JOURNAL OF COMBINATORIAL THEORY 6, 313–316 (1969)

Note

A Bound for the Number of Matroids*

DOMINIC J. A. WELSH

Merton College, Oxford, England, and the University of Michigan Ann Arbor, Michigan 48104

Communicated by R. Fulkerson

Received August 13, 1968

In this note I give a simple proof of the result of Crapo [1] that there exist at least 2^n non-isomorphic matroids on a set of n elements. In fact I prove the stronger results:

THEOREM 1. *On a set of n elements there exist at least 2^n non-isomorphic transversal matroids.*

This can clearly be proved from

THEOREM 2. *On a set of n elements there exist at least $\binom{n}{r}$ non-isomorphic transversal matroids of rank r.*

An easy consequence of the method of proof of Theorem 1 is

COROLLARY. *There exist at least 2^{n-1} non-isomorphic, loop-free, transversal matroids on a set of n elements.*

A matroid $(S, \mathscr{M})$ is a finite set S together with a family $\mathscr{M}$ of *independent* subsets of S, such that

(1) $\phi \in \mathscr{M}$.
(2) If $A \in M$ and $B \subset A$ then $B \in \mathscr{M}$.
(3) If X is any subset of S then all maximal independent subsets of X have the same cardinality.

We will denote the matroid $(S, \mathscr{M})$ by $\mathscr{M}$ where there is no possibility of confusion. A base of M is any maximal independent subset of S. Two matroids $\mathscr{M}_1$, $\mathscr{M}_2$ on S are said to be isomorphic (written $M_1 \simeq M_2$), if

* Research supported in part by a grant from the Office of Naval Research.

there is a 1-1 map f on S onto itself such that $A \in \mathcal{M}_1$ if and only if $f(A) \in \mathcal{M}_2$. A set X is *dependent* if it is not independent, and a *circuit* of a matroid is a minimal dependent set. An element x is called a *loop* of $\mathcal{M}$ if the singleton subset $\{x\}$ is not a member of $\mathcal{M}$. Whitney [2] shows that a matroid is determined up to isomorphism by its bases or by its circuits. For a fuller discussion of matroids we refer to [2] or Tutte [3].

If $A = (A_1, ..., A_m)$ is any collection of subsets of S, a subset $S = (e_1, ..., e_p)$ of distinct elements of S is a partial transversal of $\mathcal{A}$ if there exists a bijection σ on $(1, ..., p)$ into $(1, ..., m)$ such that

$$e_i \in A_{\sigma(i)} \qquad (i = 1, ..., p).$$

Edmonds and Fulkerson [4] and Mirsky and Perfect [5] prove the following important result:

If $\mathcal{A}$ is a finite collection of subsets of the finite set S then the family $\mathcal{T}(A)$ of partial transversals of $\mathcal{A}$ form the independent sets of a matroid on S. We call $\mathcal{T}(A)$ *the transversal matroid of the family* $\mathcal{A}$.

More generally an arbitrary matroid $(S, \mathcal{M})$ is called a *transversal matroid* if and only if there exists some family $\mathcal{A}$ of subsets of S such that $\mathcal{M}$ is exactly the set of partial transversals of $\mathcal{A}$. For a fuller discussion of transversal matroids we refer to [4] and [5].

PROOF OF THEOREM 1: Let S be the set of the n elements $(1, 2, ..., n)$. If $(i_1, i_2, ..., i_n)$ is any n-tuple of binary digits define $\mathcal{A}(i_1, i_2, ..., i_n)$ to be the family of sets $A_{i_1}, ..., A_{i_n}$ where

$$\begin{aligned} A_{i_k} &= (1, 2, ..., k), \qquad \text{where} \quad i_k = 1, \\ &= \phi, \text{ otherwise.} \end{aligned}$$

Let $\mathcal{M}(i_1, i_2, ..., i_n)$ denote the transversal matroid of the family $\mathcal{A}(i_1, ..., i_n)$. We assert

LEMMA. *If the sequence $(i_1, ..., i_n)$ is distinct from the sequence $(j_1, ..., j_n)$ then the transversal matroids $\mathcal{M}(i_1, ..., i_n)$ and $\mathcal{M}(j_1, ..., j_n)$ are not isomorphic.*

PROOF: The lemma clearly holds for $n = 1$. Assume it is true for $n < N$. Let

$$(i_1, ..., i_N) \neq (j_1, ..., j_N),$$

but suppose

$$\mathcal{M} = \mathcal{M}(i_1, i_2, ..., i_m) \approx \mathcal{M}' = \mathcal{M}(j_1, j_2, ..., j_n).$$

Suppose first that $i_N = j_N = 0$, so that

$$B \equiv (A_{i_1}, \ldots, A_{i_{N-1}}) \neq \mathscr{B}' \equiv (A_{j_1}, \ldots, A_{j_{N-1}}).$$

Let $\mathscr{Q}$, $\mathscr{Q}'$ be the transversal matroids of $\mathscr{B}$ and $\mathscr{B}'$, respectively, then

$$\mathscr{Q} \simeq \mathscr{M} \simeq \mathscr{M}' \simeq \mathscr{Q}',$$

which contradicts the induction hypothesis.

