Computer Graphics

Shape matching

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SAPIENZA
UNIVERSITÀ DI ROMA

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Rilevazione Opinioni Studenti

Il questionario è completamente anonimo

Istruzioni:

- Disattivare il “blocco pop-up” del browser
- Visitare il sito https://www.uniroma1.it
- Cliccare su **Studenti** per accedere a **Infostud 2.0**
- Cliccare su **Corsi di laurea**
- Nel menu di sinistra, selezionare **Opinioni studenti**
- Inserire il codice OPIS e cliccare su **Vai al questionario**

Il codice OPIS è:

85G3I2HQ

Correspondence problems

We can compare shapes, but...

$\stackrel{?}{=}$

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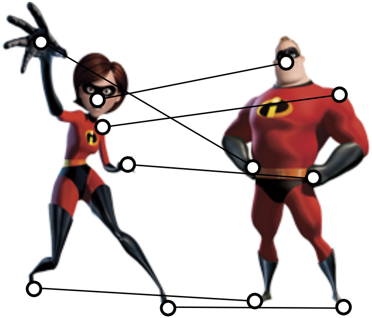
How to match them, point by point?



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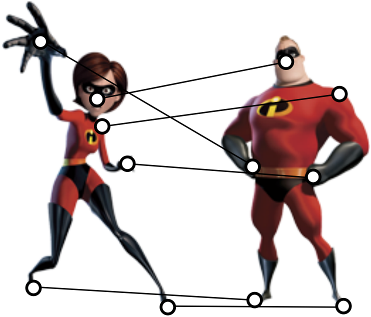
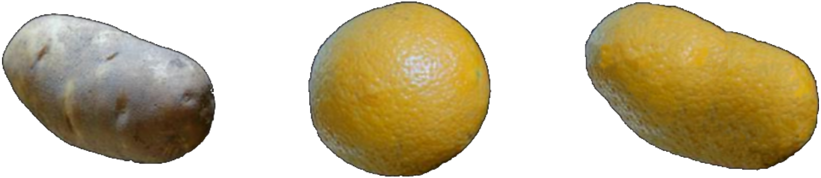


This is known as a matching, assignment, or correspondence problem

Still an open and challenging area of research!

Example: Texture transfer

With a correspondence we can transfer information (e.g. texture coordinates) across shapes



$\mathcal{X}$ $\mathcal{Y}$ texture of $\mathcal{Y}$ transferred to $\mathcal{X}$

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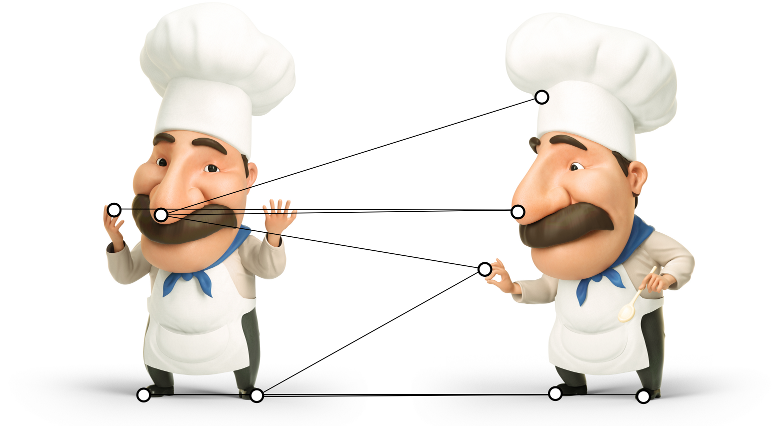

Correspondence: at least one match for each sample (that is, not necessarily 1-to-1)

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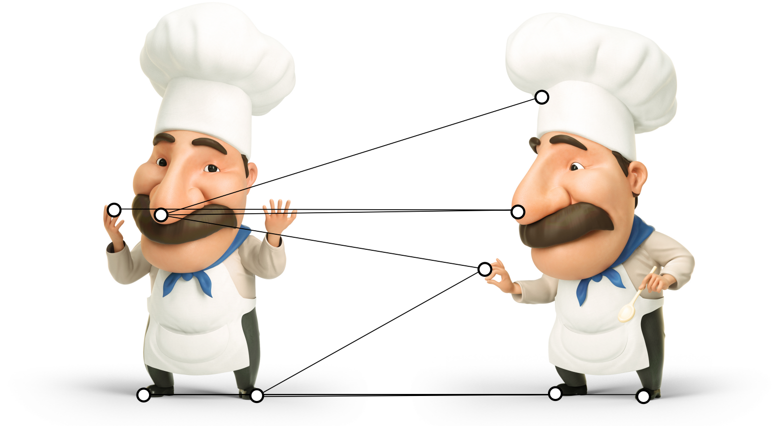

