Matroids And their Graphs

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April 2018

Abstract

Matroids are an abstraction of the concept of linear dependence which have since developed into a rich field known as matroid theory[4]. They are extremely flexible systems of sets that can be characterised in a number of diverse ways; in this paper we focus on matroids formed from graphs. In particular, we focus on the optimality of the greedy algorithm in solving optimisation problems in graph theory such as the minimal spanning tree problem which can be solved in this way when the independent sets of the graph forms a matroid. Futhermore, we show that maroids are the only hereditary systems for which this is true. The paper concludes with some insight into other set-systems which provide optimal solutions for combinatorial optimisation problems through the greedy algorithm.

1 Introduction

1.1 What is a Matroid

Matroid theory arose from the investigations of the mathematician H.Whitney [4] of the concept of linear dependence and so borrows heavily from and abstracts a significant amount of linear algebraic concepts particularly with regards matrices and vector spaces. For example, the properties that can be discussed from subsets of linearly independent columns of a given matrix. Another very natural application of matroid theory is graph theory, looking at for example the present of closed walks, or trees and in particular spanning trees of graphs. This will be the focus in this paper as we follow Oxley's survey of matroid theory[2] and textbook.[1]

We will begin by describing what it means for a set to be independent and to clarify what it means to be an independent set in a set-system and to describe the structure needed to describe our system as a matroid. This begins with the follow defintion.

Definition 1.1. An *independence system* is a pair $(E, \mathscr{S})$, where E is a finite set and $\mathscr{S}$ is a collection of sets satisfying the following conditions:

(I1) $\mathscr{S}$ is non-empty.
(I2) $\mathscr{S}$ is a hereditary subset of the power set of E.
The elements of $\mathscr{S}$ are called the *independent sets*.

These independence systems can then be extended into matroids by adding a third axiom, the *exchange axiom*. As we can see from the following defintion, all matroids are independence systems but the converse will not be true in general.

Definition 1.2. A matroid is a pair $(E, \mathscr{I})$ with finite ground set E and $\mathscr{I}$ being a collection of independent subsets of E satisfying the following conditions:
(I1): The empty set is always independent
(I2): Every subset of an independent set is independent
(I3): If A and B are two independent sets of $\mathscr{I}$ and $|A| > |B|$, then there exists $x \in A \setminus B$ such that $B \cup \{x\}$ is in $\mathscr{I}$

The *exchange axiom* can be further characterised into the following form.

Lemma 1.3. *Prove that $(E, \mathscr{I})$ is a matroid if and only if $\mathscr{I}$ satisfies $(I1), (I2)$ and the following condition:*
$(I3)'$ If I_1, I_2 are in $\mathscr{I}$ and $|I_2| = |I_1| + 1$, then there is an element $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathscr{I}$

Proof. Suppose $(E, \mathscr{I})$ is a matroid. Then by hypothesis, $\mathscr{I}$ satisies $(I1), (I2)$. Let $I_1, I_2 \in \mathscr{I}$ and $|I_2| = |I_1| + 1$ then $|I_2| > |I_1|$ and so $(I3)'$ holds trivially due to the defnition of a matroid. But by $(I2)$ there is an I_2' such that $e \in I_2'$ and $|I_2'| = |I_1| + 1$ as a matorid is hereditary and $I_2' \subset I_2$. $\implies \exists e \in I_2' \setminus I_1$ such that $I_1 \cup \{e\} \in \mathscr{I}$.
Conversely, Suppose $\mathscr{I}$ satisfies $(I2), (I1), (I3)'$
Let I_1, I_2 be in $\mathscr{I}$ such that $|I_2| = |I_1| + 1$, then there exists e in $I_2 \setminus I_1$ such that $I_1 \cup \{e\}$.
Let I_1' be in $\mathscr{I}$ where $I_1' \supseteq I_1$ then e in $I_1' \setminus I_2$ and $|I_1'| > |I_2|$. From this we can see that every independent set A with cardinality greater than an independent set B can be shrunk by removing elements while retaining independence due to $(I2)$(the hereditary property) until we have $|A| = |B| + 1$ and then we can apply $(I3)'$. Therefore the matroid property $(I3)$ is satisfied through $(I3)'$. $\square$

Note. The above defintions of the *exchange axiom*, defined by $(I3), (I3)'$ will be used interchangeably for the remainder of this paper.

The following example draws attention to an important property of matroids in their intersection. The intersection of a matroid is not gauranteed to also be a matroid. This example gives a demonstration of the structure of a matroid defined on a set E.

Example 1.4. Let M_1, M_2 be matroids on a set E. Let $E = \{1, 2, 3, 4\}$
Let $\mathscr{I}_1 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}\}$
Let $\mathscr{I}_2 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}\}$
Let $\mathscr{I}_1 \cap \mathscr{I}_2 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\, \{3, 4\}\}$
let $(E, \mathscr{I}_1 \cap \mathscr{I}_2)$ be a pair, is it a matroid?

Let $I_1 = \{1, 2\}$ and $I_2 = \{3\}$
If there exists an $e \in I_1$ such that $I_2 \cup \{e\} \in \mathscr{I}$ then we have a matroid. Otherwise we do not have a matroid.
$I_2 \cup \{1\} = \{1, 3\} \notin \mathscr{I}$,
$I_2 \cup \{2\} = \{2, 3\} \notin \mathscr{I}$
$\implies (E, \mathscr{I}_1 \cap \mathscr{I}_2)$ is not a matroid.

We will proceed through this paper discussing the various structures of matroids (i.e circuits,bases and rank), illustrating their graph theoretic counterparts and some applications that where matroids allow us to avail of some interesting properties. Concluding with a brief overview at some non-hereditary set-systems that share a common application with matroids.

