Project Summary

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Hybrid control systems are a theoretical construct that emerged over the past several decades as the result of broad adoption of computerized, digital systems in control practice on one hand, and thanks to the recognition of the necessity to formalize the role which discrete, semantically driven events play in the process executions, on the other.

While hybrid systems are becoming an indispensable tool for analysis and design in modern engineering, the development of their theory was closely following the established research paradigm of classical nonlinear control, and was underutilizing some of the more sophisticated tools that became available to applied mathematicians and engineers, especially those stemming from the domains of algebraic and differential topology, model theory, category theory.

There are no intrinsic reasons for that. Hybrid systems form a highly syncretic area, situated at a point where several branches of mathematics, control theory and theoretical computer science come together. Conceptually, the area is rife for the deployment of the most advanced tools mathematics has to offer.

The attempts to do so were initiated over the past decade, but in a patchwork way, without the benefit of a coherent unifying program: each new approach came with the built-in overhead of fitting it into the context of the existing techniques and methods.

This project aims at changing that. We will start with assembling existing approaches into a unified mosaic using the canvas of topology and algebra (using as our central tools category theory and algebraic topology), and proceed to forge a computationally effective theory for the specification and synthesis of hybrid systems behaviors.

We are relying on several guiding principles. **Tameness** postulates that most if not all aspects of the theory can be seen through an algebraic lens: all objects of hybrid control systems are located within some *o-minimal structure*. Focus on **Path Spaces** implies they should be considered as one of the key primitives of the hybrid systems, and most if not all constructions could be derived from them, rendering the distinction between open and closed systems as secondary, and simplifying many problems of compositionality. **Categorification** means an upfront investment into the underlying topological and combinatorial structures, most naturally formalized in terms of topological categories: while the resulting constructions become more abstract, they also become easier to formalize and to compose.

While the primary goals of the project are theoretical and algorithmic, we expect interactions with and rapid impact on several domains of relevance to the DoD and AF in particular. Thus, we aim at applications in *multi-agent systems* (including the problem of safe coordination of swarms of manned and unmanned vehicles, and the problem of assured coverage. *Robotic systems*, a rich source of hybrid models, will benefit from efficient formalisms developed in this project, and innovative data-driven approaches bypassing explicit modeling. *Material science and biological systems* form a class of hybrid dynamical systems in an intellectually compelling and technologically rich domain heretofore barely touched upon.

Ultimately, this project aimes to transform synergetically the research paradigm of hybrid, and, more generally, control dynamical system, leading to a new generation of tools (algorithmic pipes and libraries) that engineers across DoD relevant industries would adopt.

1 Introduction

Hybrid control systems are a syncretic notion, located at a point where several branches of mathematics, control theory and theoretical computer science come together. The feeling that a unification of this patched framework exists is prevalent in the community yet is still elusive. The patchwork nature of the field and its many disparate practitioners are starting to palpably affect progress: each new approach comes with the built-in overhead of fitting it into the context of existing techniques. We aim to start assembling existing approaches into a unified mosaic using the canvas of topology and algebra (using as our central tools category theory and algebraic topology), to forge a computationally effective theory for the specification and synthesis of hybrid systems behaviors.

Here are our guiding principles:

Tameness. The pragmatic view of models for engineered or natural systems leads one to abandoning the classical tenets of analysis as the underlying paradigm: general continuous or smooth functions do not really appear in any practical approach. Instead, one can algebraize everything and assume some intrinsic tameness: all objects of hybrid control systems are located within some *o-minimal structure*.

Path Spaces. The structure of the space of trajectories can be considered as the key primitive. All constructions would be derived from them. This renders the structure of the underlying phase or configuration space secondary, so that the traditional ontological framework of starting with the space, defining dynamics thereon is reversed: the configuration spaces are secondary to the space of trajectories. This implies that the distinction between open and closed systems becomes secondary, and many problems of compositionality become tractable questions about the topology of the space of paths.

