Matroids for solving Optimisation Problems

The Greedy algorithm as a solution

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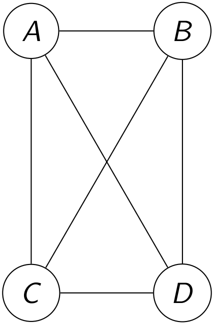
The Problem



Let G be K_4 as seen here, where the vertices correspond to towns to be linked by railway network, and the weight on each edge is the cost of providing a link between the towns correspnding to the respective vertices. In this case, the minimum weight of a spanning tree in G corrsponds to the minimum cost of providing a railway that will link all n towns.

Greedy Algorithm

Kruskal's algorithm is a greedy algorithm that finds a minimum spanning tree for a connected weighted Graph.

Algorithm:

1) Create a graph F containing just the vertices of G.
2) Create a set $S = E(G)$; the edge set of G.
3) While S is non-empty and F is not yet spanning
   3(a) Remove an edge with minimum weight from S.
   3(b) If the removed edge introduces no cycles to F then add the edge to F

Why Greedy works?

Let B_G be a spanning tree created through the greedy algorithm.

Lemma

If $(E, \mathcal{I})$ is a matroid M, then B_G is a solution to the optimization problem.

But what is a matroid?

Independence Systems and Matroids

Definition

An *independence system* is a pair $(E, \mathcal{S})$, where E is a set and $\mathcal{S}$ is a non-empty subset of the power set of E, closed under inclusion. The elements of $\mathcal{S}$ are called the *independent sets*.

Definition

A matroid is a pair $(E, \mathcal{I})$ with finite ground set E and $\mathcal{I}$ being a collection of independent subsets of E satisfying the following conditions

(I1): The empty set is always independent
(I2): Every subset of an independent set is independent
(I3): If A and B are two independent sets in $\mathcal{I}$ and $|A| = |B| + 1$, then there exists $x \in A \setminus B$ such that $B \cup \{x\}$ is in $\mathcal{I}$

Bases of a Matroid

Definition

A base is a maximally independent subset of $\mathcal{I}$.
All the maximally independent sets have the same cardinality, this is the *rank* of the matroid.

Definition

Let $\mathcal{B}$ be a set of subsets of a finite set E. Then $\mathcal{B}$ is the collection of bases of a matroid on E if and only if $\mathcal{B}$ satisfies the following conditions:
(B1) $\mathcal{B}$ is non-empty.
(B2) If B_1 and B_2 are members of $\mathcal{B}$ and $x \in B_1 \setminus B_2$, then there is an element y of $B_1 \setminus B_2$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

Spanning trees are Bases

Definition

A spanning tree T of an undirected graph G is a subgraph that is a *tree* which includes all of the vertices of G.

From this we can see, that if $\mathcal{B}$ is the collection of maximally elements of $\mathcal{I}$, then in G, $\mathcal{B}$ is the set of spanning trees of the graph.

Weight Function

The optimization problem associated with $(E, \mathcal{S})$ is the following: for a given weight function $\omega : E \to \mathbb{R}^+$, we want to find an independent set A whose weight is maximal, where

$$\omega(A) := \sum_{e \in A} \omega(e) \tag{1}$$

Demonstration of Kruskal's algorithm

```
owen@owen-ThinkPad-X201 ~/Matroids $ perl kruskal.pl
$VAR1 = {
          'AB' => 5,
          'AC' => 9,
          'AD' => 2,
          'BD' => 2,
          'BC' => 7,
          'CD' => 2
        };
$VAR1 = {
          'A' => [
                   'AD'
                 ],
          'B' => [
                   'BD'
                 ],
          'C' => [
                   'CD'
                 ],
          'D' => [
                   'AD',
                   'BD',
                   'CD'
                 ]
        };
This Graph is Acyclic
owen@owen-ThinkPad-X201 ~/Matroids $
```

Hello

Why Greedy works: The solution

Lemma

If $(E, \mathcal{I})$ is a matroid M, then B_G is a solution to the optimization problem.

Proof.

If $r(M) = r$, then $B_G = \{e_1, e_2, ..., e_r\}$ is a basis of M. Let B be another basis of M, $B = \{f_1, f_2, ..., f_r\}$ where $\omega(f_1) \geq \omega(f_2) \geq ... \geq \omega(f_r)$. This follows from the result of the next additional lemma showing that not only is B_G a maximum weight basis of M ,but is also at least as heavy as the elements of B at each step.

$\square$

Continued

Lemma

if $1 \leq j \leq r$, *then* $\omega(e_j) \geq \omega(f_j)$.

Proof.

Suppose(seeking a contradiction) that k is the least integer for which $\omega(e_k) \leq \omega(f_k)$. Take $I_1 = \{e_1, e_2, ..., e_{k-1}\}$ and $I_1 = \{f_1, f_2, ..., f_{k-1}\}$. Since $|I_1| = |I_2| + 1$ by $(I3)$ implies $I_1 \cup \{f_t\} \in \mathcal{I}$ for some $f_t \in I_2 \setminus I_1$. But this means that $\omega(f_t) \geq \omega(f_t) > \omega(e_k)$. Hence the Greedy algorithm would have chosen f_t over e_k, which gives us our contradiction. $\square$

Wrap Up

The combination of the previous lemma and our use of the greedy algorithm to find a maximal member B of $\mathcal{I}$ of maximum weight allows us to deduce that Kruskal's algorithm does generate a minimum weight spanning tree of a graph.

We have seen that greedy algorithm gives us a solution to our optimisation problem as long as we have a matroid.