

## Lecture 1: Mark and Recapture

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**Warning:** This note may contain typos and other inaccuracies which are usually discussed during class. Please do not cite this note as a reliable source. If you find mistakes, please inform me.

Suppose you are a marine biologist (Although you prefer to pretend to be an architect), and suppose you are tasked with counting the number of individuals in a huge school of fish in the middle of the Atlantic ocean. How would you go about doing that? One possible approach is called Mark and recapture. Start by catching  $k$  fish. Then, mark them somehow and release them. Then catch another group of  $k$  fish and count the number of fish that are already marked,  $Z$ . You can now guess that the number of fish in the entire school is roughly  $k^2/Z$ .

## Mark and recapture

Given a set of  $n$  elements, sample  $k$  elements without replacement twice. Count the number of identical elements in both groups,  $Z$ . Define a random variable  $z_{i,j}$  which indicates that element  $i$  in the first group is the same as element  $j$  in the second. The value of  $Z$  is therefore  $Z = \sum_{i,j} z_{i,j}$ . Let's compute the expectation of  $Z$  using linearity of expectation. Note that the  $z_{i,j}$  variables are not independent!

$$E[Z] = E\left[\sum_{i,j} z_{i,j}\right] = \sum_{i,j} E[z_{i,j}] = \sum_{i,j} 1/n = k^2/n \quad (1)$$

Lets compute the standard deviation of  $Z$ . Recall:

$$\sigma^2[Z] = E[Z - E[Z]]^2 = E[Z^2] - E[Z]^2$$

We need the use the linearity of expectation again to compute  $E[Z^2]$ :

$$E[Z^2] = E\left[\left(\sum_{i,j} z_{i,j}\right)\left(\sum_{i',j'} z_{i',j'}\right)\right] \quad (2)$$

$$= \sum_{i=i',j=j'} E[z_{i,j} z_{i',j'}] \quad (3)$$

$$+ \sum_{i=i',j \neq j'} E[z_{i,j} z_{i',j'}] + \sum_{i \neq i',j=j'} E[z_{i,j} z_{i',j'}] \quad (4)$$

$$+ \sum_{i \neq i',j \neq j'} E[z_{i,j} z_{i',j'}] \quad (5)$$

$$= \frac{k^2}{n} + 0 + 0 + \frac{k^2(k-1)^2}{n(n-1)} \quad (6)$$

Using the expression for variance  $\sigma^2[Z] = E[Z^2] - (E[Z])^2$  we get:

$$\sigma^2[Z] = \frac{k^2}{n} + \frac{k^2(k-1)^2}{n(n-1)} - \left(\frac{k^2}{n}\right)^2 \quad (7)$$

$$\leq \frac{k^2}{n} \quad (\text{for } k \leq n) \quad (8)$$

Now we invoke Chebyshev's inequality.

$$\Pr[|Z - \frac{k^2}{n}| > t] \leq \frac{\sigma^2}{t^2} \leq \frac{k^2}{nt^2} \quad (9)$$

Choosing  $t = 10k/\sqrt{n}$  we get that with probability at least 0.99

$$|Z - \frac{k^2}{n}| \leq 10k/\sqrt{n} \quad (10)$$

Which gives:

$$n \leq \frac{k^2}{Z} (1 + \frac{10\sqrt{n}}{k}) \quad (11)$$

$$n \geq \frac{k^2}{Z} (1 - \frac{10\sqrt{n}}{k}) \quad (12)$$

This gives us the following procedure: First, sample 2 groups of size  $k \geq 50\sqrt{n}$  each. Count the number of collisions  $Z$ . Estimate the size of the set as  $n_{alg} = k^2/Z$ . We are guaranteed that with probability 0.99 our estimate is within 20% accuracy.

$$\frac{5}{6}n \leq n_{alg} \leq \frac{5}{4}n \quad (13)$$