MAT-206/360: Inferencia Estadística **Pauta de Corrección**
Certamen 1. Octubre 12, 2017
Tiempo: 90 minutos Felipe Osorio

1.a) Podemos escribir

$$f(y;\theta) = \exp(\phi y\theta)\exp(c(y;\phi))/\exp(\phi b(\theta)).$$

Es decir, dado que ϕ es conocido, tenemos que $T(y) = \phi y$, $\eta(\theta) = \theta$, $a(\theta) = \exp(\phi b(\theta))$ y $h(y) = \exp(c(y;\phi))$. De este modo, Y pertenece a la FE (1-paramétrica), y anotamos $Y \sim \mathsf{FE}(\theta,\phi)$.

1.b) Debemos calcular

$$\begin{aligned} M_Y(t) = \mathrm{E}\{\exp(tY)\} &= \int \exp(ty)\exp\{\phi[y\theta - b(\theta)] + c(y;\phi)\}\,\mathrm{d}y \\ &= \int \exp\Big\{\phi\Big[\Big(\theta + \frac{t}{\phi}\Big)y - b(\theta)\Big] + c(y;\phi)\Big\}\,\mathrm{d}y \\ &= \exp\{\phi[b(\theta + t/\phi) - b(\theta)]\}\int \exp\Big\{\phi\Big[\Big(\theta + \frac{t}{\phi}\Big)y - b\Big(\theta + \frac{t}{\phi}\Big)\Big] + c(y;\phi)\Big\}\,\mathrm{d}y\,^{*} \\ &= \exp\{\phi[b(\theta + t/\phi) - b(\theta)]\}. \end{aligned}$$

Ahora,

$$\begin{aligned} \frac{\mathrm{d}\,M_Y(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\exp\{\phi[b(\theta + t/\phi) - b(\theta)]\} = \exp\{\phi[b(\theta + t/\phi) - b(\theta)]\}\frac{\phi\,\mathrm{d}\,b(\theta + t/\phi)}{\mathrm{d}t} \\ &= M_Y(t)b'(\theta + t/\phi). \end{aligned}$$

Así, para obtener $\mathrm{E}(Y)$ podemos hacer

$$\mathrm{E}(Y) = \frac{\mathrm{d}\,M_Y(t)}{\mathrm{d}t}\Big|_{t=0} = M_Y(t)\big|_{t=0}\,b'(\theta),$$

pero $M_Y(t)\big|_{t=0} = \exp\{\phi[b(\theta) - b(\theta)]\} = 1$. Por tanto, $\mathrm{E}(Y) = b'(\theta)$.

2. a) Tenemos que la función de probabilidad de X puede ser escrita como

$$p(x;\theta) = a(x)\exp\{x\log\theta - \log C(\theta)\},$$

es decir X pertenece a la FE canónica (1-paramétrica), con $T(X) = X$, $\eta = \log\theta$, $b(\eta) = \log C(e^{\eta})$ y $h(x) = a(x)$.

2. b) Como $T(X) = X$, sigue que[†]

$$\mathrm{E}(X) = \frac{\mathrm{d}}{\mathrm{d}\eta}\log C(e^{\eta}), \qquad \mathrm{var}(X) = \frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\log C(e^{\eta}).$$

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[*] Esta última integral es 1, pues el integrando corresponde a la densidad de una $\mathsf{FE}(\theta + t/\phi, \phi)$.

[†] En efecto, $\mathrm{E}(T(X)) = b'(\eta)$ y $\mathrm{var}(T(X)) = b''(\eta)$.

De este modo,

$$\mathrm{E}(X) = \frac{\mathrm{d}}{\mathrm{d}\eta} \log C(e^{\eta}) = \frac{C'(e^{\eta})e^{\eta}}{C(e^{\eta})},$$

mientras que

$$\begin{aligned}\mathrm{var}(X) &= \frac{\mathrm{d}^2}{\mathrm{d}\eta^2} \log C(e^{\eta}) = \frac{1}{C^2(e^{\eta})}\Big\{C(e^{\eta})\frac{\mathrm{d}}{\mathrm{d}\eta}C'(e^{\eta})e^{\eta} - C'(e^{\eta})e^{\eta}C'(e^{\eta})e^{\eta}\Big\}\\ &= \frac{[C''(e^{\eta})e^{2\eta} + C'(e^{\eta})e^{\eta}]C(e^{\eta}) - [C'(e^{\eta})]^2e^{2\eta}}{C^2(e^{\eta})}.\end{aligned}$$

3. Sea $\pi_m = 1 - \sum_{i=1}^{m-1} \pi_i$, $x_m = n - \sum_{i=1}^{m-1} x_i$, y

$$h(\boldsymbol{x}) = \binom{n}{x_1,\,\dots,x_m}.$$

De este modo, la función de probabilidad de la distribución multinomial puede ser escrita como

$$\begin{aligned}p(x_1,\dots,x_{m-1};\boldsymbol{\pi}) &= h(\boldsymbol{x})\exp(\log(\pi_1^{x_1}\cdots\pi_m^{x_m})) = h(\boldsymbol{x})\exp\Big(\sum_{i=1}^{m} x_i \log \pi_i\Big)\\ &= h(\boldsymbol{x})\exp\Big(\sum_{i=1}^{m-1} x_i \log \pi_i + x_m \log \pi_m\Big)\\ &= h(\boldsymbol{x})\exp\Big(\sum_{i=1}^{m-1} x_i \log \pi_i + \Big(n - \sum_{i=1}^{m-1} x_i\Big)\log\Big(1 - \sum_{i=1}^{m-1}\pi_i\Big)\Big)\\ &= h(\boldsymbol{x})\exp\Big(\sum_{i=1}^{m-1} x_i\Big(\log \pi_i - \log\Big(1 - \sum_{i=1}^{m-1}\pi_i\Big)\Big) + n\log\Big(1 - \sum_{i=1}^{m-1}\pi_i\Big)\Big)\\ &= h(\boldsymbol{x})\Big(1 - \sum_{i=1}^{m-1}\pi_i\Big)^n \exp\Big\{\sum_{i=1}^{m-1} x_i \log\Big(\frac{\pi_i}{1 - \sum_{j=1}^{m-1}\pi_j}\Big)\Big\}.\end{aligned}$$

Es decir, la distribución multinomial pertenece a la FE $(m-1)$-paramétrica con

$$A(\theta) = \Big(1 - \sum_{i=1}^{m-1}\pi_i\Big)^{-n},$$

y

$$\eta_i(\boldsymbol{\pi}) = \log\Big(\frac{\pi_i}{1 - \sum_{j=1}^{m-1}\pi_j}\Big), \qquad T_i(\boldsymbol{x}) = x_i,$$

para $i = 1,\dots,m-1$.

4. El modelo estadístico para la muestra aleatoria $X_1,\dots,X_n$ es definido como

$$\mathcal{P} = \{f(x;\theta)^{\otimes n} : \theta \in (0,\infty)\}.$$

Primeramente, se calculará $\operatorname{E}(X)$. De este modo (usando la sugerencia), sigue que:

$$\begin{aligned}
\operatorname{E}(X) &= \int_0^\infty x f(x;\theta)\,\mathrm{d}x = \frac{1}{\theta(\theta+1)}\int_0^\infty x(x+1)e^{-x/\theta}\,\mathrm{d}x \\
&= \frac{1}{\theta(\theta+1)}\Big[\int_0^\infty x^2 e^{-x/\theta}\,\mathrm{d}x + \int_0^\infty x e^{-x/\theta}\,\mathrm{d}x\Big] \\
&= \frac{1}{\theta(\theta+1)}[\theta^3\Gamma(3) + \theta^2\Gamma(2)] = \frac{\theta(2\theta+1)}{\theta+1}. \qquad (1)
\end{aligned}$$

En este caso, la función de log-verosimilitud es dada por:

$$\begin{aligned}
\ell(\theta;\boldsymbol{x}) &= \sum_{i=1}^n \log f(x_i;\theta) = \sum_{i=1}^n \Big\{\log(x_i+1) - \log\theta(\theta+1) - \frac{1}{\theta}\,x_i\Big\} \\
&= \sum_{i=1}^n \log(x_i+1) - n\log\theta(\theta+1) - \frac{1}{\theta}\sum_{i=1}^n x_i,
\end{aligned}$$

mientras que, la función score asume la forma:

$$U(\theta;\boldsymbol{x}) = \frac{\mathrm{d}\,\ell(\theta)}{\mathrm{d}\theta} = -\frac{n(2\theta+1)}{\theta(\theta+1)} + \frac{1}{\theta^2}\sum_{i=1}^n x_i.$$

Por otro lado, la segunda derivada de la función de log-verosimilitud puede ser escrita como:

$$\begin{aligned}
\frac{\mathrm{d}^2\,\ell(\theta;\boldsymbol{x})}{\mathrm{d}\theta^2} &= n\Big\{\Big(\frac{1}{\theta(\theta+1)}\Big)^2\Big[(2\theta+1)^2 - 2\theta(\theta+1)\Big] - \frac{2}{\theta^3}\,\overline{x}\Big\} \\
&= \frac{n(2\theta^2+2\theta+1)}{[\theta(\theta+1)]^2} - \frac{2}{\theta^3}\sum_{i=1}^n x_i.
\end{aligned}$$

De este modo, la información de Fisher asociada a la muestra $\boldsymbol{X} = (X_1,\dots,X_n)^\top$, resulta:

$$\mathcal{F}_n(\theta) = \operatorname{E}\Big\{-\frac{\mathrm{d}^2\,\ell(\theta;\boldsymbol{x})}{\mathrm{d}\theta^2}\Big\} = -\frac{n(2\theta^2+2\theta+1)}{[\theta(\theta+1)]^2} + \frac{2}{\theta^3}\sum_{i=1}^n \operatorname{E}(X_i),$$

como $\operatorname{E}(X_i) = \theta(2\theta+1)/(\theta+1)$, para $i = 1,\dots,n$, sigue que

$$\mathcal{F}_n(\theta) = \frac{n}{\theta^2(\theta+1)^2}\{(4\theta+2)(\theta+1) - (2\theta^2+2\theta+1)\} = \frac{2n}{\theta}\Big(\frac{\theta+2}{\theta+1}\Big).$$