Categorification. The shift to path spaces forces some significant upfront investment into the underlying topological and combinatorial structures, most naturally formalized in terms of topological categories. The resulting constructions become more abstract, yet intrinsically easier to operate with and to formalize.

We map the sponsor's Research Concentration Areas (RCA) onto the pieces of the emerging mosaic, by aligning our Theoretical constructs with the RCAs outlined in the topic description. We outline the correspondences between the teams of the PIs and the specific tasks within our program.

2 Research Thrusts

We start with a reformulation of the established notion of a hybrid system (confined, for now, to the traditional, state-space view: a broader perspective will follow).

Any state-space based modeling a real-life system starts with the *configuration space* M, describing possible instantaneous conformations of the objects comprising it. This can be the familiar C-space of robotic design class [102, 132], or the actual configuration space [57] modeling swarms of bodies subject to the exclusion constraints, or a point in the group of volume preserving diffeomorphisms, in case of soft body models [10, 109].

Over those *base configuration spaces* one forms bundles to address kinematics and dynamics: tangent bundle TM for Langrangean and cotangent bundle T^*M for Hamiltonian dynamics; higher jet spaces for more involved constructions [91]. We will be referring to the resulting construct as the *phase space* P. The base spaces already can be singular, which makes the problem of proper definition of the phase spaces nontrivial, as we discuss below.

The *hybrid spaces* reside atop of the phase spaces. The traditional definition [32, 69], with its formalism of guards and reset maps obscures the key functional property of the hybrid space: it expresses predicates (thermostat reached 451°F, left leg hit ground, molecular conformation reached a saddle point in its energetic landscape etc) over the *current state* of the system, i.e. a point of the phase space. These predicates, when true, might trigger a discontinuous action: the system switches into an alarm mode, the actuator tenses, the shape snaps into a different conformation. One can think of the discrete, execution logic driven collection of states possible over a given point of the phase space as the *logical states fibration*[1] over the phase space.

The coordinates of a point in this fibration consist, informally, of the pair (p, l), where $p \in P$ is the state of the system, and l is a well-defined propositional formula, - i.e. a logical predicate expressable in terms of definable [24] functions of the phase space p. The collection of these predicates depends on the particular stage of the execution (hitting 451°F might have very different semantics). Hence the hybrid space is in general not a trivial fibration. The collection of different logical predicates associated with a particular point of the phase state is its *logical fiber*. Due to the discrete nature of the logical fiber, one can think of it as a (partial, branching) covering of the phase space. The totality of those logical fibers over the phase spaces forms the *hybrid space* E.

Given a particular propositional formula (a particular branch of that covering) a *guard* is then the subset of the phase space where the formula is `True`; the lift of the subset to the corresponding branch of the covering defines a *guard* G as a subset of the hybrid space E.

Once an execution path (a trajectory of the hybrid dynamical system) hits a guard, it is mapped to a different point of the hybrid space by the reset map R.

This view leads to our operational view on the hybrid dynamical system through the

Definition (Hybrid System)**.** *A hybrid system $\mathscr{H}$ in the o-minimal structure*[2] *$\mathscr{C}$ is the collection of objects and morphisms*

$$(G \to E \to P \to M, F, R), \tag{1}$$

where $E \to M$ is a local covering over a stratification of P, $G \to E$ is an inclusion morphism, and $F : E \times \mathbb{R}$ is a locally defined semiflow such that all trajectories in E are either complete, or enter G. Finally, $R : G \to E$ is the reset map, a morphism preserving fibers[3] *of $E \to M$.*

We do not require that E is connected; it can have maximal strata of different dimensions.

Many phenomena befuddling the hybrid systems theory can be easily expressed in terms of the data we presented. Thus, one can prove that the Zeno phenomenon corresponds to the presences of R cycles in a fiber over M, and so on.

