

## Lecture 7: Singular Value Decomposition

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We will see that any matrix  $A \in \mathbb{R}^{m \times n}$  (w.l.o.g.  $m \leq n$ ) can be written as

$$A = \sum_{\ell=1}^m \sigma_{\ell} u_{\ell} v_{\ell}^T \quad (1)$$

$$\forall \ell \quad \sigma_{\ell} \in \mathbb{R}, \quad \sigma_{\ell} \geq 0 \quad (2)$$

$$\forall \ell, \ell' \quad \langle u_{\ell}, u_{\ell'} \rangle = \langle v_{\ell}, v_{\ell'} \rangle = \delta(\ell, \ell') \quad (3)$$

To prove this consider the matrix  $AA^T \in \mathbb{R}^{m \times m}$ . Set  $u_{\ell}$  to be the  $\ell$ 'th eigenvector of  $AA^T$ . By definition we have that  $AA^T u_{\ell} = \lambda_{\ell} u_{\ell}$ . Since  $AA^T$  is positive semidefinite we have  $\lambda_{\ell} \geq 0$ . Since  $AA^T$  is symmetric we have that  $\forall \ell, \ell' \quad \langle u_{\ell}, u_{\ell'} \rangle = \delta(\ell, \ell')$ . Set  $\sigma_{\ell} = \sqrt{\lambda_{\ell}}$  and  $v_{\ell} = \frac{1}{\sigma_{\ell}} A^T u_{\ell}$ . Now we can compute the following:

$$\langle v_{\ell}, v_{\ell'} \rangle = \frac{1}{\sigma_{\ell}^2} u_{\ell}^T A A^T u_{\ell'} = \frac{1}{\sigma_{\ell}^2} \lambda_{\ell} \langle u_{\ell}, u_{\ell'} \rangle = \delta(\ell, \ell')$$

We are only left to show that  $A = \sum_{\ell=1}^m \sigma_{\ell} u_{\ell} v_{\ell}^T$ . To do that we examine the norm or the difference multiplied by a test vector  $w = \sum_{i=1}^m \alpha_i u_i$ .

$$\begin{aligned} \|w^T(A - \sum_{\ell=1}^m \sigma_{\ell} u_{\ell} v_{\ell}^T)\| &= \|(\sum_{i=1}^m \alpha_i u_i^T)(A - \sum_{\ell=1}^m \sigma_{\ell} u_{\ell} v_{\ell}^T)\| \\ &= \|(\sum_{i=1}^m \alpha_i u_i^T A - \sum_{i=1}^m \sum_{\ell=1}^m \delta(i, \ell) \alpha_i \sigma_{\ell} v_{\ell}^T)\| \\ &= \|(\sum_{i=1}^m \alpha_i \sigma_i v_i^T - \sum_{i=1}^m \alpha_i \sigma_i v_i^T)\| = 0 \end{aligned}$$

The vectors  $u_{\ell}$  and  $v_{\ell}$  are called the left and right singular vectors of  $A$  and  $\sigma_{\ell}$  are the singular values of  $A$ . It is customary to order the singular values in descending order  $\sigma_1 \geq \sigma_2, \dots, \sigma_m \geq 0$ . Also, we will denote by  $r$  the rank of  $A$ . Here is another very convenient way to write the fact that  $A = \sum_{\ell=1}^m \sigma_{\ell} u_{\ell} v_{\ell}^T$

- Let  $\Sigma \in \mathbb{R}^{r \times r}$  be a diagonal matrix whose entries are  $\Sigma(i, i) = \sigma_i$  and  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r$ .
- Let  $U \in \mathbb{R}^{m \times r}$  be the matrix whose  $i$ 'th column is the left singular vectors of  $A$  corresponding to singular value  $\sigma_i$ .
- Let  $V \in \mathbb{R}^{n \times r}$  be the matrix whose  $i$ 'th column is the right singular vectors of  $A$  corresponding to singular value  $\sigma_i$ .

We have that  $A = U \Sigma V^T$  and that  $U^T U = V^T V = I_r$ . Note that the sum goes only up to  $r$  which is the rank of  $A$ . Clearly, not summing up zero valued singular values does not change the sum.

