Matroids And their Graphs

o.mcdonnell4@nuigalway.ie

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1 Circuit characterization of a matroid

Definition 1.1. By using (I1)–(I3) , it is not difficult to show that the collection $\mathscr{C}$ of circuits of a matroid M has the following three properties:
(C1) The empty set is not in $\mathscr{C}$
(C2) No member of $\mathscr{C}$ is a proper subset of another member of $\mathscr{C}$
(C3) if C_1 and C_2 are distinct members of C and $e \in C_1 \cap C_2$, then $(C_1 \cup C_2) \setminus \{e\}$ contains a member of $\mathscr{C}$

Theorem 1.1. *Let M be a matroid and $\mathscr{C}$ be its collection of circuits. Then $\mathscr{C}$ satisfies (C1) - (C3)*

Proof:
(C1) is obvious as by $I1$ the empty set must always be an independent set.

(C2) is also straightforward because any $C \in \mathscr{C}$ is a minimally independent set by definition. Therefore , if there exists a $C_1 \in \mathscr{C}$ such that $C_1 \subset C$ then $C_1 \in \mathscr{C}$ and C is not a minimally dependent subset of E.

(C3) Let $A, B \in \mathscr{C}$ and suppose that(Seeking a contradiction) $(A \cup B) \setminus \{e\}$ where e is $\in (A \cap B)$ does not contain a circuit.
Then $(A \cup B) \setminus \{e\}$ is independent and therefore in $\mathscr{I}$

The set $A \setminus B$ is non-empty.
Let $s \in A \setminus B \implies s \in A$
as A is in $\mathscr{C}$ it is minimally dependent. $\implies A \setminus \{s\} \in \mathscr{I}$ i.e is independent.

Let J be a maximal independent set of $(A \cup B)$ with the following properties: $S \setminus \{s\} \subset J$ and therefore $\{s\} \notin J$ but as B is a circuit there must be some element $t \in B$ that is not in J. s and t are distinct.

$\implies |J|$ must be at most equal to $|(A \cup B) \setminus \{s, t\}|$
$\implies |J| \leq |(A \cup B) \setminus \{s, t\}| = |(A \cup B)| - 2 < |(A \cup B) \setminus \{e\}|$

Now by (I3) we can subsitute elements from $|(A \cup B) \setminus \{e\}|$ into $|(A \cup B) \setminus \{s, t\}|$ that are not in $|(A \cup B) \setminus \{s, t\}|$ but the only elements that fits this condition are $\{s, t\}$ and introducing either of these elements breaks the independence of J.
Therefore, $|(A \cup B) \setminus \{e\}|$ must contain a circuit

□