Computer Graphics

Local coordinates

Emanuele Rodolà
rodola@di.uniroma1.it
SAPIENZA
UNIVERSITÀ DI ROMA

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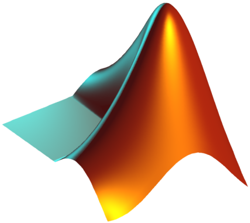
Exercises

- Farthest point sampling



Adjacency matrices as operators

We can see adjacency matrices as operators when applied to functions

For example, $\mathbf{g} = \mathbf{A}\mathbf{f}$ yields a vertex-based function g defined as:

$$g(v_i) = \sum_{e_{ij} \in E} f(v_j)$$

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On this observation, one can construct new operators such as $\mathbf{I} - \mathbf{A}$:

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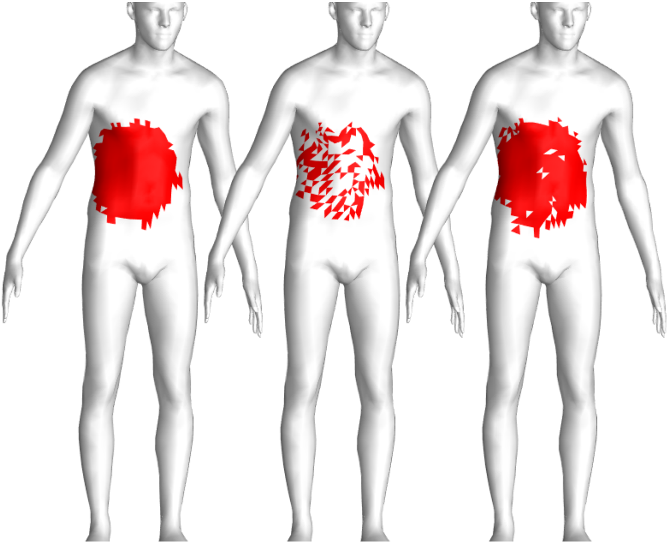
And similarly for more complex (but linear) operations

Example: Hole filling



original $\mathbf{f}$ | noisy $\tilde{\mathbf{f}}$ | denoised $\mathbf{A}\tilde{\mathbf{f}}$

Graph Laplacian

Given a mesh graph $G = (V, E)$, consider this condition on vertex v_i:

$$\mathbf{v}_i - \frac{1}{d_i} \sum_{j:(i,j) \in E} \mathbf{v}_j = 0$$

where d_i is the valence of v_i

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In matrix notation, we define the $n \times n$ matrix $\mathbf{L}$ as:

$$L_{ij} = \begin{cases} 1 & \text{if } i = j \\ -\frac{1}{d_i} & \text{if } e_{ij} \in E \\ 0 & \text{otherwise} \end{cases}$$

also known as the graph Laplacian of G

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$$\mathbf{LV} = \mathbf{0}$$

Least squares meshes

Is there an embedding satisfying $\mathbf{LV} = \mathbf{0}$?

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Sorkine and Cohen-Or, "Least-squares meshes". Proc. SMI, 2004

Least squares meshes

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Then, consider the linear system:

$$\begin{pmatrix} \mathbf{L} \\ \mathbf{A} \end{pmatrix} \mathbf{V} = \mathbf{b}$$

where

$$a_{ij} = \begin{cases} 1 & \text{if } v_j \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases} \quad , \quad b_k = \begin{cases} (0,0,0) & k \leq n \\ \mathbf{v}_{s_{k-n}} & n < k \leq n+m \end{cases}$$

Sorkine and Cohen-Or, "Least-squares meshes". Proc. SMI, 2004

Least squares meshes

$$\begin{pmatrix} \mathbf{L} \\ \mathbf{A} \end{pmatrix} \mathbf{V} \approx \mathbf{b}$$

Sorkine and Cohen-Or, "Least-squares meshes". Proc. SMI, 2004

Least squares meshes

$$\min_{\mathbf{V}\in\mathbb{R}^{n\times 3}} \| \begin{pmatrix} \mathbf{L} \\ \mathbf{A} \end{pmatrix} \mathbf{V} - \mathbf{b} \|_2^2$$

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Least squares meshes

$$\min_{\mathbf{V}\in\mathbb{R}^{n\times 3}} \|\mathbf{L}\mathbf{V}\|_2^2 + \sum_{v_i\in\mathcal{A}} \|\mathbf{v}_i - \mathbf{b}_i\|_2^2$$

