Computer Graphics

Metric spaces

Emanuele Rodolà
rodola@di.uniroma1.it

SAPIENZA
UNIVERSITÀ DI ROMA

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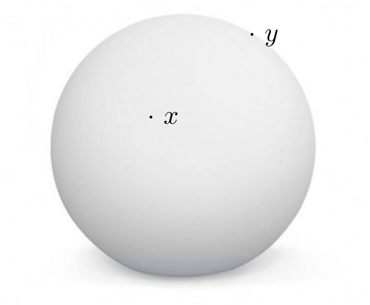
Measuring distance: surfaces

How do you measure distance between x and y in this picture?



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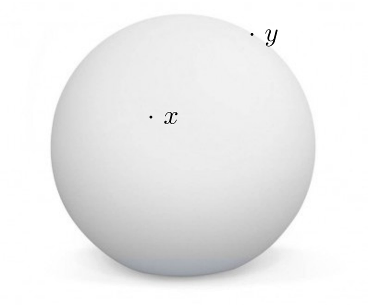


There is not a unique way!

- You can pass through the sphere with a straight line (Euclidean)
- You can walk on the surface in a "straight" path (non-Euclidean)

Measuring distance: images

How do you measure distance between x and y in this picture?

Measuring distance: images

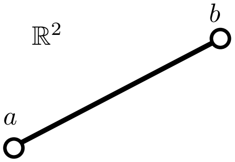
How do you measure distance between x and y in this picture?

Again, not a unique way:

- Natural Euclidean distance
- Smallest amount of pixels between x and y (Manhattan distance)
- $d(x, y) = 0$ if same color, $d(x, y) = 1$ otherwise

Euclidean distance

The Euclidean distance measures the length of a straight line connecting two points:



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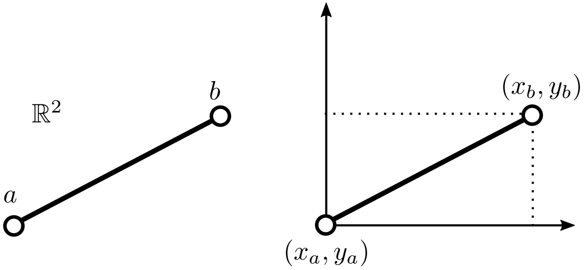

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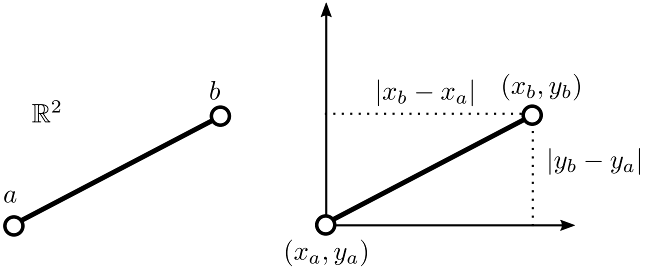

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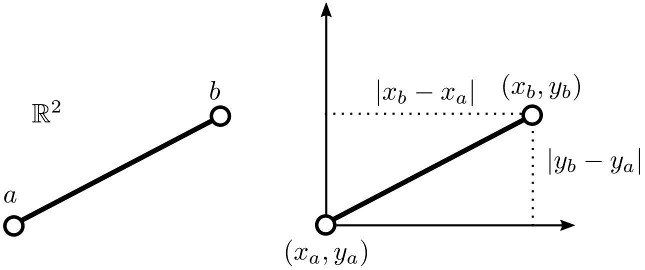

Apply Pythagoras' theorem: $d(a, b) = (|x_b - x_a|^2 + |y_b - y_a|^2)^{\frac{1}{2}}$

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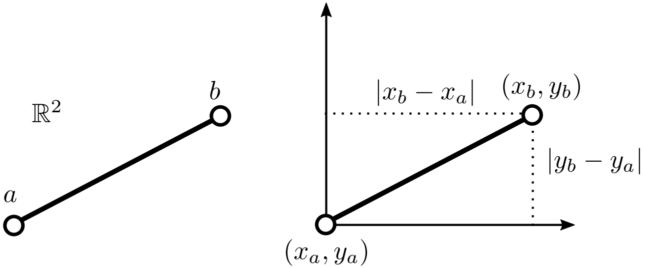

