SOME INTERPRETATIONS OF ABSTRACT LINEAR DEPENDENCE IN TERMS OF PROJECTIVE GEOMETRY.[1]

By Saunders MacLane.

1. *Introduction.* The abstract theory of linear dependence, in the form recently developed by Whitney,[2] is closely related to the study of projective configurations. For any matroid (that is, any finite system of elements for which a suitably restricted notion of "linear dependence" is given) can be interpreted as a schematic geometric figure. Such a schematic figure, like a schematic configuration, is composed of a number of points, lines, planes, etc., with certain combinatorially defined incidences. The problem of representing a matroid by a matrix then becomes simply the problem of realizing a schematic figure by some geometric figure—and the impossibility of always finding such a representation turns out to be a simple consequence of Pascal's theorem! Even when such representation is possible, it depends essentially upon the field from which the elements of the representing matrix are taken. However, only algebraic fields need be used, and hence arises a connection between certain matroids and the algebraic fields in which they can be best represented.

Matroids will be defined by axioms on "rank," as in Whitney's paper. Without loss of generality we can also assume the following two axioms:

R_4: The rank of a single element is always 1.

R_5: The rank of a pair of elements is always 2.

For example, an element e which does not satisfy R_4 may be dropped from or added to a matroid M without otherwise altering the structure of M. These two axioms are equivalent to the following conditions on "bases":

B_3: Every element belongs to at least one base.

B_4: There is no pair of elements (e_1, e_2) such that every base containing e_1 remains a base when e_1 is replaced by e_2.

These conditions on M are in turn equivalent to the following conditions on the dual matroid M^*:

C^*_3: Every element is omitted from at least one circuit complement.

[1] Presented to the American Mathematical Society, December 28, 1934.

[2] H. Whitney, "On the abstract properties of linear dependence," *American Journal of Mathematics*, vol. 57 (1935), pp. 509-533.

C^*_4: For every pair of elements (e_1, e_2) there is a circuit complement containing e_1 but not e_2.

2. *Schematic geometric figures.* A rectilinear plane figure consists of a number of points and of all the lines joining these points in pairs. The combinatorial structure of such a figure can be specified by giving for each line L the set of all those points of the figure which lie on L. These sets satisfy the following axioms:

F_1: Any pair of points belongs to one and only one line.
F_2: Every line contains at least two points.
F_3: No line contains all the points.
F_4: There are at least two points.

A system consisting of a finite number of "points" and certain sets of these points, called "lines," and satisfying these axioms will be called a *schematic plane figure*; it may or may not correspond to some actual figure. In the same way a *schematic space figure* would involve "points," "lines," and "planes," satisfying F_1 to F_4 and the following axioms:

S_1: Every triple of points belonging to no line belongs to one and only one plane.
S_2: Every plane contains three points not on a line.
S_3: No plane contains all the points.
S_4: If a plane contains two points of a line, it contains all the points of that line.

The definition of a schematic n-dimensional figure is similar; it involves "points" and "k-dimensional planes" for $k = 1, \cdots, n - 1$.

The equivalence of schematic figures and matroids may be formulated as follows:

THEOREM 1. *Every schematic n-dimensional figure is a matroid of rank $n + 1$ if the rank of a set of points A is defined as the smallest r such that all the points of A are contained in some $(r - 1)$-plane. Conversely, every matroid of rank $n + 1$ becomes a schematic n-dimensional figure if the k-planes are taken as maximal sets of elements of rank $k + 1$. This translation sets up a one-one correspondence between matroids and schematic figures.*[3]

From this theorem it follows that a schematic n-dimensional figure is

[3] The conditions R_4 and R_5 of § 1 are necessary here to exclude the geometrically meaningless cases of a point of dimension -1 or of two coincident points.

determined once the corresponding matroid M or its dual M^* is given. By Whitney's results, this dual M^* is completely determined by its circuit complements. The circuit complements in M^* correspond to maximal sets of rank $r(M) - 1$ in M. Hence

THEOREM 2. *A schematic n-dimensional figure is completely determined if its* $(n-1)$*-planes are given. If a set of "points" and certain subsets of this set are given, these subsets will be the* $(n-1)$*-planes of some figure if and only if they are the circuit complements of a matroid* M^**; that is, if and only if these subsets satisfy the above axioms* C^*_3 *and* C^*_4*, while their complements satisfy Whitney's axioms* C_1 *and* C_2 *for circuits.*

These results also show that matroids form a direct generalization of schematic configurations. A *schematic plane configuration*[4] $p_\gamma g_\pi$ consists of p "points" and g "lines," with each point on γ lines and each line on π points. Such a configuration becomes a schematic figure in the above sense if those pairs of points not already joined by lines are joined by new "diagonal" lines. Similar transformations are possible for space configurations.

