Matroids And their Graphs

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19 January 2018

1 Cardinality of maximal independent sets

Theorem 1.1. *Show that if $\mathscr{I}$ is a non-empty hereditary set of subsets of a finite set E, then $(E, \mathscr{I})$ is a matroid if and only if, for all $X \subset E$, all maximal members of $\{I : I \in \mathscr{I}$ and $I \subset X\}$ have the same number of elements*

Proof: ($\implies$) Let B_1, B_2 be maximal elements of $\{I : I \in \mathscr{I}\text{and } I \subset X\}$
And assume $|B_1| < |B_2|$ Then since $B_1, B_2 \in \mathscr{I}$
There $\exists e \in (B_2 \setminus B_1)$ such that $B_1 \cap \{e\} \in \mathscr{I}$
This contradicts our maximality of B_1
$\implies |B_1| \geq |B_2|$
Now repeat same procedure but assume $|B_2| > |B_1|$
$\implies |B_1| = |B_2|$
$\implies$ All maximal elements of the set $\{I : I \in \mathscr{I}$ and $I \subset X\}$ in out matroid M have the same cardinality i.e number of elements

$\square$