 10 11 13 13 14 16 19 19 21 28 29 29 30 30 31 36 39 39 44 46 48 48
0H	53 54 57 59 60 65 66 66 67 70 83 89 98 99
1L	10 16 28 29
1H	96 97
2L	
2H	64
3L	
3H	83

Notice how it is easy to now see that most state populations are relatively small and that there are relatively few states with larger population. Also, notice that you can use this plot to relatively easily identify minimum, maximum, and other order statistics.

Box and Whisker Diagram - visual order statistics. This graphical display identifies the "5-number-summary" associated with the minimum, quartiles, and the maximum. That is, y_1, Q_1, Q_2, Q_3, y_n. These values separate the data roughly into quarters. To distinguish these quarters connect y_1 and Q_1 with a straight line (a whisker) and do the same with Q_3 and y_n. Use a box to connect Q_1 with Q_2 and the same to connect Q_2 with Q_3. Then the boxed areas also identify the IQR.

```
from pylab import boxplot,savefig,close
@interact
```

```
def _(data =
    input_box([1,2,3,4,6,7,8,9,11,15,21],label="Enter␣
    Your␣Data:")):
     B = boxplot(data, notch=True, sym=’x’, vert=False)
     savefig("boxplot.png")
     close()
```

2.9 Exercises

Complete the online homework "Computational Measures".

Exercise 2.9.1.

Exercise 2.9.2.

Chapter 3

Counting and Combinatorics

3.1 Introduction

Discussion on the usefulness of having ways to count the number of elements in a set without having to explicitly listing all elements.

Consider counting the number of ways one can arrange Peter, Paul, and Mary with the order important. Listing the possibilities:

- Peter, Paul, Mary
- Peter, Mary, Paul
- Paul, Peter, Mary
- Paul, Mary, Peter
- Mary, Peter, Paul
- Mary, Paul, Peter

So, it is easy to see that these are all of the possible outcomes and that the total number of such outcomes is 6. What happens however if we add Simone to the list?

- Simone, Peter, Paul, Mary
- Simone, Peter, Mary, Paul
- Simone, Paul, Peter, Mary
- Simone, Paul, Mary, Peter
- Simone, Mary, Peter, Paul
- Simone, Mary, Paul, Peter
- Peter, Simone, Paul, Mary
- Peter, Simone, Mary, Paul
- Paul, Simone, Peter, Mary
- Paul, Simone, Mary, Peter
- Mary, Simone, Peter, Paul
- Mary, Simone, Paul, Peter

- Peter, Paul, Simone, Mary
- Peter, Mary, Simone, Paul
- Paul, Peter, Simone, Mary
- Paul, Mary, Simone, Peter
- Mary, Peter, Simone, Paul
- Mary, Paul, Simone, Peter
- Peter, Paul, Mary, Simone
- Peter, Mary, Paul, Simone
- Paul, Peter, Mary, Simone
- Paul, Mary, Peter, Simone
- Mary, Peter, Paul, Simone
- Mary, Paul, Peter, Simone

Notice how the list quickly grows when just adding one more choice. This illustrates how keeping track of the number of items in a set can quickly get impossible to keep up with and to count unless we can approach this problem using a more mathematical approach.

Definition 3.1.1 (Cardinality)**.** Given a set of elements A, the number of elements in the set is known as the sets cardinality and is denoted |A|. If the set has an infinite number of elements then we set |A| = ∞.

In order to "count without counting" we establish the following foundational principle.

Theorem 3.1.2 (Multiplication Principle)**.** *Given two successive events A and B, the number of ways to perform A and then B is |A||B|.*

Proof. If either of the events has infinite cardinality, then it is clear that the number of ways to perform A and then B will also be infinite. So, assume that both |A| and |B| are finite. In order to count the successive events, enumerate the elements in each set

$$A = \left\{a_1, a_2, a_3, ..., a_{|A|}\right\}$$
$$B = \left\{b_1, b_2, b_3, ..., b_{|B|}\right\}$$

and consider the function f(k,j) = (k-1)|B| + j. This function is one-to-one and onto from the set

$$\{(k,j) : 1 \leq k \leq |A|, 1 \leq j \leq |B|\}$$

onto

$$\{s : 1 \leq s \leq |A||B|\}\,.$$

Since this second set has |A| |B| elements then the conclusion follows. coordinates. □

Definition 3.1.3 (Factorial)**.** For any natural number n,

$$n! = n(n-1)(n-2)...3 \cdot 2 \cdot 1$$

Example 3.1.4 (iPad security code)**.** Consider your ipad's security. To unlock the screen you need to enter your four digit pass code. How easy is it to guess this pass code?

Using the standard 10 digit keypad, we first have two questions to consider?

1. Does the order in which the digits are entered matter?
2. Can you reuse a digit more than once?

For the ipad, the order does matter and you cannot reuse digits. In this case, the number of possible codes can be determined by considering each digit as a separate event with four such events in succession providing the right code. By successively applying the multiplication principle, you find that the number of possible codes is the number of remaining available digits at each step. Namely, $10 \times 9 \times 8 \times 7 = 5040$.

Note that if you were allowed to reuse the digits then the number of possible outcomes would be more since all 10 digits would be available for each event. Namely, $10 \times 10 \times 10 \times 10 = 10000$.

Example 3.1.5 (iPad security code with greasy fingers)**.** Reconsider your ipad's security. In this case, you like to eat chocolate bars and have greasy fingers. When you type in your passcode your fingers leave a residue over the four numbers pressed. If someone now tries to guess your passcode, how many possible attempts are necessary?

Since there are only four numbers to pick from with order important, the number of possible passcodes remaining is $4 \times 3 \times 2 \times 1 = 24$

Example 3.1.6 (National Treasure)**.** In the 2004 movie "National Treasure" Ben and Riley are attempting to guess Abagail's password to enter the room with the Declaration. They are able to determine the passphrase to get into the vault room by doing a scan that detects the buttons pushed (not due to chocolate but just due to the natural oils on fingers). They notice that the buttons pushed include the characters AEFGLORVY.

Assuming these characters are used only once each, how many possible passphrases are possible?

In this case, the order of the characters matters but all of the characters are distinct. Since we have 9 characters provided, the we can consider each character as an event with the first event as a choice from the 9, the second event as a choice from the remaining 8, etc. This gives $9 \times 8 \, times... \times 1 = 362880$ possible passphrases.

Assuming that some of the characters could be used more than once, how many passphrases need to be considered if the total length of passphrase can be at most 12 characters?

Notice, in this case you don't know which characters might be reused and so the number of possible outcomes will be much larger. What is the answer?

You can break this problem down into distinct cases:

- Using 9 characters This is the answer computed above.
- Using 10 characters In this case, 1 character can be used twice. To determine the number of possibilities, let's first pick which character can be doubled. There are 9 options for picking that character. Next, if we consider the two instances of that letter as distinct values then we can just count the number of ways to arrange unique 10 characters which is 10! However, swapping the two characters (which are actually identical) would not give a new passphrase. Since these are counted twice, let's divide these out to give $10!/2$.

- Using 11 characters In this situation we have two unique options:
  - One character is used three times and the others just once. Continuing as in the previous case, 11!/3!.Two characters are used twice and the others just once.
- Using 12 characters
  1. One letter from the nine is used four times and all the others are used once.
  2. One letter is used three times, another letter is used two times, and the others are used once.
  3. Three letters are used twice and the others are used once.

With this large collection of possible outcomes, how are the movie characters able to determine the correct "VALLEYFORGE" passphrase?

3.2 Permutations

When counting various outcomes the order of things sometimes matters. When the order of a set of elements changes we call the second a permutation (or an arrangement) of the first.

Theorem 3.2.1 (Permutations of n objects)**.** *The number of ways to arrange n distinct items is n!*

Proof. Notice that if n=1, then there is only 1 item to arrange and that there is only one possible arrangment.

By induction, assume that any set with n elements has n! arrangments and assume that

$$|A| = \{a_1, a_2, ..., a_n, a_{n+1}\}.$$

Notice that there are n+1 ways to choose 1 element from A and that in doing so leaves a set with n elements. Combining the induction hypothesis with the multiplication principle this gives (n+1)n! = (n+1)! possible outcomes. □

Theorem 3.2.2 (Permutations of n objects selecting a subset of size r without replacement)**.** *The number of ways to arrange r items from a set of n distinct items is* $P^n_r = \frac{n!}{(n-r)!}$

Proof. If $r > n$ or $r < 0$ then this is not possible and so the result would be no permuatations. Otherwise, apply the multiplication principle r times noting that there are n choices for the first selection, n-1 choices for the second selection, and with n-r+1 choices for the rth selection. This gives

$$\begin{aligned} P^n_r &= n(n-1)...(n-r+1) \\ &= n(n-1)...(n-r+1)\frac{(n-r)!}{(n-r)!} \\ &= \frac{n(n-1)...(n-r+1)(n-r)!}{(n-r)!} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

□

Theorem 3.2.3 (Permutations of n objects selecting r times with replacement)**.** *The number of ways to obtain an arrangement of r choices from a group of size n is n^r*

Proof. Use the multiplication principle r times and see that for each choice all n objects in the universe remain available. That is,

$$n \cdot n \cdot n ... n = n^r$$

□

Theorem 3.2.4 (Permutations when not all items are distinguishable and without replacement: (Multinomial Coefficients))**.** *If n items belong to s categories, n_1 in first, n_2 in second, ... , n_s in the last, the number of ways to pick all is*

$$\frac{n!}{n_1! \cdot n_2! ... n_2!}$$

3.3 Combinations

When counting various outcomes sometimes the order of things does not matter. In this case we count each different set of outcomes a combination.

Theorem 3.3.1 (Combinations of n distinct objects selecting r without replacement)**.** *The number of ways to arrange r items from a set of n distinct items is $C^n_r = \frac{n!}{r!(n-r)!}$*

Proof. Consider creating a permutation of r objects from a set of size n by first picking an unordered subset of size r and then counting the number of ways to order that subset. Using our notation and the multiplication principle,

$$P^n_r = C^n_r \cdot r!$$

Solving give the result. □

Theorem 3.3.2 (Combinations of n distinct objects selecting r with replacement)**.** *The number of ways to arrange r items from a set of n distinct items is $C^{n+r-1}_r = \frac{(r+n-1)!}{r!(n-1)!}$*

Proof. Label each item in your group in some defined order. Since order doesn't matter, as you repeatedly sample r times with replacement you can always write down your outcomes sorted from low to high placement. Finally, separate like values by some symbol, say "|", and consider each of the n distinct objects as indistinct *'s. There will be n-1 of these separators since there will be n to choose from. For example, if choosing r=6 times from the set a, b, c, d, then the outcome b, b, a, d, a, b could be collected as a, a, b, b, b, d and written in our code as **|***||* . Notice that shuffling around the identical *'s would not change the code (and similarly for the identical |'s) but swapping a * with a | would be a different outcome. Therefore, we can consider this to be a multinomial coefficient and the number of ways to rearrange this code is

$$\frac{(r+n-1)}{r!(n-1)!}.$$

□

Example 3.3.3. Revisiting your ipad's security, what happens if the order in which the digits are entered does not matter? If so, then you would be picking a combination of 4 digits without replacement from a group of 10 digits. Namely,

$$\begin{aligned}\frac{10!}{4!6!} &= \frac{10 \times 9 \times 8 \times 7 \times 6!}{4 \times 3 \times 2 \times 1 \times 6!} \\ &= \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} \\ &= \frac{5040}{24} \\ &= 210.\end{aligned}$$

Notice that the total number of options is much smaller when order does not matter.

Note that if you were allowed to reuse the digits then the number of possible outcomes would be

$$\begin{aligned}\frac{13!}{3!10!} &= \frac{13 \times 12 \times 11}{3 \times 2 \times 1} \\ &= 286\end{aligned}$$

which once again is more since numbers are allowed to repeat.

Definition 3.3.4 (Binomial Coefficients)**.** The value C_r^n is known as the binomial coefficient. It is denoted by $\binom{n}{r}$ and is read "n choose k".

Theorem 3.3.5 (Combinations when distinguishable and with replacement)**.** = *Number of ways to get unordered samples of size r from n objects.*

Lots of interesting facts about the binomial coefficients.

3.4 Exercises

Complete the online homework "Counting".

A standard deck of playing cards consists of 52 cards broken up into four "suits" known as Hearts, Spades, Diamonds, and Clubs. Each suit is broken up additionally into unique cards with "face values" from 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace and generally in that order from low to high.

1. Pick two cards without replacement one after the other from this deck and determine the following number of possible outcomes:

- The number of ways to get an Ace for both cards.
- The number of ways to get an Ace for only one of the two cards.
- The number of ways to get an Ace on the first draw and a Spade on the second draw.

2. Pick five cards without replacement one after the other from a newly shuffled full deck and determine the following number of possible outcomes:

- All cards have different faces
- "A pair". That is, two cards have the same face but the others are from three other faces.
- "Three of a kind". That is, three cards have the same face but the others are from two other faces.

- "Two Pair". That is, two cards come from one face, two other cards come from a common face that is not the same as the first two cards, and the last card comes from some other face.
- "Full House". That is, three cards have the same face and the other two come from a common face that is not the same as the first three cards.
- "Four of a Kind". That is, four cards have the same face and the other card comes from some other face.
- "Flush". That is, the five cards for a sequence in order of adjacent faces in the original list and from the same suit.
- "Royal Flush". That is, a flush but only with the cards Ace, King, Queen, Jack, 10.

Completely determine the number of possible passphrases for the National Treasure example started above. Present your answer in a report form.

Chapter 4

Probability Theory

This chapter uses relative frequency to motivate the definition of probability and then delves into the resulting consequences.

4.1 Relative Frequency

Mathematics generally focuses on providing precise answers with absolute certainty. For example, solving an equation generates specific (and non-varying) solutions. Statistics on the other hand deals with providing precise answers to questions when there is uncertainty. It might seem impossible to provide such precise answers but the focus of this text is to show how that can be done so long as the questions are properly posed and the answers properly interpreted.

People often make claims about being the biggest, best, most often recommended, etc. One sometimes even believes these claims. In this class, we will attempt to determine if such claims are reasonable by first introducing probability from a semi rigorous mathematical viewpoint using concepts developed in Calculus. We will use this framework to carefully discuss making such statistical inferences as above and in general to obtain accurate knowledge even when the known data is not complete. When attempting to precisely measure this uncertainty a few experiments are in order. When doing statistical experiments, a few terms and corresponding notation might be useful:

- S = Universal Set or Sample Space Experiment or Outcome Space. This is the collection of all possible outcomes.

- Random Experiment. A random experiment is a repeatable activity which has more than one possible outcome all of which can be specified in advance but can not be known in advance with certainty.

- Trial. Performing a Random Experiment one time and measuring the result.

- A = Event. A collection of outcomes. Generally denoted by an upper case letter such as A, B, C, etc.

- Success/Failure. When recording the result of a trial, a success for event A occurs when the outcome lies in A. If not, then the trial was a failure. There is no qualitative meaning to this term.

- Mutually Exclusive Events. Two events which share no common outcomes. Also known as disjoint events.

- |A| = Frequency. In a sequence of n events, the frequency is the number of trials which resulted in a success for event A.
- |A| / n = Relative Frequency. A proportion of successes to total number of trials.
- Histogram. A bar chart representation of data where area corresponds to the value being described.

