

Trigonometric polynomials

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1. MANAS Lecture-1 *Interpolations: Equi-spaced and Chebyshev Nodes*

For interpolating a function we need polynomials and the nodes. It is much more important to fix nodes, looking at the kind of function, from before and then use the polynomials (for ex. trigonometric polynomials e^{inx} , algebraic polynomials x^n etc.) to interpolate the function. We will start with equispaced nodes

1.1. Equispaced Nodes

1.1.1. Periodic functions

Let $f : [-\pi, \pi] \rightarrow \mathbb{C}$ be a 2π periodic function. We will try to interpolate the function f using the trigonometric polynomial $p(x)$ and equispaced nodes $\{t_j\}_{-n}^n$, where $n \in \mathbb{N}$, which are defined as follows

$$t_j = \frac{2\pi j}{2n+1} \quad \forall j \in \{-n, -n+1, \dots, n-1, n\} \quad (1.1) \quad \boxed{\text{trig equi}}$$

and

$$p(x) \in \text{span} \{e^{-inx}, e^{-i(n-1)x}, \dots, e^{i(n-1)x}, e^{inx}\} \quad \forall j \in \{-n, -n+1, \dots, n-1, n\}$$

So, $p(x)$ looks like

$$p(x) = \sum_{m=-n}^n c_m e^{imx} \quad (1.2) \quad \boxed{\text{trig equi}}$$

Remark 1.1. *These nodes are such that*

$$-\pi < t_{-n}, \quad t_n < \pi$$

*i.e., we are not interpolating the function at the **endpoints**, instead we are interpolating the function only in between of the interval $(-\pi, \pi)$. But still these nodes are helpful while interpolating provided we have got a periodic function and we are choosing the trigonometric polynomials to interpolate them.*

QUESTION 1.1. *Why are we not considering the endpoints as an option for nodes?*

ANSWER 1.1. *We can consider the endpoints as well. It's just to make things convenient we are choosing the t_j 's as mentioned above.*

To find c_m in (1.2) we exploit the following relation between f and p

$$f(t_j) = p(t_j) \quad \forall j \in \{-n, -n+1, \dots, n-1, n\},$$

i.e., they agree at the nodes. This relation can further be written as follows

$$f(t_j) = \sum_{m=-n}^n c_m e^{\frac{2\pi i j m}{2n+1}}$$

If we use the following notation $f_j = f(t_j)$ and $e_{jm} = e^{\frac{2\pi i j m}{2n+1}}$. Then we get

$$f = E \cdot c$$

where $f = (f_{-n}, \dots, f_n)$ and $E = [e_{jm}]_{-n \leq j, m \leq n}$. Since inverse of E is given as

$$E^{-1} = \frac{1}{2n+1} \left[e^{-\frac{2\pi i j m}{2n+1}} \right]_{-n \leq j, m \leq n}$$

So, we get c_m looks like

$$c_m = \frac{1}{2n+1} \sum_{j=-n}^n f(t_j) e^{-\frac{2\pi i j m}{2n+1}}$$

If we look closely then we see that the polynomial (1.2) looks like a truncated Fourier series, so is there some connection between c_m and Fourier coefficients. Before proceeding further, let us recall the definition of Fourier series

$$f(x) \simeq \sum_{m=-\infty}^{\infty} \hat{f}_m e^{imx}$$

where the coefficients are defined as

$$\hat{f}_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} dx$$

Suppose f is smooth enough for the Fourier transform to make sense, then using the following fact

$$\frac{1}{2n+1} \sum_{j=-n}^n e^{i(l-m)t_j} = \begin{cases} 1 & \text{if } l \equiv m \pmod{2n+1} \\ 0 & \text{else} \end{cases}$$

we can show that

$$c_m = \hat{f}_m + \sum_{l=-\infty, l \neq 0}^{\infty} \hat{f}_{m+l(2n+1)}. \quad (1.3)$$

{cm fourie}

We use this expression to get an error estimate as follows

$$\begin{aligned} |f(x) - p(x)| &= \left| \sum_{m=-\infty}^{\infty} \hat{f}_m e^{imx} - \sum_{\substack{k=-n \\ \text{red}}}^n \sum_{l=-\infty}^{\infty} \hat{f}_{k+l(2n+1)} e^{ikx} \right| \\ &= \left| \sum_{|m| > n} \hat{f}_m e^{imx} - \sum_{\substack{k=-n \\ \text{red}}}^n \sum_{l \neq 0} \hat{f}_{k+l(2n+1)} e^{ikx} \right| \quad \{m \text{ runs over every integer, while } k \text{ from } -n \text{ to } n\} \\ &\leq 2 \sum_{|m| > n} |\hat{f}_m| \end{aligned}$$

If suppose $f \in C^k[-\pi, \pi]$, then we have

$$\begin{aligned}\hat{f}_m &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} dx \\ &= \frac{1}{2\pi(-im)} \int_{-\pi}^{\pi} f'(x) e^{-imx} dx \\ &\vdots \\ &= \frac{1}{2\pi(-im)^k} \int_{-\pi}^{\pi} f^{(k)}(x) e^{-imx} dx\end{aligned}$$

Thus we see that

$$\hat{f}_m \sim \mathcal{O}\left(\frac{1}{m^k}\right)$$

which suggests that

$$\begin{aligned}|f(x) - p(x)| &\leq 2 \sum_{|m| > n} |\hat{f}_m| \\ &\leq 2 \sum_{|m| > n} \frac{1}{|m|^k} \\ &= 4 \sum_{j=1}^{\infty} \frac{1}{(n+j)^k}\end{aligned}$$

Using the fact that for positive, monotonically decreasing and integrable function we have

$$\sum_{j=a}^{\infty} f(j) = \int_a^{\infty} f(x) dx$$

we get

$$|f(x) - p(x)| \leq \sum_{j=1}^{\infty} \frac{1}{(n+j)^k} = \int_1^{\infty} \frac{1}{(n+x)^k} dx = \frac{k}{(n+1)^{k-1}} \sim \mathcal{O}\left(\frac{1}{n^{k-1}}\right)$$

Remark 1.2. It appears that C^1 functions cannot be approximated by this straight forward bound that we obtained, so turns out that this bound is not the sharpest possible bound, but in the case of C^1 functions whose second derivatives are in the Bounded Variation class we do get a bound of the form similar in manner to the C^2 functions, i.e., of the order $\mathcal{O}(n^{-1})$.