The requirement that the underlying structure is in $\mathscr{C}$ (a part of our *tameness* thread) has powerful implications. In particular, the assumption that the spaces and the morphisms used are semialgebraic, subanalytic, or are within some other o-minimal structure automatically implies that all the spaces admit a *proper stratifications* (e.g. satisfying the standard Whitney con-

[1] The points of the fibers correspond to the *modes* of the hybrid system.

[2] That is, a category of topological spaces where the objects and morphisms are within an o-minimal structure, such as the category of piece-wise linear [149], or semi-algebraic [49], or subanalytic [145] sets.

[3] One might argue about potential loss of generality encoded in the requirement that R is fiber-preserving, but in all models we surveyed, this requirement is satisfied.

ditions [71, 140][4]). That would imply, in particular, that all the spaces E, G, P, M can be properly *triangulated*, and the corresponding maps respect this stratification.

This requirement has two sources, one prescriptive and one phenomenological. On one hand, the existence of triangulations compatible with the structure maps (1) makes all the constructions effective and implementable as computer code. Such effectivizations of the continuous constructions by some cellular decompositions can be seen in all successful computational pipes in nonlinear dynamics and control, from Canny's cylindrical decompositions [37] to cubical approximations for effective Conley theory [108].

On the other hand, in engineering and modeling practice, whether ours, or in the existing literature, we haven't encountered a single model where the underlying topological space is not represented as a member of one of the standard triangulable topological categories, typically residing within an o-minimal structure, such as the PL category (where all composite pieces are convex polyhedra), or semi-algebraic category (where the pieces are described by polynomial equations and inequalities), etc. While the literature on hybrid (or, more generally, control) systems overwhelmingly works under the assumption that the hybrid spaces are arbitrary closed subsets of the Euclidean spaces, and their dynamics is semicontinuous, the actual *examples* and any software *implementations*[5] invariably deal with the semialgebraic, or subanalytic, or just piece-wise linear structures, see e.g. [68, 70, 137].

Further implications of the existence of the triangulations respecting the structure maps in (1), are the properties necessary for the homotopy-theoretic constructions (e.g. all inclusion maps are cofibrations etc) deployed when we reason about the topology of the hybrid spaces. These properties are on one hand strongly restrictive (all spaces are ENRs [52], and so on), on the other they give us powerful tools, that allows us to bracket out a host of familiar worries permeating the standard approaches to the dynamical systems in engineering [85, 110].

Model Systems

It is easiest to illustrate our ideas on a model example. We take a system (overused by the community, but both versatile and fundamental) of bouncing balls, coupled and constrained. Formally, the configuration space of a single ball is formed by a domain $\mathscr{D} \subset \mathbb{R}^d$ in which the ball is contained. The phase space is naturally identified with the cotangent bundle $T^*\mathbb{R}^d \cong (\mathbb{R}^d)^* \times \mathbb{R}^d$ restricted to $\mathscr{D}$.

In the ideal reflection case of *mathematical billiards* [135] this system is the archetypal model of systems with impacts, and served as the testing ground for investigations of ergodicity and integrability. When losses of energy (or, more generally, reflection operators) are introduced, the hybrid systems enter. If the boundary $\partial\mathscr{D}$ of the domain $\mathscr{D}$ is smooth, the guard is the hypersurface $(\mathbb{R}^d)^* \times \partial\mathscr{D}$ of the phase space, and the reset operator preserves the fibers of the cotangent bundle (in the case of mathematical billiards, the reset at $x \in \partial\mathscr{D}$ is just the reflection in the plane orthogonal to the covector normal to $\partial\mathscr{D}$ at x.

[4] These geometric conditions preclude various pathologies, such as a cell of the space wrapping infinitely many times around an adjacent sell, and result in many other simplifications. What is more, they are automatically satisfied in the geometric categories we work with.

[5] The reemergence of *analogue* models of computations, - be it slime, DNA-based, quantum, or any other version, - might muddle a bit the clean viewpoint we advocate here. Some of those models (quantum) are outright within our paradigm, and we hope to have a better analysis of how well other models fit our perspective, once they pass the gestation phase.