1.2 Enumeration of Matroids

Definition 1.5. Let $\mathscr{I}$ be the collection of subsets of E that do not contain all of the edges of any *cycle* of G. We get a matroid on the edge set of every graph G by defining $\mathscr{I}$ in this way. This matroid is called the *cycle matroid* of the graph G and is denoted $M(G)$.

Definition 1.6. If M_i, M_j are matroids , then there exists a bijection from the ground set of M_i to the ground set of M_j, such that a set is independent in the first matroid if and only if it is independent in the second matroid, then M_i and M_j are said to be isomorphic.

Note. A matroid that is isomorphic to the cycle matroid of some graph is called graphic. And every graphic matroid is binary

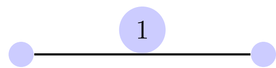
The numbers of non-isomorphic matroids and binary matroids on an n-element set for $0 \leq n \leq 8$

n	0	1	2	3	4	5	6	7	8
matroids	1	2	4	8	17	38	98	306	1724
binary matroids	1	2	4	8	16	32	68	148	342

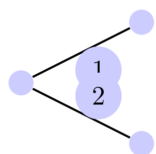
It can be seen from this table, that number of possible matroids on an n-set grows very rapidly.
Example: Let E be a set, $\{1, 2, 3\}$ then:
Show there are exactly eight non-isomorphic matroids on E. Along with the corresponding Graph of each matroid. This confirms to us the value in the previous table for $n = 3$.
Solution:

$$\{\emptyset\}$$

$$\{\{\emptyset\},\{1\}\} \cong \{\{\emptyset\},\{2\}\} \cong \{\{\emptyset\},\{3\}\}$$

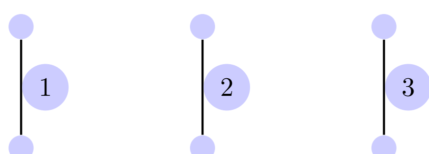


$$\{\{\emptyset\},\{1\},\{2\},\{1,2\}\} \cong \{\{\emptyset\},\{1\},\{3\},\{1,3\}\} \cong \{\{\emptyset\},\{2\},\{3\},\{2,3\}\}$$

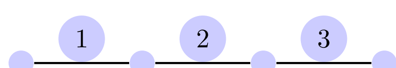


$$\{\{\emptyset\},\{1\},\{2\}\}$$

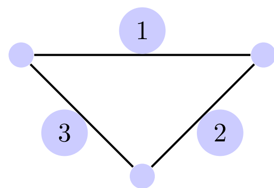
$$\{\{\emptyset\},\{1\},\{2\},\{3\}\}$$



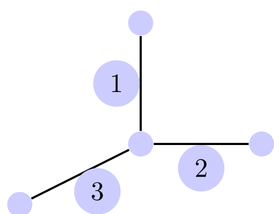
$$\{\{\emptyset\},\{1\},\{2\},\{3\},\{1,2\},\{2,3\}\}$$



$$\{\{\emptyset\},\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}$$



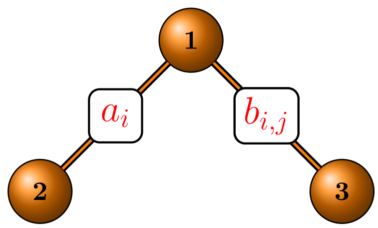
$$\{\{\emptyset\},\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$$

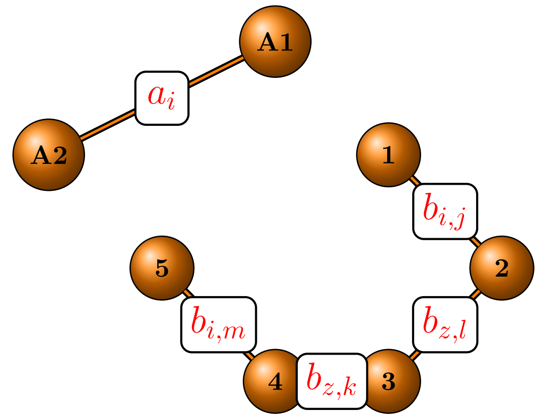


1.3 Graphs are Matroids

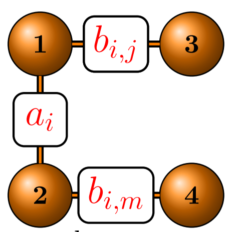
Theorem 1.7. *Let G be a graph and $\mathscr{I}$ be the set of all cyclefree subgraphs of G. Show that if we have the pair $(E, \mathscr{I})$ as defined above by our graph, we have a matroid. In other words, that the cycle matroid $M(G)$ of a graph is a matroid.*

Proof. Let $A, B \in \mathscr{I}$ with $|A| = |B| + 1$. To prove $I3$ of the definition of a *matroid*, We show that for some $a \in A$,
$B \cup \{a\} \in \mathscr{I}$, we should consider $B \cup \{a\}$ for each $a \in A$.
Now suppose $|A| > |B|$ and that $|A| = |B| + 1$
Let $|A \cap B| = s$, $|A \setminus B| = r$, $|A| = s + r$ and $|B| = s + r - 1$
So $|B \setminus A| = r - 1$
Suppose $A \setminus B = \{a_1, a_2, ..., a_r\}$
Suppose $B \cup \{a_i\} \notin \mathscr{I}$ for each $i \in \{1, 2, ...\}$
Consider a_i for $i = 1, 2, ...$ there must be a path $b_{i1}, b_{12}, ..., b_{ir}$ of edges in B such that a_i make a cycle