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In vector notation:

$$d(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\|_2$$

where $\mathbf{a} = \begin{pmatrix} x_a \\ y_a \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} x_b \\ y_b \end{pmatrix}$

L_p distance in $\mathbb{R}^k$

One can generalize to different power coefficients $p \geq 1$:

$$\|\mathbf{x} - \mathbf{y}\|_2 = \left(|x_1 - y_1|^2 + |x_2 - y_2|^2\right)^{\frac{1}{2}}$$

$$\Downarrow$$

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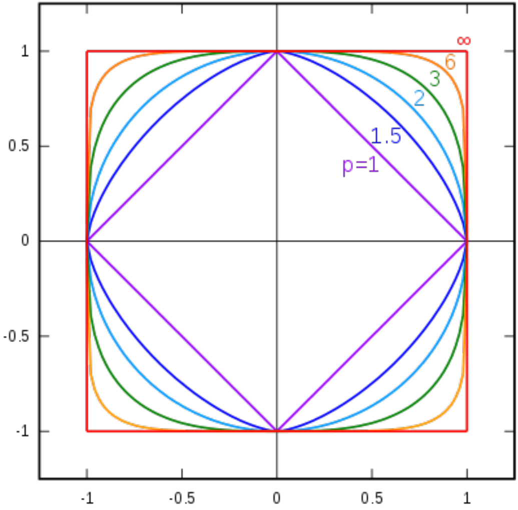
This definition gives us the L_p distance between vectors in $\mathbb{R}^k$

Examples:

- Euclidean (L_2) distance between 3D points
- Manhattan (L_1) distance between cities in a map

L_p unit balls in $\mathbb{R}^2$



L_p unit balls on meshes

L_1 L_2 L_7

Each isoline identifies points at the same distance from the source

Note that these distances are taken in $\mathbb{R}^3$, not on the surface!

Metric spaces

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We will specify a metric space as the pair $(\mathcal{M}, d_{\mathcal{M}})$

Example:

- The sphere with Euclidean distance is $(\mathbb{S}^2, d_{L_2})$
- The sphere with geodesic distance is $(\mathbb{S}^2, d_g)$

Farthest point sampling (FPS)

Given an initial seed point, each new sample is placed as far as possible from the previous ones.

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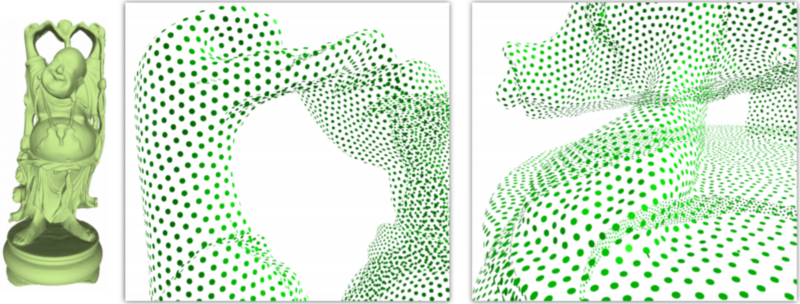

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FPS clearly depends on an underlying definition of distance function, and we can manipulate colors using a geometric language

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- $\mathcal{X} = \mathcal{A} \times \mathcal{B}\,, \quad d_{\mathcal{X}}((a_1,b_1),(a_2,b_2)) = \sqrt{d_{\mathcal{A}}(a_1,a_2)^2 + d_{\mathcal{B}}(b_1,b_2)^2}$

Ambient space and restriction

If $\mathcal{A}$ is a metric space and $\mathcal{X} \subset \mathcal{A}$, then $\mathcal{A}$ is called ambient space for $\mathcal{X}$.

$$\mathcal{A} = (\mathbb{R}^3, d_{L_2})$$

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A metric on $\mathcal{X}$ can be obtained by the restriction $d_{\mathcal{X}} = d_{\mathcal{A}|\mathcal{X}}$, such that:

$$d_{\mathcal{X}}(x, y) = d_{\mathcal{A}}(x, y)$$

for all $x, y \in \mathcal{X}$

Isometries

Let $(\mathcal{M}, d_{\mathcal{M}})$ and $(\mathcal{N}, d_{\mathcal{N}})$ be two metric spaces.