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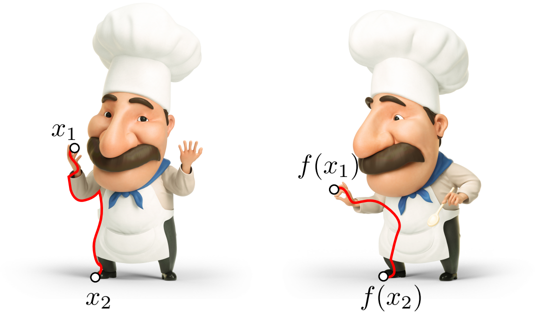
By itself, it doesn't have anything to do with distortion or semantics

Metric distortion

For a map $f : \mathcal{X} \to \mathcal{Y}$ between metric spaces $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_Y)$ we define its (absolute) distortion as:

$$\mathrm{dis}(f) = \max_{x_1, x_2 \in \mathcal{X}} |d_{\mathcal{X}}(x_1, x_2) - d_{\mathcal{Y}}(f(x_1), f(x_2))|$$



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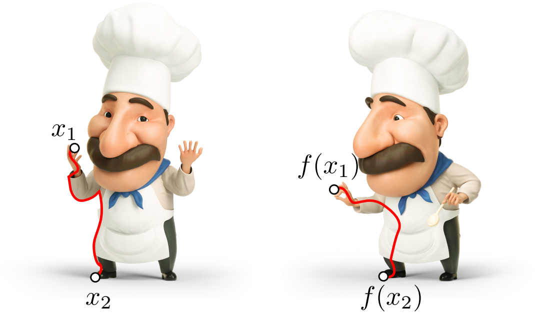


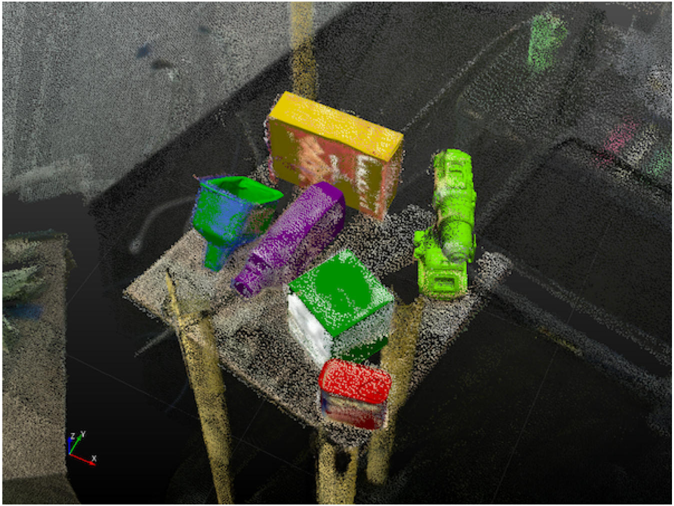
Similarly, for a correspondence $\mathcal{R}$ we define:

$$\mathrm{dis}(\mathcal{R}) = \max\{|d_\mathcal{X}(x_1, x_2) - d_\mathcal{Y}(y_1, y_2)| \ : \ (x_1, y_1), (x_2, y_2) \in \mathcal{R}\}$$

Example: Rigid correspondence

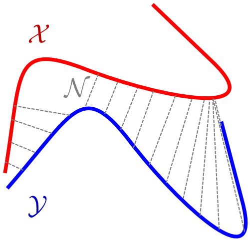
This is a simple case where $d_{\mathcal{X}} = d_{\mathcal{Y}} = \| \cdot \|_2$

