

Lecture 2: Probabilistic Inequalities

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Fact 0.1 (Markov's inequality). For any positive random variable X :

$$\Pr(X > t) \leq \frac{E[X]}{t} \quad (1)$$

Fact 0.2 (Chebyshev's inequality). For any random variable X

$$\Pr[|X - E[X]| > t] \leq \frac{\sigma^2(X)}{t^2} \quad (2)$$

Theorem 0.1 (Chernoff's bound). Let X_i be a set of **independent** random variables such that $\mathbb{E}[X_i] = 0$ and $|X_i| \leq 1$ almost surely. Also define $\sigma_i^2 = \mathbb{E}[X_i^2]$ and $\sigma^2 = \sum_i \sigma_i^2$. Then:

$$\Pr\left[\sum_i X_i \geq t\right] \leq \max(e^{-t^2/4\sigma^2}, e^{-t/2})$$

Proof.

$$\Pr\left[\sum_i X_i \geq t\right] = \Pr\left[\lambda \sum_i X_i \geq \lambda t\right] \quad (\text{for } \lambda \geq 0) \quad (3)$$

$$= \Pr[e^{\lambda \sum_i X_i} \geq e^{\lambda t}] \quad (\text{because } e^x \text{ is monotone}) \quad (4)$$

$$\leq \mathbb{E}[e^{\lambda \sum_i X_i}] / e^{\lambda t} \quad (\text{by Markov}) \quad (5)$$

$$= \prod_i \mathbb{E}[e^{\lambda X_i}] / e^{\lambda t} \quad (6)$$

Now, for $x \in [0, 1]$ we have that $e^x \leq 1 + x + x^2$ so $\mathbb{E}[e^{\lambda X_i}] \leq 1 + \mathbb{E}[\lambda X_i] + \lambda^2 \mathbb{E}[X_i^2] \leq 1 + \lambda^2 \sigma_i^2$. Now, since $1 + x \leq e^x$ we have that $1 + \lambda^2 \sigma_i^2 \leq e^{\lambda^2 \sigma_i^2}$. Combining the above we have that $\mathbb{E}[e^{\lambda X_i}] \leq e^{\lambda^2 \sigma_i^2}$

$$\prod_i \mathbb{E}[e^{\lambda X_i}] / e^{\lambda t} \leq \prod_i \mathbb{E}[e^{\lambda^2 \sigma_i^2}] / e^{\lambda t} \quad (7)$$

$$= e^{\lambda^2 \sigma^2 - \lambda t} \quad (8)$$

Now, optimizing over $\lambda \in [0, 1]$ we get that $\lambda = \min(1, t/2\sigma^2)$ which completes the proof.

0.1 Other Useful forms

Chernoff's inequality: Let X_1, \dots, X_n be independent $\{0, 1\}$ valued random variables. Each X_i takes the value 1 with probability p_i and 0 else. Let $X = \sum_{i=1}^n X_i$ and let $\mu = E[X] = \sum_{i=1}^n p_i$. Then:

$$\Pr[X > (1 + \varepsilon)\mu] \leq e^{-\mu\varepsilon^2/4}$$

$$\Pr[X < (1 - \varepsilon)\mu] \leq e^{-\mu\varepsilon^2/2}$$

Or, using the union bound:

$$\Pr[|X - \mu| > \varepsilon\mu] \leq 2e^{-\mu\varepsilon^2/4}$$

□