If all the derivatives of f are bounded, then we get

$$|f(x) - p(x)| \sim \mathcal{O}\left(\frac{1}{n^k}\right) \quad \forall k \in \mathbb{N}$$

Finally, if f is analytic in $[-\pi, \pi]$ then extend it analytically on the subset $[-\pi, \pi] \times [-a, a]$ of Complex plane \mathbb{C} . Let $D := [-\pi, \pi] \times [0, a]$. Then, by Cauchy's theorem we get

$$\int_{\partial D} f(z) e^{-imz} dz = 0$$

Due to periodicity we see that the sum of integral on the lines $x = -\pi$ and $x = \pi$ as the integral is evaluated on the opposite direction.

$$\int_0^a f(-\pi + iy) e^{-im(-\pi + iy)} dy = e^{im\pi} \int_0^a f(iy) e^{my} dy = - \int_a^0 f(\pi + iy) e^{-im(\pi + iy)} dy$$

Thus we get that

$$\int_{-\pi}^{\pi} f(x) e^{-imx} dx = \int_{-\pi}^{\pi} f(x + ia) e^{-im(x+ia)} dx$$

Hence we see that the Fourier coefficients, for $m < 0$, are given as

$$\hat{f}_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x + ia) e^{-im(x+ia)} dx = \frac{e^{ma}}{2\pi} \int_{-\pi}^{\pi} f(x + ia) e^{-imx} dx$$

which suggests that

$$\hat{f}_m \leq \|f\|_{L^\infty([- \pi, \pi] \times [a, 0])} e^{ma}$$

For $m \geq 0$, choose the rectangle that is below the real axis, i.e., $D_1 := [-\pi, \pi] \times [-a, 0]$ and doing the same we will get that

$$\hat{f}_m \sim \mathcal{O}(e^{-ma})$$

As a result of this analytic continuation, we see that for an analytic function, we have

$$\hat{f}_m \sim \mathcal{O}(e^{-|m|a})$$

Hence,

$$|f(x) - p(x)| \leq 2 \sum_{|m| > n} |\hat{f}_m| \leq \frac{Ce^{-na}}{1 - e^{-a}}$$

This bound depend upon the fact that we can extend the function f analytically. Also the bound depend upon the parameter a , which we want to be away from the pole of the function f extended analytically. To see such a fact, consider the following example

$$f(x) = \frac{1}{x + i\alpha} \quad x \in \mathbb{R}, \alpha \in \mathbb{R}$$

Remark 1.3. Runge's function which is defined as

$$R(x) = \frac{1}{1 + 25x^2}$$

cannot be interpolated using the equispaced nodes as if we choose the equispaced nodes on the interval $[-1, 1]$ and consider the Lagrange's polynomial then we will see that Lebesgue constant blows up, hence we cannot interpolate it using equispaced nodes.

Homework 1.1. Study the effect of ' a ' on the following analytic periodic function $\frac{1}{\sin(x) - i \sinh(1)}$, which will be the pole at $z = i$, hence in this case a will be i

Homework 1.2. Plot the graph of log-log for error for the case of C^k function and semilogy for the case of analytic function.

Homework 1.3. Consider the signum function in the interval $[-1, 1]$ and do the trigonometric polynomial interpolation there using equispaced nodes

1.2. Chebyshev Nodes and Polynomials

Consider a function $f : [-1, 1] \rightarrow \mathbb{R}$ and let us define a new function $g : [-\pi, \pi] \rightarrow \mathbb{R}$ as follows

$$g(t) = f(\cos(t));$$

Note that g is a 2π -periodic function and also an even function. So we can apply all the results from the section as follows.

Let t_j be as defined in (1.1) and let us consider the following trigonometric polynomial to interpolate g

$$q(t) = \sum_{m=-n}^n d_m e^{imt}$$

where d_m was defined as

$$d_m = \frac{1}{2n+1} \sum_{j=-n}^n g(t_j) e^{-im t_j}$$

Since, g is an even function, we get

$$d_m = \frac{g(t_0)}{2n+1} + \frac{2}{2n+1} \sum_{j=1}^n g(t_j) \cos(mt_j) = d_{-m}$$

So, our expression for q now becomes

$$q(t) = d_0 + 2 \sum_{m=1}^n d_m \cos(mt)$$

Since g is an even function we have that it has Fourier cosine series only

$$g(x) \simeq \hat{g}_0 + 2 \sum_{m=1}^{\infty} \hat{g}_m \cos(mx)$$

Similar to (1.3), we get that

$$d_m = \hat{g}_m + 2 \sum_{m=1}^{\infty} \hat{g}_{m+l(2n+1)}; \quad 0 \leq m \leq n$$

Now, we get similar to the previous section

$$|g(t) - q(t)| \leq 7 \sum_{m>n} |\hat{g}_m|$$

If function f is C^k we get that

$$|g(t) - q(t)| \sim \mathcal{O}\left(\frac{1}{n^{k-1}}\right)$$

Similarly, for analytic function we get that

$$|g(t) - q(t)| \sim \mathcal{O}\left(e^{-n|a|}\right)$$

Let us define the following points

$$x_j = \cos(t_j)$$

which are called as the Chebyshev nodes and the polynomial

$$T_m(x) = \cos(m \cos^{-1}(x)); \quad m \geq 0$$

is called as the Chebyshev polynomials.

