Solutions

Problem 1: Stock price over two days

1. **Outcomes, events, and probabilities**

   Let U denote "up by 2" and D denote "down by 2" on a given day. Over two days, the possible paths and resulting prices S_2 are:

Outcome	Price	$\mathbb{P}$(Outcome)
UU	$10+2+2=14$	$\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$
UD	$10+2-2=10$	$\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$
DU	$10-2+2=10$	$\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$
DD	$10-2-2=6$	$\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$

   Thus the probability mass function of S_2 is:

$$\mathbb{P}(S_2=6)=\frac{1}{4},\quad \mathbb{P}(S_2=10)=\frac{1}{2},\quad \mathbb{P}(S_2=14)=\frac{1}{4}.$$

2. **CDF of S_2**

   Define $F(x)=\mathbb{P}(S_2\le x)$. Then

$$F(x)=\begin{cases}0, & x<6,\\ \frac{1}{4}, & 6\le x<10,\\ \frac{3}{4}, & 10\le x<14,\\ 1, & x\ge 14.\end{cases}$$

   This is a step function with jumps at $x=6,10,14$.

3. **Expectation and variance of S_2**

$$\mathbb{E}[S_2]=6\cdot\frac{1}{4}+10\cdot\frac{1}{2}+14\cdot\frac{1}{4}=\frac{6+20+14}{4}=\frac{40}{4}=10.$$

   Using the fact that $\mathbb{V}[X]=\mathbb{E}[X^2]-(\mathbb{E}[X])^2$,

$$\mathbb{E}[S_2^2]=6^2\cdot\frac{1}{4}+10^2\cdot\frac{1}{2}+14^2\cdot\frac{1}{4}=\frac{36}{4}+\frac{100}{2}+\frac{196}{4}=9+50+49=108.$$

$$\mathrm{Var}(S_2)=\mathbb{E}[S_2^2]-(\mathbb{E}[S_2])^2=108-10^2=108-100=8.$$

4. **Contract payoff as a random variable and its expectation**

   At the end of day two you may buy one share at a fixed price of 11. If the market price is S_2, your profit (payoff) is

$$X=\max\{0,S_2-11\}.$$

This is a random variable: it assigns a real number (profit or loss) to each outcome of S_2.

From the distribution:

S_2	$\max\{0, S_2 - 11\}$	$\mathbb{P}$
6	0	$\frac{1}{4}$
10	0	$\frac{1}{2}$
14	3	$\frac{1}{4}$

Thus,

$$\mathbb{E}[X] = \frac{1}{4}0 + \frac{1}{2}0 + \frac{1}{4}3 = \frac{3}{4}.$$

Problem 1: Hospital capacity

Let T be the length of stay (in days) for a patient who is admitted. Conditional on admission, $T \sim \text{Unif}(0, 30)$ with CDF

$$F_T(t) = \begin{cases} 0, & t < 0, \\ \frac{t}{30}, & 0 \le t \le 30, \\ 1, & t > 30. \end{cases}$$

Let X be the length of stay for a random ER arrival, with $X = 0$ if not admitted.

(a) **Overall distribution function**

With probability $4/5$ the patient is not admitted and $X = 0$. With probability $1/5$ they are admitted and $X = T$.

For $t < 0$,

$$F_X(t) = \mathbb{P}(X \le t) = 0.$$

For $0 \le t \le 30$,

$$F_X(t) = \mathbb{P}(X = 0) + \mathbb{P}(\text{admitted and } T \le t) = \frac{4}{5} + \frac{1}{5}\mathbb{P}(T \le t) = \frac{4}{5} + \frac{1}{5} \cdot \frac{t}{30} = \frac{4}{5} + \frac{t}{150}.$$

For $t > 30$,

$$F_X(t) = 1.$$

So

$$F_X(t) = \begin{cases} 0, & t < 0, \\ \frac{4}{5} + \dfrac{t}{150}, & 0 \le t \le 30, \\ 1, & t > 30. \end{cases}$$

There is a jump of size $4/5$ at $t = 0$ (point mass at 0), and a linear increase from $4/5$ to 1 on $[0, 30]$.

