Matroids And their Graphs

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Abstract

Matroids are an abstraction of linear dependence according to H.Whitney in his seminal paper on matroid theory. They are extremely flexible systems, and can be characterised in a number of diverse ways, in this paper we focus on matroids formed on graphs. In particular, this paper focuses on the optimality of the greedy algorithm in solving optimisation problems in graph theory such as the minimal spanning tree problem which can be shown to be solved in this way when the independent sets of the graph forms a matroid. Futhermore, we show that maroids are the only hereditary systems for which this is true. The paper concludes with some insight into other set-systems which provide optimal solutions for combinatorial optimisation problems through the greedy algorithm.

1 Introduction

1.1 Matroid

Definition 1.1. An *independence system* is a pair $(E, \mathscr{S})$, where E is a finite set and $\mathscr{S}$ is a collection of sets satisfying the following conditions:
(I1) $\mathscr{S}$ is non-empty.
(I2) $\mathscr{S}$ is a hereditary subset of the power set of E.
The elements of $\mathscr{S}$ are called the *independent sets*.

Definition 1.2. A matroid is a pair $(E, \mathscr{I})$ with finite ground set E and $\mathscr{I}$ being a collection of independent subsets of E satisfying the following conditions:
(I1): The empty set is always independent
(I2): Every subset of an independent set is independent
(I3): If A and B are two independent sets of $\mathscr{I}$ and $|A| > |B|$, then there exists $x \in A \setminus B$ such that $B \cup \{x\}$ is in $\mathscr{I}$

Lemma 1.1. *Prove that $(E, \mathscr{I})$ is a matroid if and only if $\mathscr{I}$ satisfies $(I1), (I2)$ and the following condition:*
$(I3)'$ *If I_1, I_2 are in $\mathscr{I}$ and $|I_2| = |I_1| + 1$, then there is an element $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathscr{I}$*

Proof. Suppose $(E, \mathscr{I})$ is a matroid. Then by hypothesis, $\mathscr{I}$ satisies $(I1), (I2)$. Let $I_1, I_2 \in \mathscr{I}$ and $|I_2| = |I_1| + 1$ then $|I_2| > |I_1|$ and so $(I3)'$ holds trivially due to the defnition of a matroid. But by $(I2)$ there is an I_2' such that $e \in I_2'$ and $|I_2'| = |I_1| + 1$ as a matorid is hereditary and $I_2' \subset I_2$. $\implies \exists e \in I_2' \setminus I_1$ such that $I_1 \cup \{e\} \in \mathscr{I}$.

Conversely, Suppose $\mathscr{I}$ satisfies $(I2), (I1), (I3)'$

Let I_1, I_2 be in $\mathscr{I}$ such that $|I_2| = |I_1| + 1$, then there exists e in $I_2 \setminus I_1$ such that $I_1 \cup \{e\}$.

Let I_1' be in $\mathscr{I}$ where $I_1' \supseteq I_1$ then e in $I_1' \setminus I_2$ and $|I_1'| > |I_2|$. From this we can see that every independent set A with cardinality greater than an independent set B can be shrunk by removing elements while retaining independence due to $(I2)$(the hereditary property) until we have $|A| = |B| + 1$ and then we can apply $(I3)'$. Therefore the matroid property $(I3)$ is satisfied through $(I3)'$. $\square$

Example 1.1. Let M_1, M_2 be matroids on a set E. Let $E = \{1, 2, 3, 4\}$

Let $\mathscr{I}_1 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}\}$

Let $\mathscr{I}_2 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}\}$

Let $\mathscr{I}_1 \cap \mathscr{I}_2 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\{3, 4\}\}$

let $(E, \mathscr{I}_1 \cap \mathscr{I}_2)$ be a pair, is it a matroid?

Let $I_1 = \{1, 2\}$ and $I_2 = \{3\}$

If there exists an $e \in I_1$ such that $I_2 \cup \{e\} \in \mathscr{I}$ then we have a matroid. Otherwise we do not have a matroid.

$I_2 \cup \{1\} = \{1, 3\} \notin \mathscr{I}$,

$I_2 \cup \{2\} = \{2, 3\} \notin \mathscr{I}$

$\implies (E, \mathscr{I}_1 \cap \mathscr{I}_2)$ is not a matroid.

1.2 Example

Definition 1.3. Let $\mathscr{I}$ be the collection of subsets of E that do not contain all of the edges of any *cycle* of G. We get a matroid on the edge set of every graph G by defining $\mathscr{I}$ in this way. This matroid is called the *cycle matroid* of the graph G and is denoted $M(G)$.

Definition 1.4. If M_i, M_j are matroids , then there exists a bijection from the ground set of M_i to the ground set of M_j, such that a set is independent in the first matroid if and only if it is independent in the second matroid, then M_i and M_j are said to be isomorphic.