Notation: $P(b_j, b_k)$ denotes a set of edges forming path in B from the edges b_j to b_k But $P(b_j, b_k) \cap A$ is not necessarily empty. If $P(b_j, b_k) \subset A$ then $P(b_j, b_k) \cup \{a_i\}$ would be a cycle, then A would not be in $\mathscr{I}$, so at least one of the $b_i \in P(b_j, b_k)$ is contained in $B \setminus A$.
Given $A = \{a_1, ..., a_r\}$ for each a_i associate a $b_i \in B \setminus A$. Let $\hat{B} = \{b_1, ..., b_r\}$
Case 1: The b_i's are distinct
The b_i's are distinct and as shown previously each of the b_i's must be in $|B \setminus A|$ in order to avoid a circuit in A.
Therefore,$|B| \geqslant A$. Contradicting $|A| > |B|$.
Hence, I3 holds. Case 2: When the b_i's are not all distinct. Let $b_1 = b_2$.



Imagine the figure to the left in place of the graph $\{a_i, b_{i,j}\}$ above and observe how this would affect the graph of B.

We use the same argument as in Case 1 only here we need two distinct $b_i \in P(b_j, b_k)$ where $b_i \in B \setminus A$ such that $P(b_j, b_k) \cup \{a_i\}$ is a cycle. This can be seen in the diagram above, there must be another edge in the union of the paths which is in $B \setminus A$ or else we get a cycle in A. Otherwise,$P(b_j, b_k) \subset A$ then $P(b_j, b_k) \cup \{a_i\}$ would be a circuit and then $A \notin I$. Therefore, $|B| \geq |A|$, and we have a contradiction.
Hence I3 holds. $\square$

2 Cryptomorphisms

2.1 Circuit characterization of a matroid

Definition 2.1. By using (I1)–(I3) , it is not difficult to show that the collection $\mathscr{C}$ of circuits of a matroid M has the following three properties:
(C1) The empty set is not in $\mathscr{C}$

(C2) No member of $\mathscr{C}$ is a proper subset of another member of $\mathscr{C}$
(C3) if C_1 and C_2 are distinct members of C and $e \in C_1 \cap C_2$, then $(C_1 \cup C_2) \setminus \{e\}$ contains a member of $\mathscr{C}$

Theorem 2.2. *Let M be a matroid and $\mathscr{C}$ be its collection of circuits. Then $\mathscr{C}$ satisfies (C1) - (C3)*

Proof:
(C1) is obvious as by $I1$ the empty set must always be an independent set.

(C2) is also straightforward because any $C \in \mathscr{C}$ is a minimally independent set by definition. Therefore , if there exists a $C_1 \in \mathscr{C}$ such that $C_1 \subset C$ then $C_1 \in \mathscr{C}$ and C is not a minimally dependent subset of E.

(C3) Let $A, B \in \mathscr{C}$ and suppose that(Seeking a contradiction) $(A \cup B) \setminus \{e\}$ where e is $\in (A \cap B)$ does not contain a circuit.
Then $(A \cup B) \setminus \{e\}$ is independent and therefore in $\mathscr{I}$

The set $A \setminus B$ is non-empty.
Let $s \in A \setminus B \implies s \in A$
as A is in $\mathscr{C}$ it is minimally dependent. $\implies A \setminus \{s\} \in \mathscr{I}$ i.e is independent.

Let J be a maximal independent set of $(A \cup B)$ with the following properties: $S \setminus \{s\} \subset J$ and therefore $\{s\} \notin J$ but as B is a circuit there must be some element $t \in B$ that is not in J. s and t are distinct.

$\implies |J|$ must be at most equal to $|(A \cup B) \setminus \{s, t\}|$
$\implies |J| \leq |(A \cup B) \setminus \{s, t\}| = |(A \cup B)| - 2 < |(A \cup B) \setminus \{e\}|$

Now by (I3) we can subsitute elements from $|(A \cup B) \setminus \{e\}|$ into $|(A \cup B) \setminus \{s, t\}|$ that are not in $|(A \cup B) \setminus \{s, t\}|$ but the only elements that fits this condition are $\{s, t\}$ and introducing either of these elements breaks the independence of J.
Therefore, $|(A \cup B) \setminus \{e\}|$ must contain a circuit

$\square$

Theorem 2.3. *Let E be the edge sets of a graph G and let $\mathscr{C}$ be the edge sets of cycles in G.*
Then $\mathscr{C}$ is the set of circuits of a matroid.

Proof: Let $A, B \in \mathscr{C}, A \neq B$ and let $e \in A \cap B$

We must now construct a cycle of G whose edge set is contained in $(A \cup B) \setminus \{e\}$

For $i = 1, 2, 3, ...$ let P_1 be a path whose edge set is $A \setminus \{e\}$
$A \setminus \{e\} \in \mathscr{I}$ therefore P_i is not a cycle of G. This path will traverse from the edge a_j to a_u where u, j were the vertices connecting the edge e to $(A \cup B) \setminus \{e\}$ to make $A \cup B$.

Now perform the same procedure for a path P_2 whose edge set is $B \setminus \{e\}$.

P_1 and P_2 should meet at the junctions u, v, where e was removed to make $(A \cup B) \setminus \{e\}$
Therefore $P_1 \cup P_2$ should be a cycle of G.
$\implies$ (C3) holds
$\implies$ $\mathscr{C}$ is the edge sets of cycle in G.