Slightly enriching this structure by making the boundary $\partial\mathscr{D}$ dependent on parameters (modeling the "Büghler arm" juggling) one arrives at a hybrid control system, abundantly studied and well understood (see, e.g. [115]). However *composing* even two of those simple hybrid systems immediately introduces singularities semantically expressing logical ambiguities in control design and execution: the overlaps of the guards require careful analysis of how the controllers interpret small perturbations of the sensing signals; the flow can start bifurcating and so on [22]. A fully versatile modeling paradigm for hybrid systems will be, of course, even more involved.

Another class of examples is delivered by various idealizations of material bodies under exclusion conditions, the standard model for collision-avoiding multi-agent systems. Mathematical abstraction, where the objects are point-like, leads to the standard *configuration spaces* emerging in knot theory and classical mechanics [57]. (For more exotic versions addressing hierarchies of allowable collisions, and singular underlying spaces see, e.g. [18, 66].) Realistic models, where the objects under the volume exclusion constraints have non-vanishing sizes are a staple of statistical physics modeling (see, e.g. [100]), but their *topology* was recently discovered to be quite different from the conventional configuration spaces [23]. Geometrically, passing from the configuration spaces of swarms of point-like objects to swarms of *disks* replaces the *subspace arrangements* [148] with some thickenings, and the singularities and guard interactions (defined by the pairwise collision of the bodies) start to play a role, resulting in rather complicated structure of the boundary, and highly nontrivial dynamics.

As a result, the topology of the configuration spaces of swarms is highly nontrivial: for example, their Betti numbers[6] typically grow superexponentially fast (in number of the objects n in the swarm) [18]. This forces, as we argue in Section 2.4.2 the emergence of an intricate system of guards that govern the logics of the resulting hybrid system for any implementation of feedback stabilization on a particular attractor.

Currently, engineered swarms (of UAVs or other agents) deployments are small enough not to require establishing C&C hierarchies. However, already in the very near future the need to impose multi-layered control structures will become real and pressing. Remarkably, the configuration spaces come equipped with a built-in structure of the *small-cube operad* [75] action. This categorical notion proved extremely useful in many parts of topology, but also perfectly captures the intricately interwoven homological structures of small subswarms of the whole: the topological invariants of the whole can be constructed by operadic actions on the topological invariants of the parts. This connection of the small cube operads to swarm motion planning was first noted and operationalized in [14].

2.1 Composing hybrid systems (RCA1)

Composing hybrid systems presents bewildering challenges already at the level of the taxonomy of constraints on the components. One can deal with rigid, or holonomic constraints (robotic joints), one-sided constraints, typically forced by the spatial exclusion requirements, soft constraints, when the pliancy of components makes configuration space infinite-dimensional shape spaces, differential (non-holonomic) constraints etc. – let alone the puzzle of how they can be interoperably well-formed.

At the basic level of abstraction, compositions within a category can be defined using categorical pullbacks [131], reflecting the intuitive notion of products subject to compatibility con-

[6] Betti numbers are the simplest numeric topological invariant of a space: the ranks of the (co)homology groups of the space over a field. Our o-minimal setup ensures that the choice of a specific homology theory is irrelevant.

straints: say, if M_1, M_2 represent configuration spaces of some subsystems equipped with limbs connected by a common joint (with the configuration space Z), the pullback with respect to the endpoint maps $f : M_1 \to Z, g : M_2 \to Z$ would represent the composition of the subsystems connected at the joint.

In the situation of the dumbbell system described above, one would have the configurations of the balls, as well as the target space to be equal to (principal homogenous space for) the group of Euclidean motions of the plane, $M_1, M_2, Z \cong \mathtt{Eu}(2)$. The mapping f would be obtained by the applying the shift by l of the internal frame of the ball 1 along the x axis (which we can assume is aligned with the bar connector); g would be the identity on the internal frame of the ball 2. The pullback is the configuration space of the dumbbell, and the cotangent space functors (needed to formulate dynamics) can be derived without difficulty.

