1.a. La función de log-verosimilitud adopta la forma

$$\begin{aligned}\ell_n(\boldsymbol{\theta}) &= \log \prod_{i=1}^n f(x_i, y_i; \theta_1, \theta_2)\\ &= -n(2\log\theta_1 + \log\theta_2) - \sum_{i=1}^n \log y_i - \sum_{i=1}^n \Big(\frac{x_i}{\theta_1 y_i} + \frac{y_i}{\theta_1\theta_2}\Big).\end{aligned}$$

Tenemos

$$\begin{aligned}\frac{\partial \ell_n(\boldsymbol{\theta})}{\partial\theta_1} &= -\frac{2n}{\theta_1} + \sum_{i=1}^n \Big(\frac{x_i}{\theta_1^2 y_i} + \frac{y_i}{\theta_1^2\theta_2}\Big)\\ \frac{\partial \ell_n(\boldsymbol{\theta})}{\partial\theta_1} &= -\frac{n}{\theta_2} + \sum_{i=1}^n \frac{y_i}{\theta_1\theta_2^2}.\end{aligned}$$

Desde la condición de primer orden $\partial\ell_n(\boldsymbol{\theta})/\partial\theta_2 = 0$, sigue que

$$\sum_{i=1}^n \frac{y_i}{\widehat{\theta}_1\widehat{\theta}_2^2} = \frac{n}{\widehat{\theta}_2} \qquad \Rightarrow \qquad \overline{y} = \widehat{\theta}_1\widehat{\theta}_2.$$

Así, substituyendo en $\partial\ell_n(\boldsymbol{\theta})/\partial\theta_1 = 0$, obtenemos

$$2n\widehat{\theta}_1 - \sum_{i=1}^n \Big(\frac{x_i}{y_i} + \frac{y_i}{\widehat{\theta}_2}\Big) = 0,$$

de ahí que

$$\widehat{\theta}_1 = \frac{1}{n}\sum_{i=1}^n \frac{x_i}{y_i}.$$

Por $\overline{y} = \widehat{\theta}_1\widehat{\theta}_2$, finalmente obtenemos

$$\widehat{\theta}_2 = \frac{\overline{y}}{\widehat{\theta}_1} = \frac{n\overline{y}}{\sum_{i=1}^n x_i/y_i}.$$

Note que

$$\boldsymbol{\mathcal{F}}_n(\boldsymbol{\theta}) = n\boldsymbol{\mathcal{F}}_1(\boldsymbol{\theta}) = n\,\mathsf{Cov}(\boldsymbol{U}_1(\boldsymbol{\theta})),$$

donde

$$\boldsymbol{U}_1(\boldsymbol{\theta}) = \begin{pmatrix} -2/\theta_1 + (x_1/y_1 + y_1/\theta_2)/\theta_1^2 \\ -1/\theta_2 + y_1/(\theta_1\theta_2^2)\end{pmatrix}.$$

Es fácil notar que

$$\mathsf{var}(X/Y) = \mathsf{E}(X^2/Y^2) - \mathsf{E}^2(X/Y) = \theta_1^2, \qquad \mathsf{var}(Y) = \theta_1^2\theta_2^2,$$

mientras que la covarianza $\mathsf{Cov}(X/Y, Y) = 0$, pues la densidad conjunta puede ser factorizada en dos términos independientes. Así,

$$\boldsymbol{\mathcal{F}}_1(\boldsymbol{\theta}) = \begin{pmatrix} 2/\theta_1^2 & 1/(\theta_1\theta_2) \\ 1/(\theta_1\theta_2) & 1/\theta_2^2 \end{pmatrix} = \frac{1}{\theta_1\theta_2}\begin{pmatrix} 2\theta_2/\theta_1 & 1 \\ 1 & \theta_1/\theta_2 \end{pmatrix}.$$

Sabemos que, una región de confianza asintótica para $\boldsymbol{\theta} = (\theta_1, \theta_2)^\top$ adopta la forma

$$RC_n(\boldsymbol{\theta}) = \{\boldsymbol{\theta} : n(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})^\top \boldsymbol{\mathcal{F}}_1(\boldsymbol{\theta})(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}) \leq \chi^2_{1-\alpha}(2)\}.$$