$\square$

2.2 Bases

Theorem 2.4. *Show that if $\mathscr{I}$ is a non-empty hereditary set of subsets of a finite set E, then $(E, \mathscr{I})$ is a matroid if and only if, for all $X \subset E$, all maximal members of $\{I : I \in \mathscr{I}$ and $I \subset X\}$ have the same number of elements.*

Proof: ($\implies$) Let B_1, B_2 be maximal elements of $\{I : I \in \mathscr{I}$ and $I \subset X\}$
And assume $|B_1| < |B_2|$ Then since $B_1, B_2 \in \mathscr{I}$
There exists $e \in (B_2 \setminus B_1)$ such that $B_1 \cup \{e\} \in \mathscr{I}$
This contradicts our maximality of B_1.

$\implies$ All maximal elements of the set $\{I : I \in \mathscr{I}$ and $I \subset X\}$ in our matroid M have the same cardinality.

$\square$

Definition 2.5. A base is a maximally independent subset of $\mathscr{I}$.
As seen previously all maximally independent sets in a matroid have the same cardinality.

Theorem 2.6. *Let $\mathscr{B}$ be a set of subsets of a finite set E. Then $\mathscr{B}$ is the collection of bases of a matroid on E if and only if $\mathscr{B}$ satisfies the following conditions:*
(B1) $\mathscr{B}$ is non-empty.
(B2) If B_1 and B_2 are members of $\mathscr{B}$ and $x \in B_1 \setminus B_2$, then there is an element y of $B_1 \setminus B_2$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{B}$.

Proof:
By (I1) $\emptyset$ is always independet, so $\mathscr{B}$ must always contain at least the $\emptyset$, (B1) holds.
Let $B_1, B_2 \in \mathscr{B}, B_1 \neq B_2$.
$|B_1| = |B_2|$ so (I3) does not directly apply here.
Let $x \in B_1 \setminus B_2 \implies x \in B_1, x \notin B_2$
$|B_1| = |B_1 \setminus \{x\}| + 1 \implies B_1 \setminus \{x\} \in \mathscr{I}$ but not in $\mathscr{B}$
$|B_2| = |B_1 \setminus \{x\}| + 1$ so now we can use (I3)
Now $\exists y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{I}$
$|(B_1 \setminus \{x\}) \cup \{y\}| = |B_1 \setminus \{x\}| + 1 = |B_1| = ... = |B_r|$ as all maximal elements of $\mathscr{I}$ have the same cardinality $\implies (B_1 \setminus \{x\}) \cup \{y\}$ is maximal in $\mathscr{I}$
$\implies (B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{B}$, (B2) holds.

Conversely, suppose that $\mathscr{B}$ satisfies (B1) and (B2).
By B1 $\mathscr{B}$ is always non-empty which shows I1 holds.

By definition, a base $B_1 \in \mathscr{B}$ is a maximally independent subset of E. Then for all $B_i \in \mathscr{B}$ the subsets $b_{i,k} \subseteq B_i$ are independent. Therefore all the $b_{i,k}$ are in $\mathscr{I}$.
$\Longrightarrow$ (I2) holds. Showing a matroid can be generated through the bases.
Assume that (I3) fails. That, for $I_1, I_2 \in \mathscr{I}$, where $|I_1| = |I_2| + 1$,
there $\exists y \in I_1 \setminus I_2$ such that $I_2 \cup \{y\} \in \mathscr{I}$.
Let $B_1, B_2 \in \mathscr{B}, |B_1| = |B_2|$.
Let $x \in B_1 \setminus B_2$ then $B_1 \setminus \{x\} \subset B_1$.
$\Longrightarrow$ $B_1 \setminus \{x\}$ is independent.
Then there exists a $y \in B_2 \setminus (B_1 \setminus \{x\})$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{B}$ from (B2). And if $(B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{B}$ it is also in $\mathscr{I}$. A contradiction.
$\Longrightarrow$ (I3) holds and we have a matroid.

□

3 Graph Theory

3.1 Graphs and their corresponding Matroid Theory

In this section we introduce the graph theoretic deifnitions and results needed to understand the algorithms and subsequent optimisation in later sections. These definitions and results come mainly from Jungnickel's[3] text. We will also see how the subsequent properties of graphs naturally align with the properties of matroids and in particular to bases of matroids which are the cornerstone of the results in the later sections.

Definition 3.1 (Connected)**.** A graph is connected when there is a path between each pair of vertices.

Definition 3.2 (Acyclic)**.** An acyclic graph is a graph which contains no closed walks.

Definition 3.3 (Walk)**.** If there are vertices $v_{i-1}v_i$ for $i = 1, ..., n$ the sequence is called a walk. If $v_0 = v_n$ this sequence is called a closed walk.

Definition 3.4 (Tree)**.** A connected graph containing no circuits. In other words, acyclic graphs.
A forest is a disconnected graph containing no circuits

We know from *theorem 1.7*. If we let E be the edge set of our graph G. And we define $\mathscr{I}$ as the subsets of E not containing all of the edges of any cycle in G we have $M(G)$ the cycle matroid of G. We know that this is a matroid, and it is easy to see that in this case the elements of $\mathscr{I}$ correspond exactly to the trees of G.

Definition 3.5 (Spanning Tree)**.** A spanning tree T of an undirected graph G is a subgraph that is a *tree* which includes all of the vertices of G.
A disconnected graph cannot contain a spanning tree as we cannot find a walk which brings us to all of the disconnected vertices.

Proposition 3.6. *A theorem of Cayley(1889) states that the number of distinct laballed trees which can be drawn using n labelled points is* n^{n-2}.

Corollary. *The number of distinct labelled spanning trees which can be drawn using* n *labelled points is* n^2.

Remark. Determining the number of spanning trees of a graph in polynomial time is NP-hard.