Remark 1.4. *If we have the derivative of a function in a bounded variation, then we*

Homework 1.4. *Do all the homework of section 1 for this Chebyshev nodes as well?*

2. MANAS Lecture-2 Intro to Boundary Integral Operators

In this section, we try to get acquainted with the notion of Boundary integral Operators and try to prove some facts related to it.

For starters, let us consider the Laplace equation with Dirichlet Boundary Condition posed on an open and bounded domain $\Omega \subset \mathbb{R}^2$

$$\begin{cases} \Delta u = 0 & \Omega \\ u = f & \partial\Omega \end{cases} \quad (2.1)$$

We know from the polar coordinate transformation of above equation that the following Green's function

$$G(x, y) = -\frac{1}{2\pi} \log |x - y|$$

{lap eq}

is fundamental solution for the Laplacian, i.e., it solves the following equation

$$-\Delta_{\mathbf{y}} G(\mathbf{x}, \mathbf{y}) = \delta_{\mathbf{x}}(\mathbf{y}) \quad \forall \mathbf{y} \in \mathbb{R}^2$$

where $\delta_{\mathbf{x}}$ is the Dirac-Delta function with mass at \mathbf{x} .

3. Exercises

exercise 1

Exercise 3.1. For any regular enough domain Ω , show that the following limit exists and is bounded

$$\lim_{\mathbf{x} \rightarrow \mathbf{y}} \frac{(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2} \quad \forall \mathbf{x}, \mathbf{y} \in \partial\Omega$$

Proof. Consider the following parameterisation of $\partial\Omega$ near a point $\mathbf{x}_0 \in \partial\Omega$

$$\gamma(t) = (x_1(t), x_2(t)) \forall t \in (-\delta, \delta)$$

such that $\gamma(0) = \mathbf{x}_0$ and normal at any point $\gamma(t)$ is given by

$$\mathbf{n}(t) = \frac{(x_2'(t), -x_1'(t))}{|\gamma'(t)|}$$

The limit in the question now gets transforms to the following

$$\begin{aligned} \lim_{\mathbf{x} \rightarrow \mathbf{y}} \frac{(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2} &= \lim_{t \rightarrow 0} \frac{(\gamma(0) - \gamma(t)) \cdot \mathbf{n}(t)}{|\gamma(0) - \gamma(t)|^2} \\ &= \lim_{t \rightarrow 0} \frac{(-t\gamma'(0) - \frac{t^2}{2}\gamma''(0)) \cdot \mathbf{n}(t)}{t^2|\gamma'(0)|^2} \\ &= \lim_{t \rightarrow 0} \frac{-\gamma'(0) \cdot \mathbf{n}(t)}{t|\gamma'(0)|^2} - \frac{\gamma''(0) \cdot \mathbf{n}(t)}{2|\gamma'(0)|^2} \\ &= \lim_{t \rightarrow 0} \frac{-\gamma'(0) \cdot (\mathbf{n}(0) + t\mathbf{n}'(t))}{t|\gamma'(0)|^2} - \frac{\gamma''(0) \cdot \mathbf{n}(t)}{2|\gamma'(0)|^2} \\ &= \lim_{t \rightarrow 0} \frac{-\gamma'(0) \cdot \mathbf{n}'(t)}{|\gamma'(0)|^2} - \frac{\gamma''(0) \cdot \mathbf{n}(t)}{2|\gamma'(0)|^2} \\ &= \frac{-\gamma'(0) \cdot \mathbf{n}'(0)}{|\gamma'(0)|^2} - \frac{\gamma''(0) \cdot \mathbf{n}(0)}{2|\gamma'(0)|^2} \\ &= \frac{\gamma''(0) \cdot \mathbf{n}(0)}{2|\gamma'(0)|^2} \quad [\text{since } \gamma''(0) \cdot \mathbf{n}(0) = -\gamma'(0) \cdot \mathbf{n}'(0)] \end{aligned}$$

□

Exercise 3.2. Consider the following equation

$$\begin{cases} \Delta \mathbf{u} = 0; & B(0, 1) \\ \mathbf{u} = f & \partial B(0, 1) \end{cases} \quad (3.1)$$

Using the Fourier series representation for f

$$f(\theta) = \sum_{n \in \mathbb{Z}} \widehat{f}_n e^{in\theta}$$

and the polar coordinate equation for Laplacian, get the solution \mathbf{u} as follows

$$\mathbf{u}(r, \theta) = ??$$

Then, use the Double Layer representation for the solution

$$\mathbf{u} = D[\sigma]$$

and get the Boundary Integral equation with the Kernel $k(\mathbf{x}, \mathbf{y})$

$$-\frac{1}{2}\sigma(\mathbf{x}) + \int_{\partial B(0,1)} k(\mathbf{x}, \mathbf{y})\sigma(\mathbf{y})ds(\mathbf{y}) = f(\mathbf{x}); \quad \mathbf{x}, \mathbf{y} \in \partial B(0, 1). \quad (3.2)$$

{BI0_ex1}

Now we have got two tasks to prove

1. Prove the kernel k to be smooth?
2. Get the Fourier series for σ and relate it with Fourier series of f and verify that relation?
3. Discretise the equation (3.2) and get the approximate solution for u

{ **Hint:** First consider $f = e^{im\theta}$ and then prove for any function f }

Proof. First let us consider the $f = c_m e^{im\theta}$ and then considering the ansatz for u of the form

$$u(r, \theta) = R(r)T(\theta)$$

Consider the Laplacian in the polar coordinates we get

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

Substituting the ansatz into the equation reduces to the following

$$\frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{T''(\theta)}{T(\theta)}$$

From the initial condition we get that

$$R(1)T(\theta) = u(1, \theta) = c_m e^{im\theta}$$

Choosing $T(\theta) = e^{im\theta}$ and $R(1) = c_m$ gives us the following ODE

$$\begin{cases} r^2 R''(r) + rR'(r) - m^2 R(r) = 0; & r \in (0, 1) \\ R(1) = c_m \end{cases} \quad (3.3)$$

whose solution is given by

$$R(r) = cr^m + (c_m - c)r^{-m}$$

for some arbitrary constant c .

