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Introduction to Mathematical Probability and Statistics

A Calculus-based Approach

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John Travis grew up in Mississippi and had his graduate work at the University of Tennessee and Mississippi State University. As a numerical analyst, since 1988 he has been a professor of mathematics at his undergraduate alma mater Mississippi College where he currently serves as Professor and Chair of Mathematics.

You can find him playing racquetball or guitar but not generally at the same time. He is also an active supporter and organizer for the opensouce online homework system WeBWorK.

Preface

This text is intended for a one-semester calculus-based undergraduate course in probability and statistics .

A collection of WeBWorK online homework problems are available to correlate with the material in this text. Copies of these sets of problems are available by contacting the author.

WeBWorK (webwork.maa.org) is an open-source online homework system for math and science courses. WeBWorK is supported by the MAA and the NSF and comes with a Open Problem Library (OPL) of over 35,000 homework problems. Problems in the OPL target most lower division undergraduate math courses and some advanced courses. Supported courses include college algebra, discrete mathematics, probability and statistics, single and multivariable calculus, differential equations, linear algebra and complex analysis.

Sage (sagemath.org) is a free, open source, software system for advanced mathematics, which is ideal for assisting with a study of abstract algebra. Sage can be used either on your own computer, a local server, or on SageMathCloud (https://cloud.sagemath.com).

John Travis
Clinton, Mississippi 2015

Contents

Chapter 1

Review of Calculus

This chapter is a review of power series results from Calculus.

1.1 Geometric Series

Knowledge of the use of power series is very important when dealing with both probability functions.

$$S = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

as is its extension know as the negative binomial series ($n \in \mathbb{N}$).

$$NB = \sum_{k=0}^{\infty} (-1)^k \binom{-n+k-1}{k} x^k b^{-n-k} = \frac{1}{(x+b)^n}$$

In this section, we review this series, develop its properties, and explore some of its extensions.

1.1.1 Geometric Series

Theorem 1.1.1. $S = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

Proof. Consider the partial sum

$$S_n = \sum_{k=0}^{n} x^k = 1 + x + x^2 + ... + x^n$$

$$(1-x)S_n = S_n - xS_n = 1 + x + x^2 + ... + x^n - (x + x^2 + ... + x^n + x^{n+1}) = 1 - x^{n+1}$$

$$\Rightarrow S_n = \frac{1 - x^{n+1}}{1-x}$$

and so as $n \to \infty$,

$$S_n \to S = \frac{1}{1-x}$$

□

The interactive activity below shows how well the partial sums approximate $\frac{1}{1-x}$ as the number of terms increases.

```
var('x,n,k')
f = 1/(1-x)
@interact
def _(n = slider(2,20,1,2)):
        Sn = sum(x^k,k,0,n)
        pretty_print(html('$S_n(x) = %s$'%str(latex(Sn))))
        G = plot(f,x,-1,0.9,color='black')
        G += plot(Sn,x,-1,0.9,color='blue')
        G += plot(abs(f-Sn),x,-1,0.9,color='red')
        G.show(title="Partial Sums (blue) vs Infinite Series 
            (black) and Error (red)",figsize=(5,4))
```

1.1.2 Alternate Forms for the Geometric Series

Theorem 1.1.2 (Generalized Geometric Series)**.** *For* $k \in \mathbb{N}, \sum_{k=M}^{\infty} x^k = \frac{x^M}{1-x}$

Proof.

$$\begin{aligned}\sum_{k=M}^{\infty} x^k &= x^M \sum_{k=0}^{\infty} x^k \\ &= x^M \frac{1}{1-x} \\ &= \frac{x^M}{1-x}\end{aligned}$$

□

Example 1.1.3 (Integrating and Differentiating to get new Power Series)**.** The geometric power series is a nice function which is relatively easily differentiated and integrated. In doing so, one can obtain new power series which might also be very useful in their own right. Here we develop a few which are of special interest.

Let $f(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$. Then,

$$\begin{aligned}f'(x) &= \sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2} \\ f''(x) &= \sum_{k=2}^{\infty} k(k-1)x^{k-1} = \frac{2}{(1-x)^3} \\ f^{(n)}(x) &= \sum_{k=n}^{\infty} k(k-1)...(k-n+1)x^{k-n} = \frac{n!}{(1-x)^{n+1}} \\ \int f(x)dx &= \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = -ln(1-x)\end{aligned}$$

Example 1.1.4 (Playing with the base)**.**

$$\begin{aligned}\sum_{k=0}^{\infty} a^k x^k &= \sum_{k=0}^{\infty} (ax)^k \\ &= \frac{1}{1-ax}, |x| < \frac{1}{a}\end{aligned}$$

or perhaps

$$\sum_{k=0}^{\infty} (x-b)^k = \frac{1}{1-(x-b)}, |x-b| < 1$$

Example 1.1.5 (Application: Converting repeating decimals to fractional form)**.** Consider this example:

$$\begin{aligned}2.48484848... &= 2 + 0.48 + 0.0048 + 0.000048 + ... \\ &= 2 + 0.48(1 + 0.01 + 0.0001 + ...) = 2 + 0.48\sum_{k=0}^{\infty}(0.01)^k\end{aligned}$$

Therefore, applying the Geometric Series

$$\begin{aligned}2.48484848... &= 2 + 0.48\frac{1}{1-0.01} \\ &= 2 + 0.48\frac{100}{99} = 2 + \frac{48}{99}\end{aligned}$$

Example 1.1.6 (Playing around with repeating decimals)**.** Certainly most students would agree that $0.333333... = \frac{1}{3}$. So, what about 0.999999...? Simply follow the pattern above

$$\begin{aligned}0.999999... &= 0.9 + 0.09 + 0.009 + 0.0009 + ... = 0.9(1 + 0.1 + 0.1^2 + 0.1^3 + ... \\ &= 0.9\frac{1}{1-0.1} = 0.9\frac{1}{0.9} = 1\end{aligned}$$

1.2 Binomial Sums

The binomial series is also foundational. It is technically not a series since the sum$_{i}ffinitebutwewon'tbotherwiththatfornow.Itisgivenby$

1.2.1

Theorem 1.2.1 (Binomial Theorem)**.** *For $n \in \mathbb{N}$, $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$*

Proof. By induction:

Basic Step: n = 1 is trivial

Inductive Step: Assume the statement is true as given for some $n \geq 1$. Show

$(a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k}$

$$
\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n \\
&= (a+b)\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} \\
&= \sum_{k=0}^{n}\binom{n}{k}a^{k+1}b^{n-k} + \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k+1} \\
&= \sum_{k=0}^{n-1}\binom{n}{k}a^{k+1}b^{n-k} + a^{n+1} + b^{n+1} + \sum_{k=1}^{n}\binom{n}{k}a^k b^{n-k+1} \\
&= \sum_{j=1}^{n}\binom{n}{j-1}a^j b^{n-(j-1)} + a^{n+1} + b^{n+1} + \sum_{k=1}^{n}\binom{n}{k}a^k b^{n+1-k} \\
&= b^{n+1} + \sum_{k=1}^{n}\left[\binom{n}{k-1}+\binom{n}{k}\right]a^k b^{n+1-k} + a^{n+1} \\
&= b^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}a^k b^{n+1-k} + a^{n+1} \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}a^k b^{n+1-k}
\end{aligned}
$$

□

1.2.2 Binomial Series

Consider $B(a,b) = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$. This finite sum$_i sknownastheBinomialSeries.$

1.2.2.1

Show that $B(a,b) = (a+b)^n$
Show that $B(1,1) = 2^n$
Show that $B(-1,1) = 0$
Show that $B(p, 1-p) = 1$
Easily, $B(x,1) = \sum_{k=0}^{n} \binom{n}{k} a^k$

1.2.3 Trinomial Series

$$(a+b+c)^n = \sum_{k_1+k_2+k_3=n} \binom{n}{k_1,k_2,k_3} a^{k_1} b^{k_2} c^{k_3}$$

where $\binom{n}{k_1,k_2,k_3} = \frac{n!}{k_1!k_2!k_3!}$. This can be generalized to any number of terms to give what is know as a multinomial series.

1.3 Negative Binomial Series

$(a+b)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} a^k b^{-n-k}$

Theorem 1.3.1 (Alternate Form for Negative Binomial Series)**.** $(a+b)^{-n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} a^k b^{-n-k}$

Chapter 2

Representing Data

2.1 Measurement Scales

- Nominal - Mutually Exclusive and Exhaustive categories for which the numerical value has only identification significance. Ex: Male = 1, Female = -1
- Ordinal - Discrete values ranked from lowest to highest or vice versa. Ex: Class grades for GPA.
- Interval - Ordinal data where distance between data values is of significance. Ex: Heights and Weights.
- Ratio - Interval data where ratios of observations have meaning. Ex: Percentile rankings

2.2 Techniques for Representing Data

- Tabular Methods - based on the entire population yielding a global picture
  - frequency distributions
  - relative frequency distributions
  - cummulative frequency distributions
  - Stem-and-Leaf Displays
  - Box-and-Whisker Diagrams
- Summary Methods
  - Measures of the center
    1. Mean
    2. Median
    3. Mode
  - Measures of spread
    1. Range
    2. Variance and Standard Deviation
    3. Quantiles
  - Measures of Skewness - indicates the level of symmetry of the data

1. Pearson Coefficient
2. Standard Skewness
3. Bowley's Measure

- Measures of Kurtosis - indicates flatness or roundedness of the peak of the data
  1. Standard Kurtosis
  2. Coefficient of Kurtosis
- Measures of Association for Bivariate Data - indicates the likeliness of functional correlation of the data.
  1. Pearson Correlation Coefficient
  2. Spearman Rank Correlation Cooeficient
  3. Quantile-Quantile Plots
- Detection of Outliers - indicates whether abnormally large or small data distorts other techniques
  1. Z-scores
  2. Trimming
  3. Winsorizing
- Tests for Normality - indictes if the data is bell-shaped
  1. Standard Percentages relative to standard deviations from the mean
  2. Chi-square
  3. Kolmogorov-Smirnov
  4. Lilliefors
  5. Shapiro-Wilk
- Tests for Randomness - indicates whether the data has a non-systematic pattern
  1. Runs Test
  2. Mean-Square Successive Differences

Remark: Many of these measures above are relative and some are absolute.

2.3 Measures of Position

Given a collection of data, sorting the data may provide several useful descriptors. These include:

Definition 2.3.1 (Order Statistic:)**.** Given the given data set $x_1, x_2, ..., x_n$, after sorting the data label the sorted data as $y_1, y_2, ..., y_n$ where

$$y_1 \leq y_2 \leq ... \leq y_n.$$

Then, the kth order statistic is given by y_k.

For example, the age at inauguration for presidents from 1981-2016 gives the data $x_1 = 69, x_2 = 64, x_3 = 46, x_4 = 54, x_5 = 47$ (Reagan, Bush, Clinton, Bush, Obama). For this data, the order statistics are denoted $y_1 = 46, y_2 = 47, y_3 = 54, y_4 = 64, y_5 = 69$.

Definition 2.3.2 (Minimum/Maximum:)**.** The smallest and largest values in the data set. Using the notation above, minimum $= y_1$ and the maximum $= y_n$

Using the Presidential ages above, minimum $= y_1 = 46$ and maximum $= y_5 = 69$.

Definition 2.3.3 (Percentiles:)**.** A percentile is a numerical value P^p at which approximately 100p

To motivate your understanding of percentiles, consider the following data set: 2,5,8,10. The 50th percentile should be a numerical value for which approximately 50

To compute the percentile value exactly consider a percentage in the form 100p, for $0 < p < 1$, and the order statistics $y_1, y_2, ..., y_n$. Then, the 100pth percentile is given by

$$P^p = (1-r)y_m + ry_{m+1}$$

where m is the integer part of (n+1)p, namely

$$m = \lfloor (n+1)p \rfloor$$

and

$$r = (n+1)p - m,$$

the fractional part of (n+1)p. This determines a weighted average between y_m and y_{m+1} which is unique for distinct values of p provided each of the data values are distinct. Note that if some of the y-values are equal then some of these averages might be of equal numbers and will then be the common value.

Example 2.3.4 (Basic Percentiles)**.** Using the data set 2,5,8,10 with n=4 values, the 25th percentile is computed by considering

$$(n+1)p = (4+1)0.25 = 5/4 = 1.25$$

. So, m = 1 and r = 0.25. Therefore

$$P^{0.25} = 0.75 \times 2 + 0.25 \times 5 = 2.75$$

as noted above.

Similarly, the 75th percentile is given by

$$(n+1)p = (4+1)0.75 = 15/4 = 3.75$$

. So, m = 3 and r = 0.75. Therefore

$$P^{0.75} = 0.25 \times 8 + 0.75 \times 10 = 9.5$$

It is interesting to note that 3 also lies between 2 and 5 as does 2.75 and has the same percentages above (75 percent) and below (25 percent). However, it should designate a slightly larger percentile location. Indeed, going backward:

$$\begin{aligned} 3 &= (1-r) \times 2 + r \times 5 \\ \Rightarrow r &= \frac{1}{3} \\ \Rightarrow (n+1)p &= 1 + \frac{1}{3} = \frac{4}{3} \\ \Rightarrow p &= \frac{4}{15} \approx 0.267 \end{aligned}$$

and so 3 would actually be at approximately the 26.7th percentile.

