Computer Graphics

Recap of linear algebra I

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Linear algebra is the study of linear maps on finite dimensional vector spaces

Linear algebra is about matrices as much as
astronomy is about telescopes

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Example: Lists of numbers

$\mathbb{R}^n$ is defined to be the set of all n-long sequences of numbers in $\mathbb{R}$:

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_j \in \mathbb{R} \text{ for } j = 1, 2, \ldots, n\}$$

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Addition and multiplication are defined as expected:

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$
$$\lambda(x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$$

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While the additive identity can be defined as:

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With these definitions, $\mathbb{R}^n$ is a vector space

Example: Functions

Consider the set of all functions $f : [0, 1] \to \mathbb{R}$ with the standard definitions for sum and scalar product:

$$(f + g)(x) = f(x) + g(x)$$
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The above forms a vector space. In fact, any set of functions $f : S \to \mathbb{R}$ with $S \neq \emptyset$ and the definitions above forms a vector space.

Elements of a vector space (called vectors)
are not necessarily lists

A vector space is an abstract entity whose elements
might be lists, functions, or weird objects

Example: Shapes

Do shapes form a vector space?

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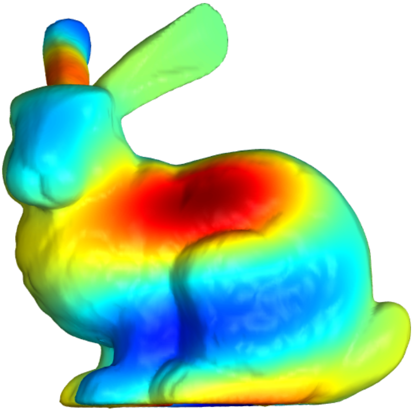
Do shapes form a vector space? Not really – try to sum two points



We will not use linear algebra to model shapes (this will be the job of differential geometry)

But we can use linear algebra to manipulate functions on shapes

Subspaces

A subset $U \subset V$ is a subspace of V if it is a vector space (using the same operations defined for V)

In particular:

- $0 \in U$
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Examples:

- $\{(x_1, x_2, 0) : x_1, x_2 \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$
- The set of piecewise-linear functions on a mesh M is a subspace of all functions $f : M \to \mathbb{R}$

Bases

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- $v_1, \ldots, v_n \in V$ are linearly independent if and only if each $v \in \mathrm{span}(v_1, \ldots, v_n)$ has only one representation as a linear combination of $v_1, \ldots, v_n$

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So every vector $v \in V$ can be expressed uniquely as a linear combination

$$v = \sum_{i=1}^{n} \alpha_i v_i$$

You can think of a basis as the minimal set of vectors that generates the entire space

Example: Bases

- $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ is a basis of $\mathbb{R}^n$ called the standard basis; its vectors are called the indicator vectors

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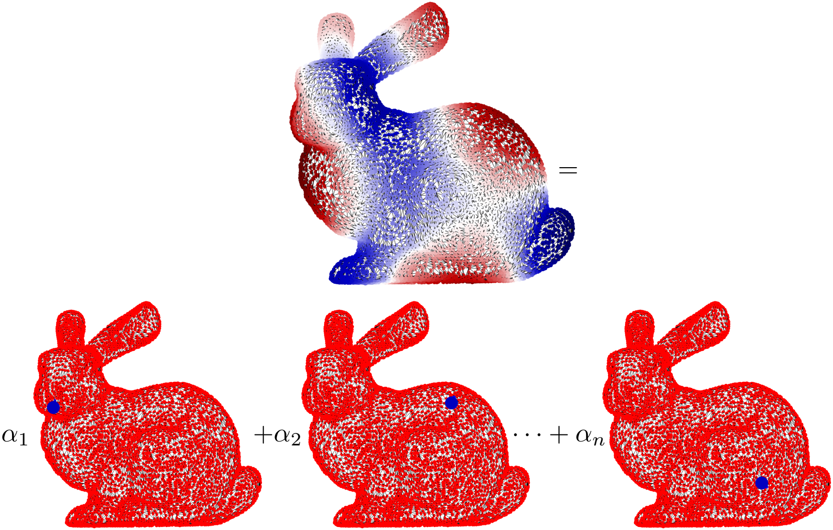
- $(1,2),(3,5.07)$ is a basis of $\mathbb{R}^2$

- $$f_1(x) = \begin{cases} 1 & \text{if } x = x_1 \\ 0 & \text{else} \end{cases}$$
  $$f_2(x) = \begin{cases} 1 & \text{if } x = x_2 \\ 0 & \text{else} \end{cases}$$
  $$\vdots$$

  is the standard basis for the set of functions $f : \mathbb{R} \to \mathbb{R}$; the basis vectors are also called indicator functions