Zero distortion means that the shapes are rigid



Example: ICP

At each step, we have both a correspondence and a rigid transformation



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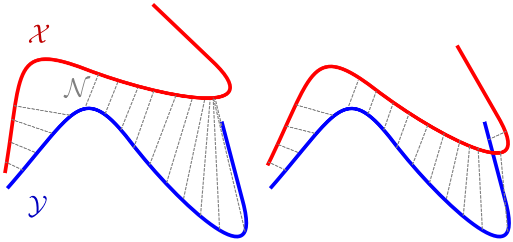

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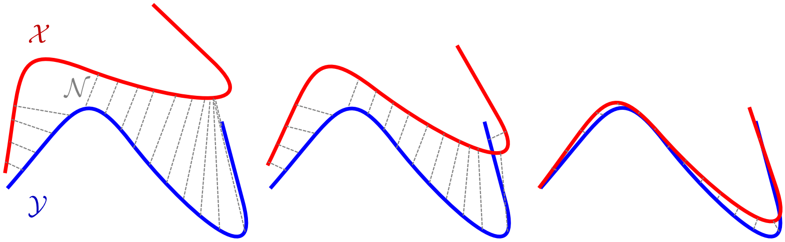

Example: ICP

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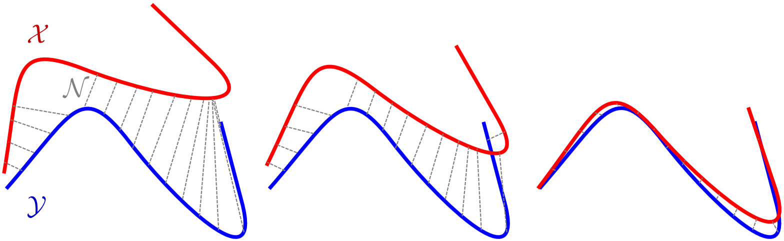

- In practice converges to local optima
- Requires a good initialization
- Only works well in the rigid setting

Example: Sort by distortion

If we know the correspondence π, then we can directly compute:

$$d(\mathcal{X}, \mathcal{Y}) = \max\{|d_{\mathcal{X}}(x_1, x_2) - d_{\mathcal{Y}}(y_1, y_2)| \ : \ (x_1, y_1), (x_2, y_2) \in \mathcal{R}\}$$

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And we can use it to sort shapes by distortion:

$\mathcal{X}$ $\mathcal{Y}_1$ $\mathcal{Y}_2$ $\mathcal{Y}_3$ $\mathcal{Y}_4$

where

$$d_{\mathcal{P}}(\mathcal{X}, \mathcal{Y}_1) \leq d_{\mathcal{P}}(\mathcal{X}, \mathcal{Y}_2) \leq d_{\mathcal{P}}(\mathcal{X}, \mathcal{Y}_3) \leq d_{\mathcal{P}}(\mathcal{X}, \mathcal{Y}_4)$$

Correspondence matrix

A common representation is the correspondence matrix $\mathbf{T} \in \{0,1\}^{m \times n}$

$$\mathbf{T} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 1 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

where $T_{i,j} = 1$ means that $(x_i, y_j) \in \mathcal{R} \subset \mathcal{X} \times \mathcal{Y}$

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$\mathbf{T}$ encodes in the standard basis a linear map between function spaces:

$$T : (\mathcal{F}(\mathcal{X}) \to \mathbb{R}) \to (\mathcal{F}(\mathcal{Y}) \to \mathbb{R})$$

In other words, it maps functions to functions

Permutation matrix

A permutation matrix $\mathbf{P} \in \{0,1\}^{n\times n}$ is a special case where each row and each column have exactly one value equal to 1:

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Permutations encode one-to-one correspondences, meaning that they map indicator functions to indicator functions

Continuous relaxation

Replace $\{0, 1\}$ with continuous values in $[0, 1]$

We now have doubly-stochastic matrices where $\mathbf{S}\mathbf{1} = \mathbf{1}, \mathbf{S}^\top\mathbf{1} = \mathbf{1}$:

$$\mathbf{S} = \begin{pmatrix} 0.2 & 0.3 & \cdots & 0.5 \\ 0.1 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0.7 & 0 & \cdots & 0.1 \end{pmatrix}$$

