Computer Graphics

Recap of linear algebra II

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Recap: Bases

A basis of V is a collection of vectors in V that is linearly independent and spans V

- $\text{span}(v_1, \ldots, v_n) = \{a_1 v_1 + \cdots a_n v_n : a_1, \ldots, a_n \in \mathbb{R}\}$
- $v_1, \ldots, v_n \in V$ are linearly independent if and only if each $v \in \text{span}(v_1, \ldots, v_n)$ has only one representation as a linear combination of $v_1, \ldots, v_n$

You can think of a basis as the minimal set of vectors that generates the entire space

Recap: Bases

Every vector $v \in V$ can be expressed uniquely as a linear combination:

$$v = \sum_{i=1}^{n} c_i v_i$$

Considering the "vector matrix" $n \times 1$ and the "basis matrix" $n \times n$

$$\mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \qquad \mathbf{V} = \begin{pmatrix} | & \dots & | \\ v_1 & \dots & v_n \\ | & \dots & | \end{pmatrix}$$

the linear combination can be in matrix notation as:

$$v = \mathbf{V}\mathbf{c}$$

Recap: Matrices

Consider a linear map $T : V \to W$, a basis $v_1, \dots, v_n \in V$ and a basis $w_1, \dots, w_m \in W$.

The matrix of T in these bases is the $m \times n$ array of values in $\mathbb{R}$

$$\mathbf{T} = \begin{pmatrix} T_{1,1} & \cdots & T_{1,n} \\ \vdots & & \vdots \\ T_{m,1} & \cdots & T_{m,n} \end{pmatrix}$$

whose entries $T_{i,j}$ are defined by

$$Tv_j = T_{1,j} w_1 + \cdots + T_{m,j} w_m$$

In other words, the matrix encodes how basis vectors are mapped, and this is enough to map all other vectors in their span, since:

$$Tv = T(\sum_j c_j v_j) = \sum_j T(c_j v_j) = \sum_j c_j Tv_j$$

Recap: Product of "map matrix" and "vector matrix"

Using the matrix of T we have:

$$Tv = \mathbf{Tc}$$

because

$$\underbrace{\begin{pmatrix} T_{1,1} & \cdots & T_{1,n} \\ \vdots & & \vdots \\ T_{m,1} & \cdots & T_{m,n} \end{pmatrix}}_{\mathbf{T}} \underbrace{\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}}_{\mathbf{c}} = \sum_{j=1}^{n} c_j \underbrace{\begin{pmatrix} T_{1,j} \\ \vdots \\ T_{m,j} \end{pmatrix}}_{\mathrm{Tv_j\ wrt\ (w_1,\ldots,w_m)}}$$

(Recall that, for bases $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_m \in W$:

$$Tv_j = T_{1,j} w_1 + \cdots + T_{m,j} w_m)$$

We see then that vector $v = \sum_j c_j v_j$ is mapped to $Tv = \sum_j c_j Tv_j$.

In other words, matrix product is behaving as expected.

Rank of a matrix

The rank of a matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is the dimension of the span of its columns

Example:

$$\mathbf{A} = \begin{pmatrix} 4 & 7 & 1 & 8 \\ 3 & 5 & 2 & 9 \end{pmatrix}$$

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The rank is the dimension of

$$\operatorname{span}\left(\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 8 \\ 9 \end{pmatrix}\right) \in \mathbb{R}^2$$

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Note that this result does not depend on a choice of basis, i.e., change of basis preserves the rank