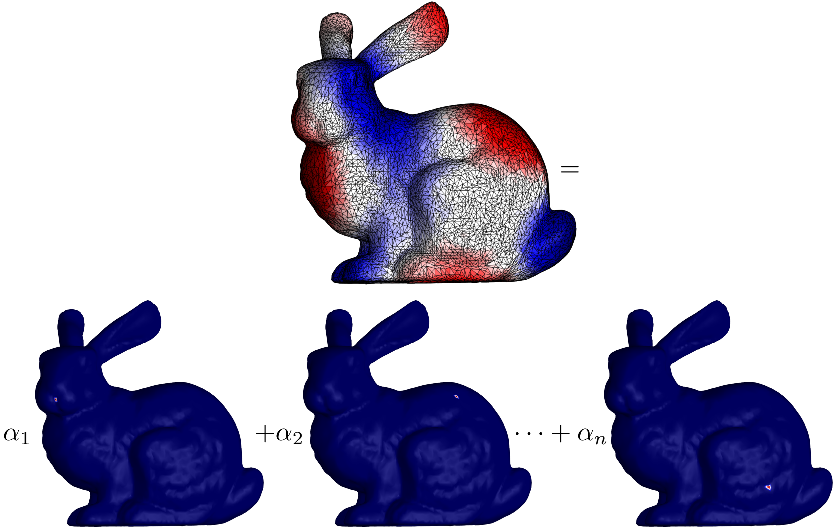
Example: Standard basis

Basis vectors are indicator functions at all vertices in the mesh



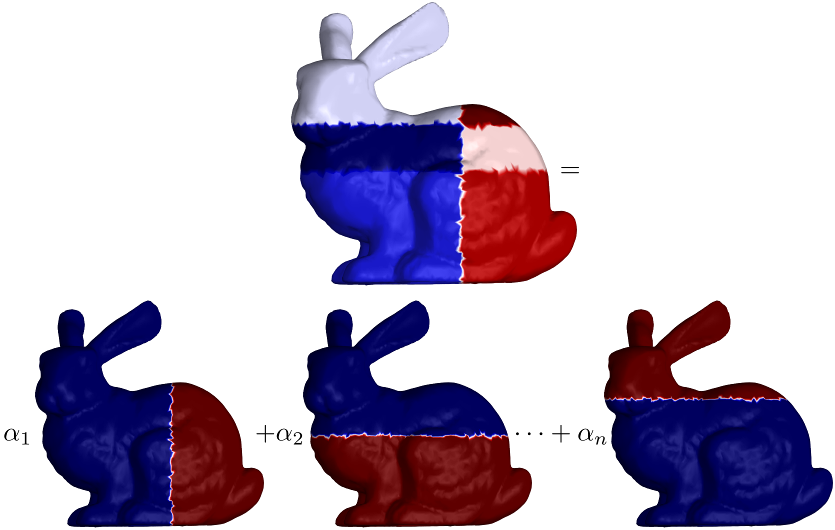
Example: Hat basis

Basis vectors are hat functions at all vertices in the mesh



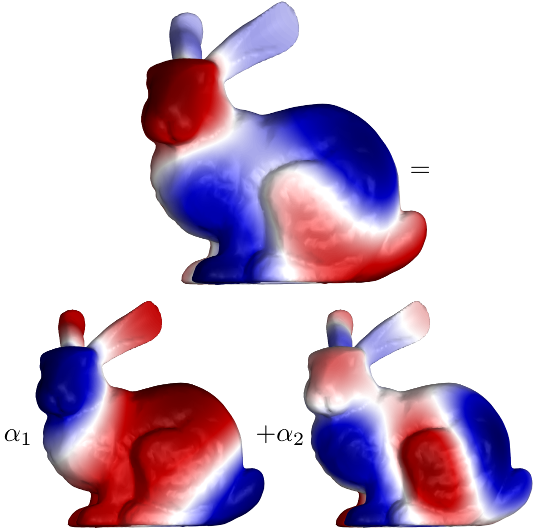
Example: Region-based basis

Basis vectors are some piecewise-constant functions on the mesh



Example: Smooth basis

Basis vectors are two random smooth functions



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$f : \mathbb{R} \to \mathbb{R}$
infinite dimensional
(functional analysis)

$f : \mathcal{A} \to \mathbb{R}$
finite dimensional
(linear algebra)

Linear maps

A linear map from V to W is a function $T : V \to W$ with the properties:

