

**Lemma 0.1.** *Let  $M$  be a matroid and  $\omega : E(M) \longleftrightarrow \mathbb{R}^k$  be a one-to-one function. Prove that  $M$  has a unique basis of maximum weight.*

*Proof.* Let  $\omega$  be an injective function, this will then allow no repetition of weights on edges. As this would mean more than one value in the domain would be mapped to one value in the range.

We want to find an independent set  $A$  whose weight is maximal, where

$$\omega(A) := \sum_{e \in A} \omega(e) \quad (1)$$

We can then arrange our edges in a set  $S$  by order of decreasing weight such that  $\omega(e_1) \geq \omega(e_2) \geq \dots \geq \omega(e_k)$ .

We have already seen in *theorem 5.1* that the greedy algorithm as described in *algorithm 2/4.1* provides a solution to this optimisation problem. And since there is no repetition in weights there is no point in the algorithm where there is more than a single choice as to the next chosen edge. Therefore there is only one possible solution when our weight function is injective.

If we lose the injectivity condition, then this is not the case and we cannot guarantee uniqueness in general.  $\square$

**Theorem 0.2.** *Let  $M$  be a matroid and  $\omega : E(M) \longleftrightarrow \mathbb{R}^k$ . When the greedy algorithm is applied to the pair  $(\mathcal{I}, \omega)$ , each iteration of the greedy algorithm involves a potential choice. Thus, in general, there are a number of different sets that the algorithm can produce as solutions to the optimisation problem  $(\mathcal{I}, \omega)$ . Let  $\mathcal{B}_G$  be the set of such sets and let  $\mathcal{B}_{max}$  be the set of maximum weight bases of  $M$ . Prove that  $\mathcal{B}_G = \mathcal{B}_{max}$ .*

*Proof.* Suppose  $\omega$  is an injective function, then we've shown there is a unique maximum weight basis for  $M$  in *theorem 5.4* and so the proof of this is trivial.

Now suppose  $\omega$  is not injective. This means maximal weight bases of  $M$  are not in general unique.

If  $r(M) = r$ , then  $B_G = \{e_1, e_2, \dots, e_r\}$  is a basis of  $M$ . Let  $B'_G$  be another basis of  $M$ ,  $B'_G = \{f_1, f_2, \dots, f_r\}$ . Both  $B_G, B'_G$  are bases generated through the greedy algorithm as described in *section 4.1*. We arrange both these bases in terms of decreasing order where  $\omega(e_1), \omega(f_1)$  are the heaviest elements in their respective bases.

As  $\omega$  is not an injective function  $B_G$  and  $B'_G$  are distinct. We also know from *theorem 5.1* that the greedy algorithm finds a maximal member  $B$  of  $\mathcal{I}$  of maximum weight.

Therefore, any base generated through the greedy algorithm is maximally weighted. Meaning, that  $\omega(B_G) = \omega(B'_G)$ . As if  $\omega(B_G) < \omega(B'_G)$  then this would mean the greedy algorithm does not find a solution to our optimisation problem, contradicting *theorem 5.1*. And therefore,  $\omega(e_j) = \omega(f_j) \forall j$  and since all these bases are maximally weighted,  $\mathcal{B}_G = \mathcal{B}_{max}$ .  $\square$