Definition 2.3.5 (Quartiles:)**.** Given a sorted data set, the first, second, and third quartiles are the values of $Q_1 = P^{0.25}, Q_2 = P^{0.5}$ and $Q_3 = P^{0.75}$.

Definition 2.3.6 (Deciles:)**.** Given a sorted data set, the first, second, ..., ninth deciles are the value of $D_1 = P^{0.1}, D_2 = P^{0.2}, ..., D_9 = P^{0.9}$

For your data set 2,5,8,10, $Q_1 = 2.75, Q_2 = 6.5$, and $Q_3 = 9.5$.

Definition 2.3.7 (5-number summary)**.** Given a set of data, the 5-number summary is a vector of the order statistics given by $<$ minimum, Q_1, Q_2, Q_3, maximum $>$.

Returning to our previous example, the five number summary would be $< 2, 2.75, 6.5, 9.5, 10 >$

2.4 Measures of the Middle

Definition 2.4.1 (Arithmetic Mean)**.** Suppose X is a discrete random variable with range $R = x_1, x_2, ..., x_n$. The arithmetic mean is given by

$$AM = \frac{x_1 + ... + x_n}{n} = \frac{\sum_{k=1}^{n} x_k}{n}.$$

If this data comes from sample data then we call it a sample mean and denote this value by $\overline{x}$. If this data comes from the entire universe of possibilities then we call it a population mean and denote this value by μ.

To illustrate, consider the previous data set: 2,5,8,10. The arithmetic mean is given by

$$\frac{2+5+8+10}{4} = \frac{25}{4} = 6.25.$$

The mean is often called the centroid in the sense that if the x values were locations of objects of equal weight, then the centroid would be the point where this system of n masses would balance.

The values can all be provided with varying weights if desired and the result is called the weighted arithmetic mean and is given by

$$\frac{m_1 x_1 + ... + m_n x_n}{m_1 + ... + m_n} = \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k}.$$

Other Means:

Definition 2.4.2 (Geometric Mean)**.**

$$GM = (x_1 x_2 ... x_n)^{1/n}$$

Again, consider 2,5,8,10. The geometric mean is given by

$$(2 \times 5 \times 8 \times 10)^{1/4} = 800^{1/4} \approx= 5.318$$

Definition 2.4.3 (Harmonic Mean)**.**

$$\frac{1}{HM} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{x_k}$$

Once again, consider 2,5,8,10. The harmonic mean is given by first computing

$$\frac{1}{4}(\frac{1}{2} + \frac{1}{5} + \frac{1}{8} + \frac{1}{10}) = 800^{1/4} \approx= 0.23125$$

and so HM $= \frac{1}{0.23125}\ approx 4.32$

Theorem 2.4.4 (Relative sizes of Means)**.** $HM \leq GM \leq AM$.

Theorem 2.4.5 (Mean Formula)**.** $AMHM = GM^2$

Definition 2.4.6 (Median:)**.** A positional measure of the middle is often utilized by finding the location of the 50th percentile. This value is also called the median and indicates the value at which approximately half the sorted data lies below and half lies above.

For data sets with an odd number of values, this is the "middle" data value if one were to successively cross off pairs from the two ends of the sorted date. For data sets with an even number of values, this is a average of the two data values left after crossing off these pairs. Using the order statistics, the median equals

$$y_{\frac{n+1}{2}}$$

if n is odd and

$$\frac{y_{\frac{n}{2}} + y_{\frac{n}{2}+1}}{2}$$

if n is even.

From the Presidential data, note that you are considering an odd number of data values and so the median is given by 54.

Definition 2.4.7 (Midrange:)**.** A mixture of the mean and median where one takes the simple average of the maximum and minimum values in the data set. Using the order statistics, this equals

$$\frac{y_1 + y_n}{2}$$

From the Presidential data, the maximum is 69 and the minimum is 46 so the midrange is 57.5, the average of these two.

Mean utilizes all of the data values so each term is important. Utilizes them all even if some of the data values might suffer from collection errors. Median ignores outliers (which might be a result of collection errors) but does not account for the relative differences between terms. Midrange is very easy to compute but ignores the relative differences for all terms but the two extremes.

Example 2.4.8 (Numerical Example of these Quantitative Measures)**.** The US Census Bureau reported the following state populations (in millions) for 2013: Spreadsheet

Notice that these are already in order so you can presume $y_1 = 0.6$ million is the minimum and $y_{50} = 38.3$ million is the maximum. Therefore, the midrange is given by

$$\frac{0.6 + 38.3}{2} = \frac{38.9}{2} = 19.45 million.$$

The mean of this data takes a bit of arithmetic but gives

$$\frac{\sum_{k=1}^{50} y_k}{50} = \frac{316.1}{50} \approx 6.33$$

million residents.

Since the number of states is even, the median is found by averaging the 25th and 26th order statistics. In this case,

$$\frac{y_{25} + y_{26}}{2} = \frac{4.4 + 4.6}{2} \approx 4.5$$

million residents.

State	Population
Wyoming	0.6
Vermont	0.6
District of Columbia	0.6
North Dakota	0.7
Alaska	0.7
South Dakota	0.8
Delaware	0.9
Montana	1
Rhode Island	1.1
New Hampshire	1.3
Maine	1.3
Hawaii	1.4
Idaho	1.6
West Virginia	1.9
Nebraska	1.9
New Mexico	2.1
Nevada	2.8
Kansas	2.9
Utah	2.9
Arkansas	3
Mississippi	3
Iowa	3.1
Connecticut	3.6
Oklahoma	3.9
Oregon	3.9
Kentucky	4.4
Louisiana	4.6
South Carolina	4.8
Alabama	4.8
Colorado	5.3
Minnesota	5.4
Wisconsin	5.7
Maryland	5.9
Missouri	6
Tennessee	6.5
Indiana	6.6
Arizona	6.6
Massachusetts	6.7
Washington	7
Virginia	8.3
New Jersey	8.9
North Carolina	9.8
Michigan	9.9
Georgia	10
Ohio	11.6
Pennsylvania	12.8
Illinois	12.9
Florida	19.6
New York	19.7
Texas	26.4
California	38.3

2.5 Measures of Spread

Definition 2.5.1 (Range:)**.** Using the order statistics,

$$y_n - y_1.$$

Easy to compute. Ignores the spread of all the data in between.

From the Presidential data, the maximum is 69 and the minimum is 46 so the range is 23, the difference of these two.

Definition 2.5.2 (Interquartile Range (IQR):)**.** $P^{0.75} - P^{0.25}$.

For the data set 2, 5, 8, 10, you have found that $Q_1 = 2.75$ and $Q_3 = 9.5$. Therefore,

$$IQR = 9.5 - 2.75 = 6.75.$$

Average Deviation from the Mean: Given a data set $x_1, x_2, ..., x_n$ with mean μ each term deviates from the mean by the value $x_k - \mu$. So, averaging these gives

$$\frac{\sum_{k=1}^{n}(x_k - \mu)}{n} = \frac{\sum_{k=1}^{n} x_k}{n} - \frac{\sum_{k=1}^{n} \mu}{n} = \mu - \mu = 0$$

which is always zero for any provided set of data. This cancellation makes this measure not useful. To avoid cancellation, perhaps removing negatives would help.

Average Absolute Deviation from the Mean:

$$\frac{\sum_{k=1}^{n} |x_k - \mu|}{n}$$

which, although nicely stated, is difficult to deal with algebraically since the absolute values do not simplify well algebraically. To avoid this algebraic roadblock, we can look for another way to nearly accomplish the same goal by squaring and then square rooting.

Average Squared Deviation from the Mean:

$$\frac{\sum_{k=1}^{n}(x_k - \mu)^2}{n}$$

which will always be non-negative but can be easily expanded using algebra. Since this is a mouthful, this measure is generally called the variance.

Using the average squared deviation from the mean, differences have been squared. Thus all values added are non-negative but very small ones have been made even smaller and larger ones have possibly been made much larger. To undo this scaling issue, one must take a square root to get things back into the right ball park.

Definition 2.5.3 (Variance and Standard Deviation)**.** The variance is the average squared deviation from the mean. If this data comes from the entire universe of possibilities then we call it a population variance and denote this value by σ^2. Therefore

$$\sigma^2 = \frac{\sum_{k=1}^{n}(x_k - \mu)^2}{n}$$

The standard deviation is the square root of the variance. If this data comes from the entire universe of possibilities then we call it a population standard deviation and denote this value by σ. Therefore

$$\sigma = \sqrt{\frac{\sum_{k=1}^{n}(x_k - \mu)^2}{n}}.$$

From the data 2,5,8,10, you have found that the mean is 6.25. Computing the variance then involves accumulating and averaging the squared differences of each data value and this mean. Then

$$\begin{aligned}
&\frac{1}{4}\left((2-6.25)^2+(5-6.25)^2+(8-6.25)^2+(10-6.25)^2\right)\\
&=\frac{18.0625+1.5625+3.0625+14.0625}{4}\\
&=\frac{36.75}{4}\\
&=9.1875.
\end{aligned}$$

If data comes from a sample of the population then we call it a sample variance and denote this value by v. Since sample data tends to reflect certain "biases" then we increase this value slightly by $\frac{n}{n-1}$ to give the sample variance

$$s^2=\frac{n}{n-1}\frac{\sum_{k=1}^{n}(x_k-\overline{x})^2}{n}=\frac{\sum_{k=1}^{n}(x_k-\overline{x})^2}{n-1}.$$

and the sample standard deviation similarly as the square root of the sample variance.

Theorem 2.5.4 (Alternate Forms for Variance)**.**

$$\begin{aligned}
\sigma^2&=\left(\frac{\sum_{k=1}^{n}x_k^2}{n}\right)-\mu^2\\
&=\left[\frac{\sum_{k=1}^{n}x_k(x_k-1)}{n}\right]+\mu-\mu^2
\end{aligned}$$

Proof. □

The Population of the individual USA states according to the 2013 Census Consider the data set

Example 2.5.5 (Numerical Example of these Quantitative Measures)**.** The US Census Bureau reported the following state populations (in millions) for 2013:

Again, you should note that these are already in order so the range is quickly found to be

$$y_n-y_1=38.3-0.6=37.7$$

million residents.

For IQR, we first must determine the quartiles. The median (found earlier) already is the second quartile so we have $Q_2=4.5$ million. For the other two, the formula for computing percentiles gives you the 25th percentiile

$$\begin{aligned}
(n+1)p&=51(1/4)=12.75\\
Q_1=P^{0.25}&=0.25\times 1.9+0.75\times 2.1=2.05
\end{aligned}$$

and the 75th percentile

$$\begin{aligned}
(n+1)p&=51(3/4)=38.25\\
Q_3=P^{0.75}&=0.75\times 7+0.25\times 8.3=7.325.
\end{aligned}$$

Hence, the IQR = 7.325 - 2.05 = 5.275 million residents.

From the computation before, the mean of this data is about 6.33 million residents. So, to determine the variance you may find it easier to compute using the theorem above.

State	Population
Wyoming	0.6
Vermont	0.6
District of Columbia	0.6
North Dakota	0.7
Alaska	0.7
South Dakota	0.8
Delaware	0.9
Montana	1
Rhode Island	1.1
New Hampshire	1.3
Maine	1.3
Hawaii	1.4
Idaho	1.6
West Virginia	1.9
Nebraska	1.9
New Mexico	2.1
Nevada	2.8
Kansas	2.9
Utah	2.9
Arkansas	3
Mississippi	3
Iowa	3.1
Connecticut	3.6
Oklahoma	3.9
Oregon	3.9
Kentucky	4.4
Louisiana	4.6
South Carolina	4.8
Alabama	4.8
Colorado	5.3
Minnesota	5.4
Wisconsin	5.7
Maryland	5.9
Missouri	6
Tennessee	6.5
Indiana	6.6
Arizona	6.6
Massachusetts	6.7
Washington	7
Virginia	8.3
New Jersey	8.9
North Carolina	9.8
Michigan	9.9
Georgia	10
Ohio	11.6
Pennsylvania	12.8
Illinois	12.9
Florida	19.6
New York	19.7
Texas	26.4
California	38.3

x	frequency
1	5
2	7
3	4
4	3
5	6

and so you get a sample variance of

$$s^2 \approx \frac{50}{49} \cdot 48.72 = 49.71$$

and a sample standard deviation of

$$s \approx \sqrt{49.71} = 7.05$$

million residents.

2.6 Grouped Data

As you considered the measures of the center and spread before, each data point was considered individually. Often, data may however be grouped into categories and perhaps expressed as a frequency distribution. In this case, rather than considering x_k to be the kth data value can take advantage of the grouping to perhaps save a bit on arithmetic.