This construction is just the first step to define the product in the category of *hybrid systems*: our definition requires to equip it with a *branched covering E* describing the logical structure of the system, and a collection of reset maps.

While the construction of the *spaces* supporting this hybrid dynamics is straightforwardly natural in categorical terms, the *dynamics* is not and is heavily dependent on the semantics (e.g. on the relative masses of the balls).

Replacing the rod connecting the rigid bodies in the dumbbell by the exclusion constraints can be achieved using the *combinatorial species calculus* [2, 25]. This would produce a categorical representation of the swarms. Interpolating between the dumbbell (the 0-degrees of freedom joint between the subsystems) and the spacial exclusion (with $D = \dim(\mathtt{Eu}(d))$-degrees of freedom, - i.e. no constraints) through the familiar prismatic, revolute etc joints (and their pliant versions) provides for a rich formalism of describing swarms of interacting objects.

Further generalizations, for example, the universal objects with respect to more complicated (and more relevant to engineering purposes) diagrams describing various compositional patterns, would yield a robust framework for characterizing compositions of hybrid systems.

Yet their application presupposes a challenge: what is the right description of the hybrid objects? How can one encode the constraints *beyond* the exclusion or coincidence (prevalent in many systems, like soft robotics, or biological computing)? How can one account for non-instantaneous, hysteresis-like constraints observed in many living organisms? Does the notion of the *hybrid space* provide the ontological framework expressive enough to accommodate the engineered and natural systems one might want to analyze, yet allowing for the generation of models that can be handled computationally?

A structure working across the stack, from abstract dynamical systems to formal specifications is missing. In a quest to build one, we plan to pursue two contrasting approaches.

2.1.1 Hybrid Spaces as Ringed Spaces

Our *algebraic* approach appeals to the venerated tradition exploiting the natural correspondence between the category of the rings of continuous functions on a topological space, and the category of topological spaces [53, 105]. Clearly, to address the dynamical systems, especially those interfacing with classical- or quantum-mechanical ones, one needs extra structures (hybrid fibration, on top of the symplectic/Poisson structures on the phase space, singular or not, flows and so on). The resulting formalisms are necessarily complicated, but powerful, and we plan to invest significant effort to build upon them.

Singular spaces that is, spaces that are not manifolds (where the local model fails to be a Euclidean space), are ubiquitous in science and engineering. They arise in two principal ways: either by factoring out the symmetries of the system (see, for example [125]) or thanks to non-generic or non-generically interacting constraints [11].

The topology of hybrid systems under equivalencies (most frequently emerging when a group of symmetries acts on the system) is of the utmost importance both for applications and from the foundations viewpoint. On one hand, symmetries, via Noether's theorem, are the key supplier of invariants in Hamiltonian systems; on the other hand, symmetries of the compositions translate into deep results about existence or nonexistence of trajectories in particular symmetry classes [50]. Systematization of hybrid equivalence classes under deformations and reparametrizations – in particular the search for moduli spaces of hybrid systems – represents a key antecedent to any sound method of hybrid systems synthesis.

Let us illustrate the emergence of singular spaces under the symmetries on the example of the swarm system on the plane. Superficially, this space is a Euclidean space $(\mathbb{R}^2)^n$, where n is the total number of agents (we ignore the exclusion constraints here). In many situations the specific function of the formation depends on the shape of the formation as a whole (compare with [86]), not to particular orientation of the shape (say, the convex hull of the agent has a prespecified area and girth). In this situation, the configuration space of the swarm would be the quotient of $(\mathbb{R}^2)^n$ by the rotation, which can be shown [83, 87] to be a cone over the $\dim_{\mathbb{C}} = (n-2)$ complex projective space. In other words, the relevant configuration space is not even a topological manifold (for $n \neq 3$).

