1 Alpha Conversion

$$\sim_\alpha\text{v}\ \frac{}{x \sim_\alpha x} \qquad \sim_\alpha\text{a}\ \frac{M \sim_\alpha M' \qquad N \sim_\alpha N'}{MN \sim_\alpha M'N'} \qquad \sim_\alpha\lambda\ \frac{\exists xs, \forall z \notin xs, (x\ z)M \sim_\alpha (y\ z)N}{\lambda xM \sim_\alpha \lambda yN}$$

2 Parallel Reduction

$$\rightrightarrows\text{v}\ \frac{}{x \rightrightarrows x} \qquad \rightrightarrows\text{a}\ \frac{M \rightrightarrows M' \qquad N \rightrightarrows N'}{MN \rightrightarrows M'N'} \qquad \rightrightarrows\lambda\ \frac{\exists xs, \forall z \notin xs, (x\ z)M \rightrightarrows (y\ z)N}{\lambda xM \rightrightarrows \lambda yN}$$

$$\rightrightarrows\beta\ \frac{\lambda xM \rightrightarrows \lambda yP' \qquad N \rightrightarrows P'' \qquad P'[y := P''] \sim_\alpha P}{(\lambda xM)N \rightrightarrows P}$$

3 New Substitution Lemmas

Lemma 1 (Swapping substitution variable)**.**

$$x\#M \Rightarrow ((x\ y)M)[x := N] \sim_\alpha M[y := N]$$

Proof. Uses α-induction principle on M term to avoid x, y and the free variables in N as binder position in the λ-case.

For some arbitrary x, y variables and term N we define the following predicate over terms:

$$P(M) \equiv x\#M \Rightarrow (x\ y)M[x := N] \sim_\alpha M[y := N]$$

We prove P is α-compatible, given N such that $M \sim_\alpha P$ and $P(M)$ we prove $P(P)$ holds. As $x\#P$ and $P \sim_\alpha M$ then $x\#M$, then:

$$\begin{array}{ccl} ((x\ y)P)[x := N] & \equiv & \{\alpha \text{ equiv. and subst. lemma}\} \\ ((x\ y)M)[x := N] & \sim_\alpha & \{\text{as } x\#M \text{ we can apply } P(M)\} \\ M[y := N] & \equiv & \{\text{subst. lemma}\} \\ P[y := N] & & \end{array}$$

var case: Hypotheses: $x\#z$ Thesis: $((x\ y)z)[x := N] \sim_\alpha z[y := N]$

As $x\#z$ then $x \neq z$.

$y = z$ case: $((x\ y)y)[x := N] = x[x := N] = N = y[y := N]$

$y \neq z$ case: $((x\ y)z)[x := N] = z[x := N] = z = z[y := N]$

app. case: Hypotheses: $x\#MP$ Thesis: $((x\ y)(MP))[x := N] \sim_\alpha (MP)[y := N]$

$$\begin{array}{ccc} ((x\ y)(MP))[x := N] & = & \\ (((x\ y)M)((x\ y)P))[x := N] & = & \\ (((x\ y)M)[x := N])(((x\ y)P)[x := N]) & \sim_\alpha & \{\text{ih}\} \\ (M[y := N])(P[y := N]) & = & \\ (MP)[y := N] & & \end{array}$$

λ case: Hypotheses: $x\#\lambda zM \wedge z \notin \{x,y\} \cup fv(N)$
Thesis: $((x\ y)(\lambda zM))[x := N] \sim_\alpha (\lambda zM)[y := N]$
As $z \notin \{x,y\} \cup fv(N)$ then $z \neq x$. Because $x\#\lambda zM$ and $z \neq x$ then $x\#M$.

$$\begin{array}{ccl}
((x\ y)(\lambda zM))[x := N] & = & \\
(\lambda((x\ y)z)((x\ y)M))[x := N] & = & \{z \notin \{x,y\} \cup fv(N)\} \\
(\lambda z((x\ y)M))[x := N] & \sim_\alpha & \{z \notin \{x,y\} \cup fv(N)\} \\
\lambda z(((x\ y)M)[x := N]) & \sim_\alpha & \{\text{ih}\} \\
\lambda z(M[y := N]) & \sim_\alpha & \{z \notin \{x,y\} \cup fv(N)\} \\
(\lambda zM)[y := N] & &
\end{array}$$

□

Lemma 2 (Substitution preserves freshness (no capture lemma))**.**

$$x\#\lambda yM \wedge x\#N \Rightarrow x\#M[y := N]$$

Proof. Uses α-induction principle on M term to avoid x, y and the free variables in N as binder position in the λ-case.

var case ($M = z$): Hypothesis: $x\#\lambda yz \wedge x\#N$ Thesis: $x\#z[y := N]$

$y = z$ case: Then $z[y := N] = y[y := N] = N$, as $x\#N$ by hypothesis, $x\#z[y := N]$.