Indeed, let's assume that data is grouped into m categories $x_1, x_2, ..., x_m$ with corresponding frequencies $f_1, f_2, ..., f_m$. Then, for example, when computing the mean rather than adding x_1 with itself f_1 times just compute $x_1 \times f_1$ for the first category and continuing through the remaining categories. This gives the following grouped data formula for the mean

$$\mu = \frac{x_1 f_1 + ... + x_m f_m}{f_1 + ... + f_m} = \frac{\sum_{k=1}^m x_k f_k}{\sum_{k=1}^m f_k}.$$

and the following grouped data formula for the variance

$$\sigma^2 = \frac{\sum_{k=1}^m (x_k - \mu)^2 f_k}{\sum_{k=1}^m f_k} = \frac{\sum_{k=1}^m x_k^2 f_k}{\sum_{k=1}^m f_k} - \mu^2$$

Consider the following data set
3, 1, 2, 2, 3, 1, 3, 4, 5, 5, 1, 4, 5, 1, 2, 4, 5, 3, 2, 5, 2, 1, 2, 2, 5
Collecting this data into a frequency distribution gives
Therefore,

$$\overline{x} = \frac{1 \times 5 + 2 \times 7 + 3 \times 4 + 4 \times 3 + 5 \times 6}{5 + 7 + 4 + 3 + 6} = \frac{5 + 14 + 12 + 12 + 30}{25} = \frac{43}{25}$$

and

$$\begin{aligned} v &= \frac{1^2 \times 5 + 2^2 \times 7 + 3^2 \times 4 + 4^2 \times 3 + 5^2 \times 6}{5 + 7 + 4 + 3 + 6} - \left(\frac{43}{25}\right)^2 \\ &= \frac{5 + 28 + 36 + 48 + 150}{25} - \left(\frac{43}{25}\right)^2 \\ &= \frac{4826}{625} \\ &\approx 7.7216 \end{aligned}$$

and so $s^2 = \frac{25}{24}\frac{4826}{625} \approx 8.043$.

2.7 Other Point Measures

Beyond measures of the middle and of spread includes a way you can determine if data is heaped up to one side or the other of the mean. One such measure is the skewness...

Definition 2.7.1 (Skewness)**.** The Skewness of $x_1, x_2, ..., x_n$ is given by

$$\frac{1}{\sigma^3}\frac{\sum_{k=1}^n (x_k - \overline{x})^3}{n}.$$

A positive skewness indicates that the positive $(x_k - \overline{x})^3$ terms overwhelm the negative terms. Therefore, this indicates data which is strung out to the right. Likewise, a negative skewness indicates data which is strung out to the left.

In addition to skewness, data might tend to be clustered around the mean and often in a "bell-shaped" manner. The kurtosis can be used o measure how closely data resembles a bell-shaped collection.

Definition 2.7.2 (Kurtosis)**.** The Kurtosis of $x_1, x_2, ..., x_n$ is given by

$$\frac{1}{\sigma^4}\frac{\sum_{k=1}^n (x_k - \overline{x})^4}{n}.$$

A kurtosis of 3 indicates that the data is perfectly bell shaped (a "normal" distribution) whereas data further away from 3 indicates data that is less bell shaped.

Theorem 2.7.3 (Alternate Formulas for Skewness and Kurtosis)**.** *Skewness =*

$$\frac{1}{s^3}\left[\frac{\sum_{k=1}^n x_k^3}{n} - 3v\overline{x} - \overline{x}^3\right]$$

and Kurtosis =

$$\frac{1}{s^4}\left[\frac{\sum_{k=1}^n x_k^4}{n} - 4\overline{x}\frac{\sum_{k=1}^n x_k^3}{n} + 6\overline{x}^2 v - 3\overline{x}^4\right]$$

Proof. For skewness, expand the cubic and break up the sum. Factoring out constants (such as $\overline{x}$) gives

$$\begin{aligned}
&\frac{\sum_{k=1}^n (x_k - \overline{x})^3}{n} \\
&= \frac{\sum_{k=1}^n x_k^3}{n} - 3\overline{x}\frac{\sum_{k=1}^n x_k^2}{n} + 3\overline{x}^2\frac{\sum_{k=1}^n x_k}{n} - \frac{\sum_{k=1}^n \overline{x}^3}{n} \\
&= \frac{\sum_{k=1}^n x_k^3}{n} - 3\overline{x}(v + \overline{x}^2) + 3\overline{x}^3 - \overline{x}^3 \\
&= \frac{\sum_{k=1}^n x_k^3}{n} - 3\overline{x}v - \overline{x}^3
\end{aligned}$$

and divide by the cube of the standard deviation to finish. Note that the first expansion in the derivation above can be used quickly if the data is collected in a table and powers easily computed.

For kurtosis, similarly expand the quartic and break up the sum as before. Note that you can extract the value of the cubic term by solving for that term in the

k	x_k
1	8
2	12
3	6
4	3
5	1
6	2

skewness formula above. Then,

$$\begin{aligned}
&\frac{\sum_{k=1}^{n}(x_k-\overline{x})^4}{n} \\
&= \frac{\sum_{k=1}^{n} x_k^4}{n} - 4\overline{x}\frac{\sum_{k=1}^{n} x_k^3}{n} + 6\overline{x}^2\frac{\sum_{k=1}^{n} x_k^2}{n} - 4\overline{x}^3\frac{\sum_{k=1}^{n} x_k}{n} + \frac{\sum_{k=1}^{n}\overline{x}^4}{n} \\
&= \frac{\sum_{k=1}^{n} x_k^4}{n} - 4\overline{x}\frac{\sum_{k=1}^{n} x_k^3}{n} + 6\overline{x}^2(v+\overline{x}^2) - 4\overline{x}^4 + \overline{x}^4 \\
&= \frac{\sum_{k=1}^{n} x_k^4}{n} - 4\overline{x}\frac{\sum_{k=1}^{n} x_k^3}{n} + 6\overline{x}^2 v - 3\overline{x}^4
\end{aligned}$$

and then divide by the fourth power of the standard deviation. Note again that the first expansion in the derivation above might also be a useful shortcut. □

2.8 Graphical Representation of Data

Data sets can range from small to very large. Visual representations of these data sets often allow you to see trends and reveal a lot about the distribution of the data values.

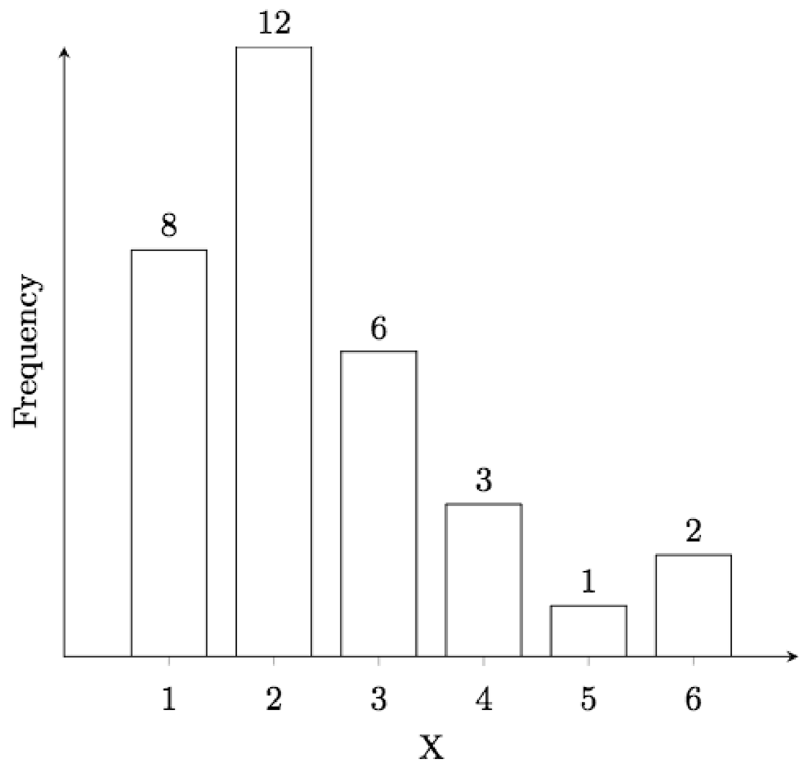
Also, probability mass functions for discrete variables can be graphed as a set of points but sometimes these points do not convey size very well. A visual representation of these functions needs to be addressed.

2.8.1 Histograms

Frequency Histograms - height matters

Consider the data set given by

A frequency histogram representing this data can be given by



Experiment with creating your own histogram by inputting your data into the interactive cell below.

```
#  This function is used to convert an input string into
    separate entries
def g(s): return str(s).replace(',',' ').replace('(','
    ').replace(')',' ').split()

@interact
def _(freq =
    input_box("1,1,1,1,2,2,2,3,3,3,3,1,5",label="Enter data
    separated by commas")):
    freq = g(freq)
    freq = [int(k) for k in freq]
    m = min(freq)
    M = max(freq)
    bn = M-m+1
    histogram( freq, range=[m-1/2,M+1/2], bins = bn,
        align="mid", linewidth=2, edgecolor="blue",
        color="yellow").show()
```

Relative Frequency Histograms - In this case, area describes your data. Notice in the interactive cell above that each bar is of width one. Therefore, frequency = area. In some instances where data may be grouped the total width of the interval may be different and so the height will need to be adjusted so that the total area of each bar corresponds to the relative frequency of that category.

Cummulative Histograms. In these a running total is presented using all values from the given point and below.

```
#  This function is used to convert an input string into
    separate entries
def g(s): return str(s).replace(',',' ').replace('(','
    ').replace(')',' ').split()

@interact
def _(freq =
    input_box("1,1,1,1,2,2,2,3,3,3,3,1,5",label="Enter data
    separated by commas")):
    freq = g(freq)
    freq = [int(k) for k in freq]
    top = len(freq)
    m = min(freq)
    M = max(freq)
    bn = M-m+1
    histogram( freq, range=[m-1/2,M+1/2], cumulative =
        "true", bins = bn, align="mid", linewidth=2,
        edgecolor="blue", color="yellow").show(ymax=top)
```

Stem-and-Leaf Plot - Histogram with data. Using the state population data above, consider organizing the data but using a "two-pass sort" where you first roughly break data up into groups based upon ranges which relate to their first digit(s). In this case, let's break up into groups according to populations corresponding to 0-4 million, 5-9 million, 10-14 million, 15-19, million, 20-24 million, 25-29 million, 30-35 million, and 35-39 million. We can represent these classes by using the stems 0L, 0H, 1L, 1H, 2L, 2H, 3L, and 3H where the L and H represent the one's digits L in 0, 1, 2, 3, 4 and H in 5, 6, 7, 8, 9. Once we group the data into these smaller groups then we can write the remaining portion of the number horizontally as leaves (in this case with one decimal place for all values.) This gives a step-and-leaf plot. If we additionally sort the data in the leaves then this gives you an ordered stem-and-leaf plot. For the state population data, the ordered stem-and-leaf plot is given by

Table 1: Stem Plot for State Populations

Stem	Leaf
0H	06 06 07 07 08 09 10 11 13 13 14 16 19 19 21 28 29 29 30 30 31 36 39 39 44 46 48 48
1L	53 54 57 59 60 65 66 66 67 70 83 89 98 99
1H	10 16 28 29
2L	96 97
2H	
3L	64
3H	83

Notice how it is easy to now see that most state populations are relatively small and that there are relatively few states with larger population. Also, notice that you can use this plot to relatively easily identify minimum, maximum, and other order statistics.

Box and Whisker Diagram - visual order statistics. This graphical display identifies the "5-number-summary" associated with the minimum, quartiles, and the maximum. That is, y_1, Q_1, Q_2, Q_3, y_n. These values separate the data roughly into quarters. To distinguish these quarters connect y_1 and Q_1 with a straight line (a whisker) and do the same with Q_3 and y_n. Use a box to connect Q_1 with Q_2 and the same to connect Q_2 with Q_3. Then the boxed areas also identify the IQR.

```
from pylab import boxplot,savefig,close
@interact
```

```
def _(data =
    input_box([1,2,3,4,6,7,8,9,11,15,21],label="Enter␣Your␣
    Data:")):
     B = boxplot(data, notch=True, sym='x', vert=False)
     savefig("boxplot.png")
     close()
```

2.9 Exercises

Complete the online homework "Computational Measures".

Create a data set with about 10 elements. For your data set, compute each of the measures from this chapter and present your data using a frequency histogram.

Find a "real-world" data set (similar perhaps to the Census data presented above.) Compute each of the measures from this chapter. Interpret and present your conclusions in an electronic report which can include an excel spreadsheet.

Chapter 3

Counting and Combinatorics

3.1 Introduction

Discussion on the usefulness of having ways to count the number of elements in a set without having to explicitly listing all elements.