To investigate these terms and to motivate our discussion of probability, consider flipping coins using the interactive cell below. Notice in this case, the sample space S = Heads, Tails and the random experiment consists of flipping a fair coin one time. Each trial results in either a Head or a Tail. Since we are measuring both Heads and Tails then we will not worry about which is a success or failure. Further, on each flip the outcomes of Heads or Tails are mutually exclusive events. We count the frequencies and compute the relative frequencies for a varying number of trials selected by you as you move the slider bar. Results are displayed using a histogram.

Question 1: What do you notice as the number of flips increases?

Question 2: Why do you rarely (if even) get exactly the same number of Heads and Tails? Would you not "expect" that to happen?

```
coin = ["Heads", "Tails"]
@interact
def _(num_rolls = slider([5..5000],label="Number␣of␣
    Flips")):
        rolls = [choice(coin) for roll in
            range(num_rolls)]
        show(rolls)
        freq = [0,0]
        for outcome in rolls:
                if (outcome==’Tails’):
                        freq[0] = freq[0]+1
                else:
                        freq[1] = freq[1]+1
        print("\nThe␣frequency␣of␣tails␣=␣"+
            str(freq[0]))+"␣and␣heads␣=␣"+
            str(freq[1])+"."
        rel = [freq[0]/num_rolls,freq[1]/num_rolls]
        print("\nThe␣relative␣frequencies␣for␣Tails␣and␣
            Heads:"+str(rel))
        show(bar_chart(freq,axes=False,ymin=0))     # A
            histogram of the results
```

Notice that as the number of flips increases, the relative frequency of Heads (and Tails) stabilized around 0.5. This makes sense intuitively since there are two options for each individual flip and 1/2 of those options are Heads while the other 1/2 is Tails.

Let’s try again by doing a random experiment consisting of rolling a single die one time. Note that the sample space in this case will be the outcomes S = 1, 2, 3, 4, 5, 6.

Question 1: What do you notice as the number of rolls increases?

Question 2: What do you expect for the relative frequencies and why are they not all exactly the same?

```
@interact
def _(num_rolls = slider([20..5000],label=’Number␣of␣
    rolls’),Number_of_Sides = [4,6,8,12,20]):
        die = list((1..Number_of_Sides))
        rolls = [choice(die) for roll in
            range(num_rolls)]
        show(rolls)

        freq = [rolls.count(outcome) for outcome in
            set(die)]  # count the numbers for each
            outcome
        print 'The␣frequencies␣of␣each␣outcome␣is␣
            '+str(freq)

        print 'The␣relative␣frequencies␣of␣each␣outcome:'
        rel_freq = [freq[outcome-1]/num_rolls for
            outcome in set(die)]  # make frequencies
            relative
        print rel_freq
        fs = []
        for f in rel_freq:
                fs.append(f.n(digits=4))
        print fs
        show(bar_chart(freq,axes=False,ymin=0))
```

Notice in this instance that there are a larger number of options (for example 6 on a regular die) but once again the relative frequencies of each outcome was close to 1/n (i.e. 1/6 for the regular die) as the number of rolls increased.

In general, this suggests a rule: if there are n outcomes and each one has the same chance of occurring on a given trial then on average on a large number of trials the relative frequency of that outcome is 1/n. In general, if a number of outcomes are "equally likely" then this is a good model for measuring the proportion of outcomes that would be expected to have any given outcome. However, it is not always true that outcomes are equally likely. Consider rolling two die and measuring their sum:

```
@interact
def _(num_rolls = slider([20..5000],label='Number␣of␣
    rolls'),num_sides = slider(4,20,1,6,label='Number␣of␣
    sides')):
    die = list((1..num_sides))
    dice = list((2..num_sides*2))
    rolls = [(choice(die),choice(die)) for roll in
        range(num_rolls)]
    sums = [sum(rolls[roll]) for roll in
        range(num_rolls)]
    show(rolls)

    freq = [sums.count(outcome) for outcome in
        set(dice)]  # count the numbers for each outcome
    print 'The␣frequencies␣of␣each␣outcome␣is␣'+str(freq)

    print 'The␣relative␣frequencies␣of␣each␣outcome:'
    rel_freq = [freq[outcome-2]/num_rolls for outcome in
        set(dice)]  # make frequencies relative
    print rel_freq
    show(bar_chart(freq,axes=False,ymin=0))     #  A
        histogram of the results
    print "Relative␣Frequence␣of␣",dice[0],"␣is␣about␣
        ",rel_freq[0].n(digits=4)
    print "Relative␣Frequence␣of␣",dice[num_sides-1],"␣
        is␣about␣",rel_freq[num_sides-1].n(digits=4)
```

Notice, not only are the answers not the same but they are not even close. To understand why this is different from the examples before, consider the possible outcomes from each pair of die. Since we are measuring the sum of the dice then (for a pair of standard 6-sided dice) the possible sums are from 2 to 12. However, there is only one way to get a 2–namely from a (1,1) pair–while there are 6 ways to get a 7–namely from the pairs (1,6), (2,5), (3,4), (4,3), (5,2), and (6,1). So it might make some sense that the likelihood of getting a 7 is 6 times larger than that of getting a 2. Check to see if that is the case with your experiment above.

4.2 Definition of Probability

Relative frequency gives a way to measure the proportion of "successful" outcomes when doing an experimental approach. From the interactive applications above, it appears that the relative frequency does jump around as the experiment is repeated but that the amount of variation decreases as the number of experiments increases. This is known to be true in general and leads to what is known as the "Law of Large Numbers". We would like to formalize what these relative frequencies seem to be approaching and will call this theoretical limit the "probability" of the outcome. In doing so, we will do our best to model our definition so that it follow the behavior of relative frequency.

4.2.1 Motivating the Definition

Using the ideas from our examples above, let's consider how we might formally define a way to measure the expectation from similar experiments. Before doing so, we need a little notation:

Definition 4.2.1. The Cardinality of the set A is the number of elements in A. This will be denoted |A| (similar to the idea of frequency of an outcome noted earlier.) If a set has a infinite number of elements, then we will say it's cardinality is also infinite and write $|\text{A}| = \infty$

Definition 4.2.2 (Pairwise Disjoint Sets)**.** $\{A_1, A_2, ..., A_n\}$ are pairwise disjoint provided $A_k \cap A_j = \emptyset$ so long as $k \neq j$.

To model the behavior above, consider how we might create a definition for our expectation of a given outcome by following the ideas uncovered above. To do so, first consider a desired collection of outcomes A. If each outcome in A is equally likely then we might follow the concept behind relative frequency and consider a measure of expectation be |A|/|S|. Indeed, on a standard 6-sided die, the expectation of the outcome A=2 from the collection S = 1,2,3,4,5,6 should be |A|/|S| = 1/6.

From the example where we take the sum of two die, the outcome A=4,5 from the collection S = 2,3,4,...,12 would be

$$|A| = |(1,3),(2,2),(3,1),(1,4),(2,3),(3,2),(4,1)| = 7$$
$$|S| = |(1,1),...,(1,6),(2,1),...,(2,6),...,(6,1),...,(6,6)| = 36$$

and so the expected relative frequency would be |A|/|S| = 7/36. Compare this theoretical value with the sum of the two outcomes from your experiment above.

We are ready to now formally give a name to the theoretical measure of expectation for outcomes from an experiment. Taking our cue from the ideas related to equally likely outcomes, we make our definition have the following basic properties:

1. Relative frequency cannot be negative, since cardinality cannot be negative

2. Relative frequencies for disjoint events should sum to one

3. Relative frequencies for collections of disjoint outcomes should equal the sum of the individual relative frequencies

4.2.2 Probability

Based upon these we give the following:

Definition 4.2.3. The probability P(A) of a given outcome A is a set function which satisfies:

1. (Nonnegativity) P(A) ≥ 0

2. (Totality) P(S) $= 1$

3. (Subadditivity) If A $\cap$ B $= \emptyset$, then P(A $\cup$ B) $=$ P(A) $+$ P(B). In general, if A_k are pairwise disjoint then $P(\cup_k A_k) = \sum_k P(A_k)$.

4.2.3 Basic Probability Theorems

Based upon this definition we can immediately establish a number of results.

Theorem 4.2.4 (Probability of Complements)**.** *For any event A, $P(A) + P(A^c) = 1$*

Proof. Let A be any event and note that $A \cap A^c = \emptyset$. But $A \cup A^c = S$. So, by subadditivity $1 = P(S) = P(A \cup A^c) = P(A) + P(A^c)$ as desired. □

Theorem 4.2.5. *$P(\emptyset) = 0$*

Proof. Note that $\emptyset^c = S$. So, by the theorem above, $1 = P(S) + P(\emptyset) \Rightarrow 1 = 1 + P(\emptyset)$. Cancelling the 1 on both sides gives $P(\emptyset) = 0$. □

Theorem 4.2.6. *For events A and B with $A \subset B, P(A) \leq P(B)$.*

Proof. Assume sets A and B satisfy $A \subset B$. Then, notice that $A \cap (B - A) = \emptyset$ and $B = A \cup (B - A)$. Therefore, by subadditivity and nonnegativity

$$\begin{gathered} 0 \leq P(B - A) \\ P(A) \leq P(A) + P(B - A) \\ P(A) \leq P(B) \end{gathered}$$

□

Theorem 4.2.7. *For any event A, $P(A) \leq 1$*

Proof. Notice $A \subset S$. By the theorem above $P(A) \leq P(S) = 1$ □

Theorem 4.2.8. *For any sets A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$*

Proof. Notice that we can write $A \cup B$ as the disjoint union

$$A \cup B = (A - B) \cup (A \cap B) \cup (B - A).$$

We can also write disjointly

$$\begin{aligned} A &= (A - B) \cup (A \cap B) \\ B &= (A \cap B) \cup (B - A) \end{aligned}$$

Hence,

$$\begin{aligned} &P(A) + P(B) - P(A \cap B) \\ &\qquad = [P(A - B) + P(A \cap B)] + [P(A \cap B) + P(B - A)] - P(A \cap B) \\ &\qquad = P(A - B) + P(A \cap B) + P(B - A) \\ &\qquad = P(A \cup B) \end{aligned}$$

□

This result can be extended to more that two sets using a property known as inclusion-exclusion. The following two theorems illustrate this property and are presented without proof.

Corollary 4.2.9. *For any sets A, B and C,*

$$\begin{aligned} P(A \cup B \cup C) = {} & P(A) + P(B) + P(C) \\ & - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ & + P(A \cap B \cap C) \end{aligned}$$

Corollary 4.2.10. *For any sets A, B, C and D,*

$$\begin{aligned} P(A \cup B \cup C \cup D) = {} & P(A) + P(B) + P(C) + P(D) \\ & - P(A \cap B) - P(A \cap C) - P(A \cap D) - P(B \cap C) - P(B \cap D) - P(C \cap D) \\ & + P(A \cap B \cap C) + P(A \cap B \cap D) + P(A \cap C \cap D) + P(B \cap C \cap D) \\ & - P(A \cap B \cap C \cap D) \end{aligned}$$

4.2.4 Equally Likely Outcomes

Many times, you will be dealing with making selections from a sample space where each item in the space has an equal chance of being selected. This may happen (for example) when items in the sample space are of equal size or when selecting a card from a completely shuffled deck or when coins are flipped or when a normal fair die is rolled.

It is important to notice that not all outcomes are equally likely–even in times when there are only two of them. Indeed, it is generally not an equally likely situation when picking the winner of a football game which pits, say, the New Orleans Saints professional football team with the New Orleans Home School Saints. Even though there are only two options the probability of the professional team winning is much greater than the chances that the high school will prevail.

When items are equally likely (sometimes also called "randomly selected") then each individual event has the same chance of being selected as any other. In this instance, determining the probability of a collection of outcomes is relatively simple.

Theorem 4.2.11 (Probability of Equally Likely Events)**.** *If outcomes in S are equally likely, then for* $A \subset S, P(A) = \frac{|A|}{|S|}$

Proof. Enumerate S = $x_1, x_2, ..., x_{|S|}$ and note $P(\{x_k\}) = c$ for some constant c since each item is equally likely. However, using each outcome as a disjoint event and the definition of probability,

$$\begin{aligned}1 = P(S) &= P(\{x_1\} \cup \{x_2\} \cup ... \cup \{x_{|S|}\})\\ &= P(\{x_1\}) + P(\{x_2\}) + ... + P(\{x_{|S|}\})\\ &= c + c + ... + c = |S| \times c\end{aligned}$$

and so $c = \frac{1}{|S|}$. Therefore, $P(\{x_k\}) = \frac{1}{|S|}$.

Hence, with A = $a_1, a_2, ..., a_{|A|}$, breaking up the disjoint probabilities as above gives

$$\begin{aligned}P(A) &= P(\{a_1\} \cup \{a_2\} \cup ... \cup \{a_{|A|}\})\\ &= P(\{a_1\}) + P(\{a_2\}) + ... + P(\{a_{|A|}\})\\ &= \frac{1}{|S|} + \frac{1}{|S|} + ... + \frac{1}{|S|}\\ &= \frac{|A|}{|S|}\end{aligned}$$

as desired. □

4.2.5 HOMEWORK

A. Determine the probabilities associated with the various 5-card hands.

B. Determine the 36 possible outcomes related to the rolling a pair of fair dice. Justify why each of these outcomes is equally likely. Determine the probabilities associated with each possible sum.

C. Suppose you have one die which only has three possible sides labeled 1, 2, or 3. Suppose a second die has twelve equally likely sides with labels 1,2,3,4,4,5,5,6,6,7,8,9. Justify that the probabilities associate with each possible sum is the same as the probabilities when using two normal 6-sided dice.

D. Analyze the game of "craps".

4.3 Conditional Probability

When finding the probability of an event, sometimes you may need to consider past history and how it might affect things. Indeed, you might think that when the local station forecasts rain than it the probability of it actually raining should be greater than if they forecast fair skies. At least that is the hope. :) In this section, you will develop a way to deal with the probability of some event that might change dependent upon the occurence or not of some other event. Consider a box with three balls: one Red, one White, and one Blue. Using an equally likely assumption, the probability of randomly pulling out a Red ball should be 1/3. That is P(Red) = 1/3. However, suppose that for a first trial you pull out the White ball and set it aside. Attempting to pull out another ball leaves you with only two options and so the probability of randomly pulling out a Red ball is 1/2. Notice that the probability changed for the second trial dependent on the outcome of the first trial.

Consider a deck of 52 standard playing cards and a success occurs when a Heart is selected from the deck. When extracting one card randomly, the probability of that card being a Heart is then P(Heart) = 13/52. Now, assume that one card has already been extracted and setaside. Now, prepare to extract another. If the first card drawn was a Heart, then there are only 12 Hearts left

Enrollment	Male	Female	Totals
STEM	420	510	930
Business	320	270	590
Other	610	710	1320
Totals	1350	1490	2840

for the second draw. However, if the first card drawn was not a Heart, then there are 13 Hearts available for the second draw. To compute this probability correctly, one need to formulate the question so that subadditivity can be utilized.

To do this, consider P(Heart on 2nd draw) = P([Heart on 1st draw $\cap$ Heart on 2nd draw] $\cup$ [Not Heart on 1st draw $\cap$ Heart on 2nd draw]) = P(Heart on 1st draw $\cap$ Heart on 2nd draw) + P(Not Heart on 1st draw $\cap$ Heart on 2nd draw) = | Heart on 1st draw $\cap$ Heart on 2nd draw | / | Number of ways to get two cards | + | Not Heart on 1st draw $\cap$ Heart on 2nd draw / | Number of ways to get two cards | = (13 12) / (52 51) + (39 13) / (52 51) = 12 / (4 51) + (3 13) / (4 51) =

Definition 4.3.1 (Conditional Probability)**.** P(B | A) = P(A $\cap$ B) / P(A), provided P(A)$>$ 0.

