$$F(x) = \frac{1}{2} x^T \begin{bmatrix} 6 & -2 \\ -2 & 6 \end{bmatrix} x + \begin{bmatrix} -1 & -1 \end{bmatrix} x$$

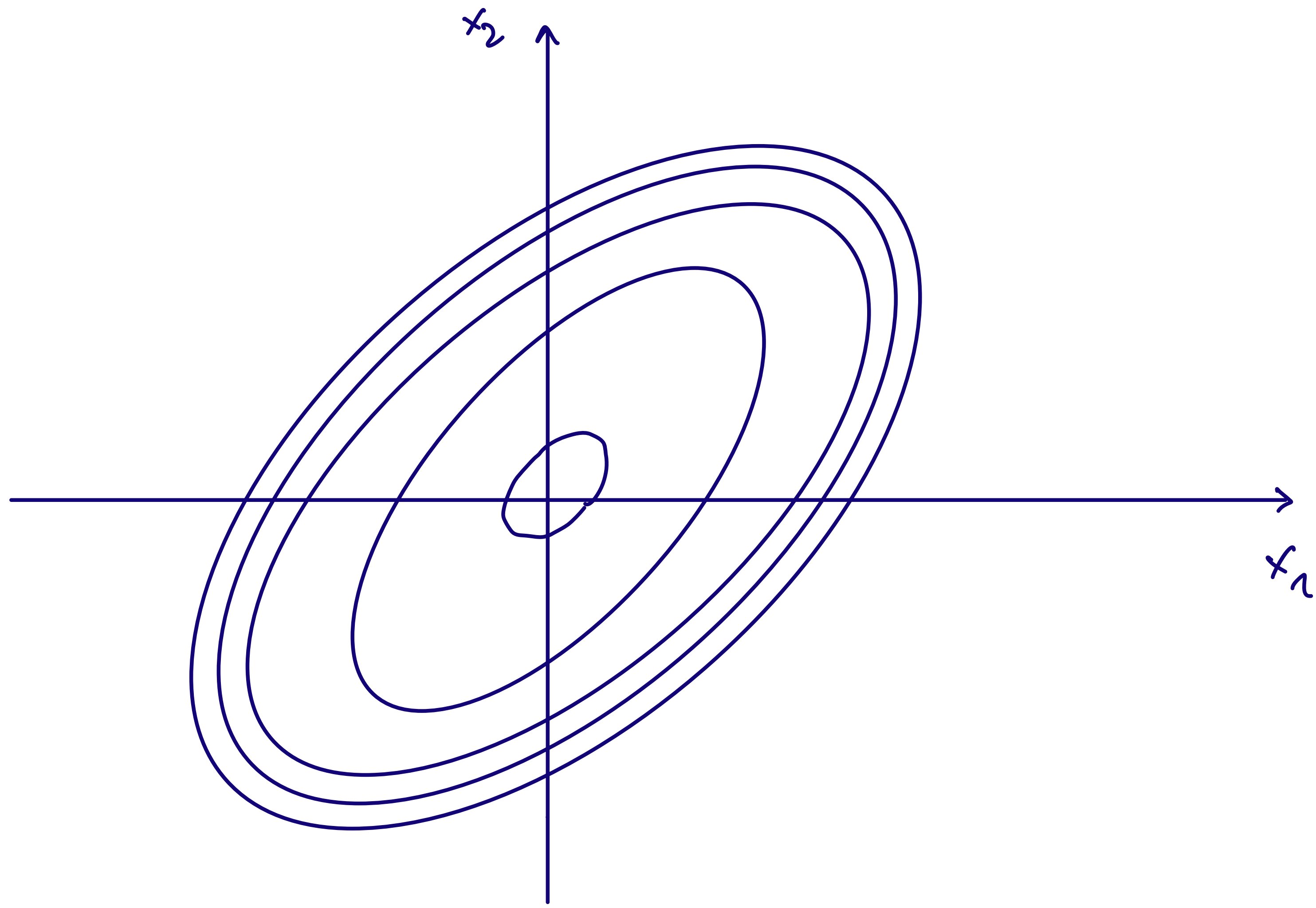
$$F(x) = \frac{1}{2}\left(6x_1^2 + 6x_2^2 - 4x_1 x_2 \right) + (-x_1 - x_2)$$

$$= 3x_1^2 + 3x_2^2 - 2x_1 x_2 - x_1 - x_2$$

(i) Contour plot

$$F(x_1, x_2) = \underline{3}x_1^2 + \underline{3}x_2^2 - \underline{2}x_1 x_2 - \underline{x_1} - \underline{x_2}$$

These will be tilted "ellipses" with one locus at the minimum and the axis of symmetry is the line $x_1 = x_2$.