## Applications of the SVD

1. Determining range, null space and rank (also numerical rank).
2. Matrix approximation.
3. Inverse and Pseudo-inverse: If  $A = U\Sigma V^T$  and  $\Sigma$  is full rank, then  $A^{-1} = V\Sigma^{-1}U^T$ . If  $\Sigma$  is singular, then its pseudo-inverse is given by  $A^\dagger = V\Sigma^\dagger U^T$ , where  $\Sigma^\dagger$  is formed by replacing every nonzero entry by its reciprocal.
4. Least squares: If we need to solve  $Ax = b$  in the least-squares sense, then  $x_{LS} = V\Sigma^\dagger U^T b$ .
5. Denoising – Small singular values typically correspond to noise. Take the matrix whose columns are the signals, compute SVD, zero small singular values, and reconstruct.
6. Compression – We have signals as the columns of the matrix  $S$ , that is, the  $i$  signal is given by

$$S_i = \sum_{j=1}^r (\sigma_j v_{ij}) u_j.$$

If some of the  $\sigma_i$  are small, we can discard them with small error, thus obtaining a compressed representation of each signal. We have to keep the coefficients  $\sigma_j v_{ij}$  for each signal and the dictionary, that is, the vectors  $u_i$  that correspond to the retained coefficients.

SVD and eigen-decomposition are related but there are quite a few differences between them.

1. Not every matrix has an eigen-decomposition (not even any square matrix). Any matrix (even rectangular) has an SVD.
2. In eigen-decomposition  $A = X\Lambda X^{-1}$ , that is, the eigen-basis is not always orthogonal. The basis of singular vectors is always orthogonal.
3. In SVD we have two singular-spaces (right and left).
4. Computing the SVD of a matrix is more numerically stable.

## Rank-k approximation in the spectral norm

The following will claim that the best approximation to  $A$  by a rank deficient matrix is obtained by the top singular values and vectors of  $A$ . More accurately:

**Fact 0.1.** *Set*

$$A_k = \sum_{j=1}^k \sigma_j u_j v_j^T.$$

*Then,*

$$\min_{\substack{B \in \mathbb{R}^{m \times n} \\ \text{rank}(B) \leq k}} \|A - B\|_2 = \|A - A_k\|_2 = \sigma_{k+1}.$$

*Proof.*

$$\|A - A_k\| = \left\| \sum_{j=1}^r \sigma_j u_j v_j^T - \sum_{j=1}^k \sigma_j u_j v_j^T \right\| = \left\| \sum_{j=k+1}^r \sigma_j u_j v_j^T \right\| = \sigma_{k+1}$$

and thus  $\sigma_{k+1}$  is the largest singular value of  $A - A_k$ . Alternatively, look at  $U^T A_k V = \text{diag}(\sigma_1, \dots, \sigma_k, 0, \dots, 0)$ , which means that  $\text{rank}(A_k) = k$ , and that

$$\|A - A_k\|_2 = \|U^T(A - A_k)V\|_2 = \|\text{diag}(0, \dots, 0, \sigma_{k+1}, \dots, \sigma_r)\|_2 = \sigma_{k+1}.$$

Let  $B$  be an arbitrary matrix with  $\text{rank}(B_k) = k$ . Then, it has a null space of dimension  $n - k$ , that is,

$$\text{null}(B) = \text{span}(w_1, \dots, w_{n-k}).$$

A dimension argument shows that

$$\text{span}(w_1, \dots, w_{n-k}) \cap \text{span}(v_1, \dots, v_{k+1}) \neq \{0\}.$$

Let  $w$  be a unit vector from the intersection. Since

$$Aw = \sum_{j=1}^{k+1} \sigma_j (v_j^T w) u_j,$$

we have

$$\|A - B\|_2^2 \geq \|(A - B)w\|_2^2 = \|Aw\|_2^2 = \sum_{j=1}^{k+1} \sigma_j^2 |v_j^T w|^2 \geq \sigma_{k+1}^2 \sum_{j=1}^{k+1} |v_j^T w|^2 = \sigma_{k+1}^2,$$

since  $w \in \text{span}\{v_1, \dots, v_{n+1}\}$ , and the  $v_j$  are orthogonal.  $\square$

## Rank-k approximation in the Frobenius norm

The same theorem holds with the Frobenius norm.

