Matroids for solving Optimisation Problems

The Greedy algorithm as a solution

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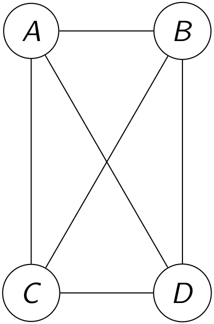
The Problem



Let G be K_4 as seen here, where the vertices correspond to towns to be linked by railway network, and the weight on each edge is the cost of providing a link between the towns correspnding to the respective vertices. In this case, the minimum weight of a spanning tree in G corrsponds to the minimum cost of providing a railway that will link all n towns.

Greedy Algorithm

Kruskal's algorithm is a greedy algorithm that finds a minimum spanning tree for a connected weighted Graph.

Algorithm:

1) Create a graph F(a set of trees) where each vertex of G is a separate tree.
2) Create a set S conatining all the edges of the graph.
3) While S is non-empty and F is not yet spanning
    3(a) Remove an edge with minimum weight from S.
    3(b) If the removed edge connects two different trees then add it to the forest F, combining two trees into a single tree.

Why Greedy works?

Lemma

If $(E, \mathcal{I})$ is a matroid M, then B_G is a solution to the optimization problem.

Where B_G is a spanning tree created through the greedy algorithm.

But what is a matroid?

Independence Systems and Matroids

Definition

An *independence system* is a pair $(E, \mathcal{S})$, where E is a set and $\mathcal{S}$ is a non-empty subset of the power set of E, closed under inclusion. The elements of $\mathcal{S}$ are called the *independent sets*.

Definition

A matroid is a pair $(E, \mathcal{I})$ with finite ground set E and $\mathcal{I}$ being a collection of independent subsets of E satisfying the following conditions

(I1): The empty set is always independent
(I2): Every subset of an independent set is independent
(I3): If A and B are two independent sets in $\mathcal{I}$ and $|A| = |B| + 1$, then there exists $x \in A \setminus B$ such that $B \cup \{x\}$ is in $\mathcal{I}$

Bases of a Matroid

Definition

A base is a maximally independent subset of $\mathcal{I}$.
All the maximally independent sets have the same cardinality, this is the *rank* of the matroid.

Definition

Let $\mathcal{B}$ be a set of subsets of a finite set E. Then $\mathcal{B}$ is the collection of bases of a matroid on E if and only if $\mathcal{B}$ satisfies the following conditions:
(B1) $\mathcal{B}$ is non-empty.
(B2) If B_1 and B_2 are members of $\mathcal{B}$ and $x \in B_1 \setminus B_2$, then there is an element y of $B_1 \setminus B_2$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

Spanning trees are Bases

Definition

A spanning tree T of an undirected graph G is a subgraph that is a *tree* which includes all of the vertices of G, with minimum possible number of edges.
Also T does not contain any cycles, but adding any further edge yields a cycle, so T is maximal in G.

Lemma

Any acyclic graph on n vertices has at most $n-1$ edges. And a spanning tree has exactly $n-1$ edges.

From this we can see, that if $\mathcal{B}$ is the collection of maximally elements of $\mathcal{I}$, then in G, $\mathcal{B}$ is the set of spanning trees of the graph.

Weight Function

The optimization problem associated with $(E, \mathcal{S})$ is the following: for a given weight function $\omega : E \to \mathbb{R}^+$, we want to find an independent set A whose weight,

$$\omega(A) := \sum_{e \in A} \omega(e) \tag{1}$$

is maximal.

Greedy Weight funtion

Lemma

If $(E, \mathcal{I})$ is a matroid M, then B_G is a solution to the optimization problem.

Proof.

If $r(M) = r$, then $B_G = \{e_1, e_2, ..., e_r\}$ is a basis of M. Let B be another basis of M, $B = \{f_1, f_2, ..., f_r\}$ where $\omega(f_1) \geq \omega(f_2) \geq ... \geq \omega(f_r)$. This follows from the result of the next additional lemma showing that not only is B_G a maximum weight basis of M ,but is also at least as heavy as the elements of B at each step. $\square$

Continued

Lemma

if $1 \leq j \leq r$, *then* $\omega(e_j) \geq \omega(f_j)$.

Proof.

Suppose(seeking a contradiction) that k is the least integer for which $\omega(e_k) \leq \omega(f_k)$. Take $I_1 = \{e_1, e_2, ..., e_{k-1}\}$ and $I_1 = \{f_1, f_2, ..., f_{k-1}\}$. Since $|I_1| = |I_2| + 1$ by $(I3)$ implies $I_1 \cup \{f_t\} \in \mathcal{I}$ for some $f_t \in I_2 \setminus I_1$. But this means that $\omega(f_t) \geq \omega(f_t) > \omega(e_k)$. Hence the Greedy algorithm would have chosen f_t over e_k, which gives us our contradiction. $\square$

Wrap Up

The combination of the previous lemma and our use of the greedy algorithm to find a maximal member B of $\mathcal{I}$ of maximum weight allows us to deduce that Kruskal's algorithm does generate a minimum weight spanning tree of a graph.
We have seen that greedy algorithm gives us a solution to our optimisation problem as long as we have a matroid.