

## Lecture 1: Preliminaries

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## 1 Preliminaries

A variable  $X$  is a random variable if it assumes different values according to probability distribution. For example,  $X$  can denote the outcome of a three sided die throw and  $X$  taking the values  $x = 1, 2, 3$  with equal probabilities. The expectation of  $X$  is the sum over the possible values times the probability of the events.

$$E[X] = \sum_{x=1}^3 x \Pr(X = x) = 1 \frac{1}{3} + 2 \frac{1}{3} + 3 \frac{1}{3} = 2 \quad (1)$$

Another example is a continuous variable  $Y$  taking the values  $[0, 1]$  uniformly. Meaning that the probability of  $Y$  being in the interval  $[t, t + dt]$  is exactly  $dt$ . And so the expectation of  $Y$  is:

$$E[Y] = \int_{t=0}^1 dt \cdot t = \frac{1}{2} t^2 \Big|_0^1 = 1/2 \quad (2)$$

### 1.1 Dependence and Independence

A variable  $X$  is said to be *dependent* on  $Y$  if given the value of  $Y$  changes the probability distribution of  $X$ . For example. Assume the variable  $X$  takes the value 1 if  $Y$  takes a value of less than  $1/3$  and the values 2 or 3 with equal probability otherwise ( $1/2$  each).

Clearly, the probability of  $X$  assuming each of its values is still  $1/3$ . however, if we know that  $Y$  is  $0.7234$  the probability of  $X$  assuming the value 1 is zero.

This is denoted by  $E(X|Y)$  (read: expectation of  $X$  given  $Y$ ).

$$E(X|Y) = \sum_{x=1}^3 x \Pr(X = x|Y \leq 1/3) = 1 \cdot 1 \quad (3)$$

$$E(X|Y) = \sum_{x=1}^3 x \Pr(X = x|Y > 1/3) = 1 \cdot 0 + 2 \frac{1}{2} + 3 \frac{1}{2} = 2.5 \quad (4)$$

Note that  $E(X|Y)$  is a function of  $y$ !!  $E(X|Y) = 1$  for  $y \in [0, 1/3]$  and  $E(X|Y) = 2.5$  for  $y \in (1/3, 1]$ .

**Definition 1.1.** Two variables are said to be *Independent* if:

$$\forall y, \quad E[X|Y = y] = E[X].$$

They are dependent otherwise.

**Fact 1.1.** For any two random variables (even if they are dependent):

$$E_Y E_X [X|Y] = E[X] \quad (5)$$

Checking this for our example:

$$E_Y E_X[X|Y] = \int_{y=0}^1 E_X[X|Y=y] dy = \int_{y=0}^{1/3} 1 dy + \int_{y=1/3}^1 2.5 dy = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 2.5 = 2 \quad (6)$$

**Fact 1.2** (Linearity of expectation 1). *For any random variable and any constant  $\alpha$ :*

$$E[\alpha X] = \alpha E[X]$$

**Fact 1.3** (Linearity of expectation 2). *For any random variable and any constant  $\alpha$ :  $E[\alpha X] = \alpha E[X]$  For any two random variables (even if they are dependent):*

$$E_{X,Y}[X + Y] = E[X] + E[Y] \quad (7)$$

Given the previous fact we can convince ourselves that this is true.  $E_{X,Y}[x + y] = E_Y E_X[X + Y] = E_Y[X|Y] + E_Y[Y] = E_X[X] + E_Y[Y]$ .

**Fact 1.4** (Markov's inequality). *For any positive random variable  $X$ :*

$$\Pr(X > t) \leq \frac{E[X]}{t} \quad (8)$$

**Definition 1.2.** *The variance of a random variable  $x$  is:*

$$\sigma^2[X] = E[(X - E[X])^2] = E[X^2] - E[X]^2 \quad (9)$$

**Fact 1.5** (Chebyshev's inequality). *For any random variable  $X$*

$$\Pr[|X - E[X]| > t] \leq \frac{\sigma^2(X)}{t^2} \quad (10)$$