Consider counting the number of ways one can arrange Peter, Paul, and Mary with the order important. Listing the possibilities:

- Peter, Paul, Mary
- Peter, Mary, Paul
- Paul, Peter, Mary
- Paul, Mary, Peter
- Mary, Peter, Paul
- Mary, Paul, Peter

So, it is easy to see that these are all of the possible outcomes and that the total number of such outcomes is 6. What happens however if we add Simone to the list?

- Simone, Peter, Paul, Mary
- Simone, Peter, Mary, Paul
- Simone, Paul, Peter, Mary
- Simone, Paul, Mary, Peter
- Simone, Mary, Peter, Paul
- Simone, Mary, Paul, Peter
- Peter, Simone, Paul, Mary
- Peter, Simone, Mary, Paul
- Paul, Simone, Peter, Mary
- Paul, Simone, Mary, Peter
- Mary, Simone, Peter, Paul
- Mary, Simone, Paul, Peter

- Peter, Paul, Simone, Mary
- Peter, Mary, Simone, Paul
- Paul, Peter, Simone, Mary
- Paul, Mary, Simone, Peter
- Mary, Peter, Simone, Paul
- Mary, Paul, Simone, Peter
- Peter, Paul, Mary, Simone
- Peter, Mary, Paul, Simone
- Paul, Peter, Mary, Simone
- Paul, Mary, Peter, Simone
- Mary, Peter, Paul, Simone
- Mary, Paul, Peter, Simone

Notice how the list quickly grows when just adding one more choice. This illustrates how keeping track of the number of items in a set can quickly get impossible to keep up with and to count unless we can approach this problem using a more mathematical approach.

Definition 3.1.1 (Cardinality)**.** Given a set of elements A, the number of elements in the set is known as the sets cardinality and is denoted |A|. If the set has an infinite number of elements then we set $|A| = \infty$.

In order to "count without counting" we establish the following foundational principle.

Theorem 3.1.2 (Multiplication Principle)**.** *Given two successive events A and B, the number of ways to perform A and then B is |A||B|.*

Proof. If either of the events has infinite cardinality, then it is clear that the number of ways to perform A and then B will also be infinite. So, assume that both |A| and |B| are finite. In order to count the successive events, enumerate the elements in each set

$$A = \left\{a_1, a_2, a_3, ..., a_{|A|}\right\}$$
$$B = \left\{b_1, b_2, b_3, ..., b_{|B|}\right\}$$

and consider the function f(k,j) = (k-1)|B| + j. This function is one-to-one and onto from the set

$$\{(k, j) : 1 \leq k \leq |A|, 1 \leq j \leq |B|\}$$

onto

$$\{s : 1 \leq s \leq |A||B|\}.$$

Since this second set has |A| |B| elements then the conclusion follows. coordinates. □

Definition 3.1.3 (Factorial)**.** For any natural number n,

$$n! = n(n-1)(n-2)...3 \cdot 2 \cdot 1$$

Example 3.1.4 (iPad security code)**.** Consider your ipad's security. To unlock the screen you need to enter your four digit pass code. How easy is it to guess this pass code?

Using the standard 10 digit keypad, we first have two questions to consider?

1. Does the order in which the digits are entered matter?

2. Can you reuse a digit more than once?

For the ipad, the order does matter and you cannot reuse digits. In this case, the number of possible codes can be determined by considering each digit as a separate event with four such events in succession providing the right code. By successively applying the multiplication principle, you find that the number of possible codes is the number of remaining available digits at each step. Namely, $10 \times 9 \times 8 \times 7 = 5040$.

Note that if you were allowed to reuse the digits then the number of possible outcomes would be more since all 10 digits would be available for each event. Namely, $10 \times 10 \times 10 \times 10 = 10000$.

Example 3.1.5 (iPad security code with greasy fingers)**.** Reconsider your ipad's security. In this case, you like to eat chocolate bars and have greasy fingers. When you type in your passcode your fingers leave a residue over the four numbers pressed. If someone now tries to guess your passcode, how many possible attempts are necessary?

Since there are only four numbers to pick from with order important, the number of possible passcodes remaining is $4 \times 3 \times 2 \times 1 = 24$

Example 3.1.6 (National Treasure)**.** In the 2004 movie "National Treasure" Ben and Riley are attempting to guess Abagail's password to enter the room with the Declaration. They are able to determine the passphrase to get into the vault room by doing a scan that detects the buttons pushed (not due to chocolate but just due to the natural oils on fingers). They notice that the buttons pushed include the characters AEFGLORVY.

Assuming these characters are used only once each, how many possible passphrases are possible?

In this case, the order of the characters matters but all of the characters are distinct. Since we have 9 characters provided, the we can consider each character as an event with the first event as a choice from the 9, the second event as a choice from the remaining 8, etc. This gives $9 \times 8\ times ... \times 1 = 362880$ possible passphrases.

Assuming that some of the characters could be used more than once, how many passphrases need to be considered if the total length of passphrase can be at most 12 characters?

Notice, in this case you don't know which characters might be reused and so the number of possible outcomes will be much larger. What is the answer?

You can break this problem down into distinct cases:

- Using 9 characters This is the answer computed above.

- Using 10 characters In this case, 1 character can be used twice. To determine the number of possibilities, let's first pick which character can be doubled. There are 9 options for picking that character. Next, if we consider the two instances of that letter as distinct values then we can just count the number of ways to arrange unique 10 characters which is 10! However, swapping the two characters (which are actually identical) would not give a new passphrase. Since these are counted twice, let's divide these out to give $10!/2$.

- Using 11 characters In this situation we have two unique options:

  - One character is used three times and the others just once. Continuing as in the previous case, $11!/3!$.Two characters are used twice and the others just once.

- Using 12 characters

  1. One letter from the nine is used four times and all the others are used once.
  2. One letter is used three times, another letter is used two times, and the others are used once.
  3. Three letters are used twice and the others are used once.

With this large collection of possible outcomes, how are the movie characters able to determine the correct "VALLEYFORGE" passphrase?

3.2 Permutations

When counting various outcomes the order of things sometimes matters. When the order of a set of elements changes we call the second a permutation (or an arrangement) of the first.

Theorem 3.2.1 (Permutations of n objects)**.** *The number of ways to arrange n distinct items is n!*

Proof. Notice that if n=1, then there is only 1 item to arrange and that there is only one possible arrangment.

By induction, assume that any set with n elements has n! arrangments and assume that

$$|A| = \{a_1, a_2, ..., a_n, a_{n+1}\}.$$

Notice that there are n+1 ways to choose 1 element from A and that in doing so leaves a set with n elements. Combining the induction hypothesis with the multiplication principle this gives (n+1)n! = (n+1)! possible outcomes. □

Theorem 3.2.2 (Permutations of n objects selecting r)**.** *The number of ways to arrange r items from a set of n distinct items is* $P_r^n = \frac{n!}{(n-r)!}$

Proof. Apply the multiplication principle r times noting that there are n choices for the first selection, n-1 choices for the second selection, and with n-r+1 choices for the rth selection. This gives

$$\begin{aligned} P_r^n &= n(n-1)...(n-r+1) \\ &= n(n-1)...(n-r+1)\frac{(n-r)!}{(n-r)!} \\ &= \frac{n(n-1)...(n-r+1)(n-r)!}{(n-r)!} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

□

Theorem 3.2.3 (Permutations when Not all items are distinguishable and without replacement: (Multinomial Coefficients))**.** *If n items belong to s categories, n1 in first, n2 in second, ... , ns in the last, the number of ways to pick all is !*

3.3 Combinations

When counting various outcomes sometimes the order of things does not matter. In this case we count each different set of outcomes a combination.

Theorem 3.3.1 (Combinations of n distinct objects selecting r without replacement)**.** *The number of ways to arrange r items from a set of n distinct items is* $C_r^n = \frac{n!}{r!(n-r)!}$

Proof. Consider creating a permutation of r objects from a set of size n by first picking an unordered subset of size r and then counting the number of ways to order that subset. Using our notation and the multiplication principle,

$$P_r^n = C_r^n \cdot r!$$

Solving give the result. □

Theorem 3.3.2 (Combinations of n distinct objects selecting r with replacement)**.** *The number of ways to arrange r items from a set of n distinct items is* $C_r^{n+r-1} = \frac{n+r-1!}{r!(n-1)!}$

Proof. blah □

Example 3.3.3. Revisiting your ipad's security, what happens if the order in which the digits are entered does not matter? If so, then you would be picking a combination of 4 digits without replacement from a group of 10 digits. Namely,

$$\begin{aligned}\frac{10!}{4!6!} &= \frac{10\times 9\times 8\times 7\times 6!}{4\times 3\times 2\times 1\times 6!}\\ &= \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1}\\ &= \frac{5040}{24}\\ &= 210.\end{aligned}$$

Notice that the total number of options is much smaller when order does not matter.

Note that if you were allowed to reuse the digits then the number of possible outcomes would be

$$\begin{aligned}\frac{13!}{3!10!} &= \frac{13\times 12\times 11}{3\times 2\times 1}\\ &= 286\end{aligned}$$

which once again is more since numbers are allowed to repeat.

Definition 3.3.4 (Binomial Coefficients)**.** The value C_r^n is known as the binomial coefficient. It is denoted by $\binom{n}{r}$ and is read "n choose k".

Theorem 3.3.5 (Combinations when distinguishable and with replacement)**.** = *Number of ways to get unordered samples of size r from n objects.*

Lots of interesting facts about the binomial coefficients.

3.4 Exercises

Complete the online homework "Counting".

Completely determine the number of possible passphrases for the National Treasure example started above. Present your answer in a report form.

Chapter 4

Probability and Probability Functions

This chapter is a definitions of probability, consequences, and probability functions.

4.1 Relative Frequency

Mathematics generally focuses on providing precise answers with absolute certainty. For example, solving an equation generates specific (and non-varying) solutions. Statistics on the other hand deals with providing precise answers to questions when there is uncertainty. It might seem impossible to provide such precise answers but the focus of this text is to show how that can be done so long as the questions are properly posed and the answers properly interpreted.

People often make claims about being the biggest, best, most often recommended, etc. One sometimes even believes these claims. In this class, we will attempt to determine if such claims are reasonable by first introducing probability from a semi rigorous mathematical viewpoint using concepts developed in Calculus. We will use this framework to carefully discuss making such statistical inferences as above and in general to obtain accurate knowledge even when the known data is not complete.

When attempting to precisely measure this uncertainty a few experiments are in order. When doing statistical experiments, a few terms and corresponding notation might be useful:

- S = Universal Set or Sample Space Experiment or Outcome Space. This is the collection of all possible outcomes.
- Random Experiment. A random experiment is a repeatable activity which has more than one possible outcome all of which can be specified in advance but can not be known in advance with certainty.
- Trial. Performing a Random Experiment one time and measuring the result.
- A = Event. A collection of outcomes. Generally denoted by an upper case letter such as A, B, C, etc.
- Success/Failure. When recording the result of a trial, a success for event A occurs when the outcome lies in A. If not, then the trial was a failure. There is no qualitative meaning to this term.

- Mutually Exclusive Events. Two events which share no common outcomes. Also known as disjoint events.
- |A| = Frequency. In a sequence of n events, the frequency is the number of trials which resulted in a success for event A.
- |A| / n = Relative Frequency. A proportion of successes to total number of trials.
- Histogram. A bar chart representation of data where area corresponds to the value being described.

To investigate these terms and to motivate our discussion of probability, consider flipping coins using the interactive cell below. Notice in this case, the sample space S = Heads, Tails and the random experiment consists of flipping a fair coin one time. Each trial results in either a Head or a Tail. Since we are measuring both Heads and Tails then we will not worry about which is a success or failure. Further, on each flip the outcomes of Heads or Tails are mutually exclusive events. We count the frequencies and compute the relative frequencies for a varying number of trials selected by you as you move the slider bar. Results are displayed using a histogram.

Question 1: What do you notice as the number of flips increases?

Question 2: Why do you rarely (if even) get exactly the same number of Heads and Tails? Would you not "expect" that to happen?

```
coin = ["Heads", "Tails"]
@interact
def _(num_rolls = slider([5..5000],label="Number␣of␣Flips")):
        rolls = [choice(coin) for roll in range(num_rolls)]
        show(rolls)
        freq = [0,0]
        for outcome in rolls:
                if (outcome==’Tails’):
                        freq[0] = freq[0]+1
                else:
                        freq[1] = freq[1]+1
        print("\nThe␣frequency␣of␣tails␣=␣"+ str(freq[0]))+"␣
            and␣heads␣=␣"+ str(freq[1])+"."
        rel = [freq[0]/num_rolls,freq[1]/num_rolls]
        print("\nThe␣relative␣frequencies␣for␣Tails␣and␣
            Heads:"+str(rel))
        show(bar_chart(freq,axes=False,ymin=0))      #  A
            histogram of the results
```

Notice that as the number of flips increases, the relative frequency of Heads (and Tails) stabilized around 0.5. This makes sense intuitively since there are two options for each individual flip and 1/2 of those options are Heads while the other 1/2 is Tails.

Let's try again by doing a random experiment consisting of rolling a single die one time. Note that the sample space in this case will be the outcomes S = 1, 2, 3, 4, 5, 6.