Now, we have to see one thing that the solution does not blow up as we go to the end points, if $r \rightarrow 1$ we see that $R(r)$ behaves well, but depending on the sign of m we see that as $r \rightarrow 0$ the solution $R(r)$ blows up and hence we cannot choose any random constant c instead we need to take care of this fact into mind. So, we see that solution looks like

$$R(r) = \begin{cases} c_m r^m e^{im\theta}; & m > 0 \\ c_m r^{-m} e^{im\theta}; & m < 0 \\ c_0; & m = 0 \end{cases}$$

Thus we see that for any general f the solution looks like

$$u(r, \theta) = \hat{f}_0 + \sum_{m < 0} \hat{f}_m r^{-m} e^{im\theta} + \sum_{m > 0} \hat{f}_m r^m e^{im\theta}$$

Now, the kernel k looks like of the following form

$$k(x, y) = \frac{1}{2\pi} \frac{(x - y) \cdot n(y)}{|x - y|^2}$$

Since, $n(y) = y$ in this particular case, we get that

$$\begin{aligned}
k(x, y) &= \frac{1}{2\pi} \cdot \frac{(x - y) \cdot n(y)}{|x - y|^2} \\
&= \frac{1}{2\pi} \cdot \frac{(x - y) \cdot y}{|x - y|^2} \\
&= \frac{1}{2\pi} \cdot \frac{x \cdot y - |y|^2}{|x|^2 + |y|^2 - 2x \cdot y} \\
&= \frac{1}{2\pi} \cdot \frac{x \cdot y - 1}{2(1 - x \cdot y)} \quad [\text{as } x, y \text{ lie on the boundary, hence } |x| = 1 \text{ and } |y| = 1] \\
&= -\frac{1}{4\pi}
\end{aligned}$$

Thus k turns out to be constant and hence is smooth. So, from the following expression

$$-\frac{1}{2}\sigma(x) + \int_{\partial B(0,1)} k(x, y)\sigma(y)ds(y) = f(x); \quad x, y \in \partial B(0, 1).$$

we get that

$$-\frac{1}{2}\sigma(x) - \frac{1}{4\pi} \int_{\partial B(0,1)} \sigma(y)ds(y) = f(x);$$

which means that we get the following

$$(I + K)(\sigma)(x) = -2f(x); \quad \partial B(0, 1) \quad (3.4) \quad \boxed{\text{BIO ex1}}$$

Let us now consider the following Fourier Series representation of σ which is given by

$$\sigma(\theta) = \sum_{n \in \mathbb{Z}} \hat{\sigma}_n e^{in\theta}$$

Putting in (3.4) gives us

$$\sum_{n \in \mathbb{Z}} \hat{\sigma}_n e^{in\theta} + \frac{1}{2\pi} \int_{\partial B(0,1)} \sum_{n \in \mathbb{Z}} \hat{\sigma}_n e^{in\theta} ds(y) = -2 \sum_{n \in \mathbb{Z}} \hat{f}_n e^{in\theta}$$

which on intergating and assuming that we can change the integral and the summation gives us the following

$$\sum_{n \in \mathbb{Z}} \hat{\sigma}_n e^{in\theta} + \sum_{n \in \mathbb{Z}} \delta_{0,n} \hat{\sigma}_n = -2 \sum_{n \in \mathbb{Z}} \hat{f}_n e^{in\theta}$$

which can be re-written as

$$2\hat{\sigma}_0 + \sum_{n \neq 0} \hat{\sigma}_n e^{in\theta} = -2\hat{f}_0 - 2 \sum_{n \neq 0} \hat{f}_n e^{in\theta}$$

which gives us the following expression of Fourier coefficients of σ in terms of Fourier coefficients of f which is

$$\hat{\sigma}_n = \begin{cases} -2\hat{f}_n; & n \neq 0 \\ -\hat{f}_0; & n = 0 \end{cases}$$

□

Exercise 3.3. Repeat the same exercise as above, but now with different domains say ellipse which is paramterised as follows

$$\gamma(t) = (a \cos(t), b \sin(t)) \quad -\pi \leq t \leq \pi$$

which represents the following ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or the following geometry for $a < 1$ and $m \in \mathbb{Z}$

$$\gamma(t) = (1 + a \cos(mt))(\cos(t), \sin(t))$$

Proof. In this case only the Kernel changes, rest all things remains same. First let us start with the ellipse whose

normal at any point is given by

$$\mathbf{n}(\mathbf{t}) = (\mathbf{b} \cos(\mathbf{t}), \mathbf{a} \sin(\mathbf{t}))$$

Now the kernel will change as follows

$$\begin{aligned} \int_{\partial\Omega} k(\mathbf{x}, \mathbf{y}) \sigma(\mathbf{y}) d\mathbf{s}(\mathbf{y}) &= \int_{\partial\Omega} \frac{(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2} \sigma(\mathbf{y}) d\mathbf{s}(\mathbf{y}) \\ &= \int_{-\pi}^{\pi} \frac{(\mathbf{x} - \gamma(\mathbf{t})) \cdot \mathbf{n}(\mathbf{t})}{|\mathbf{x} - \gamma(\mathbf{t})|^2} \cdot \sigma(\gamma(\mathbf{t})) \frac{d\mathbf{s}(\mathbf{t})}{d\mathbf{t}} d\mathbf{t} \\ &= \int_{-\pi}^{\pi} \frac{(\mathbf{x} - \gamma(\mathbf{t})) \cdot \mathbf{n}(\mathbf{t})}{|\mathbf{x} - \gamma(\mathbf{t})|^2} \cdot \sigma(\gamma(\mathbf{t})) |\gamma'(\mathbf{t})| d\mathbf{t} \\ &\simeq \sum_{j=-n}^n \frac{(\mathbf{x} - \gamma_j) \cdot \mathbf{n}_j}{|\mathbf{x} - \gamma_j|^2} \cdot \sigma_j |\gamma'_j| \frac{2\pi}{2n+1} \\ &= \sum_{j=-n}^n \frac{(\mathbf{x} - \gamma_j) \cdot \mathbf{n}_j}{|\mathbf{x} - \gamma_j|^2} \cdot \sigma_j w_j \end{aligned}$$

where we have used the following notations

$$\gamma_j = \gamma(\mathbf{t}_j), \quad \mathbf{n}_j = \mathbf{n}(\mathbf{t}_j), \quad \sigma_j = \sigma(\gamma(\mathbf{t}_j)), \quad \gamma'_j = \gamma'(\mathbf{t}_j), \quad w_j = |\gamma'_j| \cdot \frac{2\pi}{2n+1}$$