Theorem 4.3.2. *Conditional Probability satisfies all of the requirements of regular probability.*

Proof. By definition, for any event probability must be nonnegative. Therefore $P(A \cap B) \geq 0$. Therefore, P(B | A) $\geq$ 0.

Further, P (S | A) = P(A $\cap$ S)/P(A) = P(A)/P(A) = 1. □

Theorem 4.3.3 (Multiplication Rule)**.**

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$$

Proof. Unravel the definition of conditional probably by taking the denominator to the other side. Also note that you can write $A \cap B = B \cap A$. □

4.3.1 HOMEWORK

A. Given P(A) = 0.43, P(B) = 0.72, and $P(A \cap B) = 0.29$, determine

1. $P(A \cup B)$
2. $P(B|A)$
3. $P(A|B)$
4. $P(A^c \cap B^c)$

B. The table below classifies students at your university according to gender and according to major.

Determine the following:

1. P(STEM major)
2. P(STEM | Female)
3. P(Female | STEM)

4. P(Female | Not STEM)

C. You are in a probability and statistics class with a teacher who has predetermined that only one student can make an A for the course. To be "fair", he places a number of slips of paper in a bowl equal to the number of students in the course with one of the slips having an A designation. Students in the course each can pick once randomly from the bowl and without replacement to see if they can get the lucky slip. Determine the following:

1. If there are 15 students in your course, determine the probabilities of getting an A in the course if you pick first and if you pick last.
2. Since the teacher likes you the most, she will give you the option of deciding whether to pick at any position. If so, determine the position that would give you the best likelihood of getting the A slop.
3. Suppose again that the teacher was feeling more generous and decided instead to allow for two A's. Determine how that changes your likelihood of winning and on what position you would like to choose.
4. Continue as above except that only one slip does not have an A on it.
5. Discuss how your choice is affected by the number of students in the course or the number of A slips included.

Using the normal equally-likely definition, $P(\text{first}) = \frac{1}{15}$.

To get the A on the last pick requires that all of the previous picks to be something else. You don't get the opportunity to pick the A if it has already been selected. So, if L stands for losing (not getting the A), then

$$P(\text{last}) = P(\text{LLLLLLLLLLLLLLA}) = \frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = \frac{1}{15}.$$

Therefore, it is the same probability of getting the A whether you pick first or last. In general, to win on the kth pick gives

$$P(\text{kth}) = P(\text{LL...LA}) = \frac{14 \cdot 13 \cdot ... \cdot (15-k) \cdot 1}{15 \cdot 14 ... \cdot (16-k) \cdot (15-k)} = \frac{1}{15}$$

Hence, it is the same probability regardless of when you get to pick.

If there are two A's possible, then the options for person k in include either receiving the first of the two slips or the second. The probability for determining the first of the two is computed in a manner similar to above except that there is one more A and one less other.

$$P(\text{kth as first}) = P(\text{LL...LA}) = \frac{13 \cdot 12 \cdot ... \cdot (15-k) \cdot 2}{15 \cdot 14 ... \cdot (16-(k+1)) \cdot (16-k)} = \frac{2 \cdot (15-k)}{15 \cdot 14}$$

The probability of getting the second A means exactly one of the previous k-1 selections also picked the other A. There are k-1 ways that this could happen. Computing for one of the options and multiplying by k-1 gives

$$P(\text{kth as second}) = P(\text{LL...LAA}) = (k-1) \cdot \frac{13 \cdot 12 \cdot ... \cdot (15-k) \cdot 2 \cdot 1}{15 \cdot 14 ... \cdot (16-k) \cdot (15-k)} = \frac{2 \cdot (k-1)}{15 \cdot 14}.$$

Adding these two together gives

$$P(\text{getting an A when there are two}) = \frac{2 \cdot (15-k) + 2 \cdot (k-1)}{15 \cdot 14} = \frac{28}{15 \cdot 14} = \frac{2}{15}.$$

For example, if k = 5,

$$P(\text{5th as first}) = P(\text{LLLLA}) = \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 2}{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11} = \frac{20}{15 \cdot 14}$$

$$P(\text{5th as second}) = P(\text{LL...LAA}) = 4 \cdot \frac{13 \cdot 12 \cdot \cdot 11 \cdot 2 \cdot 1}{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11} = \frac{8}{15 \cdot 14}.$$

Adding these together yields the general result. So, once again, it doesn't matter which pick you use since the likelihood of getting an A is the same for all positions.

D. In this problem, you want to consider how many people are necessary in order to have an even chance of finding two or more who share a common birthday. Toward that end, assuming a year has exactly 365 equally likely days let r be the number of people in a sample and consider the following:

1. Determine the number of different outcomes of birthdays when order matters and birthdays are allowed to be repeated.

2. Determine the number of different outcomes when birthdays are not allowed to be repeated.

3. Determine the probability that two or more of your r students have the same birthday.

4. Prepare a spreadsheet with the probabilities found above from r=2 to r=50. Determine the value of r for which this probability is closest to 0.5.

5. As best as you can, sample two groups of the size found above and gather birthday information. For each group, determine if there is a shared birthday or not. Compare your results with others in the class to check whether the sampling validates that about half of the samples should have a shared birthday group.

The correct sample size to get past a probability of 0.5 is 23 people. You should justify this numerically by justifying the following probabilities:

```
# P(Match)
1 0
2 0.0027
3 0.0082
4 0.0164
5 0.0271
6 0.0405
7 0.0562
8 0.0743
9 0.0946
10 0.1169
11 0.1411
12 0.1670
13 0.1944
14 0.2231
15 0.2529
16 0.2836
17 0.3150
18 0.3469
19 0.3791
```

```
20 0.4114
21 0.4437
22 0.4757
23 0.5073
24 0.5383
25 0.5687
26 0.5982
27 0.6269
28 0.6545
29 0.6810
30 0.7063
```

E. This one is from an internet meme: Two fair 6-sided dice are rolled together and you are told that at least one of the dice is a 6. Given that a 6 will be removed, determine the probability that the other die is a 6.

In this case, you are presented with an outcome where the possible choices consist of (1,6), (2,6), (3,6), (4,6), (5,6), (6,6), (6,5), (6,4), (6,3), (6,2), (6,1). Each of these would satisfy the condition that at least one of the dice is a 6. From this group, the only success that satisfies being a 6, given that another 6 has already been removed, is the (6,6) outcome. Therefore, the conditional probability is 1/11.

It is interesting to note that if the question instead was posed so that one of the dice was a 6 and it was removed, then the probability of the other dice showing a 6 would be 1/6.

F. This is a famous problem. 100 people are in line, boarding an airplane with 100 seats, one at a time. They are in no particular order. The first person has lost his boarding pass, so he sits in a random seat. The second person does the following:

- Goes to his seat (the one it says to go to on the boarding pass). If unoccupied, sit in it.

- If occupied, find a random seat to sit in.

Everyone else behind him does the same. What is the probability that the last person sits in his correct seat?

To get the idea, consider what happens with only 2 people, then only 3. Generalize.

The answer is 1/2. To obtain this, you can define recursively the probability that the kth person sits in their own set as f(k). Consider the first traveler's and your seats. Then you get the following cases:

- P(first guy sits in his own seat and you sit in yours) $= 1\frac{}{k\cdot 1 P(firstguysitsinyourseatandyoudonotsitinyours)=\frac{1}{k}\cdot 0}$

- P(other k-2 travelers make their choices) $= (k-2)\frac{1}{k}f(k-1)$

$$f(k) = 1/k + 0 + (k-2)/kf(k-1)$$

with f(2) = 1/2.

For example, f(3) = 1/3 + f(2)/3 = 1/3 + 1/6 = 1/2. f(4) = 1/4 + 2/4 f(3) = 1/4 + 1/2 1/2 = 1/2. f(5) = 1/5 + 3/5 1/2 = 1/2. f(6) = 1/6 + 4/6 1/2 = 1/2. Etc.

4.4 Bayes Theorem

Conditional probabilities can be computed using the methods developed above if the appropriate information is available. Some times you will however have some information available, such as $P(A|B)$ but need $P(B|A)$. The ability to "play around with history" by switching what has been presumed to occur leads to the following.

Theorem 4.4.1 (Bayes Theorem)**.** *Let $S = \{S_1, S_2, ..., S_m\}$ where the S_k are pairwise disjoint and $S_1 \cup S_2 \cup ... \cup S_m = S$ (i.e. a partition of the space S). Then for any $A \subset S$*

$$P(S_j|A) = \frac{P(S_j)P(A|S_j)}{\sum_{k=1}^{m} P(S_k)P(A|S_k)}.$$

The conditional probability $P(S_j|A)$ is called the posterior probability of S_k.

Proof. Notice, by the definition of conditional probability and the multiplication rule

$$P(S_j|A) = \frac{P(S_j \cap A)}{P(A)} = \frac{P(S_j)P(A|S_j)}{P(A)}.$$

But using the disjointness of the partition

$$\begin{aligned} P(A) &= P((A \cap S_1) \cup (A \cup S_2) \cup ... \cup (A \cup S_m)) \\ &= P(A \cap S_1) + P(A \cup S_2) + ... + P(A \cup S_m) \\ &= P(S_1 \cap A) + P(S_2 \cup A) + ... + P(S_m \cup A) \\ &= P(S_1)P(A|S_1) + P(S_2)P(A|S_2) + ... + P(S_m)P(A|S_m) \\ &= \sum_{k=1}^{m} P(S_k)P(A|S_k) \end{aligned}$$

Put these two expansions together to obtain the desired result. □

To illustrate this result, from the web site http://stattrek.com/probability/bayes-theorem.aspx consider the following problem:

Exercise 4.4.2.

The interactive cell below can be used to easily compute all of the conditional probabilities associated with Bayes's Theorem. Notice how the relative size of the pie-shaped partition changes when you presume that an event in the space has already occurred.

```
#  This function is used to convert an input string into
    separate entries
def g(s): return str(s).replace(',','␣').replace('(','␣
    ').replace(')','␣').split()

@interact
def
    _(Partition_Probabilities=input_box('0.35,0.25,0.40',label="$P(B_1),P(B_2
        Conditional_Probabilities=input_box('0.02,0.01,0.03',label='$P(A|B_1)
        print_numbers=checkbox(True,label='Numerical␣
            Results␣on␣Graphs?'),
        auto_update=False):

     Partition_Probabilities = g(Partition_Probabilities)
    Conditional_Probabilities =
       g(Conditional_Probabilities)
    n = len(Partition_Probabilities)
    n0 = len(Conditional_Probabilities)

    # below needs to be n not equal to n0 but mathbook
       xml will not let me get the other
    if (n > n0):
        pretty_print("You␣must␣have␣the␣same␣number␣of␣
            partition␣probabilities␣and␣conditional␣
            probabilities.")

    else:                               # input data
       streams now are the same size!
        colors = rainbow(n)
        accum = float(0)                # to test
           whether partition probs sum to one
        ends = [0]                      # where the
           graphed partition sectors change in pie chart
        mid = []                        # middle of each
           pie chart sector used for placement of text
        p_Bk_given_A = []               # P( B_k | A )
        pA = 0                          # P(A)
        PP=[]                           # array to hold
           the numerical Partition Probabilities
        CP=[]                           # array to hold
           the numerical Conditional Probabilities
        for k in range(n):
            PP.append(float(Partition_Probabilities[k]))
            CP.append(float(Conditional_Probabilities[k]))
            p_Bk_given_A.append(PP[k]*CP[k] )
            pA += p_Bk_given_A[k]
            accum = accum + PP[k]
            ends.append(accum)
            mid.append((ends[k]+accum)/2)
#
#  Marching along from 0 to 1, saving angles for each
   partition sector boundary.
#  Later, we will multiple these by 2*pi to get actual
   sector boundary angles.
#
        if abs(accum-float(1))>0.0000001:      #  Due to
           roundoff issues, this should be close enough.
            pretty_print("Sum␣of␣probabilities␣should␣
               equal␣1.")

        else:                           # probability
           data is sensible


#
#  Draw the Venn diagram by drawing sectors from the
   angles determined above
#  First, create a circle of radius 1 to illustrate the
   the sample space S
#  Then draw each sector with varying colors and print
   out their names on the edge
#
            G = circle((0,0), 1,
               rgbcolor=’black’,fill=False,
               alpha=0.4,aspect_ratio=True,axes=False,thickness=5)
            for k in range(n):
                G += disk((0,0), 1, (ends[k]*2*pi,
                    ends[k+1]*2*pi),
                    color=colors[mod(k,10)],alpha = 0.2)
                G +=
                    text('$B_'+str(k+1)+'$',(1.1*cos(mid[k]*2*pi),
                    1.1*sin(mid[k]*2*pi)),
                    rgbcolor='black')

            G += circle((0,0), 0.6, facecolor='yellow',
               fill = True, alpha = 0.1,
               thickness=5,edgecolor='black')

#  Print the probabilities corresponding to each
   particular region as a list and on the graphs
            if print_numbers:

                html("$P(A)␣=␣%s$"%(str(pA),))
                for k in range(n):
                    html("$P(B_{%s}␣|␣A)$"%(str(k+1))+"$␣
                       =␣%s$"%str(p_Bk_given_A[k]/pA))

                    G +=
                       text(str(p_Bk_given_A[k]),(0.4*cos(mid[k]*2*pi),
                       0.4*sin(mid[k]*2*pi)),
                       rgbcolor='black')
                    G += text(str(PP[k] -
                       p_Bk_given_A[k]),(0.8*cos(mid[k]*2*pi),
                       0.8*sin(mid[k]*2*pi)),
                       rgbcolor='black')

#  This is essentially a repeat of some of the above
   code but focused only on creating the smaller inner
   circle dealing
#  with the set A so that the sectors now correspond in
   area to the Bayes Theorem probabilities


            accum = float(0)
            ends = [0]                     # where the
               graphed partition sectors change in pie
               chart
            mid = []                       # middle of
               each pie chart sector used for placement
               of text
            for k in range(n):
                accum += float(p_Bk_given_A[k]/pA)
                ends.append(accum)
                mid.append((ends[k]+accum)/2)
            H = circle((0,0), 1,
               rgbcolor='black',fill=False,
               alpha=0,aspect_ratio=True,axes=False,thickness=0)
            H += circle((0,0), 0.6,
               facecolor='yellow',fill=True,
               alpha=0.1,aspect_ratio=True,axes=False,thickness=5,edgecolor=

            for k in range(n):
                H += disk((0,0), 0.6, (ends[k]*2*pi,
```

Age	Proportion of Insured	Probability of Accident
16-20	0.05	0.08
21-25	0.06	0.07
26-55	0.49	0.02
55-65	0.25	0.03
over 65	0.15	0.04

```
      ends[k+1]*2*pi),
      color=colors[mod(k,10)],alpha = 0.2)
    H +=
      text('$B_'+str(k+1)+'|A$',(0.7*cos(mid[k]*2*pi),
      0.7*sin(mid[k]*2*pi)),
      rgbcolor='black')

#  Now, print out the bayesian probabilities
  using the smaller set A only

  if print_numbers:
    for k in range(n):
      H += text(str(
        N(p_Bk_given_A[k]/pA,digits=4)
        ),(0.4*cos(mid[k]*2*pi),
        0.4*sin(mid[k]*2*pi)),
        rgbcolor='black')

  G.show(title='Venn␣diagram␣of␣partition␣with␣
    A␣in␣middle')
  print
  H.show(title='Venn␣diagram␣presuming␣A␣has␣
    occured')
```

4.4.1 HOMEWORK

A. Your automobile insurance company uses past history to determine how to set rates by measuring the number of accidents caused by clients in various age ranges. The following table summarizes the proportion of those insured and the corresponding probabilities by age range:

One of your family friends insured by this company has an accident.

