

## Lecture 2: Sampling

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**Warning:** This note may contain typos and other inaccuracies which are usually discussed during class. Please do not cite this note as a reliable source. If you find mistakes, please inform me.

Sampling is an important concept and a great algorithmic tool if applied correctly. Today, we go over only a basic use which is subset size estimation. For example, say we have a large query log (a list of all queries made to a search engine) and say we wanted to measure how many queries contain the word “Adele”. This is easy, we just count. However, we can also take roughly  $10^{-6}$  fraction of the queries and count only them. Then, multiplying our resulting count by  $10^6$  should give us a good approximate answer. We will also see that we can ‘count’ many different subsets of queries on the same sample. To make these claims accurate we learn about:

- The Chernoff bound (which we will also prove)
- The union bound
- Reservoir sampling (and Fisher Yates shuffling)

## 1 Sampling

**Problem:** Given a large set of elements  $U$ , ( $|U| = N$ ) select a subset of elements  $\hat{U}$  ( $|\hat{U}| \approx n$ ) such that from  $\hat{U}$  the size of any subset  $S \subset U$  can be estimated.

**Approach:** Pick each element from  $U$  independently into our set  $\hat{U}$  with probability  $p = n/N$ . Let the variable  $X_i$  be 1 if element  $i$  is picked and 0 else. The number of picked elements is  $\sum_{i=1}^N X_i$  and its expectation is

$$E\left[\sum_{i=1}^N X_i\right] = \sum_{i=1}^N E[X_i] = \sum_{i=1}^N \frac{n}{N} = n$$

Let  $\hat{S}$  be the set of elements both in  $S$  and in  $\hat{U}$ , i.e.,  $\hat{S} = S \cap \hat{U}$ . Equivalently, let  $s_i$  be 1 if item  $i$  is in  $S$  and zero else. Let  $Z = \frac{N}{n} |\hat{S}|$  be our estimator of  $|S|$ :

$$E[Z] = \frac{N}{n} \sum_{i=1}^N E[X_i s_i] = \frac{N}{n} \sum_{j=1}^{|S|} \frac{n}{N} = |S|$$

The question is: how close is  $Z$  to  $|S|$ ? For this we need the Chernoff bound.

**Lemma 1.1** (Chernoff bound). *Let  $X_1, \dots, X_n$  be independent Bernoulli trials  $\Pr[X_i = 1] = p_i$ . And let  $X = \sum_{i=1}^n X_i$  and  $\mu = E[X] = \sum_{i=1}^n p_i$ .*

$$\Pr[X > (1 + \varepsilon)\mu] \leq e^{-\mu\varepsilon^2/4} \tag{1}$$

$$\Pr[X < (1 - \varepsilon)\mu] \leq e^{-\mu\varepsilon^2/2} \tag{2}$$

We will prove this soon but let's first use it to solve our problem.

In our case, we go over the elements of  $S$  and count how many of them were sampled into  $\hat{U}$ . Since each element is taken independently with probability  $p = n/N$  we have that  $\mu = E[X] = |S|n/N$ . Substituting into the Chernoff bound we get:

$$\Pr[X > (1 + \varepsilon)|S|n/N] \leq e^{-|S|n\varepsilon^2/4N} \quad (3)$$

$$\Pr[X < (1 - \varepsilon)|S|n/N] \leq e^{-|S|n\varepsilon^2/4N} \quad (4)$$

Or, using the union bound (below) and substituting  $Z = XN/n$ :

$$\Pr[|Z - |S|| > \varepsilon|S|] \leq 2e^{-|S|n\varepsilon^2/4N}$$

By demanding that  $e^{-|S|n\varepsilon^2/4N} \leq \delta$  (the failure probability be less than  $\delta$ ) we get:

$$n \geq \frac{4}{\varepsilon^2} \frac{N}{|S|} \log\left(\frac{2}{\delta}\right)$$

For example, if  $|S|$  is the size of  $10^{-5}N$  and we want to have a 10% accuracy with probability at least 0.99, we must keep a sample of roughly  $10^8$  elements, regardless of  $N$ . That might not sound like a small number but consider that fact that this is less than the number of search queries to some search engines in 1 hour (according to published reports).