Now our next job is to calculate the kernel $k(\mathbf{x}, \mathbf{y})$ for the ellipse

$$\begin{aligned} k(\mathbf{x}, \mathbf{y}) &= \frac{(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2} \\ &= \frac{(\mathbf{x}(\mathbf{t}) - \mathbf{y}) \cdot \mathbf{n}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2} \end{aligned}$$

□

Exercise 3.4. Consider the following equation

$$\begin{cases} \Delta u + \kappa^2 u = 0; & B(0, 1) \\ u = f & \partial B(0, 1) \end{cases} \quad (3.5) \quad \boxed{\{\text{helm}\}}$$

In this case fundamental solution is given by

$$\Phi(\mathbf{x}) = \frac{\iota}{4} H_0^{(1)}(\kappa|\mathbf{x}|)$$

where $\iota = \sqrt{-1}$. Using this information obtain the Double Layer for the Helmholtz equation (3.5) which is basically

$$G_D(\mathbf{x} - \mathbf{y}) = \frac{\partial}{\partial \mathbf{n}_y} \Phi(\mathbf{x} - \mathbf{y}) \quad \forall \mathbf{y} \in \partial B(0, 1), \mathbf{x} \in \overline{B(0, 1)}$$

and then consider the following representation for the solution

$$u(\mathbf{x}) = D[\sigma](\mathbf{x}) = \int_{\partial\Omega} G_D(\mathbf{x} - \mathbf{y}) \sigma(\mathbf{y}) d\mathbf{s}(\mathbf{y})$$

After which obtain a Boundary Integral equation with the Kernel $\Phi(\mathbf{x} - \mathbf{y})$ and then do the following

1. Find the following limit on the boundary

$$\lim_{\substack{\mathbf{x} \rightarrow \mathbf{y} \\ \mathbf{x}, \mathbf{y} \in \partial\Omega}} G_D(\mathbf{x} - \mathbf{y})$$

2. What are the singularities of Φ and G_D ?
3. Obtain an asymptotic for the Φ ?

4. Discretise the equation (3.2) and get the approximate solution for \mathbf{u}

Proof. Firstly let us try to calculate the gradient of $\Phi(\mathbf{x}, \mathbf{y})$

$$\begin{aligned}\frac{\partial}{\partial \mathbf{x}} \Phi(\mathbf{x}, \mathbf{y}) &= \frac{\iota}{4} \frac{\partial}{\partial \mathbf{x}} H_0^{(1)} \left(\kappa \sqrt{\mathbf{x}^2 + \mathbf{y}^2} \right) \\ &= -\frac{\iota \kappa \mathbf{x}}{4r} H_1^{(1)} \left(\kappa \sqrt{\mathbf{x}^2 + \mathbf{y}^2} \right) \quad \left[\text{where } r := \sqrt{\mathbf{x}^2 + \mathbf{y}^2} \right]\end{aligned}$$

As a result, we obtain

$$\nabla \Phi(\mathbf{x}, \mathbf{y}) = -\frac{\iota \kappa}{4r} H_1^{(1)} \left(\kappa \sqrt{\mathbf{x}^2 + \mathbf{y}^2} \right) (\mathbf{x}, \mathbf{y})$$

Let us now try to calculate the normal derivative of $\Phi(\mathbf{x} - \mathbf{y})$

$$\begin{aligned}G_D(\mathbf{x} - \mathbf{y}) &= \frac{\partial}{\partial \mathbf{n}_y} \Phi(\mathbf{x} - \mathbf{y}) = \nabla_y \Phi(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}_y \\ &= \iota \kappa H_1^{(1)}(\kappa |\mathbf{x} - \mathbf{y}|) \frac{(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}_y}{4|\mathbf{x} - \mathbf{y}|}\end{aligned}$$

Let us come to the singularities of Φ , which is same as studying the behaviour of Hankel function $H_0^{(1)}$ near zero, which looks like

$$H_0^{(1)}(z) = \frac{2\iota}{\pi} \ln(z) \quad \text{as } z \rightarrow 0$$

Thus, the singularities of Hankel functions is \log –singularity, i.e., the Single layer of the Helmholtz Green has a \log –singularity. Coming to the singularity of Double Layer, we need the behaviour of $G_D(\mathbf{x} - \mathbf{y})$ as $\mathbf{x} \rightarrow \mathbf{y}$

$$H_1^{(1)}(z) = -\left(\frac{\iota}{\pi} \Gamma(1)\right) \frac{2}{z} \quad \text{for } z > 0, \text{ as } z \rightarrow 0$$

Thus, the Single layer of the Helmholtz Green has a $\frac{1}{z}$ –singularity.