A bijective map $f : \mathcal{M} \to \mathcal{N}$ is called an isometry if:

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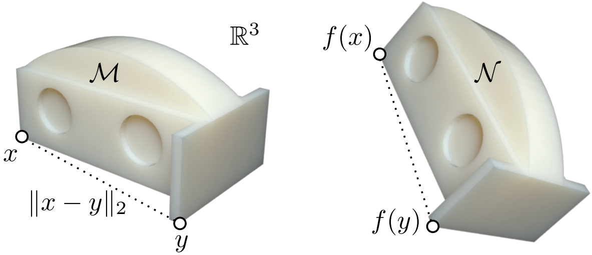

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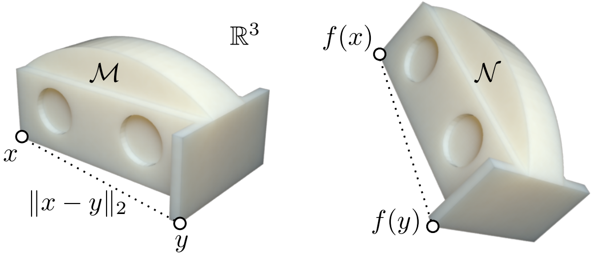
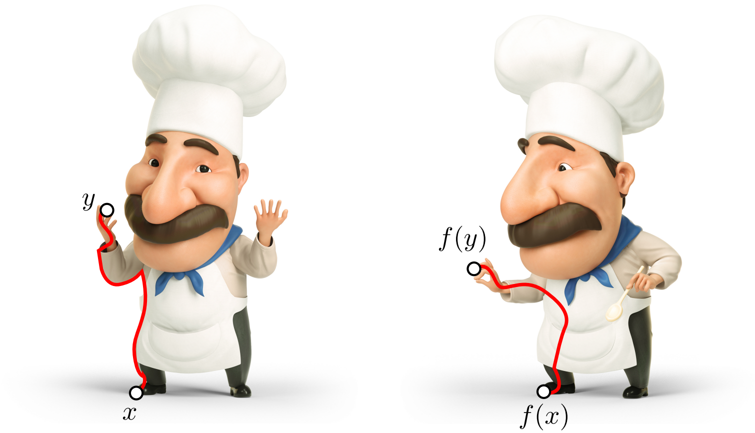


If $d_{\mathcal{M}} = \| \cdot \|_2$ and $d_{\mathcal{N}} = \| \cdot \|_2$ we say "rigid isometry"

Example: Non-rigid "quasi"-isometries



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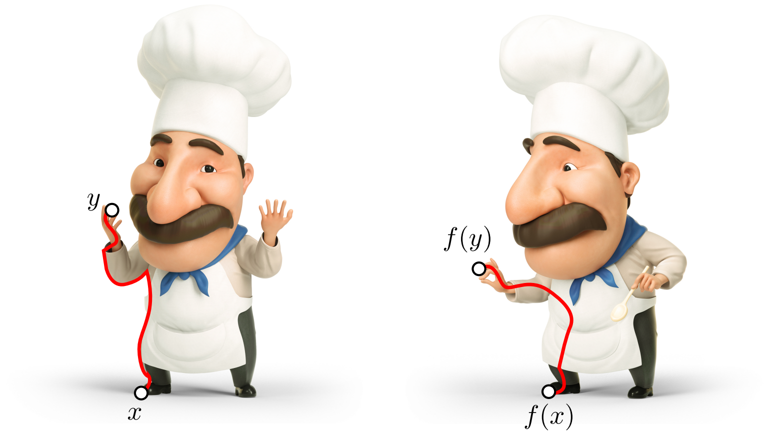

$$d_{\mathcal{M}}(x,y) \approx d_{\mathcal{N}}(f(x), f(y))$$

(here $d_{\mathcal{M}}, d_{\mathcal{N}}$ are geodesic distance functions)