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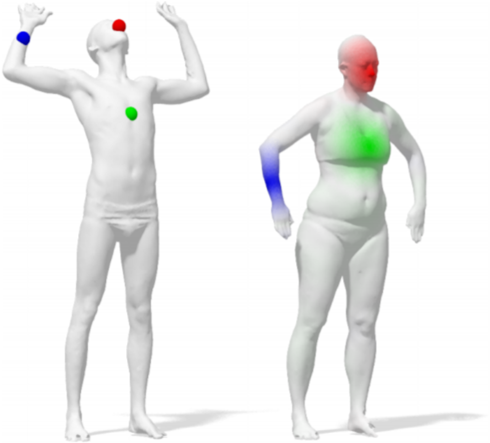
This is also called a soft map

Doubly-stochastic matrices are convex combinations of permutations:

$$\mathbf{S} = t\mathbf{P}_1 + (1-t)\mathbf{P}_2\,, \quad t \in [0,1]$$

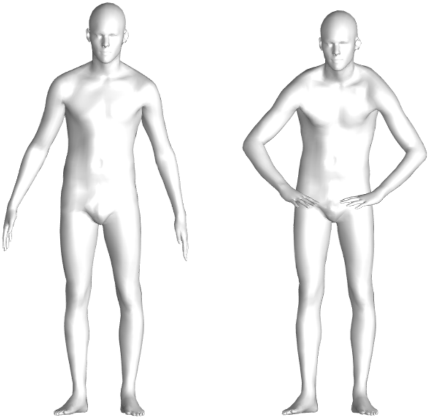
Example: Soft maps

Each row / column of a soft map can be interpreted as a probability distribution on the shape:



Visualizing correspondence

How to visualize correspondence between shapes?



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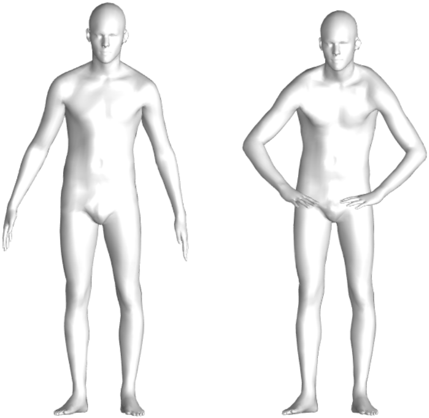
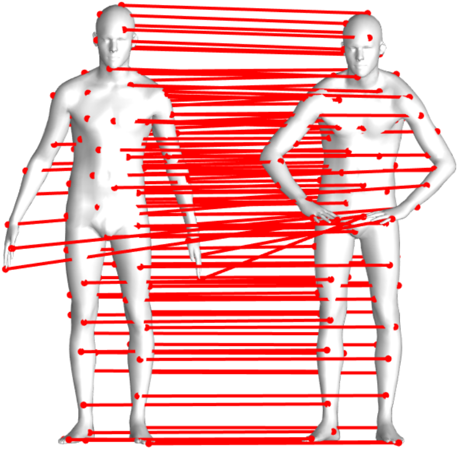


- It might be not one-to-one
- It might be sparse

Visualizing correspondence: Points and lines



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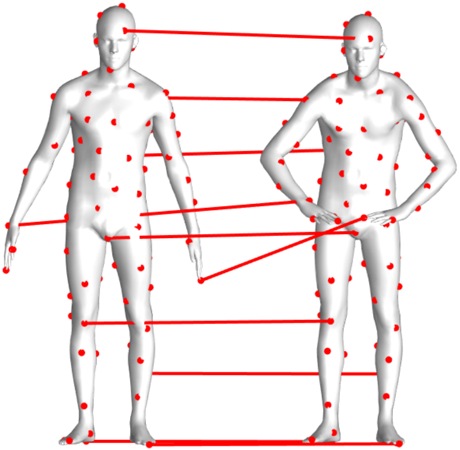
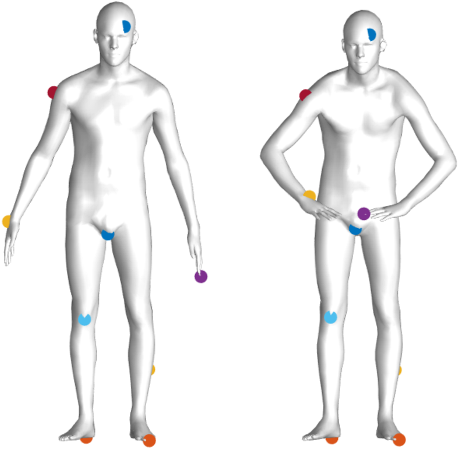