**Theorem 0.1.** *Set*

$$A_k = \sum_{j=1}^k \sigma_j u_j v_j^T.$$

*Then,*

$$\min_{\substack{B \in \mathbb{R}^{m \times n} \\ \text{rank}(B) \leq k}} \|A - B\|_F = \|A - A_k\|_F = \sqrt{\sum_{i=k+1}^m \sigma_i^2}.$$

*Proof.* Suppose  $A = U\Sigma V^T$ . Then

$$\min_{\text{rank}(B) \leq k} \|A - B\|_F^2 = \min_{\text{rank}(B) \leq k} \|U\Sigma V^T - UU^T B V V^T\|_F^2 = \min_{\text{rank}(B) \leq k} \|\Sigma - U^T B V\|_F^2.$$

Now,

$$\|\Sigma - U^T B V\|_F^2 = \sum_{i=1}^n (\Sigma_{ii} - (U^T B V)_{ii})^2 + \text{off-diagonal terms}.$$

If  $B$  is the best approximation matrix and  $U^T B V$  is not diagonal, then write  $U^T B V = D + O$ , where  $D$  is diagonal and  $O$  contains the off-diagonal elements. Then the matrix  $B = U D V^T$  is a better approximation, which is a contradiction.

Thus,  $U^T B V$  must be diagonal. Hence,

$$\|\Sigma - D\|_F^2 = \sum_{i=1}^n (\sigma_i - d_i)^2 = \sum_{i=1}^k (\sigma_i - d_i)^2 + \sum_{i=k+1}^n \sigma_i^2,$$

and this is minimal when  $d_i = \sigma_i$ ,  $i = 1, \dots, k$ . The best approximating matrix is  $A_k = U D V^T$ , and the approximation error is  $\sqrt{\sum_{i=k+1}^n \sigma_i^2}$ .  $\square$

# Data mining applications of the SVD

## Linear regression in the least-squared loss

In Linear regression we aim to find the best linear approximation to a set of observed data. For the  $m$  data points  $\{x_1, \dots, x_m\}$ ,  $x_i \in \mathbb{R}^n$ , each receiving the value  $y_i$ , we look for the weight vector  $w$  that minimizes:

$$\sum_{i=1}^n (x_i^T w - y_i)^2 = \|Aw - y\|_2^2$$

Where  $A$  is a matrix that holds the data points as rows  $A_i = x_i^T$ .

**Proposition 0.1.** *The vector  $w$  that minimizes  $\|Aw - y\|_2^2$  is  $w = A^\dagger y = V\Sigma^\dagger U^T y$  for  $A = U\Sigma V^T$  and  $\Sigma_{ii}^\dagger = 1/\Sigma_{ii}$  if  $\Sigma_{ii} > 0$  and 0 else.*

Let us define  $U_\parallel$  and  $U_\perp$  as the parts of  $U$  corresponding to positive and zero singular values of  $A$  respectively. Also let  $y_\parallel = 0$  and  $y_\perp$  be two vectors such that  $y = y_\parallel + y_\perp$  and  $U_\parallel y_\perp = 0$  and  $U_\perp y_\parallel = 0$ .

Since  $y_\parallel$  and  $y_\perp$  are orthogonal we have that  $\|Aw - y\|_2^2 = \|Aw - y_\parallel - y_\perp\|_2^2 = \|Aw - y_\parallel\|_2^2 + \|y_\perp\|_2^2$ . Now, since  $y_\parallel$  is in the range of  $A$  there is a solution  $w$  for which  $\|Aw - y_\parallel\|_2^2 = 0$ . Namely,  $w = A^\dagger y = V\Sigma^\dagger U^T y$  for  $A = U\Sigma V^T$ . This is because  $U\Sigma V^T V\Sigma^\dagger U^T y = y_\parallel$ . Moreover, we get that the minimal cost is exactly  $\|y_\perp\|_2^2$  which is independent of  $w$ .