Example: Rotation in color space

Rotating a color histogram does not change its pairwise distances

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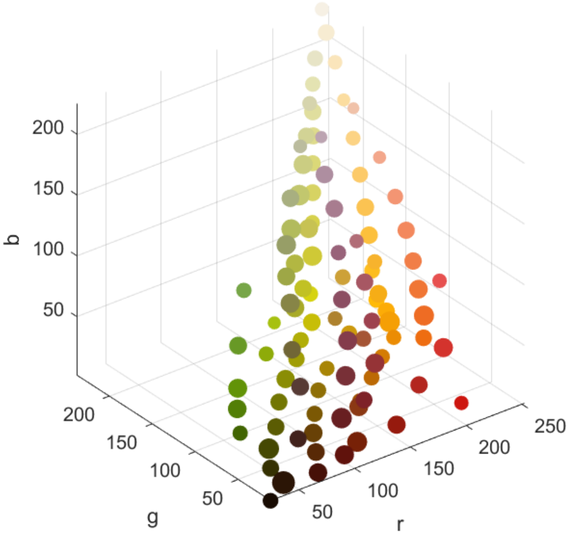

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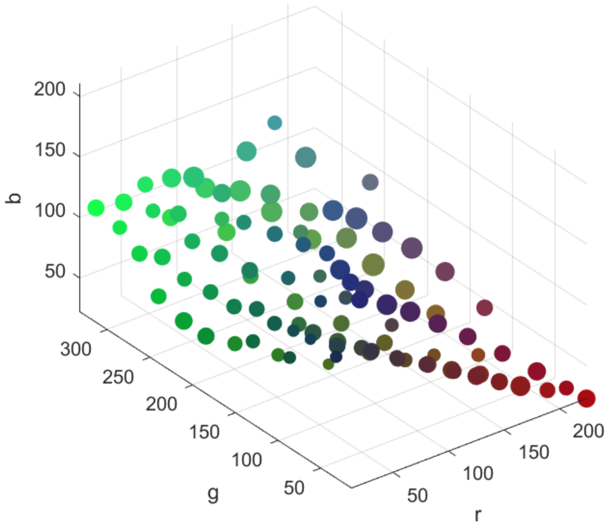

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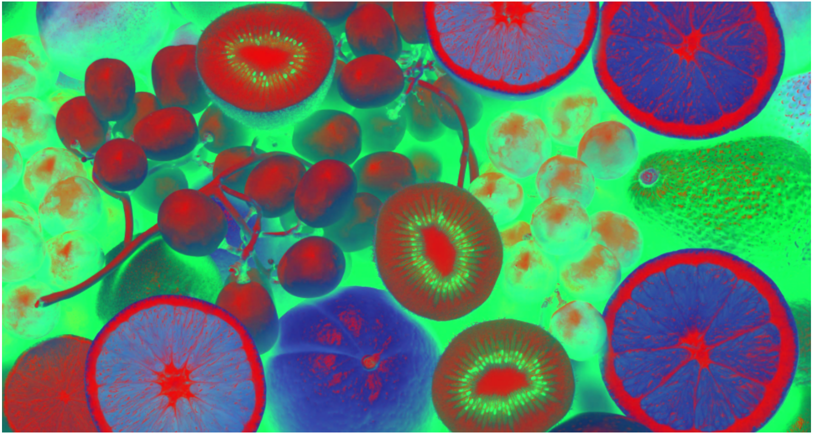

Isometry as equivalence

"Being isometric" is an equivalence relation, since it is:

- reflective $(a = a)$
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In this sense, we think of isometric shapes as being the same shape:

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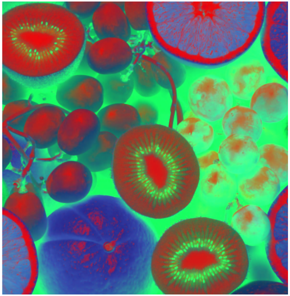
In this sense, we think of isometric images as being the same image:

$=$



Distance

We now have a notion of equivalence between shapes/images. Can we also establish a notion of distance between shapes/images?