Useful for visualizing sparse matches

Visualizing correspondence: Points and colors



Useful for visualizing sparse matches (e.g., between landmarks)

Visualizing correspondence: Dense colors

Dense matches ($\sim$all points) can be used to color the entire mesh

- Create a colormap with one color for each point ($n \times 3$ matrix)

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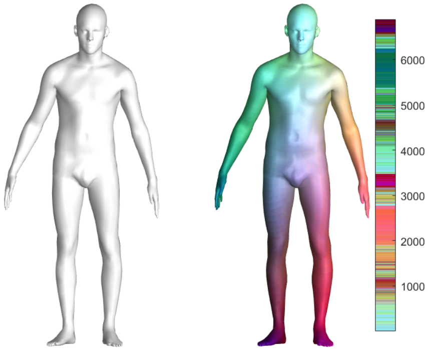

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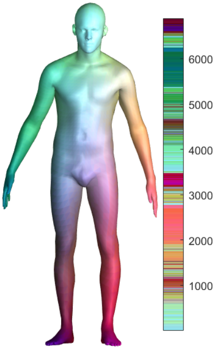

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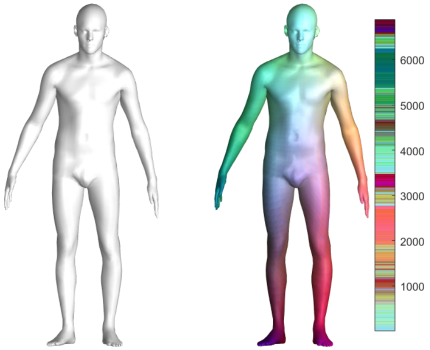
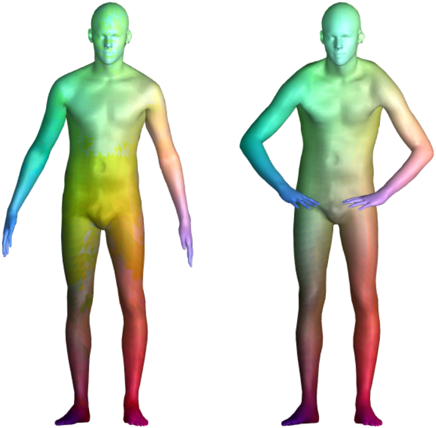


Warning: New versions of Matlab require shading `flat` (not `interp`)

Visualizing correspondence: Dense colors

Note: The map may be not 1-to-1

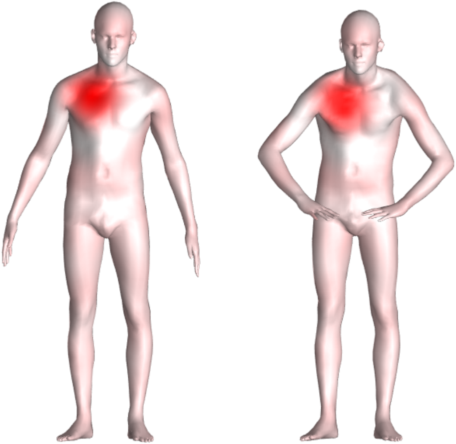
Trick: Obtain the source colormap by pulling back the target colormap



This hides the lack of surjectivity in the correspondence

Visualizing correspondence: Function mapping

The correspondence can be used to map functions to functions



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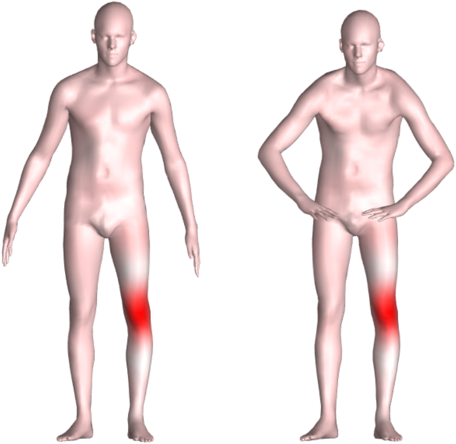