1. Determine the conditional probability that the driver was in the 16-20 age range.

2. Compare this to the probability that the driver was in the 18-20 age range. Discuss the difference.

3. Determine how much more the company should charge for someone in the 16-20 age range compared to someone in the 26-55 age range.

B. Congratulations...your family is having a baby! As part of the prenatal care, some testing is part of the normal procedure including one for spinal bifida (which is a condition in which part of the spinal cord may be exposed.) Indeed, measurement of maternal serum AFP values is a standard tool used in obstetrical care to identify pregnancies that may have an increased risk for this disorder. You want to make plans for the new child's care and want to know how serious to take the test results. However, some times the test indicates that the child has the disorder when in actuality it does not (a false positive) and likewise may indicate that the child does not have the disorder when in fact it does (a false negative.)

The combined accuracy rate for the screen to detect the chromosomal abnormalities mentioned above is approximately 85

- Approximately 85 out of every 100 babies affected by the abnormalities addressed by the screen will be identified. (Positive Positive)
- Approximately 5

1. Given that your test came back negative, determine the likelihood that the child will actually have spinal bifida.
2. Given that your test came back negative, determine the likelihood that the child will not have spina bifida
3. Given that a positive test means you have a $1/100$ to $1/300$ chance of experiencing one of the abnormalities, determine the likelihood of spinal bifida in a randomly selected child.

4.5 Independence

You have seen when repeatedly sampling without replacement leads to a change the the likelihood of some event in successive trials. Indeed, this is what conditional probabilities above illustrate. However, when sampling with replacement you may find a different situation arises. Indeed, you easily notice that when flipping a coin, P(Heads) $= 1/2$ regardless of the outcome of any previous flip. In situations such as this where the probability of an event is not affected by the occurrence (or lack of occurrence) of some other event determining the probability of compound events can be greatly simplified.

Definition 4.5.1 (Independent Events)**.** Events A and B are independent provided

$$P(A \cap B) = P(A)P(B)$$

Corollary 4.5.2 (Independence and Conditional Probability)**.** *Given independent events A and B,*

$$P(B|A) = P(B)$$

and

$$P(A|B) = P(A).$$

Proof. By the multiplication rule and the definition of independence, for any events A and B

$$P(A) \cdot P(B) = P(A \cap B) = P(A) \cdot P(B|A).$$

Therefore, if P(A) is non-zero, canceling yields the first result. Switching around notation provides the second. □

Corollary 4.5.3 (Independence and Mutual Exclusivity)**.** *If events A and B are both independent and mutually exclusive, then at least one of them has zero probability.*

Proof. By independence, $P(A \cap B) = P(A) \cdot P(B)$. However, by mutually exclusivity, $A \cap B = \emptyset \Rightarrow P(A \cap B) = 0$ gives

$$P(A) \cdot P(B) = 0.$$

Hence, one or the other (or both) must be zero. □

Corollary 4.5.4 (Successive Independent Events)**.** *Given a sequence of independent events $A_1, A_2, A_3, ...,$*

$$P(\cap_{k \in R} A_k) = \prod_{k \in R} P(A_k)$$

4.5.1 HOMEWORK

A. Given P(A) = 0.43, P(B) = 0.72, and $P(A \cap B) = 0.29$, verify that A and B are not independent.

B. Given A, B, and C are independent events, with P(A) = 2/5, P(B) = 3/4, and P(C) = 1/6, determine:

1. $P(A \cap B \cap C)$

2. $P(A^c \cap B^c \cap C)$

3. $P(A \cup B \cup C)$

C. Suppose for a pair of dice you want to consider the events A = rolling a 7 or 11 and B = otherwise. Rolling the dice 5 times, determine

1. P(AABBB)

2. P(BBBAA)

3. The probability of getting A on exactly two rolls of the dice.

D. To help "insure" the success of a mission, you propose several redundant components so that the mission is a success if one or more succeed. Supposing that these separate components act independently of each other and that each component has a 75

1. The probability of failure if you utilize 2 components.

2. The probability of failure if you utilize 5 components.

3. The number of components needed to insure that the probability of success is at least 99

E. Again, from an internet meme: Two fair 6-sided dice are rolled together and you are told that at least one of the dice is a 6. A 6 is removed and you are presented with the other die. Determine the probability that it is a 6.

For this setting, notice that the outcomes from each of the two dice are independent of each other. Removing one of the dice, regardless of it's value, does not affect the other. The question in this case does not ask for a conditional probability.

F. Consider a n=4 team single-elimination tournament where the teams are "seeded" from 1 (the best team) to 4 (the worst team). For this tournament, team 1 plays team 4 and team 2 plays team 3. The winner of each play each other to determine the final winner. When teams j and k play, set P(j wins) $= \frac{k}{j+k}$ and similarly for team k. Assuming separate games are independent of each other, determine the probability that team 4 wins the tournament. What about with 8 teams? What about 64 teams?

P(4 wins) = P(4 beats 1) P(4 beats the winner of the other bracket)
P(4 wins) = (1/5) * P(4 beats 2 | 2 beats 3) + P(4 beats 3 | 3 beats 2)
P(4 wins) = 1/5 [(3/5)(2/6) + (2/5)(3/7)] = 78/1050 = 0.0742
For the other teams:
P(1 wins) = 4/5 [(3/5)(2/3) + (2/5)(3/4)] = 0.56
P(2 wins) = 3/5 [(4/5)(1/3) + (1/5)(4/6)] = 0.24
P(3 wins) = 2/5 [(4/5)(1/4) + (1/5)(4/7)] = 0.1257

Chapter 5

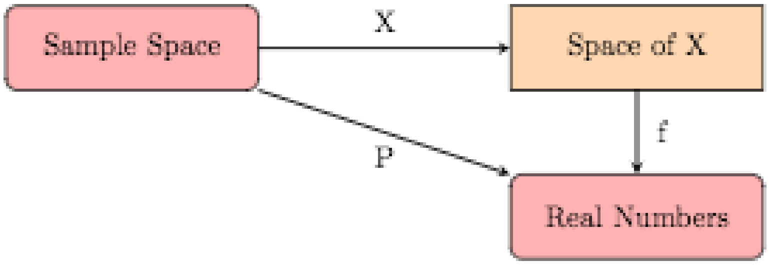
Probability Functions

This chapter is a definitions general probability functions.

5.1 Random Variables

For a given set of events, we might have difficulty doing mathematics since the outcomes are not numerical. In order to accomodate our desire to convert to numerical measures we want to assign numerical values to all outcomes. The process of doing this creates what is known as a random variable.

Definition 5.1.1 (Random Variable)**.** Given a random experiment with sample space S, a function X mapping each element of S to a unique real number is called a random variable. For each element s from the sample space S, denote this function by X(s) = x and call the range of X the space of X: R= x : X(s)=x, for some s in S



We will make various restrictions on the range of the random variable to fit different generalized problems. Then, we will be able to work on a problem (which may be inherently non-numerical) by using the random variable in subsequent calculations.

Example 5.1.2 (Success vs Failure)**.** When dealing with only two outcomes, one might use

$$S = \text{ success, failure } .$$

Choose

$$X(success) = 1$$
$$X(failure) = 0.$$

Then, R=0,1.

Example 5.1.3 (Standard Dice Pairs)**.** When gambling with a pair of dice, one might use S=ordered pairs of all possible rolls. Then

$$S = \text{ (a,b): a=die 1 outcome, b=die 2 outcome.}$$

Choose

$$X((a,b)) = a + b.$$

Then, R=2, 3, 4, 5, ..., 12.

Example 5.1.4 (Other Dice Options)**.** When rolling dice in a board game (like RISK), one might use

$$S = \text{(a,b): a=die 1 outcome, b=die 2 outcome}$$

Choose

$$X((a,b)) = \text{maxa,b.}$$

Then, R=1, 2, 3, 4, 5, 6.

Definition 5.1.5. R contains a countable number of points if either R is finite or there is a one to one correspondence between R and the positive integers. Such a set will be called discrete. We will see that often the set R is not countable. If R consists of an interval of points (or a union of intervals), then we call X a continuous random variable.

5.1.1 HOMEWORK

A. You flip three coins and measure the number of heads obtained. Determine the space R for the corresponding random variable X. From the eight possible outcomes, determine all outcomes corresponding to X=2. Identify the random variable as discrete or continuous.

B. You flip one coin repeatedly until you get a second head. Determine the space R for the corresponding random variable X. From the possibilities, determine all outcomes corresponding to X=4. Identify the random variable as discrete or continuous.

C. Now you want to measure the time between accidents at a particular intersection in town. Determine the space R for the corresponding random variable X. Describe all outcomes corresponding to $X < 1$. Be purposeful in the problem to describe the units you are using to measure time. Identify the random variable as discrete or continuous.

5.2 Probability Functions

In the formulas below, we will presume that we have a random variable X which maps the sample space S onto some range of real numbers R. From this set, we then can define a probability function f(x) which acts on the numerical values in R and returns another real number. We attempt to do so to obtain (for discrete values) P(sample space value s)= $f(X(s))$. That is, the probability of a given outcome s is equal to the composition which takes s to a numerical value x which is then plugged into f to get the same final values.

Definition 5.2.1 (Probability "Mass" Function)**.** Given a discrete random variable X on a space R, a probability mass function on X is given by a function $f : R \to \mathbb{R}$ such that:

$$\begin{aligned}
&\forall x \in R, f(x) > 0 \\
&\sum_{x \in R} f(x) = 1 \\
&A \subset R \Rightarrow P(X \in A) = \sum_{x \in A} f(x)
\end{aligned}$$

For $x \notin R$, you can use the convention f(x)=0.

Definition 5.2.2 (Probability "Density" Function)**.** Given a continuous random variable X on a space R, a probability density function on X is given by a function $f : R \to \mathbb{R}$ such that:

$$\begin{aligned}
&\forall x \in R, f(x) > 0 \\
&\int_R f(x)dx = 1 \\
&A \subset R \Rightarrow P(X \in A) = \int_A f(x)dx
\end{aligned}$$

For $x \notin R$, you can use the convention f(x)=0.

For the purposes of this book, we will use the term "Probability Function" to refer to either of these options.

Example 5.2.3 (Discrete Probability Function)**.** Consider $f(x) = x/10$ over R = 1,2,3,4. Then, f(x) is obviously positive for each of the values in R and certainly $\sum_{x \in R} f(x) = f(1)+f(2)+f(3)+f(4) = 1/10+2/10+3/10+4/10 = 1$. Therefore, f(x) is a probability mass function over the space R.

```
# Combining all of the above into one interactive cell
@interact
def _(D =
    input_box([1,2,3,5,6,8,9,11,12,14],label="Enter␣
    domain␣R␣(in␣brackets):"),
        Probs =
            input_box([1/20,1/20,1/20,3/20,1/20,4/20,4/20,1/20,1/20,3/20],label="Enter␣
            corresponding␣f(x)␣(in␣brackets):"),
        n_samples=slider(100,10000,100,100,label="Number␣
            of␣times␣to␣sample␣from␣this␣distribution:")):
    n = len(D)
    R = range(n)
    one_huh = sum(Probs)
    pretty_print('\n\nJust␣to␣be␣certain,␣we␣should␣
        check␣to␣make␣certain␣the␣probabilities␣sum␣to␣
        1\n')
    pretty_print(html('$\sum_{x\epsilon␣R}␣f(x)␣=␣
        %s$'%str(one_huh)))

    G = Graphics()
    if len(D)==len(Probs):
        f = zip(D,Probs)
        meanf = 0
        variancef = 0
        for k in R:
```

```
        meanf += D[k]*Probs[k]
        variancef += D[k]^2*Probs[k]
        G +=
            line([(D[k],0),(D[k],Probs[k])],color='green')
    variancef = variancef - meanf^2
    sd = sqrt(variancef)
    G += points(f,color='blue',size=50)
    G +=
        point((meanf,0),color='yellow',size=60,zorder=3)
    G +=
        line([(meanf-sd,0),(meanf+sd,0)],color='red',thickness=5)

    g = DiscreteProbabilitySpace(D,Probs)
    pretty_print('␣␣␣␣␣mean␣=␣%s'%str(meanf))
    pretty_print('␣variance␣=␣%s'%str(variancef))

    #  perhaps to add mean and variance for pmf here
else:
    print 'Domain␣D␣and␣Probabilities␣Probs␣must␣be␣
        lists␣of␣the␣same␣size'

#  Now, let's sample from the distribution given
    above and see how a random sampling matches up

counts = [0] * len(Probs)
X = GeneralDiscreteDistribution(Probs)
sample = []

for _ in range(n_samples):
    elem = X.get_random_element()
    sample.append(D[elem])
    counts[elem] += 1
Empirical = [1.0*x/n_samples for x in counts] #
    random

samplemean = mean(sample)
samplevariance = variance(sample)
sampdev = sqrt(samplevariance)

E = points(zip(D,Empirical),color='orange',size=40)
E +=
    point((samplemean,0.005),color='brown',size=60,zorder=3)
E +=
    line([(samplemean-sampdev,0.005),(samplemean+sampdev,0.005)],color='o
(G+E).show(ymin=0,figsize=(8,5))
```

Example 5.2.4 (Continuous Probability Function)**.** Consider $f(x) = x^2/c$ for some positive real number c and presume R = [-1,2]. Then f(x) is nonnegative (and only equals zero at one point). To make f(x) a probability density function, we must have

$$\int_{x \in R} f(x) = 1.$$

In this instance you get

$$1 = \int_{-1}^{2} x^2/c = x^3/(3c)|_{-1}^{2} = \frac{8}{3c} - \frac{-1}{3c} = \frac{3}{c}$$

Therefore, f(x) is a probability density function over R provided = 3.