There are many!

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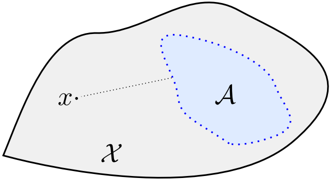
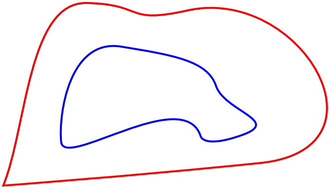
We start by defining the distance from a point x to a set $\mathcal{A} \subseteq (\mathcal{X}, d_{\mathcal{X}})$:

$$\mathrm{dist}_{\mathcal{X}}(x, \mathcal{A}) = \min_{y \in \mathcal{A}} d_{\mathcal{X}}(x, y)$$



Hausdorff distance

Consider two subsets $\mathcal{X}, \mathcal{Y} \subset (\mathcal{Z}, d_{\mathcal{Z}})$.



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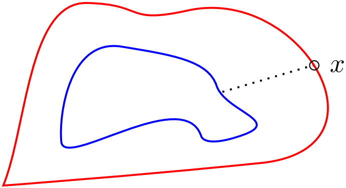
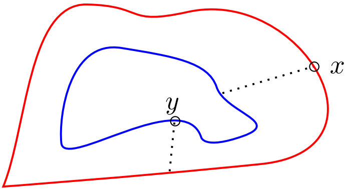


Hausdorff distance

Consider two subsets $\color{red}{\mathcal{X}}, \color{blue}{\mathcal{Y}} \subset (\mathcal{Z}, d_{\mathcal{Z}})$.

$$\max_{x \in \mathcal{X}} \operatorname{dist}_{\mathcal{Z}}(x, \mathcal{Y}), \max_{y \in \mathcal{Y}} \operatorname{dist}_{\mathcal{Z}}(y, \mathcal{X})$$



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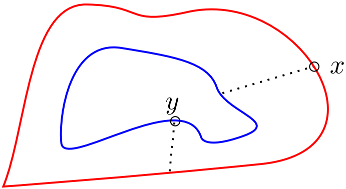

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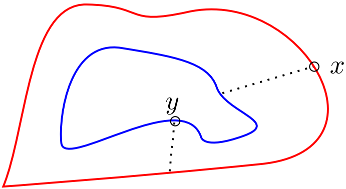

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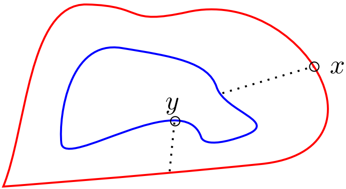
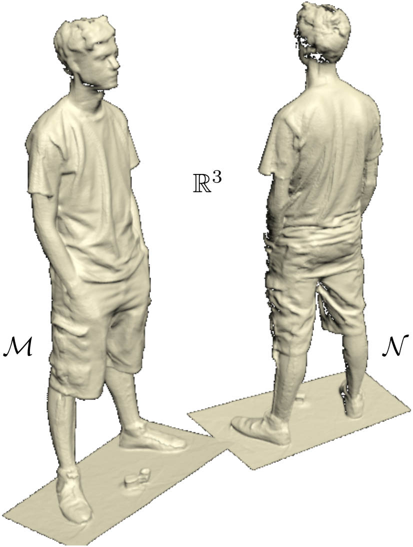


The Hausdorff distance is defined between subsets of a metric space

Note that perturbing one single point can make $d_{\mathcal{H}}^{\mathcal{Z}}$ arbitrarily large

Example: Hausdorff distance, rigid case

$d_{\mathcal{H}}^{\mathbb{R}^3}(\mathcal{M}, \mathcal{N})$ is minimum when $\mathcal{M}$ and $\mathcal{N}$ are aligned



