Computer Graphics

Shape retrieval I

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Motivation: A 3D version of Google

The challenge

Still an open problem!

Every year there are dedicated workshops and contests on shape retrieval

- 3D Object Retrieval Conference (**3DOR**)
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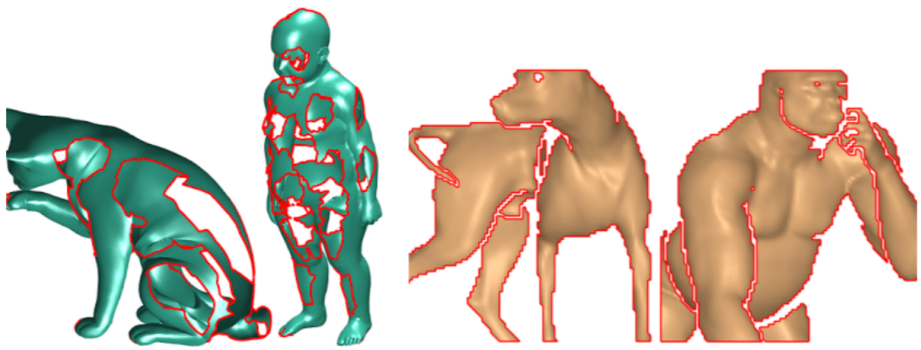
man-made objects

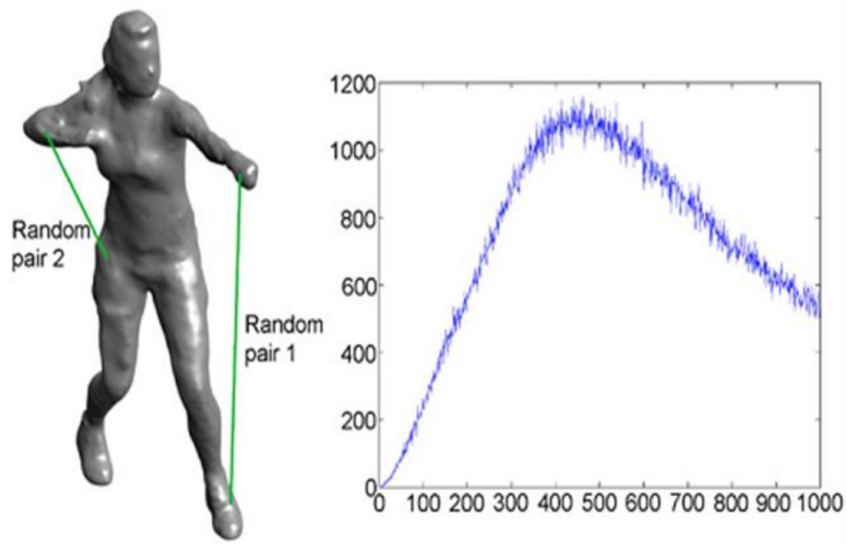


deformable objects with missing parts

General approach

Baseline algorithm:

- Compute a global descriptor for each shape (e.g., aggregate local descriptors)
- Search for nearest neighbors in descriptor space



Heat diffusion

For a surface $\mathcal{X} \in \mathbb{R}^3$ the diffusion of heat is described by the heat equation:

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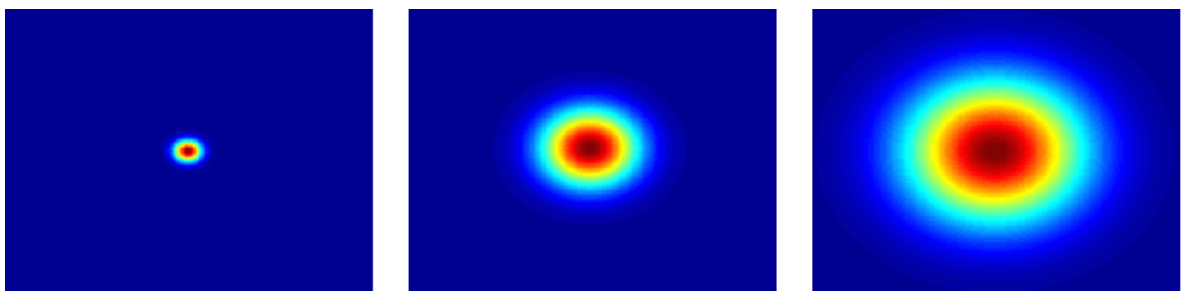

