And now similar to with circuits, we can characterise a matroid by way of it's bases as proved below.

Definition 0.1. A base is a maximally independent subset of $\mathscr{I}$.
As seen previously all maximally independent sets in a matroid have the same cardinality.

Definition 0.2. The rank of a matroid is equal to the cardinality of it's bases. The rank is denoted as $r(M)$ for a matroid M.

Theorem 0.3. *Let $\mathscr{B}$ be a set of subsets of a finite set E. Then $\mathscr{B}$ is the collection of bases of a matroid on E if and only if $\mathscr{B}$ satisfies the following conditions:*
(B1) $\mathscr{B}$ is non-empty.
(B2) If B_1 and B_2 are members of $\mathscr{B}$ and $x \in B_1 \setminus B_2$, then there is an element y of $B_1 \setminus B_2$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{B}$.

Proof. By (I1) $\emptyset$ is always independent, so $\mathscr{B}$ must always contain at least the $\emptyset$, (B1) holds.
Let $B_1, B_2 \in \mathscr{B}, B_1 \neq B_2$.
$|B_1| = |B_2|$ so (I3) does not directly apply here, as any element of cardinality larger that $|B_i|$, where $B_i \in \mathscr{B}$ will be contained in $\mathscr{C}$(the set of circuits).
Let $x \in B_1 \setminus B_2 \implies x \in B_1, x \notin B_2$.
$|B_1| = |B_1 \setminus \{x\}| + 1 \implies B_1 \setminus \{x\} \in \mathscr{I}$ but not in $\mathscr{B}$.
$|B_2| = |B_1 \setminus \{x\}| + 1$ so now we can apply (I3).
Therefore, there exists a $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{I}$
$|(B_1 \setminus \{x\}) \cup \{y\}| = |B_1 \setminus \{x\}| + 1 = |B_1| = ... = |B_r|$ where the B_i are maximal elements of $\mathscr{I}$ and have the same cardinality.
$\implies (B_1 \setminus \{x\}) \cup \{y\}$ is maximal in $\mathscr{I}$.
$\implies (B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{B}$, (B2) holds.

Conversely, suppose that $\mathscr{B}$ satisfies (B1) and (B2). This direction is much more difficult and appears in Oxley's text[**?**]
By (B1), $\mathscr{B}$ is always non-empty which shows (I1) holds.
By definition, a base $B_1 \in \mathscr{B}$ is a maximally independent subset of E. Then for all $B_i \in \mathscr{B}$ the subsets $b_{i,k} \subseteq B_i$ are independent. Therefore all the $b_{i,k}$ are in $\mathscr{I}$.
$\implies$ (I2) holds. Showing a matroid can be generated through the bases.
Next, assume that (I3) fails. That, for $I_1, I_2 \in \mathscr{I}$, where $|I_1| < |I_2|$,
there does not exist a $y \in I_1 \setminus I_2$ such that $I_2 \cup \{y\} \in \mathscr{I}$.
Let $B_1, B_2 \in \mathscr{B}, |B_1| = |B_2|$ such that $I_1 \subseteq B_1$ and $I_2 \subseteq B_2$.
Assume B_2 is chosen in such a way as to ensure that $|B_2 \setminus (I_2 \cup B_1)|$ is minimal.
By above choice of I_1, I_2 we have:

$$I_2 \setminus B_1 = I_2 \setminus I_1 \tag{1}$$

Now suppose that $B_2 \setminus (I_2 \cup B_1)$ is non-empty. Then we can choose an element x from this resulting set and apply $(B2)$, which says there is an element

y in $B_1 \setminus B_2$ such that $(B_2 \setminus \{x\}) \cup \{y\} \in \mathscr{B}$. But then

$$|((B_2 \setminus \{x\}) \cup \{y\}) \setminus (I_2 \cup B_1)| < |B_2 \setminus (I_2 \cup B_1)| \tag{2}$$

And thus the choice of B_2 is contradicted. Hence $B_2 \setminus (I_2 \cup B_1)$ is empty and so $B_2 \setminus B_1 = I_2 \setminus B_1$. Therefore,

$$B_2 \setminus B_1 = I_2 \setminus I_1 \tag{3}$$

Next we must show that $B_1 \setminus (I_1 \cup B_2)$ is empty. If not then $(B2)$ can be applied and $(B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{B}$.
Now, $I_1 \cup \{y\} \subseteq (B_1 \setminus \{x\}) \cup \{y\}$ so $I_1 \cup \{y\} \in \mathscr{I}$. Since $y \in B_2 \setminus B_1$, it follows by (3) that $y \in I_2 \setminus I_1$ and so we reach a contradiction to our inital assumption, We conclude that $B_1 \setminus (I_1 \cup B_2)$ is empty. Hence $B_1 \setminus B_2 = I_1 \setminus B_2$.
Since the last set is contained in $I_2 \setminus I_1$, it follows that

$$I_2 \setminus B_1 \subseteq I_2 \setminus I_1 \tag{4}$$

As all the bases have the same cardinality so by (3) and (4) we see that
$I_1 \setminus I_2 \geq I_2 \setminus I_1$ so $|I_1| \geq |I_2|$. This contradicts our choice of independent sets. And so we have that the pair $(E, \mathscr{I})$ is a matroid. $\square$