En nuestro caso tenemos que la región de confianza del $100(1-\alpha)\%$ para $\boldsymbol{\theta}$ es dada por

$$RC_n(\boldsymbol{\theta}) = \Big\{(\theta_1, \theta_2) : \frac{2n}{\widehat{\theta}_1\widehat{\theta}_2}\Big(\frac{\widehat{\theta}_2}{\widehat{\theta}_1}(\widehat{\theta}_1 - \theta_1)^2 + (\widehat{\theta}_1 - \theta_1)(\widehat{\theta}_2 - \theta_2) + \frac{\widehat{\theta}_1}{2\widehat{\theta}_2}(\widehat{\theta}_2 - \theta_2)^2\Big) \leq \chi^2_{1-\alpha}(2)\Big\}.$$

1.b. El estadístico de razón de verosimilitudes es dado por

$$\begin{aligned} LR_n &= 2\big(\ell_n(\widehat{\theta}_1, \widehat{\theta}_2) - \ell_n(\theta_1^0, \theta_2^0)\big) \\ &= -2n(2\log\widehat{\theta}_1 + \log\widehat{\theta}_2) - 2\sum_{i=1}^n \Big(\frac{x_i}{\widehat{\theta}_1 y_i} + \frac{y_i}{\widehat{\theta}_1\widehat{\theta}_2}\Big) \\ &\quad + 2n(2\log\theta_1^0 + \log\theta_2^0) + 2\sum_{i=1}^n \Big(\frac{x_i}{\theta_1^0 y_i} + \frac{y_i}{\theta_1^0\theta_2^0}\Big). \end{aligned}$$

Usando la definición de los estimadores ML, obtenemos

$$LR_n = 2n\Big(2\log\frac{\theta_1^0}{\widehat{\theta}_1} + \log\frac{\theta_2^0}{\widehat{\theta}_2}\Big) - 4n + 2\sum_{i=1}^n\Big(\frac{x_i}{\theta_1^0 y_i} + \frac{y_i}{\theta_1^0\theta_2^0}\Big).$$

Finalmente, rechazamos $H_0 : (\theta_1, \theta_2) = (\theta_1^0, \theta_2^0)$ si

$$LR_n \geq \chi^2_{1-\alpha}(2),$$

donde $\chi^2_{1-\alpha}(2)$ representa un valor cuantil $1-\alpha$ de la distribución chi-cuadrado con 2 grados de libertad.

2.a. Tenemos

$$\ell_1(\boldsymbol{\theta}) = x\log\theta_1 + y\log\theta_2 + z\log\theta_3 - \theta_1 - \theta_2 - \theta_3 - \log(x!y!z!).$$

De este modo,

$$\frac{\partial \ell_1(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^\top} = \begin{pmatrix} -x/\theta_1^2 & 0 & 0 \\ 0 & -y/\theta_2^2 & 0 \\ 0 & 0 & -z/\theta_3^2 \end{pmatrix}.$$

Así, la matriz de información de Fisher asume la forma $\boldsymbol{\mathcal{F}}_1(\boldsymbol{\theta}) = \operatorname{diag}(1/\theta_1, 1/\theta_2, 1/\theta_3)$. Notando que la función de log-verosimilitud (para toda la muestra) es dada por:

$$\begin{aligned} \ell_n(\boldsymbol{\theta}) = \Big(\sum_{i=1}^n x_i\Big)\log\theta_1 + \Big(\sum_{i=1}^n y_i\Big)\log\theta_2 + \Big(\sum_{i=1}^n z_i\Big)\log\theta_3 \\ - n(\theta_1 + \theta_2 + \theta_3) - \sum_{i=1}^n \log x_i!y_i!z_i! \end{aligned}$$

y maximizando $\ell_n(\boldsymbol{\theta})$, obtenemos

$$\widehat{\theta}_1 = \overline{X}, \qquad \widehat{\theta}_2 = \overline{Y}, \qquad \widehat{\theta}_3 = \overline{Z}.$$