Singular spaces also appear unavoidably in hybrid systems when there are no symmetries to quotient by: namely, when the intersections of the guards are singular or, even worse, the guards themselves are singular. Extant hybrid dynamical systems literature largely ignores dealing with the singularities adopting a flabbier C^0-category to work in, sacrificing the access to the homotopy-equivalent structures that is key to our approach.

However, to build the theory of dynamics, calculus of variations, optimal control, one needs the differential geometric apparatus of vector fields, flows, differential forms, jet bundles and tensors adapted to the setting of singular spaces.

Our approach relies on the *differential spaces* in the sense of Sikorski (see [126] and references therein) that is very general, yielding an axiomatic, type-theory-flavored constructions which allows for potentially efficient codification and verification protocols.

A differential space is a topological space with a sheaf of C^∞-rings. (A C^∞-ring is an algebra over a certain Lawvere theory [30].) Compositionality is hardwired into the C^∞-ring as it by definition respects compositions with functions from $\mathbb{R}^n$ to $\mathbb{R}$, for and n.

Differential spaces form a category. One can define vector fields on differential spaces, with unique maximal integral curves [126] (under minor conditions). The differential calculus, however, is not fully formalized: contractions, Lie derivatives and the rest of the Cartan package [43] still need to be defined, as of now.

Working out Cartan package would help with defining operational versions of Lagrangians and Hamiltonians, the Euler-Lagrange and Hamiltonian vector fields, and related concepts for the differential spaces, the key for using these abstractions in our definitional pipeline.

One might question the adoption of the differential spaces as our foundational formalism given these deficiencies. We believe that overcoming them is well within reach, and that once done, the resulting apparatus would be a powerful counterpart to the topological affordances provided by the tameness.

To achieve that we plan to deploy Dubuc and Kock's theory of Kähler differentials for C^∞-rings [54] and develop the C^∞-algebraic analogue of Grothendieck's algebraic differential forms on differential spaces. (Our construction will be *conservative*: in the case of manifolds with corners we will simply get ordinary differential forms.) A version of Stokes' theorem would follow, and by reformulating the ordinary Cartan package in terms of sheaves we will be able to obtain the desired toolbox for differential spaces, vastly expanding the kinds of phase spaces where continuous time dynamics can take place, on guards, on reset maps and on the interactions of the guards.

Our toolbox would also enable the *Lagrangian formalism*, and a principled approach to variational problems on hybrid spaces leading to geometric mechanics, geometric control and optimal control in a vastly more general setting.

To illustrate on our model example, recall that the configuration spaces of linkages have been studied intensely for centuries [41, 82]. These spaces can be arbitrarily complicated [80], so the emergence of singularities there is all but assured. There is a body of work on *kinematics* on such spaces, but dynamics, precisely for the singular nature of the configuration spaces (precluding a natural formation of the phase space), is lacking.

The construction of the cotangent bundle of differential spaces out of Dubuc-Kock Kähler differentials would be opening the door to the coordinate-free derivation of Euler-Lagrange equations and optimal control techniques, and their computational implementations.

2.1.2 Hybrid Spaces from Trajectories

Our alternative, data-driven topological formalism for describing, analyzing and synthesizing hybrid systems abandons their carrying space entirely, adopting instead the empirical samples of input/output trajectory fragments as the primitive of the construction, alongside their automatically registered gluing properties. Such adjacency-structured observations now serve as alternative system descriptors, respecting which desired properties can be deduced and manipulated without the need for any parametric model. This can be viewed as a highly nonlinear analogy to classical realization theory relating internal and input/output systems models.

The "fundamental lemma" of [106, 147] asserts that observing a generic trajectory of a (controllable) linear system would be enough to reconstruct the system itself (up to a linear transformation). This simple yet powerful idea is extremely enticing from the viewpoint of the data-driven system modeling, as it promises a precise recovery of the model from partial observations (from, perhaps, even different trajectories). This apparatus, in line with the general ideology of Willems' behavioral approach [154] is essentially confined to the linear category.