$t = 0$ $t = 1.3$ $t = 3.7$

The surface Laplacian

On surfaces we have the Laplace-Beltrami operator Δ, discretized as

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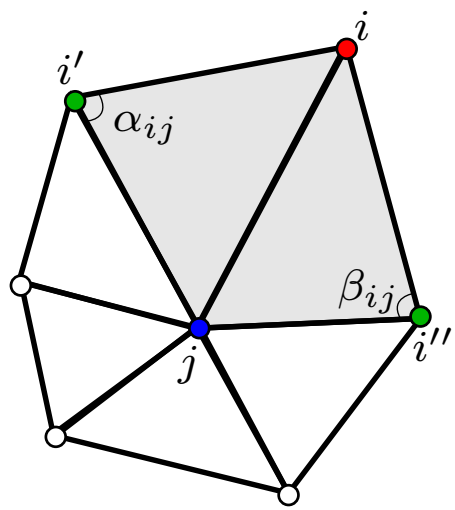
$\mathbf{S} \in \mathbb{R}^{n \times n}$ is the stiffness matrix with values

$$\mathbf{S}_{ij} = \begin{cases} -\frac{1}{2}(\cot\alpha_{ij} + \cot\beta_{ij}) & \text{if } e_{ij} \in E \\ -\sum_{k \neq i} s_{ik} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$



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- $\mathbf{S}$ has the same structure as the vertex adjacency matrix
- Compared with the graph Laplacian, $\mathbf{S}$ has different off-diag values

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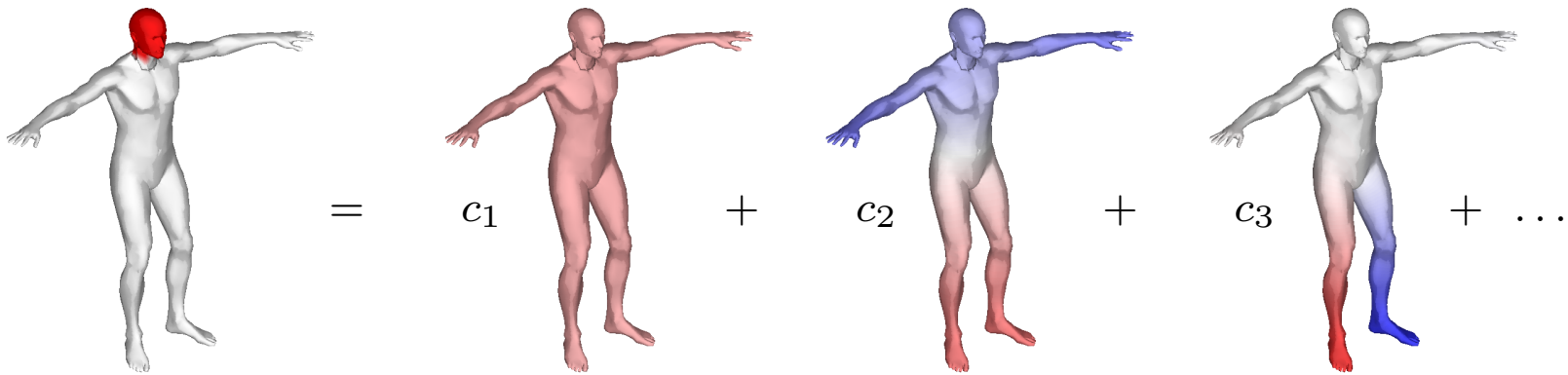
$$\mathbf{L} = \mathbf{D} - \mathbf{A}$$

