

## Lecture 5: Estimating Frequency Moments

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**Warning:** This note may contain typos and other inaccuracies which are usually discussed during class. Please do not cite this note as a reliable source. If you find mistakes, please inform me.

Assume we have a stream  $A$ , of length  $N$  which is composed of  $m$  different types of items  $a_1, \dots, a_m$  each of which repeats itself  $n_1, \dots, n_m$  times (in arbitrary order) We define the frequency moments  $f_k$  as:

$$f_k = \sum_{i=1}^m n_i^k$$

Our aim to to process the stream in one element at a time and attain an  $(\epsilon, \delta)$ -approximation. This means, that our estimate is up to multiplicative factor  $(1 \pm \epsilon)$  with probability at least  $1 - \delta$ . Note that  $f_0$  is the number of distinct elements in the stream  $m$  and that  $f_1$  is the number of elements  $N$ .  $f_2$  is also an important quantity which represents how "skewed" the distribution of the elements in stream is.

Let's first assume that we know  $N$  in advance. This is not necessary and we will fix it later. But for now, it makes our analysis simpler.

Let us first define a random variable  $X$ . We choose an index  $q \in [1, \dots, N]$  uniformly at random. Let  $a$  be the element in place  $q$  in the stream, i.e.  $a = A_q$ . Define by  $r$  the number of times  $a$  appears in the stream after location  $q$ , including. In other words  $r = |\{i | A_i = a, i \geq q\}|$ . We define  $X$ :

$$X = N(r^k - (r-1)^k)$$

We claim that  $E[X] = f_k$ . Let us define the variable  $e_{i,j}$  which indicates the event that the index  $q$  is such that  $A_q = a_i$  and  $a_i$  appears exactly  $j$  times after the location  $q$ . Note that the events  $e_{i,j}$  are disjoint and that if  $e_{i,j}$  happens then  $r$  takes the value  $j$ . Therefore,  $X = \sum_{i,j} e_{i,j} N(j^k - (j-1)^k)$ . Moreover,  $\Pr[e_{i,j}] = \frac{n_i}{N} \frac{1}{n_i} = \frac{1}{N}$  since the probability of choosing  $a_i$  is  $\frac{n_i}{N}$  and given that this happens the probability of each index (out of the locations of  $a_i$ ) is equal to  $\frac{1}{n_i}$ .

$$\begin{aligned} E[X] &= \sum_{i,j} E[e_{i,j} N(j^k - (j-1)^k)] \\ &= \sum_{i=1}^m \sum_{j=1}^{n_i} \Pr[e_{i,j}] N(j^k - (j-1)^k) \\ &= \sum_{i=1}^m \sum_{j=1}^{n_i} (j^k - (j-1)^k) \\ &= \sum_{i=1}^m n_i^k = f_k. \end{aligned}$$

It is somewhat complicated and tedious to compute the variance of  $X$ . Citing from the paper [1] we use the fact that

$$\text{Var}[X] \leq km^{1-1/k} f_k^2.$$

We define  $Y$  as the mean of  $s$  different copies of  $X$ ,  $Y = \frac{1}{s} \sum_{i=1}^s X_i$ . Clearly,  $E[Y] = E[X] = f_k$  and  $\text{Var}[Y] \leq \text{Var}[X]/s = km^{1-1/k} f_k^2 / s$ . Using Chebyshev's inequality we have that

$$\Pr[|Y - f_k| > \epsilon f_k] \leq \frac{\text{Var}[Y]}{\epsilon^2 f_k^2} \leq \frac{km^{1-1/k}}{\epsilon^2 s} \leq \delta.$$

where the last inequality holds if  $s \geq \frac{km^{1-1/k}}{\varepsilon^2 \delta}$ .

## Estimating $f_0$

This bound is not the most efficient algorithm for approximating the zero'th frequency moment (which is the number of distinct elements,  $m$ ). Here we will describe a more efficient algorithm which is a merging of ideas from [ ] and [ ].

First, assume a hash function  $h : a \rightarrow [0, 1]$  uniformly. Let us define a random variable  $X = \min_i h(a_i)$ . Intuitively,  $X$  should be roughly  $1/m$  and therefore  $1/X$  should be a fair estimate of  $m$ . This is almost true. In what comes next we make this into an exact statement.

Let us first compute the expectation of  $X$ . The distribution function  $f_X$  of the  $X$  is  $f_X(x) = m(1-x)^{m-1}$ . This is because, we have  $m$  different choices for the minimal element and for every value it takes,  $x$ , all the rest  $m-1$  values need to be higher than it (w.p  $(1-x)^{m-1}$ ). Therefore, u:

$$\begin{aligned} E[X] &= \int_0^1 xm(1-x)^{m-1}dx \\ &= \int_0^1 (1-y)my^{m-1}dy \\ &= \int_0^1 my^{m-1}dy - \int_0^1 my^m dy \\ &= 1 - \frac{m}{m+1} = \frac{1}{m+1} \end{aligned}$$

This is after the substitution  $y = 1 - x$ . We now compute the variance of  $X$ . For that we first compute  $E[X^2]$ .

$$\begin{aligned} E[X^2] &= \int_0^1 x^2 m(1-x)^{m-1}dx \\ &= \int_0^1 (1-y)^2 my^{m-1}dy \\ &= \int_0^1 my^{m-1}dy - \int_0^1 2my^m dy + \int_0^1 my^{m+1}dy \\ &= 1 - \frac{2m}{m+1} + \frac{m}{m+2} \leq \frac{3}{(m+1)^2} \end{aligned}$$

Thus, the standard deviation of  $\sigma(X)$  is in the same order of magnitude as its expectation  $E[X]$ . To reduce this ratio we again define  $Y = \frac{1}{s} \sum_{i=1}^s X_i$  for which  $E[Y] = \frac{1}{m+1}$ . and  $Var[Y] \leq \frac{2}{s(m+1)^2}$ .