Lemma 3.7. *[3] Any acyclic graph on* n *vertices has at most* $n-1$ *edges.*

Proof: Let G be an acyclic graph with n vertices.
if n = 1 then we have no edges hence nothing to prove.
Assumse $n > 1$, let e be an edge in G connecting two vertices ab in the vertex set of G.
Let $H = G\backslash\{e\}, H$ has one more connected component than G. H has two maximal acyclic connected componenets, and thus can be decomposedinto acyclic connected graphs $H_1, H_2, ..., H_k wherek \geq 2$.
By induction, we can assume each graph H_i contains at most $n_i - 1$ edges where n_i is the number of vertices of H_i.
Then G has at most $n-1$ edges.
$(n_1 - 1) + ... + (n_k - 1) + 1 = (n_1 + ... + n_k) - (k-1) \leq n-1$ edges.

□

Definition 3.8 (Bridge)**.** A bridge(or cut edge) is an edge of a graph whose deletion increases the numbe rof connected components. Equivalently, an edge is a bridge is and only if it is not contained in any cycle.

The above definition is mainly useful for us in proving the next lemma, which comes from Jungnickel's text and will help us visualise the characteristics of a spanning tree of a graph. In particular condition (1) and (2), will allow us to see that the spanning trees of a graph directly correspond to the bases of a matroid.

Lemma 3.9. *[3] Let* G *be a graph. Then the following conditions are equivalent:*
1) G *is a tree.*
2) G *does not contain any cycles, but adding any further edge yields a cycle.*
3) Any two vertices of G *are connected by a unique path.*
4) G *is connected, and any edge of* G *is a bridge.*

Proof: $(1) \implies (2)$
Suppose that G is a tree,then G is a connected graph with no circuits. Let e be a new edge in G with $e = g_i g_k$ where g_i, g_k are in the vertex set of G. Then as $G \cup \{e\}$ must be connected, there exists a walk between any pair of vertices of G. So there is a walk K from $g_j \to g_k$ and there is also a walk L from $g_k \to g_j$ where K does not traverse e and L does traverse e and so we have a cycle.

Proof: $(2) \implies (3)$
Let u, v be vertices of G. If there was not path joining uv in G then $e = uv$ does not create a cycle in G. Thus G must be connected.

Suppose G contained two different paths W_1, W_2 from u to v.
Then $u \longrightarrow v \longrightarrow u$ would be a closed walk in G.
$\implies$ G contains a cycle. Which is a contradiction.

Proof: (3) $\implies$ (4)
G is connected by hypothesis. Let $e = uv$ be an edge in G.
Suppose e is not a bridge, then $G \setminus \{e\}$ is still connected. But then we have two distinct path from u to v in G.

Proof: (4) $\implies$ (1)
G is connected by hypotheis.
Suppose G contains a cycle K. Then any edge of K could be ommited from G, and the resulting graph would still be connected. In other words, no edge of K would be a bridge, a contradiction.

□

The vertices of a graph/network can be labelled and referred to as nodes. Information may then be recorded in them along with a cost, penalty or probability associated with each edge. For example, the problem of joining all nodes in a graph by the minimum length using our respective metric to a tree known as a *minimum spanning tree.*
Weights can be assigned using a process as detailed below. We will later redefine this process in a more suitable way to take advantage of matroid properties in order to determine these minimum spanning trees in graphs.

Definition 3.10. let (G, ω) be a network. For any subset T of the edge set of G, ω is called the weight of T.

$$\omega(T) = \sum_{e \in T} \omega(T) \tag{1}$$

Definition 3.11 (Minimal Spanning Tree)**.** A spanning tree is a *minimal* spanning tree if its weight is minimal of all the weights of spanning trees. A forest can be considered by finding a minimal spanning tree for each connected component of G.

Remark. If the weight ω is constant, any spanning tree is minimal.
In this case, determining a minimal spanning tree could be done using a breadth-first search.

4 Greedy Algorithm

The greedy algorithm is an algorithmic paradigm. It does not always provide a solution in general, not least an optimal solution. As such the greedy algorithm is a description of a problem solving heuristic. It says that when trying to solve a problem we should choose the local optimum at each iteration in the hope of finding a global optimum. In this section, we will introduce the generic procedure that makes the gredy algorithm and showing how it can be modified in a variety of ways, including compatibility with matroids.

Algorithm 1 Greedy algorithm

Let $(E, \mathscr{S})$ be an independence system and $\omega : E \longrightarrow \mathbb{R}^+$

procedure GREEDY($E, \mathscr{S}, \omega, T$)
  order the elements of E according to their weight
  $E = \{e_1, ..., e_m\}$ with $\omega(e_1) \geq \omega(e_2) \geq ... \geq \omega(e_m)$
  $T \leftarrow \emptyset$
  for $k = 1$ to m **do**
    if $T \cup \{e_k\} \in \mathscr{S}$ **then**
      append e_k to T

We can see in the above pseudocode that this is a sequential iterative algorithm. Our data is kept in a list sorted by weight,going from heaviest to lightest and then sequentially selecting the heaviest entry in our list at each iteration. By selecting that element and then adding it to our initial empty variable T. By the end of the process we should have joined a certain number of the elements to our varibale T. Where T at the time of termination is the greedy algorithm solution. This is how the algorithm looks in the most abstract sense, later we will see it used as a way of solving the *minimal spanning tree* problem mentioned in *definition 3.11*.
First we will see how this corresponds to matroids. The greedy algorithm above constructed a maximal weight element from our list called T. Depending on the structure of our system and the termination clause this element's cardinality can differ. It is our hope to show that when the greedy algorithm is applied to a matroid we get an optimal solution, meaning that we generate a base of the matroid. However, for now it is enough to show that the greedy algorithm always produces a solution in general. Which is illustrated by the below pseudocode.