Laplacian eigenfunctions

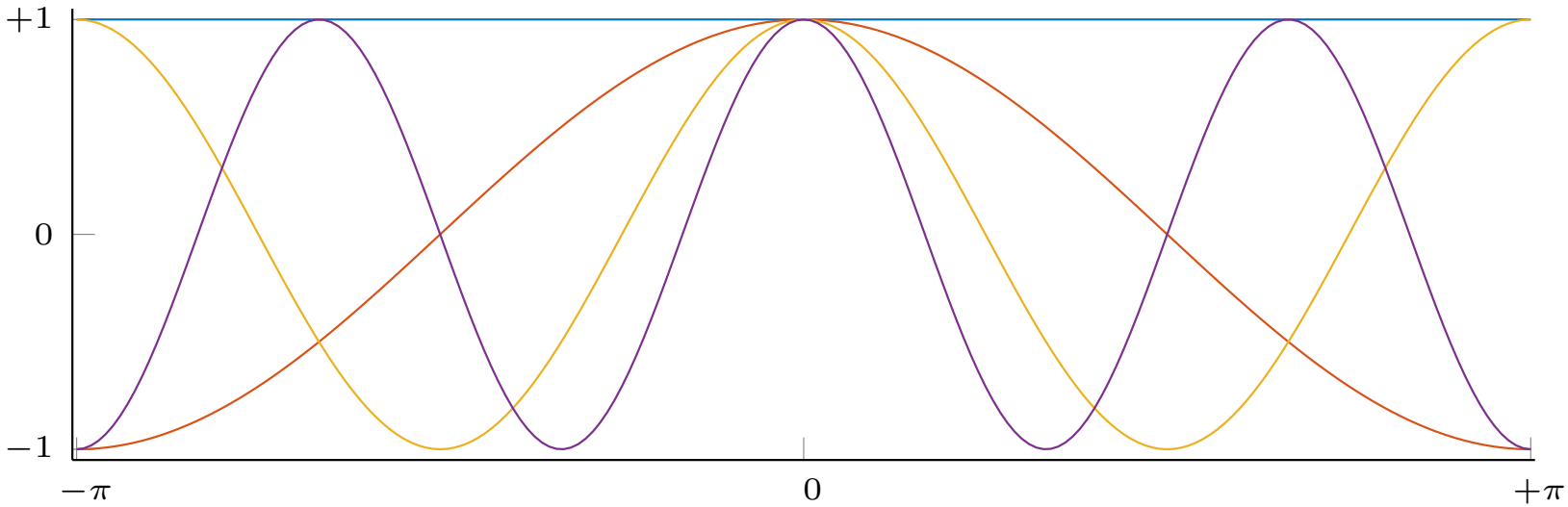
$$\Delta\phi_i = \lambda_i\phi_i$$

The eigenfunctions $\{\phi_i\}_{i=1,\ldots,k}$ form an orthogonal basis for the vector space of functions $f : \mathcal{X} \to \mathbb{R}$