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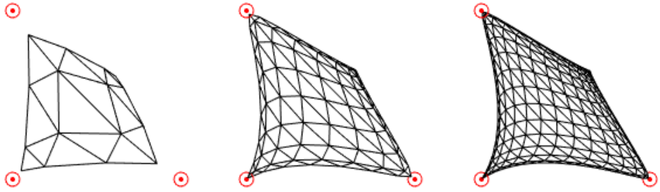
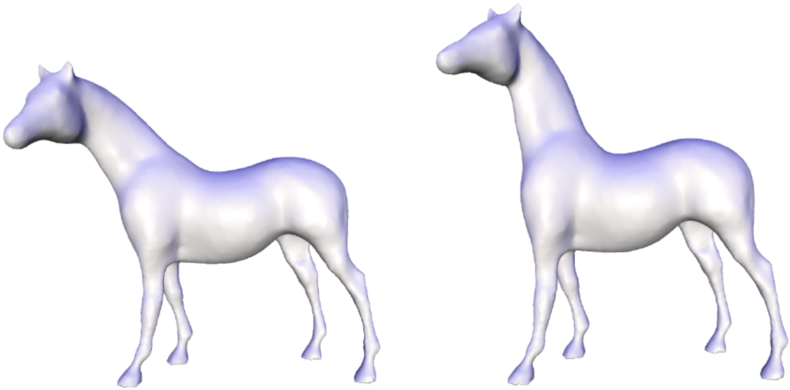
$$\min_{\mathbf{V}\in\mathbb{R}^{n\times 3}} \|\mathbf{LV}\|_2^2 + \sum_{v_i\in\mathcal{A}} \|\mathbf{v}_i - \mathbf{b}_i\|_2^2$$



- Anchor constraints are not satisfied exactly
- At higher resolution, error distributes better among the constraints

Sorkine and Cohen-Or, "Least-squares meshes". Proc. SMI, 2004

Least squares meshes



Move the anchor positions to do shape modeling

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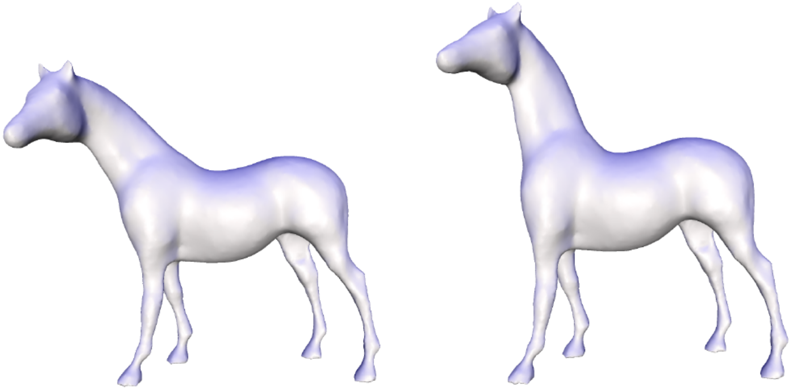

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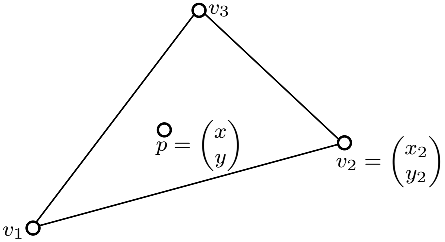
Barycentric coordinates

How to express p in terms of v_1, v_2, v_3?



Barycentric coordinates

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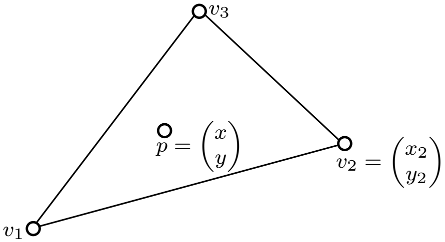

In other words, we want a new coordinate system where

$$v_1 = (1, 0, 0)$$
$$v_2 = (0, 1, 0)$$
$$v_3 = (0, 0, 1)$$

and

$$p = (\lambda_1, \lambda_2, \lambda_3)$$

Barycentric coordinates

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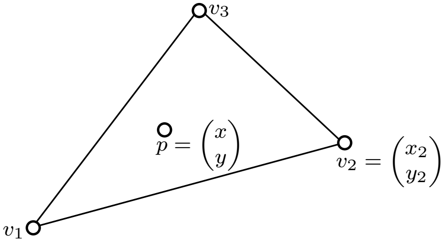

In particular, we want a convex combination

$$p = \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$$

with

$$\lambda_1, \lambda_2, \lambda_3 \geq 0\,, \quad \lambda_1 + \lambda_2 + \lambda_3 = 1$$

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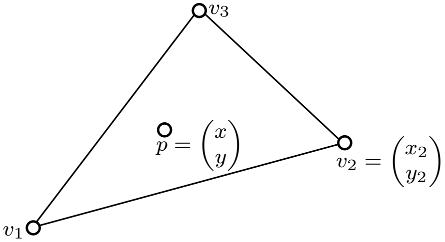


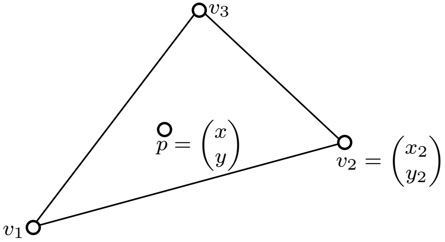
We come to the linear system

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

which has a unique solution

Barycentric coordinates

How to express p in terms of v_1, v_2, v_3?