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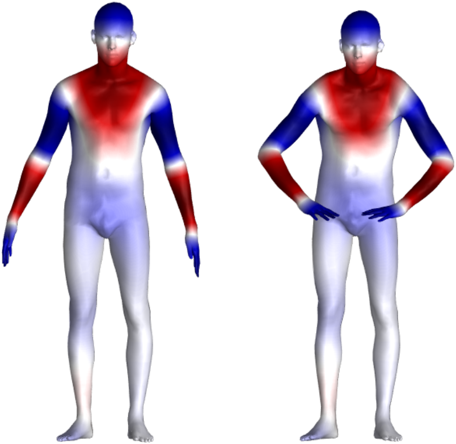

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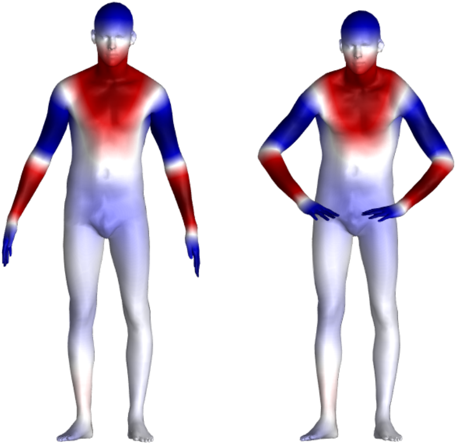

This also works for non-surjective and in general soft maps

Maps in the Laplacian eigenbases

Consider two isometric shapes $\mathcal{X}, \mathcal{Y}$ with Laplacian eigenfunctions $\{\phi_i\}, \{\psi_j\}$ respectively spanning the functional spaces $\mathcal{F}(\mathcal{X}), \mathcal{F}(\mathcal{Y})$.

Maps in the Laplacian eigenbases

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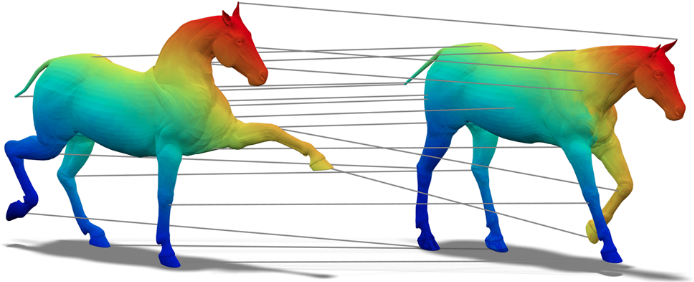
Now let $T : \mathcal{F}(\mathcal{X}) \to \mathcal{F}(\mathcal{Y})$ be the ground-truth map.



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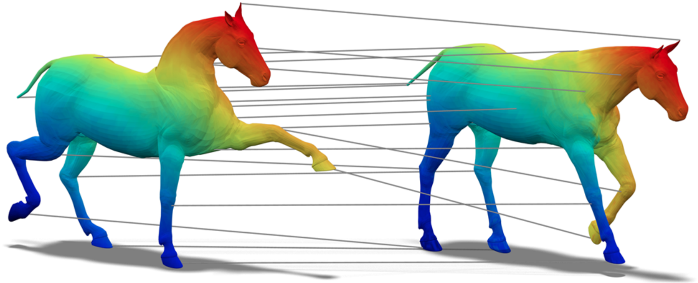

How does the matrix representation of T look like, in the Laplacian eigenbases?

Maps in arbitrary (orthogonal) bases

$f : \mathcal{X} \to \mathbb{R}$

$$g = Tf$$

$g : \mathcal{Y} \to \mathbb{R}$

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$$\mathbf{g} = \mathbf{T}\mathbf{f}$$

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ortho. basis $\mathbf{A}$

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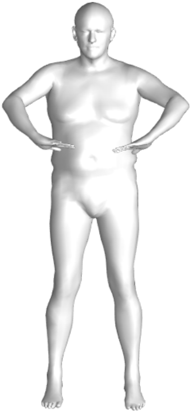

$g : \mathcal{Y} \to \mathbb{R}$

ortho. basis $\mathbf{B}$

Maps in arbitrary (orthogonal) bases

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ortho. basis $\mathbf{A}$

$$(\mathbf{B}\mathbf{B}^\top \mathbf{g}) = \mathbf{T}(\mathbf{A}\mathbf{A}^\top \mathbf{f})$$