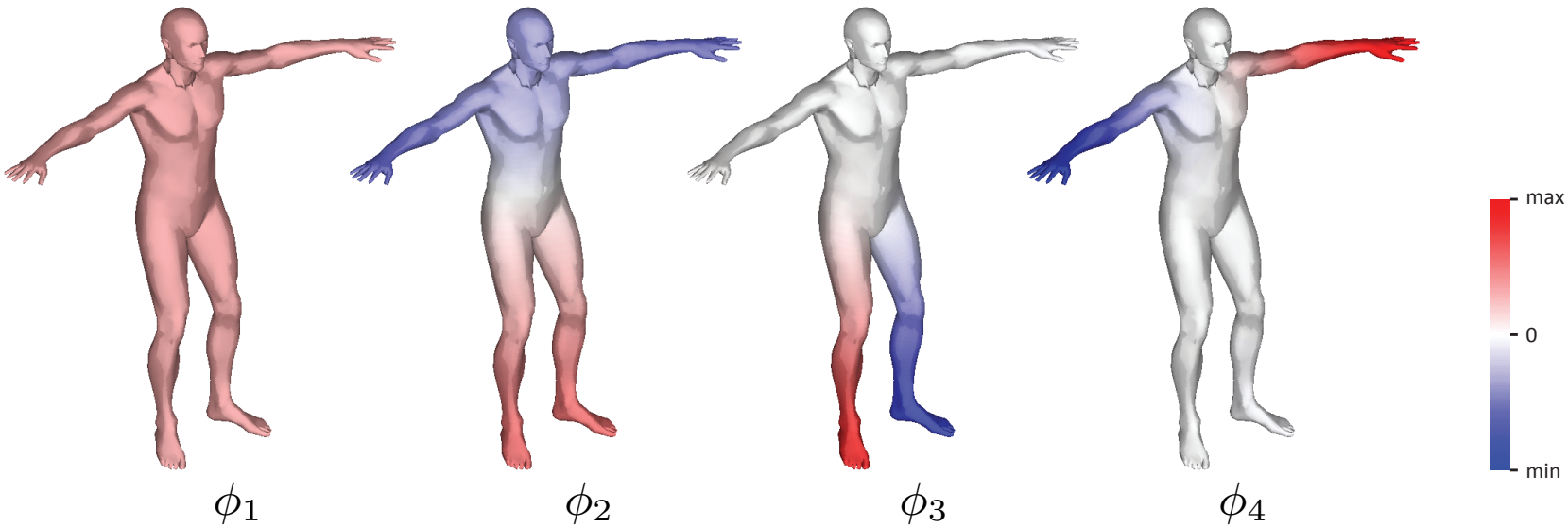
f ϕ_1 ϕ_2 ϕ_3

Laplacian eigenfunctions: Euclidean



First eigenfunctions of 1D Euclidean Laplacian = standard Fourier basis

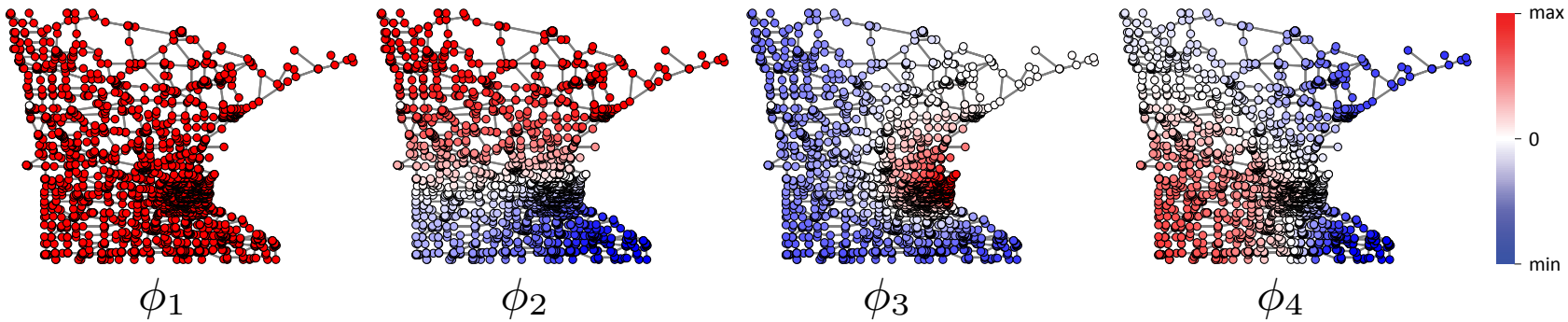
Laplacian eigenfunctions: manifold



ϕ_1 ϕ_2 ϕ_3 ϕ_4

First eigenfunctions of a manifold Laplacian

Laplacian eigenfunctions: graph



First eigenfunctions of a graph Laplacian

Laplacian eigenfunctions: discrete case

We consider the generalized eigenvalue problem:

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Note: The inner product between functions is area-weighted:

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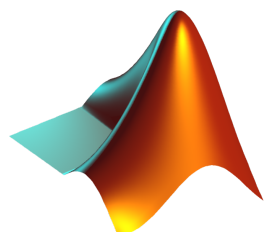
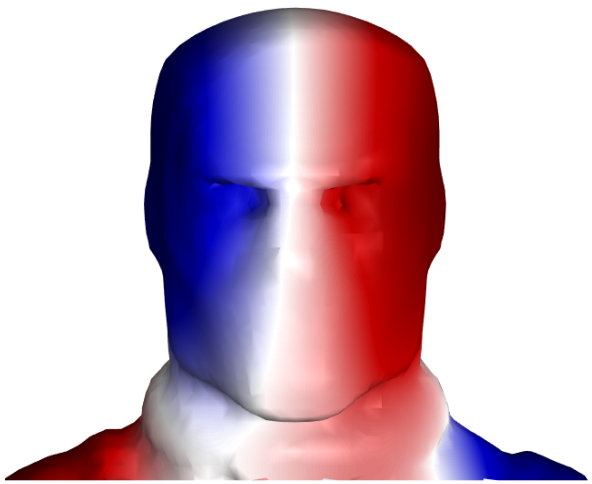