Question 1: What do you notice as the number of rolls increases?

Question 2: What do you expect for the relative frequencies and why are they not all exactly the same?

```
@interact
def _(num_rolls = slider([20..5000],label=’Number␣of␣
    rolls’),Number_of_Sides = [4,6,8,12,20]):
        die = list((1..Number_of_Sides))
        rolls = [choice(die) for roll in range(num_rolls)]
```

```
        show(rolls)

        freq = [rolls.count(outcome) for outcome in
            set(die)]  # count the numbers for each outcome
        print 'The␣frequencies␣of␣each␣outcome␣is␣'+str(freq)

        print 'The␣relative␣frequencies␣of␣each␣outcome:'
        rel_freq = [freq[outcome-1]/num_rolls for outcome in
            set(die)]  # make frequencies relative
        print rel_freq
        fs = []
        for f in rel_freq:
                fs.append(f.n(digits=4))
        print fs
        show(bar_chart(freq,axes=False,ymin=0))
```

Notice in this instance that there are a larger number of options (for example 6 on a regular die) but once again the relative frequencies of each outcome was close to 1/n (i.e. 1/6 for the regular die) as the number of rolls increased.

In general, this suggests a rule: if there are n outcomes and each one has the same chance of occurring on a given trial then on average on a large number of trials the relative frequency of that outcome is 1/n. In general, if a number of outcomes are "equally likely" then this is a good model for measuring the proportion of outcomes that would be expected to have any given outcome. However, it is not always true that outcomes are equally likely. Consider rolling two die and measuring their sum:

```
@interact
def _(num_rolls = slider([20..5000],label='Number␣of␣
    rolls'),num_sides = slider(4,20,1,6,label='Number␣of␣
    sides')):
    die = list((1..num_sides))
    dice = list((2..num_sides*2))
    rolls = [(choice(die),choice(die)) for roll in
        range(num_rolls)]
    sums = [sum(rolls[roll]) for roll in range(num_rolls)]
    show(rolls)

    freq = [sums.count(outcome) for outcome in set(dice)]  #
        count the numbers for each outcome
    print 'The␣frequencies␣of␣each␣outcome␣is␣'+str(freq)

    print 'The␣relative␣frequencies␣of␣each␣outcome:'
    rel_freq = [freq[outcome-2]/num_rolls for outcome in
        set(dice)]  # make frequencies relative
    print rel_freq
    show(bar_chart(freq,axes=False,ymin=0))     #  A
        histogram of the results
    print "Relative␣Frequence␣of␣",dice[0],"␣is␣about␣
        ",rel_freq[0].n(digits=4)
    print "Relative␣Frequence␣of␣",dice[num_sides-1],"␣is␣
        about␣",rel_freq[num_sides-1].n(digits=4)
```

Notice, not only are the answers not the same but they are not even close. To understand why this is different from the examples before, consider the possible outcomes from each pair of die. Since we are measuring the sum of the dice then (for a pair of standard 6-sided dice) the possible sums are from 2 to 12. However, there is only one way to get a 2–namely from a (1,1) pair–while there are 6 ways to get a 7–namely from the pairs (1,6), (2,5), (3,4), (4,3), (5,2), and (6,1). So it might make some sense that the likelihood of getting a 7 is 6 times larger than that of getting a 2. Check to see if that is the case with your experiment above.

4.2 Definition of Probability

4.2.1 Motivating the Definition

Using the ideas from our examples above, let's consider how we might formally define a way to measure the expectation from similar experiments. Before doing so, we need a little notation:

Definition 4.2.1. The Cardinality of the set A is the number of elements in A. This will be denoted |A| (similar to the idea of frequency of an outcome noted earlier.) If a set has a infinite number of elements, then we will say it's cardinality is also infinite and write $|A| = \infty$

Definition 4.2.2 (Pairwise Disjoint Sets)**.** $\{A_1, A_2, ..., A_n\}$ are pairwise disjoint provided $A_k \cap A_j = \emptyset$ so long as $k \neq j$.

To model the behavior above, consider how we might create a definition for our expectation of a given outcome by following the ideas uncovered above. To do so, first consider a desired collection of outcomes A. If each outcome in A is equally likely then we might follow the concept behind relative frequency and consider a measure of expectation be |A|/|S|. Indeed, on a standard 6-sided die, the expectation of the outcome A=2 from the collection S = 1,2,3,4,5,6 should be |A|/|S| = 1/6.

From the example where we take the sum of two die, the outcome A=4,5 from the collection S = 2,3,4,...,12 would be

$$|A| = |(1,3),(2,2),(3,1),(1,4),(2,3),(3,2),(4,1)| = 7$$
$$|S| = |(1,1),...,(1,6),(2,1),...,(2,6),...,(6,1),...,(6,6)| = 36$$

and so the expected relative frequency would be |A|/|S| = 7/36. Compare this theoretical value with the sum of the two outcomes from your experiment above.

We are ready to now formally give a name to the theoretical measure of expectation for outcomes from an experiment. Taking our cue from the ideas related to equally likely outcomes, we make our definition have the following basic properties:

1. Relative frequency cannot be negative, since cardinality cannot be negative
2. Relative frequencies for disjoint events should sum to one
3. Relative frequencies for collections of disjoint outcomes should equal the sum of the individual relative frequencies

4.2.2 Probability

Based upon these we give the following:

Definition 4.2.3. The probability P(A) of a given outcome A is a set function which satisfies:

1. (Nonnegativity) $P(A) \geq 0$
2. (Totality) $P(S) = 1$
3. (Subadditivity) If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$. In general, if A_k are pairwise disjoint then $P(\cup_k A_k) = \sum_k P(A_k)$.

4.2.3 Basic Probability Theorems

Based upon this definition we can immediately establish a number of results.

Theorem 4.2.4 (Probability of Complements)**.** *For any event A, $P(A)+P(A^c) = 1$*

Proof. Let A be any event and note that $A \cap A^c = \emptyset$. But $A \cup A^c = S$. So, by subadditivity $1 = P(S) = P(A \cup A^c) = P(A) + P(A^c)$ as desired. □

Theorem 4.2.5. $P(\emptyset) = 0$

Proof. Note that $\emptyset^c = S$. So, by the theorem above, $1 = P(S) + P(\emptyset) \Rightarrow 1 = 1 + P(\emptyset)$. Cancelling the 1 on both sides gives $P(\emptyset) = 0$. □

Theorem 4.2.6. *For events A and B with $A \subset B, P(A) \leq P(B)$.*

Proof. Assume sets A and B satisfy $A \subset B$. Then, notice that $A \cap (B - A) = \emptyset$ and $B = A \cup (B - A)$. Therefore, by subadditivity and nonnegativity

$$\begin{aligned} 0 &\leq P(B - A) \\ P(A) &\leq P(A) + P(B - A) \\ P(A) &\leq P(B) \end{aligned}$$

□

Theorem 4.2.7. *For any event A, $P(A) \leq 1$*

Proof. Notice $A \subset S$. By the theorem above $P(A) \leq P(S) = 1$ □

Theorem 4.2.8. *For any sets A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$*

Proof. Notice that we can write $A \cup B$ as the disjoint union

$$A \cup B = (A - B) \cup (A \cap B) \cup (B - A).$$

We can also write disjointly

$$\begin{aligned} A &= (A - B) \cup (A \cap B) \\ B &= (A \cap B) \cup (B - A) \end{aligned}$$

Hence,

$$\begin{aligned} &P(A) + P(B) - P(A \cap B) \\ &\quad = [P(A - B) + P(A \cap B)] + [P(A \cap B) + P(B - A)] - P(A \cap B) \\ &\quad = P(A - B) + P(A \cap B) + P(B - A) \\ &\quad = P(A \cup B) \end{aligned}$$

□

This result can be extended to more that two sets using a property known as inclusion-exclusion. The following two theorems illustrate this property and are presented without proof.

Corollary 4.2.9. *For any sets A, B and C,*

$$\begin{aligned} P(A \cup B \cup C) = &P(A) + P(B) + P(C) \\ &- P(A \cap B) - P(A \cap C) - P(B \cap C) \\ &+ P(A \cap B \cap C) \end{aligned}$$

Corollary 4.2.10. *For any sets A, B, C and D,*

$$\begin{aligned} P(A \cup B \cup C \cup D) = &P(A) + P(B) + P(C) + P(D) \\ &- P(A \cap B) - P(A \cap C) - P(A \cap D) - P(B \cap C) - P(B \cap D) - P(C \cap D) \\ &+ P(A \cap B \cap C) + P(A \cap B \cap D) + P(A \cap C \cap D) + P(B \cap C \cap D) \\ &- P(A \cap B \cap C \cap D) \end{aligned}$$

4.3 Conditional Probability

Example 4.3.1 (Changing Sample Space - Balls)**.** Consider a box with three balls: one Red, one White, and one Blue. Using an equally likely assumption, the probability of randomly pulling out a Red ball should be 1/3. That is P(Red) = 1/3. However, suppose that for a first trial you pull out the White ball and set it aside. Attempting to pull out another ball leaves you with only two options and so the probability of randomly pulling out a Red ball is 1/2. Notice that the probability changed for the second trial dependent on the outcome of the first trial.

Example 4.3.2 (Changing Sample Space - Cards)**.** Consider a deck of 52 standard playing cards and a success occurs when a Heart is selected from the deck. When extracting one card randomly, the probability of that card being a Heart is then P(Heart) = 13/52. Now, assume that one card has already been extracted and setaside. Now, prepare to extract another. If the first card drawn was a Heart, then there are only 12 Hearts left for the second draw. However, if the first card drawn was not a Heart, then there are 13 Hearts available for the second draw. To compute this probability correctly, one need to formulate the question so that subadditivity can be utilized.

To do this, consider P(Heart on 2nd draw) = P([Heart on 1st draw ∩ Heart on 2nd draw] ∪ [Not Heart on 1st draw ∩ Heart on 2nd draw]) = P(Heart on 1st draw ∩ Heart on 2nd draw) + P(Not Heart on 1st draw ∩ Heart on 2nd draw) = | Heart on 1st draw ∩ Heart on 2nd draw | / | Number of ways to get two cards | + | Not Heart on 1st draw ∩ Heart on 2nd draw / | Number of ways to get two cards | = (13 12) / (52 51) + (39 13) / (52 51) = 12 / (4 51) + (3 13) / (4 51) =

Definition 4.3.3 (Conditional Probability)**.** P(B | A) = P(A ∩ B) / P(A), provided P(A)> 0.

Theorem 4.3.4. *Conditional Probability satisfies all of the requirements of regular probability.*

Proof. By definition, for any event probability must be nonnegative. Therefore $P(A \cap B) \geq 0$. Therefore, P(B | A) ≥ 0.

Further, P (S | A) = P(A ∩ S)/P(A) = P(A)/P(A) = 1. □

Theorem 4.3.5 (Multiplication Rule)**.**

$$P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$$

Proof. Unravel the definition of conditional probably by taking the denominator to the other side. Also note that you can write $A \cap B = B \cap A$. □

4.4 Bayes Theorem

Theorem 4.4.1 (Bayes Theorem)**.** *Let $S = \{S_1, S_2, ..., S_m\}$ where the S_k are pairwise disjoint and $S_1 \cup S_2 \cup ... \cup S_m = S$ (i.e. a partition of the space S). Then for any $A \subset S$*

$$P(S_j|A) = \frac{P(S_j)P(A|S_j)}{\sum_{k=1}^{m} P(S_k)P(A|S_k)}.$$

The conditional probability $P(S_j|A)$ is called the posterior probability of S_k.