(b) **Given admitted: expectation and variance**
If admitted, $T \sim \text{Unif}(0, 30)$. For $U \sim \text{Unif}(a, b)$,

$$\mathbb{E}[U] = \frac{a+b}{2}, \quad \text{Var}(U) = \frac{(b-a)^2}{12}.$$

Here $a = 0$, $b = 30$, so

$$\mathbb{E}[T] = \frac{0+30}{2} = 15, \quad \text{Var}(T) = \frac{30^2}{12} = \frac{900}{12} = 75.$$

(c) **Expected stay for a random ER arrival**
Let A denote the event "admitted". Then

$$\mathbb{E}[X] = \mathbb{E}[X \mid A]\mathbb{P}(A) + \mathbb{E}[X \mid A^c]\mathbb{P}(A^c).$$

We have $\mathbb{P}(A) = \frac{1}{5}$, $\mathbb{P}(A^c) = \frac{4}{5}$,

$$\mathbb{E}[X \mid A] = \mathbb{E}[T] = 15, \quad \mathbb{E}[X \mid A^c] = 0.$$

Thus

$$\mathbb{E}[X] = 15 \cdot \frac{1}{5} + 0 \cdot \frac{4}{5} = 3 \text{ days}.$$

(d) **Expected number admitted and expected total bed-days**
If n patients arrive tomorrow and each is admitted independently with probability $1/5$, then the number admitted

$$N \sim \text{Binomial}\left(n, \frac{1}{5}\right),$$

so

$$\mathbb{E}[N] = n \cdot \frac{1}{5} = \frac{n}{5}.$$

For total bed-days, each ER arrival contributes an expected $\mathbb{E}[X] = 3$ days. Let X_i be the stay length for patient i. Then the total bed-days is $S = \sum_{i=1}^{n} X_i$, and

$$\mathbb{E}[S] = \sum_{i=1}^{n} \mathbb{E}[X_i] = n \cdot 3 = 3n.$$

Problem 2: Softmax choice over three options

We have three options a, b, c with values $v_a > v_b > v_c$, and probability mass function

$$\mathbb{P}[a] = \frac{e^{v_a/\sigma}}{Z}, \quad \mathbb{P}[b] = \frac{e^{v_b/\sigma}}{Z}, \quad \mathbb{P}[c] = \frac{e^{v_c/\sigma}}{Z},$$

where

$$Z = e^{v_a/\sigma} + e^{v_b/\sigma} + e^{v_c/\sigma}, \quad \sigma > 0.$$

(a) **Axioms of probability**

Each probability is nonnegative because exponentials are positive and $Z > 0$:

$$e^{v_i/\sigma} > 0, \quad Z = \sum_i e^{v_i/\sigma} > 0 \;\Rightarrow\; \mathbb{P}[i] = \frac{e^{v_i/\sigma}}{Z} \geq 0.$$

The probabilities sum to 1:

$$\mathbb{P}[a] + \mathbb{P}[b] + \mathbb{P}[c] = \frac{e^{v_a/\sigma} + e^{v_b/\sigma} + e^{v_c/\sigma}}{Z} = \frac{Z}{Z} = 1.$$

(b) **Independence of irrelevant alternatives (ratio $p[a]/p[b]$)**

With all three options present,

$$\frac{\mathbb{P}[a]}{\mathbb{P}[b]} = \frac{\frac{e^{v_a/\sigma}}{Z}}{\frac{e^{v_b/\sigma}}{Z}} = \frac{e^{v_a/\sigma}}{e^{v_b/\sigma}} = e^{(v_a - v_b)/\sigma}.$$

If we remove option c, the updated probabilities are

$$\mathbb{P}'[a] = \frac{e^{v_a/\sigma}}{e^{v_a/\sigma} + e^{v_b/\sigma}}, \quad \mathbb{P}'[b] = \frac{e^{v_b/\sigma}}{e^{v_a/\sigma} + e^{v_b/\sigma}}.$$

Then

$$\frac{\mathbb{P}'[a]}{\mathbb{P}'[b]} = \frac{\frac{e^{v_a/\sigma}}{e^{v_a/\sigma} + e^{v_b/\sigma}}}{\frac{e^{v_b/\sigma}}{e^{v_a/\sigma} + e^{v_b/\sigma}}} = \frac{e^{v_a/\sigma}}{e^{v_b/\sigma}} = e^{(v_a - v_b)/\sigma}.$$

Thus

$$\frac{\mathbb{P}[a]}{\mathbb{P}[b]} = \frac{\mathbb{P}'[a]}{\mathbb{P}'[b]},$$

so the ratio of a to b does not depend on whether c is present.