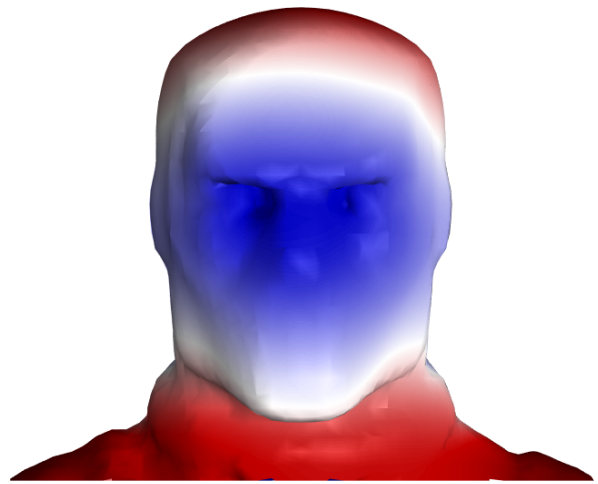
Inner product on a surface

Given two scalar functions $f, g : \mathcal{X} \to \mathbb{R}$, their inner product as:

$$\langle f, g \rangle_{\mathcal{X}} = \int_{\mathcal{X}} f(x) g(x) dx$$



$f : \mathcal{X} \to \mathbb{R}$



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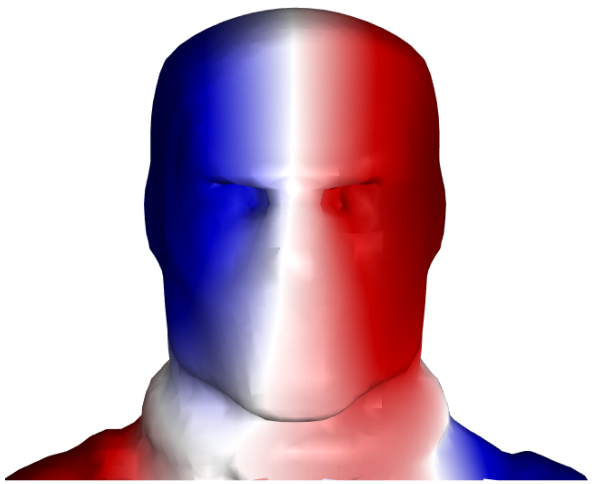
When discretized on a triangle mesh of n vertices, this boils down to:

$$\mathbf{f}^\top \mathrm{diag}(a_i) \mathbf{g}$$

where a_i with $i = 1, \ldots, n$ are the local area elements at each vertex



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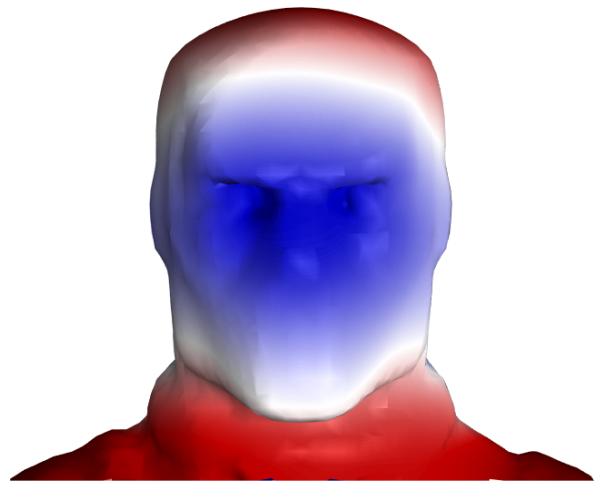

