

Matroids And their Graphs

o.mcdonnell4@nuigalway.ie

19 January 2018

1 Circuit characterization of a matroid

Definition 1.1. By using (I1)–(I3), it is not difficult to show that the collection \mathcal{C} of circuits of a matroid M has the following three properties:

(C1) The empty set is not in \mathcal{C}

(C2) No member of \mathcal{C} is a proper subset of another member of \mathcal{C}

(C3) if C_1 and C_2 are distinct members of \mathcal{C} and $e \in C_1 \cap C_2$, then $(C_1 \cap C_2) \setminus \{e\}$ contains a member of \mathcal{C}

Theorem 1.1. *Let M be a matroid and \mathcal{C} be its collection of circuits. Then \mathcal{C} satisfies (C1) - (C3)*

Proof:

(C1) is obvious as by (I1) the empty set must always be an independent set.

(C2) is also straightforward because any $C \in \mathcal{C}$ is a minimally independent set by definition. Therefore, if there exists a $C_1 \in \mathcal{C}$ such that $C_1 \subset C$ then $C_1 \in \mathcal{C}$ and C is not a minimally independent subset of E .

(C3) Let $A, B \in \mathcal{C}$ and suppose that (Seeking a contradiction) $(A \cap B) \setminus \{e\}$ where $e \in (A \cap B)$ does not contain a circuit.

Then $(A \cap B) \setminus \{e\}$ is independent and therefore in \mathcal{I}

The set $A \setminus B$ is non-empty.

Let $s \in A \setminus B \implies s \in A$

as A is in \mathcal{C} then it is minimally dependent. $\implies A \setminus \{s\} \in \mathcal{I}$ i.e is independent.

Let J be a maximal independent set of $(A \cap B)$ with the following properties: $s \in J$ and therefore $\{s\} \subseteq J$ but as B is a circuit there must be some element $t \in B$ that is not in J . s and t are distinct.

$\implies |J|$ must be at most equal to $|(A \cap B) \setminus \{s, t\}|$

$\implies |J| \leq |(A \cap B) \setminus \{s, t\}| = |(A \cap B)| - 2 < |(A \cap B) \setminus \{e\}|$

Now by (I3) we can substitute elements from $|(A \cap B) \setminus \{e\}|$ into $|(A \cap B) \setminus \{s, t\}|$ that are not in $|(A \cap B) \setminus \{s, t\}|$ but the only elements that fits this condition are $\{s, t\}$ and introducing either of these elements breaks the independence of J .

Therefore, $|(A \cap B) \setminus \{e\}|$ must contain a circuit

□