1 Types

$$\alpha, \beta ::= \tau \mid \alpha \to \beta$$

1.1 First Element of a List Satisfying a Property

$$\text{here}\ \frac{P(x)}{\text{First}\ P\ (x : xs)} \qquad \text{there}\ \frac{\neg P(x) \qquad \text{First}\ P\ xs}{\text{First}\ P\ (x : xs)}$$

A context is a list of pairs of atoms and types:

$$Context \equiv [(a, \alpha)]$$

, where for any atom a and context Γ we denote as $a \in \Gamma$ when:

$$\text{First}\ ((\_\equiv\_\ b) \circ \pi_1)\ \Gamma$$

Given evidence that an atom a is a context Γ, of type $a \in \Gamma$. Note this evidence always points to its first ocurrence by definition of First predicate. We denote the function that returns the asociated type for the atom a by simple concatenation:

$$\_\ \_ : (\Gamma : Context) \to a \in \Gamma \to \alpha$$

To make the notation more classic, we will just concatenate the context directly with the atom, not the evidence that the atom is in the context, but in all aplications of this function we will ask in the premises the required evidence.

We say a context Γ is included in other Δ, denoted as $\Gamma \subseteq \Delta$, if:

$$\forall a \in \Gamma, a \in \Delta \wedge \Gamma x \equiv \Delta x$$

Extending or overriding a context Γ with a new association atom a with a type α is denoted by $\Gamma, a : \alpha$.

1.2 Typing Relation

$$\vdash\text{v}\ \frac{x \in \Gamma}{\Gamma \vdash x : \Gamma x} \qquad \vdash\text{a}\ \frac{\Gamma \vdash M : \alpha \to \beta \qquad \Gamma \vdash N : \alpha}{\Gamma \vdash MN : \beta}$$

$$\vdash\lambda\ \frac{\Gamma, x : \alpha \vdash M : \beta}{\Gamma \vdash \lambda x M : \alpha \to \beta}$$

1.3 Weakening Lemmas

Lemma 1 (Weakening Lemma)**.**

$$\Gamma \subseteq \Delta \wedge \Gamma \vdash M : \alpha \Rightarrow \Delta \vdash M : \alpha$$

Proof. Induction on the typing relation.

var case: Hypothesis : $\vdash\mathrm{v}\ \dfrac{x \in \Gamma}{\Gamma \vdash x : \Gamma x}$ Thesis : $\Delta \vdash x : \Delta x$

$$\vdash\mathrm{v}\ \dfrac{\dfrac{x \in \Gamma \qquad \Gamma \subseteq \Delta}{x \in \Delta \wedge \Gamma x = \Delta x}}{\Delta \vdash x : \Delta x = \Gamma x}$$

app. case: Hypothesis: $\vdash\mathrm{a}\ \dfrac{\Gamma \vdash M : \alpha \to \beta \qquad \Gamma \vdash N : \alpha}{\Gamma \vdash MN : \beta}$ Thesis: $\Delta \vdash MN : \beta$

$$\vdash\mathrm{a}\ \dfrac{ih\ \dfrac{\Gamma \subseteq \Delta \qquad \Gamma \vdash M : \alpha \to \beta}{\Delta \vdash M : \alpha \to \beta} \qquad ih\ \dfrac{\Gamma \subseteq \Delta \qquad \Gamma \vdash N : \alpha}{\Delta \vdash N : \beta}}{\Delta \vdash MN : \beta}$$

abs. case: Hypothesis: $\vdash\lambda\ \dfrac{\Gamma, x : \alpha \vdash M : \beta}{\Gamma \vdash \lambda x M : \alpha \to \beta}$

Thesis: $\Delta \vdash \lambda x M : \alpha \to \beta$

$$\mathrm{ih}\ \dfrac{\dfrac{\Gamma \subseteq \Delta}{\Gamma, x : \beta \subseteq \Delta, x : \beta} \qquad \Gamma, x : \beta \vdash M : \alpha}{\vdash\lambda\ \dfrac{\Delta, x : \beta \vdash M : \alpha}{\Delta \vdash \lambda x M : \alpha}}$$

□

Lemma 2 (Weakening Fresh Atoms)**.**

$$x\#M \wedge \Gamma \vdash M : \alpha \Rightarrow \Gamma, x : \beta \vdash M : \alpha$$

