0368-3248-01-Algorithms in Data Mining **Fall 2011**

Lecture 13: Algorithms In Data Mining - Exam Answers

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Warning: *This note may contain typos and other inaccuracies which are usually discussed during class. Please do not cite this note as a reliable source. If you find mistakes, please inform me.*

General Info

1. Solve 3 out of 4 questions.
2. Each correct answer is worth 33.3 points.
3. If you have solved more than three questions, please indicate which three you would like to be checked.
4. The exam's duration is 3 hours. If you need more time please ask the attending professor.
5. Good luck!

Useful facts

1. For any vector $x \in \mathbb{R}^d$ we define the p-norm of x as follows:

$$||x||_p = [\sum_{i=1}^{d} (x(i))^p]^{1/p}$$

2. **Markov's inequality:** For any *non-negative* random variable X:

$$\Pr[X > t] \leq E[X]/t.$$

3. **Chebyshev's inequality:** For any random variable X:

$$\Pr[|X - E[X]| > t] \leq \mathrm{Var}[X]/t^2.$$

4. **Chernoff's inequality:** Let $x_1, \ldots, x_n$ be independent $\{0,1\}$ valued random variables. Each x_i takes the value 1 with probability p_i and 0 else. Let $X = \sum_{i=1}^{n} x_i$ and let $\mu = E[X] = \sum_{i=1}^{n} p_i$. Then:

$$\begin{aligned} \Pr[X > (1+\varepsilon)\mu] &\leq e^{-\mu\varepsilon^2/4} \\ \Pr[X < (1-\varepsilon)\mu] &\leq e^{-\mu\varepsilon^2/2} \end{aligned}$$

   Or in a another convenient form:

$$\Pr[|X - \mu| > \varepsilon\mu] \leq 2e^{-\mu\varepsilon^2/4}$$

5. **Hoeffding's inequality:** Let $x_1, \ldots, x_n$ be independent random variables taking values in $\{+1, -1\}$ each with probability $1/2$, then:

$$\Pr[|\sum_{i=1}^{n} x_i a_i| > t] \leq 2e^{-\frac{t^2}{\sum_{i=1}^{n} a_i^2}}.$$

6. For any $x \geq 2$ we have:

$$e^{-1} \geq (1 - \frac{1}{x})^x \geq \frac{2}{3}e^{-1}$$

7. For convenience:

$$\frac{3}{5} \leq 1 - e^{-1} \approx 0.632 \leq \frac{2}{3} \quad \text{and} \quad \frac{3}{4} \leq 1 - \frac{2}{3}e^{-1} \approx 0.754 \leq \frac{4}{5}$$

1 Probabilistic inequalities

setup

In this question you will be asked to derive the three most used probabilistic inequalities for a specific random variable. Let $x_1, \ldots, x_n$ be independent $\{-1, 1\}$ valued random variables. Each x_i takes the value 1 with probability $1/2$ and -1 else. Let $X = \sum_{i=1}^{n} x_i$.

questions

1. Let the random variable Y be defined as $Y = |X|$. Prove that Markov's inequality holds for Y. Hint: note that Y takes integer values. Also, there is no need to compute $\Pr[Y = i]$.

2. Prove Chebyshev's inequality for the above random variable X. You can use the fact that Markov's inequality holds for any positive variable regardless of your success (or lack of if) in the previous question. Hint: $\mathrm{Var}[X] = E[(X - E[X])^2]$.

3. Argue that
$$\Pr[X > a] = \Pr[\Pi_{i=1}^{n} e^{\lambda x_i} > e^{\lambda a}] \leq \frac{E[\Pi_{i=1}^{n} e^{\lambda x_i}]}{e^{\lambda a}}$$
for any $\lambda \in [0, 1]$. Explain each transition.

4. Argue that:
$$\frac{E[\Pi_{i=1}^{n} e^{\lambda x_i}]}{e^{\lambda a}} = \frac{\Pi_{i=1}^{n} E[e^{\lambda x_i}]}{e^{\lambda a}} = \frac{(E[e^{\lambda x_1}])^n}{e^{\lambda a}}$$
What properties of the random variables x_i did you use in each transition?

