Matroids And their Graphs

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1 Cardinality of maximal independent sets

Theorem 1.1. *Show that if $\mathscr{I}$ is a non-empty hereditary set of subsets of a finite set E, then $(E, \mathscr{I})$ is a matroid if and only if, for all $X \subset E$, all maximal members of $\{I : I \in \mathscr{I}$ and $I \subset X\}$ have the same number of elements.*

Proof: $(\implies)$ Let B_1, B_2 be maximal elements of $\{I : I \in \mathscr{I} \text{and } I \subset X\}$
And assume $|B_1| < |B_2|$ Then since $B_1, B_2 \in \mathscr{I}$
There exists $e \in (B_2 \setminus B_1)$ such that $B_1 \cup \{e\} \in \mathscr{I}$
This contradicts our maximality of B_1.

$\implies$ All maximal elements of the set $\{I : I \in \mathscr{I}$ and $I \subset X\}$ in our matroid M have the same cardinality.

$\square$