Proof. Induction on the typing relation.

var case: Hypothesis : $\vdash\mathrm{v}\ \dfrac{y \in \Gamma}{\Gamma \vdash y : \Gamma y}$ Thesis : $\Gamma, x : \beta \vdash y : \Gamma y$

$$\vdash\mathrm{v}\ \dfrac{\dfrac{\dfrac{x\#y}{x \neq y} \qquad \dfrac{\Gamma \vdash y : \Gamma y}{y \in \Gamma}}{y \in (\Gamma, x : \beta) \wedge (\Gamma, x : \beta) y = \Gamma y}}{\Gamma, x : \beta \vdash y : (\Gamma, x : \beta) y = \Gamma y}$$

app. case: Hypothesis: $\vdash\mathrm{a}\ \dfrac{\Gamma \vdash M : \gamma \to \alpha \qquad \Gamma \vdash N : \gamma}{\Gamma \vdash MN : \alpha}$

Thesis: $\Gamma, x : \beta \vdash MN : \alpha$

$$\dfrac{ih\ \dfrac{\dfrac{x\#MN}{x\#M} \qquad \Gamma \vdash M : \gamma \to \alpha}{\Gamma, x : \beta \vdash M : \gamma \to \alpha} \qquad ih\ \dfrac{\dfrac{x\#MN}{x\#N} \qquad \Gamma \vdash N : \gamma}{\Gamma, x : \beta \vdash N : \gamma}}{\Gamma, x : \beta \vdash MN : \alpha}$$

abs. case: Hypothesis: $\vdash\lambda\ \dfrac{\Gamma, y : \gamma \vdash M : \alpha}{\Gamma \vdash \lambda y M : \gamma \to \alpha}$

Thesis: $\Gamma, x : \beta \vdash \lambda y M : \gamma \to \alpha$

$$x = y \text{ subcase: } \quad \vdash\lambda\;\dfrac{\text{lem. 1}\;\dfrac{\Gamma, x:\gamma \subseteq \Gamma, x:\beta, x:\gamma \qquad \Gamma, x:\gamma \vdash M:\alpha}{\Gamma, x:\beta, x:\gamma \vdash M:\alpha}}{\Gamma, x:\beta \vdash \lambda x M : \gamma \to \alpha}$$

$$x \neq y \text{ subcase: } \quad \vdash\lambda\;\dfrac{\dfrac{\text{ih}\;\dfrac{\dfrac{x\#\lambda yM \qquad x \neq y}{x\#M} \qquad \Gamma, y:\gamma \vdash M:\alpha}{\Gamma, y:\gamma, x:\beta \vdash M:\alpha} \qquad x \neq y}{\Gamma, x:\beta, y:\gamma \vdash M:\alpha}}{\Gamma, x:\beta \vdash \lambda y M : \gamma \to \alpha}$$

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1.4 Strengthening Lemma

Lemma 3 (Strengthening Fresh Variables)**.**

$$x\#M \wedge \Gamma, x:\beta \vdash M:\alpha \Rightarrow \Gamma \vdash M:\alpha$$

Proof. Induction on the typing relaton.

var case: Hypothesis : $\vdash\text{v}\;\dfrac{y \in \Gamma, x:\beta}{\Gamma, x:\beta \vdash y : (\Gamma, y:\beta)y}$ Thesis : $\Gamma \vdash y : (\Gamma, x:\beta)y$

$$\vdash\text{v}\;\dfrac{\dfrac{\dfrac{x\#y}{x \neq y} \qquad y \in \Gamma, x:\beta}{y \in \Gamma \wedge (\Gamma, x:\beta)y = \Gamma y}}{\Gamma \vdash y : \Gamma y = (\Gamma, x:\beta)y}$$

app. case: Hypothesis: $\vdash\text{a}\;\dfrac{\Gamma, x:\beta \vdash M:\gamma\to\alpha \qquad \Gamma, x:\beta \vdash N:\gamma}{\Gamma, x:\beta \vdash MN:\alpha}$

