

Theorem 0.1. Let \mathcal{I} be a collection of subsets of a set E . Then (E, \mathcal{I}) is a matroid if and only if \mathcal{I} satisfies the following conditions:

- (I1) $\emptyset \in \mathcal{I}$
- (I2) If $I \in \mathcal{I}$ and $I' \subset I$ then $I' \in \mathcal{I}$
- (G) For all weight functions $\omega : E \rightarrow \mathbb{R}^+$, the greedy algorithm produces a maximal member of \mathcal{I} of maximum weight.

Proof. Suppose (E, \mathcal{I}) is a matroid. Then $\emptyset \in \mathcal{I}$ and (I2) holds trivially. And by *theorem 5.1* we know that greedy algorithm can find a maximal $B \in \mathcal{B}$ of maximum weight if (E, \mathcal{I}) is a matroid.

Conversely, suppose (E, \mathcal{I}) is a pair satisfying (I1), (I2) and (G). Need to prove \mathcal{I} satisfies (I3) in order to have a matroid.

Suppose that (seeking a contradiction), that is $I_1, I_2 \in \mathcal{I}$ with $|I_2| > |I_1|$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

Now, $|I_1 \setminus I_2| < |I_2 \setminus I_1|$ and $I_1 \setminus I_2$ is non-empty.

So we can choose an $\epsilon > 0$ such that

$$0 < (1 + \epsilon)(|I_1 \setminus I_2|) < |I_2 \setminus I_1| \quad (1)$$

Define $\omega : E \rightarrow \mathbb{R}^+$ by:

$$\omega(e) = \begin{cases} 2 & \text{if } e \in I_1 \cap I_2 \\ \frac{1}{|I_1 \setminus I_2|} & \text{if } e \in I_1 \setminus I_2 \\ \frac{1+\epsilon}{|I_2 \setminus I_1|} & \text{if } e \in I_2 \setminus I_1 \\ 0, & \text{otherwise.} \end{cases}$$

We need the greedy algorithm to fail for only one weight function to get our contradiction.

- The greedy algorithm will choose all the elements of $I_1 \cap I_2$ first as they are the heaviest elements.
- Then it will choose all the elements of $I_1 \setminus I_2$.
- By assumption, it cannot then pick any element of $I_2 \setminus I_1$. Thus the remaining elements of B_G will be in $E \setminus (I_1 \cup I_2)$.

Hence,

$$\omega(B_G) = 2|I_1 \cap I_2| + |I_1 \setminus I_2| \left(\frac{1}{|I_1 \setminus I_2|} \right) = 2|I_1 \cap I_2| + 1$$

But by (I2), I_2 is contained in a maximal member B_2 of \mathcal{I} and, $I_2 \subset B_2$.

$$\omega(B_2) \geq \omega(I_2) = 2|I_1 \cap I_2| + |I_2 \setminus I_1| \left(\frac{1+\epsilon}{|I_2 \setminus I_1|} \right) > 2|I_1 \cap I_2| + 1 = \omega(B_G)$$

$$\implies \omega(B_2) > \omega(B_G)$$

Which means the greedy algorithm does not find a solution to our optimisation problem shown by *theorem x.y*, so the greedy algorithm fails for this weight function. We have a contradiction.

\implies (I3) holds.

$\implies (E, \mathcal{I})$ is a matroid.

□