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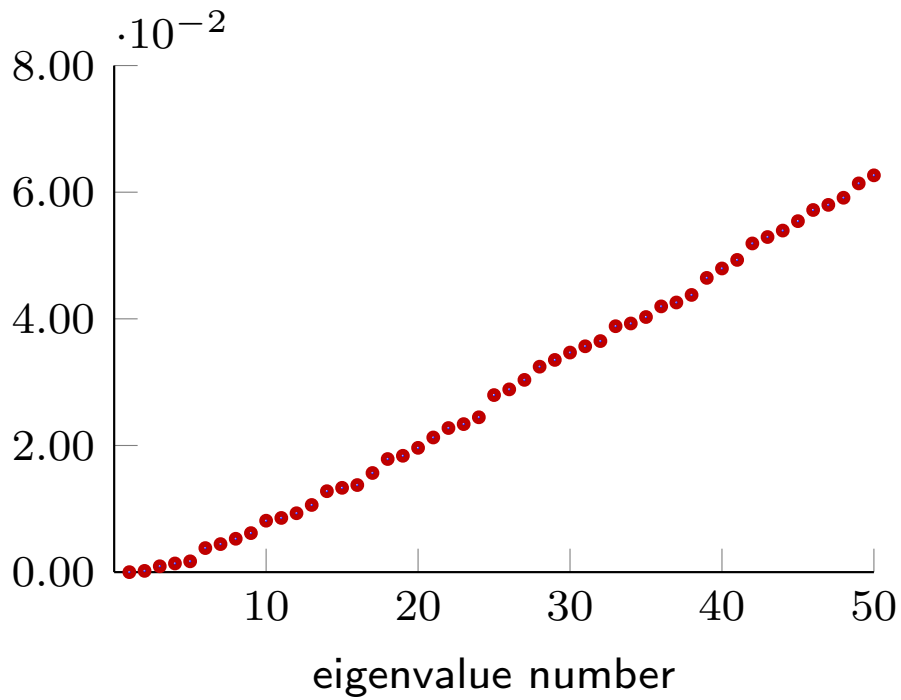

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They follow Weyl's asymptotic law:

$$\lambda_i \approx \frac{\pi}{\int_{\mathcal{X}} da} i \qquad \text{for } i \to \infty$$

That is, they grow linearly with growth rate that is inversely proportional to surface area