$g : \mathcal{Y} \to \mathbb{R}$

ortho. basis $\mathbf{B}$

Maps in arbitrary (orthogonal) bases

$$\mathbf{B}^\top \mathbf{g} = (\mathbf{B}^\top \mathbf{T} \mathbf{A}) \mathbf{A}^\top \mathbf{f}$$

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Maps in arbitrary (orthogonal) bases

$f : \mathcal{X} \to \mathbb{R}$

ortho. basis $\mathbf{A}$

$$\mathbf{B}^\top \mathbf{g} = \mathbf{C}\mathbf{A}^\top \mathbf{f}$$

matrix $\mathbf{C} = \mathbf{B}^\top \mathbf{T}\mathbf{A}$
is map T wrt $\mathbf{A}, \mathbf{B}$ bases

$g : \mathcal{Y} \to \mathbb{R}$

ortho. basis $\mathbf{B}$

Maps in arbitrary (orthogonal) bases

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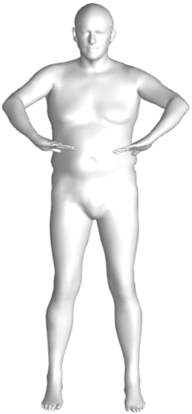
matrix $\mathbf{T} = \mathbf{B}\mathbf{C}\mathbf{A}^\top$
goes back to the std bases

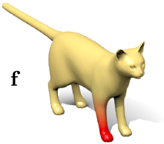


$g : \mathcal{Y} \to \mathbb{R}$

ortho. basis $\mathbf{B}$

Maps in the Laplacian eigenbases

$\mathbf{f}$



$\downarrow$

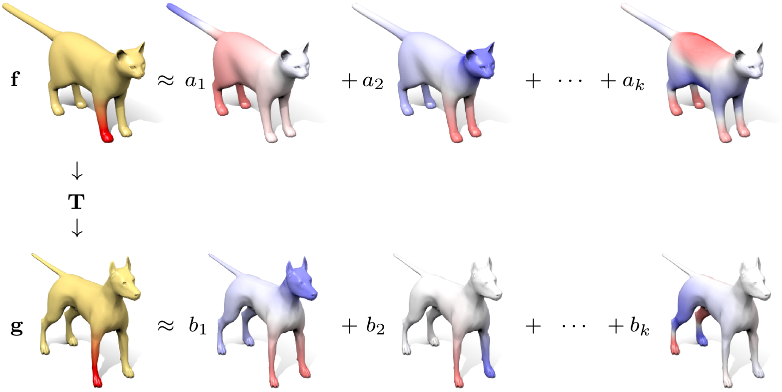
$\mathbf{T}$

$\downarrow$

$\mathbf{g}$

Ovsjanikov et al. 2012

Maps in the Laplacian eigenbases



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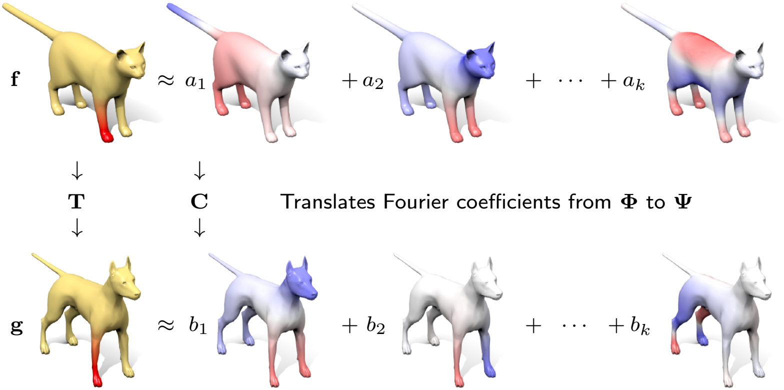