## PCA, Optimal squared loss dimension reduction

Given a set of  $n$  vectors  $x_1, \dots, x_n$  in  $\mathbb{R}^m$ . We look for a rank  $k$  projection matrix  $P \in \mathbb{R}^{m \times m}$  that minimizes:

$$\sum_{i=1}^n \|Px_i - x_i\|_2^2$$

If we denote by  $A$  the matrix whose  $i$ 'th column is  $x_i$  then this is equivalent to minimizing  $\|PA - A\|_F^2$ . Since the best possible rank  $k$  approximation to the matrix  $A$  is  $A_k = \sum_{i=1}^k \sigma_i u_i v_i^T$  the best possible solution would be a projection  $P$  for which  $PA = A_k$ . This is achieved by  $P = U_k U_k^T$  where  $U_k$  is the matrix corresponding to the first  $k$  left singular vectors of  $A$ .

If we define  $y_i = U_k^T x_i$  we see that the values of  $y_i \in \mathbb{R}^k$  are optimally fitted to the set of points  $x_i$  in the sense that they minimize:

$$\min_{y_1, \dots, y_n} \min_{\Psi \in \mathbb{R}^{k \times m}} \sum_{i=1}^n \|\Psi y_i - x_i\|_2^2$$

The mapping of  $x_i \rightarrow U_k^T x_i = y_i$  thus reduces the dimension of any set of points  $x_1, \dots, x_n$  in  $\mathbb{R}^m$  to a set of points  $y_1, \dots, y_n$  in  $\mathbb{R}^k$  optimally in the squared loss sense. This is commonly referred to as Principal Component Analysis (PCA).

## Closest orthogonal matrix

The SVD also allows to find the orthogonal matrix that is closest to a given matrix. Again, suppose that  $A = U\Sigma V^T$  and  $W$  is an orthogonal matrix that minimizes  $\|A - W\|_F^2$  among all orthogonal matrices. Now,

$$\|U\Sigma V^T - W\|_F^2 = \|U\Sigma V^T - U U^T W V V^T\| = \|\Sigma - \tilde{W}\|,$$

where  $\tilde{W} = U^T W V$  is another orthogonal matrix. We need to find the orthogonal matrix  $\tilde{W}$  that is closest to  $\Sigma$ . Alternatively, we need to minimize  $\|\tilde{W}^T \Sigma - I\|_F^2$ .

If  $U$  is orthogonal and  $D$  is diagonal and positive, then

$$\begin{aligned}
\text{trace}(UD) &= \sum_{i,k} u_{ik} d_{ki} \leq \sum_i \left( \left( \sum_k u_{ik}^2 \right)^{1/2} \left( \sum_k d_{ki}^2 \right)^{1/2} \right) \\
&= \sum_i \left( \sum_k d_{ki}^2 \right)^{1/2} = \sum_i (d_{ii}^2)^{1/2} = \sum_i d_{ii} = \text{trace}(D).
\end{aligned} \tag{4}$$

Now

$$\begin{aligned}
\|\tilde{W}^T \Sigma - I\|_F^2 &= \text{trace} \left( (\tilde{W}^T \Sigma - I) (\tilde{W}^T \Sigma - I)^T \right) \\
&= \text{trace} \left( (\tilde{W}^T \Sigma - I) (\Sigma \tilde{W} - I) \right) \\
&= \text{trace} (\tilde{W}^T \Sigma^2 \tilde{W}) - \text{trace} (\tilde{W}^T \Sigma) - \text{trace} (\Sigma \tilde{W}) + n \\
&= \text{trace} \left( (\Sigma \tilde{W})^T (\Sigma \tilde{W}) \right) - 2 \text{trace} (\Sigma \tilde{W}) + n \\
&= \|\Sigma \tilde{W}\|_F^2 - 2 \text{trace} (\Sigma \tilde{W}) + n \\
&= \|\Sigma\|_F^2 - 2 \text{trace} (\Sigma \tilde{W}) + n.
\end{aligned}$$

Thus, we need to maximize  $\text{trace} (\Sigma \tilde{W})$ . But this is maximized by  $\tilde{W} = I$  by (4). Thus, the best approximating matrix is  $W = UV^T$ .