3. *Matrix representations of matroids.* The columns of a matrix stand in relations of rank and thus form a matroid. The question whether every matroid can be represented in this way by a matrix is clarified by the equivalence of matroids and schematic figures. Thus Whitney has constructed a matroid of rank 3 which cannot be represented as a matrix. This matroid has 9 elements $1, 2, \cdots, 9$ and the following 20 circuit complements:

$$712,\ 814,\ 923,\ 734,\ 836,\ 945,\ 756,\ 825;$$
$$16,\ 19,\ 69,\ 13,\ 15,\ 24,\ 26,\ 35,\ 46,\ 78,\ 79,\ 89.$$

Any attempt to represent this matroid yields a figure in which the lines 16, 19, and 69 coalesce into one line 169. A geometric representation reveals at once that this is simply Pascal's theorem for the hexagon 723845 inscribed in the degenerate conic composed of the two lines 743 and 825. The points 1, 6, and 9 are the intersections of opposite sides of the hexagon. In exactly the same way the theorem of Desargues may be used to construct a matroid with ten elements which has no matrix representation. Furthermore, the matroid arising from Pascal's theorem can be generalized to the case of $2m+3$ elements, which we denote by $1, 2, \cdots, 2m, \alpha, \beta, \gamma$. The circuit complements are:

[4] F. Levi, *Geometrische Konfigurationen*, p. 4.

$$12\alpha,\ 34\alpha, \cdots, (2m-3,\ 2m-2,\ \alpha),\ (2m-1,\ 2m,\ \alpha),$$
$$14\beta, 36\beta, \cdots, (2m-3, 2m, \beta), (2m-1, 2, \beta),$$
$$23\gamma,\ 45\gamma, \cdots, (2m-2,\ 2m-1,\ \gamma).$$

together with all the pairs of elements not included in one of these triples. No matrix representation is possible, for any attempt to construct one yields a matroid with the additional circuit complement $(2m, 1, \gamma)$.

These matroids fail to be matrices because of the presence of too few circuit complements. Failure is also possible for the opposite reason. Thus the plane figures (matroids) formed by finite projective geometries [5] can be represented only by matrices of elements from a finite field. Another important special case of the matrix representation of matroids is the problem of constructing a geometric realization for schematic plane configurations. Here it is well-known that a configuration (p_γ, g_π) cannot in general be realized if [6]

$$2(p+g)-p\gamma-8<0.$$

The use of geometric figures also simplifies the investigation of the conditions for the representability of individual matroids. Thus for a matroid M of rank 3 we need only find three homogeneous coördinates for each element (point) of the matroid, such that when three points lie on a line (i. e., are contained in a circuit complement of the dual matroid), then the determinant of the corresponding coördinates is zero, and conversely. This application of the usual theorems of analytic geometry can replace Whitney's Theorem 32.

4. *Representation in finite algebraic fields.* The configuration of eight elements which can be represented in the complex but not in the real plane [7] suggests that the representability of a matroid depends essentially on the field from which the elements of the representing matrix are taken. Another similar example can be constructed for the field $R(2^{1/2})$, where R is the field of rational numbers. We need only take a point with coördinates $(1, 2^{1/2}, 0)$ and carry out the constructions in the von Staudt algebra of throws corresponding to

$$(2^{1/2})(2^{1/2})=1+1.$$

The resulting figure (matroid) consists of 11 points, $1, 2, \cdots, 9, 0, \alpha$, the following sets of points being collinear:

$$1279\alpha,\ 2356,\ 1380,\ 248,\ 347,\ 578,\ 549,\ 690,\ 50\alpha,\ 68\alpha.$$

[5] Veblen and Young, *Projective Geometry*, vol. I, p. 3 and p. 201.

[6] E. Steinitz, *Encyklopädie der mathematischen Wiss.*, III A B 5a, p. 485.

[7] F. Levi, *loc. cit.*, pp. 98-102.

Any attempt to represent this matroid by a matrix leads to a matrix whose elements generate a field containing $R(2^{1/2})$. This matroid is thus a sort of geometric analog of the irreducible equation for $2^{1/2}$. Generalization yields:

THEOREM 3. *Let K be a finite algebraic field over the field of rational numbers. Then there exists a matroid M of rank 3 which can be represented by a matrix with elements in K, while any other representation of M by a matrix with elements in a number-field K_1 requires $K_1 \supset K$.*

Such finite fields are sufficient for the representation of all representable matroids, in the following sense.

THEOREM 4. *Let the matroid M be representable by a matrix of complex numbers. Then M can also be represented by a matrix with elements from an algebraic field of finite degree.*

For let the matroid M have rank n and consist of p points, and let these points be assigned the indeterminate homogeneous coördinates a_{ij}, for $i = 1, \cdots, n;\ j = 1, \cdots, p$. Each circuit complement of the dual of M requires the vanishing of a number of determinants of the a_{ij}, and thus corresponds to a number of algebraic equations for these quantities. The set of all values of the a_{ij} giving at least the required circuit complements thus constitutes an algebraic manifold N_1 in the np-dimensional space of all coördinates a_{ij}. The set of those coördinates yielding additional undesired circuit complements forms another manifold N_2. Since the original matrix was representable, there exists a point of $N_1 - N_2$. The parametric representation of irreducible algebraic manifolds [8] thus makes possible the construction of a point with algebraic coördinates in $N_1 - N_2$.

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[8] B. L. van der Waerden, *Moderne Algebra*, vol. 2, p. 51 ff.