By the contrast, the general paradigm of Cohen-Jones-Segal (CJS) [44] is intrinsically nonlinear. Their setup deals with a gradient-like system (M, f, v), where M is a compact smooth manifold, f a smooth function (with isolated critical points), and v is a vector field quasi-gradient with respect to the function f.

These data engender a (topological) category $\mathbf{C}(M, f)$ whose objects are the critical points of f, and the morphisms are (the closures of) the families of the trajectories connecting the corresponding critical points. The composition of the trajectories is defined in the obvious way.

This modest construction reveals, quite spectacularly, the structure of the underlying configuration space. Namely, if one forms the *classifying space* of this category $\mathscr{B}\mathbf{C}(M, f)$ [122] using the standard tools of homotopy limits (essentially glueing together the spaces corresponding to morphisms, and identifying them in the way dictated by the combinatorics of the arrow compositions). This classifying space turns out to be homotopy equivalent to the underlying phase space M, and even homeomorphic to it, if the flow v is generic enough [44].

The applied, engineering interpretation of this result is, of course, to consider it as the full-throttle non-linear analogue of the Willems' fundamental lemma[7], allowing one to reconstruct the phase space from observations. Remarkably, the classifying space constructions on which the CJS apparatus relies can be made fully operational within computational topology framework (such as [55] or [51]), allowing one to reason about the topological spaces, for example using the apparatus of topological data analysis and persistency, making classifying spaces for the topological category of the trajectories of a Morse gradient field manifestly computable (as long as one is able to recover the spaces of the trajectories $\mathcal{T}(p, p')$).

This latter step is certainly feasible, if one deploys, in conjunction with this apparatus of homotopy limits, the ideas of persistency. In the persistency setting one works with filtrations of simplicial complexes (derived using an efficient computational pipe from the observations, usually of a point cloud) which are fed through the machinery of persistent homology, serving as a *topological filter*, and allowing one to deduce the topological invariants of the space [8] that survive small perturbation.

Of course, as stated, CJS construction is covering only a limited kind of dynamical systems, the quasi-gradient closed loop dynamics. The successful adaptation of this construction to the needs of the hybrid system theory would require the generalizations *recurrent, non-gradient* systems. The apparatus of gradient 1-forms developed in [60,112] is the most natural to use. The incorporation of the hybrid dynamics extensions, involving guards and jumps should present little difficulties.

We argue that all of these features can be incorporated into the CJS framework, resulting in a novel, powerful suite of algorithms (and theorems supporting their usage) that would derive from empirical data (observations of the trajectories of a hybrid system) the topological model of the underlying configuration space (state model) and a robust representation of the dynamics on that configuration set.

Conceptually, the observed (fragments) of the trajectories should be represented as a point cloud X: each datapoint is a observed time series. Using, if necessary, the Takens' embedding approach [136], one can create the proximity relation between subfragments of those trajectories, and build a Rips (or Čech) complex over those trajectories. The long bars [124] of the resulting filtered simplicial complex can then be interpreted as the homologies of the underlying configuration space.

The resulting persistence structure comes with several additional structures that would facilitate the analysis: the *bi*-filtration (where the parameters are the proximity and the duration of the time of proximity). Further, the resulting simplicial space carries a structure of the discrete

[7] Presumed disconnect from the Willems' result, which deals with the *control* system disappears once one realizes that the Fundamental Lemma remains valid if one restricts consideration to the *constant* controls, as long as the sample in the (state space, control space) pair is spanning. A direct analogue in our nonlinear setting would be observations of sufficiently many trajectories of rich enough family of *switched* trajectories.

[8] We use the notion of topological invariants somewhat loosely here, as the desired level of the morphisms the invariants are supposed to survive depend on the context of the problem. At a bare minimum we assume homotopy invariants like singular homologies; quite often one can extract invariants with respect to homeomorphisms (like the dimension) or even more subtle characteristics, like the products and other homological operations.

vector field [121].