Solving the system, we get the closed form expressions:

$$\lambda_1 = \frac{(y_2 - y_3)(x - x_3) + (x_3 - x_2)(y - y_3)}{(y_2 - y_3)(x_1 - x_3) + (x_3 - x_2)(y_1 - y_3)}$$

$$\lambda_2 = \frac{(y_3 - y_1)(x - x_3) + (x_1 - x_3)(y - y_3)}{(y_2 - y_3)(x_1 - x_3) + (x_3 - x_2)(y_1 - y_3)}$$

$$\lambda_3 = 1 - \lambda_1 - \lambda_2$$

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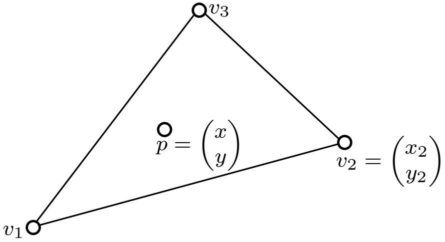

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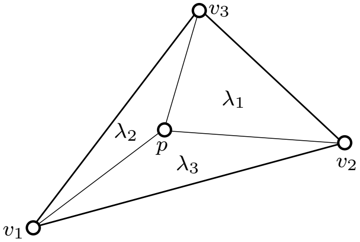
If p is outside the triangle, we get at least one negative coordinate!

Barycentric coordinates: interpretation

The barycentric coordinates are proportional to the triangle areas



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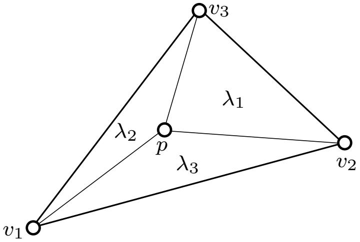
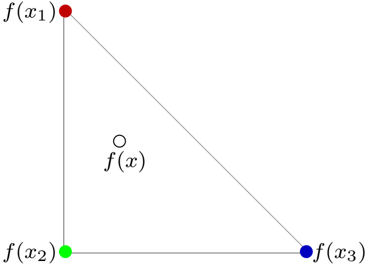
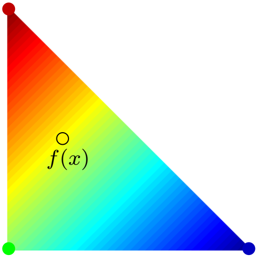


The triangle centroid (or barycenter) has coordinates $(0.33, 0.33, 0.33)$

Example: linear interpolation

Barycentric coordinates can be used to compute linear interpolation of function values:



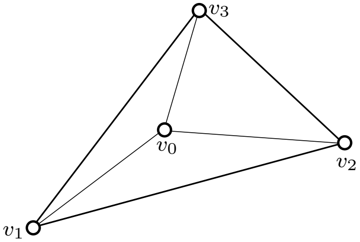


Given function values $f(x_1), f(x_2), f(x_3)$ at the 3 vertices, the function values $f(x)$ inside the triangle are obtained by barycentric coordinates

See lesson 5 - "Visualization I" (Mar 13, 2019)

Mean value coordinates

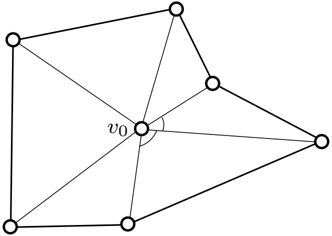
This can also be seen as a triangle mesh with 4 vertices:



So we are asking: How to express any vertex v_0 as a combination of its neighbors?

Mean value coordinates

We consider the more general case with a non-convex polygon



Where we look for coordinates such that

$$v_0 = \sum_i \lambda_i v_i \,, \quad \lambda_i > 0 \,, \quad \sum_i \lambda_i = 1$$

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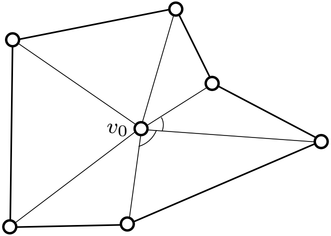

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$$v_0 = \sum_i \lambda_i v_i \,, \quad \lambda_i > 0 \,, \quad \sum_i \lambda_i = 1$$

There is no unique solution!

Mean value coordinates

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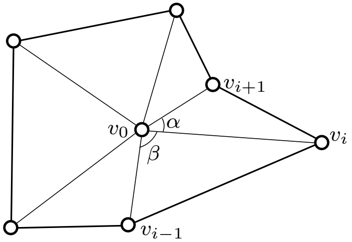

One possible solution is given by Floater's mean value coordinates:

$$w_i = \frac{\tan(\alpha/2) + \tan(\beta/2)}{\|v_i - v_0\|}\,, \quad \lambda_i = \frac{w_i}{\sum_j w_j}$$

Floater, "Mean value coordinates". CAGD 20(1), 2003

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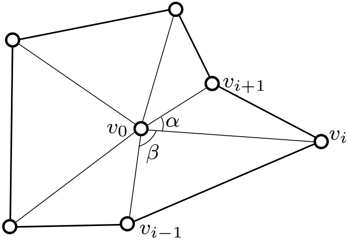

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