$y \neq z$ case: $z[y := N] = z$ so we have to prove that $x \neq z$. As $x\#\lambda yz$ then $x = y$ or the desired result $x \neq z$. Finally, for the pending case, if $x = y$ as $y \neq z$ then $x \neq z$.

app. case ($M = PQ$): Hypothesis: $x\#\lambda y(PQ) \wedge x\#N$ Thesis: $x\#(PQ)[y := N]$

$$\#\text{a}\ \dfrac{\text{ih}\ \dfrac{\dfrac{x\#\lambda y(PQ)}{x\#\lambda yP \wedge x\#\lambda yP} \qquad x\#N}{x\#P[x := N] \wedge x\#Q[x := N]}}{x\#(P[x := N])(Q[x := N]) = (PQ)[x := N]}$$

λ case ($M = \lambda zM$): Hypothesis: $z \notin \{x,y\} \cup fv(N) \wedge x\#\lambda y(\lambda zM) \wedge x\#N$
Thesis: $x\#(\lambda zM)[y := N]$

$$\dfrac{\dfrac{\text{ih}\ \dfrac{\dfrac{z \neq x \qquad x\#\lambda y(\lambda zM)}{x\#\lambda yM} \qquad x\#N}{x\#M[y := N]}}{x\#\lambda z(M[y := N])} \qquad \dfrac{\dfrac{z \notin fv(N)}{z\#M}}{\lambda z(M[y := N]) \sim_\alpha (\lambda zM)[y := N]}}{x\#(\lambda zM)[y := N]}$$

□

4 Parallel Relation Lemmas

Lemma 3 (Parallel is equivariant)**.**

$$M \rightrightarrows N \Rightarrow \pi M \rightrightarrows \pi N$$

Proof. Induction on the parallel relation.

var. rule: Direct.

app. rule: Direct.

λ rule: Hypotheses: $\rightrightarrows\lambda\ \dfrac{\exists xs, \forall z \notin xs, (x\ z)M \rightrightarrows (y\ z)N}{\lambda x M \rightrightarrows \lambda y N}$

Thesis: $\lambda(\pi\ x)(\pi\ M) \rightrightarrows \lambda(\pi\ y)(\pi\ N)$

Proof:

$$(\pi\ z) \equiv z\ as\ z \notin dom(\pi)\ \dfrac{\text{ih}\ \dfrac{\dfrac{\exists xs, \forall z \notin xs, (x\ z)M \rightrightarrows (y\ z)N}{\forall z \notin xs +\!\!+\, dom(\pi), (x\ z)M \rightrightarrows (y\ z)N}}{\forall z \notin xs +\!\!+\, dom(\pi), \pi((x\ z)M) \rightrightarrows \pi((y\ z)N)}}{\dfrac{\forall z \notin xs +\!\!+\, dom(\pi), ((\pi\ x)\ (\pi\ z))(\pi\ M) \rightrightarrows ((\pi\ y)\ (\pi\ z))(\pi\ N)}{\rightrightarrows\lambda\ \dfrac{\forall z \notin xs +\!\!+\, dom(\pi), ((\pi\ x)\ z)(\pi\ M) \rightrightarrows ((\pi\ y)\ z)(\pi\ N)}{\lambda(\pi\ x)(\pi\ M) \rightrightarrows \lambda(\pi\ y)(\pi\ N)}}}$$

β rule: Hypotheses: $\rightrightarrows\beta\ \dfrac{\lambda x M \rightrightarrows \lambda y P' \qquad N \rightrightarrows P'' \qquad P \sim_\alpha P'[y := P'']}{(\lambda x M)N \rightrightarrows P}$

Thesis: $(\lambda(\pi\ x)(\pi\ M))(\pi\ N) \rightrightarrows \pi\ P$

Proof:

$$\rightrightarrows\beta\ \dfrac{\text{ih}\ \dfrac{\lambda x M \rightrightarrows \lambda y P'}{\lambda(\pi\ x)(\pi\ M) \rightrightarrows \lambda(\pi\ y)(\pi\ P')} \qquad \text{ih}\ \dfrac{N \rightrightarrows P''}{\pi\ N \rightrightarrows \pi\ P''} \qquad \alpha\ \text{ind.}\ \dfrac{\alpha\ \text{equiv.}\ \dfrac{P \sim_\alpha P'[y := P'']}{\pi\ P \sim_\alpha \pi\ (P'[y := P''])}}{\pi\ P \sim_\alpha (\pi\ P')[(\pi\ y) := (\pi\ P}}{(\lambda(\pi\ x)(\pi\ M))(\pi\ N) \rightrightarrows \pi\ P}$$

$\square$

Lemma 4 (Parallel is right α-equivalent)**.**

$$M \rightrightarrows N \wedge N \sim_\alpha P \Rightarrow M \rightrightarrows P$$

Proof. Trivial induction on the parallel relation.

var. rule: Direct.

app. rule: Direct.