Example: Reduced bases

Consider the $\mathbb{R}^{n\times k}$ matrix

$$\mathbf{V} = \begin{pmatrix} | & \cdots & \cdots & | \\ \mathbf{v}_1 & \cdots & \cdots & \mathbf{v}_k \\ | & \cdots & \cdots & | \end{pmatrix}$$

containing basis vectors as its columns, and the $\mathbb{R}^{n\times k'}$ matrix

$$\mathbf{V}' = \begin{pmatrix} | & \cdots & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_{k'} \\ | & \cdots & | \end{pmatrix}$$

obtained by truncating $\mathbf{V}$ to the first $k' < k$ columns

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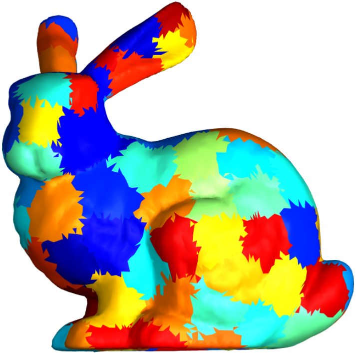
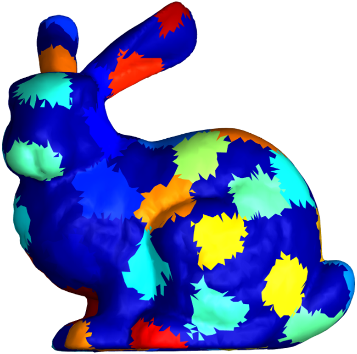
Then, $k = \mathrm{rank}(\mathbf{V}) > \mathrm{rank}(\mathbf{V}') = k'$

The rank reflects the expressive power of the full $(\mathbf{V})$ and reduced $(\mathbf{V}')$ bases

Example: Reduced bases



full basis
$\mathrm{rank}(\mathbf{V}) = k$



reduced basis
$\mathrm{rank}(\mathbf{V}') = k' < k$

Invariant subspaces

A subspace U of V is called invariant under $T : V \to V$ if:

$$u \in U \quad \text{implies} \quad Tu \in U$$

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Conversely, if $Tv = \lambda v$ for some $\lambda \in \mathbb{R}$, then $\mathrm{span}(v)$ is a 1-dimensional subspace of V invariant under T

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If the equation holds for m distinct eigenvalues and eigenvectors:

$$Tv_1 = \lambda_1 v_1$$
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Then it can be proven that $v_1, \ldots, v_m$ are linearly independent, and it must be:

$$m \leq \dim(V)$$

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Additional notes:

- This decomposition only makes sense for $T : V \to V$
- If V is a function space, eigenvectors are called eigenfunctions

Eigenspaces

If distinct eigenvectors $E = (v_1, \ldots, v_m)$ correspond to the same eigenvalue λ, then E spans an eigenspace of T

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- Certain linear maps $T : V \to V$ induce invariant subspaces on V

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- Certain linear maps $T : V \to V$ induce invariant subspaces on V
- These subspaces are spanned by eigenvectors of T
- Subspaces might be high-dimensional and are called eigenspaces
- Eigenspaces provide a form of decomposition of V

Inner product

We want to be able to measure lengths and angles among vectors

Note: We assume our vectors are real-valued

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To do so, we define the inner product as a function $\langle u, v \rangle : V \times V \to \mathbb{R}$ with the properties:

- **non-negativity:** $\langle v, v \rangle \geq 0$ for all $v \in V$

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- **symmetry:** $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$

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Examples: Inner products

- **Lists:**

  The Euclidean inner product (or dot product) is defined by

  $$\langle (u_1, \ldots, u_n), (v_1, \ldots, v_n) \rangle = u_1 v_1 + \cdots u_n v_n$$

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- **Functions:**

  On the vector space of continuous functions $f : [-1, 1] \to \mathbb{R}$

  $$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx$$

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Each inner product determines a norm:

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For $u, v \in \mathbb{R}^2$, it can be shown that

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where $\theta \in \mathbb{R}$ is the angle between u, v if we think of them as arrows with initial point at the origin

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From this, we can think of the inner product as encoding a general notion of angle between two vectors:

$$\theta = \arccos \frac{\langle u, v \rangle}{\|u\| \|v\|}$$

For example, we can now think of "angle between two functions"