En nuestro caso, tenemos que $\delta = \theta_1 + \theta_2 - \theta_3$ puede ser escrito como $\delta = \boldsymbol{a}^\top\boldsymbol{\theta} = 0$ con $\boldsymbol{a} = (1,1,-1)^\top$. Notando que $\partial \boldsymbol{a}^\top\boldsymbol{\theta}/\partial\boldsymbol{\theta} = \boldsymbol{a}^\top$, sigue:

$$IC_n(\delta) = \big[\widehat{\delta}_n - z_{1-\alpha/2}\widehat{\mathsf{SE}}(\widehat{\delta}), \widehat{\delta}_n + z_{1-\alpha/2}\widehat{\mathsf{SE}}(\widehat{\delta})\big],$$

con $\widehat{\delta}_n = \widehat{\theta}_1 + \widehat{\theta}_2 - \widehat{\theta}_3 = \overline{X} + \overline{Y} - \overline{Z}$, y

$$\widehat{\mathsf{SE}}(\widehat{\delta}_n) = \sqrt{\boldsymbol{a}^\top \boldsymbol{\mathcal{F}}_1^{-1}(\widehat{\boldsymbol{\theta}})\boldsymbol{a}/n}$$

como

$$\boldsymbol{a}^\top \boldsymbol{\mathcal{F}}_1^{-1}(\widehat{\boldsymbol{\theta}})\boldsymbol{a} = (1,1,-1)\begin{pmatrix} \widehat{\theta}_1 & 0 & 0 \\ 0 & \widehat{\theta}_2 & 0 \\ 0 & 0 & \widehat{\theta}_3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \widehat{\theta}_1 + \widehat{\theta}_2 + \widehat{\theta}_3.$$

De ahí que,

$$IC_n(\delta) = \Bigg[\overline{X} + \overline{Y} - \overline{Z} \mp z_{1-\alpha/2}\sqrt{\frac{\overline{X} + \overline{Y} + \overline{Z}}{n}}\Bigg].$$

2.b. Bajo $H_0: \theta_1 + \theta_2 = \theta_3$, debemos obtener $\widetilde{\theta}_1$ y $\widetilde{\theta}_2$ maximizando

$$\begin{aligned}\ell_n(\theta_1, \theta_2, \theta_1 + \theta_2) = {} & \Big(\sum_{i=1}^n x_i\Big)\log\theta_1 + \Big(\sum_{i=1}^n y_i\Big)\log\theta_2 + \Big(\sum_{i=1}^n z_i\Big)\log(\theta_1 + \theta_2) \\ & - n\theta_1 - n\theta_2 - n(\theta_1 + \theta_2) - \sum_{i=1}^n \log x_i! y_i! z_i!\end{aligned}$$

De ahí que

$$\widetilde{\theta}_1 = \frac{\overline{X}}{2}\Big(\frac{\overline{X} + \overline{Y} + \overline{Z}}{\overline{X} + \overline{Y}}\Big), \qquad \widetilde{\theta}_2 = \frac{\overline{Y}}{2}\Big(\frac{\overline{X} + \overline{Y} + \overline{Z}}{\overline{X} + \overline{Y}}\Big), \qquad \widetilde{\theta}_3 = \frac{\overline{X} + \overline{Y} + \overline{Z}}{2}.$$

Es fácil notar que la función score adopta la forma

$$\boldsymbol{U}_n(\boldsymbol{\theta}) = \frac{\partial \ell_n(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \Big(\frac{n\overline{X}}{\theta_1}, \frac{n\overline{Y}}{\theta_2}, \frac{n\overline{Z}}{\theta_3}\Big)^\top,$$

de ahí que

$$\begin{aligned} R_n &= \frac{1}{n}\boldsymbol{U}_n^\top(\widetilde{\boldsymbol{\theta}})\{\boldsymbol{\mathcal{F}}_1(\widetilde{\boldsymbol{\theta}})\}^{-1}\boldsymbol{U}_n(\widetilde{\boldsymbol{\theta}}) = \frac{1}{n}\Big(\frac{n^2\overline{X}^2}{\widetilde{\theta}_1} + \frac{n^2\overline{Y}^2}{\widetilde{\theta}_2} + \frac{n^2\overline{Z}^2}{\widetilde{\theta}_3}\Big) \\ &= \frac{2n}{\overline{X} + \overline{Y} + \overline{Z}}\Big((\overline{X} + \overline{Y})^2 + \overline{Z}^2\Big). \end{aligned}$$

Finalmente, rechazamos $H_0: \theta_1 + \theta_2 = \theta_3$ si

$$R_n \geq \chi^2_{1-\alpha}(1),$$

con $\chi^2_{1-\alpha}(1)$ el valor cuantil $1-\alpha$ de la distribución chi-cuadrado con 1 grado de libertad.