Proof. Notice, by the definition of conditional probability and the multiplication rule

$$P(S_j|A) = \frac{P(S_j \cap A)}{P(A)} = \frac{P(S_j)P(A|S_j)}{P(A)}.$$

But using the disjointness of the partition

$$\begin{aligned}
P(A) &= P((A \cap S_1) \cup (A \cup S_2) \cup ... \cup (A \cup S_m)) \\
&= P(A \cap S_1) + P(A \cup S_2) + ... + P(A \cup S_m) \\
&= P(S_1 \cap A) + P(S_2 \cup A) + ... + P(S_m \cup A) \\
&= P(S_1)P(A|S_1) + P(S_2)P(A|S_2) + ... + P(S_m)P(A|S_m) \\
&= \sum_{k=1}^{m} P(S_k)P(A|S_k)
\end{aligned}$$

Put these two expansions together to obtain the desired result. $\square$

```
#  This function is used to convert an input string into
    separate entries
def g(s): return str(s).replace(',','␣').replace('(','␣
    ').replace(')','␣').split()

@interact
def
    _(Partition_Probabilities=input_box('0.35,0.25,0.40',label="$P(B_1),P(B_2),...$"),
         Conditional_Probabilities=input_box('0.02,0.01,0.03',label='$P(A|B_1),P(A|B_2),.
         print_numbers=checkbox(True,label='Numerical␣Results␣
             on␣Graphs?'),
         auto_update=False):

    Partition_Probabilities = g(Partition_Probabilities)
    Conditional_Probabilities = g(Conditional_Probabilities)
    n = len(Partition_Probabilities)
    n0 = len(Conditional_Probabilities)

    # below needs to be n not equal to n0 but mathbook xml
        will not let me get the other
    if (n > n0):
         pretty_print("You␣must␣have␣the␣same␣number␣of␣
             partition␣probabilities␣and␣conditional␣
             probabilities.")

    else:                               # input data streams
        now are the same size!
         colors = rainbow(n)
         accum = float(0)               # to test whether
             partition probs sum to one
         ends = [0]                     # where the graphed
             partition sectors change in pie chart
         mid = []                       # middle of each pie
             chart sector used for placement of text
         p_Bk_given_A = []              # P( B_k | A )
         pA = 0                         # P(A)
         PP=[]                          # array to hold the
             numerical Partition Probabilities
         CP=[]                          # array to hold the
             numerical Conditional Probabilities
         for k in range(n):
             PP.append(float(Partition_Probabilities[k]))
             CP.append(float(Conditional_Probabilities[k]))
             p_Bk_given_A.append(PP[k]*CP[k] )
             pA += p_Bk_given_A[k]
             accum = accum + PP[k]
```

```
            ends.append(accum)
            mid.append((ends[k]+accum)/2)
#
#  Marching along from 0 to 1, saving angles for each
   partition sector boundary.
#  Later, we will multiple these by 2*pi to get actual
   sector boundary angles.
#
        if abs(accum-float(1))>0.0000001:      #  Due to
            roundoff issues, this should be close enough.
            pretty_print("Sum␣of␣probabilities␣should␣equal␣
                1.")

        else:                               # probability data
            is sensible

#
#  Draw the Venn diagram by drawing sectors from the angles
   determined above
#  First, create a circle of radius 1 to illustrate the the
   sample space S
#  Then draw each sector with varying colors and print out
   their names on the edge
#
            G = circle((0,0), 1,
                rgbcolor='black',fill=False,
                alpha=0.4,aspect_ratio=True,axes=False,thickness=5)
            for k in range(n):
                G += disk((0,0), 1, (ends[k]*2*pi,
                    ends[k+1]*2*pi),
                    color=colors[mod(k,10)],alpha = 0.2)
                G +=
                    text('$B_'+str(k+1)+'$',(1.1*cos(mid[k]*2*pi),
                    1.1*sin(mid[k]*2*pi)), rgbcolor='black')

            G += circle((0,0), 0.6, facecolor='yellow', fill
                = True, alpha = 0.1,
                thickness=5,edgecolor='black')

#  Print the probabilities corresponding to each particular
   region as a list and on the graphs
            if print_numbers:

                html("$P(A)␣=␣%s$"%(str(pA),))
                for k in range(n):
                    html("$P(B_{%s}␣|␣A)$"%(str(k+1))+"$␣=␣
                        %s$"%str(p_Bk_given_A[k]/pA))

                    G +=
                        text(str(p_Bk_given_A[k]),(0.4*cos(mid[k]*2*pi),
                        0.4*sin(mid[k]*2*pi)),
                        rgbcolor='black')
                    G += text(str(PP[k] -
                        p_Bk_given_A[k]),(0.8*cos(mid[k]*2*pi),
                        0.8*sin(mid[k]*2*pi)),
                        rgbcolor='black')

#  This is essentially a repeat of some of the above code
   but focused only on creating the smaller inner circle
```

```
    dealing
#   with the set A so that the sectors now correspond in area
    to the Bayes Theorem probabilities


            accum = float(0)
            ends = [0]                        # where the
                graphed partition sectors change in pie chart
            mid = []                          # middle of each
                pie chart sector used for placement of text
            for k in range(n):
                accum += float(p_Bk_given_A[k]/pA)
                ends.append(accum)
                mid.append((ends[k]+accum)/2)
            H = circle((0,0), 1,
                rgbcolor='black',fill=False,
                alpha=0,aspect_ratio=True,axes=False,thickness=0)
            H += circle((0,0), 0.6,
                facecolor='yellow',fill=True,
                alpha=0.1,aspect_ratio=True,axes=False,thickness=5,edgecolor='black')

            for k in range(n):
                H += disk((0,0), 0.6, (ends[k]*2*pi,
                    ends[k+1]*2*pi),
                    color=colors[mod(k,10)],alpha = 0.2)
                H +=
                    text('$B_'+str(k+1)+'|A$',(0.7*cos(mid[k]*2*pi),
                    0.7*sin(mid[k]*2*pi)), rgbcolor='black')

        #   Now, print out the bayesian probabilities using
            the smaller set A only

            if print_numbers:
                for k in range(n):
                    H += text(str(
                        N(p_Bk_given_A[k]/pA,digits=4)
                        ),(0.4*cos(mid[k]*2*pi),
                        0.4*sin(mid[k]*2*pi)),
                        rgbcolor='black')

            G.show(title='Venn diagram of partition with A
                in middle')
            print
            H.show(title='Venn diagram presuming A has
                occured')
```

4.5 Independence

Definition 4.5.1 (Independent Events)**.** Events A and B are independent provided $P(A \cap B) = P(A)P(B)$

4.6 Random Variables

For a given set of events, we might have difficulty doing mathematics since the outcomes are not numerical. In order to accomodate our desire to convert to numerical measures we want to assign numerical values to all outcomes. The process of doing this creates what is known as a random variable.

Definition 4.6.1 (Random Variable)**.** Given a random experiment with sample space S, a function X mapping each element of S to a unique real number is called a random variable. For each element s from the sample space S, denote this function by X(s) = x and call the range of X the space of X: R= x : X(s)=x, for some s in S

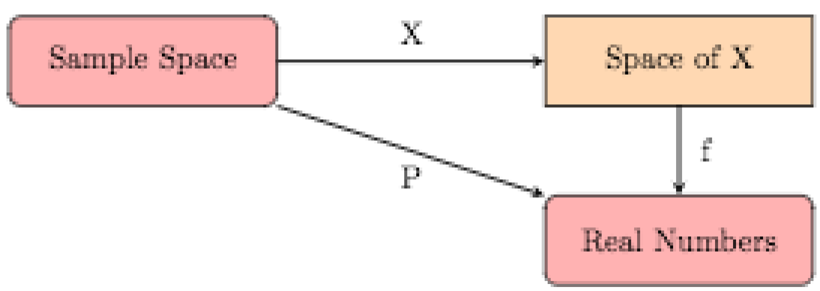
We will make various restrictions on the range of the random variable to fit different generalized problems. Then, we will be able to work on a problem (which may be inherently non-numerical) by using the random variable in subsequent calculations.

Example 4.6.2 (Success vs Failure)**.** When dealing with only two outcomes, one might use S = success, failure. Choose X(success)=1, X(failure)=0. Then, R=0,1.

Example 4.6.3 (Standard Dice Pairs)**.** When gambling with a pair of dice, one might use S=ordered pairs of all possible rolls = (a,b): a=die 1 outcome, b=die 2 outcome. Choose X((a,b)) = a+b. Then, R=2, 3, 4, 5, ..., 12.

Example 4.6.4 (Other Dice Options)**.** When rolling dice in a board game (like RISK), one might use S=(a,b): a=die 1 outcome, b=die 2 outcome Choose X((a,b)) = maxa,b. Then, R=1, 2, 3, 4, 5, 6

Definition 4.6.5. R contains a countable number of points if either R is finite or there is a one to one correspondence between R and the positive integers. Such a set will be called discrete. We will see that often the set R is not countable. If R consists of an interval of points (or a union of intervals), then we call X a continuous random variable.



4.7 Probability Functions

In the formulas below, we will presume that we have a random variable X which maps the sample space S onto some range of real numbers R. From this set, we then can define a probability function f(x) which acts on the numerical values in R and returns another real number. We attempt to do so to obtain (for discrete values) P(sample space value s)= $f(X(s))$. That is, the probability of a given outcome s is equal to the composition which takes s to a numerical value x which is then plugged into f to get the same final values.

Definition 4.7.1 (Probability Mass Function)**.** Given a discrete random variable X on a space R, a probability mass function on X is given by a function $f : R \to \mathbb{R}$ such that:

$$\forall x \in R, f(x) > 0$$
$$\sum_{x \in R} f(x) = 1$$
$$A \subset R \Rightarrow P(X \in A) = \sum_{x \in A} f(x)$$

Definition 4.7.2 (Probability Density Function)**.** Given a continuous random variable X on a space R, a probability density function on X is given by a function $f : R \to \mathbb{R}$ such that:

$$\forall x \in R, f(x) > 0$$
$$\int_R f(x) = 1$$
$$A \subset R \Rightarrow P(X \in A) = \int_A f(x)dx$$

Example 4.7.3 (Discrete Probability Function)**.** Consider $f(x) = x/10$ over R $=$ 1,2,3,4. Then, f(x) is obviously positive for each of the values in R and certainly $\sum_{x \in R} f(x) = f(1) + f(2) + f(3) + f(4) = 1/10 + 2/10 + 3/10 + 4/10 = 1$. Therefore, f(x) is a probability mass function over the space R.

```
# Combining all of the above into one interactive cell
@interact
def _(D = input_box([1,2,3,5,6,8,9,11,12,14],label="Enter␣
    domain␣R␣(in␣brackets):"),
        Probs =
            input_box([1/20,1/20,1/20,3/20,1/20,4/20,4/20,1/20,1/20,3/20],label="Enter␣
            corresponding␣f(x)␣(in␣brackets):"),
        n_samples=slider(100,10000,100,100,label="Number␣of␣
            times␣to␣sample␣from␣this␣distribution:")):
    n = len(D)
    R = range(n)
    one_huh = sum(Probs)
    pretty_print('\n\nJust␣to␣be␣certain,␣we␣should␣check␣to␣
        make␣certain␣the␣probabilities␣sum␣to␣1\n')
    pretty_print(html('$\sum_{x\epsilon␣R}␣f(x)␣=␣
        %s$'%str(one_huh)))

    G = Graphics()
    if len(D)==len(Probs):
        f = zip(D,Probs)
        meanf = 0
        variancef = 0
        for k in R:
            meanf += D[k]*Probs[k]
            variancef += D[k]^2*Probs[k]
            G +=
                line([(D[k],0),(D[k],Probs[k])],color='green')
        variancef = variancef - meanf^2
        sd = sqrt(variancef)
        G += points(f,color='blue',size=50)
        G += point((meanf,0),color='yellow',size=60,zorder=3)
```

```
        G +=
            line([(meanf-sd,0),(meanf+sd,0)],color='red',thickness=5)

        g = DiscreteProbabilitySpace(D,Probs)
        pretty_print('     mean = %s'%str(meanf))
        pretty_print(' variance = %s'%str(variancef))

        #  perhaps to add mean and variance for pmf here
    else:
        print 'Domain D and Probabilities Probs must be 
            lists of the same size'

    #  Now, let's sample from the distribution given above
        and see how a random sampling matches up

    counts = [0] * len(Probs)
    X = GeneralDiscreteDistribution(Probs)
    sample = []

    for _ in range(n_samples):
        elem = X.get_random_element()
        sample.append(D[elem])
        counts[elem] += 1
    Empirical = [1.0*x/n_samples for x in counts] # random

    samplemean = mean(sample)
    samplevariance = variance(sample)
    sampdev = sqrt(samplevariance)

    E = points(zip(D,Empirical),color='orange',size=40)
    E +=
        point((samplemean,0.005),color='brown',size=60,zorder=3)
    E +=
        line([(samplemean-sampdev,0.005),(samplemean+sampdev,0.005)],color='oran
    (G+E).show(ymin=0,figsize=(8,5))
```

Example 4.7.4 (Continuous Probability Function)**.** Consider $f(x) = x^2/c$ for some positive real number c and presume R = [-1,2]. Then f(x) is nonnegative (and only equals zero at one point). To make f(x) a probability density function, we must have

$$\int_{x \in R} f(x) = 1.$$

In this instance you get

$$1 = \int_{-1}^{2} x^2/c = x^3/(3c)|_{-1}^{2} = \frac{8}{3c} - \frac{-1}{3c} = \frac{3}{c}$$

Therefore, f(x) is a probability density function over R provided $= 3$.