Thesis: $\Gamma \vdash MN:\alpha$

$$\dfrac{ih\;\dfrac{\dfrac{x\#MN}{x\#M} \qquad \Gamma, x:\beta \vdash M:\gamma\to\alpha}{\Gamma \vdash M:\gamma\to\alpha} \qquad ih\;\dfrac{\dfrac{x\#MN}{x\#N} \qquad \Gamma, x:\beta \vdash N:\gamma}{\Gamma \vdash N:\gamma}}{\Gamma \vdash MN:\alpha}$$

abs. case: Hypothesis: $\vdash\lambda\;\dfrac{\Gamma, x:\beta, y:\gamma \vdash M:\alpha}{\Gamma, x:\beta \vdash \lambda y M : \gamma\to\alpha}$

Thesis: $\Gamma \vdash \lambda yM : \gamma \to \alpha$

$$x = y \text{ subcase: } \quad \vdash\lambda\;\dfrac{\text{lem. 1}\;\dfrac{\Gamma, x:\gamma \subseteq \Gamma, x:\beta, x:\gamma \qquad \Gamma, x:\gamma \vdash M:\alpha}{\Gamma, x:\beta, x:\gamma \vdash M:\alpha}}{\Gamma, x:\beta \vdash \lambda x M : \gamma \to \alpha}$$

$$x \neq y \text{ subcase: } \quad \vdash\lambda\;\dfrac{\dfrac{\text{ih}\;\dfrac{\dfrac{x\#\lambda yM \qquad x \neq y}{x\#M} \qquad \Gamma, y:\gamma \vdash M:\alpha}{\Gamma, y:\gamma, x:\beta \vdash M:\alpha} \qquad x \neq y}{\Gamma, x:\beta, y:\gamma \vdash M:\alpha}}{\Gamma, x:\beta \vdash \lambda y M : \gamma \to \alpha}$$

∎

1.5 Equivariant

Lemma 4 (Typing Relation is Equivariant)**.**

$$\Gamma \vdash M : \alpha \Rightarrow (x\ y)\Gamma \vdash (x\ y)M : \alpha$$

Proof. Induction on the typing relation.

var case: Hypothesis : $\vdash\mathrm{v}\ \dfrac{z \in \Gamma}{\Gamma \vdash z : \Gamma z}$ Thesis : $(x\ y)\Gamma \vdash (x\ y)z : \Gamma z$

$$\vdash\mathrm{v}\ \dfrac{\dfrac{z \in \Gamma}{(x\ y)z \in (x\ y)\Gamma \wedge \Gamma z = ((x\ y)\Gamma)((x\ y)z)}}{(x\ y)\Gamma \vdash (x\ y)z : ((x\ y)\Gamma)((x\ y)z) = \Gamma z}$$

app. case: Hypothesis: $\vdash\mathrm{a}\ \dfrac{\Gamma \vdash M : \beta \to \alpha \qquad \Gamma \vdash N : \beta}{\Gamma \vdash MN : \alpha}$

Thesis: $(x\ y)\Gamma \vdash (x\ y)(MN) : \alpha$

$$\dfrac{\dfrac{ih\ \dfrac{\Gamma \vdash M : \beta \to \alpha}{(x\ y)\Gamma \vdash (x\ y)M : \beta \to \alpha} \qquad ih\ \dfrac{\Gamma \vdash N : \gamma}{(x\ y)\Gamma \vdash (x\ y)N : \gamma}}{(x\ y)\Gamma \vdash ((x\ y)M)((x\ y)N) : \alpha}}{(x\ y)\Gamma \vdash (x\ y)(MN) : \alpha}$$

abs. case: Hypothesis: $\vdash\lambda\ \dfrac{\Gamma, z : \beta \vdash M : \alpha}{\Gamma \vdash \lambda z M : \beta \to \alpha}$

Thesis: $(x\ y)\Gamma \vdash (x\ y)(\lambda z M) : \beta \to \alpha$

$$\vdash\lambda\ \dfrac{\dfrac{\dfrac{\mathrm{ih}\ \dfrac{\Gamma, z : \beta \vdash M : \alpha}{(x\ y)(\Gamma, z : \beta) \vdash (x\ y)M : \alpha}}{((x\ y)\Gamma), (x\ y)z : \beta \vdash (x\ y)M : \alpha}}{(x\ y)\Gamma \vdash \lambda((x\ y)z)((x\ y)M) : \beta \to \alpha}}{(x\ y)\Gamma \vdash (x\ y)(\lambda z M) : \beta \to \alpha}$$