λ rule: Hypotheses:

$$\rightrightarrows\lambda\ \dfrac{\exists xs, \forall z \notin xs, (x\ z)M \rightrightarrows (y\ z)N}{\lambda x M \rightrightarrows \lambda y N} \sim_\alpha\text{a}\ \dfrac{\exists ys, \forall w \notin xs, (y\ w)N \sim_\alpha (z\ w)P}{\lambda y N \sim_\alpha \lambda z P}$$

Thesis: $\lambda x M \rightrightarrows \lambda z P$

Proof:
$$\text{ih}\ \dfrac{\forall w \notin xs +\!\!+\, ys, (x\ w)M \rightrightarrows (y\ w)N \qquad \forall w \notin xs +\!\!+\, ys, (y\ w)N \sim_\alpha (z\ w)P}{\rightrightarrows\lambda\ \dfrac{\forall w \notin xs +\!\!+\, ys, (x\ w)M \rightrightarrows (z\ w)P}{\lambda x M \rightrightarrows \lambda z P}}$$

β rule: Hypotheses:

$$\rightrightarrows\beta\ \dfrac{\lambda x M \rightrightarrows \lambda y P' \qquad N \rightrightarrows P'' \qquad P \sim_\alpha P'[y := P'']}{(\lambda x M)N \rightrightarrows P} \qquad P \sim_\alpha Q$$

Thesis: $(\lambda x M)N \rightrightarrows Q$

Proof:

$$\rightrightarrows\beta\ \dfrac{\lambda x M \rightrightarrows \lambda y P' \qquad N \rightrightarrows P'' \qquad \dfrac{Q \sim_\alpha P \sim_\alpha P'[y := P'']}{Q \sim_\alpha P'[y := P'']}}{(\lambda x M)N \rightrightarrows Q}$$

□

Lemma 5 (Parallel is left α-equivalent)**.**

$$M \sim_\alpha N \wedge N \rightrightarrows P \Rightarrow M \rightrightarrows P$$

Proof. Trivial induction on the parallel relation, analog to previous one as rules are symetric except from the β rule that we discuss next.

β rule: Hypotheses:

$$Q \sim_\alpha (\lambda x M)N \qquad \rightrightarrows\beta\ \dfrac{\lambda x M \rightrightarrows \lambda y P' \qquad N \rightrightarrows P'' \qquad P \sim_\alpha P'[y := P'']}{(\lambda x M)N \rightrightarrows P}$$

Thesis: $Q \rightrightarrows P$

Proof:

$$Q \sim_\alpha (\lambda x M)N \Rightarrow Q \equiv (\lambda y Q')Q'' \wedge \lambda z Q' \sim_\alpha \lambda x M \wedge Q'' \sim_\alpha N$$

$$\rightrightarrows\beta\ \dfrac{\text{hi}\ \dfrac{\lambda z Q' \sim_\alpha \lambda x M \qquad \lambda x M \rightrightarrows \lambda y P'}{\lambda z Q' \rightrightarrows \lambda y P'} \qquad \text{hi}\ \dfrac{Q'' \sim_\alpha N \qquad N \rightrightarrows P''}{Q'' \rightrightarrows P''} \qquad P \sim_\alpha P'[y := P'']}{Q \rightrightarrows P}$$

□

Lemma 6 (Parallel relation preserves freshness)**.**

$$x \# M \wedge M \rightrightarrows N \Rightarrow x \# N$$

Proof. α-induction on the term M. For any atom x we define the following predicate:

$$P(M) = \forall N, x \# M \wedge M \rightrightarrows N \Rightarrow x \# N$$

We prove P is α-compatible as a direct consequence of $\#$ and $\rightrightarrows$ relations are α-compatible on the M term also. That is, given term P such that $M \sim_\alpha P$, $x\#P$, $P \rightrightarrows N$, and assuming $P(M)$ holds, we prove that $x\#P$:

$$P(M)\ \dfrac{\dfrac{M \sim_\alpha P \qquad x\#P}{x\#M} \quad \text{lem. } 5\ \dfrac{M \sim_\alpha P \qquad P \rightrightarrows N}{M \rightrightarrows N}}{x\#N}$$

We use the α-induction principle with permutations to prove $\forall M, P(M)$ assuming for the λ-case that the binder position is distinct from x.

var case: Direct.

app. case: Hipothesis: $x\#MP \wedge MP \rightrightarrows N$ Thesis: $x\#N$

$\rightrightarrows$a subcase: $$\rightrightarrows\text{a}\ \dfrac{M \rightrightarrows N' \qquad P \rightrightarrows N''}{MP \rightrightarrows \underbrace{N'N''}_{=N}}$$

$$\#\text{a}\ \dfrac{\text{ih}\ \dfrac{\dfrac{x\#MP}{x\#M} \qquad M \rightrightarrows N'}{x\#N'} \qquad \text{ih}\ \dfrac{\dfrac{x\#MP}{x\#P} \qquad P \rightrightarrows N''}{x\#N''}}{x\#N'N'' = N}$$