## Computing the SVD: The power method

We give a simple algorithm for computing the Singular Value Decomposition of a matrix  $A \in \mathbb{R}^{m \times n}$ . We start by computing the first singular value  $\sigma_1$  and left and right singular vectors  $u_1$  and  $v_1$  of  $A$ , for which  $\min_{i < j} \log(\sigma_i / \sigma_j) \geq \lambda$ :

1. Generate  $x_0$  such that  $x_0(i) \sim \mathcal{N}(0, 1)$ .
2.  $s \leftarrow \log(4 \log(2n/\delta) / \varepsilon \delta) / 2\lambda$
3. for  $i$  in  $[1, \dots, s]$ :
4.  $x_i \leftarrow A^T A x_{i-1}$
5.  $v_1 \leftarrow x_i / \|x_i\|$
6.  $\sigma_1 \leftarrow \|A v_1\|$
7.  $u_1 \leftarrow A v_1 / \sigma_1$
8. return  $(\sigma_1, u_1, v_1)$

Let us prove the correctness of this algorithm. First, write each vector  $x_i$  as a linear combination of the right singular values of  $A$  i.e.  $x_i = \sum_j \alpha_j^i v_j$ . From the fact that  $v_j$  are the eigenvectors of  $A^T A$  corresponding to eigenvalues  $\sigma_j^2$  we get that  $\alpha_j^i = \alpha_j^{i-1} \sigma_j^2$ . Thus,  $\alpha_j^s = \alpha_j^0 \sigma_j^{2s}$ . Looking at the ratio between the coefficients of  $v_1$  and  $v_i$  for  $x_s$  we get that:

$$\frac{|\langle x_s, v_1 \rangle|}{|\langle x_s, v_i \rangle|} = \frac{|\alpha_1^0|}{|\alpha_i^0|} \left( \frac{\sigma_1}{\sigma_i} \right)^{2s}$$

Demanding that the error in the estimation of  $\sigma_1$  is less than  $\varepsilon$  gives the requirement on  $s$ .

$$\frac{|\alpha_1^0|}{|\alpha_i^0|} \left( \frac{\sigma_1}{\sigma_i} \right)^{2s} \geq \frac{n}{\varepsilon} \quad (5)$$

$$s \geq \frac{\log(n|\alpha_i^0|/\varepsilon|\alpha_1^0|)}{2\log(\sigma_1/\sigma_i)} \quad (6)$$

From the two-stability of the gaussian distribution we have that  $\alpha_i^0 \sim \mathcal{N}(0, 1)$ . Therefore,  $\Pr[\alpha_i^0 > t] \leq e^{-t^2}$  which gives that with probability at least  $1 - \delta/2$  we have for all  $i$ ,  $|\alpha_i^0| \leq \sqrt{\log(2n/\delta)}$ . Also,  $\Pr[|\alpha_1^0| \leq \delta/4] \leq \delta/2$  (this is because  $\Pr[|z| < t] \leq \max_r \Psi_z(r) \cdot 2t$  for any distribution and the normal distribution function at zero takes its maximal value which is less than 2). Thus, with probability at least  $1 - \delta$  we have that for all  $i$ ,  $\frac{|\alpha_1^0|}{|\alpha_i^0|} \leq \frac{\sqrt{\log(2n/\delta)}}{\delta/4}$ . Combining all of the above we get that it is sufficient to set  $s = \log(4n \log(2n/\delta)/\varepsilon\delta)/2\lambda = O(\log(n/\varepsilon\delta)/\lambda)$  in order to get  $\varepsilon$  precision with probability at least  $1 - \delta$ .

We now describe how to extend this to a full SVD of  $A$ . Since we have computed  $(\sigma_1, u_1, v_1)$ , we can repeat this procedure for  $A - \sigma_1 u_1 v_1^T = \sum_{i=2}^n \sigma_i u_i v_i^T$ . The top singular value and vectors of which are  $(\sigma_2, u_2, v_2)$ . Thus, computing the rank- $k$  approximation of  $A$  requires  $O(mnks) = O(mnk \log(n/\varepsilon\delta)/\lambda)$  operations. This is because computing  $A^T A x$  requires  $O(mn)$  operations and for each of the first  $k$  singular values and vectors this is performed  $s$  times.

The main problem with this algorithm is that its running time is heavily influenced by the value of  $\lambda$ . Other variants of this algorithm are much less sensitive to the value of this parameter, but are out of the scope of this class.