(c) **Effect of increasing v_a**

Let v_b, v_c, σ be fixed and consider v_a as a variable.

First, note

$$\mathbb{P}[a] = \frac{e^{v_a/\sigma}}{e^{v_a/\sigma} + e^{v_b/\sigma} + e^{v_c/\sigma}}.$$

Differentiate with respect to v_a:

$$\frac{\partial \mathbb{P}[a]}{\partial v_a} = \frac{\frac{1}{\sigma} e^{v_a/\sigma}\left(e^{v_a/\sigma} + e^{v_b/\sigma} + e^{v_c/\sigma}\right) - e^{v_a/\sigma} \cdot \frac{1}{\sigma} e^{v_a/\sigma}}{\left(e^{v_a/\sigma} + e^{v_b/\sigma} + e^{v_c/\sigma}\right)^2}.$$

This simplifies to

$$\frac{\partial \mathbb{P}[a]}{\partial v_a} = \frac{\frac{1}{\sigma} e^{v_a/\sigma} \left(e^{v_b/\sigma} + e^{v_c/\sigma} \right)}{\left(e^{v_a/\sigma} + e^{v_b/\sigma} + e^{v_c/\sigma} \right)^2} > 0,$$

because all terms are positive. Thus $\mathbb{P}[a]$ increases as v_a increases.
For b and c, note that $\mathbb{P}[a] + \mathbb{P}[b] + \mathbb{P}[c] = 1$, so

$$\frac{\partial \mathbb{P}[b]}{\partial v_a} + \frac{\partial \mathbb{P}[c]}{\partial v_a} = -\frac{\partial \mathbb{P}[a]}{\partial v_a} < 0.$$

By symmetry, direct calculation also shows

$$\frac{\partial \mathbb{P}[b]}{\partial v_a} < 0, \quad \frac{\partial \mathbb{P}[c]}{\partial v_a} < 0.$$

Hence as v_a increases, $\mathbb{P}[a]$ goes up, while $\mathbb{P}[b]$ and $\mathbb{P}[c]$ go down.

(d) **Limits as $\sigma \to \infty$ and $\sigma \to 0$**
As $\sigma \to \infty$, for each fixed v_i,

$$\frac{v_i}{\sigma} \to 0 \quad \Rightarrow \quad e^{v_i/\sigma} \to e^0 = 1.$$

Therefore,

$$\mathbb{P}[a] \to \frac{1}{1+1+1} = \frac{1}{3}, \quad \mathbb{P}[b] \to \frac{1}{3}, \quad \mathbb{P}[c] \to \frac{1}{3},$$

so the probabilities become uniform.
As $\sigma \to 0^+$, divide numerator and denominator of $\mathbb{P}[a]$ by $e^{v_a/\sigma}$:

$$\mathbb{P}[a] = \frac{1}{1 + e^{(v_b - v_a)/\sigma} + e^{(v_c - v_a)/\sigma}}.$$

Since $v_b < v_a$ and $v_c < v_a$, we have $v_b - v_a < 0$ and $v_c - v_a < 0$, so for $\sigma \to 0^+$,

$$e^{(v_b - v_a)/\sigma} \to 0, \quad e^{(v_c - v_a)/\sigma} \to 0.$$

Thus

$$\lim_{\sigma \to 0^+} \mathbb{P}[a] = \frac{1}{1+0+0} = 1, \quad \lim_{\sigma \to 0^+} \mathbb{P}[b] = 0, \quad \lim_{\sigma \to 0^+} \mathbb{P}[c] = 0.$$

So in the limit $\sigma \to 0^+$, softmax selects the option with the highest value (here a) with probability 1; this is the "hard max" limit.