Definition 4.7.5 (Distribution Function)**.** Given a random variable X on a space R, a probability distribution function on X is given by a function $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) = P(X \leq x)$

Example 4.7.6 (Discrete Distribution Function)**.** Using $f(x) = x/10$ over R = 1,2,3,4 again, note that F(x) will only change at these four domain values. We get

X	F(x)
$x < 1$	0
$1 \leq x < 2$	1/10
$2 \leq x < 3$	3/10
$3 \leq x < 4$	6/10
$4 \leq x$	1

X	F(x)
$x < -1$	0
$-1 \leq x < 2$	$x^3/9 + 1/9$
$2 \leq x$	1

Example 4.7.7 (Continuous Distribution Function)**.** Consider $f(x) = x^2/3$ over R = [-1,2]. Then, for $-1 \leq x \leq 2$,

$$F(x) = \int_{-1}^{x} u^2/3du = x^3/9 + 1/9.$$

Notice, F(-1) = 0 since nothing has yet been accumulated over values smaller than -1 and F(2)=1 since by that time everything has been accumulated. In summary:

4.8 Properties of the Distribution Function

Theorem 4.8.1. $F(x) = 0, \forall x \leq \inf(R)$

Proof. □

Theorem 4.8.2. $F(x) = 1, \forall x \geq \sup(R)$

Proof. □

Theorem 4.8.3. *F is non-decreasing*

Proof. Case 1: R discrete

$$\forall x_1, x_2 \in \mathbb{Z} \ni x_1 < x_2$$

$$\begin{aligned} F(x_2) &= \sum_{x \leq x_2} f(x) \\ &= \sum_{x \leq x_1} f(x) + \sum_{x_1 < x \leq x_2} f(x) \\ &\geq \sum_{x \leq x_1} f(x) = F(x_1) \end{aligned}$$

Case 2: R continuous

$$\forall x_1, x_2 \in \mathbb{R} \ni x_1 < x_2$$

$$\begin{aligned} F(x_2) &= \int_{-\infty}^{x_2} f(x)dx \\ &= \int_{-\infty}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx \\ &\geq \int_{-\infty}^{x_1} f(x)dx \\ &= F(x_1) \end{aligned}$$

□

Theorem 4.8.4 (Using Discrete Distribution Function to compute probabilities). *for* $x \in R, f(x) = F(x) - F(x-1)$

Theorem 4.8.5 (Using Continuous Distribution function to compute probabilities). *for* $a < b, (a,b) \in R, P(a < X < b) = F(b) - F(a)$

Corollary 4.8.6. *For continuous distributions,* $P(X = a) = 0$

4.9 Standard Units

Any distribution variable can be converted to "standard units" using the linear translation $z = \frac{x-\mu}{\sigma}$. In doing so, then values of z will always represent the number of standard deviations x is from the mean and will provide "dimensionless" comparisons.

4.10 Expected Value

Blaise Pascal was a 17th century mathematician and philosopher who was accomplished in many areas but may likely be best known to you for his creation of what is now known as Pascal's Triangle. As part of his philosophical pursuits, he proposed what is known as "Pascal's wager". It suggests two mutually exclusive outcomes: that God exists or that he does not. His argument is that a rational person should live as though God exists and seek to believe in God. If God does not actually exist, such a person will have only a finite loss (some pleasures, luxury, etc.), whereas they stand to receive infinite gains as represented by eternity in Heaven and avoid an infinite losses of eternity in Hell. This type of reasoning is part of what is known as "decision theory".

You may not confront such dire payouts when making your daily decisions but we need a formal method for making these determinations precise. The procedure for doing so is what we call expected value.

Definition 4.10.1 (Expected Value). Given a random variable X over space R, corresponding probability function f(x) and "value function" u(x), the expected value of u(x) is given by

$$E = E[u(X)] = \sum_{x \in R} u(x) f(x)$$

provided X is discrete, or

$$E = E[u(X)] = \int_R u(x) f(x) dx$$

provided X is continuous.

Example 4.10.2 (Discrete Expected Value). Consider $f(x) = x/10$ over R = 1,2,3,4 where the payout is 10 euros if x=1, 5 euros if x=2, 2 euros if x=3 and -7 euros if x = 4. Then your value function would be u(1)=10, u(2) = 5, u(3)=2, and u(4) = -7. Computing the expect payout gives

$$E = 10 \times 1/10 + 5 \times 2/10 + 2 \times 3/10 - 7 \times 4/10 = -2/10$$

Therefore, the expected payout is actually negative due to a relatively large negative payout associated with the largest likelihood outcome and the larger positive payout only associated with the least likely outcome.

Example 4.10.3 (Continuous Expected Value)**.** Consider $f(x) = x^2/3$ over R = [-1,2] with value function given by $u(x) = e^x - 1$. Then, the expected value for u(x) is given by

$$E = \int_{-1}^{2} (e^x - 1) \cdot x^2/3 = -1/9 \cdot (e + 15) \cdot e^{-1} + 2/3 \cdot e^2 - 8/9 \approx 3.3129$$

So, going back to Pascal's wager, let X = 0 represent disbelief when God doesn't exist and X = 1 represent disbelief when God does exist, X = 2 represent belief when God does exist, and X = 3 represent belief when God does not exist. Let p be the likelihood that God exists. Then you can compute the expected value of disbelief and the expect value of belief by first creating a value function. Below, for argument sake we are somewhat randomly assign a value of one million to disbelief if God doesn't exist. The conclusions are the same if you choose any other finite number...

$$\begin{gathered} u(0) = 1,000,000, f(0) = 1 - p \\ u(1) = -\infty, f(1) = p \\ u(2) = \infty, f(2) = p \\ u(3) = 0, f(3) = 1 - p \end{gathered}$$

Then,

$$\begin{aligned} E[disbelief] &= u(0)f(0) + u(1)f(1) \\ &= 1000000 \times (1 - p) - \infty \times p \\ &= -\infty \end{aligned}$$

if p>0. On the other hand,

$$\begin{aligned} E[belief] &= u(2)f(2) + u(3)f(3) \\ &= \infty \times p + 0 \times (1 - p) \\ &= \infty \end{aligned}$$

if p>0. So Pascal's conclusion is that if there is even the slightest chance that God exists then belief is the smart and scientific choice.

Chapter 5

Binomial, Geometric, and Negative Binomial Distributions

Distributions relating number of successes to number of trials with one of these variable and the other fixed.

5.1 Binomial Distribution

Consider the situation where one can observe a sequence of n independent trials with the likelihood of a success on each individual trial stays constant from trial to trial. Call this likelihood the probably of "success" and denote its value by p where $0 < p < 1$. If we let the variable X measure the number of successes obtained when doing a fixed number of trials n, then the resulting distribution of probabilities is called a Binomial Distribution.

5.1.1 Derivation of Binomial Probability Function

Since successive trials are independent, then the probability of X successes occurring within n trials is given by $P(X = x) = \binom{n}{x} P(SS...SFF...F) = \binom{n}{x} p^x(1-p)^{n-x}$

5.1.2 Binomial Distribution mean

$$\begin{aligned}
\mu = E[X] &= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} x \frac{n(n-1)!}{x(x-1)!(n-x)!} p^x (1-p)^{n-x} \\
&= np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!((n-1)-(x-1))!} p^{x-1} (1-p)^{(n-1)-(x-1)}
\end{aligned}$$

Using the change of variables $k = x - 1$ and $m = n - 1$ yields a binomial series

$$\begin{aligned}
&= np \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} p^k (1-p)^{m-k} \\
&= np(p + (1-p))^m = np
\end{aligned}$$

5.1.3 Binomial Distribution variance

$$\begin{aligned}\sigma^2 = E[X(X-1)] + \mu - \mu^2 &= \sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x (1-p)^{n-x} + np - n^2p^2 \\ &= \sum_{x=2}^{n} x(x-1)\frac{n(n-1)(n-2)!}{x(x-1)(x-2)!(n-x)!} p^x (1-p)^{n-x} + np - n^2p^2 \\ &= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!((n-2)-(x-2))!} p^{x-2}(1-p)^{(n-2)-(x-2)} + np - n^2\end{aligned}$$

Using the change of variables $k = x - 2$ and $m = n - 2$ yields a binomial series

$$\begin{aligned}&= n(n-1)p^2 \sum k = 0^m \frac{m!}{k!(m-k)!} p^k (1-p)^{m-k} + np - n^2p^2 \\ &= n(n-1)p^2 + np - n^2p^2 = np - np^2 = np(1-p)\end{aligned}$$

5.2 Geometric Distribution

Consider the situation where one can observe a sequence of independent trials where the likelihood of a success on each individual trial stays constant from trial to trial. Call this likelihood the probably of "success" and denote its value by p where $0 < p < 1$. If we let the variable X measure the number of trials needed in order to obtain the first success, then the resulting distribution of probabilities is called a Geometric Distribution.

5.2.1 Derivation of Geometric Probability Function

Since successive trials are independent, then the probability of the first success occurring on the mth trial presumes that the previous m-1 trials were all failures. Therefore the desired probability is given by

$$f(x) = P(X = m) = P(FF...FS) = (1-p)^{m-1}p$$

5.2.2 Properties of the Geometric DistributionGeometric Distribution sums to 1

$$k = 1\infty f(x) = \sum_{k=1}^{\infty} (1-p)^{k-1}p = p\sum_{j=0}^{\infty} (1-p)^j = p\frac{1}{1-(1-p)} = 1$$

5.2.3 Derivation of Geometric Mean

$$\begin{aligned}
\mu = E[X] &= \sum_{k=0}^{\infty} k(1-p)^{k-1}p \\
&= p\sum_{k=1}^{\infty} k(1-p)^{k-1} \\
&= p\frac{1}{(1-(1-p))^2} \\
&= p\frac{1}{p^2} = \frac{1}{p}
\end{aligned}$$

5.2.4 Derivation of Geometric Variance

$$\begin{aligned}
\sigma^2 &= E[X(X-1)] + \mu - \mu^2 \\
&=_{k=0}^{\infty} k(k-1)(1-p)^{k-1}p + \mu - \mu^2 \\
&= (1-p)p\sum_{k=2}^{\infty} k(k-1)(1-p)^{k-2} + \frac{1}{p} - \frac{1}{p^2} \\
&= (1-p)p\frac{2}{(1-(1-p))^3} + \frac{1}{p} - \frac{1}{p^2} \\
&= \frac{1-p}{p^2}
\end{aligned}$$

5.2.5 Derivation of Geometric Distribution Function

Consider the accumulated probabilities over a range of values...

$$\begin{aligned}
P(X \leq a) &= 1 - P(X > a) \\
&= 1 - \sum_{k=a+1}^{\infty} (1-p)^{k-1}p \\
&= 1 - p\frac{(1-p)^a}{1-(1-p)} \\
&= 1 - (1-p)^a
\end{aligned}$$

5.3 Negative Binomial

Consider the situation where one can observe a sequence of independent trials where the likelihood of a success on each individual trial stays constant from trial to trial. Call this likelihood the probably of "success" and denote its value by p where $0 < p < 1$. If we let the variable X measure the number of trials needed in order to obtain the rth success, $r \geq 1$ then the resulting distribution of probabilities is called a Geometric Distribution.

Note that r=1 gives the Geometric Distribution.

5.3.1 Negative Binomial Series

Theorem 5.3.1. $\frac{1}{(a+b)^n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} a^k b^{-n-k}$

Proof. First, convert the problem to a slightly different form: $\frac{1}{(a+b)^n} = \frac{1}{b^n} \frac{1}{(\frac{a}{b}+1)^n} = \frac{1}{b^n} \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} \left(\frac{a}{b}\right)^k$

So, let's replace $\frac{a}{b} = x$ and ignore for a while the term factored out. Then, we only need to show

$$\sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k = \left(\frac{1}{1+x}\right)^n$$

However

$$\begin{aligned}\left(\frac{1}{1+x}\right)^n &= \left(\frac{1}{1-(-x)}\right)^n \\ &= \left(\sum_{k=0}^{\infty} (-1)^k x^k\right)^n\end{aligned}$$

This infinite sum raised to a power can be expanded by distributing terms in the standard way. In doing so, the various powers of x multiplied together will create a series in powers of x involving $x^0, x^1, x^2, \ldots$. To detemine the final coefficients notice that the number of time m^k will appear in this product depends upon the number of ways one can write k as a sum of nonnegative integers.

For example, the coefficient of x^3 will come from the n ways of multiplying the coefficients $x^3, x^0, \ldots, x^0$ and $x^2, x^1, x^0, \ldots, x^0$ and $x^1, x^1, x^1, x^0, \ldots, x^0$. This is equivalent to finding the number of ways to write the number k as a sum of nonnegative integers. The possible set of nonnegative integers is 0,1,2,...,k and one way to count the combinations is to separate k *'s by n-1 |'s. For example, if k = 3 then *||** means $x^1x^0x^2 = x^3$. Similarly for k = 5 and |**|*|**| implies $x^0x^2x^1x^2x^0 = x^5$. The number of ways to interchange the identical *'s among the idential |'s is $\binom{n+k-1}{k}$.

Furthermore, to obtain an even power of x will require an even number of odd powers and an odd power of x will require an odd number of odd powers. So, the coefficient of the odd terms stays odd and the coefficient of the even terms remains even. Therefore,

$$\left(\frac{1}{1+x}\right)^n = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k$$

Similarly, $\left(\frac{1}{1-x}\right)^n = \left(\sum_{k=0}^{\infty} x^k\right)^n = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$ □

5.3.2 Negative Binomial Distribution FormulasNegative Binomial Distribution Sums to 1

Consider the situation where one can observe a sequence of independent trials with the likelihood of a success on each individual trial p where $0 < p < 1$. For a positive integer r, let the variable X measure the number of trials needed in order to obtain the rth success. Then the resulting distribution of probabilities is called a Negative Binomial Distribution.

