0.1 Graphs are Matroids

Theorem 0.1. *Let G be a graph and $\mathscr{I}$ be the set of all cyclefree subgraphs of G. Show that if we have the pair $(E, \mathscr{I})$ as defined above by our graph, we have a matroid. In other words, that the cycle matroid $M(G)$ of a graph is a matroid.*

Proof. Let $A, B \in \mathscr{I}$ with $|A| = |B| + 1$. To prove $I3$ of the definition of a *matroid*, We show that for some $a \in A$,
$B \cup \{a\} \in \mathscr{I}$, we should consider $B \cup \{a\}$ for each $a \in A$.
Now suppose $|A| > |B|$ and that $|A| = |B| + 1$
Let $|A \cap B| = s$, $|A \setminus B| = r$, $|A| = s + r$ and $|B| = s + r - 1$
So $|B \setminus A| = r - 1$
Suppose $A \setminus B = \{a_1, a_2, ..., a_r\}$
Suppose $B \cup \{a_i\} \notin \mathscr{I}$ for each $i \in \{1, 2, ...\}$
Consider a_i for $i = 1, 2, ...$ there must be a path $b_{i1}, b_{12}, ..., b_{ir}$ of edges in B such that a_i make a cycle

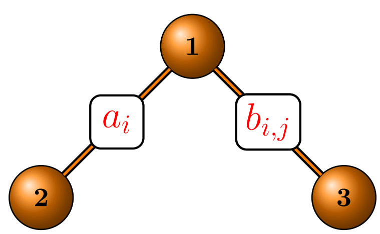


Figure:1.4.1

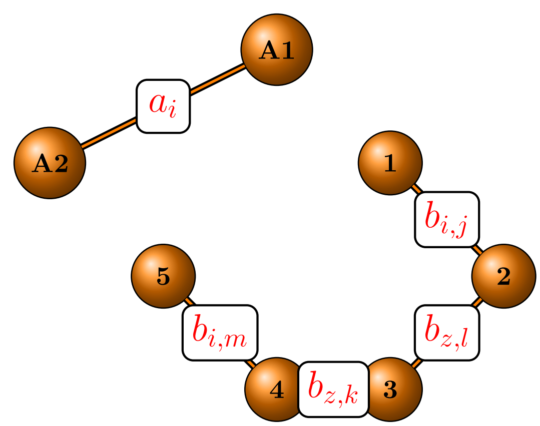
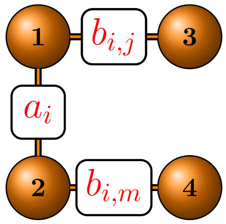


Figure: 1.4.2

Notation: $P(b_j, b_k)$ denotes a set of edges forming path in B from the edges b_j to b_k But $P(b_j, b_k) \cap A$ is not necessarily empty. If $P(b_j, b_k) \subset A$ then $P(b_j, b_k) \cup \{a_i\}$ would be a cycle, then A would not be in $\mathscr{I}$, so at least one of the $b_i \in P(b_j, b_k)$ is contained in $B \setminus A$.
Given $A = \{a_1, ..., a_r\}$ for each a_i associate a $b_i \in B \setminus A$. Let $\hat{B} = \{b_1, ..., b_r\}$
Case 1: The b_i's are distinct
The b_i's are distinct and as shown previously each of the b_i's must be in $|B \setminus A|$ in order to avoid a circuit in A.
Therefore,$|B| \geqslant A$. Contradicting $|A| > |B|$.
Hence, I3 holds.
Case 2: When the b_i's are not all distinct. Let $b_1 = b_2$.



Picture *figure 1.4.3* in place of *figure 1.4.1* above and observe how this would affect the graph of B in *figure 1.4.2*

Figure: 1.4.3

We use the same argument as in Case 1 only here we need two distinct $b_i \in P(b_j, b_k)$ where $b_i \in B \setminus A$ such that $P(b_j, b_k) \cup \{a_i\}$ is a cycle. This can be seen in the diagram above, there must be another edge in the union of the paths which is in $B \setminus A$ or else we get a cycle in A. Otherwise,$P(b_j, b_k) \subset A$ then $P(b_j, b_k) \cup \{a_i\}$ would be a circuit and then $A \notin I$. Therefore, $|B| \geq |A|$, and we have a contradiction.

Hence I3 holds. $\square$