Algorithm 2 Greedy algorithm

The *greedy algorithm* for the pair $(\mathscr{I}, \omega)$ is as follows:

procedure GREEDY($\mathscr{I}, \omega$)
  Set $x_0 = \emptyset$ and $j = 0$
  if $\exists e \in E \setminus x_j$ such that $x \cup \{e\} \in \mathscr{I}$ **then**
    Choose such an element e_{j+1} of maximum weight,
    let $x_{j+1} = x_j \cup \{e_{j+1}\}$ and
    GREEDY($\mathscr{I}, \omega$)
  else
    Let $x_j = B_G$
    return x_j
  $j{+}{+}$

The procedure described above is as follows: Select the emptyset as your first

element denoted x_0, as the empty set is always independent. we want to build the set x. Now as before, order your elements from heaviest to lightest.
Then select the heaviest weighted element possible such that $x \cup \{e\} \in \mathscr{I}$ where e is the selected element. Otherwise we want to return x, as x is then equal to the solution of the greedy algorithm, i.e B_G (the base generated by the greedy algorithm). The element e should then be removed from further selection, and the process should be called recursively until there is no possible element that can be added to the set x without inducing a circuit.

Remark. It is interesting to note that in this way we should find a maximal member B_{max} of $\mathscr{B}$ (the collection of bases of a matroid) and we will prove that this is certainly the case later. But if we negate this process in the following way we can use this exact procedure to find a minimal member B_{min} of $\mathscr{B}$. This works as follows:
Let $\omega : E \longrightarrow \mathbb{R}^-$ be the weight function, we want to find an independent set A whose weight is maximal, where

$$\omega(A) := \sum_{e \in A} |\omega(e)| \tag{2}$$

Due to the fact that our weights are now negative real numbers, finding the maximal element at each iteration corresponds to finding the value with the minimal absolute value. And so through completing the greedy algorithm process we should successfully find our minimal element B_{min} of $\mathscr{B}$. Assuming that our algorithm is correct.

5 Optimization Problems

5.1 Optimisation Example: Kruskal's Algorithm

Example 5.1. Suppose we have a country containing an n number of cities that are currently isolated from each other. As the new minister for transport it is your idea to correct this tansport issue and to lay a railroad which should connect each city to any other city by a unique path. However, you have a budget. Each railway line will cost a certain predefined amount to lay (with no difficulties or unforeseen costs). How will you decide which city-links are the optimal ones to lay railtracks on?
We will soon see that this can be done by finding a minimal spanning tree. Which can be found through a greedy algorithm process. A suitable example of this kind of algorithm which should solve our problem is Kruskal's Algorithm. Which is detailed below.

Algorithm 3 Kruskal's algorithm

Let G be a connected graph with vertex set $V = \{1, ..., n\}$ and $\omega : E \longrightarrow \mathbb{R}^+$ a weight function. The edges of G are ordered according to their weight, that is, $E = \{e_1, ..., e_m\}$ and $\omega(e_1) \leq ... \leq \omega(e_m)$.

```
procedure KRUSKAL(G, ω, T)
    T ← ∅
    for k = 1 to m do
        if ACYCLIC(T ∪ {e_k}) then
            append e_k to T
```

This process is very similar to our prior description of the generic greedy algorithm. Although this time, we sort the elements of our edge set from lightest to heaviest. And then we iterate through. At each iteration, we check if the created set created by through $T \cup \{e_k\}$ at each iteration remains an independent and if so we add e_k to the generated set and if not we ignore that element and move to the next in the list E.

For graphs, the process above is equivalent to this semi-pseudocode. That you make a forest containing just the vertices of G. And then we iteratively add the desired elements to that forest as long as no cycles are induces in the resulting subgraph of G.

1) Create a graph F containing just the vertices of G.
2) Create a set $S = E(G)$; the edge set of G.
3) While S is non-empty and F is not yet spanning
3(a) Remove an edge with minimum weight from S.
3(b) If the removed edge introduces no cycles to F
then add the edge to F

In the next section we will prove the correctness of algorithms such as this. It is also notable to mention that Kruskal's algorithm will terminate after excatly $n-1$ iterations, where n is the number of vertices in a graph. This corresponds to the *lemma 3.7* which says a spanning tree has exactly $n-1$ edges. Any more edges added at that point will induce a cycle and so kruskal's algorithm terminates.

5.2 Proofs of correctness

Problem: *Find a maximal member B of $\mathscr{I}$ of maximum weight.*

Note. Let B_G be a base of a matroid generated by the greedy algorithm.

Theorem 5.2. *If $(E, \mathscr{I})$ is a matroid M, then B_G is a solution to the optimization problem.*

Proof. If $r(M) = r$, then $B_G = \{e_1, e_2, ..., e_r\}$ is a basis of M. Let B be another basis of M, $B = \{f_1, f_2, ..., f_r\}$ where $\omega(f_1) \geq \omega(f_2) \geq ... \geq \omega(f_r)$. We claim that $\omega(e_j) \geq \omega(f_f)$ $\forall j$, then it follows that $\omega(B_G) \geq \omega(B)$ for any other basis in $\mathscr{B}$. $\square$

Lemma 5.3. *If $1 \leq j \leq r$, then $\omega(e_j) \geq \omega(f_j)$.*

Proof. Suppose (seeking a contradiction) that k is the least integer for which $\omega(e_k) < \omega(f_k)$. Take $I_1 = \{e_1, e_2, ..., e_{k-1}\}$ and $I_2 = \{f_1, f_2, ..., f_k\}$. Since $|I_2| = |I_1| + 1$ $(I3)$ implies $I_1 \cup \{f_t\} \in \mathscr{I}$ for some $f_t \in I_2 \setminus I_1$. But this means that $\omega(f_t) \geq \omega(f_k) > \omega(e_k)$
And hence the Greedy algorithm would have chosen f_t over e_k, which gives us our contradiction. $\square$