□

1.6 α-compatible

Lemma 5 (Typing relation is α-compatible)**.**

$$\Gamma \vdash M : \alpha \wedge M \sim_\alpha N \Rightarrow \Gamma \vdash N : \alpha$$

Proof. Permutation induction on M.

var case: Direct.

app. case: Direct.

abs. case: Hypotheses: $\vdash\lambda\ \dfrac{\Gamma, x:\alpha \vdash M:\beta}{\Gamma \vdash \lambda x M : \alpha \to \beta} \sim_\alpha \lambda\ \dfrac{\exists xs, \forall z \notin xs, (x\ z)M \sim_\alpha (y\ z)N}{\lambda x M \sim_\alpha \lambda y N}$

Inductive Hypothesis:

$$\forall u, v, N, \Gamma, \gamma, \quad \left.\begin{array}{ll} \Gamma \vdash (u\ v)M : \gamma & \wedge \\ (u\ v)M \sim_\alpha (u\ v)N & \end{array}\right\} \Rightarrow \Gamma \vdash (u\ v)N : \gamma$$

Thesis: $\Gamma \vdash \lambda y N : \beta \to \alpha$

Let be $z \notin \{y\} \cup xs \cup dom(\Gamma) \cup fv(N)$.

$$\text{lem. 4}\ \dfrac{\text{ih}\ \dfrac{\text{lem. 1}\ \dfrac{\dfrac{\text{lem. 4}\ \dfrac{\Gamma, x:\alpha \vdash M:\beta}{(x\ z)(\Gamma, x:\alpha) \vdash (x\ z)M:\beta}}{(x\ z)\Gamma, z:\alpha \vdash (x\ z)M:\beta} \quad \dfrac{z \notin dom(\Gamma)}{(x\ z)\Gamma, z:\alpha \subseteq \Gamma, z:\alpha}}{\Gamma, z:\alpha \vdash (x\ z)M:\beta} \quad \dfrac{z \notin xs}{(x\ z)M \sim_\alpha (y\ z)N}}{\dfrac{\Gamma, z:\alpha \vdash (y\ z)N:\beta}{(y\ z)(\Gamma, z:\alpha) \vdash (y\ z)(y\ z)N:\beta}}}{(y\ z)\Gamma, y:\alpha \vdash N:\beta}$$

$$\text{lem. 3}\ \dfrac{z\#N \quad \dfrac{(y\ z)\Gamma, y:\alpha, z:\alpha \subseteq \Gamma, y:\alpha, z:\alpha \quad \text{lem. 2}\ \dfrac{z\#N \quad (y\ z)\Gamma, y:\alpha \vdash N:\beta}{(y\ z)\Gamma, y:\alpha, z:\alpha \vdash N:\beta}}{\Gamma, y:\alpha, z:\alpha \vdash N:\beta}}{\vdash\lambda\ \dfrac{\Gamma, y:\alpha \vdash N:\beta}{\Gamma \vdash \lambda y N : \alpha \to \beta}}$$

□

1.7 Substitution Lemma for Typing Relation

Lemma 6 (Substitution Lemma for Typing Relation)**.**

$$\Gamma, x:\beta \vdash M:\alpha \wedge \Gamma \vdash N:\beta \Rightarrow \Gamma \vdash M[x:=N]:\alpha$$

Proof. α-induction principle on M term to avoid x and free variables in N as bound variable in the abstraction case. The predicate is α-compatible on M as direct consequence of the substitution lemma for α-convertible terms.

var case: Hypothesis : $\vdash\text{v}\ \dfrac{y \in (\Gamma, x:\beta)}{\Gamma, x:\beta \vdash y : (\Gamma, x:\beta)y}$

Thesis : $\Gamma \vdash y[x:=N] : (\Gamma, x:\beta)y$

case $x = y$: $\text{hyp.}\ \dfrac{\Gamma \vdash x[x:=N] : (\Gamma, x:\beta)x}{\Gamma \vdash N:\beta}$

case $x \neq y$: $\vdash\text{v}\ \dfrac{\dfrac{y \in (\Gamma, x:\beta) \qquad x \neq y}{y \in \Gamma}}{\Gamma \vdash \underbrace{y}_{=y[x:=N]} : \underbrace{\Gamma y}_{=(\Gamma, x:\beta)y}}$

app. case: Hypothesis: $\vdash$a $\dfrac{\Gamma, x:\beta \vdash M:\gamma\to\alpha \qquad \Gamma, x:\beta\vdash P:\gamma}{\Gamma, x:\beta \vdash MP:\alpha}$