X	F(x)
$x < 1$	0
$1 \leq x < 2$	1/10
$2 \leq x < 3$	3/10
$3 \leq x < 4$	6/10
$4 \leq x$	1

X	F(x)
$x < -1$	0
$-1 \leq x < 2$	$x^3/9 + 1/9$
$2 \leq x$	1

Definition 5.2.5 (Distribution Function)**.** Given a random variable X on a space R, a probability distribution function on X is given by a function $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) = P(X \leq x)$

Example 5.2.6 (Discrete Distribution Function)**.** Using $f(x) = x/10$ over R = 1,2,3,4 again, note that F(x) will only change at these four domain values. We get

Example 5.2.7 (Continuous Distribution Function)**.** Consider $f(x) = x^2/3$ over R = [-1,2]. Then, for $-1 \leq x \leq 2$,

$$F(x) = \int_{-1}^{x} u^2/3du = x^3/9 + 1/9.$$

Notice, F(-1) = 0 since nothing has yet been accumulated over values smaller than -1 and F(2)=1 since by that time everything has been accumulated. In summary:

Theorem 5.2.8. *$F(x) = 0, \forall x < \inf(R)$*

Proof. Let a = inf(R). Then, for

$$x < a, F(x) = P(X \leq x) \leq P(X < a) = 0$$

since none of the x-values in this range are in R. □

Theorem 5.2.9. *$F(x) = 1, \forall x \geq \sup(R)$*

Proof. Let b = sup(R). Then, for

$$x \geq b, F(x) = P(X \leq x) = P(X \leq b) + P(b < Xlex) = P(Xleb) = 1$$

since all of the x-values in this range are in R and therefore will either sum over or integrate over all of R. □

Theorem 5.2.10. *F is non-decreasing*

Proof. Case 1: R discrete

$$\forall x_1, x_2 \in \mathbb{Z} \ni x_1 < x_2$$

$$\begin{aligned} F(x_2) &= \sum_{x \leq x_2} f(x) \\ &= \sum_{x \leq x_1} f(x) + \sum_{x_1 < x \leq x_2} f(x) \\ &\geq \sum_{x \leq x_1} f(x) = F(x_1) \end{aligned}$$

Case 2: R continuous

$$
\begin{aligned}
\forall x_1, x_2 \in \mathbb{R} \ni x_1 < x_2 \\
F(x_2) &= \int_{-\infty}^{x_2} f(x)dx \\
&= \int_{-\infty}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx \\
&\geq \int_{-\infty}^{x_1} f(x)dx \\
&= F(x_1)
\end{aligned}
$$

□

Theorem 5.2.11 (Using Discrete Distribution Function to compute probabilities)**.** *for $x \in R, f(x) = F(x) - F(x-1)$*

Proof. Assume $x \in R$ for some discrete R. Then,

$$F(x) - F(x-1) = \sum_{u \leq x} f(u) - \sum_{u < x} f(u) = f(x)$$

□

Theorem 5.2.12 (Using Continuous Distribution function to compute probabilities)**.** *for $a < b, (a, b) \in R, P(a < X \leq b) = F(b) - F(a)$*

Proof. For a and b as noted, consider

$$
\begin{aligned}
F(b) - F(a) &= \int_{-\infty}^{b} f(x)dx - \int_{-\infty}^{a} f(x)dx \\
&= \int_{a}^{b} f(x)dx \\
&= P(a < x \leq b)
\end{aligned}
$$

□

Corollary 5.2.13. *For continuous distributions, P(X = a) = 0*

Proof. We will assume that F(x) is a continuous function. With that assumption, note

$$P(a - \epsilon < x \leq a) = \int_{a-\epsilon}^{a} f(x)dx = F(a) - F(a - \epsilon)$$

Take the limit as $\epsilon \to 0^+$ to get the result noting that □

5.2.1 HOMEWORK

A. Consider the random variable from the previous section where you flip three coins and measure the number of heads obtained. Determine f(0), f(1), f(2), and f(3) and the corresponding distribution function F(x). These can be expressed in a table format. Generalize your answer to the case when you flip a n coins where n is a fixed natural number.

B.

5.3 Expected Value

Blaise Pascal was a 17th century mathematician and philosopher who was accomplished in many areas but may likely be best known to you for his creation of what is now known as Pascal's Triangle. As part of his philosophical pursuits, he proposed what is known as "Pascal's wager". It suggests two mutually exclusive outcomes: that God exists or that he does not. His argument is that a rational person should live as though God exists and seek to believe in God. If God does not actually exist, such a person will have only a finite loss (some pleasures, luxury, etc.), whereas they stand to receive infinite gains as represented by eternity in Heaven and avoid an infinite losses of eternity in Hell. This type of reasoning is part of what is known as "decision theory".

You may not confront such dire payouts when making your daily decisions but we need a formal method for making these determinations precise. The procedure for doing so is what we call expected value.

Definition 5.3.1 (Expected Value)**.** Given a random variable X over space R, corresponding probability function f(x) and "value function" u(x), the expected value of u(x) is given by

$$E = E[u(X)] = \sum_{x \in R} u(x)f(x)$$

provided X is discrete, or

$$E = E[u(X)] = \int_R u(x)f(x)dx$$

provided X is continuous.

Theorem 5.3.2 (Expected Value is a Linear Operator)**.**

1. *E[c] = c*
2. *E[c u(X)] = c E[u(X)]*
3. *E[u(X) + v(X)] = E[u(X)] + E[v(X)]*

Proof. Each of these follows by utilizing the corresponding linearity properties of the summation and integration operations. For example, to verify part three in the continuous case:

$$\begin{aligned} E[u(X) + v(X)] &= \int_{x \in R} [u(x) + v(x)]f(x)dx \\ &= \int_{x \in R} u(x)f(x)dx + \int_{x \in R} u(x)f(x)dx \\ &= E[u(X)] + E[v(X)]. \end{aligned}$$

□

Example 5.3.3 (Discrete Expected Value)**.** Consider $f(x) = x/10$ over R = 1,2,3,4 where the payout is 10 euros if x=1, 5 euros if x=2, 2 euros if x=3 and -7 euros if x = 4. Then your value function would be u(1)=10, u(2) = 5, u(3)=2, and u(4) = -7. Computing the expect payout gives

$$E = 10 \times 1/10 + 5 \times 2/10 + 2 \times 3/10 - 7 \times 4/10 = -2/10$$

Therefore, the expected payout is actually negative due to a relatively large negative payout associated with the largest likelihood outcome and the larger positive payout only associated with the least likely outcome.

X	f(x)
0	0.10
1	0.25
2	0.40
4	0.15
7	0.10

Example 5.3.4 (Continuous Expected Value)**.** Consider $f(x) = x^2/3$ over R = [-1,2] with value function given by $u(x) = e^x - 1$. Then, the expected value for u(x) is given by

$$E = \int_{-1}^{2} (e^x - 1) \cdot x^2/3 = -1/9 \cdot (e + 15) \cdot e^{-1} + 2/3 \cdot e^2 - 8/9 \approx 3.3129$$

Definition 5.3.5 (Theoretical Measures)**.** Given a random variable with probability function f(x) over space R

1. The mean of X = $\mu = E[x]$
2. The variance of X = $\sigma^2 = E[(x-\mu)^2]$
3. The skewness of X = $\gamma_1 = \frac{E[(x-\mu)^3]}{\sigma^3}$
4. The kurtosis of X = $\gamma_2 = \frac{E[(x-\mu)^4]}{\sigma^4}$

Theorem 5.3.6 (Alternate Formulas for Theoretical Measures)**.**

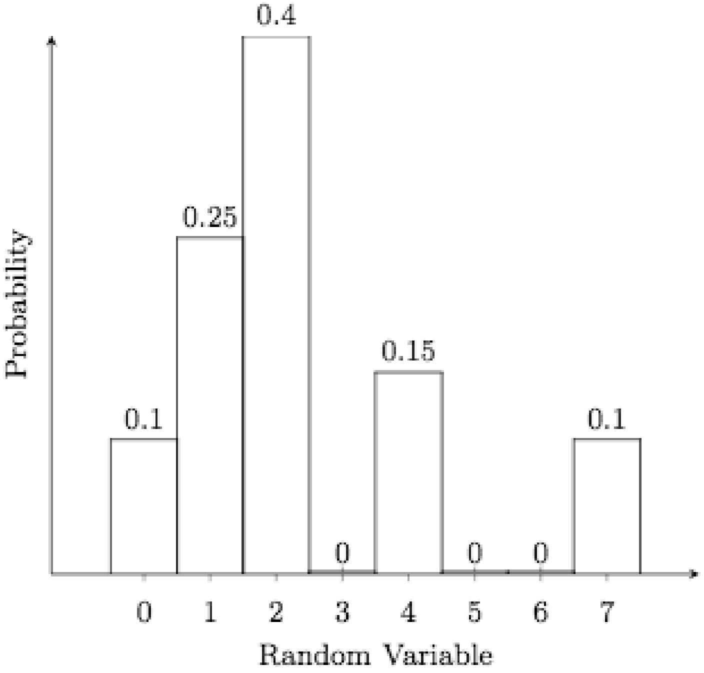
1. $\sigma^2 = E[x^2] - \mu^2 = E[X(x-1)] + \mu - \mu^2$

2. $\gamma_1 = \frac{1}{\sigma^3} \cdot \left[E[X^3] - 3\mu E[X^2] + 2\mu^3\right]$

3. $\gamma_2 = \frac{1}{\sigma^4} \cdot \left[E[X^4] - 4\mu E[X^3] + 6\mu^2 E[X^2] - 3\mu^4\right]$

Proof. In each case, expand the binomial inside and use the linearity of expected value. □

Consider the following example when computing these statistics for a discrete variable. In this case, we will utilize a variable with a relatively small space so that the summations can be easily done by hand. Indeed, consider



Using the definition of mean as a sum,

$$\begin{aligned}\mu &= 0\cdot 0.10 + 1\cdot 0.25 + 2\cdot 0.40 + 4\cdot 0.15 + 7\cdot 0.10\\ &= 0 + 0.25 + 0.80 + 0.60 + 0.70\\ &= 2.35\end{aligned}$$

Notice where this lies on the probability histogram for this distribution.

For the variance

$$\begin{aligned}\sigma^2 &= E[X^2] - \mu^2\\ &= \left[0^2\cdot 0.10 + 1^2\cdot 0.25 + 2^2\cdot 0.40 + 4^2\cdot 0.15 + 7^2\cdot 0.10\right] - 2.35^2\\ &= 0 + 0.25 + 1.60 + 2.40 + 4.90 - 5.5225\\ &= 9.15 - 5.225\\ &= 3.6275\end{aligned}$$

and so the standard deviation $\sigma = \sqrt{3.6275} \approx 1.90$. Notice that 4 times this value encompasses almost all of the range of the distribution.

For the skewness

$$\begin{aligned}\text{Numerator} &= E[X^3] - 3\mu E[X^2] + 2\mu^3\\ &= \left[0^3\cdot 0.10 + 1^3\cdot 0.25 + 2^3\cdot 0.40 + 4^3\cdot 0.15 + 7^3\cdot 0.10\right] - 3\cdot 2.35\cdot 9.15 + 2\cdot 2.35^3\\ &\approx 0 + 0.25 + 3.20 + 9.60 + 34.3 - 64.5075 + 25.96\\ &= 47.35 - 64.5075 + 25.96\\ &\approx 8.80\end{aligned}$$

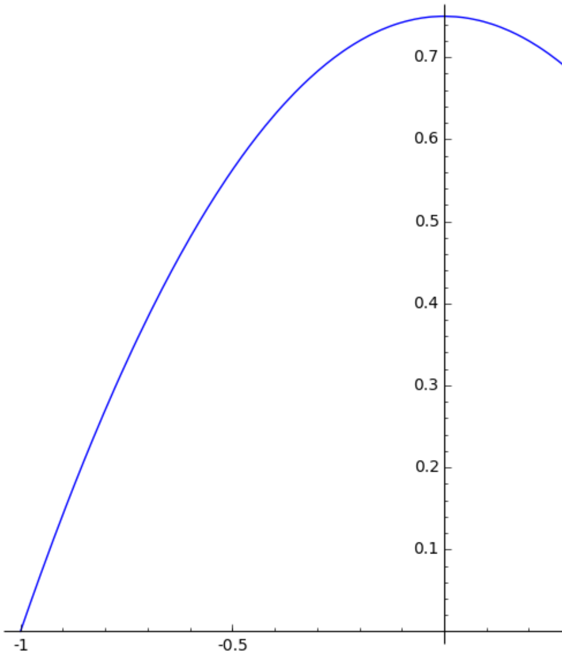
which yields a skewness of $\gamma_1 = 8.80/\sigma^3 \approx 1.27$. This indicates a slight skewness to the right of the mean. You can notice the 4 and 7 entries on the histogram illustrate a slight trailing off to the right.

Finally, for kurtosis

$$
\begin{aligned}
\text{Numerator} &= E[X^4] - 4\mu E[X^3] + 6\mu^2 E[X^2] - 3\mu^4 \\
&= \left[0^4 \cdot 0.10 + 1^4 \cdot 0.25 + 2^4 \cdot 0.40 + 4^4 \cdot 0.15 + 7^4 \cdot 0.10\right] - 4 \cdot 2.35 \cdot 47.35 + 6 \cdot 2.35^2 \cdot 9.15 \\
&\approx 0 + 0.25 + 6.40 + 38.4 + 240.1 - 445.09 + 303.19 - 91.49 \\
&\approx 285.15 - 445.09 + 303.19 - 91.49 \\
&\approx 51.75
\end{aligned}
$$

which yields a kurtosis of $\gamma_2 = 51.75/\sigma^4 \approx 3.93$ which also notes that the data appears to have a modestly bell-shaped distribution.

Consider the following example when computing these statistics for a continuous variable. Let $f(x) = \frac{3}{4} \cdot (1 - x^2)$ over R = [-1,1].



Then for the mean

$$
\begin{aligned}
\mu &= \int_{-1}^{1} x \cdot \frac{3}{4} \cdot (1 - x^2) dx \\
&= \int_{-1}^{1} \frac{3}{4} \cdot (x - x^3) dx \\
&= \frac{3}{4} \cdot (x^2/2 - x^4/4)\Big|_{-1}^{1} \\
&= \frac{3}{4} \cdot [(1/2) - (1/4)] - [(1/2) - (1/4)] \\
&= 0
\end{aligned}
$$

as expected since the probability function is symmetric about x=0.

For the variance

$$\begin{aligned}\sigma^2 &= \int_{-1}^{1} x^2 \cdot \frac{3}{4} \cdot (1-x^2)dx - \mu^2 \\ &= \int_{-1}^{1} \cdot \frac{3}{4} \cdot (x^2 - x^4)dx - 0 \\ &= \frac{3}{4} \cdot (x^3/3 - x^5/5)\Big|_{-1}^{1} \\ &= \frac{3}{4} \cdot 2 \cdot (1/3 - 1/5) \\ &= \frac{3}{4} \cdot \frac{4}{15} \\ &= \frac{1}{5}\end{aligned}$$

and taking the square root gives a standard deviation slightly less than 1/2. Notice that four times this value encompasses almost all of the range of the distribution.

For the skewness, notice that the graph is symmetrical about the mean and so we would expect a skewness of 0. Just to check it out

$$\begin{aligned}\text{Numerator} &= E[X^3] - 3\mu E[X^2] + 2\mu^3 \\ &= \int_{-1}^{1} x^3 \cdot \frac{3}{4} \cdot (1-x^2)dx - 3E[X^2] \cdot 0 + 0^3 \\ &= \int_{-1}^{1} \cdot \frac{3}{4} \cdot (x^3 - x^5)dx \\ &= \frac{3}{4} \cdot (x^4/4 - x^6/6)\Big|_{-1}^{1} \\ &= 0\end{aligned}$$

as expected without having to actually complete the calculation by dividing by the cube of the standard deviation.

Finally, note that the probability function in this case is modestly close to a bell shaped curve so we would expect a kurtosis in the vicinity of 3. Indeed, noting that (conveniently) $\mu = 0$ gives

$$\begin{aligned}\text{Numerator} &= E[X^4] - 4\mu E[X^3] + 6\mu^2 E[X^2] - 3\mu^4 \\ &= \int_{-1}^{1} x^4 \cdot \frac{3}{4} \cdot (1-x^2)dx \\ &= \frac{3}{4} \cdot (x^5/5 - x^7/7)\Big|_{-1}^{1} \\ &= \frac{3}{4} \cdot 2(1/5 - 1/7) \\ &= \frac{3}{35}\end{aligned}$$

and so by dividing by $\sigma^4 = \sqrt{\frac{1}{5}}^{\,4} = \frac{1}{25}$ gives a kurtosis of

$$\gamma_2 = \frac{3}{35} / \frac{1}{25} = \frac{75}{35} \approx 2.14.$$

Going back to Pascal's wager, let X = 0 represent disbelief when God doesn't exist and X = 1 represent disbelief when God does exist, X = 2 represent belief when God does exist, and X = 3 represent belief when God does not exist. Let p be the likelihood that God exists. Then you can compute the expected value of disbelief and the expect value of belief by first creating a value function. Below, for argument sake we are somewhat randomly assign a value of one million to disbelief if God doesn't exist. The conclusions are the same if you choose any other finite number...

$$\begin{gathered} u(0) = 1{,}000{,}000, f(0) = 1 - p \\ u(1) = -\infty, f(1) = p \\ u(2) = \infty, f(2) = p \\ u(3) = 0, f(3) = 1 - p \end{gathered}$$

Then,

$$\begin{aligned} E[\text{disbelief}] &= u(0)f(0) + u(1)f(1) \\ &= 1000000 \times (1 - p) - \infty \times p \\ &= -\infty \end{aligned}$$

if p>0. On the other hand,

$$\begin{aligned} E[\text{belief}] &= u(2)f(2) + u(3)f(3) \\ &= \infty \times p + 0 \times (1 - p) \\ &= \infty \end{aligned}$$

if p>0. So Pascal's conclusion is that if there is even the slightest chance that God exists then belief is the smart and scientific choice.