## 2 The union bound

**Lemma 2.1.** *For any set of  $m$  events  $A_1, \dots, A_m$ :*

$$\Pr[\cup_{i=1}^m A_i] \leq \sum_{i=1}^m \Pr A_i.$$

In words, the probability that one or more events happen is at most the sum of the individual event probabilities.

This simple notion is going to become very handy. Given the above setup, assume we want to estimate size of  $m$  different subsets,  $S_1, \dots, S_m$ . Also, we are not willing to compromise on the quality. This means that we demand:

$$\forall i \quad |Z^i - |S^i|| \leq \varepsilon|S^i|$$

Let  $f_i$  be the event that we fail in estimating the size of  $S^i$ , i.e.,  $|Z^i - |S^i|| > \varepsilon|S^i|$ .

$$\Pr[\cup_{i=1}^m f_i] \leq \sum_{i=1}^m \Pr[f_i] \leq m \max(\Pr[f_i])$$

Or, the probability of failure in at least one event out of  $m$  is at most  $m$  times the maximal probability of failure. By demanding that  $m \max(\Pr[f_i]) \leq \delta$  we succeed in all events with probability at least  $1 - \delta$ .

$$\max_i 2e^{-|S_i|n\varepsilon^2/4N} \leq \delta/m \rightarrow n \geq \frac{4}{\varepsilon^2} \frac{N}{\min |S_i|} \log\left(\frac{2m}{\delta}\right)$$

Note that we only “pay” a logarithmic factor in  $m$ . Thus, sampling only a slightly larger set guarantees us accuracy for a very large number of different sets simultaneously.

### 3 Uniformly Sampling from a stream

When we are given a stream of elements, we cannot sample items with probability  $n/N$  because we do not know  $N$ , the length of the stream. However, we can sample item  $i$  with probability  $p_i = \min(1, n/i)$ . Since at all points  $i \leq N$  we are, in fact, oversampling. This, however is trivial to correct for. After the stream is consumed, for every element  $i$  that we picked we keep it with probability  $p_i/N$ . Thought exercise 1: show that the total amount of space used by the algorithm is not much more than  $n$  with high probability. Thought exercise 2: can you think of a way to modify this sampling technique such that it avoids the extra space? Can you achieve this without an increase in processing time per item?

### 4 Fisher Yates shuffling

Suppose you need to shuffle a very long stream of elements. How would you go about doing that? The algorithm goes as follows: go over the elements one by one. In step  $i$  generate a random number  $j$  between 0 and  $i$  and switch elements  $a_i$  and  $a_j$  (if  $i = j$  then  $a_i$  remains in place). To prove this gives a truly uniform shuffle of the vector, we prove that each element is placed in any location with probability exactly  $1/n$ . We prove this by recursion. Assume that after  $k$  steps all the elements  $\{a_1, \dots, a_k\}$  are uniformly shuffled, i.e.  $\Pr[a_i \text{ is in position } j] = 1/k$  for all  $i, j \leq k$ . Now, advance one step in the algorithm. Element  $a_i$  is in position  $j \leq k$  only if it was there in the last step of the algorithm (w.p.  $1/k$ ) and if it was not swapped with  $a_{k+1}$ , w.p.  $k/(k+1)$ . Multiplying the two gives  $1/(k+1)$ . The element occupying the last position and the position of  $a_{k+1}$  are trivial. This completes the proof by induction.

### 5 Reservoir Sampling

In reservoir sampling, we want to keep exactly  $n$  elements out of a stream of  $N$  uniformly at random. The trivial thing to do is to shuffle the stream and take the first  $n$  elements. This is clearly correct but can we do any better?

We can simulate the Fisher-Yates algorithm and discard any element which is out of the range of  $[0, \dots, n-1]$ . In other words, do the following: First, take the first  $n$  elements in the stream. For element  $a_i$  in the stream, generate a random number  $r$  between 0 and  $i$ . If  $r \geq n$  discard the element. If  $r < n$ , discard element  $r$  from your collection and insert element  $a_i$ . Question for thought, how come we can simply discard elements? Can we be sure Fisher Yates will never be cycled them back into the sample?