Matroids And their Graphs

o.mcdonnell4@nuigalway.ie

February 2018

1 Base characterisation of a matroid

Theorem 1.1. *Let $\mathscr{B}$ be a set of subsets of a finite set E. Then $\mathscr{B}$ is the collection of bases of a matroid on E if and only if $\mathscr{B}$ satisfies the following conditions:*
(B1) $\mathscr{B}$ is non-empty.
(B2) If B_1 and B_2 are member of $\mathscr{B}$ and $x \in B_1 \setminus B_2$, then there is an element y of $B_1 \setminus B_2$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{B}$.

Definition 1.1. A base is a maximally independent subset of $\mathscr{I}$.
A seen previously all maximally independent sets in a matroid have the same cardinality.

Proof: (B2)
($\implies$) Let $B_1, B_2 \in \mathscr{B}, B_1 \neq B_2$
$|B_1| = |B_2|$ so I3 does not directly apply here.
Let $x \in B_1 \setminus B_2 \implies x \in B_1, x \notin B_2$
$|B_1| = |B_1 \setminus \{x\}| + 1 \implies B_1 \setminus \{x\} \in \mathscr{I}$ but not in $\mathscr{B}$
$|B_2| = |B_1 \setminus \{x\}| + 1$ so now we can use I3
Now $\exists y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{I}$
$|(B_1 \setminus \{x\}) \cup \{y\}| = |B_1 \setminus \{x\}| + 1 = |B_1| = ... = |B_r|$ as all maximal elements of $\mathscr{I}$ have the same cardinality $\implies (B_1 \setminus \{x\}) \cup \{y\}$ is maximal in $\mathscr{I}$
$\implies (B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{B}$

($\Longleftarrow$) If $\mathscr{B}$ satisfies (B1) and (B2) then we have a matroid.
By B1 $\mathscr{B}$ is always non-empty i.e it contains at least the null set which shows I1 holds.

By definition a base $B_1 \in \mathscr{B}$ is a maximally independent subset of E. $\forall B_i \in \mathscr{B}$ all the subsets b_k of B_i are also independent since B_i is a maximally independent set. Therefore all those b_k are in $\mathscr{I}$. $\implies$ (I2) holds. This shows a matroid can be generated through it's bases.

Assume that (I3) fails. meaning that, for $I_1, I_2 \in \mathscr{I}$, where $|I_1| = |I_2| + 1$, there $\exists y \in I_1 \setminus I_2$ such that $I_2 \cup \{y\} \in \mathscr{I}$.
Let $B_1, B_2 \in \mathscr{B}, |B_1| = |B_2|$
Let $x \in B_1 \setminus B_2$ then $B_1 \setminus \{x\} \subset B_1 \implies B_1 \setminus \{x\}$ is independent.
Then there exists a $y \in B_2 \setminus (B_1 \setminus \{x\})$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{B}$ due to (B2). And if $(B_1 \setminus \{x\}) \cup \{y\} \in \mathscr{B}$ then its is also in $\mathscr{I}$. Which is a contradiction.
$\implies$ (I3) holds.

$\square$