Mark and Recapture

Edo Liberty
Algorithms in Data mining

Suppose you are a marine biologist (Although you prefer to pretend to be an architect), and suppose you are tasked with counting the number of individuals in a huge school of tune fish in the middle of the atlantic ocean. How would you go about doing that? One possible approach is called Mark and recapture. Start by catching k fish. Then, mark them somehow and release them. Then catch another group of k fish and count the number of fish that are already marked, Z. You can now guess that the number of fish in the entire school is roughly k^2/Z.

Mark and recapture

Given a set of n elements, sample k elements without replacement twice. Count the number of identical elements in both groups, Z. Define a random variable $z_{i,j}$ which indicates that element i in the first group is the same as element j in the second. The value of Z is therefore $Z = \sum_{i,j} z_{i,j}$. Let's compute the expectation of Z using linearity of expectation. Note that the $z_{i,j}$ variables are not independent!

$$E[Z] = E[\sum_{i,j} z_{i,j}] = \sum_{i,j} E[z_{i,j}] = \sum_{i,j} 1/n = k^2/n \tag{1}$$

Lets compute the standard deviation of Z. Recall:

$$\sigma^2[Z] = E[Z - E[Z]]^2 = E[Z^2] - E[Z]^2$$

We need the use the linearity of expectation again to compute $E[Z^2]$:

$$
\begin{aligned}
E[Z^2] &= E[(\sum_{i,j} z_{i,j})(\sum_{i',j'} z_{i',j'})] && (2)\\
&= \sum_{i=i',j=j'} E[z_{i,j} z_{i',j'}] && (3)\\
&\quad + \sum_{i=i',j\neq j'} E[z_{i,j} z_{i',j'}] + \sum_{i\neq i',j=j'} E[z_{i,j} z_{i',j'}] && (4)\\
&\quad + \sum_{i\neq i',j\neq j'} E[z_{i,j} z_{i',j'}] && (5)\\
&= \frac{k^2}{n} + 0 + 0 + \frac{k^2(k-1)^2}{n(n-1)} && (6)
\end{aligned}
$$

Using the expression for variance $\sigma^2[Z] = \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2$ we get:

$$
\begin{aligned}
\sigma^2[Z] &= \frac{k^2}{n} + \frac{k^2(k-1)^2}{n(n-1)} - \left(\frac{k^2}{n}\right)^2 && (7)\\
&\leq \frac{k^2}{n} \text{ (for } k \leq n) && (8)
\end{aligned}
$$

Now we invoke Chebyshev's inequality.

$$
\Pr[|Z - \frac{k^2}{n}| > t] \leq \frac{\sigma^2}{t^2} \leq \frac{k^2}{nt^2} \tag{9}
$$

Choosing $t = 10k/\sqrt{n}$ we get that with probability at least 0.99

$$
|Z - \frac{k^2}{n}| \leq 10k/\sqrt{n} \tag{10}
$$

Which gives:

$$
\begin{aligned}
n &\leq \frac{k^2}{Z}(1 + \frac{10\sqrt{n}}{k}) && (11)\\
n &\geq \frac{k^2}{Z}(1 - \frac{10\sqrt{n}}{k}) && (12)
\end{aligned}
$$

This gives us the following procedure: First, sample 2 groups of size $k \geq 50\sqrt{n}$ each. Count the number of collision Z. Estimate the size of the set as $n_{alg} = k^2/Z$. We are guarantied that with probability 0.99 our estimate is within 20% accuracy.

$$
\frac{5}{6}n \leq n_{alg} \leq \frac{5}{4}n \tag{13}
$$