Let us now try to see the behaviour of $G_D(\mathbf{x} - \mathbf{y})$ when $\mathbf{x}, \mathbf{y} \in \partial\Omega$ and $\mathbf{x} \rightarrow \mathbf{y}$

$$\begin{aligned}\lim_{\mathbf{x} \rightarrow \mathbf{y}} G_D(\mathbf{x} - \mathbf{y}) &= \lim_{\mathbf{x} \rightarrow \mathbf{y}} \iota \kappa H_1^{(1)}(\kappa |\mathbf{x} - \mathbf{y}|) \frac{(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}_y}{4|\mathbf{x} - \mathbf{y}|} \\ &= \frac{\iota \kappa}{4} \lim_{\mathbf{x} \rightarrow \mathbf{y}} H_1^{(1)}(\kappa |\mathbf{x} - \mathbf{y}|) \frac{(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}_y}{|\mathbf{x} - \mathbf{y}|} \\ &= \frac{\iota \kappa}{4} \lim_{\mathbf{x} \rightarrow \mathbf{y}} \frac{-2\iota \Gamma(1)}{\kappa \pi |\mathbf{x} - \mathbf{y}|} \frac{(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}_y}{|\mathbf{x} - \mathbf{y}|} \quad \{\text{using the asymptotic of } H_1^{(1)}(z) \text{ as } z \rightarrow 0\} \\ &= \frac{1}{2\pi} \lim_{\mathbf{x} \rightarrow \mathbf{y}} \frac{(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}_y}{|\mathbf{x} - \mathbf{y}|^2} \\ &= \frac{\gamma''(0) \cdot \mathbf{n}(0)}{4\pi |\gamma'(0)|^2} \quad \{\text{from Exercise(3.1)}\}\end{aligned}$$

Thus, on the boundary we have that Double Layer is bounded.

Observe that the Single Layer for Helmholtz behaves in a same manner as that of Laplacian, which is on the boundary the Single Layer has a \log –blow up. In fact thanks to this exercise we observe that on the boundary the Single Layer and Double Layer behaves in the same fashion. \square

Exercise 3.5. Use Trapezoidal Rule to evaluate

$$I := \int_{-\pi}^{\pi} \exp\left(-\left(\frac{t}{0.5}\right)^2\right) t^2 \log(t) \quad \left\{ \text{as } \exp\left(-\left(\frac{t}{0.5}\right)^2\right) \text{ is numerically periodic} \right\}$$

MATLAB CODE for the integral evaluation is "integral". Make the sequence

$$I_n := \sum_{m=-n}^n \sin(t_m^2 \log(t_m)) w_m$$

and see the error $(I_n - I)$. Another alternative is that for a large enough $m \gg 1$, calculate I_m and then calculate

the error $(I_n - I_m)$, this also works for the error evaluation.

Exercise 3.6. SVD Try to see the physical interpretation of SVD and learn what it is trying to say under the hood and look for the Young-Ekart theorem and first solve the following problem yourself

$$\min \|A - B\|$$

$$\text{subject to } \text{rank}(B) = 1$$

Proof. Let U, Σ, V represent the singular value decomposition of A . First let us consider the case when the matrix B is of the form

$$B = u_1 \alpha^t \quad \alpha \in \mathbb{R}^n$$

Then we want to minimise $\|A - u_1 \alpha^t\|$. So let us choose $x \in \mathbb{R}^n$ which can be written as

$$x = \sum_{i=1}^n \beta_i v_i$$

and calculate $\|Ax - u_1 \alpha^t x\|$

$$\begin{aligned} \|Ax - u_1 \alpha^t x\|^2 &= \left\| A \sum_{i=1}^n \beta_i v_i - u_1 \alpha^t \sum_{i=1}^n \beta_i v_i \right\|^2 \\ &= \left\| \sum_{i=1}^n \beta_i \sigma_i u_i - \sum_{i=1}^n \beta_i \alpha^t v_i u_1 \right\|^2 \\ &= \left\| \sum_{i=1}^n \beta_i \sigma_i u_i - \sum_{i=1}^n \beta_i \langle \alpha, v_i \rangle u_1 \right\|^2 \\ &= \left\| \left(\beta_1 \sigma_1 - \sum_{i=1}^n \beta_i \langle \alpha, v_i \rangle \right) u_1 + \sum_{i>1} \beta_i \sigma_i u_i \right\|^2 \\ &= \left| \beta_1 \sigma_1 - \sum_{i=1}^n \beta_i \langle \alpha, v_i \rangle \right|^2 + \sum_{i>1} |\beta_i \sigma_i|^2 \end{aligned}$$

which suggests that the first term above if vanishes then we get the minimum which we are after, and thus giving an obvious choice for α which is

$$\langle \alpha, v_i \rangle = \begin{cases} \sigma_1; & i = 1 \\ 0; & i \neq 1 \end{cases} \quad (3.6)$$

$\{\alpha\}$

Thus, $\alpha = \sigma_1 v_1$ is the choice for which we will attain the minimum. Another version of the proof is given by using the fact that V is orthogonal matrix and hence we can map unit disc $B(0, 1)$ to unit disc under the multiplication by V . So, it's given as

$$\begin{aligned} \|AVx - u_1 \alpha^t Vx\|^2 &= \|U_1 \Sigma x - u_1 \alpha^t Vx\|^2 \\ &= \|[\sigma_1 u_1 \ \sigma_2 u_2 \ \cdots \ \sigma_n u_n]x - (\alpha^t Vx)u_1\|^2 \\ &= \left\| \sum_{i=1}^n x_i \sigma_i u_i - \sum_{i=1}^n x_i \langle \alpha, v_i \rangle u_1 \right\|^2 \\ &= \left| x_1 \sigma_1 - \sum_{i=1}^n x_i \langle \alpha, v_i \rangle \right|^2 + \sum_{i>1} |x_i \sigma_i|^2 \end{aligned}$$

Now, the same analysis as above will help us to get the desired α which is defined in (3.6) □

Exercise 3.7. Define the following kernel

$$K(z; x, y) := n(y) \nabla_y H_0^1(z|x - y|) = G_D(z; x - y)$$

and do the following

H1. Compute $\frac{d}{dz}K(z; \mathbf{x}, \mathbf{y})$ and find asymptotic behaviour as $\mathbf{x} \rightarrow \mathbf{y}$.