5.4 Standard Units

Any distribution variable can be converted to "standard units" using the linear translation $z = \dfrac{x - \mu}{\sigma}$. In doing so, then values of z will always represent the number of standard deviations x is from the mean and will provide "dimensionless" comparisons.

Chapter 6

Uniform and Hypergeometric Distributions

When motivating our definition of probability you may have noticed that we modeled our definition on the relative frequency of equally-likely outcomes. In this chapter you will develop the theoretical formulas which can be used to model equally-likely outcomes.

6.1 Discrete Uniform Distribution

Assume that you have a variable with space R = 1, 2, 3, ..., n so that the likelihood of each value is equally likely. Then, the probability function satisfies $f(x) = c$ for any $x \in R$. As before, since $\sum_{x \in R} f(x) = 1$, then

$$f(x) = \frac{1}{n}$$

is the probability function.

```
# Uniform distribution over 1 .. n
pretty_print("Discrete␣Uniform␣Distribution␣over␣the␣set␣
    1,␣2,␣...,␣n")
var(’x’)
@interact
def _(n=slider(2,10,1,2)):
    np1 = n+1
    R = range(1,np1)
    f(x) = 1/n
    pretty_print(html(’Density␣Function:␣$f(x)␣
        =%s$’%str(latex(f(x)))+’␣over␣the␣space␣$R␣=␣
        %s$’%str(R)))
    points((k,f(x=k)) for k in R).show()
    for k in R:
        pretty_print(html(’$f(%s’%k+’)␣=␣%s’%f(x=k)+’␣
            \\approx␣%s$’%f(x=k).n(digits=5)))
```

Theorem 6.1.1 (Properties of the Discrete Uniform Probability Function)**.**

1. *$f(x) = \frac{1}{n}$ over R = 1, 2, 3, ..., n satisfies the properties of a discrete probability function*

2. $\mu = \frac{1+n}{2}$

3. $\sigma^2 = \frac{n^2-1}{12}$

4. $\gamma_1 = 0$

5. $\gamma_2 = \frac{6}{5}\frac{1+n^2}{1-n^2}$

6. Distribution function F(x) = fracxn for x ∈ R.

Proof.

1. Trivially, by construction you get

$$\sum_{k=1}^{n} \frac{1}{n} = 1$$

Also, 1/n is positive for all x values.

2. To determine the mean,

$$\begin{aligned}\mu &= \sum_{k=1}^{n} x \cdot \frac{1}{n}\\ &= \frac{1}{n}\sum_{k=1}^{n} x\\ &= \frac{1}{n}\frac{n(n+1)}{2}\\ &= \frac{1+n}{2}\end{aligned}$$

3. To determine the variance,

$$\begin{aligned}\sigma^2 &= \sum_{k=1}^{n} x^2 \cdot \frac{1}{n} - \mu^2\\ &= \frac{1}{n}\sum_{k=1}^{n} x^2 - \left(\frac{1+n}{2}\right)^2\\ &= \frac{1}{n}\frac{n(n+1)(2n+1)}{6} - \frac{1+2n+n^2}{4}\\ &= \frac{(2n^2+3n+1)}{6} - \frac{1+2n+n^2}{4}\\ &= \frac{(4n^2+6n+2)}{12} - \frac{3+6n+3n^2}{12}\\ &= \frac{(n^2-1)}{12}\end{aligned}$$

4. For skewness,

$$\begin{aligned}\gamma_1 &= \sum_{k=1}^{n} x^3 \cdot \frac{1}{n} - 3\mu\sum_{k=1}^{n} x^2 \cdot \frac{1}{n} + 2\mu^3\\ &= \frac{n^2(n+1)^2}{4n} - 3\frac{(n(n+1))}{2}\frac{1+n}{2} + 2\left(\frac{1+n}{2}\right)^3\\ &=\end{aligned}$$

5. For Kurtosis, use the fourth moment and simplify...the algebra is performed using Sage in the active cell below this proof.

6.

□

Sage can also do the algebra for you to determine each of these measures. Notice, as n increases the Kurtosis approaches $\frac{6}{5}$ which indicates that there is (obviously) no tend toward central tendency over time.

```
var(’x,n’)
f = 1/n
mu = sum(x*f,x,1,n).factor()
pretty_print(’Mean␣=␣’,mu)
mu = (1+n)/2
v = sum((x-mu)^2*f, x, 1, n)
pretty_print(’Variance␣=␣’,v.factor())
stand = sqrt(v)
pretty_print(’Skewness␣=␣␣’,(sum((x-mu)^3*f, x, 1,
    n)/stand^3))
kurt = sum((x-mu)^4*f, x, 1, n)/stand^4
pretty_print(’Kurtosis␣=␣’,(kurt-3).factor(),’␣+␣3’)
```

Example 6.1.2 (Rolling one die)**.** When you consider rolling a regular, fair, single 6-sided die, each side is equally likely. The sample space consists of the 6 sides, each with a unique number of physical dots. Let the random variable X correspond each side with the number corresponding to the number of dots. Then, R = 1, 2, 3, 4, 5, 6. Since each side is equally likely then f(x) = 1/6.

Further, the probability of getting an outcome in A=2,3 would be f(2)+f(3) = 1/6 + 1/6 = 2/6.

6.2 Continuous Uniform Distribution

Modeling the idea of "equally-likely" in a continuous world requires a slightly different perspective since there are obviously infinitely many outcomes to consider. Instead, you should consider requiring that intervals in the domain which are of equal width should have the same probability regardless of where they are in that domain. This behaviour suggests $P(u < X < v) = P(u+w < X < v+w)$. In integral notation you obtain the following:

$$\begin{aligned}\int_u^v f(x)dx &= \int_{u+w}^{v+w} f(x)dx \\ F(v) - F(u) &= F(v+w) - F(u+w) \\ F(u+w) - F(u) &= F(v+w) - F(v)\end{aligned}$$

which is true regardless of w so long as you stay in the domain of interest. This only happens if F is linear and therefore f must be constant. Say, f(x)=c. In many situations, the space of X will be a single interval with R = [a,b]. Unless otherwise noted, this will be our assumption as well.

Theorem 6.2.1 (Properties of the Continuous Uniform Probability Function)**.**

1. *$f(x) = \frac{1}{b-a}$ satisfies the properties of a probability function over $R =$ [a,b].*

2. *$\mu = \frac{a+b}{2}$*

3. *$\sigma^2 = \frac{b^2-a^2}{12}$*

4. $\gamma_1 = 0$

5. $\gamma_2 = \frac{9\left(a^5 - 5\,a^4 b + 10\,a^3 b^2 - 10\,a^2 b^3 + 5\,a b^4 - b^5\right)(a-b)}{5\,(a^3 - 3\,a^2 b + 3\,a b^2 - b^3)^2}$

```
# Continous uniform distribution statistics derivation
reset()
var('x,a,b')


f = 1/(b-a)

mu = integrate(x*f,x,a,b).factor()
pretty_print('Mean␣=␣',mu)

v = integrate((x-mu)^2*f, x, a, b)

pretty_print('Variance␣=␣',v.factor())
stand = sqrt(v)
sk = (integrate((x-mu)^3*f, x, a, b)/stand^3)
pretty_print('Skewness␣=␣␣',sk)
kurt = (integrate((x-mu)^4*f, x, a, b)/stand^4)
pretty_print('Kurtosis␣=␣',kurt)

pretty_print('Several␣Examples')
a1=0
for b1 in range(2,7):
    pretty_print('Using␣[',a1,',',b1,']:')
    pretty_print('␣␣␣␣mean␣=␣',mu(a=a1,b=b1))
    pretty_print('variance␣=␣',v(a=a1,b=b1))
    pretty_print('skewness␣=␣',sk(a=a1,b=b1))
    pretty_print('kurtosis␣=␣',kurt(a=a1,b=b1))
```

Example 6.2.2 (Occurence of exactly one event randomly in a given interval)**.** Suppose you know that only one person showed up at the counter of a local business in a given 30 minute interval of time. Then, R=[0,30] given f(x) = 1/30.

Further, the probability that the person arrived within the first 6 minutes would be $\int_0^6 \frac{1}{30} dx = 0.2$.

Theorem 6.2.3 (Distribution Function for Continuous Uniform)**.** *For* $x \in [a,b], F(x) = \frac{x-a}{b-a}$

Proof. For x in this range,

$$F(x) = \int_a^x \frac{1}{b-a} du = \frac{u}{b-a}\Big|_a^x = \frac{x-a}{b-a}.$$

□

6.3 Hypergeometric Distribution

For the discrete uniform distribution, the presumption is that you will be making a selection one time from the collection of items. However, if you want to take a larger sample without replacement from a distribution in which originally all are equally likely then you will end up with something which will not be uniform.

Indeed, consider a collection of n items from which you want to take a sample of size r without replacement. If n_1 of the items are "desired" and the remainder are not, let the random variable X measure the number of items from the first group in your sample. The resulting collection of probabilities is called a Hypergeometric Distribution.

Since you are sampling without replacement and trying only measure the number of items from your desired group in the sample, then the space of X will include R = 0, 1, ..., r assuming $n_1 \geq r$ and $n - n_1 \geq r$. In the case when r is too large for either of these, the formulas below will follow noting that binomial coefficients are zero if the top is smaller than the bottom or if the bottom is negative.

So f(x) = P(X = x) = P(x from the sample are from the target group and the remainder are not). Breaking these up gives

$$f(x) = \frac{\binom{n_1}{x}\binom{n-n_1}{r-x}}{\binom{n}{r}}$$

Theorem 6.3.1 (Properties of the Hypergeometric Distribution)**.**

1. *$f(x) = \frac{\binom{n_1}{x}\binom{n-n_1}{r-x}}{\binom{n}{r}}$ satisfies the properties of a probability function.*
2. *$\mu = r\frac{n_1}{n}$*
3. *$\sigma^2 = r\frac{n_1}{n}\frac{n_2}{n}\frac{n-r}{n-1}$*
4. *$\gamma_1 = \frac{(n-2n_1)\sqrt{n-1}(n-2r)}{rn_1(n-n_1)\sqrt{n-r}(n-2)}$*
5. *$\gamma_2 = \frac{n(n+1)-6n(n-r)}{n_1(n-n_1)} + \frac{3r(n-r)(n+6)}{n^2} - 6$*

Proof.

1.

$$\begin{aligned}\sum_{x=0}^{n}\binom{n}{x}y^x &= (1+y)^n, \text{ by the Binomial Theorem}\\ &= (1+y)^{n_1}\cdot(1+y)^{n_2}\\ &= \sum_{x=0}^{n_1}\binom{n_1}{x}y^x\cdot\sum_{x=0}^{n_2}\binom{n_2}{x}y^x\\ &= \sum_{x=0}^{n}\sum_{t=0}^{r}\binom{n_1}{r}\binom{n_2}{r-t}y^x\end{aligned}$$

Equating like coefficients for the various powers of y gives

$$\binom{n}{r} = \sum_{t=0}^{r}\binom{n_1}{r}\binom{n_2}{r-t}.$$

Dividing gives

$$1 = \sum_{x=0}^{r} f(x).$$

2. For the mean

$$\begin{aligned}
\sum_{x=0}^{n} x\frac{\binom{n_1}{x}\binom{n-n_1}{r-x}}{\binom{n}{r}} &= \frac{1}{\binom{n}{r}}\sum_{x=1}^{n}\frac{n_1(n_1-1)!}{(x-1)!(n_1-x)!}\binom{n-n_1}{r-x} \\
&= \frac{n_1}{\binom{n}{r}}\sum_{x=1}^{n}\frac{(n_1-1)!}{(x-1)!((n_1-1)-(x-1))!}\binom{n-n_1}{r-x} \\
&= \frac{n_1}{\frac{n(n-1)!}{r!(n-r)!}}\sum_{x=1}^{n}\binom{n_1-1}{x-1}\binom{n-n_1}{r-x}
\end{aligned}$$

Consider the following change of variables for the summation:

$$\begin{aligned}
y &= x-1 \\
n_3 &= n_1-1 \\
s &= r-1 \\
m &= n-1
\end{aligned}$$

Then, this becomes

$$\begin{aligned}
\mu = \sum_{x=0}^{n} x\frac{\binom{n_1}{x}\binom{n-n_1}{r-x}}{\binom{n}{r}} &= r\frac{n_1}{n}\sum_{y=0}^{m}\frac{\binom{n_3}{y}\binom{m-n_3}{s-y}}{\binom{m}{s}} \\
&= r\frac{n_1}{n}\cdot 1
\end{aligned}$$

noting that the summation is in the same form as was show yields 1 above.

3. The proof of the variance formula is similar and uses E(X(X-1)+ - 2. The proof of skewness and kurtosis are messy and we won't bother with them for this distribution!

□

Note, if r=1 then you are back at a regular discrete uniform model. Indeed,

$$P(\text{desired item}) = 1\cdot\frac{n_1}{n} = \mu.$$

which is indeed what you might expect when selecting once.

Consider the Hypergeometric distribution for various values of n_1, n_2, and r using the interactive cell below. Notice what happens when you start with relatively small values of n_1, n_2, and r (say, start with $n_1 = 5, n_2 = 8$, and r = 4 and then doubling then all again and again. Consider the likely skewness and kurtosis of the graph as the values get larger.

```
# Hypergeometric distribution over 0 .. N
# Size of classes N1 and N2 must be given as well as
    subset size r
var('x')
@interact
def _(N1=slider(1,40,1,10,label='$N_1$'),
    N2=slider(1,40,1,10,label='$N_2$'),
    r=slider(1,40,1,10,label='$r$')):
    N = N1 + N1
    R = range(r+1)
    if (r > N1)|(r > N2):
        pretty_print(’When␣r␣is␣bigger␣than␣N1␣or␣N2,␣
            special␣consideration␣must␣be␣made’)
    else:
        f(x) =
            binomial(N1,x)*binomial(N2,r-x)/binomial(N,r)
        pretty_print(html(’Density␣Function:␣$f(x)␣
            =%s$’%str(latex(f(x)))))
        pretty_print(html(’over␣the␣space␣$R␣=␣
            %s$’%str(R)))
        points((k,f(x=k)) for k in R).show()
        for k in R:
           print (html(’$f(%s’%k+’)␣=␣
               %s’%latex(f(x=k))+’␣\\approx␣
               %s$’%f(x=k).n(digits=5)))
```

6.4 Exercises

Exercise 6.4.1 (- The Proverbial Urn Problem).

Exercise 6.4.2 (- Playing Cards).

Exercise 6.4.3 (- Starting Seniors).

