

## Lecture 3: Assignment 1 answers

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**Warning:** This note may contain typos and other inaccuracies which are usually discussed during class. Please do not cite this note as a reliable source. If you find mistakes, please inform me.

## 1 Approximating the size of a tree

### setup

In this question we will try to approximate the number of leaves in a tree. A binary tree is a graph consisting of internal nodes and  $n$  leaves. Each internal node,  $u$ , has two children. A left child  $l(u)$  and a right child  $r(u)$ . The only node which does not have a parent is the root of the tree  $u_{root}$ . For each node we also denote by  $d(u)$  its depth in the tree which is the distance from the root. For example  $d(u_{root}) = 0$  and  $d(r(u_{root})) = 1$ .

We define a random walk on a tree as the process of starting at the root and then randomly moving to one of the children until we hit a leaf. More precisely:

1.  $u \leftarrow u_{root}$
2. while  $u$  is an internal node
3.     w.p.  $1/2$
4.      $u \leftarrow l(u)$
5.     otherwise
6.      $u \leftarrow r(u)$
7. return  $u$

### questions

1. Let the leaf  $u$  be at depth  $d(u)$ . Calculate the probability,  $p(u)$ , that the random walk outputs  $u$ ?
2. Let  $x$  be the output leaf of a random walk and let  $f(x) = 2^{d(x)}$  be a function defined on the leaves. Compute the value of:

$$E_{x \sim w}[f(x)]$$

where  $x \sim w$  denotes that  $x$  is chosen according to the distribution on the leaves generated by the random walk.

3. We say that a tree is  $c$ -balanced if  $d(u) \leq \log_2 n + c$  for all leaves in the tree. Show that for a  $c$ -balanced tree

$$\text{Var}_{x \sim w}[f(x)] \leq 2^c n^2$$

4. Let  $Y = \frac{1}{s} \sum_{i=1}^s f(x_i)$  where  $x_i$  are output nodes of  $s$  independent random walks on the tree. Compute  $E[Y]$  and show that  $\text{Var}[Y] \leq 2^c n^2 / s$ .

5. Use Chebyshev's inequality to find a value for  $s$  such that for two constants  $\varepsilon \in [0, 1]$  and  $\delta \in [0, 1]$ :

$$\Pr[|Y - n| > \varepsilon n] < \delta.$$

$s$  should be a function of  $c$ ,  $\varepsilon$  and  $\delta$ .

## 2 Answers

- 1.
- 2.
- 3.
- 4.
- 5.

## 3 Approximate histograms

### setup

We are given a stream of elements  $x_1, \dots, x_N$  where  $x_i \in \{a_1, \dots, a_n\}$ . Let  $n_i$  denote the number of times element  $a_i$  appeared in the stream, i.e.,  $n_i = |\{j | x_j = a_i\}|$ . Our goal is to estimate  $n_i$  for all frequent elements. Let the sub stream  $y$  include every element in the stream  $x$  with probability  $p$ . let  $\hat{n}_i = |\{j | y_j = a_i\}|$  be the number of times  $a_i$  appears in  $y$ .

### questions

1. Let  $z_i = \hat{n}_i/p$ , compute  $\mathbb{E}[z_i]$
2. Assume  $a_1$  is such that  $n_1 \geq \theta N$  for some fixed  $\theta$ . Compute a value for  $p$  (as low as possible) which guaranties that  $n_1(1 + \varepsilon) \geq z_1 \geq n_1(1 - \varepsilon)$  w.p. at least  $1/2$ .
3. Assume  $a_1$  is such that  $n_1 < \theta N(1 - 2\varepsilon)$  for some fixed  $\theta$ . Compute a value for  $p$  (as low as possible) which gives that  $z_1 \leq \theta N(1 - \varepsilon)$  w.p. at least  $1/2$ .
4. Use the union bound to specify a value for  $p$  which guaranties that for every  $i$ , if  $n_i \geq \theta N$  then  $n_i(1 + \varepsilon) \geq z_i \geq n_i(1 - \varepsilon)$  and if  $n_i < \theta N(1 - 2\varepsilon)$  then  $z_i \leq \theta N(1 - \varepsilon)$  with probability at least  $1 - \delta$ .
5. Compare this result with the algorithm described in class for approximately counting frequent items in streams, which is better under what circumstances?

## 4 Answers

- 1.
- 2.
- 3.
- 4.
- 5.