0.1 Matroid

Definition 0.1. An *independence system* is a pair $(E, \mathscr{S})$, where E is a set and $\mathscr{S}$ is a collection of sets satisfying:
(I1) $\mathscr{S}$ is non-empty.
(I2) $\mathscr{S}$ is a hereditary subset of the power set of E.
The elements of $\mathscr{S}$ are called the *independent sets*.

Definition 0.2. A matroid is a pair $(E, \mathscr{I})$ with finite ground set E and $\mathscr{I}$ being a collection of independent subsets of E satisfying the following conditions:

(I1): The empty set is always independent
(I2): Every subset of an independent set is independent
(I3): If A and B are two independent sets of $\mathscr{I}$ and $|A| > |B|$, then there exists $x \in A \setminus B$ such that $B \cup \{x\}$ is in $\mathscr{I}$

Lemma 0.1. *Prove that $(E, \mathscr{I})$ is a matroid if and only if $\mathscr{I}$ satisfies $(I2)$ and the following two conditions:*
$(I1)' \mathscr{I} \neq \emptyset$
$(I3)'$ If I_1, I_2 are in $\mathscr{I}$ and $|I_2| = |I_1| + 1$, then there is an element $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathscr{I}$

Proof. Suppose $(E, \mathscr{I})$ is a matroid. Then by hypothesis, $\mathscr{I}$ satisies $(I2)$.
By $(I1)$ the empty set is laways contained in $\mathscr{I}$, so $\mathscr{I}$ is always non-empty.
$\implies$ $(I1)'$ holds.
Let $I_1, I_2 \in \mathscr{I}$ and $|I_2| > |I_1|$ then by $(I3)$ there exists $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathscr{I}$ by $(I3)$.
But by $(I2)$ there is an I_2' such that $e \in I_2'$ and $|I_2'| = |I_1| + 1$ as a matorid is hereditary and $I_2' \subset I_2$. $\implies \exists e \in I_2' \setminus I_1$ such that $I_1 \cup \{e\} \in \mathscr{I}$.
Conversely, Suppose $\mathscr{I}$ satisfies $(I2), (I1)', (I3)'$
By (I1)', $\mathscr{I}$ is always non-empty. (I2) and (I3) hold by hypothesis. □

Example 0.1. Let M_1, M_2 be matroids on a set E. Let $E = \{1, 2, 3, 4\}$
Let $\mathscr{I}_1 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}\}$
Let $\mathscr{I}_2 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}\}$
let $(E, \mathscr{I}_1 \cap \mathscr{I}_2)$ be a pair, is it a matroid?
Let $I_1 = \{1, 2\}$ and $I_2 = \{3\}$
If $\exists e \in I_1$ such that $I_2 \cup \{e\} \in \mathscr{I}$ then we have a matroid.
$I_2 \cup \{1\} = \{1, 3\} \notin \mathscr{I}$,
$I_2 \cup \{2\} = \{2, 3\} \notin \mathscr{I}$
$\implies$ $(E, \mathscr{I}_1 \cap \mathscr{I}_2)$ is not a matroid.