Matroids And their Graphs

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1 Graph Theory Definitions

Definition 1.1 (Connected)**.** A graph is connected when there is a path between each pair of vertices.

Definition 1.2 (Tree)**.** A connected graph with no circuits.
All subsets of E in $\mathscr{I}$ in M(G) the cycle matroid of G are the trees of the graph G.

Definition 1.3 (Forest)**.** A disconnected graph with no circuits

Proposition 1.1. *A theorem of Cayley(1889) states that the number of distinct laballed trees which can be drawn using n labelled points is $n^(n-2)$.*

Definition 1.4 (Rooted tree)**.** A rooted tree has one vertex designated as an origin and called the root.

Definition 1.5 (Binary tree)**.** A rooted tree in which the root has degree 2 and every other vertex has degree 1 or 3.

The vertices are often labelled and referred to as nodes. Information may be recorded in them and a cost, penalty or probability may be associated with each edge. For example, the problem of joining all nodes in a graph by the minimum length of cable leads to a tree known as a *minimum spanning tree.*

Definition 1.6 (Spanning Tree)**.** A spanning tree T of an undirected graph G is a subgraph that is a *tree* which includes all of the vertices of G, with minimum possible number of edges.
A disconnected graph cannot contain a spanning tree as we cannot find a walk which brings us to all of the disconnected vertices.

Definition 1.7 (Acyclic)**.** An acyclic graph is a graph which contains no closed walks.

Definition 1.8 (Walk)**.** If there are vertices $v_{i-1}v_i$ for $i = 1, ..., n$ the sequence is called a walk.

Definition 1.9 (Closed Walk)**.** If $v_0 = v_n$ this sequence is called a closed walk.

Definition 1.10 (Bridge)**.** A bridge(or cut edge) is an edge of a graph whose deletion increases the numbe rof connected components. Equivalently, an edge is a bridge is and only if it is not contained in any cycle.

Lemma 1.2. *Any acyclic graph on n vertices has at most $n-1$ edges.*

Proof: Let G be an acyclic graph with n vertices.
if n = 1 then we have no edges hence nothing to prove.
Assumse $n > 1$, let e be an edge in G connecting two vertices ab in the vertex set of G.
Let $H = G \setminus \{e\}, H$ has one more connected component than G. H has two maximal acyclic connected componenets, and thus can be decomposedinto acyclic connected graphs $H_1, H_2, ..., H_k where k \geq 2$.
By induction, we can assume each graph H_i contains at most $n_i - 1$ edges where n_i is the number of vertices of H_i.
Then G has at most $n-1$ edges.
$(n_1 - 1) + ... + (n_k - 1) + 1 = (n_1 + ... + n_k) - (k-1) \leq n - 1$ edges.

□

Lemma 1.3. *Let G be a graph. Then the following conditions are equivalent:*
1) G is a tree.
2) G does not contain any cycles, but adding any further edge yields a cycle.
3) Any two vertices of G are connected by a unique path.
4) G is connected, and any edge of G is a bridge.

Proof: $(1) \implies (2)$
Suppose that G is a tree,then G is a connected graph with no circuits. Let e be a new edge in G with $e = g_i g_k$ where g_i, g_k are in the vertex set of G. Then as $G \cup \{e\}$ must be connected, there exists a walk between any pair of vertices of G. So there is a walk K from $g_j \rightarrow g_k$ and there is also a walk L from $g_k \rightarrow g_j$ where K does not traverse e and L does traverse e and so we have a cycle.

Proof: $(2) \implies (3)$
Let u, v be vertices of G. If there was not path joining uv in G then $e = uv$ does not create a cycle in G. Thus G must be connected.
Suppose G contained two different paths W_1, W_2 from u to v.
Then $u \longrightarrow v \longrightarrow u$ would be a closed walk in G.
$\implies$ G contains a cycle. Which is a contradiction.

Proof: $(3) \iff (4)$
G is connected by hypothesis. Let $e = uv$ be an edge in G.
Suppose e is not a bridge, then $G \setminus \{e\}$ is still connected. But then we have two distinct path from u to v in G.

Proof: $(4) \implies (1)$
G is connected by hypotheis.
Suppose G contains a cycle K. Then any edge of K could be ommited from G, and the resulting graph would still be connected. In other words, no edge of K would be a bridge, a contradiction.

□

Definition 1.11. let (G, ω) be a network. For any subset T of the edge set of G,

$$\omega(T) = \sum_{e \in T} \omega(T)$$

is called the weight of T.

Definition 1.12 (Minimal Spanning Tree)**.** A spanning tree is a *minimal* spanning tree if its weight is minimal of all the weights of spanning trees.
A forest can be considered by finding a minimal spanning tree for each connected component of G.

Remark. If the weight ω is constant, any spanning tree is minimal.
In this case, determining a minimal spanning tree could be done using a breadth-first search.

Remark. Determining the number of spanning trees of a graph in polynomial time is NP-hard.

Note. Let T be a tree and e an edge not in T. According to lemma 1.3 the graph arising from $T \cup \{e\}$ contains a unique cycle. We denote this cycle by $C_T(e)$

Theorem 1.4 (Without Proof)**.** *Let (G, ω) be a network, where G is a connected graph. A spanning tree T of G is minimal if and only if, for each edge e in $G \backslash T$, we have,*

$$\omega(e) \geq \omega(f) \forall \text{ edges } f \text{ in } C_T(e).$$

Another characterisation of minimal spanning trees exist. But the following defintions are required first.

Definition 1.13 (Cut)**.** Let G be a graph having vertex set V. A *cut* is a partition $S = X \dot{\cup} X'$ of V into two non-empty subsets.

Definition 1.14 (Cocycle)**.** By $E(s)$ or $E(X, X')$ we denote the set of all edges incident with one vertex in X and one vertex in X'. Any such edge set is called a cocycle.

Remark. We are concerned with cocycles that are constructed from trees.