Lemma 5.4. *Let M be a mtroid and $\omega : E(M) \longleftrightarrow \mathbb{R}^{\nvdash}$ be a one-to-one function. Prove that M has a unique basis of maximum weight.*

Proof. Let ω be an injective function, this will then allow no repition of weights on edges. As this would mean more than one value in the domain would be mapped to one value in the range.
We want to find an independent set A whose weight is maximal, where

$$\omega(A) := \sum_{e \in A} \omega(e) \tag{3}$$

We can then arrange our edges in a set S by order of decreasing weight such that $\omega(e_1) \geq \omega(e_2) \geq ... \geq \omega(e_k)$.
We have already seen in *theorem 5.1* that the greedy algorithm as described in *algorithm 2/4.1* provides a solution to this optimisation problem. And since there is no repition in weights there is no point in the algorithm where there is more than a single choice as to the next chosen edge. Therefore there is only one possible soltuion when out weight function is injective.

If we lose the injectivity condition, then this is not the case and we cannot gaurantee uniqueness in general. $\square$

Theorem 5.5. *Let M be a mtroid and $\omega : E(M) \longleftrightarrow \mathbb{R}^{\nvdash}$. When the greedy algorithm is applied to the pair $(\mathscr{I}, \omega)$, each iteration of the greedy algorithm involves a potential choice. Thus, in general, there are a number of different set that the algorithm can produce as solutions to the optimisation problem $(\mathscr{I}, \omega)$. Let $\mathscr{B}_G$ be the set of such sets and let $\mathscr{B}_{max}$ be the set of maximum weight bases of M. Prove that $\mathscr{B}_G = \mathscr{B}_{max}$.*

Proof. Suppose ω is an injective function, then we've show there is a unique maximum weight basis for M in *theorem 5.4* and so the proof of this trivial.

Now suppose ω is not injective.This means maximal weight bases of M are not in general unique.
If $r(M) = r$, then $B_G = \{e_1, e_2, ..., e_r\}$ is a basis of M. Let B'_G be another basis of M, $B'_G = \{f_1, f_2, ..., f_r\}$. Both B_g, B'_G are bases generated through the greedy algorithm as described in *section 4.1.* We arrange both these bases in terms of decreasing order where $\omega(e_1), \omega(f_1)$ are the heaviest elements in their

respective bases.
As ω is not an injective function B_g and B'_G are distinct. We also know from *theorem 5.1* that the greedy algorithm finds a maxiaml member B of $\mathscr{I}$ of maximum weight.

Therefore, any base generated through the greedy algorithm is maximally weighted. Meaning, that $\omega(B_G) = \omega(B'_G)$. As if $\omega(B_G) < \omega(B'_G)$ then this would mean the greedy algorithm does not find a solution to out optimisation problem, contradicting *theorem 5.1.* And therefore, $\omega(e_j) = \omega(f_j)\ \forall j$ and since all these bases are maximally weighted, $\mathscr{B}_G = \mathscr{B}_{max}$. □

5.3 Cryptomorphism: Greedy

Theorem 5.6. *Let $\mathscr{I}$ be a collection of subsets of a set E. Then $(E, \mathscr{I})$ is a matroid if and only if $\mathscr{I}$ satisfies the following conditions:*
$(I1)$ $\emptyset \in \mathscr{I}$
$(I2)$ *If $I \in \mathscr{I}$ and $I' \subset I$ then $I' \in \mathscr{I}$*
(G) *For all weight functions $\omega : E \longrightarrow \mathbb{R}^+$, the greedy algorithm produces a maximal member of $\mathscr{I}$ of maximum weight.*

Proof. Suppose $(E, \mathscr{I})$ is a matroid. Then $\emptyset \in \mathscr{I}$ and (I2) holds trivially. And by *theorem 5.1* we know that greedy algorithm cna find a maximal $B \in \mathscr{B}$ of maximum weight if $(E, \mathscr{I})$ is a matroid.
Conversely, suppose $(E, \mathscr{I})$ is a pair satisfying $(I1), (I2)$ and (G). Need to prove $\mathscr{I}$ satisfies $(I3)$ in order to have a matroid.
Suppose that (seeking a contradiction), that is $I_1, T_2 \in \mathscr{I}$ with $|I_2| > |I_1|$ such that $I_1 \cup \{e\} \in \mathscr{I}$.
Now, $|I_1 \setminus I_2| < |I_2 \setminus I_1|$ and $I_1 \setminus I_2$ is non-empty.
So we can choose an $\epsilon > 0$ such that

$$0 < (1+\epsilon)(|I_1 \setminus I_2|) < |I_2 \setminus I_1| \tag{4}$$

Define $\omega : E \longrightarrow \mathbb{R}^+$ by:

$$\omega(e) = \begin{cases} 2 if e \in I_1 \cap I_2 \\ \frac{1}{|I_1 \setminus I_2|} if e \in I_1 \setminus I_2 \\ \frac{1+\epsilon}{|I_2 \setminus I_1|} if e \in I_2 \setminus I_1 \\ 0, otherwise. \end{cases}$$

We need the greedy algorithm to fail for only one weihgt function to get our contradiction.

- The greedy algorithm will choose all the elements of $I_1 \cap I_2$ first as they are the heaviest elements.
- Then it will choose all the elements of $I_1 \setminus I_2$.

