Matroids And their Graphs

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February 2018

1 Base characterisation of a matroid

Theorem 1.1. *Let $\mathscr{B}$ be a set of subsets of a finite set E. Then $\mathscr{B}$ is the collection of bases of a matroid on E if and only if $\mathscr{B}$ satisfies the following conditions:*
(B1) $\mathscr{B}$ is non-empty.
(B2) If B_1 and B_2 are members of $\mathscr{B}$ and $x \in B_1 \setminus B_2$, then there is an element y of $B_1 \setminus B_2$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{B}$.

Definition 1.1. A base is a maximally independent subset of $\mathscr{I}$.
As seen previously all maximally independent sets in a matroid have the same cardinality.

Proof:
By (I1) $\emptyset$ is always independet, so $\mathscr{B}$ must always contain at least the $\emptyset$, (B1) holds.
Let $B_1, B_2 \in \mathscr{B}, B_1 \neq B_2$.
$|B_1| = |B_2|$ so (I3) does not directly apply here.
Let $x \in B_1 \setminus B_2 \implies x \in B_1, x \notin B_2$
$|B_1| = |B_1 \setminus \{x\}| + 1 \implies B_1 \setminus \{x\} \in \mathscr{I}$ but not in $\mathscr{B}$
$|B_2| = |B_1 \setminus \{x\}| + 1$ so now we can use (I3)
Now $\exists y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{I}$
$|(B_1 \setminus \{x\}) \cup \{y\}| = |B_1 \setminus \{x\}| + 1 = |B_1| = ... = |B_r|$ as all maximal elements of $\mathscr{I}$ have the same cardinality $\implies (B_1 \setminus \{x\}) \cup \{y\}$ is maximal in $\mathscr{I}$
$\implies (B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{B}$, (B2) holds.

Conversely, suppose that $\mathscr{B}$ satisfies (B1) and (B2).
By B1 $\mathscr{B}$ is always non-empty which shows I1 holds.
By definition, a base $B_1 \in \mathscr{B}$ is a maximally independent subset of E. Then for all $B_i \in \mathscr{B}$ the subsets $b_{i,k} \subseteq B_i$ are independent. Therefore all the $b_{i,k}$ are in $\mathscr{I}$.
$\implies$ (I2) holds. Showing a matroid can be generated through the bases.
Assume that (I3) fails. That, for $I_1, I_2 \in \mathscr{I}$, where $|I_1| = |I_2| + 1$,
there $\exists y \in I_1 \setminus I_2$ such that $I_2 \cup \{y\} \in \mathscr{I}$.
Let $B_1, B_2 \in \mathscr{B}, |B_1| = |B_2|$.
Let $x \in B_1 \setminus B_2$ then $B_1 \setminus \{x\} \subset B_1$.
$\implies B_1 \setminus \{x\}$ is independent.
Then there exists a $y \in B_2 \setminus (B_1 \setminus \{x\})$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{B}$ from (B2). And if $(B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{B}$ it is also in $\mathscr{I}$. A contradiction.
$\implies$ (I3) holds and we have a matroid.

□