

Lecture 1: Recap of basic probability

Lecturer: Edo Liberty

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1 Preliminaries

A variable X is a random variable if it assumes different values according to a probability distribution. For example, X can denote the outcome of a three sided die throw. The variable X takes the values $x = 1, 2, 3$ with equal probabilities.

The expectation of X is the sum over the possible values times the probability of the events.

$$\mathbb{E}[X] = \sum_{x=1}^3 x \Pr(X = x) = 1 \frac{1}{3} + 2 \frac{1}{3} + 3 \frac{1}{3} = 2 \quad (1)$$

Continuous random variable are described by their distribution function φ .

$$\Pr[Y \in [a, b]] = \int_a^b \varphi(t) dt.$$

For a function φ to be a valid distribution we must have:

$$\forall t, \varphi(t) \geq 0 \quad (2)$$

$$\int_a^b \varphi(t) dt \quad \text{is well defined for all } a \text{ and } b \quad (3)$$

$$\int_{-\infty}^{\infty} \varphi(t) dt = 1 \quad (4)$$

For example consider the continuous variable Y taking values in $[0, 1]$ uniformly. That means $\varphi(t) = 1$ if $t \in [0, 1]$ and zero else. This means that the probability of Y being in the interval $[t, t + dt]$ is exactly dt . And so the expectation of Y is:

$$\mathbb{E}[Y] = \int_{t=0}^1 t \varphi(t) dt = \int_{t=0}^1 t \cdot dt = \frac{1}{2} t^2 \Big|_0^1 = 1/2 \quad (5)$$

Remark 1.1. *Strictly speaking, distributions are not necessarily continuous or bounded functions. In fact, they can even not be a function at all. For example, the distribution of X above includes three Dirac- δ functions which are not valid functions.*

1.1 Dependence and Independence

A variable X is said to be *dependent* on Y if the distribution of X given Y is different than the distribution of X . For example. Assume the variable X takes the value 1 if Y takes a value of less than $1/3$ and the values 2 or 3 with equal probability otherwise ($1/2$ each). Clearly, the probability of X assuming each of its

values is still $1/3$. however, if we know that Y is 0.7234 the probability of X assuming the value 1 is zero. Let us calculate the expectation of X given Y as an exercise.

$$\mathbb{E}(X|Y) = \sum_{x=1}^3 x \Pr(X = x|Y \leq 1/3) = 1 \cdot 1 \quad (6)$$

$$\mathbb{E}(X|Y) = \sum_{x=1}^3 x \Pr(X = x|Y > 1/3) = 1 \cdot 0 + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{2} = 2.5 \quad (7)$$

$E(X|Y) = 1$ for $y \in [0, 1/3]$ and $E(X|Y) = 2.5$ for $y \in (1/3, 1]$.

Remember: $\mathbb{E}(X|Y)$ is a function of y !

Definition 1.1 (Independence). *Two variables are said to be Independent if:*

$$\forall y, \Pr[X = x|Y = y] = \Pr[X = x].$$

They are dependent otherwise.

Fact 1.1. *If two variables X and Y are Independent then so are $f(X)$ and $g(Y)$ for any functions f and g .*

Fact 1.2 (Linearity of expectation 1). *For any random variable and any constant α :*

$$\mathbb{E}[\alpha X] = \alpha \mathbb{E}[X] \quad (8)$$

Fact 1.3 (Linearity of expectation 2). *For any two random variables*

$$\mathbb{E}_{X,Y}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] \quad (9)$$

even if they are dependent.

Fact 1.4 (Multiplication of random variables). *For any two **independent** random variables*

$$\mathbb{E}_{X,Y}[XY] = \mathbb{E}[X]\mathbb{E}[Y] \quad (10)$$

This does not necessarily hold if they are dependent.

Definition 1.2 (Variance). *For a random variable X we have*

$$\text{Var}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \quad (11)$$

The standard deviation σ of X is defined to be $\sigma(X) \equiv \sqrt{\text{Var}[X]}$.

Definition 1.3 (Additivity of variances). *For any two **independent** variables X and Y we have*

$$\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y] \quad (12)$$

Fact 1.5 (Markov's inequality). *For any positive random variable X :*

$$\Pr(X > t) \leq \frac{\mathbb{E}[X]}{t} \quad (13)$$

Fact 1.6 (Chebyshev's inequality). *For any random variable X*

$$\Pr[|X - \mathbb{E}[X]| > t] \leq \frac{\sigma^2(X)}{t^2} \quad (14)$$