Example: Hausdorff distance, non-rigid case

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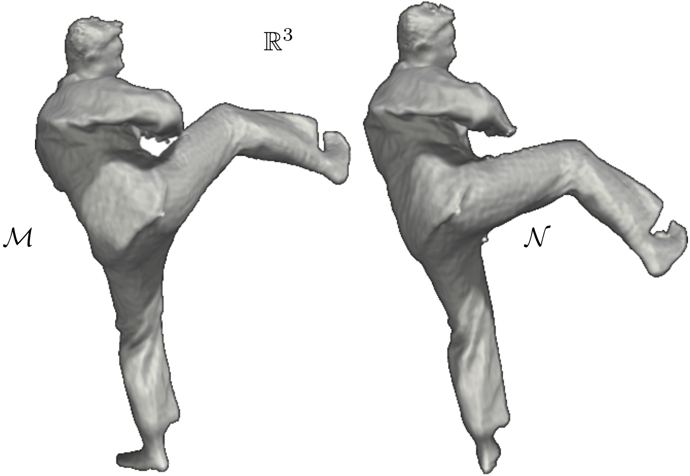

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Example: evaluate quality of level-of-detail

Gears of War 2 · ZBrush Decimation Master

Shortest paths

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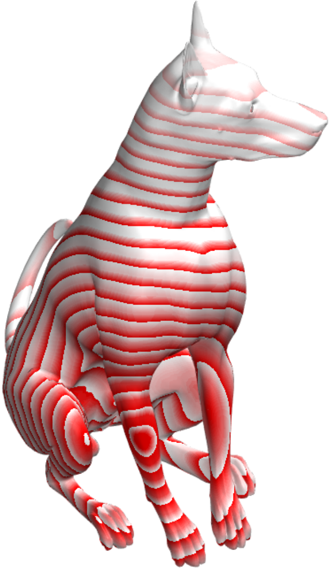
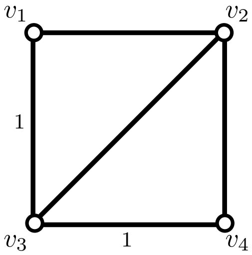


Euclidean

Geodesic

Shortest paths on a graph



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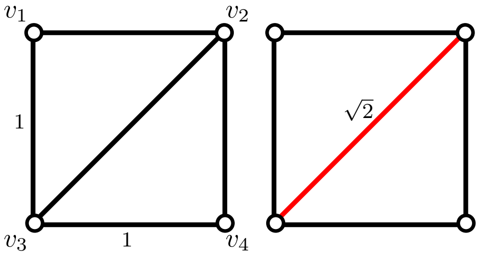

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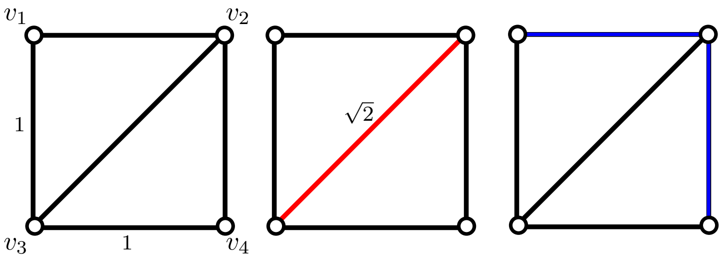

$$\sqrt{2} = d(v_2, v_3) \quad \neq \quad d(v_1, v_4) = 2$$

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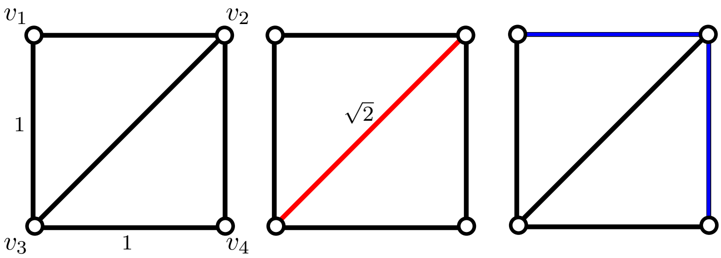