Exercise 6.4.4 (- Old Faithful).

Exercise 6.4.5 (- Continuous Uniform Random Variable Scenarios).

Exercise 6.4.6 (- Louisiana Mega Millions Lottery).

Chapter 7

Binomial, Geometric, and Negative Binomial Distributions

Many practical problems involve dealing with success vs failure where . In these situations, "success" should not be interpreted as having any moral or subjective meaning but only construed to mean that something you are looking for actually occurs.

In situation where a single trial is performed and the result is determined only to be a success or failure is called a Bernoulli event. Indeed, one could create a corresponding probability function using a random variable X over the space R = 0,1 mapping X(success) = 1 and X(failure)=0. If p = P(success) then

$$f(x) = p^x \cdot (1-p)^{1-x}$$

would be a formula but which only related to two values P(failure) = f(0) = (1-p) and P(Success) = f(1) = p.

Notice that p=0 means that you will always get a failure and that p=1 means that you will always get a success. In these cases, X would no longer be a random variable since the outcome for X could be predicted with certainty. Therefore, we will always assume that $0 < p < 1$.

The Bernoulli distribution on its own is not extremely useful but serves as a starting point for several others that are useful. Indeed, in this chapter you will investigate distributions that relate some number of successes in multiple trials to some number of independent trials. The difference between these distributions will be that one of these variables will be fixed and the other one will be variable.

7.1 Binomial Distribution

Consider a sequence of n independent Bernoulli trials with the likelihood of a success p on each individual trial stays constant from trial to trial with $0 < p < 1$. If we let the variable X measure the number of successes obtained when doing a fixed number of trials n, then the resulting distribution of probabilities is called a Binomial Distribution.

```
# Binomial distribution over 0 .. n
# Probability of success on one independent trial = p
    must also be given
```

```
var('x')
@interact
def _(n=slider(3,50,1,3),p=slider(1/20,19/20,1/20,1/2)):
    np1 = n+1
    R = range(np1)
    f(x) =
        factorial(n)/(factorial(x)*factorial(n-x))*p^x*(1-p)^(n-x)
    pretty_print(html('Density␣Function:␣$f(x)␣
        =%s$'%str(latex(f(x)))))
    pretty_print(html('over␣the␣space␣$R␣=␣%s$'%str(R)))
    G = points((k,f(x=k)) for k in R)
    G.show()
    R = [k for k in R]
    probs = [f(x=k) for k in R]
#     H = histogram( R, weights = probs, align="mid",
    linewidth=2, edgecolor="blue", color="yellow")
#     H.show()
    for k in R:
        pretty_print(html('$f(%s'%k+')␣=␣
            %s'%latex(f(x=k))+'␣\\approx␣
            %s$'%f(x=k).n(digits=5)))
```

Theorem 7.1.1 (Derivation of Binomial Probability Function)**.** *For R = 0, 1, ..., n,*

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

Proof. Since successive trials are independent, then the probability of X successes occurring within n trials is given by

$$P(X = x) = \binom{n}{x} P(SS...SFF...F) = \binom{n}{x} p^x (1-p)^{n-x}$$

□

Theorem 7.1.2 (Verification of Binomial Distribution Formula)**.**

$$\sum_{x \in R} f(x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = 1.$$

Proof. Using the Binomial Theorem with a = p and b = 1-p yields

$$\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} = (p + (1-p))^n = 1$$

□

Theorem 7.1.3 (Binomial Distribution mean)**.**

$$\mu = np$$

Proof.

$$\begin{aligned}
\mu &= E[X] \\
&= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} x \frac{n(n-1)!}{x(x-1)!(n-x)!} p^x (1-p)^{n-x} \\
&= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!((n-1)-(x-1))!} p^{x-1} (1-p)^{(n-1)-(x-1)}
\end{aligned}$$

Using the change of variables $k = x - 1$ and $m = n - 1$ yields a binomial series

$$\begin{aligned}
&= np \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} p^k (1-p)^{m-k} \\
&= np(p + (1-p))^m = np
\end{aligned}$$

□

Theorem 7.1.4 (Binomial Distribution variance).

$$\sigma^2 = np(1-p)$$

Proof.

$$\begin{aligned}
\sigma^2 &= E[X(X-1)] + \mu - \mu^2 \\
&= \sum_{x=0}^{n} x(x-1) \binom{n}{x} p^x (1-p)^{n-x} + np - n^2p^2 \\
&= \sum_{x=2}^{n} x(x-1) \frac{n(n-1)(n-2)!}{x(x-1)(x-2)!(n-x)!} p^x (1-p)^{n-x} + np - n^2p^2 \\
&= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!((n-2)-(x-2))!} p^{x-2} (1-p)^{(n-2)-(x-2)} + np - n^2p^2
\end{aligned}$$

Using the change of variables $k = x - 2$ and $m = n - 2$ yields a binomial series

$$\begin{aligned}
&= n(n-1)p^2 \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} p^k (1-p)^{m-k} + np - n^2p^2 \\
&= n(n-1)p^2 + np - n^2p^2 = np - np^2 = np(1-p)
\end{aligned}$$

□

The following uses Sage to determine the general formulas for the Binomial distribution.

```
var('x,n,p')
assume(x,'integer')
f(x) = binomial(n,x)*p^x*(1-p)^(n-x)
mu = sum(x*f,x,0,n)
M2 = sum(x^2*f,x,0,n)
M3 = sum(x^3*f,x,0,n)
M4 = sum(x^4*f,x,0,n)
```

```
pretty_print('Mean␣=␣',mu)

v = (M2-mu^2).factor()
pretty_print('Variance␣=␣',v)
stand = sqrt(v)

sk = ((M3 - 3*M2*mu + 2*mu^3)).factor()/stand^3
pretty_print('Skewness␣=␣',sk)

kurt = (M4 - 4*M3*mu + 6*M2*mu^2
    -3*mu^4).factor()/stand^4
pretty_print('Kurtosis␣=␣',(kurt-3).factor(),'+3')
```

Flipping Coins

Suppose you flip a coin exactly 20 times. Determine the probability of getting exactly 10 heads and then determine the probability of getting 10 or fewer heads.

This is binomial with n = 20, p = 1/2 and you are looking for f(10). With these values

$$f(10) = \binom{20}{10} \cdot \left(\frac{1}{2}\right)^{10} \cdot \left(\frac{1}{2}\right)^{20-10} = \frac{46189}{262144} \approx 0.176$$

Notice, the mean for this distribution is also 10 so one might expect 10 heads in general. Next, to determine the probability for 10 or fewer heads requires F(10) = f(0) + f(1) + ... + f(10). There is no "nice" formula for F but this calculation can be performed using a graphing calculator, such as the TI-84 with F(x) = binomcdf(n,p,x). In this case, F(10) = binomcdf(20,1/2,10) = 0.588.

7.2 Geometric Distribution

Consider the situation where one can observe a sequence of independent trials where the likelihood of a success on each individual trial stays constant from trial to trial. Call this likelihood the probably of "success" and denote its value by p where $0 < p < 1$. If we let the variable X measure the number of trials needed in order to obtain the first success, then the resulting distribution of probabilities is called a Geometric Distribution.

Since successive trials are independent, then the probability of the first success occurring on the mth trial presumes that the previous m-1 trials were all failures. Therefore the desired probability is given by

$$f(x) = P(X = m) = P(FF...FS) = (1-p)^{m-1}p$$

Theorem 7.2.1 (Geometric Distribution sums to 1)**.**

$$f(x) = (1-p)^{m-1}p$$

sums to 1 over $R = \{1, 2, ...\}$

Proof.

$$\sum_{k=1}^{\infty} f(x) = \sum_{k=1}^{\infty} (1-p)^{k-1}p = p\sum_{j=0}^{\infty} (1-p)^{j} = p\frac{1}{1-(1-p)} = 1$$

□

```
# Geometric distribution over 0 .. n
# Probability of success on one independent trial = p
    must also be given
var('x')
# n = 50 by default. actually should be infinite
@interact
def _(p=input_box(0.1,label='p␣=␣
    '),n=[25,50,75,100,200]):
    np1 = n+1
    R = range(1,np1)
    f(x) = (1-p)^(x-1)*p
    F(x) = 1 - (1-p)^x
    pretty_print(html('Density␣Function:␣$f(x)␣
        =%s$'%str(latex(f(x)))+'␣over␣the␣space␣$R␣=␣
        %s$'%str(R)))
    points((k,f(x=k)) for k in
        R).show(title="Probability␣Function")
    print
    points((k,F(x=k)) for k in
        R).show(title="Distribution␣Function")
    if (n == 25):
        for k in R:
            pretty_print(html('$f(%s'%k+')␣=␣
                %s'%latex(f(x=k))+'␣\\approx␣
                %s$'%f(x=k).n(digits=5)))
```

Theorem 7.2.2 (Geometric Mean)**.** *For the geometric distribution,*

$$\mu = 1/p$$

Proof.

$$\begin{aligned}
\mu = E[X] &= \sum_{k=0}^{\infty} k(1-p)^{k-1}p \\
&= p\sum_{k=1}^{\infty} k(1-p)^{k-1} \\
&= p\frac{1}{(1-(1-p))^2} \\
&= p\frac{1}{p^2} = \frac{1}{p}
\end{aligned}$$

□

Theorem 7.2.3 (Geometric Variance)**.** *For the geometric distribution*

$$\sigma^2 = \frac{1-p}{p^2}$$

Proof.

$$\begin{aligned}
\sigma^2 &= E[X(X-1)] + \mu - \mu^2 \\
&=^{\infty}_{k=0} k(k-1)(1-p)^{k-1}p + \mu - \mu^2 \\
&= (1-p)p\sum_{k=2}^{\infty} k(k-1)(1-p)^{k-2} + \frac{1}{p} - \frac{1}{p^2} \\
&= (1-p)p\frac{2}{(1-(1-p))^3} + \frac{1}{p} - \frac{1}{p^2} \\
&= \frac{1-p}{p^2}
\end{aligned}$$

□

Theorem 7.2.4 (Geometric Distribution Function)**.**

$$F(a) = 1 - (1-p)^a$$

Proof. Consider the accumulated probabilities over a range of values...

$$\begin{aligned}
P(X \leq a) &= 1 - P(X > a) \\
&= 1 - \sum_{k=a+1}^{\infty} (1-p)^{k-1}p \\
&= 1 - p\frac{(1-p)^a}{1-(1-p)} \\
&= 1 - (1-p)^a
\end{aligned}$$

□

Theorem 7.2.5 (Statistics for Geometric Distribution)**.** *Mean, Variance, Skewness, Kurtosis computed by Sage.*

```
var('x,n,p')
assume(x,'integer')
f(x) = p*(1-p)^(x-1)
mu = sum(x*f,x,0,oo)
M2 = sum(x^2*f,x,0,oo)
M3 = sum(x^3*f,x,0,oo)
M4 = sum(x^4*f,x,0,oo)

pretty_print('Mean␣=␣',mu)

v = (M2-mu^2).factor()
pretty_print('Variance␣=␣',v)
stand = sqrt(v)

sk = ((M3 - 3*M2*mu + 2*mu^3)).factor()/stand^3
pretty_print('Skewness␣=␣',sk)

kurt = (M4 - 4*M3*mu + 6*M2*mu^2
    -3*mu^4).factor()/stand^4
pretty_print('Kurtosis␣=␣',(kurt-3).factor(),'+3')
```

Theorem 7.2.6 (The Geometric Distribution yields a memoryless model.)**.** *If X has a geometric distribution and a and b are nonnegative integers, then*

$$P(X > a + b | X > b) = P(X > a)$$

Proof. Using the definition of conditional probability,

$$\begin{aligned} P(X > a+b|X > b) &= P(X > a+b \cap X > b)\ P(X > b) \\ &= P(X > a+b)/P(X > b) \\ &= (1-p)^{a+b}/(1-p)^b \\ &= (1-p)^a \\ &= P(X > a) \end{aligned}$$

□

7.3 Negative Binomial

Consider the situation where one can observe a sequence of independent trials where the likelihood of a success on each individual trial stays constant from trial to trial. Call this likelihood the probably of "success" and denote its value by p where $0 < p < 1$. If we let the variable X measure the number of trials needed in order to obtain the rth success, $r \geq 1$ then the resulting distribution of probabilities is called a Geometric Distribution.

Note that r=1 gives the Geometric Distribution.

Theorem 7.3.1 (Negative Binomial Series)**.**

$$\frac{1}{(a+b)^n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} a^k b^{-n-k}$$

Proof. First, convert the problem to a slightly different form: $\frac{1}{(a+b)^n} = \frac{1}{b^n}\frac{1}{(\frac{a}{b}+1)^n} = \frac{1}{b^n}\sum_{k=0}^{\infty}(-1)^k\binom{n+k-1}{k}\left(\frac{a}{b}\right)^k$

So, let's replace $\frac{a}{b} = x$ and ignore for a while the term factored out. Then, we only need to show

$$\sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k = \left(\frac{1}{1+x}\right)^n$$

However

$$\begin{aligned} \left(\frac{1}{1+x}\right)^n &= \left(\frac{1}{1-(-x)}\right)^n \\ &= \left(\sum_{k=0}^{\infty}(-1)^k x^k\right)^n \end{aligned}$$

This infinite sum raised to a power can be expanded by distributing terms in the standard way. In doing so, the various powers of x multiplied together will create a series in powers of x involving $x^0, x^1, x^2, ...$ To detemine the final coefficients notice that the number of time m^k will appear in this product depends upon the number of ways one can write k as a sum of nonnegative integers.

For example, the coefficient of x^3 will come from the n ways of multiplying the coefficients $x^3, x^0, ..., x^0$ and $x^2, x^1, x^0, ..., x^0$ and $x^1, x^1, x^1, x^0, ..., x^0$. This is equivalent to finding the number of ways to write the number k as a sum of nonnegative integers. The possible set of nonnegative integers is 0,1,2,...,k and one way to count the combinations is to separate k *'s by n-1 |'s. For example, if k = 3 then *||** means $x^1x^0x^2 = x^3$. Similarly for k = 5 and |**|*|**| implies

$x^0x^2x^1x^2x^0 = x^5$. The number of ways to interchange the identical *'s among the idential $|$'s is $\binom{n+k-1}{k}$.

Furthermore, to obtain an even power of x will require an even number of odd powers and an odd power of x will require an odd number of odd powers. So, the coefficient of the odd terms stays odd and the coefficient of the even terms remains even. Therefore,

$$\left(\frac{1}{1+x}\right)^n = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k$$

Similarly, $\left(\frac{1}{1-x}\right)^n = \left(\sum_{k=0}^{\infty} x^k\right)^n = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$ □

Consider the situation where one can observe a sequence of independent trials with the likelihood of a success on each individual trial p where $0 < p < 1$. For a positive integer r, let the variable X measure the number of trials needed in order to obtain the rth success. Then the resulting distribution of probabilities is called a Negative Binomial Distribution.

Theorem 7.3.2 (Derivation of Negative Binomial Probability Function)**.**

$$f(x) = \binom{x-1}{r-1}(1-p)^{x-r}p^r,$$

for $x \in R = \{r, r+1, ...\}$.

