

## Lecture 2: Frequent Items in Streams

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**Warning:** This note may contain typos and other inaccuracies which are usually discussed during class. Please do not cite this note as a reliable source. If you find mistakes, please inform me.

## 1 Approximated histograms

In this section we will describe a simple modification of the algorithm described in [1]. Say we are given a stream of elements  $X = [x_1, \dots, x_N]$  where  $x_i \in \{a_1, \dots, a_n\}$ . Let  $n_i$  denote the number of times element  $a_i$  appeared in the stream, i.e.,  $n_i = |\{j | x_j = a_i\}|$ . Our goal is to estimate  $n_i$  for all frequent elements. This can be solved exactly by keeping a counter for each element  $\{a_1, \dots, a_n\}$ . Alas, this might require,  $\Theta(n)$  memory. Another approach is to sample a large enough fraction of the stream and compute count the frequencies in the sample (see homework question). Here we suggest a deterministic algorithm.

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**Algorithm 1** Frequent items counter
 

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input:  $\varepsilon, \theta \in (0, 1], X = [x_1, \dots, x_N]$ 
 $C \leftarrow \{\}$ 
for  $x \in X$  do
  if  $x \in C$  then
     $C[x]++$ 
  else if  $\text{size}(C) < 1/\varepsilon\theta$  then
     $C[x] = 1$ 
  else
    for  $a \in C$  do
       $C[a]--$ 
      if  $C[a] == 0$  then
         $\text{del}(C[a])$ 
      end if
    end for
  end if
end for
for  $a \in C$  do
  if  $C[a] \leq N\theta(1 - \varepsilon)$  then
     $\text{del}(C[a])$ 
  end if
end for

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**Claim 1.1.** For elements  $a_i$  for which  $n_i \leq N\theta(1 - \varepsilon)$  we have  $n_i \notin C$ .

This is easy to see since we add 1 to the counter of  $C[a]$  every time we encounter  $a$ . So, clearly  $C[a_i] \leq n_i \leq N\theta(1 - \varepsilon)$ . Therefore, in the last loop of the algorithm it will be deleted.

**Claim 1.2.** For elements  $a_i$  for which  $n_i \geq N\theta$  we have  $n_i \geq C[a_i] \geq n_i(1 - \varepsilon)$ .

This is slightly less obvious. Notice that every time we decrease the counters in the map  $C$  we have that  $\text{size}(C) \geq 1/\varepsilon\theta$ . That means that we decrement at least  $1/\varepsilon\theta$  different counters simultaneously. If we let  $t$  denote the the number of times this step is performed, we have  $t/\varepsilon\theta \leq N$  because we could not have deleted more items than the entire stream. Using the observation that  $C[a_i] \geq n_i - t$  we have  $C[a_i] \geq n_i - N\varepsilon\theta \geq n_i(1 - \varepsilon)$ .

**Remarks:** note that this algorithm uses  $O(1)$  memory (assuming  $\varepsilon$  and  $\theta$  are constants).

## Count Sketches

Here we learn about a structure names CountSketch which was suggested in [2]. It will allow us to estimate the frequency of the  $k$  most frequent items in a stream even if it is less than a constant fraction of the stream. There will, however, be other limitations.

We denote the elements by  $o_1, \dots, o_m$  having each appeared  $n_1 \geq \dots \geq n_m$  (the names of the elements are ordered according to their frequency). Before describing the CountSketch structure, let us first analyze one of its building blocks. For lack of a more creative name, we will call it  $B$ .  $B$  is an array of length  $b$  which is associated with two hash functions:  $h : o \rightarrow [1, \dots, b]$  and  $s : o \rightarrow [-1, 1]$ .

We define two function for  $B$  one for adding elements into it.

1. define  $Add(o)$ :
2.  $B[h(o)] = B[h(o)] + s(o)$ .

and one for returning an estimate for  $n_i$  given  $o_i$

1. define  $Query(o)$ :
2. return  $B[h(o)]s(o)$ .

In order to compute the expectation of  $B[h(o)]s(o)$  we need to define the “inverse” of  $h$ . Let  $h^{-1}(o_i) = \{o_j | h(o_j) = h(o_i)\}$ . In words,  $h^{-1}(o_i)$  is the set of all elements for  $h(o_i) = h(o_j)$ . Since each element in  $o_j \in h^{-1}(o_i)$  is encountered exactly  $n_j$  times and for each of those  $s(o_j)$  is added to  $B[h(o)]$  we have that  $B[h(o_i)] = \sum_{o_j \in h^{-1}(o_i)} n_j s(o_j)$ . Let us compute the expected result of a query.

$$\begin{aligned} \mathbb{E}[B[h(o_i)]s(o_i)] &= \mathbb{E}\left[\sum_{o_j \in h^{-1}(o_i)} n_j s(o_j) s(o_i)\right] \\ &= n_i + \mathbb{E}\left[\sum_{o_j \in h^{-1}(o_i), o_i \neq o_j} n_j s(o_j) s(o_i)\right] = n_i \end{aligned}$$