Note. A matroid that is isomorphic to the cycle matroid of some graph is called graphic. And every graphic matroid is binary

The numbers of non-isomorphic matroids and binary matroids on an n-element set for $0 \leq n \leq 8$

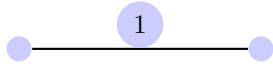| n	0	1	2	3	4	5	6	7	8
matroids	1	2	4	8	17	38	98	306	1724
binary matroids	1	2	4	8	16	32	68	148	342

Example: Let E be a set, $\{1, 2, 3\}$ then:

Show there are exactly eight non-isomorphic matroids on E. Along with the corresponding Graph of each matroid. This confirms to us the value in the previous table for $n = 3$.
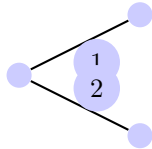
Solution:

$$\{\emptyset\}$$

$$\{\{\emptyset\}, \{1\}\} \cong \{\{\emptyset\}, \{2\}\} \cong \{\{\emptyset\}, \{3\}\}$$
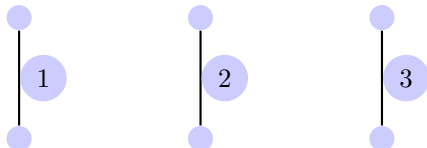

$$\{\{\emptyset\}, \{1\}, \{2\}, \{1, 2\}\} \cong \{\{\emptyset\}, \{1\}, \{3\}, \{1, 3\}\} \cong \{\{\emptyset\}, \{2\}, \{3\}, \{2, 3\}\}$$
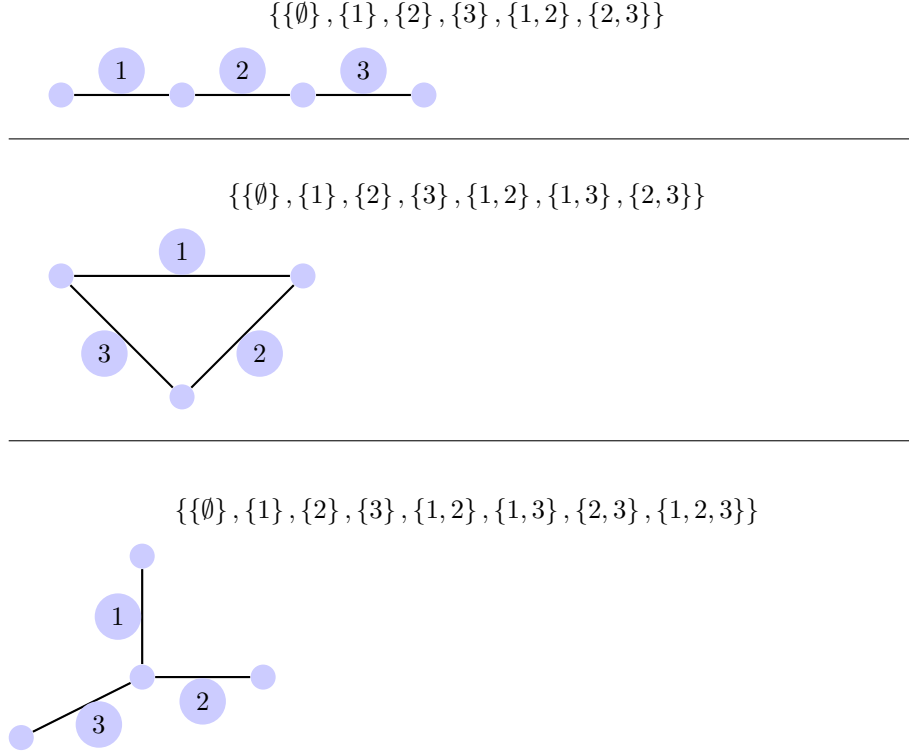