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Note that any orthogonal basis can be made orthonormal by rescaling each basis vector:

$$v_i \mapsto \frac{v_i}{\|v_i\|}$$

Orthonormal bases

Given an orthonormal basis, $v \in V$ can be written as a linear combination:

$$v = \langle v, v_1 \rangle v_1 + \cdots + \langle v, v_n \rangle v_n$$

So the combination coefficients are simply given by inner products

Inner product in matrix notation

For vectors $u, v \in V$ in the standard basis $\{e_i\}$, we can write:

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In matrix notation, we can thus write

$$\langle u, v \rangle = \mathbf{u}^\top \mathbf{v}$$

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  - $\mathbf{Wu}$ gives the coefficients of u in the standard basis (and similarly for v)

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$$\begin{aligned} \langle u, v \rangle &= \sum_{i,j} u_i v_j \underbrace{\langle w_i, w_j \rangle}_{\mathbf{w}_i^\top \mathbf{w}_j} \\ &= \mathbf{u}^\top \mathbf{W}^\top \mathbf{W} \mathbf{v} \end{aligned}$$

where $\mathbf{W}$ contains the basis vectors $\mathbf{w}_i$ as its columns

- If $\{w_i\}$ is orthonormal, then $\mathbf{W}^\top \mathbf{W} = \mathbf{I}$
- Another way to see the derivation above is as follows:
  - $\mathbf{Wu}$ gives the coefficients of u in the standard basis (and similarly for v)
  - Once we are in the standard basis, we can write

$$\langle u, v \rangle = (\mathbf{Wu})^\top (\mathbf{Wv})$$

Inner product in matrix notation

For vectors $u, v \in V$ in some other basis $\{w_i\}$, we can write:

$$\langle u, v \rangle = \sum_{i,j} u_i v_j \underbrace{\langle w_i, w_j \rangle}_{\mathbf{w}_i^\top \mathbf{w}_j}$$
$$= \mathbf{u}^\top \mathbf{W}^\top \mathbf{W} \mathbf{v}$$

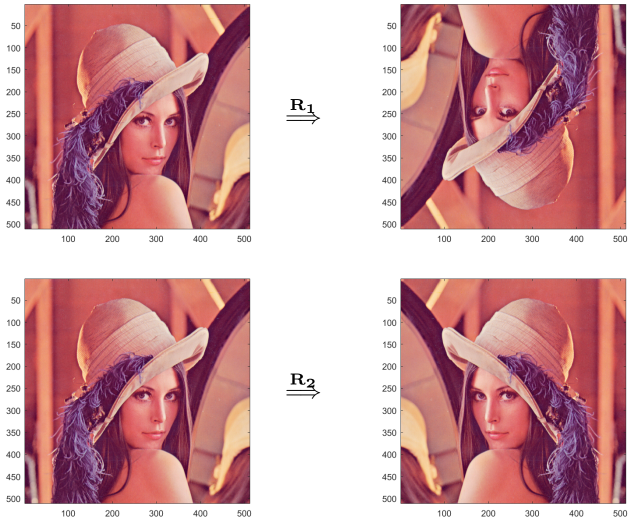
where $\mathbf{W}$ contains the basis vectors $\mathbf{w}_i$ as its columns

- If $\{w_i\}$ is orthonormal, then $\mathbf{W}^\top \mathbf{W} = \mathbf{I}$
- Another way to see the derivation above is as follows:
  - $\mathbf{Wu}$ gives the coefficients of u in the standard basis (and similarly for v)
  - Once we are in the standard basis, we can write

$$\langle u, v \rangle = (\mathbf{Wu})^\top (\mathbf{Wv}) = \mathbf{u}^\top \mathbf{W}^\top \mathbf{W} \mathbf{v}$$

Exercise

Find the two linear maps $\mathbf{R_1}$ and $\mathbf{R_2}$ that produce the following reflections:



Suggested reading

See sections 3.F, 5.A – 6.B of:

S. Axler, “Linear algebra done right – 3rd edition”. Springer, 2015