Problem 2: Weather Markov chain

States: Clear (C) and Rainy (R). Transition matrix (column-stochastic) is

$$P = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix},$$

where $p_{i \leftarrow j}$ is the probability of going from state j to state i.

Let the forecast vector be a column vector

$$v = \begin{pmatrix} \mathbb{P}(\text{Clear}) \\ \mathbb{P}(\text{Rainy}) \end{pmatrix}.$$

The update rule is $v_{\text{next}} = Pv_{\text{current}}$.

Assume today is clear, so

$$v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(a) **Forecast for tomorrow**

$$v_1 = Pv_0 = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}.$$

So tomorrow: 60% chance Clear, 40% chance Rainy.

(b) **Forecast for the day after tomorrow**

$$v_2 = Pv_1 = P^2 v_0.$$

First compute P^2:

$$P^2 = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix} \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix} = \begin{pmatrix} 0.36 + 0.16 & 0.24 + 0.24 \\ 0.24 + 0.24 & 0.16 + 0.36 \end{pmatrix} = \begin{pmatrix} 0.52 & 0.48 \\ 0.48 & 0.52 \end{pmatrix}.$$

Then

$$v_2 = P^2 v_0 = \begin{pmatrix} 0.52 \\ 0.48 \end{pmatrix}.$$

So on the second day: 52% Clear, 48% Rainy.

(c) **Forecast for the third day**

$$v_3 = Pv_2 = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix} \begin{pmatrix} 0.52 \\ 0.48 \end{pmatrix} = \begin{pmatrix} 0.6 \cdot 0.52 + 0.4 \cdot 0.48 \\ 0.4 \cdot 0.52 + 0.6 \cdot 0.48 \end{pmatrix} = \begin{pmatrix} 0.504 \\ 0.496 \end{pmatrix}.$$

So on the third day: 50.4% Clear, 49.6% Rainy.

(d) **Show** $(0.5, 0.5)$ **is stationary**

Let

$$v^* = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}.$$

Then

$$Pv^* = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.6 \cdot 0.5 + 0.4 \cdot 0.5 \\ 0.4 \cdot 0.5 + 0.6 \cdot 0.5 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = v^*.$$

Thus if the current forecast is $(0.5, 0.5)$, the next forecast remains $(0.5, 0.5)$; this is a stationary distribution.

Problem 3: Poisson model and MLE

We have $X \sim \text{Poisson}(\lambda)$ with

$$\mathbb{P}[X = k] = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots, \ \lambda > 0.$$

(a) **Poisson pmf obeys probability axioms**

Non-negativity: for all k, $\lambda^k \geq 0$, $e^{-\lambda} > 0$, $k! > 0$, so

$$\mathbb{P}[X = k] = \frac{\lambda^k e^{-\lambda}}{k!} \geq 0.$$

Sum to 1:

$$\sum_{k=0}^{\infty} \mathbb{P}[X = k] = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1,$$

using the given hint.

(b) **Likelihood for sample** $x_1 = 1, x_2 = 2, x_3 = 3$

Assuming X_1, X_2, X_3 are independent $\text{Poisson}(\lambda)$, the likelihood is

$$L(\lambda) = \prod_{i=1}^{3} \mathbb{P}[X_i = x_i] = \prod_{i=1}^{3} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}.$$

With $x_1 = 1, x_2 = 2, x_3 = 3$,

$$L(\lambda) = \frac{\lambda^1 e^{-\lambda}}{1!} \cdot \frac{\lambda^2 e^{-\lambda}}{2!} \cdot \frac{\lambda^3 e^{-\lambda}}{3!} = \frac{\lambda^6 e^{-3\lambda}}{1!\,2!\,3!}.$$

(c) **Maximum likelihood estimator**

Log-likelihood:

$$\ell(\lambda) = \log L(\lambda) = 6 \log \lambda - 3\lambda - \log(1!\,2!\,3!).$$