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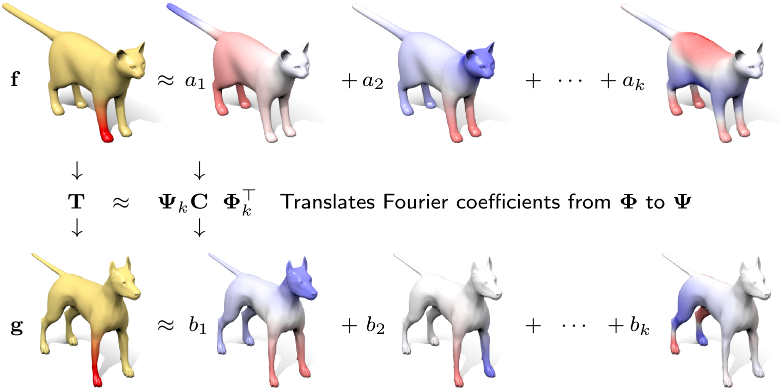


$$\boldsymbol{\Psi}_k^\top \mathbf{g} = \mathbf{C} \boldsymbol{\Phi}_k^\top \mathbf{f}$$

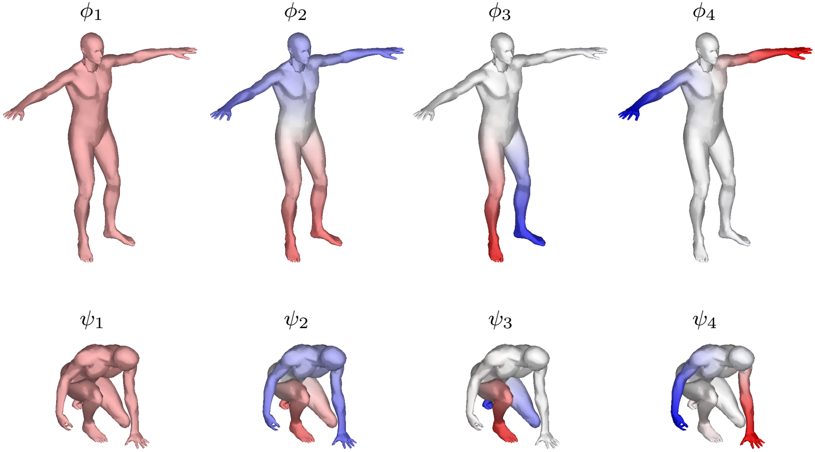
where $\boldsymbol{\Phi}_k = (\phi_1, \ldots, \phi_k)$, $\boldsymbol{\Psi}_k = (\psi_1, \ldots, \psi_k)$ are LB eigenbases

Note: The bases are truncated to the first k eigenfunctions

Ovsjanikov et al. 2012

Laplacian eigenfunctions

The Laplacian is invariant to isometries



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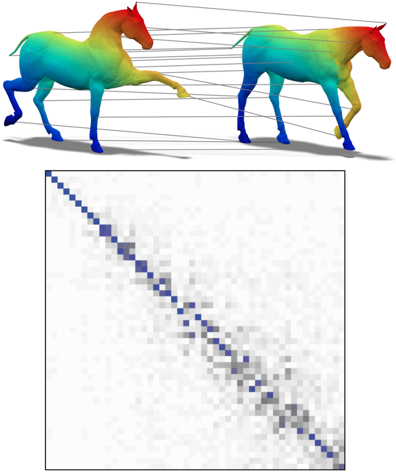

$$\mathbf{C} = \mathbf{\Psi}_k^\top \mathbf{T} \mathbf{\Phi}_k \Rightarrow c_{ij} = \langle \psi_i, T\varphi_j \rangle$$

For isometric shapes, $\mathbf{C}$ is diagonal since $\boldsymbol{\psi}_i = \pm \mathbf{T} \boldsymbol{\phi}_i$

Rank of a map

In the standard basis, a one-to-one correspondence is written as a permutation matrix in $\mathbb{R}^{n\times n}$

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Each column is a basis vector, so $\operatorname{rank}(\mathbf{P}) = n$

Rank of a map

In the Laplacian eigenbasis, a correspondence is written as a generic matrix in $\mathbb{R}^{k\times k}$

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1k} \\ c_{21} & c_{22} & \cdots & c_{2k} \\ \vdots & & & \vdots \\ c_{k1} & c_{k2} & \cdots & c_{kk} \end{pmatrix}$$

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Functions mapped via $\mathbf{C}$ span a subspace of those mapped via $\mathbf{P}$

Solving for a map

$$\mathbf{C}\mathbf{\Phi}_k^\top \mathbf{f} = \mathbf{\Psi}_k^\top \mathbf{g}$$

Where f, g are corresponding functions (possibly noisy)

For example: GPS, HKS, indicators at corresponding points, etc.

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