5. Conclude that $\Pr[X > a] \leq e^{-\frac{a^2}{2n}}$ by showing that:
$$\exists \ \lambda \in [0, 1] \ \ s.t. \ \ \frac{(E[e^{\lambda x_1}])^n}{e^{\lambda a}} \leq e^{-\frac{a^2}{2n}}$$
Hint: For the hyperbolic cosine function we have $\cosh(x) = \frac{1}{2}(e^x + e^{-x}) \leq e^{x^2/2}$ for $x \in [0, 1]$.

answers

1.

$$\begin{aligned} E[Y] &= \sum_{i=0}^{n} \Pr[Y=i]\cdot i \\ &= \sum_{i=0}^{t} \Pr[Y=i]\cdot i + \sum_{i=t+1}^{n} \Pr[Y=i]\cdot i \\ &\geq \sum_{i=t+1}^{n} \Pr[Y=i]\cdot i \\ &\geq \sum_{i=t+1}^{n} \Pr[Y=i]\cdot t \\ &= t\cdot \Pr[Y>t] \end{aligned}$$

Therefore, $E[Y] \geq t \cdot \Pr[Y > t]$ which is Markov's inequality.

2. This is identical to the general proof of Chebyshev's inequality. We define $Z = (X - E[X])^2$. Since Z is positive we can use Markov's inequality for it and get:

$$\Pr[|X - E[X]| > t] = \Pr[Z > t^2] \leq \frac{E[Z]}{t^2} = \frac{\mathrm{Var}[X]}{t^2}$$

Here we used that $E[Z] = E[(X - E[X])^2] = \mathrm{Var}[X]$.

3. First transition:

$$\Pr[X > a] = \Pr[\lambda X > \lambda a] = \Pr[e^{\lambda X} > e^{\lambda a}] = \Pr[e^{\lambda \sum x_i} > e^{\lambda a}] = \Pr[\Pi_{i=1}^{n} e^{\lambda x_i} > e^{\lambda a}]$$

These hold due to the monotonicity of multiplication by a positive constant and exponentiation. Now, using Markov's inequality on the last inequality we get:

$$\Pr[\Pi_{i=1}^{n} e^{\lambda x_i} > e^{\lambda a}] \leq \frac{E[\Pi_{i=1}^{n} e^{\lambda x_i}]}{e^{\lambda a}}$$

4. The first transition is true due to the independence of the variables x_i. This means that $e^{\lambda x_i}$ are independent. The second transition is due to all expectations of $e^{\lambda x_i}$ being equal which stems from x_i being identically distributed.

5. First, we compute the expectation of $e^{\lambda x_i}$

$$E[e^{\lambda x_i}] = \frac{1}{2}e^{\lambda} + \frac{1}{2}e^{-\lambda} = \cosh(\lambda) \leq e^{\lambda^2/2}$$

From the above we have that $\Pr[X > a] \leq e^{n\lambda^2/2 - \lambda a}$. Setting $\lambda = a/n$ we get $e^{n\lambda^2/2 - \lambda a} = e^{-\frac{a^2}{2n}}$ which concludes the proof.

2 Approximating the size of a graph

setup

In this question we will try to approximate the size of a graph. A graph $G(V, E)$ is a set of nodes $|V| = n$ and a set of edges $|E| = m$. Each edge $e \in V \times V$ is a set of two nodes which support it. We assume the graph is simple which means there are no duplicate edges and no self loops (i.e. an edge $e = (u, u)$). The degree of a node, $\deg(u)$, is the number of edges which it supports. More formally $\deg(u) = |\{e \in E | u \in e\}|$. The degree of each node in the graph is at least 1. The question refers to the following sampling procedure:

1. $e = (u, v) \leftarrow$ an edge uniformly at random from E.
2. with probability 1/2
3. return u
4. else
5. return v

Throughout this question we assume that *i*) we can sample edges uniformly from the graph *ii*) that the value of m in known *iii*) that given a node u we can compute $\deg(u)$. The value of n, however, is unknown.

questions

1. Let $p(u)$ denote the probability that the sampling procedure returns a specific node, u. Compute $p(u)$ as a function of $\deg(u)$ and m. (Note: $\sum_{u \in V} \deg(u) = 2m$)
2. Let $f(u) = \frac{2m}{\deg(u)}$. Compute:
$$E_{x \sim smp}[f(x)]$$
where $x \sim smp$ denotes that x is chosen according to the distribution on the nodes generated by the above sampling procedure.
3. We say that a graph is d-degree-bounded if $\max_{u \in V} \deg(u) \leq d$. Show that for a d-degree-bounded graph:
$$\mathrm{Var}_{x \sim smp}[f(x)] \leq dn^2$$
4. Let $Y = \frac{1}{s} \sum_{i=1}^{s} f(x_i)$ where x_i are nodes chosen independently from the graph according to the above sampling procedure. Compute $E[Y]$ **and** show that $\mathrm{Var}[Y] \leq dn^2/s$.
5. Use Chebyshev's inequality to find a value for s such that for for any d-degree-bounded graph and any two constants $\varepsilon \in [0, 1]$ and $\delta \in [0, 1]$:
$$\Pr[|Y - n| > \varepsilon n] < \delta.$$
s should be a function of d, ε and δ.

answers

1. A node is chosen only if an edge it is adjacent to is picked with probability and then it is the node picked between the two. The first event happens with probability $deg(u)/m$ since the edges a re chosen uniformly at random. The second event happens with probability $1/2$ independently of the first event. This gives $p(u) = \frac{deg(u)}{m}\frac{deg(2)}{2} = \frac{deg(u)}{2m}$.

2. By the definition to the expectation:

$$E_{x\sim smp}[f(x)] = \sum_{u\in V} p(u)f(u) = \sum_{u\in V} \frac{deg(u)}{2m}\frac{2m}{\deg(u)} = \sum_{u\in V} 1 = n$$

3. We say that a graph is d-degree-bounded if $\max_{u\in V}\deg(u) \leq d$. Show that for a d-degree-bounded graph:

$$\mathrm{Var}_{x\sim smp}[f(x)] \leq E_{x\sim smp}[f^2(x)] = \sum_{u\in V} \frac{deg(u)}{2m}(\frac{2m}{\deg(u)})^2 = \sum_{u\in V} \frac{2m}{\deg(u)}$$

   Since $deg(u) \geq 1$ then $\sum_{u\in V}\frac{2m}{\deg(u)} \leq \sum_{u\in V}\frac{2m}{1} = 2mn$. Also, since the graph is d-degree-bounded $2m = \sum_{u\in V} deg(u) \leq nd$ thus $2mn \leq dn^2$.

4. Y is the average of s independent copies of $f(x)$ and therefore, by linearity of the expectation, we have that $E[Y] = E[f] = n$. Moreover, Since the nodes x_i are chosen independently we have that $\mathrm{Var}[Y] = \frac{1}{s^2}\sum_{i=1}^{s}\mathrm{Var}[f(x_i)]$. Since $f(x_i)$ distribute identically and substituting $\mathrm{Var}(x) \leq dn^2$ we get $\frac{1}{s^2}\sum_{i=1}^{s}\mathrm{Var}[f(x_i)] \leq \frac{s}{s^2}dn^2 = dn^2/s$.

5. Since $E[Y] = n$ we get that the above holds if

$$\Pr[|Y - E[n]| > \varepsilon n] < \frac{\mathrm{Var}[Y]}{\varepsilon^2 n^2} \leq \frac{dn^2/s}{\varepsilon^2 n^2} = \frac{d}{s\varepsilon^2}$$

   The condition that $\frac{d}{s\varepsilon^2} \leq \delta$ holds for $s \geq \frac{d}{\delta\varepsilon^2}$

3 Approximate median

setup

Given a list A of n numbers $a_1, \ldots, a_n$, we define the rank of an element $r(a_i)$ as the number of elements which are smaller than it. For example, the smallest number has rank zero and the largest has rank $n-1$. Equal elements are ordered arbitrarily. The median of A is an element a such that $r(a) = n/2$ (rounded either up or down). An α-approximate-median is a number a such that:

$$n(1/2 - \alpha) \leq r(a) \leq n(1/2 + \alpha)$$

In this question we sample k elements uniformly at random *with replacement* from the list A. Let the samples be $\{x_1, \ldots, x_k\} = X$. You will be asked to show that the median of X is an α-approximate-median of A.

questions

1. What is the probability the a randomly chosen element x is such that:

$$r(x) > n(1/2 + \alpha)$$

2. Let us define $X_{>\alpha}$ as the set of samples whose rank is greater than $n(1/2 + \alpha)$. More precisely, $X_{>\alpha} = \{x_i \in X | r(x_i) > n(1/2 + \alpha)\}$. Similarly we define $X_{<\alpha} = \{x_i \in X | r(x_i) < n(1/2 - \alpha)\}$. Prove that if $|X_{>\alpha}| < k/2$ and $|X_{<\alpha}| < k/2$ then the median of X is an α-approximate-median of A.

3. Let $Z = |X_{>\alpha}|$. Find t for which:

$$\Pr[Z \geq k/2] = \Pr[Z \geq (1+t)E[Z]]$$

4. Bound from above the probability that $Z \geq k/2$ as tightly as possible. If you do so using a probabilistic inequality, justify your choice.

5. Compute the minimal value for k which will guarantee that $|X_{>\alpha}| < k/2$ **and** $|X_{<\alpha}| < k/2$ with probability at least $1 - \delta$.

answers

1. There are $n(1/2-\alpha)$ elements for which $r(x) > n(1/2+\alpha)$. Since the element is chosen uniformly, the probability of that happening is $(1/2-\alpha)$.

2. First we note that the median of X cannot be either in $X_{>\alpha}$ or in $X_{<\alpha}$. This is simply because each of them includes less than half of the elements in X. Moreover, by the definitions of $X_{>\alpha}$ and $X_{<\alpha}$ we have:
$$n(1/2-\alpha) \leq r(median(X)) \quad \text{and} \quad r(median(X)) \leq n(1/2+\alpha)$$
which means that $median(X)$ is an α-approximate-median of A.

3. Since the probability of a sample being in $X_{>\alpha}$ is exactly $1/2-\alpha$ and since we have k independent samples, $E[Z] = E[|X_{>\alpha}|] = k(1/2-\alpha)$. Solving for t we get
$$(1+t)E[Z] = k/2 \;\rightarrow\; (1+t)(1/2-\alpha) = 1/2 \;\rightarrow\; t = \frac{2\alpha}{1-2\alpha}$$

4. Since the value of Z is the sum of independent indicator variables we can apply Chernoff's inequality. Denoting $\mu = E[Z] = k(1/2-\alpha)$ and $t = \frac{2\alpha}{1-2\alpha}$ we have:
$$\Pr[Z \geq k/2] = \Pr[Z \geq (1+t)\mu] \leq e^{-\mu t^2/4}$$

5. Similarly to the the above we can argue that
$$\Pr[|X_{<\alpha}| \geq k/2] \leq e^{-\mu t^2/4}$$
From the union bound we have that the probability of the event that $|X_{<\alpha}| \geq k/2$ or that $|X_{>\alpha}| \geq k/2$ is at most the sum of their probabilities.
$$\Pr\left[|X_{<\alpha}| \geq k/2 \cup |X_{>\alpha}| \geq k/2\right] \leq \Pr\left[|X_{<\alpha}| \geq k/2\right] + \Pr[|X_{>\alpha}| \geq k/2] \leq 2e^{-\mu t^2/4}$$
Demanding that this failure probability is less than δ we guarantee success with probability at least $1-\delta$. Substituting $\mu = k(1/2-\alpha)$ and $t = \frac{2\alpha}{1-2\alpha}$ this is achieved for
$$2e^{-\mu t^2/4} < \delta \quad \rightarrow \quad k > \frac{4\log(2/\delta)(1/2-\alpha)}{\alpha^2}$$

4 Soulmate search

setup

In this question you will be asked to derive a search algorithm for a unique nearest neighbor given a local sensitive hash function family. We assume a universe of n objects $x_1, \ldots, x_n$ and a distance function d. For any pair of points $0 \leq d(x_i, x_j) \leq 1$. Moreover, each point x_i has exactly one soulmate point x_j such that $d(x_i, x_j) \leq r$, r is known constant. For all other points in the universe $d(x_i, x_{j'}) > 2r$. You are given a family H of hash functions such that $\Pr_{h \sim H}[h(x_i) = h(x_j)] = \frac{1}{1+d(x_i,x_j)}$ for any pair ($h \sim H$ means that h is chosen uniformly from H). We also define a bucketing hash function g which accepts an element x and returns a list of hash values.

$$g(x) = [h_1(x), \ldots, h_k(x)]$$

where each of the hush functions $h_1, \ldots, h_k$ was chosen uniformly and independently from the family H. We say that x_i and x_j are in love if $g(x_i) = g(x_j)$.

questions

1. What is the probability of two points, whose distance is $d(x_i, x_j)$, falling in love?
2. Compute a value for k for which the probability that x_i and x_j who are not soulmates ($d(x_i, x_j) \geq 2r$) of falling in love is at most $1/n$. Or, find k for which the following holds:

$$\Pr[g(x_i) = g(x_j) \mid d(x_i, x_j) \geq 2r] \leq 1/n$$

3. For this value of k, what is the probability that x_i falls in love with her soulmate? That means $\Pr[g(x_i) = g(x_j) \mid d(x_i, x_j) \leq r]$. Help: you can use the approximation $\frac{\log(1+r)}{\log(1+2r)} \approx \frac{1}{2}$.
4. We now create m independent copies of g, $g_1, \ldots, g_m$. We say that x_i finds x_j if $g_\ell(x_i) = g_\ell(x_j)$ for at least one function g_ℓ. Give a bound on the value of m which insures that **all** x_i find their soulmates with probability at least $1 - \delta$?
5. Given the above value for m, bound from above the **expected** number of points x_j that x_i fell in love with which were not her soulmates.

answers

1. Since each of the hash functions was chosen independently, we have that for each $\Pr[h_\ell(x_i) = h_\ell(x_j)] = \frac{1}{1+d(x_i,x_j)}$. For x_i and x_j to be in love this must hold for all k hash functions which happens with probability $\frac{1}{(1+d(x_i,x_j))^k}$.

2. Using the expression from the previous question for two points for which $d(x_i, x_j) \geq 2r$ we have:

$$\begin{aligned} \frac{1}{(1+d(x_i,x_j))^k} &\leq \frac{1}{(1+2r)^k} \leq 1/n \\ k &\geq \frac{\log(n)}{\log(1+2r)} \end{aligned}$$

3. Substituting $d(x_i, x_j) \leq r$ and $k = \frac{\log(n)}{\log(1+2r)}$ we get:

$$\begin{aligned} \Pr[g(x_i) = g(x_j)] &= \frac{1}{(1+d(x_i,x_j))^k} \geq \frac{1}{(1+r)^k} \\ &= (1+r)^{-\frac{\log(n)}{\log(1+2r)}} = n^{-\frac{\log(1+r)}{\log(1+2r)}} \approx n^{-1/2} \end{aligned}$$

   which uses the approximation $\frac{\log(1+r)}{\log(1+2r)} \approx \frac{1}{2}$.

4. For a point to fail in finding her soulmate, it must fail in falling in love m consecutive times. The probability of one point failing is therefore $(1-n^{-1/2})^m$. By the union bound, the probability of any of the n points failing is at most $n(1-n^{-1/2})^m$. demanding that this is bounded by δ yields:

$$n(1-n^{-1/2})^m \approx ne^{-m/\sqrt{n}} \leq \delta \qquad \rightarrow \qquad m \geq \sqrt{n}\log(n/\delta)$$

5. We can denote by $Z_{i,j,\ell}$ the event that point x_i and x_j are such that $d(x_i, x_j) > 2r$ and $g_\ell(x_i) = g_\ell(x_j)$. The number of points that x_i falls in love with is bounded by $\sum_{i=1}^{n} \cup_{\ell=1}^{k} Z_{i,j,\ell}$. Using the linearity of expectation and the fact that $\Pr[Z_{i,j,\ell} = 1] \leq 1/\text{n}$ we have that:

$$E[\sum_{i=1}^{n} \cup_{\ell=1}^{m} Z_{i,j,\ell}] \leq E[\sum_{i=1}^{n}\sum_{\ell=1}^{m} Z_{i,j,\ell}] = \sum_{i=1}^{n}\sum_{\ell=1}^{m} E[Z_{i,j,\ell}] \leq \sum_{i=1}^{n}\sum_{\ell=1}^{m} 1/n = m$$