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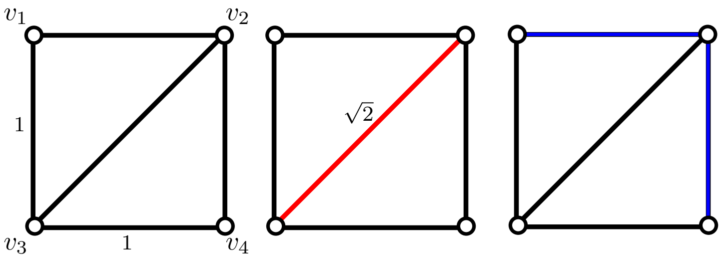
Shortest paths along edges provide upper bounds to exact geodesics

Shortest paths on a graph



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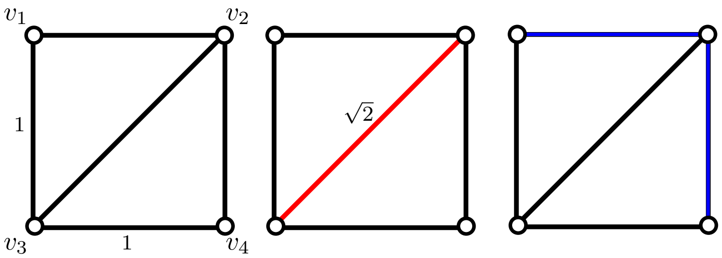

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Shortest paths along edges provide upper bounds to exact geodesics

- Still useful with high resolution meshes or for local distances
- Solved by Dijkstra's algorithm on the mesh graph

Voronoi decomposition

For a given sampling $\mathcal{S}$, the associated Voronoi regions are defined as:

$$V_i(\mathcal{S}) = \{x \in \mathcal{X} \ : \ d_{\mathcal{X}}(x, x_i) < d_{\mathcal{X}}(x, x_j), x_{j \neq i} \in \mathcal{S}\}$$

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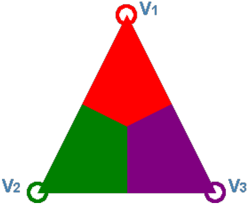
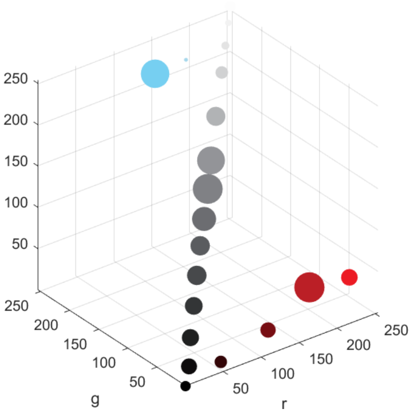
In the triangle example:



Three Voronoi regions, associated to the sampling $\mathcal{S} = \{v_1, v_2, v_3\}$

Examples: Voronoi decomposition

heat diffusion metric
on the surface



Euclidean metric in color space
(result of k-means)

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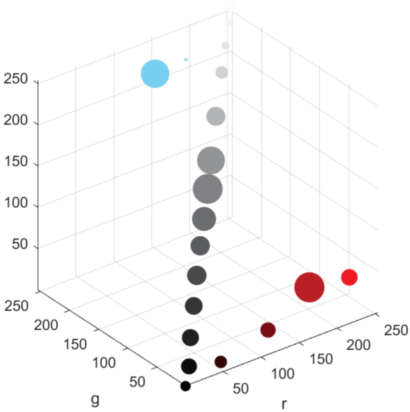

Euclidean metric in color space
(result of k-means)

Exercise: Farthest point sampling

Implement FPS using this algorithm:

- Fix n and let $\mathcal{S}^{(0)} = \{y\}$ for some $y \in \mathcal{X}$
- Proceed recursively:
    - At step k, given $\mathcal{S}^{(k-1)}$, select $x \in (\mathcal{X}, d_{\mathcal{X}})$ such that

$$x = \arg\max_{x \in \mathcal{X}} d_{\mathcal{X}}(x, \mathcal{S}^{(k-1)})$$

    - Set $\mathcal{S}^{(k)} = \mathcal{S}^{(k-1)} \cup x$
    - Repeat until $k = n$
- Test with different starting points y
- Test with a fixed starting point and gradually increasing n

Use FPS to define the initial seed for k-means in color space.