Proof. Since successive trials are independent, then the probability of the rth success occurring on the x-th trial presumes that in the previous x-1 trials were r-1 successes and x-r failures. You can arrange these indistinguishable successes (and failures) in $\binom{x-1}{r-1}$ unique ways. Therefore the desired probability is given by

$$P(X = m) = \binom{x-1}{r-1}(1-p)^{x-r}p^r$$

□

Theorem 7.3.3 (Negative Binomial Distribution Sums to 1)**.**

$$\sum_{x=r}^{\infty} \binom{x-1}{r-1}(1-p)^{x-r}p^r = 1$$

Proof.

$$\sum_{x=r}^{\infty} \binom{x-1}{r-1}(1-p)^{x-r}p^r = p^r \sum_{x=r}^{\infty} \binom{x-1}{r-1}(1-p)^{x-r}$$

and by using $k = x - r$

$$\begin{aligned} &= p^r \sum_{k=0}^{\infty} \binom{r+k-1}{k}(1-p)^k \\ &= p^r \frac{1}{(1-(1-p))^r} \\ &= 1 \end{aligned}$$

□

Theorem 7.3.4 (Statistics for Negative Binomial Distribution)**.** *For the Negative Binomial Distribution,*

$$\mu = r\frac{1-p}{p}$$

$$\sigma^2 = r\frac{1-p}{p^2}$$

$$\gamma_1 = \frac{2-p}{\sqrt{r(1-p)}}$$

$$\gamma_2 = \frac{p^2-6p+6}{r(1-p)}$$

```
var('x,n,p,r')
assume(x,'integer')
@interact
def _(r=[2,3,4,5,6]):
    f(x) = binomial(x-1,r-1)*p^r*(1-p)^(x-1)
    mu = sum(x*f,x,r,oo).factor()
    M2 = sum(x^2*f,x,r,oo).factor()
    M3 = sum(x^3*f,x,r,oo).factor()
    M4 = sum(x^4*f,x,r,oo).factor()

    pretty_print('Mean␣=␣',mu)

    v = (M2-mu^2).factor()
    pretty_print('Variance␣=␣',v)
    stand = sqrt(v)

    sk = ((M3 - 3*M2*mu + 2*mu^3)).factor()/stand^3
    pretty_print('Skewness␣=␣',sk)

    kurt = (M4 - 4*M3*mu + 6*M2*mu^2
        -3*mu^4).factor()/stand^4
    pretty_print('Kurtosis␣=␣',(kurt-3).factor(),'+3')
```

7.4 Exercises

Exercise 7.4.1 (- Gallup Consumer Confidence Polling)**.**

Exercise 7.4.2 (- Rolling Dice)**.**

Exercise 7.4.3 (- Collecting Kids Meal Prizes)**.**

Chapter 8

Poisson, Exponential, and Gamma Distributions

8.1 Poisson Distribution

8.2 Exponential Distribution

8.3 Gamma Distribution

Chapter 9

Normal Distributions

9.1 Properties of the Normal Distribution

You have seen that most distributions become "bell shaped" as certain parameters are allowed to increase. The question might arise regarding whether this always must happen or is it just a happy coincidence. The amazing answer is that if you interpret the question in the correct way then this is always true.

Definition 9.1.1 (The Normal Distribution)**.** Given two parameters μ and σ a random variable X with density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\left(\frac{x-\mu}{\sigma}\right)^2/2}$$

Theorem 9.1.2. *If $\mu = 0$ and $\sigma = 1$, then we say X has a standard normal distribution and often use Z as the variable name. In this case, the density function reduces to*

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{z^2/2}$$

Proof. Convert to "standard units" using the conversion $z = \frac{x-\mu}{\sigma} = \frac{x-0}{1} = x$. □

9.2 Theorems

9.3 Chi-Square Distribution

9.4 Central Limit Theorem

Often, when one wants to solve various scientific problems, several assumptions will be made regarding the nature of the underlying setting and base their conclusions on those assumptions. Indeed, if one is going to use a Binomial Distribution or a Negative Binomial Distribution, an assumption on the value of p is necessary. For Poisson and Exponential Distributions, one must know the mean. For Normal Distributions, one must assume values for both the mean and the standard deviation. Where do these values come from? Often, one may perform a preliminary study and obtain a sample statistic...such as a sample mean or a relative frequency and use these values for or p.

But what is the underlying distribution of these sample statistics? The Central Limit Theorem gives the answer...

To motivate this discussion, consider the following two interactive experiments. For the first graph below, a sequence of N random samples, each of size r, ranging from 0 to "Range" is generated and graphed as small data points. As the number of samples N and the sample size r increase, notice that the data seems to cover the entire range of possible values relatively uniformly. (For this scatter plot note that each row represents the data for one sample of size r. The larger the N, the greater the number of rows.) Each row is averaged and that mean value is plotted on the graph as a red circle. If you check the "Show$_{M}ean” box, themeanofthesecirclesisindicatedbythegreenlineinthemiddleoftheplot.$

For the second graph below, the means are collected and the relative frequency of each is plotted. As N increases, you should see that the results begin to show an interesting tendency. As you increase the data range, you may notice this graph has a larger number of data values. Smoothing groups this data into intervals of length two for perhaps a graph with less variability.

Consider each of the following:

- As N increases with single digit values of r, what appears to happen to the mean and range of the means? How does increasing the data range from 1-100 to 1-200 or 1-300 affect these results?

- As N increases (say, for a middle value of r), what appears to happen to the means? How does increasing the data range from 1-100 to 1-200 or 1-300 affect these results?

- As r increases (say, for a middle value of N), what appears to happen to the range of the averages? Does your conclusion actually depend upon the value of N? (Look at the graph and don't worry about the actual numerical values.) How does increasing N for the second graph affect the skewness and kurtosis of that graph? Do things change significantly as r is increased?

```
var('n,k')
from sage.finance.time_series import TimeSeries

@interact(layout=dict(top=[['Range'],['Show_Mean',
    'Smoothing']],
bottom=[['N'],['r']]))

def
    _(Range=[100,200,300,500],N=slider(5,200,2,2,label="N␣
    =␣Number␣of␣Samples"),r=slider(3,200,1,2,label="r␣=␣
    Sample␣Size"),Show_Mean=False,Smoothing=False):
    R=[1..N]      #  R ranges over the number of
        samples...will point to the list of averages
    rangemax = Range

    data = random_matrix(ZZ,N,r,x=rangemax)
    datapoints = []
    avg_values = []
    avg_string = []
    averages = []
    for n in range(N):
        temp = 0
        for k in range(r):
            datapoints += [(data[n][k],n)]
            temp += data[n][k]
        avg_values.append(round(temp/r))
        if Smoothing:
            avg_string.append(str(2*round((temp/r)/2)))
        else:
            avg_string.append(str(round(temp/r)))

        averages += [(round(temp/r),n)]    #  make these
            averages integers for use in grouping later
    SCAT =
        scatter_plot(datapoints,markersize=2,edgecolor='red',figsize=(10,4),axes_labels
        Values','Sample␣Number'])
    AVGS =
        scatter_plot(averages,markersize=50,edgecolor='blue',marker='o',figsize=(7,4))

    freqslist =
        frequency_distribution(avg_string,1).function().items()


# compute sample statistics for the raw data as well as
    for the N averages
    Mean_data = (sum(sum(data))/(N*r)).n()
#     STD_data = sqrt(sum(sum( (data-Mean_data)^2
    ))/(N*r)).n()
    Mean_averages = mean(avg_values).n()
#     STD_averages = sqrt(variance(avg_values).n())
#     print "Data mean =",Mean_data," vs Mean of the
    averages =",Mean_averages
#     print "Data STD = ",STD_data," vs Standard Dev of
    avgs =", STD_averages
    if Show_Mean:
        avg_line =
            line([(Mean_data,0),(Mean_data,N-1)],rgbcolor='green',thickness=10)
        avg_text =
            text('xbar',(Mean_data,N),horizontal_alignment='right',rgbcolor='green')
    else:
        avg_line = Graphics()
        avg_text = Graphics()

#  Plot a scatter plot exhibiting uniformly random data
    and the collection of averages
    print(html("The␣random␣data␣plot␣on␣the␣left␣with␣
        each␣row␣representing␣a␣sample␣with␣size␣
        determined␣by\n"+
          "the␣slider␣above␣and␣each␣circle␣representing␣
              the␣average␣for␣that␣particular␣sample.\n"+
          "First,␣keep␣sample␣size␣relatively␣low␣and␣
              increase␣the␣number␣of␣samples.␣␣Then,␣\n"+
          "watch␣what␣happens␣when␣you␣slowly␣increase␣
              the␣sample␣size."))


#  Plot the relative frequencies of the grouped sample
    averages
    print(html("Now,␣the␣averages␣(ie.␣the␣circles)␣from␣
        above␣are␣collected␣and␣counted\n"+
          "with␣the␣relative␣frequency␣of␣each␣average␣
              graphed␣below.␣␣For␣a␣relatively␣large␣
              number␣of\n"+
          "samples,␣notice␣what␣seems␣to␣happen␣to␣these␣
              averages␣as␣the␣sample␣size␣increases."))
    if Smoothing:
        binRange = Range//2
    else:
        binRange = Range

    # normed=True  # if you want to have relative
        frequencies below

    his_low = 2*rangemax/7
    his_high = 5*rangemax/7

    T =
       histogram(avg_values,normed=False,bins=binRange,range=(his_low,his_hig
       Averages','Frequency'])
    #T =
       TimeSeries(avg_values).plot_histogram(axes_labels=['Sample
       Averages','Frequency'])

    pretty_print('Scatter␣Plot␣of␣random␣data.␣␣
       Horizontal␣is␣number␣of␣samples.')
    (SCAT+AVGS+avg_line+avg_text).show()
    pretty_print('Histogram␣of␣Sample␣Averages')
    T.show(figsize=(5,2))
```

```
var('n,k')
from sage.finance.time_series import TimeSeries

@interact(layout=dict(top=[['Range'],['Show_Mean',
   'Smoothing']],
bottom=[['N'],['r']]))

def
   _(Range=[100,200,300,500],N=slider(5,200,2,2,label="N␣
   =␣Number␣of␣Samples"),r=slider(3,200,1,2,label="r␣=␣
   Sample␣Size"),Show_Mean=False,Smoothing=False):
    R=[1..N]      #  R ranges over the number of
        samples...will point to the list of averages
    rangemax = Range

    data = random_matrix(ZZ,N,r,x=rangemax)
    datapoints = []
    avg_values = []
    avg_string = []
    averages = []
    for n in range(N):
        temp = 0
        for k in range(r):
            datapoints += [(data[n][k],n)]
            temp += data[n][k]
        avg_values.append(round(temp/r))
        if Smoothing:
            avg_string.append(str(2*round((temp/r)/2)))
        else:
            avg_string.append(str(round(temp/r)))

        averages += [(round(temp/r),n)]   #  make these
            averages integers for use in grouping later
    SCAT =
       scatter_plot(datapoints,markersize=2,edgecolor='red',figsize=(10,4),a
        Values','Sample␣Number'])
    AVGS =
        scatter_plot(averages,markersize=50,edgecolor='blue',marker='o',figsize=(7,4))

    freqslist =
        frequency_distribution(avg_string,1).function().items()


# compute sample statistics for the raw data as well as
    for the N averages
    Mean_data = (sum(sum(data))/(N*r)).n()
#     STD_data = sqrt(sum(sum( (data-Mean_data)^2
    ))/(N*r)).n()
    Mean_averages = mean(avg_values).n()
#     STD_averages = sqrt(variance(avg_values).n())
#     print "Data mean =",Mean_data," vs Mean of the
    averages =",Mean_averages
#     print "Data STD = ",STD_data," vs Standard Dev of
    avgs =", STD_averages
    if Show_Mean:
        avg_line =
            line([(Mean_data,0),(Mean_data,N-1)],rgbcolor='green',thickness=10)
        avg_text =
            text('xbar',(Mean_data,N),horizontal_alignment='right',rgbcolor='green')
    else:
        avg_line = Graphics()
        avg_text = Graphics()

#  Plot a scatter plot exhibiting uniformly random data
    and the collection of averages
    print(html("The␣random␣data␣plot␣on␣the␣left␣with␣
        each␣row␣representing␣a␣sample␣with␣size␣
        determined␣by\n"+
         "the␣slider␣above␣and␣each␣circle␣representing␣
             the␣average␣for␣that␣particular␣sample.\n"+
         "First,␣keep␣sample␣size␣relatively␣low␣and␣
             increase␣the␣number␣of␣samples.␣␣Then,␣\n"+
         "watch␣what␣happens␣when␣you␣slowly␣increase␣
             the␣sample␣size."))


#  Plot the relative frequencies of the grouped sample
    averages
    print(html("Now,␣the␣averages␣(ie.␣the␣circles)␣from␣
        above␣are␣collected␣and␣counted\n"+
         "with␣the␣relative␣frequency␣of␣each␣average␣
             graphed␣below.␣␣For␣a␣relatively␣large␣
             number␣of\n"+
         "samples,␣notice␣what␣seems␣to␣happen␣to␣these␣
             averages␣as␣the␣sample␣size␣increases."))
    if Smoothing:
        binRange = Range//2
    else:
        binRange = Range

    # normed=True  # if you want to have relative
        frequencies below

    his_low = 2*rangemax/7
    his_high = 5*rangemax/7

    T =
        histogram(avg_values,normed=False,bins=binRange,range=(his_low,his_hig
        Averages','Frequency'])
    #T =
        TimeSeries(avg_values).plot_histogram(axes_labels=['Sample
        Averages','Frequency'])

    pretty_print('Scatter␣Plot␣of␣random␣data.␣␣
        Horizontal␣is␣number␣of␣samples.')
    (SCAT+AVGS+avg_line+avg_text).show()
    pretty_print('Histogram␣of␣Sample␣Averages')
    T.show(figsize=(5,2))
```

9.4.1

Theorem

9.4.2

Approximating distributions Limiting distributions

Chapter 10

Estimating Data using Intervals

You should have noticed by now that repeatedly sampling from a given distribution will yield a variety of sample statistics such as $\overline{x}$ as an estimate perhaps for the population mean μ or $\frac{Y}{n}$ as an estimate for the population likelihood of success p. In this section, you will see how these sample "point estimators" are actually the best possible choices.

In creating these point estimates repeatedly, you have noticed that the results will change somewhat over time. Indeed, flip a coin 20 times and you might expect 10 heads. However, in practice it is likely to 9 or 12 out of 20 and possible to get any of the other possible outcomes. This natural variation makes the point estimates noted above to almost certainly be in error. However one would expect that they should be close and the Central Limit Theorem does indicate that the distribution of sample means should be approximately normally distributed. Thus, instead of relying just on the value of the point estimate, you might want to investigate a way to determine a reasonable interval centered on the sample statistic in which you have some confidence the actual population statistic should belong. This leads to a discussion of interval estimates known as confidence intervals (using calculational tools) and statistical tolerance intervals (using order statistics).

10.1 Point Estimates

Equally likely point estimates.

For Binomial, Geometric, what is p? For exponential, what is lambda? For normal, what is mu and sigma?

10.2 Chebyshev

An interval centered on the mean in which at least a certain proportion of the actual data must lie.

Theorem 10.2.1 (Chebyshev's Theorem). *Given a random variable X with given mean and standard deviation , for k>1 at least 2 1 1 k of the observations lie within k standard deviations from the mean.*

10.3 Measures of Spread

Measures of spread: • Average Deviation from the Mean – always zero for any distribution • Average Absolute Deviation from the Mean – difficult to deal with algebra when absolute values are used • Average Squared Deviation from the Mean – always non-negative and good with algebra and calculus

Definition 10.3.1 (Variance)**.** The variance is a measure of spread found by using the average squared deviation from the mean 2 = ( ))(2 1 k n k k xfx = if this value exists and is also denoted by Var(X). The positive square root of the variance is called the standard deviation and is denoted by .