$\rightrightarrows\beta$ subcase: $$\rightrightarrows\beta\ \dfrac{\lambda xM' \rightrightarrows \lambda yN' \qquad P \rightrightarrows N'' \qquad N'[y := N''] \sim_\alpha N}{\underbrace{(\lambda xM')}_{=M} P \rightrightarrows N}$$

$$\dfrac{\text{lem. } 2\ \dfrac{\text{ih}\ \dfrac{\dfrac{x\#(\lambda xM)'P}{x\#\lambda xM'} \qquad \lambda xM' \rightrightarrows \lambda yN'}{x\#\lambda yN'} \qquad \text{ih}\ \dfrac{\dfrac{x\#MP}{x\#P} \qquad P \rightrightarrows N''}{x\#N''}}{x\#N'[y := N'']} \qquad N'[y := N''] \sim_\alpha N}{x\#N}$$

λ case: Hypothesis: $x \neq y \wedge x\#\lambda yM \wedge \lambda yM \rightrightarrows \lambda zN$ Thesis: $x\#\lambda zN$.

$x \neq y \wedge x\#\lambda yM$ then $x\#M$.

$$\rightrightarrows\lambda\ \dfrac{\forall u, u \notin xs, (y\ u)M \rightrightarrows (z\ u)N}{\lambda yM \rightrightarrows \lambda zN}$$

Let be $u \notin \{x\} \cup xs \cup fv(N)$ then $(y\ u)M \rightrightarrows (z\ u)N$

$$\dfrac{\dfrac{\text{ih}\ \dfrac{\dfrac{\dfrac{x \neq y \qquad x\#\lambda yM}{x\#M} \qquad x \neq y, u}{x\#(y\ u)M} \qquad (y\ u)M \rightrightarrows (z\ u)N}{x\#(z\ u)N}}{x\#\lambda u(z\ u)N} \qquad \dfrac{\dfrac{u \notin fv(N)}{u\#N}}{\lambda u(z\ u)N \sim_\alpha \lambda zN}}{x\#\lambda zN}$$

$\square$

Lemma 7 (Parallel relation λ-elimination)**.**

$$\lambda xM \rightrightarrows M' \Rightarrow (\exists M'')(M \rightrightarrows M'', \lambda xM \rightrightarrows \lambda xM'', M' \sim_\alpha \lambda xM'')$$

$$\begin{tikzcd} & \lambda x M \arrow[dl] \arrow[dr, dashed] & & M \arrow[d, dashed] \\ M' & \sim_\alpha & \lambda x M'' & M'' \end{tikzcd}$$

Proof. We prove that $M'' = (y\ x)M'$ satifies all conjunction assertions in the thesis.

As $\lambda xM \rightrightarrows M'$ by parallel elimination we know:

- $M'' = \lambda yM'$
- $\exists xs, \forall z \notin xs, (x\ z)M \rightrightarrows (y\ z)M'$

Next prove the thesis conjunctions:

- $$\text{lem.4}\,\dfrac{\text{swap idem.}\,\dfrac{\rightrightarrows\text{equiv.}\,\dfrac{\dfrac{\text{be } z \notin xs \cup fv(\lambda yM') \qquad \forall z \notin xs, (x\ z)M \rightrightarrows (y\ z)M'}{(x\ z)M \rightrightarrows (y\ z)M'}}{(x\ z)(x\ z)M \rightrightarrows (x\ z)(y\ z)M'}}{M \rightrightarrows (x\ z)(y\ z)M'} \qquad \dfrac{x\#\lambda yM' \qquad \dfrac{z \notin fv(\lambda yM')}{z\#\lambda yM'}}{(x\ z)(y\ z)M' \sim_\alpha (x\ y)M'}}{M \rightrightarrows (x\ y)M'}$$
- $$\text{lem.4}\,\dfrac{\lambda xM \rightrightarrows \lambda yM' \qquad \lambda yM' \sim_\alpha \lambda x((x\ y)M')}{\lambda xM \rightrightarrows \lambda x((x\ y)M')}$$
- $$\dfrac{\text{lem.6}\,\dfrac{x\#\lambda xM \qquad \lambda xM \rightrightarrows \lambda yM'}{x\#\lambda yM'}}{\lambda yM' \sim_\alpha \lambda x((y\ x)M')}$$