H2. Implement $\frac{d}{dz}K(z; \mathbf{x}, \mathbf{y})$ as kernel in `chunkIE` (similar to `ostokes`) which will call as "freq_diff".

H3. Define the following function

$$A'(z)[\sigma] = \int_{\partial\Omega} \frac{d}{dz}K(z; \mathbf{x}, \mathbf{y})\sigma(\mathbf{y})d\mathbf{s}(\mathbf{y})$$

and verify the following identity

$$|A(z + \mathbf{h})[\sigma] - A(z - \mathbf{h})[\sigma] - A'(z)[\sigma]| \simeq \mathcal{O}(h^2) \quad (3.7) \quad \boxed{\text{diff opti}}$$

where

$$A(z)[\sigma] = \int_{\partial\Omega} G_D(z; \mathbf{x} - \mathbf{y})\sigma(\mathbf{y})d\mathbf{s}(\mathbf{y})$$

i.e. Double Layer corresponding to the Helmholtz equation with respect to $\kappa = z$, i.e., the wave number is given by z . So, the question here is using the "chunkmat" try to prove the order (3.7).

H4. Find root finding algorithm and look for order of convergence of these methods.

H5. Do Complex Muller method with correct initial guess and try to see can we reduce the number of iterations.

Proof. H1 Let us first consider the derivative of K with respect to the variable z

$$\begin{aligned} \frac{d}{dz}K(z; \mathbf{x}, \mathbf{y}) &= \frac{d}{dz}\mathbf{n}(\mathbf{y})\nabla_{\mathbf{y}}H_0^{(1)}(z|\mathbf{x} - \mathbf{y}|) \\ &= \mathbf{n}(\mathbf{y})\nabla_{\mathbf{y}}\left[\frac{d}{dz}H_0^{(1)}(z|\mathbf{x} - \mathbf{y}|)\right] \\ &= \mathbf{n}(\mathbf{y})\nabla_{\mathbf{y}}\left[H_1^{(1)}(z|\mathbf{x} - \mathbf{y}|)|\mathbf{x} - \mathbf{y}|\right] \\ &= \mathbf{n}(\mathbf{y})\left\{\left[\nabla_{\mathbf{y}}H_1^{(1)}(z|\mathbf{x} - \mathbf{y}|)\right]|\mathbf{x} - \mathbf{y}| + H_1^{(1)}(z|\mathbf{x} - \mathbf{y}|)\nabla_{\mathbf{y}}[|\mathbf{x} - \mathbf{y}|]\right\} \\ &= \mathbf{n}(\mathbf{y})\left\{\left[\left\{H_0^{(1)}(z|\mathbf{x} - \mathbf{y}|) - \frac{1}{z|\mathbf{x} - \mathbf{y}|}H_1^{(1)}(z|\mathbf{x} - \mathbf{y}|)\right\}\frac{z(\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|}\right]|\mathbf{x} - \mathbf{y}| + H_1^{(1)}(z|\mathbf{x} - \mathbf{y}|)\frac{(\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|}\right\} \\ &= \mathbf{n}(\mathbf{y})\left\{zH_0^{(1)}(z|\mathbf{x} - \mathbf{y}|)(\mathbf{x} - \mathbf{y}) - H_1^{(1)}(z|\mathbf{x} - \mathbf{y}|)\frac{(\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|} + H_1^{(1)}(z|\mathbf{x} - \mathbf{y}|)\frac{(\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|}\right\} \\ &= zH_0^{(1)}(z|\mathbf{x} - \mathbf{y}|)(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) \end{aligned}$$

To find the asymptotic behaviour we need to know about the asymptotic of the $H_0^{(1)}$. We know that

$$H_0^{(1)}(z) = \frac{2i}{\pi} \ln(z) \quad \text{as } z \rightarrow 0$$

We obtain the following inequality

$$\begin{aligned} \left|zH_0^{(1)}(z|\mathbf{x} - \mathbf{y}|)(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}(\mathbf{y})\right| &\leq \left|H_0^{(1)}(z|\mathbf{x} - \mathbf{y}|)\right| |z| |\mathbf{x} - \mathbf{y}| \\ &\leq C|z| |\mathbf{x} - \mathbf{y}| \log(z|\mathbf{x} - \mathbf{y}|) \\ &\rightarrow 0 \quad \text{as } \mathbf{x} \rightarrow \mathbf{y} \end{aligned}$$

Since, the asymptotic of $H_0^{(1)}$ looks like

$$\begin{aligned} H_0^{(1)}(z) &= J_0^{(1)}(z) + iY_0^{(1)}(z) \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{z^2}{4}\right)^k}{k!\Gamma(k+1)} + i \left(\frac{2}{\pi} \left(\ln\left(\frac{z}{2}\right) + \gamma \right) J_0(z) + \frac{2}{\pi} \sum_{k=1}^{\infty} (-1)^{k+1} \left(\sum_{m=1}^k \frac{1}{m} \right) \frac{\left(\frac{z^2}{4}\right)^k}{(k!)^2} \right) \end{aligned}$$

which suggests that asymptotic of $zH_0^{(1)}(z)$ is of the form $\mathcal{O}(z \ln(z))$

□

Exercise 3.8. Do the Newton-Raphson for the Fredholm determinant using the fact that

$$\frac{d}{dz}(\det A(z)) = \text{tr}(A'(z))$$

and thus our objective function will turn out to be

$$z_{n+1} = z_n - \frac{\det(A, z_n)}{\text{tr}(A'(z_n))}$$

Remark 3.9. Memory complexity is given by **memory allotted to an object**, like an $n \times n$ matrix will take around n^2 memory and an vector will take around n memory and thus to store a matrix and vector we will need roughly around $\mathcal{O}(n^2)$ memory.

