0.1 Cryptomorphism: Greedy

The following theorem appears in Oxley's text[**?**] which gives us another cryptomorphism of matroids. From this proof we can see that the greedy algorithm is not optimal for hereditary systems unless that system is also a matroid.

Theorem 0.1. *Let $\mathscr{I}$ be a collection of subsets of a set E. Then $(E, \mathscr{I})$ is a matroid if and only if $\mathscr{I}$ satisfies the following conditions:*
$(I1)$ $\emptyset \in \mathscr{I}$
$(I2)$ If $I \in \mathscr{I}$ and $I' \subset I$ then $I' \in \mathscr{I}$
(G) For all weight functions $\omega : E \longrightarrow \mathbb{R}^+$, the greedy algorithm produces a maximal member of $\mathscr{I}$ of maximum weight.

Proof. Suppose $(E, \mathscr{I})$ is a matroid. Then $\emptyset \in \mathscr{I}$ and (I2) holds trivially. And by *theorem 5.2* we know that the greedy algorithm can find a maximal member $B \in \mathscr{B}$ of maximum weight if $(E, \mathscr{I})$ is a matroid.
Conversely, suppose $(E, \mathscr{I})$ is a pair satisfying $(I1), (I2)$ and (G). Need to prove $\mathscr{I}$ satisfies $(I3)$ in order to have a matroid.
Suppose that (seeking a contradiction), if $I_1, I_2 \in \mathscr{I}$ with $|I_2| > |I_1|$ where there does not exist an $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathscr{I}$.
Now, $|I_1 \setminus I_2| < |I_2 \setminus I_1|$ and $I_1 \setminus I_2$ is non-empty.
So we can choose an $\epsilon > 0$ such that

$$0 < (1 + \epsilon)(|I_1 \setminus I_2|) < |I_2 \setminus I_1| \tag{1}$$

Define $\omega : E \longrightarrow \mathbb{R}^+$ by:

$$\omega(e) = \begin{cases} 2, \text{ if } e \in I_1 \cap I_2 \\ \frac{1}{|I_1 \setminus I_2|}, \text{ if } e \in I_1 \setminus I_2 \\ \frac{1+\epsilon}{|I_2 \setminus I_1|}, \text{ if } e \in I_2 \setminus I_1 \\ 0, \text{ otherwise.} \end{cases}$$

We need the greedy algorithm to fail for only one weight function to get our contradiction.

- The greedy algorithm will choose all the elements of $I_1 \cap I_2$ first as they are the heaviest elements.
- Then it will choose all the elements of $I_1 \setminus I_2$.
- By assumption, it cannot then pick any element of $I_2 \setminus I_1$. Thus the remaining elements of B_G will be in $E \setminus (I_1 \cup I_2)$.

Hence,

$$\omega(B_G) = 2|I_1 \cap I_2| + |I_1 \setminus I_2| \left(\frac{1}{|I_1 \setminus I_2|} \right) = 2|I_1 \cap I_2| + 1$$

But by (I2), I_2 is contained in a maximal member B_2 of $\mathscr{I}$ and, $I_2 \subset B_2$.

$$\omega(B_2) \geq \omega(I_2) = 2|I_1 \cap I_2| + |I_2 \setminus I_1| \left(\frac{1+\epsilon}{|I_2 \setminus I_1|} \right) > 2|I_1 \cap I_2| + 1 = \omega(B_G)$$

$\implies \omega(B_2) > \omega(B_G)$
Which means the greedy algorithm does not find a solution to our optimisation problem shown by *theorem 5.2*, so the greedy algorithm fails for this weight function. We have a contradiction.
$\implies$ (I3) holds.
$\implies$ $(E, \mathscr{I})$ is a matroid. $\square$