$\square$

Lemma 8 (Parallel relation β-elimination)**.**

$$\rightrightarrows\beta\,\dfrac{\lambda xM \rightrightarrows \lambda yM' \quad N \rightrightarrows N' \quad M'[y := N'] \sim_\alpha P}{(\lambda xM)N \rightrightarrows P}$$

$$\Downarrow$$

$$(\exists M'')(\lambda xM \rightrightarrows \lambda xM'', M''[x := N'] \sim_\alpha P)$$

$$\begin{tikzcd} & \lambda x M \arrow[dl] \arrow[dr, dashed] & & M \arrow[d, dashed] & N \arrow[d] \\ \lambda y M' & \sim_\alpha & \lambda x M'' & M'' & N' \end{tikzcd} \qquad \begin{tikzcd} & (\lambda x M)N \arrow[d] \arrow[ddl] \arrow[ddr] & \\ & P \arrow[dl, phantom, "\sim_\alpha"] \arrow[dr, phantom, "\sim_\alpha"] & \\ M'[y := N'] & \sim_\alpha & M''[x := N'] \end{tikzcd}$$

Proof. We prove that $M'' = (y\ x)M'$ satifies the thesis:

- $$\text{lem.4}\ \dfrac{\lambda x M \rightrightarrows \lambda y M' \qquad \dfrac{\text{lem.6}\ \dfrac{x\#\lambda x M \qquad \lambda x M \rightrightarrows \lambda y M'}{x\#\lambda y M'}}{\lambda y M' \sim_\alpha \lambda x((x\ y)M')}}{\lambda x M \rightrightarrows \lambda x((y\ x)M')}$$

- $$\begin{array}{lll} ((y\ x)M')[x := N'] & \sim_\alpha & \\ ((x\ y)M')[x := N'] & \sim_\alpha & \text{lemma 3(proved by alpha induction) using as } x\#\lambda y M' \\ M'[y := N'] & \sim_\alpha & \text{hypothesis} \\ P & & \end{array}$$

□

5 Binder Relation

$$\notin_b\text{v}\ \dfrac{}{x \notin_b y} \qquad \notin_b\text{a}\ \dfrac{x \notin_b M \qquad x \notin_b N}{x \notin_b MN} \qquad \notin_b\lambda\ \dfrac{x \neq y \qquad x \notin_b M}{x \notin_b \lambda y M}$$

6 Another Alpha Primitive Induction Principle

If for some predicate P over terms exists a finite set of atoms A then:

$$\left.\begin{array}{ll} P\ \alpha\text{-compatible} & \wedge \\ \forall a, P(a) & \wedge \\ \forall M\ N, (\forall b \in A, b \notin_b MN) \wedge P(M) \wedge P(N) \Rightarrow P(MN) & \wedge \\ \forall M\ a,\ (\forall b \in A, b \notin_b \lambda a M) \wedge P(M) \Rightarrow P(\lambda a M) & \end{array}\right\} \Rightarrow \forall M, P(M)$$

As in Barendregt convention, this induction principle enable us to assume binders position fresh enough from a given finite context of variables A through the entire term induction (not only in the abstraction case). The following results uses this principle to avoid some binders in the β case of the application.

Lemma 9 (Parallel relation substitution lemma)**.**

$$M \rightrightarrows M' \wedge N \rightrightarrows N' \Rightarrow M[x := N] \rightrightarrows M'[x := N']$$

Proof. Uses previously introduced α induction principle gave in 6 over M term. Given some N, N' terms such that $N \rightrightarrows N'$, and x atom, we define the following predicate over terms:

$$P(M) \equiv \forall M', M \rightrightarrows M' \Rightarrow M[x := N] \rightrightarrows M'[x := N']$$

Next we prove that P is α-compatible, that is, $P(M) \wedge M \sim_\alpha N \Rightarrow P(N)$.

$$\text{congruence}\ \dfrac{\text{P(M)}\ \dfrac{\text{lem. 5}\ \dfrac{M \sim_\alpha N \qquad N \rightrightarrows M'}{M \rightrightarrows M'} \qquad N \rightrightarrows N'}{M[x := N] \rightrightarrows M'[x := N']} \qquad \text{subst.lemma}\ \dfrac{M \sim_\alpha N}{M[x := N] \equiv N[x := N]}}{N[x := N] \rightrightarrows M'[x := N']}$$

We take as $\{x\} \cup fv(N) \cup fv(N')$ the set of binders to avoid in the α-induction.

var case: $\forall a, P(a)$

$x \equiv a$ subcase: $x[x := N] \equiv N \rightrightarrows N' \equiv x[x := N']$

$x \not\equiv a$ subcase: $a[x := N] \equiv a \rightrightarrows a \equiv a[x := N']$

app. case: $\forall P\ Q, (\forall b \in \{x\} \cup fv(N) \cup fv(N'), b \notin_b PQ) \wedge P(P) \wedge P(Q) \Rightarrow P(PQ)$