$$\{\{\emptyset\}, \{1\}, \{2\}\}$$

$$\{\{\emptyset\}, \{1\}, \{2\}, \{3\}\}$$

$$\{\{\emptyset\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}\}$$

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  1       2       3
●————●————●————●
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$$\{\{\emptyset\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$$

```
        1
  ●————————————●
   3          2
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$$\{\{\emptyset\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

1.3 Graphic Matroid

Theorem 1.2. *Let G be a graph and $\mathscr{I}$ be the set of all cyclefree subgraphs of G. Show that if we have the pair $(E, \mathscr{I})$ as defined above by our graph, we have a matroid. In other words, that the cycle matroid $M(G)$ of a graph is a matroid.*

Proof. Let $A, B \in \mathscr{I}$ with $|A| = |B| + 1$. To prove $I3$ of the definition of a matroid, We show that for some $a \in A$,
$B \cup \{a\} \in \mathscr{I}$, we should consider $B \cup \{a\}$ for each $a \in A$.
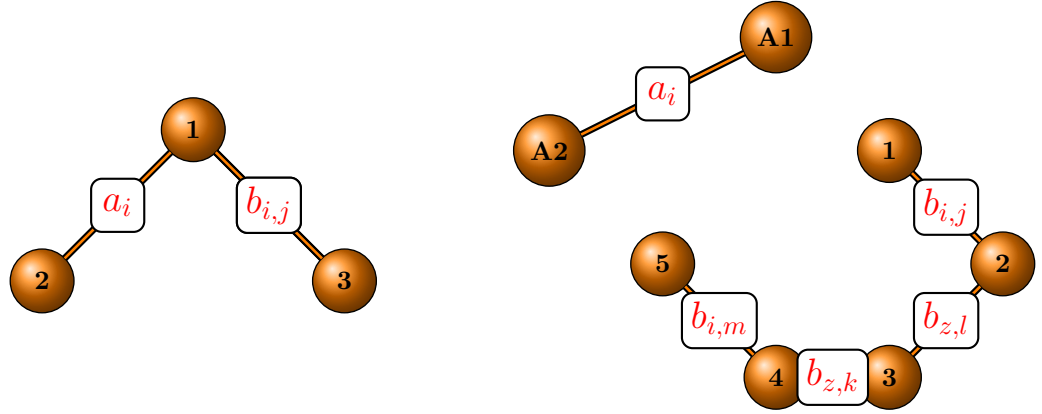Now suppose $|A| > |B|$ and that $|A| = |B| + 1$
Let $|A \cap B| = s$, $|A \setminus B| = r$, $|A| = s + r$ and $|B| = s + r - 1$
So $|B \setminus A| = r - 1$
Suppose $A \setminus B = \{a_1, a_2, ..., a_r\}$
Suppose $B \cup \{a_i\} \notin \mathscr{I}$ for each $i \in \{1, 2, ...\}$
Consider a_i for $i = 1, 2, ...$ there must be a path $b_{i1}, b_{12}, ..., b_{ir}$ of edges in B such that a_i make a cycle

Notation: $P(b_j, b_k)$ denotes a set of edges forming path in B from the edges b_j to b_k But $P(b_j, b_k) \cap A$ is not necessarily empty. If $P(b_j, b_k) \subset A$ then $P(b_j, b_k) \cup \{a_i\}$ would be a cycle, then A would not be in $\mathscr{I}$, so at least one of the $b_i \in P(b_j, b_k)$ is contained in $B \setminus A$.
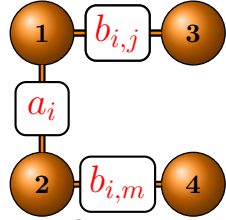
Given $A = \{a_1, ..., a_r\}$ for each a_i associate a $b_i \in B \setminus A$. Let $\hat{B} = \{b_1, ..., b_r\}$
Case 1: The b_i's are distinct
The b_i's are distinct and as shown previously each of the b_i's must be in $|B \setminus A|$ in order to avoid a circuit in A.
Therefore,$|B| \geqslant A$. Contradicting $|A| > |B|$.
Hence, I3 holds. Case 2: When the b_i's are not all distinct. Let $b_1 = b_2$.

Imagine the figure to the left in place of the graph $\{a_i, b_{i,j}\}$ above and observe how this would affect the graph of B.

We use the same argument as in Case 1 only here we need two distinct $b_i \in P(b_j, b_k)$ where $b_i \in B \setminus A$ such that $P(b_j, b_k) \cup \{a_i\}$ is a cycle. This can be seen in the diagram above, there must be another edge in the union of the paths which is in $B \setminus A$ or else we get a cycle in A. Otherwise,$P(b_j, b_k) \subset A$ then $P(b_j, b_k) \cup \{a_i\}$ would be a circuit and then $A \notin I$. Therefore, $|B| \geq |A|$, and we have a contradiction.
Hence I3 holds. $\qquad\square$

2 Cryptomorphisms

2.1 Circuit characterization of a matroid

Definition 2.1. By using (I1)–(I3) , it is not difficult to show that the collection $\mathscr{C}$ of circuits of a matroid M has the following three properties:
(C1) The empty set is not in $\mathscr{C}$

(C2) No member of $\mathscr{C}$ is a proper subset of another member of $\mathscr{C}$

(C3) if C_1 and C_2 are distinct members of C and $e \in C_1 \cap C_2$, then $(C_1 \cup C_2) \setminus \{e\}$ contains a member of $\mathscr{C}$

Theorem 2.1. *Let M be a matroid and $\mathscr{C}$ be its collection of circuits. Then $\mathscr{C}$ satisfies (C1) - (C3)*

Proof:
(C1) is obvious as by $I1$ the empty set must always be an independent set.

(C2) is also straightforward because any $C \in \mathscr{C}$ is a minimally independent set by definition. Therefore , if there exists a $C_1 \in \mathscr{C}$ such that $C_1 \subset C$ then $C_1 \in \mathscr{C}$ and C is not a minimally dependent subset of E.

(C3) Let $A, B \in \mathscr{C}$ and suppose that(Seeking a contradiction) $(A \cup B) \setminus \{e\}$ where e is $\in (A \cap B)$ does not contain a circuit.
Then $(A \cup B) \setminus \{e\}$ is independent and therefore in $\mathscr{I}$

The set $A \setminus B$ is non-empty.
Let $s \in A \setminus B \implies s \in A$
as A is in $\mathscr{C}$ it is minimally dependent. $\implies A \setminus \{s\} \in \mathscr{I}$ i.e is independent.

Let J be a maximal independent set of $(A \cup B)$ with the following properties: $S \setminus \{s\} \subset J$ and therefore $\{s\} \notin J$ but as B is a circuit there must be some element $t \in B$ that is not in J. s and t are distinct.

$\implies |J|$ must be at most equal to $|(A \cup B) \setminus \{s,t\}|$
$\implies |J| \leq |(A \cup B) \setminus \{s,t\}| = |(A \cup B)| - 2 < |(A \cup B) \setminus \{e\}|$

Now by (I3) we can subsitute elements from $|(A \cup B) \setminus \{e\}|$ into $|(A \cup B) \setminus \{s,t\}|$ that are not in $|(A \cup B) \setminus \{s,t\}|$ but the only elements that fits this condition are $\{s,t\}$ and introducing either of these elements breaks the independence of J.
Therefore, $|(A \cup B) \setminus \{e\}|$ must contain a circuit

$\square$

Theorem 2.2. *Let E be the edge sets of a graph G and let $\mathscr{C}$ be the edge sets of cycles in G.*
Then $\mathscr{C}$ is the set of circuits of a matroid.

Proof: Let $A, B \in \mathscr{C}, A \neq B$ and let $e \in A \cap B$

We must now construct a cycle of G whose edge set is contained in $(A \cup B) \setminus \{e\}$

For $i = 1, 2, 3, ...$ let P_1 be a path whose edge set is $A \setminus \{e\}$
$A \setminus \{e\} \in \mathscr{I}$ therefore P_i is not a cycle of G. This path will traverse from the edge a_j to a_u where u, j were the vertices connecting the edge e to $(A \cup B) \setminus \{e\}$ to make $A \cup B$.

Now perform the same procedure for a path P_2 whose edge set is $B \setminus \{e\}$.

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P_1 and P_2 should meet at the junctions u, v, where e was removed to make $(A \cup B) \setminus \{e\}$

Therefore $P_1 \cup P_2$ should be a cycle of G.

$\implies$ (C3) holds

$\implies \mathscr{C}$ is the edge sets of cycle in G.

$\square$

2.2   Bases

Theorem 2.3. *Show that if $\mathscr{I}$ is a non-empty hereditary set of subsets of a finite set E, then $(E, \mathscr{I})$ is a matroid if and only if, for all $X \subset E$, all maximal members of $\{I : I \in \mathscr{I} \text{ and } I \subset X\}$ have the same number of elements.*

Proof: ($\implies$) Let B_1, B_2 be maximal elements of $\{I : I \in \mathscr{I} \text{and } I \subset X\}$

And assume $|B_1| < |B_2|$ Then since $B_1, B_2 \in \mathscr{I}$

There exists $e \in (B_2 \setminus B_1)$ such that $B_1 \cup \{e\} \in \mathscr{I}$

This contradicts our maximality of B_1.

$\implies$ All maximal elements of the set $\{I : I \in \mathscr{I} \text{ and } I \subset X\}$ in our matroid M have the same cardinality.

$\square$

Definition 2.2. A base is a maximally independent subset of $\mathscr{I}$.

As seen previously all maximally independent sets in a matroid have the same cardinality.

Theorem 2.4. *Let $\mathscr{B}$ be a set of subsets of a finite set E. Then $\mathscr{B}$ is the collection of bases of a matroid on E if and only if $\mathscr{B}$ satisfies the following conditions:*

(B1) $\mathscr{B}$ is non-empty.

(B2) If B_1 and B_2 are members of $\mathscr{B}$ and $x \in B_1 \setminus B_2$, then there is an element y of $B_1 \setminus B_2$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{B}$.

Proof:

By (I1) $\emptyset$ is always independet, so $\mathscr{B}$ must always contain at least the $\emptyset$, (B1) holds.

Let $B_1, B_2 \in \mathscr{B}, B_1 \neq B_2$.

$|B_1| = |B_2|$ so (I3) does not directly apply here.

Let $x \in B_1 \setminus B_2 \implies x \in B_1, x \notin B_2$

$|B_1| = |B_1 \setminus \{x\}| + 1 \implies B_1 \setminus \{x\} \in \mathscr{I}$ but not in $\mathscr{B}$

$|B_2| = |B_1 \setminus \{x\}| + 1$ so now we can use (I3)

Now $\exists y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{I}$

$|(B_1 \setminus \{x\}) \cup \{y\}| = |B_1 \setminus \{x\}| + 1 = |B_1| = ... = |B_r|$ as all maximal elements of $\mathscr{I}$ have the same cardinality $\implies (B_1 \setminus \{x\}) \cup \{y\}$ is maximal in $\mathscr{I}$

$\implies (B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{B}$, (B2) holds.

Conversely, suppose that $\mathscr{B}$ satisfies (B1) and (B2).

By B1 $\mathscr{B}$ is always non-empty which shows I1 holds.

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By definition, a base $B_1 \in \mathscr{B}$ is a maximally independent subset of E. Then for all $B_i \in \mathscr{B}$ the subsets $b_{i,k} \subseteq B_i$ are independent. Therefore all the $b_{i,k}$ are in $\mathscr{I}$.

$\implies$ (I2) holds. Showing a matroid can be generated through the bases.

Assume that (I3) fails. That, for $I_1, I_2 \in \mathscr{I}$, where $|I_1| = |I_2| + 1$,

there $\exists y \in I_1 \setminus I_2$ such that $I_2 \cup \{y\} \in \mathscr{I}$.

Let $B_1, B_2 \in \mathscr{B}, |B_1| = |B_2|$.

Let $x \in B_1 \setminus B_2$ then $B_1 \setminus \{x\} \subset B_1$.

$\implies B_1 \setminus \{x\}$ is independent.

Then there exists a $y \in B_2 \setminus (B_1 \setminus \{x\})$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{B}$ from (B2). And if $(B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{B}$ it is also in $\mathscr{I}$. A contradiction.

$\implies$ (I3) holds and we have a matroid.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3 Graph Theory

3.1 Graph Theory Definitions

Definition 3.1 (Connected). A graph is connected when there is a path between each pair of vertices.

Definition 3.2 (Tree). A connected graph with no circuits.
All subsets of E in $\mathscr{I}$ in $M(G)$ the cycle matroid of G are the trees of the graph G.

Definition 3.3 (Forest). A disconnected graph with no circuits

Proposition 3.1. *A theorem of Cayley(1889) states that the number of distinct laballed trees which can be drawn using n labelled points is n^{n-2}.*

Corollary. *The number of distinct labelled spanning trees which can be drawn using n labelled points is n^2.*

Remark. Determining the number of spanning trees of a graph in polynomial time is NP-hard.

The vertices are often labelled and referred to as nodes. Information may be recorded in them and a cost, penalty or probability may be associated with each edge. For example, the problem of joining all nodes in a graph by the minimum length of cable leads to a tree known as a *minimum spanning tree.*

Definition 3.4 (Spanning Tree). A spanning tree T of an undirected graph G is a subgraph that is a *tree* which includes all of the vertices of G.
A disconnected graph cannot contain a spanning tree as we cannot find a walk which brings us to all of the disconnected vertices.

Definition 3.5 (Acyclic). An acyclic graph is a graph which contains no closed walks.

Definition 3.6 (Walk)**.** If there are vertices $v_{i-1}v_i$ for $i = 1, ..., n$ the sequence is called a walk.

Definition 3.7 (Closed Walk)**.** If $v_0 = v_n$ this sequence is called a closed walk.

Definition 3.8 (Bridge)**.** A bridge(or cut edge) is an edge of a graph whose deletion increases the numbe rof connected components. Equivalently, an edge is a bridge is and only if it is not contained in any cycle.

Lemma 3.2. *Any acyclic graph on n vertices has at most $n - 1$ edges.*

Proof: Let G be an acyclic graph with n vertices.
if n = 1 then we have no edges hence nothing to prove.
Assumse $n > 1$, let e be an edge in G connecting two vertices ab in the vertex set of G.
Let $H = G \backslash \{e\}$, H has one more connected component than G. H has two maximal acyclic connected componenets, and thus can be decomposedinto acyclic connected graphs $H_1, H_2, ..., H_k wherek \geq 2$.
By induction, we can assume each graph H_i contains at most $n_i - 1$ edges where n_i is the number of vertices of H_i.
Then G has at most $n - 1$ edges.
$(n_1 - 1) + ... + (n_k - 1) + 1 = (n_1 + ... + n_k) - (k - 1) \leq n - 1$ edges.
$\square$

Lemma 3.3. *Let G be a graph. Then the following conditions are equivalent:*
1) G is a tree.
2) G does not contain any cycles, but adding any further edge yields a cycle.
3) Any two vertices of G are connected by a unique path.
4) G is connected, and any edge of G is a bridge.

Proof: $(1) \Longrightarrow (2)$
Suppose that G is a tree,then G is a connected graph with no circuits. Let e be a new edge in G with $e = g_i g_k$ where g_i, g_k are in the vertex set of G. Then as $G \cup \{e\}$ must be connected, there exists a walk between any pair of vertices of G. So there is a walk K from $g_j \to g_k$ and there is also a walk L from $g_k \to g_j$ where K does not traverse e and L does traverse e and so we have a cycle.

Proof: $(2) \Longrightarrow (3)$
Let u, v be vertices of G. If there was not path joining uv in G then $e = uv$ does not create a cycle in G. Thus G must be connected.
Suppose G contained two different paths W_1, W_2 from u to v.
Then $u \longrightarrow v \longrightarrow u$ would be a closed walk in G.
$\Longrightarrow$ G contains a cycle. Which is a contradiction.

Proof: $(3) \Longrightarrow (4)$
G is connected by hypothesis. Let $e = uv$ be an edge in G.
Suppose e is not a bridge, then $G \backslash \{e\}$ is still connected. But then we have two distinct path from u to v in G.

Proof: $(4) \implies (1)$

G is connected by hypotheis.

Suppose G contains a cycle K. Then any edge of K could be ommited from G, and the resulting graph would still be connected. In other words, no edge of K would be a bridge, a contradiction.

□

Definition 3.9. let (G, ω) be a network. For any subset T of the edge set of G, ω is called the weight of T.

$$\omega(T) = \sum_{e \in T} \omega(T) \tag{1}$$

Definition 3.10 (Minimal Spanning Tree)**.** A spanning tree is a *minimal* spanning tree if its weight is minimal of all the weights of spanning trees.

A forest can be considered by finding a minimal spanning tree for each connected component of G.

Remark. If the weight ω is constant, any spanning tree is minimal.

In this case, determining a minimal spanning tree could be done using a breadth-first search.

Theorem 3.4 (Without Proof)**.** *Let (G, ω) be a network, where G is a connected graph. A spanning tree T of G is minimal if and only if, for each edge e in $G \backslash T$, we have,*

$\omega(e) \geq \omega(f) \ \forall \ edges \ f \ in \ C_T(e).$

4 Algorithms

4.1 Greedy Algorithm

Algorithm 1 Greedy algorithm

Let $(E, \mathscr{S})$ be an independence system and $\omega : E \longrightarrow \mathbb{R}^+$

1: **procedure** GREEDY$(E, \mathscr{S}, \omega, T)$
2:   order the elements of E according to their weight
3:   $E = \{e_1, ..., e_m\}$ with $\omega(e_1) \geq \omega(e_2) \geq ... \geq \omega(e_m)$
4:   $T \leftarrow \emptyset$
5:   **for** $k = 1$ to m **do**
6:     **if** $T \cup \{e_k\} \in \mathscr{S}$ **then**
7:       append e_k to T

Algorithm 2 Greedy algorithm

The *greedy algorithm* for the pair $(\mathscr{I}, \omega)$ is as follows:

1: **procedure** GREEDY$(\mathscr{I}, \omega)$
2:      Set $x_0 = \emptyset$ and $j = 0$
3:      **if** $\exists e \in E \setminus x_j$ such that $x \cup \{e\} \in \mathscr{I}$ **then**
4:           Choose such an element e_{j+1} of maximum weight,
5:           let $x_{j+1} = x_j \cup \{e_{j+1}\}$ and **Break**
6:      **else**
7:           Let $x_j = B_G$
8:           **return** x_j
9:      $j++$

4.2 Kruskal's Algorithm

Algorithm 3 Kruskal's algorithm

Let G be a connected graph with vertex set $V = \{1, ..., n\}$ and $\omega : E \longrightarrow \mathbb{R}^+$ a weight function. The edges of G are ordered according to their weight, that is, $E = \{e_1, ..., e_m\}$ and $\omega(e_1) \leq ... \leq \omega(e_m)$.

1: **procedure** KRUSKAL(G, ω, T)
2:      $T \leftarrow \emptyset$
3:      **for** $k = 1$ to m **do**
4:           **if** ACYCLIC$(T \cup \{e_k\})$ **then**
5:           append e_k to T

---

1) Create a graph F containing just the vertices of G.
2) Create a set $S = E(G)$; the edge set of G.
3) While S is non-empty and F is not yet spanning
3(a) Remove an edge with minimum weight from S.
3(b) If the removed edge introduces no cycles to F
then add the edge to F

4.3 Depth-first Search

Algorithm 4 DFS

Let G be a graph with vertex set $V = \{1, ..., n\}$

1: **procedure** DFS(G, V)
2:     label v as discovered
3:     **for all** edges from v to w **in** G.adjacentEdges(V) **do**
4:         **if** (vertex w is not labelled as discovered) **then**
5:             recursively call DFS(G, w)

This algorithm allows you to find the connected components of a disconnected graph. Then using the following algorithm we can check if our forest at each step of our algorithm is acyclic.

Algorithm 5 Acyclic Check

Let G be a graph with the set of connected componenets C as found by DFS(G,v) where v is an arbitrary vertex in G.

1: **procedure** ACYCLIC(G, C)
2:     **for all** i **in** C **do**
3:         **if** i.edgeCount$() > n - 1$ **then return** False
        return True

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5 Optimization Problems

Problem: *Find a maximal member B of $\mathscr{I}$ of maximum weight.*

Note. Let B_G be a base of a matroid generated by the greedy algorithm.

Theorem 5.1. *If $(E, \mathscr{I})$ is a matroid M, then B_G is a solution to the optimization problem.*

Proof. If $r(M) = r$, then $B_G = \{e_1, e_2, ..., e_r\}$ is a basis of M. Let B be another basis of M, $B = \{f_1, f_2, ..., f_r\}$ where $\omega(f_1) \geq \omega(f_2) \geq ... \geq \omega(f_r)$. We claim that $\omega(e_j) \geq \omega(f_f) \; \forall j$, then it follows that $\omega(B_G) \geq \omega(B)$ for any other basis in $\mathscr{B}$. □

Lemma 5.2. *If $1 \leq j \leq r$, then $\omega(e_j) \geq \omega(f_j)$.*

Proof. Suppose (seeking a contradiction) that k is the least integer for which $\omega(e_k) < \omega(f_k)$. Take $I_1 = \{e_1, e_2, ..., e_{k-1}\}$ and $I_2 = \{f_1, f_2, ..., f_k\}$. Since $|I_2| = |I_1| + 1$ (I3) implies $I_1 \cup \{f_t\} \in \mathscr{I}$ for some $f_t \in I_2 \setminus I_1$. But this means that
$\omega(f_t) \geq \omega(f_k) > \omega(e_k)$
And hence the Greedy algorithm would have chosen f_t over e_k, which gives us our contradiction. □

Lemma 5.3. *Let M be a mtroid and $\omega : E(M) \longleftrightarrow \mathbb{R}^{\Vdash}$ be a one-to-one function. Prove that M has a unique basis of maximum weight.*

Proof. Let ω be an injective function, this will then allow no repition of weights on edges. As this would mean more than one value in the domain would be mapped to one value in the range.
We want to find an independent set A whose weight is maximal, where

$$\omega(A) := \sum_{e \in A} \omega(e) \tag{2}$$

We can then arrange our edges in a set S by order of decreasing weight such that $\omega(e_1) \geq \omega(e_2) \geq ... \geq \omega(e_k)$.
We have already seen in *theorem 5.1* that the greedy algorithm as described in *algorithm 2/4.1* provides a solution to this optimisation problem. And since there is no repition in weights there is no point in the algorithm where there is more than a single choice as to the next chosen edge. Therefore there is only one possible soltuion when out weight function is injective.

If we lose the injectivity condition, then this is not the case and we cannot gaurantee uniqueness in general. $\qquad \square$

Theorem 5.4. *Let M be a mtroid and $\omega : E(M) \longleftrightarrow \mathbb{R}^{\Vdash}$. When the greedy algorithm is applied to the pair $(\mathscr{I}, \omega)$, each iteration of the greedy algorithm involves a potential choice. Thus, in general, there are a number of different set that the algorithm can produce as solutions to the optimisation problem $(\mathscr{I}, \omega)$. Let $\mathscr{B}_G$ be the set of such sets and let $\mathscr{B}_{max}$ be the set of maximum weight bases of M. Prove that $\mathscr{B}_G = \mathscr{B}_{max}$.*

Proof. Suppose ω is an injective function, then we've show there is a unique maximum weight basis for M in *theorem 5.4* and so the proof of this trivial.

Now suppose ω is not injective.This means maximal weight bases of M are not in general unique.
If $r(M) = r$, then $B_G = \{e_1, e_2, ..., e_r\}$ is a basis of M. Let B_G' be another basis of M, $B_G' = \{f_1, f_2, ..., f_r\}$. Both B_g, B_G' are bases generated through the greedy algorithm as described in *section 4.1*. We arrange both these bases in terms of decreasing order where $\omega(e_1), \omega(f_1)$ are the heaviest elements in their respective bases.
As ω is not an injective function B_g and B_G' are distinct. We also know from *theorem 5.1* that the greedy algorithm finds a maxiaml member B of $\mathscr{I}$ of maximum weight.

Therefore, any base generated through the greedy algorithm is maximally weighted. Meaning, that $\omega(B_G) = \omega(B_G')$. As if $\omega(B_G) < \omega(B_G')$ then this would mean the greedy algorithm does not find a solution to out optimisation problem, contradicting *theorem 5.1*. And therefore, $\omega(e_j) = \omega(f_j) \ \forall j$ and since all these bases are maximally weighted, $\mathscr{B}_G = \mathscr{B}_{max}$. $\qquad \square$

Theorem 5.5. *Let $\mathscr{I}$ be a collection of subsets of a set E. Then $(E, \mathscr{I})$ is a matroid if and only if $\mathscr{I}$ satisfies the following conditions:*

$(I1)$ $\emptyset \in \mathscr{I}$

$(I2)$ If $I \in \mathscr{I}$ and $I' \subset I$ then $I' \in \mathscr{I}$

(G) *For all weight functions $\omega : E \longrightarrow \mathbb{R}^+$, the greedy algorithm produces a maximal member of $\mathscr{I}$ of maximum weight.*

Proof. Suppose $(E, \mathscr{I})$ is a matroid. Then $\emptyset \in \mathscr{I}$ and (I2) holds trivially. And by *theorem 5.1* we know that greedy algorithm cna find a maximal $B \in \mathscr{B}$ of maximum weight if $(E, \mathscr{I})$ is a matroid.

Conversely, suppose $(E, \mathscr{I})$ is a pair satisfying $(I1), (I2)$ and (G). Need to prove $\mathscr{I}$ satisfies $(I3)$ in order to have a matroid.

Suppose that (seeking a contradiction), that is $I_1, T_2 \in \mathscr{I}$ with $|I_2| > |I_1|$ such that $I_1 \cup \{e\} \in \mathscr{I}$.

Now, $|I_1 \setminus I_2| < |I_2 \setminus I_1|$ and $I_1 \setminus I_2$ is non-empty.

So we can choose an $\epsilon > 0$ such that

$$0 < (1 + \epsilon)(|I_1 \setminus I_2|) < |I_2 \setminus I_1| \tag{3}$$

Define $\omega : E \longrightarrow \mathbb{R}^+$ by:

$$\omega(e) = \begin{cases} 2 \, if \, e \in I_1 \cap I_2 \\ \frac{1}{|I_1 \setminus I_2|} if \, e \in I_1 \setminus I_2 \\ \frac{1+\epsilon}{|I_2 \setminus I_1|} if \, e \in I_2 \setminus I_1 \\ 0, otherwise. \end{cases}$$

We need the greedy algorithm to fail for only one weihgt function to get our contradiction.

- The greedy algorithm will choose all the elements of $I_1 \cap I_2$ first as they are the heaviest elements.

- Then it will choose all the elements of $I_1 \setminus I_2$.

- By assumption, it cannot then pick any element of $I_2 \setminus I_1$. Thus the remaining elements of B_G will be in $E \setminus (I_1 \cup I_2)$.

Hence,

$$\omega(B_G) = 2|I_1 \cap I_2| + |I_1 \setminus I_2|(\frac{1}{|I_1 \setminus I_2|}) = 2|I_1 \cap I_2| + 1$$

But by (I2), I_2 is contained in a maximal member B_2 of $\mathscr{I}$ and, $I_2 \subset B_2$.

$$\omega(B_2) \geq \omega(I_2) = 2|I_1 \cap I_2| + |I_2 \setminus I_1|(\frac{1+\epsilon}{|I_2 \setminus I_1|}) > 2|I_1 \cap I_2| + 1 = \omega(B_G)$$

$\implies \omega(B_2) > \omega(B_G)$

Which means the greedy algorithm does not find a solution to our optimisation problem shown by *theorem x.y*, so the greedy algorithm fails for this weight

function. We have a contradiction.

$\implies$ (I3) holds.

$\implies$ $(E, \mathscr{I})$ is a matroid. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

6 Other Optimizable set-systems

6.1 Accesible Set System

Definition 6.1. An *accesible set-system* is a pair $(E, \mathscr{S})$ where E is a finite ground set and $\mathscr{S}$ is a non-empty subset of the power set of E.

Satisfying the follow *accessibility axiom*:

(A) For any non-empty feasible set $X \in \mathscr{S}$. There exists an element $e \in X$ such that $X \setminus \{e\} \in \mathscr{S}$. Elements of $\mathscr{S}$ are called the *feasible sets* of M. Maximal feasible sets are also called *bases*.

This axiom is needed due to the process of the greedy algorithm. We require the ability to sequentially select a single element and then union it to our constructed solution at each step so an arbitrary set system is useless.

Remark. Matroid $\supseteq$ Indepence System $\subseteq$ Accesible Set-System.

All matroids are indepence systems and all indepence systems are accesible set-systems but the converse is not true in general.

Our generalised problem is now:

Proposition 6.1. *For any accesible set-system M and any weight function $\omega : E \longrightarrow \mathbb{R}^+$, the optimisation problem is:*

Maximise $\omega(B)$ such that B is a basis of M.

We can now apply a modified greedy algorithm in order to find a solution to this generalised version of our matroid/independence system problem.

Algorithm 6 Greedy algorithm for accessible set-systems

Let $M = (E, \mathscr{S})$ be an accessible set-system and $\omega : E \longrightarrow \mathbb{R}^+$ a weight function.

1: **procedure** GREEDY$(E, \mathscr{S}, \omega, T)$
2:      $T \leftarrow \emptyset$, $X \leftarrow E$
3:      **while** there $\exists x \in X$ with $T \cup \{x\} \in \mathscr{S}$ **do**
4:          choose some $x \in X$ with $T \cup \{x\} \in \mathscr{S}$ and
5:          $\omega(x) \geq \omega(y) \ \forall y \in X$ with $T \cup \{y\} \in \mathscr{S}$
6:          $T \leftarrow T \cup \{x\}$, $X \leftarrow X \setminus \{x\}$

Using our above definitions and algorithm we can now begin finding solutions to our problem for any accessible set-system. However, we are interested in learning about the charactersations of these set-systems that lead to optimal solutions to our greedy algorithm as seen in **section 5.1**.

One such caracterisation is the concept of a *greedoid*. A greedoid is an accessible set-system satisfying (I3). Formally this means,

Definition 6.2. A greedoid is a pair $(E, \mathscr{S})$ where E is a finite ground set and $\mathscr{S}$ is a collection of the feasible subsets of E satisfying the following conditions:
(I1): $\mathscr{S}$ is non-empty, $\emptyset \in \mathscr{S}$.
(A): For any non-empty feasible set $X \in \mathscr{S}$. There exists an element $e \in X$ such that $X \setminus \{e\} \in \mathscr{S}$.
(I3): If A and B are two independent sets of $\mathscr{I}$ and $|A| > |B|$, then there exists $x \in A \setminus B$ such that $B \cup \{x\}$ is in $\mathscr{I}$.

Unfortunately the greedy algorithm while providing solutions does not gaurantee optimal solutions for all greedoids. To characterise the greedoids that do result in optimal solutions when the greedy algorithm is applied we must add to our existing machinery an additional axiom. This is called the *strong exchange axiom*. This axiom is a strong version of (I3) the exchange axiom of a matroid.

Proposition 6.2. *Let $M = (E, \mathscr{S})$ be a greedoid. Then the above modified greedy algorithm finds an optimal soltuion to our problem for any weight function $\omega : E \longleftrightarrow \mathbb{R}^+$ if and only if M satisfies the follwing axiom:*
(SE): For $A, B \in \mathscr{S}$ with $|A| = |B| + 1$, there always exists some $e \in A \setminus B$ such that $B \cup \{e\}$ and $A \setminus \{e\}$ are contained in $\mathscr{S}$.

Remark. It can be seen that this axiom holds trivially for matroids due to the hereditary condition.

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