

Matroids for solving Optimisation Problems

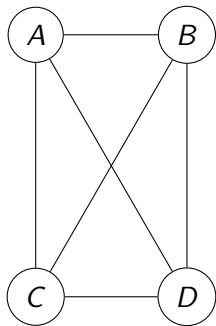
The Greedy algorithm as a solution

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The Problem



Hello

Greedy Algorithm

Kruskal's algorithm is a greedy algorithm that finds a minimum spanning tree for a connected weighted Graph. Kruskal's algorithm is a special case of our optimization problem.

Algorithm:

- 1) Create a graph F (a set of trees) where each vertex of G is a separate tree.
 - 2) Create a set S containing all the edges of the graph.
 - 3) While S is non-empty and F is not yet spanning
 - 3(a) Remove an edge with minimum weight from S .
 - 3(b) If the removed edge connects two different trees then add it to the forest F , combining two trees into a single tree.
- At the termination of the algorithm, the forest forms a minimum spanning tree of the graph.

Why Greedy works?

Example when Greedy fails.

Independence Systems

Definition

An *independence system* is a pair (E, \mathcal{S}) , where E is a set and \mathcal{S} is a non-empty subset of the power set of E , closed under inclusion.

The elements of \mathcal{S} are called the *independent sets*.

The optimization problem associated with (E, \mathcal{S}) is the following: for a given weight function $\omega : E \rightarrow \mathbb{R}^+$, we want to find an independent set A whose weight,

$$\omega(A) := \sum_{e \in A} \omega(e) \tag{1}$$

is maximal.

Matroid

Definition

A matroid is a pair (E, \mathcal{I}) with finite ground set E and \mathcal{I} being a collection of independent subsets of E satisfying the following conditions

(I1): The empty set is always independent

(I2): Every subset of an independent set is independent

(I3): If A and B are two independent sets in \mathcal{I} and $|A| = |B| + 1$, then there exists $x \in A \setminus B$ such that $B \cup \{x\}$ is in \mathcal{I}

Bases of a Matroid

Definition

A base is a maximally independent subset of \mathcal{I} .

All the maximally independent sets have the same cardinality, this is the *rank* of the matroid.

Definition

Let \mathcal{B} be a set of subsets of a finite set E . Then \mathcal{B} is the collection of bases of a matroid on E if and only if \mathcal{B} satisfies the following conditions:

(B1) \mathcal{B} is non-empty.

(B2) If B_1 and B_2 are members of \mathcal{B} and $x \in B_1 \setminus B_2$, then there is an element y of $B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

Spanning trees are Bases

Definition

A spanning tree T of an undirected graph G is a subgraph that is a *tree* which includes all of the vertices of G , with minimum possible number of edges.

Also T does not contain any cycles, but adding any further edge yields a cycle, so T is maximal in G .

Lemma

Any acyclic graph on n vertices has at most $n - 1$ edges. And a spanning tree has exactly $n - 1$ edges.

From this we can see, that if \mathcal{B} is the collection of maximally elements of \mathcal{I} , then in G , \mathcal{B} is the set of spanning trees of the graph.

Weight Function

Greedy Weight function

Lemma

If (E, \mathcal{I}) is a matroid M , then B_G is a solution to the optimisation problem.

Proof.

If $r(M) = r$, then $B_G = \{e_1, e_2, \dots, e_r\}$ is a basis of M . Let B be another basis of M , $B = \{f_1, f_2, \dots, f_r\}$ where $\omega(f_1) \geq \omega(f_2) \geq \dots \geq \omega(f_r)$. This follows from the result of the next additional lemma showing that not only is B_G a maximum weight basis of M , but is also at least as heavy as the elements of B at each step.



Continued

Lemma

if $1 \leq j \leq r$, then $\omega(e_j) \geq \omega(f_j)$.

Proof.

Suppose (seeking a contradiction) that k is the least integer for which $\omega(e_k) \leq \omega(f_k)$. Take $I_1 = \{e_1, e_2, \dots, e_{k-1}\}$ and $I_2 = \{f_1, f_2, \dots, f_{k-1}\}$. Since $|I_1| = |I_2| + 1$ by (I3) implies $I_1 \cup \{f_t\} \in \mathcal{I}$ for some $f_t \in I_2 \setminus I_1$. But this means that $\omega(f_t) \geq \omega(f_t) > \omega(e_k)$. Hence the Greedy algorithm would have chosen f_t over e_k , which gives us our contradiction. □

Wrap Up

The combination of the previous lemma and our use of the greedy algorithm to find a maximal member B of \mathcal{I} of maximum weight allows us to deduce that Kruskal's algorithm does generate a minimum weight spanning tree of a graph.

We have seen that greedy algorithm gives us a solution to our optimisation problem as long as we have a matroid.