(iii) $\nabla F(x) = Ax + d$

$$x_{k+1} = x_k - \alpha_k g_k \qquad \alpha > 0$$

$$x_0 = [0, 0]^T$$

$$g_0 = d$$

$$Ad = \begin{pmatrix} 6 & -2 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \end{pmatrix} = 4 \begin{pmatrix} -1 \\ -1 \end{pmatrix} = 4d$$

* $x_1 = x_0 - \alpha g_0 = -\alpha d$

* $x_2 = x_1 - \alpha g_1$

$$g_1 = A(-\alpha d) + d$$
$$= -\alpha (Ad) + d$$
$$= -\alpha \, 4d + d$$
$$= (1 - 4\alpha) d$$

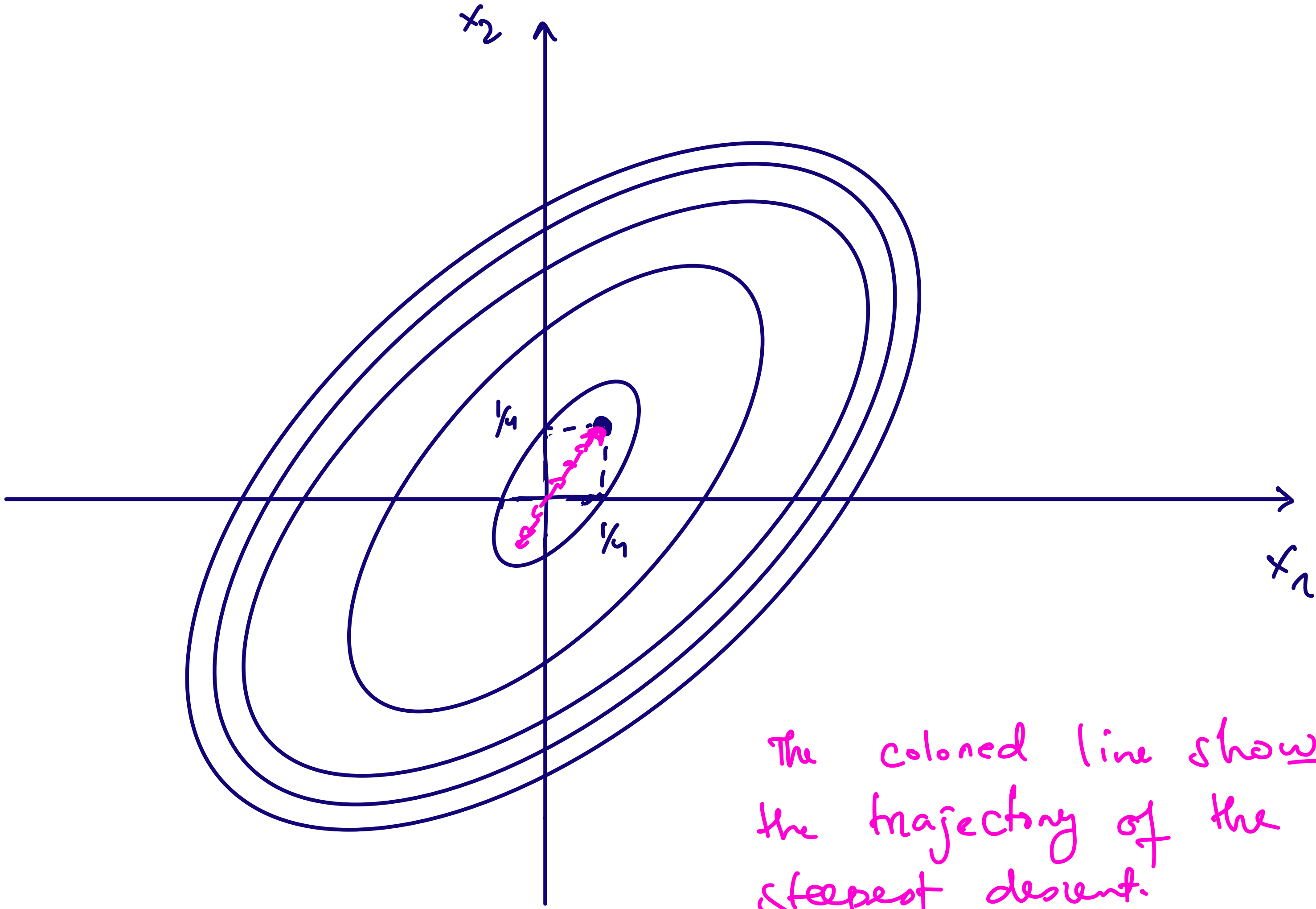
$$x_2 = -\alpha d + (1 - 4\alpha) d$$
$$= (1 - 5\alpha) d$$

Since d is an eigenvector of A, the steepest descent will keep moving a line parallel to d until it reaches the minimum. Say we reach the minimum in one step! $\Rightarrow$

$$-\alpha = 1 - 5\alpha$$
$$4\alpha = 1$$
$$\alpha = \frac{1}{4}$$

So, the minimum is at $x_{min} = -\frac{1}{4}\begin{pmatrix}-1\\-1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1\\1\end{pmatrix}$ //

We need to assume very small learning rate



The colored line shows the trajectory of the steepest descent.

We sometimes go forward and sometimes backwards but we stay in the line //

(iii) We already found

$$x_0 = (0,0)^T$$
$$x_1 = -\alpha d$$
$$x_2 = (1-5\alpha) d$$

setting $\alpha = 0.1$:

$$x_0 = [0,0]^T$$
$$x_1 = -0.1 \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix}$$

$$x_2 = (1-0.5) \begin{pmatrix} -1 \\ -1 \end{pmatrix} = 0.5 \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -0.5 \\ -0.5 \end{pmatrix}$$

It is interesting that it goes forward and then backward.