Since successive trials are independent, then the probability of the rth success occurring on the mth trial presumes that in the previous m-1 trials were r-1 successes and m-r failures. Therefore the desired probability is given by

$$P(X = m) = \binom{m-1}{r-1}(1-p)^{m-r}p^r$$

$m = r\infty\binom{m-1}{r-1}(1-p)^{m-r}p^r = p^r m = r\infty\binom{m-1}{r-1}(1-p)^{m-r}$
and by using $k = m - r$

$$\begin{aligned}
&= p^r \sum_{k=0}^{\infty} \binom{r+k-1}{k}(1-p)^k \\
&= p^r \frac{1}{(1-(1-p))^r} \\
&= 1
\end{aligned}$$

Chapter 6

Poisson, Exponential, and Gamma Distributions

6.1 Poisson Distribution

6.2 Exponential Distribution

6.3 Gamma Distribution

Chapter 7

Normal Distributions

7.1 Properties of the Normal Distribution

You have seen that most distributions become "bell shaped" as certain parameters are allowed to increase. The question might arise regarding whether this always must happen or is it just a happy coincidence. The amazing answer is that if you interpret the question in the correct way then this is always true.

Definition 7.1.1 (The Normal Distribution)**.** Given two parameters μ and σ a random variable X with density function

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\left(\frac{x-\mu}{\sigma}\right)^2/2}$$

Theorem 7.1.2. *If $\mu = 0$ and $\sigma = 1$, then we say X has a standard normal distribution and often use Z as the variable name. In this case, the density function reduces to*

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{z^2/2}$$

Proof. Convert to "standard units" using the conversion $z = \frac{x-\mu}{\sigma} = \frac{x-0}{1} = x$. $\square$

7.2 Theorems

7.3 Chi-Square Distribution

7.4 Central Limit Theorem

Often, when one wants to solve various scientific problems, several assumptions will be made regarding the nature of the underlying setting and base their conclusions on those assumptions. Indeed, if one is going to use a Binomial Distribution or a Negative Binomial Distribution, an assumption on the value of p is necessary. For Poisson and Exponential Distributions, one must know the mean. For Normal Distributions, one must assume values for both the mean and the standard deviation. Where do these values come from? Often, one may perform a preliminary study and obtain a sample statistic...such as a sample mean or a relative frequency and use these values for or p.

But what is the underlying distribution of these sample statistics? The Central Limit Theorem gives the answer...

To motivate this discussion, consider the following two interactive experiments. For the first graph below, a sequence of N random samples, each of size r, ranging from 0 to "Range" is generated and graphed as small data points. As the number of samples N and the sample size r increase, notice that the data seems to cover the entire range of possible values relatively uniformly. (For this scatter plot note that each row represents the data for one sample of size r. The larger the N, the greater the number of rows.) Each row is averaged and that mean value is plotted on the graph as a red circle. If you check the "Show$_M ean”box, themeanofthesecirclesisindicatedbythegreenlineinthemi$

For the second graph below, the means are collected and the relative frequency of each is plotted. As N increases, you should see that the results begin to show an interesting tendency. As you increase the data range, you may notice this graph has a larger number of data values. Smoothing groups this data into intervals of length two for perhaps a graph with less variability.

Consider each of the following:

- As N increases with single digit values of r, what appears to happen to the mean and range of the means? How does increasing the data range from 1-100 to 1-200 or 1-300 affect these results?

- As N increases (say, for a middle value of r), what appears to happen to the means? How does increasing the data range from 1-100 to 1-200 or 1-300 affect these results?

- As r increases (say, for a middle value of N), what appears to happen to the range of the averages? Does your conclusion actually depend upon the value of N? (Look at the graph and don't worry about the actual numerical values.) How does increasing N for the second graph affect the skewness and kurtosis of that graph? Do things change significantly as r is increased?

```
var('n,k')
from sage.finance.time_series import TimeSeries

@interact(layout=dict(top=[['Range'],['Show_Mean',
    'Smoothing']],
bottom=[['N'],['r']]))

def _(Range=[100,200,300,500],N=slider(5,200,2,2,label="N␣=␣
    Number␣of␣Samples"),r=slider(3,200,1,2,label="r␣=␣Sample␣
    Size"),Show_Mean=False,Smoothing=False):
    R=[1..N]      #  R ranges over the number of
        samples...will point to the list of averages
    rangemax = Range

    data = random_matrix(ZZ,N,r,x=rangemax)
    datapoints = []
    avg_values = []
    avg_string = []
    averages = []
    for n in range(N):
        temp = 0
        for k in range(r):
            datapoints += [(data[n][k],n)]
            temp += data[n][k]
        avg_values.append(round(temp/r))
        if Smoothing:
            avg_string.append(str(2*round((temp/r)/2)))
        else:
            avg_string.append(str(round(temp/r)))
```

```
        averages += [(round(temp/r),n)]    #  make these
            averages integers for use in grouping later
    SCAT =
        scatter_plot(datapoints,markersize=2,edgecolor='red',figsize=(10,4),axes_labels=
        Values','Sample␣Number'])
    AVGS =
        scatter_plot(averages,markersize=50,edgecolor='blue',marker='o',figsize=(7,4))

    freqslist =
        frequency_distribution(avg_string,1).function().items()


# compute sample statistics for the raw data as well as for
    the N averages
    Mean_data = (sum(sum(data))/(N*r)).n()
#     STD_data = sqrt(sum(sum( (data-Mean_data)^2
    ))/(N*r)).n()
    Mean_averages = mean(avg_values).n()
#     STD_averages = sqrt(variance(avg_values).n())
#     print "Data mean =",Mean_data," vs Mean of the averages
    =",Mean_averages
#     print "Data STD = ",STD_data," vs Standard Dev of avgs
    =", STD_averages
    if Show_Mean:
        avg_line =
            line([(Mean_data,0),(Mean_data,N-1)],rgbcolor='green',thickness=10)
        avg_text =
            text('xbar',(Mean_data,N),horizontal_alignment='right',rgbcolor='green')
    else:
        avg_line = Graphics()
        avg_text = Graphics()

#  Plot a scatter plot exhibiting uniformly random data and
    the collection of averages
    print(html("The␣random␣data␣plot␣on␣the␣left␣with␣each␣
        row␣representing␣a␣sample␣with␣size␣determined␣by\n"+
          "the␣slider␣above␣and␣each␣circle␣representing␣the␣
              average␣for␣that␣particular␣sample.\n"+
          "First,␣keep␣sample␣size␣relatively␣low␣and␣
              increase␣the␣number␣of␣samples.␣␣Then,␣\n"+
          "watch␣what␣happens␣when␣you␣slowly␣increase␣the␣
              sample␣size."))


#  Plot the relative frequencies of the grouped sample
    averages
    print(html("Now,␣the␣averages␣(ie.␣the␣circles)␣from␣
        above␣are␣collected␣and␣counted\n"+
          "with␣the␣relative␣frequency␣of␣each␣average␣
              graphed␣below.␣␣For␣a␣relatively␣large␣number␣
              of\n"+
          "samples,␣notice␣what␣seems␣to␣happen␣to␣these␣
              averages␣as␣the␣sample␣size␣increases."))
    if Smoothing:
        binRange = Range//2
    else:
        binRange = Range

    # normed=True  # if you want to have relative
```

```
        frequencies below

    his_low = 2*rangemax/7
    his_high = 5*rangemax/7

    T =
        histogram(avg_values,normed=False,bins=binRange,range=(his_low,his_high)
        Averages','Frequency'])
    #T =
        TimeSeries(avg_values).plot_histogram(axes_labels=['Sample
        Averages','Frequency'])

    pretty_print('Scatter␣Plot␣of␣random␣data.␣␣Horizontal␣
        is␣number␣of␣samples.')
    (SCAT+AVGS+avg_line+avg_text).show()
    pretty_print('Histogram␣of␣Sample␣Averages')
    T.show(figsize=(5,2))
```

```
var('n,k')
from sage.finance.time_series import TimeSeries

@interact(layout=dict(top=[['Range'],['Show_Mean',
    'Smoothing']],
bottom=[['N'],['r']]))

def _(Range=[100,200,300,500],N=slider(5,200,2,2,label="N␣=␣
    Number␣of␣Samples"),r=slider(3,200,1,2,label="r␣=␣Sample␣
    Size"),Show_Mean=False,Smoothing=False):
    R=[1..N]      #  R ranges over the number of
        samples...will point to the list of averages
    rangemax = Range

    data = random_matrix(ZZ,N,r,x=rangemax)
    datapoints = []
    avg_values = []
    avg_string = []
    averages = []
    for n in range(N):
        temp = 0
        for k in range(r):
            datapoints += [(data[n][k],n)]
            temp += data[n][k]
        avg_values.append(round(temp/r))
        if Smoothing:
            avg_string.append(str(2*round((temp/r)/2)))
        else:
            avg_string.append(str(round(temp/r)))

        averages += [(round(temp/r),n)]    #  make these
            averages integers for use in grouping later
    SCAT =
        scatter_plot(datapoints,markersize=2,edgecolor='red',figsize=(10,4),axes
        Values','Sample␣Number'])
    AVGS =
        scatter_plot(averages,markersize=50,edgecolor='blue',marker='o',figsize=

    freqslist =
        frequency_distribution(avg_string,1).function().items()
```

```
# compute sample statistics for the raw data as well as for
    the N averages
    Mean_data = (sum(sum(data))/(N*r)).n()
#     STD_data = sqrt(sum(sum( (data-Mean_data)^2
    ))/(N*r)).n()
    Mean_averages = mean(avg_values).n()
#     STD_averages = sqrt(variance(avg_values).n())
#     print "Data mean =",Mean_data," vs Mean of the averages
    =",Mean_averages
#     print "Data STD = ",STD_data," vs Standard Dev of avgs
    =", STD_averages
    if Show_Mean:
        avg_line =
            line([(Mean_data,0),(Mean_data,N-1)],rgbcolor='green',thickness=10)
        avg_text =
            text('xbar',(Mean_data,N),horizontal_alignment='right',rgbcolor='green')
    else:
        avg_line = Graphics()
        avg_text = Graphics()

#  Plot a scatter plot exhibiting uniformly random data and
    the collection of averages
    print(html("The␣random␣data␣plot␣on␣the␣left␣with␣each␣
        row␣representing␣a␣sample␣with␣size␣determined␣by\n"+
         "the␣slider␣above␣and␣each␣circle␣representing␣the␣
            average␣for␣that␣particular␣sample.\n"+
         "First,␣keep␣sample␣size␣relatively␣low␣and␣
            increase␣the␣number␣of␣samples.␣␣Then,␣\n"+
         "watch␣what␣happens␣when␣you␣slowly␣increase␣the␣
            sample␣size."))


#  Plot the relative frequencies of the grouped sample
    averages
    print(html("Now,␣the␣averages␣(ie.␣the␣circles)␣from␣
        above␣are␣collected␣and␣counted\n"+
         "with␣the␣relative␣frequency␣of␣each␣average␣
            graphed␣below.␣␣For␣a␣relatively␣large␣number␣
            of\n"+
         "samples,␣notice␣what␣seems␣to␣happen␣to␣these␣
            averages␣as␣the␣sample␣size␣increases."))
    if Smoothing:
        binRange = Range//2
    else:
        binRange = Range

    # normed=True  # if you want to have relative
        frequencies below

    his_low = 2*rangemax/7
    his_high = 5*rangemax/7

    T =
        histogram(avg_values,normed=False,bins=binRange,range=(his_low,his_high),axes_lab
        Averages','Frequency'])
    #T =
        TimeSeries(avg_values).plot_histogram(axes_labels=['Sample
        Averages','Frequency'])
```

```
pretty_print('Scatter␣Plot␣of␣random␣data.␣␣Horizontal␣
    is␣number␣of␣samples.')
(SCAT+AVGS+avg_line+avg_text).show()
pretty_print('Histogram␣of␣Sample␣Averages')
T.show(figsize=(5,2))
```

7.4.1

Theorem

7.4.2

Approximating distributions Limiting distributions

Chapter 8

Estimating Data using Intervals

8.1 Chebyshev

An interval centered on the mean in which at least a certain proportion of the actual data must lie.

Theorem 8.1.1 (Chebyshev's Theorem)**.** *Given a random variable X with given mean and standard deviation , for k>1 at least 2 1 1 k of the observations lie within k standard deviations from the mean.*

8.2 Measures of Spread

Measures of spread: • Average Deviation from the Mean – always zero for any distribution • Average Absolute Deviation from the Mean – difficult to deal with algebra when absolute values are used • Average Squared Deviation from the Mean – always non-negative and good with algebra and calculus

Definition 8.2.1 (Variance)**.** The variance is a measure of spread found by using the average squared deviation from the mean 2 = ( ))(2 1 k n k k xfx = if this value exists and is also denoted by Var(X). The positive square root of the variance is called the standard deviation and is denoted by .