$\rightrightarrows$a subcase: Hypotheses: $\rightrightarrows\text{a}\ \dfrac{P \rightrightarrows P' \qquad Q \rightrightarrows Q'}{PQ \rightrightarrows P'Q'}$ Thesis: $(PQ)[x := N] \rightrightarrows (P'Q')[x := N']$

$$\rightrightarrows\text{a}\ \dfrac{P(P)\ \dfrac{P \rightrightarrows P'}{P[x := N] \rightrightarrows P'[x := N']} \qquad P(Q)\ \dfrac{Q \rightrightarrows Q'}{Q[x := N] \rightrightarrows Q'[x := N']}}{(PQ)[x := N] \rightrightarrows (P'Q')[x := N']}$$

β subcase: Hypotheses: $\rightrightarrows\beta\ \dfrac{\lambda yP \rightrightarrows \lambda zP' \qquad Q \rightrightarrows Q' \qquad P'[z := Q'] \sim_\alpha R}{(\lambda yP)Q \rightrightarrows R}$

Thesis: $((\lambda yP)Q)[x := N] \rightrightarrows R[x := N']$

$$\text{lem. 8}\ \dfrac{(\lambda yP)Q \rightrightarrows R}{\exists P'', \lambda yP \rightrightarrows \lambda yP'' \wedge P''[y := Q'] \sim_\alpha R}$$

$$\dfrac{\dfrac{\forall b \in \{x\} \cup fv(N) \cup fv(N'), b \notin_b (\lambda yP)Q}{y \notin \{x\} \cup fv(N)}}{(\lambda yP)[x := N]) \sim_\alpha \lambda y(P[x := N])}$$

Analog

$$\dfrac{\dfrac{\forall b \in \{x\} \cup fv(N) \cup fv(N'), b \notin_b (\lambda yP)Q}{y \notin \{x\} \cup fv(N')}}{(\lambda yP'')[x := N'] \sim_\alpha \lambda y(P''[x := N'])}$$

So

$$\sim_\alpha\text{a}\ \dfrac{\dfrac{(\lambda yP)[x := N]) \sim_\alpha \lambda y(P[x := N]) \qquad Q[x := N] \sim_\alpha Q[x := N]}{((\lambda yP)[x := N])(Q[x := N]) \sim_\alpha (\lambda y(P[x := N]))(Q[x := N])}}{((\lambda yP)Q)[x := N] \sim_\alpha (\lambda y(P[x := N]))(Q[x := N])}$$

Finally,

$$\rightrightarrows\beta\ \dfrac{\dfrac{P(\lambda yP)\ \dfrac{\lambda yP \rightrightarrows \lambda yP''}{(\lambda yP)[x := N] \rightrightarrows (\lambda yP'')[x := N']}}{\lambda y(P[x := N]) \rightrightarrows \lambda y(P''[x := N'])} \qquad P(Q)\ \dfrac{Q \rightrightarrows Q'}{Q[x := N] \rightrightarrows Q'[x := N']} \qquad \dfrac{y \notin \{x\} \cup fv(N')}{P''[x := N'][y := Q'[x := N']] \sim_\alpha P''[y := Q'][x := N'] \equiv R[x := N']}}{\underbrace{(\lambda y(P[x := N]))(Q[x := N])}_{\sim_\alpha((\lambda yP)Q)[x:=N]} \rightrightarrows R[x := N']}$$

λ case: $\forall M\ a,\ (\forall b \in \{x\} \cup fv(N) \cup fv(N'), b \notin_b \lambda aM) \wedge PM \Rightarrow P(\lambda aM)$

Hypotheses: $\lambda yP \rightrightarrows Q$ Thesis: $(\lambda yP)[x := N] \rightrightarrows Q[x := N']$

$$\text{lem. 7}\ \dfrac{(\lambda yP) \rightrightarrows Q}{\exists Q', P \rightrightarrows Q' \wedge \lambda yP \rightrightarrows \lambda yQ' \wedge Q \sim_\alpha \lambda yQ'}$$

$$\dfrac{\dfrac{y \notin \{x\} \cup fv(N)}{(\lambda yP)[x := N]) \sim_\alpha \lambda y(P[x := N])} \qquad \dfrac{P(P)\ \dfrac{P \rightrightarrows Q'}{P[x := N] \rightrightarrows Q'[x := N']}}{\lambda y(P[x := N]) \rightrightarrows \lambda y(Q'[x := N'])}}{(\lambda yP)[x := N] \rightrightarrows \lambda y(Q'[x := N'])} \qquad \dfrac{y \notin \{x\} \cup fv(N')}{\lambda y(Q'[x := N']) \sim_\alpha (\lambda yQ')[x := N']}$$

$$\dfrac{(\lambda yP)[x := N] \rightrightarrows \lambda y(Q'[x := N']) \qquad \lambda y(Q'[x := N']) \sim_\alpha (\lambda yQ')[x := N']}{(\lambda yP)[x := N] \rightrightarrows \underbrace{(\lambda yQ')[x := N']}_{\equiv Q[x:=N'] \text{ as } Q \sim_\alpha \lambda yQ'}}$$

□

Lemma 10 (Diamond property of parallel relation)**.** $M \rightrightarrows N \wedge M \rightrightarrows P \Rightarrow \exists Q, N \rightrightarrows Q \wedge P \rightrightarrows Q$

$$\begin{tikzcd} & M \arrow[dl] \arrow[dr] & \\ N \arrow[dr, dashed] & & P \arrow[dl, dashed] \\ & Q & \end{tikzcd}$$

Proof. Induction on the M term.

var case: Direct.

app. case: Cases on both parallel reductions.