As a reminder, we are interested in the frequencies  $n_1, \dots, n_k$ , for the top  $k$  most items. We see that if  $b > 8k$  we have that  $|h^{-1}(o_i) \cap \{o_1, \dots, o_k\}| = 0$  with probability at least  $7/8$ . In other words, the element  $o_i$  does not map under  $h$  to the same cell in  $B$  with any of the top  $k$  frequency items. We will define  $h_{>k}^{-1} = h^{-1}(o_i) \cap \{o_{k+1}, \dots, o_m\}$ . We will assume from this point on that  $h^{-1}(o_i) \subset \{o_{k+1}, \dots, o_m\}$  or in other words that  $h_{>k}^{-1} = h^{-1}(o_i)$ .

Now, let us bound the variance of  $B[h(o_i)]s(o_i)$ .

$$\begin{aligned}
\text{Var}(B[h(o_i)]s(o_i)) &\leq E[B[h(o_i)]^2 s(o_i)^2] \\
&= E[(\sum_{o_j \in h_{>k}^{-1}(o_i)} n_j s(o_j))(\sum_{o_{j'} \in h_{>k}^{-1}(o_i)} n_{j'} s(o_{j'}))] \\
&= E_h \sum_{o_j \in h_{>k}^{-1}(o_i)} \sum_{o_{j'} \in h_{>k}^{-1}(o_i)} E_s[n_j n_{j'} s(o_j) s(o_{j'})] \\
&= E_h \sum_{o_j \in h_{>k}^{-1}(o_i)} n_j^2 \\
&= \sum_{j=k+1}^m n_j^2 / b
\end{aligned}$$

Note that we have both an expectation over the choice of the hash function  $s$  and over the hash function  $h$ .

Using this bound on the variance of  $B[h(o_i)]s(o_i)$  and Chebyshev's inequality we attain that:

$$\Pr \left[ |B[h(o_i)]s(o_i) - n_i| > \sqrt{8 \sum_{j=k+1}^m n_j^2 / b} \right] \leq 1/8$$

However, note that we also demanded that none of the top  $k$  elements map to the same cell as  $o_i$  which only happened with probability  $7/8$ . Using the union bound on these two events we get:

$$\Pr[|\hat{n}_i - n_i| \leq \gamma] \geq 3/4$$

where we denote  $\hat{n}_i = B[h(o_i)]s(o_i)$  and  $\gamma = \sqrt{8 \sum_{j=k+1}^m n_j^2 / b}$ .

Note that this happens for every elements individually only with constant probability. We would like to get that this holds with probability  $1 - \delta$  for all elements simultaneously. We do that by repeating this entire structure  $t$  times creating the CountSketch  $B_1, \dots, B_t$ . When inserting an element we insert it into all  $t$  arrays  $B_i$  and above. When querying the CountSketch we return  $query(o_i) = \text{Median}(\hat{n}_i^1, \dots, \hat{n}_i^t)$  where  $\hat{n}_i^\ell$  is the estimator  $\hat{n}_i$  from  $B_\ell$ .

Because  $\Pr[|\hat{n}_i - n_i| \leq \gamma] \geq 3/4$  we get from Chernoff's inequality that at least half the values  $\hat{n}_i^\ell$  will be such that  $|\hat{n}_i^\ell - n_i| \leq \gamma$  (including the median) for all  $m$  elements with probability at least  $1 - \delta$  for  $t \in O(\log(m/\delta))$ .

The only thing left to do is set the correct value for  $b$  (the length of  $B$ ). We will demand that  $\gamma \leq \epsilon n_k$ . This gives  $b \geq 8 \sum_{i=k+1}^m n_i^2 / \epsilon^2 n_k^2$ . Therefore, for  $t = O(\log(m/\delta))$  and  $b \geq 8 \max(k, \frac{\sum_{i=k+1}^m n_i^2}{\epsilon^2 n_k^2})$  with probability at least  $1 - \delta$  for each element in the stream  $|\hat{n}_i - n_i| \leq \epsilon n_k$ .

The algorithm for finding the most frequent items is therefore to go over the stream and keep a CountSketch of all the elements seen this far. When we process an element, we also estimate its frequency  $\hat{n}$  and keep the top  $k$  most frequent items in estimated frequencies. These are guaranteed to contain all elements  $o_i$  for which  $n_i > (1 + 2\epsilon)n_k$  and not to contain any element  $o_i$  for which  $n_i < (1 - 2\epsilon)n_k$ .

## References

- [1] Richard M. Karp, Christos H. Papadimitriou, and Scott Shenker. A simple algorithm for finding frequent elements in streams and bags. *ACM Transactions on Database Systems*, 28:2003, 2003.
- [2] Moses Charikar, Kevin Chen, and Martin Farach-colton. Finding frequent items in data streams. pages 693–703, 2002.