Similarly, time complexity is given by **time required to do an operation**, which is called as flops, such as to add two numbers we do one operation, i.e., one flop and similarly to do multiplication of two numbers we need to do one flops. To compute the inner product we need to do the following

$$\langle v, u \rangle = \sum_{i=1}^n v_i u_i$$

we need to do n multiplications and $n - 1$ addition, so we need to do $2n - 1$ which is same as saying that we need to do $\mathcal{O}(n)$ operations. To calculate the matrix vector calculations we need to do the following operations

$$A.x = \left[\sum_{j=1}^n a_{ij} x_j \right]_{1 \leq i \leq n}$$

For each column of $A.x$ we need to do n multiplications and $n - 1$ additions which means that for each column we need to do $2n - 1$ operations and since there are total n number of columns we do exactly $2n^2 - n$ operations which means we need to do the $\mathcal{O}(n^2)$ operations. So, to calculate $\langle v, Au \rangle$ we need to do the following

$$\langle v, Au \rangle = \sum_{i=1}^n v_i \left(\sum_{j=1}^n a_{ij} x_j \right) \quad (3.8)$$

{inner prod

each term in above inner product requires multiplication of a number v_i by a sum of $\sum_{j=1}^n a_{ij} x_j$, this sum requires n multiplications and $n - 1$ additions and after the sum calculations we need to do the multiplications by a number which is just one operation and done for n terms will result in total of $n(2n - 1)$ operations. So far we have accounted for the total number of operations requires to calculate the terms inside (3.8) after which adding them requires $n - 1$ operations so in total we need $(n - 1)n(2n - 1)$ which is roughly $\mathcal{O}(n^3)$ operations. This can be written in short as to compute inner product we need $\mathcal{O}(n^2)$ operations and for each term in inner product we need $(2n - 1 + 1 = 2n)$ operations, so in total we need $\mathcal{O}(n^2 \times n)$, i.e., $\mathcal{O}(n^3)$ operations.

Exercise 3.10. In `chunkIE`, we create the chunker using `chunkIE` and divide it into panels and to each panel we associate 16 many Chebyshev points. When we introduce perforations we introduce Chebyshev points on each panels that arises out of the division of the perforations by panels. Try to do this calculations on your and get a feel for how they are calculated, how is `chunkIE` evaluating the Chebyshev points on my [get_chunker.m](#)?

Exercise 3.11. Go to the code [chunkie/demo/demo_flam.m](#), location is found using, "`git grep rskelf_sv`" and see how they are calculating the $A^{-1}v$ and try to understand Fast Algorithm(FLAM) for that, which in our case we are doing using BACKSLASH. Try to implement this to our Optimisation function $\langle v, A^{-1}u \rangle$ (remember we are using the $A \backslash v$ right now to compute $A^{-1}u$) and see how it works(try mimicking it in the optimisation case)?

Proof. **DONE in MATLAB!!**

□

Exercise 3.12 (Building an Algorithm). We know how to evaluate the following for any vector $\mathbf{q} \in \mathbb{R}^d$

$$f(\mathbf{x}_i) = \sum_{j=1}^d H_0^{(1)}(\kappa|\mathbf{x}_i - \mathbf{x}_j|) \mathbf{q}(\mathbf{x}_j)$$

Build an Algorithm to evaluate the following

$$g(\mathbf{x}_i) = \kappa \sum_{j=1}^d H_0^{(1)}(\kappa|\mathbf{x}_i - \mathbf{x}_j|) (\mathbf{x}_i - \mathbf{x}_j) \cdot \mathbf{n}(\mathbf{x}_j) \mathbf{q}(\mathbf{x}_j)$$

Proof. Define the following vector

$$\mathbf{r}^l(\mathbf{x}_j) = \mathbf{q}(\mathbf{x}_l) \delta_{jl} \quad 1 \leq l \leq d$$

Using this \mathbf{r}_l we define the following matrix

$$F_{i,l} = \sum_{j=1}^d H_0^{(1)}(\kappa|\mathbf{x}_i - \mathbf{x}_j|) \mathbf{r}^l(\mathbf{x}_j) = H_0^{(1)}(\kappa|\mathbf{x}_i - \mathbf{x}_l|) \mathbf{q}(\mathbf{x}_l)$$

Thus now our function g looks like

$$g(\mathbf{x}_i) = \kappa \sum_{j=1}^d F_{i,j} \langle \mathbf{x}_i - \mathbf{x}_j, \mathbf{n}(\mathbf{x}_j) \rangle$$

We can again define matrices M^x and M^y as follows

$$M_{ij}^x = (\mathbf{x}_i^{(1)} - \mathbf{x}_j^{(1)}) F_{i,j}, \quad M_{ij}^y = (\mathbf{x}_i^{(2)} - \mathbf{x}_j^{(2)}) F_{i,j}$$

$$\mathbf{n}^x(\mathbf{x}_j) = \mathbf{n}_1(\mathbf{x}_j), \quad \mathbf{n}^y(\mathbf{x}_j) = \mathbf{n}_2(\mathbf{x}_j)$$

where we have used following notation $\mathbf{x} = (\mathbf{x}^{(1)}, \mathbf{x}^{(2)})$ and $\mathbf{n}(\mathbf{x}) = (\mathbf{n}_1(\mathbf{x}), \mathbf{n}_2(\mathbf{x}))$. So, now $g(\mathbf{x}_i)$ looks like

$$g(\mathbf{x}_i) = \kappa \sum_{j=1}^d M_{ij}^x \mathbf{n}_1(\mathbf{x}_j) + \kappa \sum_{j=1}^d M_{ij}^y \mathbf{n}_2(\mathbf{x}_j)$$

Also, observe that

$$\left| H_0^{(1)}(\kappa|\mathbf{x} - \mathbf{y}|) (\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) \right| \leq |\mathbf{x} - \mathbf{y}|^2 |\log(\kappa|\mathbf{x} - \mathbf{y}|)| \longrightarrow 0 \text{ as } \mathbf{x} \rightarrow \mathbf{y}$$

□

References