Using Chebyshev's inequality we get that

$$\Pr[|Y - \frac{1}{m+1}| \geq \frac{\varepsilon/2}{m+1}] \leq \frac{8}{\varepsilon^2 s} \leq \delta$$

if  $s \geq \frac{8}{\varepsilon^2 \delta}$ . Therefore, multiplying this procedure  $\frac{8}{\varepsilon^2 \delta}$  different hash function and taking their mean minimal value guaranties that with probability at least  $1 - \delta$  we have  $\frac{1}{m+1}(1 - \varepsilon/2) \leq Y \leq \frac{1}{m+1}(1 + \varepsilon/2)$ . In other words:  $(m+1)\frac{1}{1+\varepsilon/2} \leq \frac{1}{Y} \leq (m+1)\frac{1}{1-\varepsilon/2}$ . But, since  $\frac{1}{1-\varepsilon/2} \leq 1 + \varepsilon$  and  $1 - \varepsilon \leq \frac{1}{1+\varepsilon/2}$  we get the desired results that  $(m+1)(1 - \varepsilon) \leq \frac{1}{Y} \leq (m+1)(1 + \varepsilon)$

## Estimating $f_2$

We will give here a better estimator of  $f_2$ . Assume a hash function  $h : a \rightarrow \{-1, 1\}$  with probability  $1/2$  each. Define  $Z = \sum_{i=1}^N h(A_i) = \sum_{i=1}^m n_i h(a_i)$ .

Consider the variable  $X = Z^2$ . As usual, we will begin with computing the expectation and variance of  $X$ .

$$\begin{aligned}
E[X] &= E[Z^2] = E\left[\sum_{i=1}^m n_i h(a_i)^2\right] \\
&= E\left[\left(\sum_{i=1}^m n_i h(a_i)\right)\left(\sum_{i'=1}^m n_{i'} h(a_{i'})\right)\right] \\
&= \sum_{i=1}^m \sum_{i'=1}^m n_i n_{i'} E[h(a_i)h(a_{i'})] \\
&= \sum_{i=1}^m n_i^2 = f_2
\end{aligned}$$

Similarly,

$$\begin{aligned}
E[X^2] &= E[Z^4] = \sum_{i=1}^m n_i^4 + 6 \sum_{1 \leq i < i' \leq m} n_i^2 n_{i'}^2 \\
\text{Var}[X] &= E[X^2] - E^2[X] \leq 4 \sum_{1 \leq i < i' \leq m} n_i^2 n_{i'}^2 \leq 2f_2
\end{aligned}$$

Finally, defining  $Y = \frac{1}{s} \sum_{i=1}^s X_i$ , where each  $X_i$  is an independent copy of  $X$  we have that:

$$\Pr[|Y - f_2| \geq \varepsilon f_2] \leq \delta$$

if  $s \geq \frac{2}{\varepsilon^2 \delta}$ .

## Connection to random projections (next class)

Consider the  $s$  hash functions  $h_i : a \rightarrow \{-1, 1\}$  we used in estimating the second frequency moment. Consider the matrix  $H \in \mathbb{R}^{s \times m}$  such that  $H(i, j) = h_i(j)$ . Also, consider representing each input element  $a_i$  by  $\vec{a}_i$ , the  $i$ 'th standard basis vector in  $\mathbb{R}^m$  (the vector whose  $i$ 'th entry is equal to 1 and the rest are zero). Analogously,  $\vec{A}_i$  is the vector representing the  $i$ 'th element in the stream. Remember that our estimate  $Y$  of  $f_2$  was  $\frac{1}{s} \sum_{i=1}^s Z_i^2 = \|\frac{1}{\sqrt{s}} \vec{Z}\|^2$ . Moreover, from the definition of  $\vec{Z}$ ,  $H$ , and  $\vec{A}_i$  we have that  $\vec{Z} = \sum_{i=1}^N H \vec{A}_i = H \sum_{i=1}^N \vec{A}_i = \frac{1}{\sqrt{s}} H \vec{A}$ . Here,  $\vec{A} = \sum_{i=1}^N \vec{A}_i = [n_1, n_2, \dots, n_m]$ . Note however, that  $f_2 = \|\vec{A}\|^2$  by definition of the second frequency moment. We get that for any stream and any element frequencies  $\|\frac{1}{\sqrt{s}} H \vec{A}\|^2 \approx_{(\varepsilon, \delta)} \|\vec{A}\|^2$ . In other words, multiplying any vector  $\vec{A}$  by the matrix  $\frac{1}{\sqrt{s}} H$  is very likely to preserve its norm. We will see that this phenomenon is in fact more overreaching and has some serious consequences on point ensembles in high dimensional euclidian spaces.