Suppose now that $i_N = 1$, $j_N = 0$. Then clearly N is a loop in $\mathscr{M}'$. However $N \in A_{i_N}$ and therefore is independent in $\mathscr{M}$. Hence since $\mathscr{M} \simeq \mathscr{M}'$ there must exist some i such that i is independent in $\mathscr{M}'$ but is a loop in $\mathscr{M}$. But since $A_{i_N} = \{1, 2, \ldots, N\}$, i is independent in $\mathscr{M}$ for all i. Thus we have a contradiction.

Finally let $i_N = j_N = 1$. Let k be the maximum integer for which $i_k \neq j_k$. Take $i_k = 1$, $j_k = 0$. Consider the sets $A_{i_k}, A_{i_{k-1}}, \ldots, A_{i_N}$. For notational reasons suppose that the non-null members of this collection are

$$(1, 2, \ldots, k), (1, 2, \ldots, k_1), \ldots \ldots, (1, 2, \ldots, k_p = N),$$

where $k < k_1 < k_2 < \cdots < k_p = N$.

Clearly a set (x, y) of two elements is dependent in $\mathscr{M}$ if and only if

$$k_{p-1} < x, y \leqslant k_p .$$

Similarly a set of three elements (x, y, z) is dependent in $\mathscr{M}$ if and only if

$$k_{p-2} < x, y, z \leqslant k_p ,$$

and so on. Hence the matroids $\mathscr{M}$ and $\mathscr{M}'$ have exactly the same number of dependent sets of cardinality n for $1 \leqslant n \leqslant p - 1$. Hence the circuits of $\mathscr{M}$ which have cardinality exactly p are those p-subsets of $k + 1$, $k + 2, \ldots, n$ which properly contain no circuit of $\mathscr{M}$. But since $j_k = 0$, the set of circuits of $\mathscr{M}'$ of cardinality p contains all p-subsets of $(k, k + 1, \ldots, n)$ which properly contain no circuit of $\mathscr{M}$. Since $\mathscr{M}$ and $\mathscr{M}'$ have the same number of circuits of cardinality strictly less than p we see that $\mathscr{M}$ has fewer circuits of cardinality p than does $\mathscr{M}'$. Hence $\mathscr{M}$ and $\mathscr{M}'$ cannot be isomorphic. This completes the proof of the lemma.

Since the number of distinct n-tuples of binary digits is 2^n the theorem follows.

Proof of Theorem 2: Let $(i_1, \ldots, i_n)$ be any n-tuple of zeros and ones containing exactly r non-zero entries. It is easy to see therefore that if J is *any* subset of the non-zero entries in $(i_1, \ldots, i_n)$ then

$$(1, 2, \ldots, J) \subset \bigcup_{k \in J} A_k ,$$

which, by Hall's theorem [6], implies that the non-null members of the family $\mathscr{A} \equiv (A_{i_1}, \ldots, A_{i_n})$ has a partial transversal of length r or, equivalently, the non-null members of $\mathscr{A}$ have a transversal. Thus all n-tuples with exactly r non-zero entries give rise to transversal matroids of rank r. By the previous lemma these are all non-isomorphic.

The proof of the corollary is very easy. Consider any n-tuple-$(i_1, \ldots, i_n)$. Then an element j, $(1 \leqslant j \leqslant n)$, can only be a loop if none of the family $A_{i_1}, \ldots, A_{i_n}$ contains j. Thus the n-tuple gives rise to a matroid with no loops if and only if $A_{i_n} = \phi$.

I close this note with the following conjectures. Let $f(n)$ denote the number of non-isomorphic matroids on a set of n elements. It can be verified that the following holds, for small n.

n	1	2	3	4	5
$f(n)$	2	4	8	17	38

I conjecture that

(1) $f(m+n) \geqslant f(m)\, f(n)$,

(2) $\lim\limits_{n\to\infty} n^{-1} \log f(n) = \theta < \infty$.

By a fundamental theorem for subadditive functions, (2) will follow from (1) if it can be shown that $f(n) \leqslant k^n$ for some $k < \infty$.

References

1. H. H. Crapo, Single-Element Extensions of Matroids, *J. Res. Nat. Bur. Standards Sect. B* **69** (1965), 55–65.
2. H. Whitney, On the Abstract Properties of Linear Dependence, *Amer. J. Math.* **57** (1935), 509–533.
3. W. T. Tutte, Lectures on Matroids, *J. Res. Nat. Bur. Standards Sect. B* **69** (1965), 1–48.
4. J. Edmonds and D. R. Fulkerson, Transversals and Matroid Partitions, *J. Res. Nat. Bur. Standards Sect. B* **69** (1965), 147–153.
5. L. Mirsky and H. Perfect, Applications of the Notion of Independence to Problems of Combinatorial Analysis, *J. Combinatorial Theory* **2** (1967), 327–357.
6. P. Hall, On Representatives of Subsets, *J. London Math. Soc.* **10** (1935), 26–30.