0.1 Cryptomorphism: Greedy

Theorem 0.1. Let \mathscr{I} be a collection of subsets of a set E. Then (E,\mathscr{I}) is a matroid if and only if \mathscr{I} satisfies the following conditions:

- $(I1) \emptyset \in \mathscr{I}$
- (I2) If $I \in \mathscr{I}$ and $I' \subset I$ then $I' \in \mathscr{I}$
- (G) For all weight functions $\omega : E \longrightarrow \mathbb{R}^+$, the greedy algorithm produces a maximal member of \mathscr{I} of maximum weight.

Proof. Suppose (E, \mathscr{I}) is a matroid. Then $\emptyset \in \mathscr{I}$ and (I2) holds trivially. And by theorem 5.1 we know that greedy algorithm cna find a maximal $B \in \mathscr{B}$ of maximum weight if (E, \mathscr{I}) is a matroid.

Conversely, suppose (E, \mathscr{I}) is a pair satisfying (I1), (I2) and (G). Need to prove \mathscr{I} satisfies (I3) in order to have a matroid.

Suppose that (seeking a contradiction), that is $I_1, T_2 \in \mathscr{I}$ with $|I_2| > |I_1|$ such that $I_1 \cup \{e\} \in \mathscr{I}$.

Now, $|I_1 \setminus I_2| < |I_2 \setminus I_1|$ and $I_1 \setminus I_2$ is non-empty.

So we can choose an $\epsilon > 0$ such that

$$0 < (1 + \epsilon)(|I_1 \setminus I_2|) < |I_2 \setminus I_1| \tag{1}$$

Define $\omega: E \longrightarrow \mathbb{R}^+$ by:

$$\omega(e) = \begin{cases} 2ife \in I_1 \cap I_2 \\ \frac{1}{|I_1 \setminus I_2|} ife \in I_1 \setminus I_2 \\ \frac{1+\epsilon}{|I_2 \setminus I_1|} ife \in I_2 \setminus I_1 \\ 0, otherwise. \end{cases}$$

We need the greedy algorithm to fail for only one weihgt function to get our contradiction.

- The greedy algorithm will choose all the elements of $I_1 \cap I_2$ first as they are the heaviest elements.
- Then it will choose all the elements of $I_1 \setminus I_2$.
- By assumption, it cannot then pick any element of $I_2 \setminus I_1$. Thus the remaining elements of B_G will be in $E \setminus (I_1 \cup I_2)$.

Hence,

$$\omega(B_G) = 2|I_1 \cap I_2| + |I_1 \setminus I_2| \left(\frac{1}{|I_1 \setminus I_2|}\right) = 2|I_1 \cap I_2| + 1$$

But by (I2), I_2 is contained in a maximal member B_2 of \mathscr{I} and, $I_2 \subset B_2$.

$$\omega(B_2) \ge \omega(I_2) = 2|I_1 \cap I_2| + |I_2 \setminus I_1| \left(\frac{1+\epsilon}{|I_2 \setminus I_1|}\right) > 2|I_1 \cap I_2| + 1 = \omega(B_G)$$

 $\implies \omega(B_2) > \omega(B_G)$

Which means the greedy algorithm does not find a solution to our optimisation problem shown by *theorem* x.y, so the greedy algorithm fails for this weight function. We have a contradiction.

 \implies (I3) holds.

 $\implies (E, \mathscr{I})$ is a matroid.