Thesis: $\Gamma \vdash (MP)[x:=N]:\alpha$

$$\vdash\text{a}\ \dfrac{\text{ih.}\ \dfrac{\Gamma, x:\beta\vdash M:\gamma\to\alpha}{\Gamma\vdash M[x:=N]:\gamma\to\alpha} \qquad \text{ih.}\ \dfrac{\Gamma, x:\beta\vdash P:\gamma}{\Gamma\vdash P[x:=N]:\gamma}}{\Gamma\vdash \underbrace{(M[x:=N])(P[x:=N])}_{=(MP)[x:=N]}:\alpha}$$

abs. case: Hypothesis: $\vdash\lambda\ \dfrac{\Gamma, x:\beta, y:\gamma\vdash M:\alpha}{\Gamma, x:\beta\vdash \lambda yM:\gamma\to\alpha}$

Thesis: $\Gamma\vdash(\lambda yM)[x:=N]:\gamma\to\alpha$

We can asume $y\notin\{x\}\cup fv(N)$.

$$\vdash\lambda\ \dfrac{\text{ih}\ \dfrac{\text{lem. 1}\ \dfrac{\Gamma, x:\beta, y:\gamma\vdash M:\alpha \qquad \dfrac{y\neq x}{\Gamma, x:\beta, y:\gamma\subseteq\Gamma, y:\gamma, x:\beta}}{\Gamma, y:\gamma, x:\beta\vdash M:\alpha} \qquad \text{lem. 2}\ \dfrac{\Gamma\vdash N:\beta \qquad y\#N}{\Gamma, y:\gamma\vdash N:\beta}}{\Gamma, y:\gamma\vdash M[x:=N]:\alpha}}{\Gamma\vdash\lambda y(M[x:=N]):\gamma\to\alpha}$$

Finally, by the α-compatible property of typing relation and due to freshness of binder variable we end up with the naive case of substitution case.

$$\text{lem. 5}\ \dfrac{\Gamma\vdash\lambda y(M[x:=N]):\gamma\to\alpha \qquad \dfrac{y\notin\{x\}\cup fv(N)}{\lambda y(M[x:=N])\sim_\alpha(\lambda yM)[x:=N]}}{\Gamma\vdash(\lambda yM)[x:=N]:\gamma\to\alpha}$$

□

2 Subject Reduction

$$\triangleright\beta\ \dfrac{}{(\lambda xM)N\triangleright M[x:=N]}$$

Corollary 1 (Typing judge is preserved by β-contraction)**.**

$$\Gamma\vdash M:\alpha, M\triangleright N\Rightarrow\Gamma\vdash N:\alpha$$

Proof. We have to prove that $\Gamma\vdash M[x:=N]:\alpha$. By hypotheses $\Gamma\vdash(\lambda xM):\alpha$, then by typing judge $\Gamma, x:\alpha\vdash M:\beta$ and $\Gamma\vdash N:\alpha$, using last lemma 6 we end up with the desired result. □

We define the β-reduction relation as the contextual clousure of the β-contraction. We denote it as $\to_\beta$.

Corollary 2. $\Gamma\vdash M:\alpha, M\to_\beta N\Rightarrow\Gamma\vdash N:\alpha$

Proof. Direct induction on the contextual clousure relation. For the β-contraction case we directly use last result. □

Subject redution is proved by induction on the reflexive-transitive closure of $\rightarrow_\alpha(\rightarrow_\beta \cup \sim_\alpha)$ relation, using directly the last result 2 and the α-compatible of the typing relation lemma 5 for the $\rightarrow_\alpha$ and $\sim_\alpha$ cases respectively.

Lemma 7 (Subject Reduction)**.** $\Gamma \vdash M : \alpha, M \rightarrow^*_\beta N \Rightarrow \Gamma \vdash N : \alpha$