- By assumption, it cannot then pick any element of $I_2 \setminus I_1$. Thus the remaining elements of B_G will be in $E \setminus (I_1 \cup I_2)$.

Hence,

$$\omega(B_G) = 2|I_1 \cap I_2| + |I_1 \setminus I_2|(\frac{1}{|I_1 \setminus I_2|}) = 2|I_1 \cap I_2| + 1$$

But by (I2), I_2 is contained in a maximal member B_2 of $\mathscr{I}$ and, $I_2 \subset B_2$.

$$\omega(B_2) \geq \omega(I_2) = 2|I_1 \cap I_2| + |I_2 \setminus I_1|(\frac{1+\epsilon}{|I_2 \setminus I_1|}) > 2|I_1 \cap I_2| + 1 = \omega(B_G)$$

$\implies \omega(B_2) > \omega(B_G)$
Which means the greedy algorithm does not find a solution to our optimisation problem shown by *theorem x.y*, so the greedy algorithm fails for this weight function. We have a contradiction.
$\implies$ (I3) holds.
$\implies$ $(E, \mathscr{I})$ is a matroid. □

6 Other Optimizable set-systems

6.1 Accesible Set System

Definition 6.1. An *accesible set-system* is a pair $(E, \mathscr{S})$ where E is a finite ground set and $\mathscr{S}$ is a non-empty subset of the power set of E.
Satisfying the follow *accessibility axiom*:
(A) For any non-empty feasible set $X \in \mathscr{S}$. There exists an element $e \in X$ such that $X \setminus \{e\} \in \mathscr{S}$. Elements of $\mathscr{S}$ are called the *feasible sets* of M. Maximal feasible sets are also called *bases*.

This axiom is needed due to the process of the greedy algorithm. We require the ability to sequentially select a single element and then union it to our constructed solution at each step so an arbitrary set system is useless.

Remark. Matroid $\supseteq$ Indepence System $\subseteq$ Accesible Set-System.
All matroids are indepence systems and all indepence systems are accesible set-systems but the converse is not true in general.

Our generalised problem is now:

Proposition 6.2. *For any accesible set-system M and any weight function $\omega : E \longrightarrow \mathbb{R}^+$, the optimisation problem is:*
Maximise $\omega(B)$ such that B is a basis of M.

We can now apply a modified greedy algorithm in order to find a solution to this generalised version of our matroid/independence system problem.

Algorithm 4 Greedy algorithm for accessible set-systems

Let $M = (E, \mathscr{S})$ be an accessible set-system and $\omega : E \longrightarrow \mathbb{R}^+$ a weight function.

```
procedure GREEDY(E, S, ω, T)
    T ← ∅, X ← E
    while there ∃x ∈ X with T ∪ {x} ∈ S do
        choose some x ∈ X with T ∪ {x} ∈ S and
        ω(x) ≥ ω(y) ∀y ∈ X with T ∪ {y} ∈ S
        T ← T ∪ {x}, X ← X \ {x}
```

Using our above definitions and algorithm we can now begin finding solutions to our problem for any accessible set-system. However, we are interested in learning about the charactersations of these set-systems that lead to optimal solutions to our greedy algorithm as seen in **section 5.1**.
One such caracterisation is the concept of a *greedoid*. A greedoid is an accessible set-system satisfying (I3). Formally this means,

Definition 6.3. A greedoid is a pair $(E, \mathscr{S})$ where E is a finite ground set and $\mathscr{S}$ is a collection of the feasible subsets of E satisfying the following conditions:
(I1): $\mathscr{S}$ is non-empty, $\emptyset \in \mathscr{S}$.
(A): For any non-empty feasible set $X \in \mathscr{S}$. There exists an element $e \in X$ such that $X \setminus \{e\} \in \mathscr{S}$.
(I3): If A and B are two independent sets of $\mathscr{I}$ and $|A| > |B|$, then there exists $x \in A \setminus B$ such that $B \cup \{x\}$ is in $\mathscr{I}$.

Unfortunately the greedy algorithm while providing solutions does not gaurantee optimal solutions for all greedoids. To characterise the greedoids that do result in optimal solutions when the greedy algorithm is applied we must add to our existing machinery an additional axiom. This is called the *strong exchange axiom*. This axiom is a strong version of (I3) the exchange axiom of a matroid.

Proposition 6.4. *Let $M = (E, \mathscr{S})$ be a greedoid. Then the above modified greedy algorithm finds an optimal soltuion to our problem for any weight function $\omega : E \longleftrightarrow \mathbb{R}^+$ if and only if M satisfies the follwing axiom:*
(SE): For $A, B \in \mathscr{S}$ with $|A| = |B| + 1$, there always exists some $e \in A \setminus B$ such that $B \cup \{e\}$ and $A \setminus \{e\}$ are contained in $\mathscr{S}$.

Remark. It can be seen that this axiom holds trivially for matroids due to the hereditary condition.

7 Appendix

7.1 Depth-first Search

Algorithm 5 DFS

Let G be a graph with vertex set $V = \{1, ..., n\}$

procedure DFS(G, V)
  label v as discovered
  for all edges from v to w **in** G.adjacentEdges(V) **do**
    if (vertex w is not labelled as discovered) **then**
      recursively call DFS(G, w)

This algorithm allows you to find the connected components of a disconnected graph. Then using the following algorithm we can check if our forest at each step of our algorithm is acyclic.

Algorithm 6 Acyclic Check

Let G be a graph with the set of connected componenets C as found by DFS(G,v) where v is an arbitrary vertex in G.

procedure ACYCLIC(G, C)
  for all i **in** C **do**
    if i.edgeCount() $> n - 1$ **then return** False
  return True

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