$\rightrightarrows$a-$\rightrightarrows$a subcase: Hypotheses: $\rightrightarrows\text{a}\ \dfrac{M \rightrightarrows N \qquad M' \rightrightarrows N'}{MM' \rightrightarrows NN'} \qquad \rightrightarrows\text{a}\ \dfrac{M \rightrightarrows P \qquad M' \rightrightarrows P'}{MM' \rightrightarrows PP'}$

Thesis: $\exists Q, NN' \rightrightarrows Q \wedge PP' \rightrightarrows Q$

$$\rightrightarrows\text{a}\ \dfrac{\text{ih}\ \dfrac{M \rightrightarrows N \qquad M \rightrightarrows P}{\exists Q, N \rightrightarrows Q \wedge P \rightrightarrows Q} \qquad \text{ih}\ \dfrac{M' \rightrightarrows N' \qquad M' \rightrightarrows P'}{\exists Q', N' \rightrightarrows Q' \wedge P' \rightrightarrows Q'}}{\exists (QQ'), NN' \rightrightarrows QQ' \wedge PP' \rightrightarrows QQ'}$$

$\rightrightarrows\beta$-$\rightrightarrows\beta$ subcase: Hypotheses: $\rightrightarrows\beta\ \dfrac{\lambda xM \rightrightarrows \lambda yN' \qquad M' \rightrightarrows N'' \qquad N'[y := N''] \sim_\alpha N}{(\lambda xM)M' \rightrightarrows N}$

$$\rightrightarrows\beta\ \dfrac{\lambda xM \rightrightarrows \lambda zP' \qquad M' \rightrightarrows P'' \qquad P'[z := P''] \sim_\alpha P}{(\lambda xM)M' \rightrightarrows P}$$

Thesis: $\exists Q, N \rightrightarrows Q \wedge P \rightrightarrows Q$

$$\text{lem. 8}\ \dfrac{(\lambda xM)M' \rightrightarrows N}{\exists N''', \lambda xM \rightrightarrows \lambda xN''' \wedge N'''[x := N''] \sim_\alpha N}$$

$$\text{lem. 8}\ \dfrac{(\lambda xM)M' \rightrightarrows P}{\exists P''', \lambda xP \rightrightarrows \lambda xP''' \wedge P'''[x := P''] \sim_\alpha P}$$

$$\text{ih}\ \dfrac{\lambda xM \rightrightarrows \lambda xN''' \qquad \lambda xM \rightrightarrows \lambda xP'''}{\exists R, \lambda xN''' \rightrightarrows R \wedge \lambda xP''' \rightrightarrows S}$$

$$\text{ih}\ \dfrac{M' \rightrightarrows N'' \qquad M' \rightrightarrows P''}{\exists S, N'' \rightrightarrows S \wedge P'' \rightrightarrows S}$$

$$\text{lem. 7}\ \frac{\lambda x N''' \rightrightarrows R}{\exists R', N''' \rightrightarrows R' \wedge \lambda x N''' \rightrightarrows \lambda x R' \wedge R \sim_\alpha \lambda x R'}$$

$$\text{lem. 7}\ \frac{\lambda x P''' \rightrightarrows R}{\exists R'', P''' \rightrightarrows R'' \wedge \lambda x P''' \rightrightarrows \lambda x R'' \wedge R \sim_\alpha \lambda x R''}$$

Finally,

$\exists R'[x := S], N \rightrightarrows R'[x := S], P \rightrightarrows R'[x := S]$

Because:

$$\frac{N \sim_\alpha N'''[x := N''] \qquad \text{lemma 9}\ \dfrac{N''' \rightrightarrows R' \qquad N'' \rightrightarrows S}{N'''[x := N''] \rightrightarrows R'[x := S]}}{N \rightrightarrows R[x := S]}$$

$$\frac{P \sim_\alpha P'''[x := P''] \qquad \text{lemma 9}\ \dfrac{\dfrac{P''' \rightrightarrows R'' \qquad \dfrac{R \sim_\alpha \lambda x R' \qquad R \sim_\alpha \lambda x R''}{R'' \sim_\alpha R'}}{P''' \rightrightarrows R'} \qquad P'' \rightrightarrows S}{P'''[x := P''] \rightrightarrows R'[x := S]}}{P \rightrightarrows R'[x := S]}$$

$\rightrightarrows$a-$\rightrightarrows\beta$ subcase: Hypotheses: $(\rightrightarrows\text{a})\ \dfrac{\lambda x M \rightrightarrows N' \qquad M' \rightrightarrows N''}{(\lambda x M)M' \rightrightarrows N'N''}$

$$(\rightrightarrows\beta)\ \frac{\lambda x M \rightrightarrows \lambda y P' \qquad M' \rightrightarrows P'' \qquad P'[y := P''] \sim_\alpha P}{(\lambda x M)M' \rightrightarrows P}$$

Thesis: $\exists Q, N'N'' \rightrightarrows Q \wedge P \rightrightarrows Q$

Applying λ and β elimination.

$$\text{lem. 7}\ \frac{\lambda x M \rightrightarrows N'}{\exists N''', M \rightrightarrows N''' \wedge \lambda x M \rightrightarrows \lambda x N''' \wedge N' \sim_\alpha \lambda x N'''}$$

$$\text{lem. 8}\ \frac{(\lambda x M)M' \rightrightarrows P}{\exists P''', \lambda x M \rightrightarrows \lambda x P''' \wedge P'''[x := P''] \sim_\alpha P}$$

Applying inductive hypothesis.

$$\text{ih}\ \frac{\lambda x M \rightrightarrows \lambda x N''' \qquad \lambda x M \rightrightarrows \lambda x P'''}{\exists Q, \lambda x N''' \rightrightarrows Q \wedge \lambda x P''' \rightrightarrows Q}$$

$$\text{ih}\ \frac{M' \rightrightarrows N'' \qquad M' \rightrightarrows P''}{\exists R, N'' \rightrightarrows R \wedge P'' \rightrightarrows R}$$

Applying λ-elimination again.

$$\text{lem. 7}\ \frac{\lambda x N''' \rightrightarrows Q}{\exists S', N''' \rightrightarrows S' \wedge \lambda x N''' \rightrightarrows \lambda x S' \wedge Q \sim_\alpha \lambda x S'}$$

$$\text{lem. 7}\ \frac{\lambda x P''' \rightrightarrows Q}{\exists S'', P''' \rightrightarrows S'' \wedge \lambda x P''' \rightrightarrows \lambda x S'' \wedge Q \sim_\alpha \lambda x S''}$$

Finally $\exists S'[x := R], N'N'' \rightrightarrows S'[x := R] \wedge P \rightrightarrows S'[x := R]$ as proved next.

$$\text{lem.}\ \frac{\rightrightarrows\text{a}\ \dfrac{N' \sim_\alpha \lambda x N''' \qquad N'' \sim_\alpha N''}{N'N'' \sim_\alpha (\lambda x N''')N''} \qquad \rightrightarrows\beta\ \dfrac{\lambda x N''' \rightrightarrows \lambda x S' \qquad N'' \rightrightarrows R}{(\lambda x N''')N'' \rightrightarrows S'[x := R]}}{N'N'' \rightrightarrows S'[x := R]}$$

$$
\dfrac{P \sim_\alpha P'''[x := P''] \qquad \text{lemma 9}\,\dfrac{\dfrac{P''' \rightrightarrows S'' \qquad \dfrac{Q \sim_\alpha \lambda x S' \qquad Q \sim_\alpha \lambda x S''}{S'' \sim_\alpha S'}}{P''' \rightrightarrows S'} \qquad P'' \rightrightarrows R}{P'''[x := P''] \rightrightarrows S'[x := R]}}{P \rightrightarrows S'[x := R]}
$$

$\rightrightarrows\beta$-$\rightrightarrows$a subcase: Analog to previous one.

λ case: Hypotheses: $\lambda y M \rightrightarrows N \wedge \lambda y M \rightrightarrows P$ Thesis: $\exists Q, N \rightrightarrows Q \wedge P \rightrightarrows Q$

$$
\text{lem.5}\,\dfrac{\dfrac{\text{ih}\,\dfrac{\text{lem. 7}\,\dfrac{(\lambda y M) \rightrightarrows N}{\exists N', M \rightrightarrows N' \wedge N \sim_\alpha \lambda y N'} \qquad \text{lem. 7}\,\dfrac{(\lambda y M) \rightrightarrows P}{\exists P', M \rightrightarrows P' \wedge P \sim_\alpha \lambda y P'}}{\exists Q, N' \rightrightarrows Q \wedge P' \rightrightarrows Q}}{\exists \lambda y Q, \lambda y N' \rightrightarrows \lambda y Q \wedge \lambda y P' \rightrightarrows \lambda y Q}}{\exists \lambda y Q, N \rightrightarrows \lambda y Q \wedge P \rightrightarrows \lambda y Q}
$$

$\square$