Differentiate and set to zero:

$$\ell'(\lambda) = \frac{6}{\lambda} - 3 = 0 \quad \Rightarrow \quad \frac{6}{\lambda} = 3 \quad \Rightarrow \quad \hat{\lambda} = 2.$$

(d) **Probability $X \le 2$ under fitted model**
Under the fitted model, $X \sim \text{Poisson}(2)$. Then

$$\mathbb{P}[X \le 2] = \mathbb{P}[X = 0] + \mathbb{P}[X = 1] + \mathbb{P}[X = 2] = \sum_{k=0}^{2} \frac{2^k e^{-2}}{k!}.$$

Compute:

$$\mathbb{P}[X \le 2] = e^{-2}\left(\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!}\right) = e^{-2}(1 + 2 + 2) = 5e^{-2}.$$

Problem 3: Categorical model and MLE

We observe 4 independent draws from a categorical distribution over labels $\{a, b, c\}$:

$$a \text{ once}, \quad b \text{ twice}, \quad c \text{ once}.$$

Let the probabilities be p_a, p_b, p_c.

(a) **Axioms of probability**
For a valid pmf over $\{a, b, c\}$, the probabilities must satisfy:

$$p_a \ge 0, \quad p_b \ge 0, \quad p_c \ge 0, \quad p_a + p_b + p_c = 1.$$

(b) **Likelihood for the data**
Let the counts be $n_a = 1$, $n_b = 2$, $n_c = 1$, with total $n = 4$.
Up to a combinatorial constant, the likelihood is

$$L(p_a, p_b, p_c) = p_a^{n_a} p_b^{n_b} p_c^{n_c} = p_a^1 p_b^2 p_c^1.$$

(If we include the multinomial coefficient, L is $\frac{4!}{1!\,2!\,1!} p_a p_b^2 p_c$, but this factor does not affect the maximization.)

(c) **Maximum likelihood estimators**
We maximize L subject to $p_a + p_b + p_c = 1$, $p_i \ge 0$.
Use the hint $p_c = 1 - p_a - p_b$ and write

$$L(p_a, p_b) = p_a p_b^2 (1 - p_a - p_b),$$

for $p_a \ge 0$, $p_b \ge 0$, $p_a + p_b \le 1$.
It is convenient to maximize the log-likelihood:

$$\ell(p_a, p_b) = \log L = \log p_a + 2 \log p_b + \log(1 - p_a - p_b),$$

for $0 < p_a, p_b, p_a + p_b < 1$.
Compute partial derivatives:

$$\frac{\partial \ell}{\partial p_a} = \frac{1}{p_a} - \frac{1}{1 - p_a - p_b}, \quad \frac{\partial \ell}{\partial p_b} = \frac{2}{p_b} - \frac{1}{1 - p_a - p_b}.$$

Set these to zero for an interior maximum:

$$\frac{1}{p_a} = \frac{1}{1-p_a-p_b} \;\Rightarrow\; p_a = 1-p_a-p_b \;\Rightarrow\; 2p_a+p_b = 1,$$

$$\frac{2}{p_b} = \frac{1}{1-p_a-p_b} \;\Rightarrow\; 2(1-p_a-p_b) = p_b \;\Rightarrow\; 2-2p_a-2p_b = p_b \;\Rightarrow\; 2-2p_a = 3p_b.$$

From $2p_a+p_b = 1$ we get $p_b = 1-2p_a$. Substitute into $2-2p_a = 3p_b$:

$$2-2p_a = 3(1-2p_a) = 3-6p_a \;\Rightarrow\; 2-2p_a = 3-6p_a \;\Rightarrow\; 4p_a = 1 \;\Rightarrow\; p_a = \frac{1}{4}.$$

Then

$$p_b = 1-2p_a = 1-\frac{2}{4} = \frac{1}{2},$$

and

$$p_c = 1-p_a-p_b = 1-\frac{1}{4}-\frac{1}{2} = \frac{1}{4}.$$

Thus the MLEs are

$$\hat{p}_a = \frac{1}{4}, \quad \hat{p}_b = \frac{1}{2}, \quad \hat{p}_c = \frac{1}{4},$$

which coincide with the observed relative frequencies.