- **additivity**: $T(u + v) = Tu + Tv$ for all $u, v \in V$
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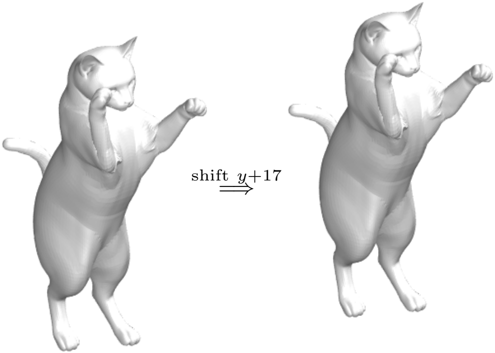
- from $\mathbb{R}^n$ to $\mathbb{R}^m$, defined as

$$T(x_1, \dots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \dots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

Example: Shape translation

Q: Does the following translation operation define a linear transformation?

$$T : \mathbb{R}^3 \to \mathbb{R}^3 \,, \quad T(x, y, z) = (x, y, z) + (0, 17, 0)$$



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Q: Does the following translation operation define a linear transformation? No – linear maps take 0 to 0

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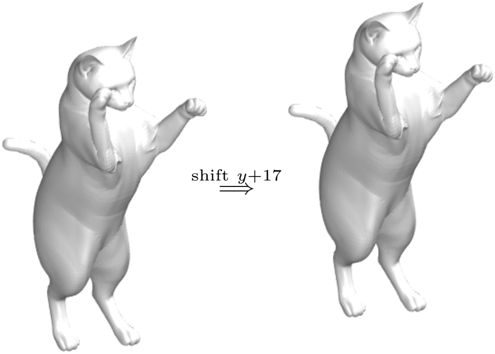

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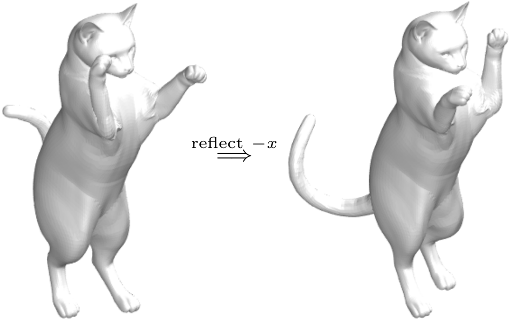

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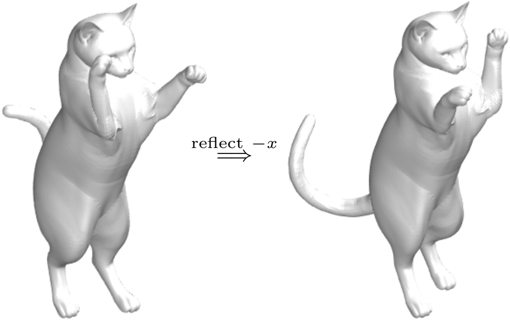

The matrix is a representation for a linear map, but it depends on the choice of bases

Linear maps as matrices

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The matrix of T in these bases is the $m \times n$ array of values in $\mathbb{R}$

$$\mathbf{T} = \begin{pmatrix} T_{1,1} & \cdots & T_{1,n} \\ \vdots & & \vdots \\ T_{m,1} & \cdots & T_{m,n} \end{pmatrix}$$

whose entries $T_{i,j}$ are defined by

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Hence each column of $\mathbf{T}$ contains the linear combination coefficients for the image via T of a basis vector from V

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In other words, the matrix encodes how basis vectors are mapped. and this is enough to map all other vectors in their span, since:

$$Tv = T(\sum_j \alpha_j v_j) = \sum_j T(\alpha_j v_j) = \sum_j \alpha_j Tv_j$$

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We call $\mathbb{R}^{m \times n}$ the vector space of all $m \times n$ matrices with values in $\mathbb{R}$

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- **scalar multiplication**: given $\lambda \in \mathbb{R}$, the matrix for λT is given by λ times the matrix of T

In fact, we have just shown that matrices form a vector space (Q1: what is the additive identity?) (Q2: what is the vector space dimension?)

We call $\mathbb{R}^{m \times n}$ the vector space of all $m \times n$ matrices with values in $\mathbb{R}$

- **product**: the matrix for ST can be computed by the matrix product between $\mathbf{S}$ and $\mathbf{T}$; in fact, the matrix product is defined precisely to make this work

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  Q3: is matrix product commutative?

  Q4: do we need the same bases for $S : U \to V$ and $T : V \to W$?