Laplacian eigenvalues: Properties

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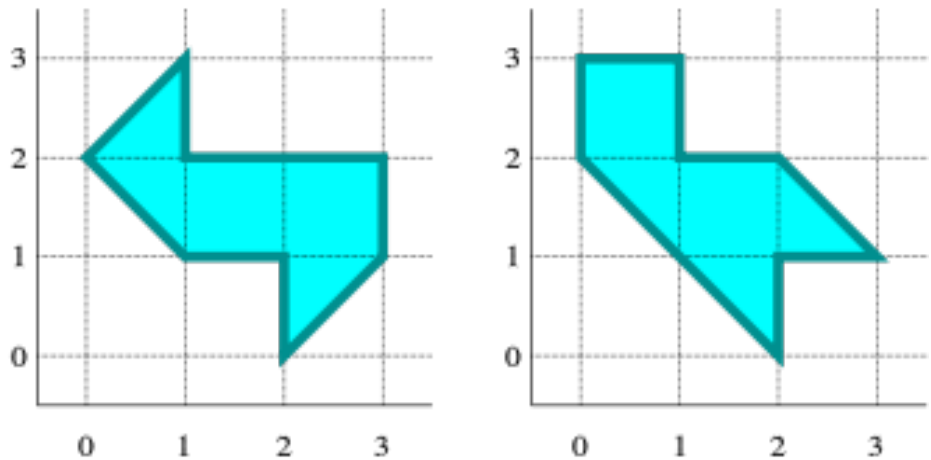
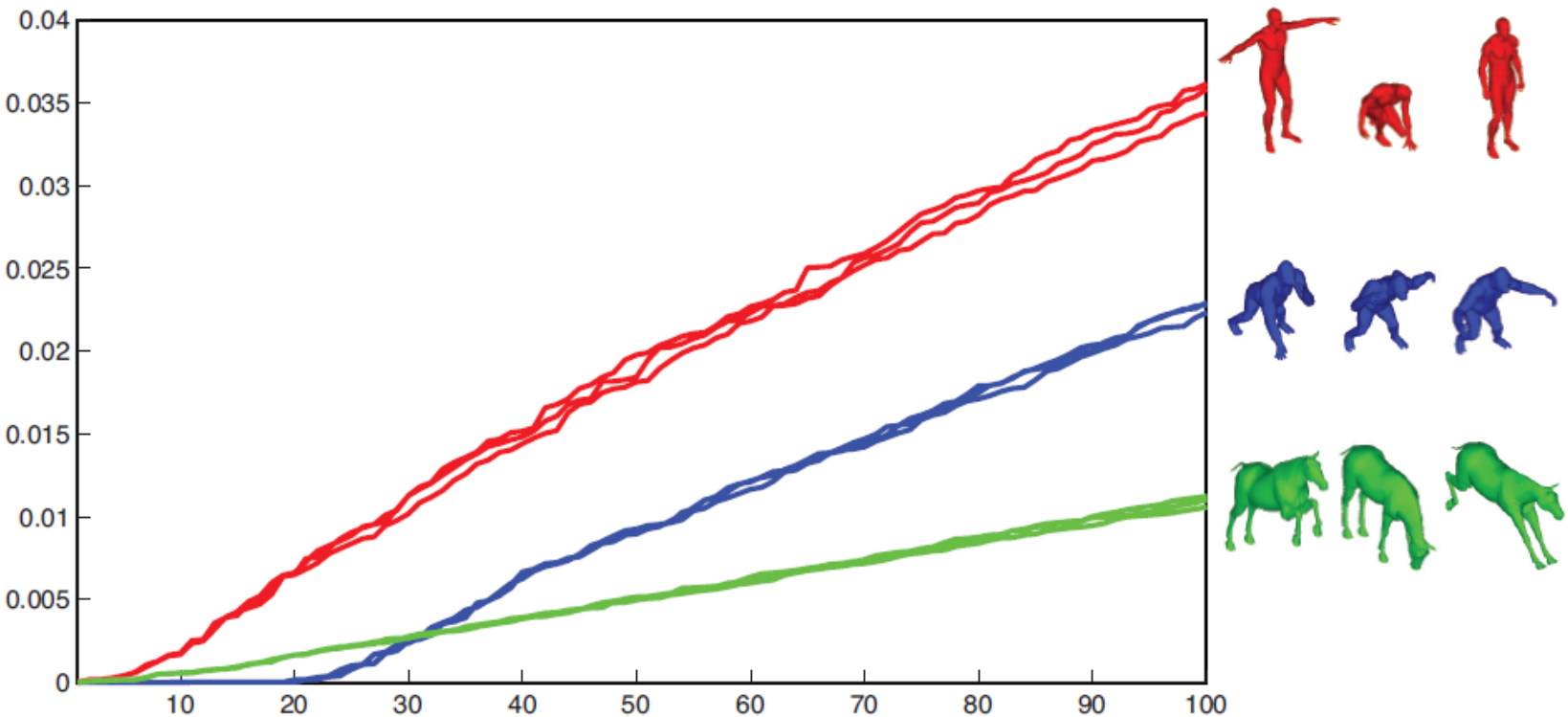
However, one "cannot hear the shape of the drum"



Shape DNA

We can use the Laplacian spectrum as a global shape descriptor



Global Point Signature (GPS)

The Laplacian eigenfunctions can be directly used to define local point descriptors:

$$\mathrm{GPS}(x_i) = \left(\frac{\phi_1(x_i)}{\sqrt{\lambda_1}}, \frac{\phi_2(x_i)}{\sqrt{\lambda_2}}, \ldots, \frac{\phi_k(x_i)}{\sqrt{\lambda_k}}\right) \in \mathbb{R}^k$$

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$$\mathrm{GPS}(\mathcal{X}) = \int_{\mathcal{X}} \mathrm{GPS}(x)dx \Longrightarrow \mathbf{H} = (\mathbf{1}^\top \mathbf{M}\mathbf{G})^\top$$

Exercise: Laplace-Beltrami operator

Implement the surface Laplacian

Test it by reproducing the following behavior on a shape of your choice: