Matroids And their Graphs

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1 Graphic Matroids

Question: 1 Why is G not a matroid(graphical hint?)?

Question: 1 Is P(E) a graphic matroid(Where P(E) is the power set of he ground set E)?

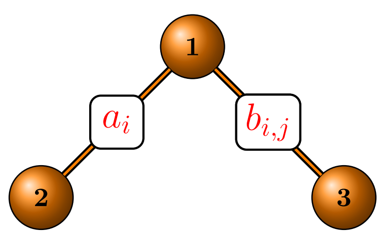
Let G be a graph and I be the set of all cyclefree subgraphs of G
Let $A, B \in I$ with $|A| = |B| + 1$

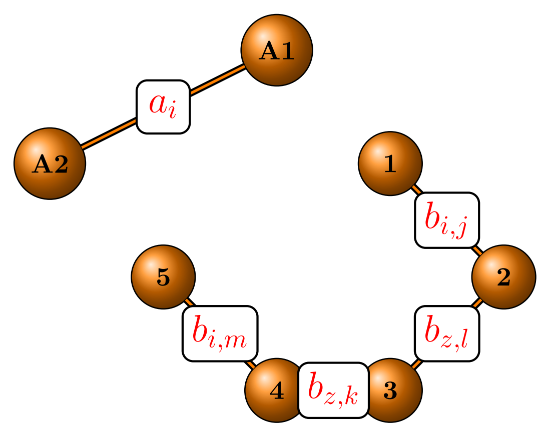
To prove $I3$ of the definition of a *matroid*, We show that for some $a \in A$, $B \cup \{a\} \in I$ we should consider $B \cup \{a\}$ for each $a \in A$.

Proof:
Now suppose $|A| > |B|$ and that $|A| = |B| + 1$
Let $|A \cap B| = s$, $|A \setminus B| = r$, $|A| = s + r$ and $|B| = s + r - 1$
So $|B \setminus A| = r - 1$

Suppose $A \setminus B = \{a_1, a_2, ..., a_r\}$
Suppose $B \cup \{a_i\} \notin I$ for each $i \in \{1, 2, ...\}$

Consider a_i for $i = 1, 2, ...$ there must be a path $b_{i1}, b_{12}, ..., b_{ir}$ of edges in B such that a_i make a cycle





Notation: $P(b_j, b_k)$ denotes a path in B from edges b_j to b_k

But $P(b_j, b_k) \cap A$ is not necessarily disjoint
if $P(b_j, b_k) \subset A$ then $P(b_j, b_k) \cup \{a_i\}$ would be a cycle
and then A would not be in $\mathcal{I}$, so at least one of the $b_i \in P(b_j, b_k) \in B \setminus A$

Given $A = \{a_1, ..., a_r\}$ for each a_i associate a $b_i \in B \setminus A$. Let $\hat{B} = \{b_1, ..., b_r\}$

Case 1: The b_i's are distinct
The b_i's are distinct and as shown previously each of the b_i's must be in $|B \setminus A|$ in order to avoid a circuit in A.
Therefore, $|B| \geqslant A$
Contradicting $|A| > |B|$

Hence, I3 holds
Case 2: When the b_i's are not all distinct
Let $b_1 = b_2$.

Again demo graph to be added depicting the two separate graphs and then the joined version highligting b1=b2
We use the same argument as in Case 1 only here we need two distinct $b_i \in P(b_j, b_k)$ where $b_i \in B \setminus A$ such that $P(b_j, b_k) \cup \{a_i\}$ is a cycle. This can be seen in the diagram above, there must be another edge in the union of the paths which is in $B \setminus A$ or else we get a cycle in A

Otherwise, $P(b_j, b_k) \subset A$ then $P(b_j, b_k) \cup \{a_i\}$ would be a circuit and then $A \notin I$. As now, $|B| \geq |A|$, and we have a contradiction.
Hence I3 holds.

$\square$