Matrix of a vector

Suppose $v \in V$ is an arbitrary vector, while $v_1, \ldots, v_n$ is a basis of V. The matrix of coordinates ("vector matrix") of v w.r.t. this basis is the

$n \times 1$ matrix:

$$\mathbf{v} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

so that

$$v = c_1 v_1 + \cdots c_n v_n$$

Note that the expansion coefficients depend on the choice of basis for V !

Product of "map matrix" and "vector matrix"

Consider a linear map $T : V \to W$, a basis $v_1, \ldots, v_n \in V$ and a basis $w_1, \ldots, w_m \in W$.

From the definition of matrix product, one can show that it operates on a vector matrix as expected:

$$\mathbf{Tv} = \mathbf{w} \quad \Leftrightarrow \quad Tv = w$$

where $\mathbf{Tv}$ is the matrix product of $\mathbf{T}$ and $\mathbf{v}$, while Tv simply denotes the function evaluation $T(v)$

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Remember: $\mathbf{T}, \mathbf{v}, \mathbf{w}$ must follow a coherent choice of bases in order for the above to make sense. $\mathbf{v}$ can not be expressed in basis $(\tilde{v}_1, \dots, \tilde{v}_n)$ if $\mathbf{T}$ only knows how to map basis vectors $(v_1, \dots, v_n)$.

$$Tv_j = T_{1,j}w_1 + \cdots + T_{m,j}w_m$$

Product of "map matrix" and "vector matrix"

$$\underbrace{\begin{pmatrix} T_{1,1} & \cdots & T_{1,n} \\ \vdots & & \vdots \\ T_{m,1} & \cdots & T_{m,n} \end{pmatrix}}_{\mathbf{T}} \underbrace{\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}}_{\mathbf{c}} = \sum_{j=1}^{n} c_j \underbrace{\begin{pmatrix} T_{1,j} \\ \vdots \\ T_{m,j} \end{pmatrix}}_{\mathrm{Tv_j \ wrt \ (w_1,\ldots,w_m)}}$$

Because recall that, for bases $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_m \in W$:

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We see then that vector $c = \sum_j c_j v_j$ is mapped to $Tc = \sum_j c_j T v_j$

In other words, matrix product is behaving as expected

Linear maps as a vector space

Linear maps $T : V \to W$ form a vector space, with addition and multiplication (Q: what is the additive identity?) defined as:

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In addition, we also have a useful definition of product between linear maps. This is kind of a special situation, since in general it makes no sense to multiply vectors.

If $T : U \to V$ and $S : V \to W$, their product $ST : U \to W$ is defined by

$$(ST)(u) = S(Tu)$$

In other words, ST is just the usual composition $S \circ T$ of two functions

Algebraic properties of products of linear maps

- **associativity**: $(T_1 T_2)T_3 = T_1(T_2 T_3)$
- **identity**: $TI = IT = T$
- **distributive properties**: $(S_1 + S_2)T = S_1 T + S_2 T$ and $S(T_1 + T_2) = ST_1 + ST_2$

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Keep in mind that composition of linear maps is not commutative, i.e.

$$ST \neq TS$$

in general (although there are special cases)

Example: Take $Sf = f'$ and $(Tf)(x) = x^2 f(x)$

Example: Reflections

In the standard basis, reflection matrices for points in $\mathbb{R}^3$ have the form:

$$\mathbf{R}_x = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{R}_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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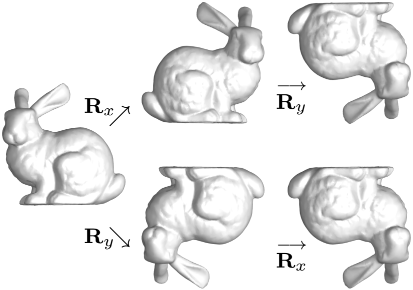
In particular, reflecting along the x coordinate and then on the y coordinate has the same effect as the viceversa:



which means that in this case the two underlying linear maps commute, i.e. $\mathbf{R}_x\mathbf{R}_y = \mathbf{R}_y\mathbf{R}_x$

Exercise: grid basis

For lena.png (download from course website):

- Divide the image in $k = 1024$ equal square regions (32×32 grid)

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- Interpret the color vectors of the image as scalar functions defined on the original pixel grid
- Express these functions in the grid basis by solving, for each function, the linear system

$$\mathbf{Bc} \approx \mathbf{p}$$

where $\mathbf{B}$ contains the basis functions as its columns, and $\mathbf{p}$ is the matrix representation of each color in the standard basis

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- Go back to the standard basis by computing $\mathbf{Bc}$ and plotting the result as a new image

Suggested reading

Most of the material from this lecture was selected from sections 1.A – 3.D of the following excellent textbook:

S. Axler, "Linear algebra done right – 3rd edition". Springer, 2015