We remark that the algorithmic side of generating the simplicial complexes and the filtrations is well understood: on one side, by the thriving community of researchers building the persistence-oriented software; on the other side, by the extremely efficient algorithms of the identifying similar fragments of the trajectories [58].[9]

2.1.3 Merging Path and State Spaces

A glimpse of synthesis between the path space approach of Section 2.1.2 and the aimed for formalization of Lagrangian dynamics for the robotic artifacts can be seen in the example of the partial but effective formalism developed recently in [46].

A body of theoretical and experimental work for both steady state [47, 101, 118] and transitional [48, 79, 138] regimes in robotic locomotion has been encompassed recently by a class of hybrid systems [78], ensuring consistent (i.e. non-blocking, deterministic) executions A version of this class of systems has been incorporated in [46]. Here a hybrid dynamical system, or rather a version of Morse-Conley decomposition of it is considered as a small category, with the objects being the repellors and attractors. Together they form a certain *double category* whose vertical morphisms represent hybrid semi-conjugacies (which inter alia admit a formalization of the Template/Anchor hierarchical compositions [56,61,74]), while the horizontal morphisms formalize and generalize the notion of sequential composition [36].

While the relation of the (homotopy type of the) classifying space of the 2-category constructed in [46] to the underlying hybrid space is by design almost tautological, the existence of that structure provides a preliminary proof of concept and a testing ground for the synthesis of the path spaces and the differential spaces thread.

These empirically meaningful compositions, while only recently formalized within in a coherent theory, together capture extensive efforts of empirical and provably correct descriptors of robot behaviors resting on the systematic design and control of dynamical attractor basins [88]. With certain caveats, such hybrid systems admit a version of Conley's fundamental theorem [93], vindicating the intuition that physically constructed attractor basins represent a broadly generalizable sound step toward the grounding of software abstractions in the sensorimotor streams of empirical hardware [89]. We anticipate that resetting the physically inspired category within the framework of differential spaces will provide an effective opportunity for doing so.

From either perspective, a properly reformulated version of double categories offers a tantalizing formalization of the decades of physically grounded theorizing and behavior building for our expanding fleet of dynamical robotic machines [84] that begs for generalization to the far broader class of Lagrangian – and perhaps the even more general "port-Hamiltonian" – systems [146].

2.2 Invariants of hybrid (sub)systems (RCA2)

At some level of abstraction, invariants of dynamical systems (open or closed) can be defined as simulations relative to archetypal models: trivial dynamics for integrals of motions; attractive dynamics on the real line for Lyapunov functions, and so on. However, upgrading even these baseline models to their hybrid versions incurs complications (e.g., failure of conservation laws

[9] As a side remark, the proximity detection pipe would require a thorough analysis of the metric on the hybrid trajectory spaces; probably the popular "graphical convergence" should be replaced by more natural (and classical) Skorokhod topology [76].

consequent upon energy pumped in or out at the resets). Generalizing these results to open systems and more intricate invariants is challenging using the existing hybrid system formalisms.

2.2.1 Category Theory Tools for Stability and Safety

The categorical lens offers a systematic way to address long term behavior of the controlled dynamics, encompassing classical properties, such as stability, or avoidance (safe region containment), generalized to open hybrid systems [6, 73], along with such constructs as Control Barrier functions [9], etc.

Classically, Lyapunov theory provides a powerful method for certifying the stability of nonlinear dynamical systems and for the synthesis of controllers via Control Lyapunov Functions (CLFs), the control variant of Lyapunov functions [127]. CLFs have also proven powerful for hybrid systems [7], in particular for hybrid system models of walking robots wherein they led to provably stable walking that can be realized experimentally [63, 104, 113]. Lyapunov functions can be best understood as a mapping from the dynamical system of interest to a simplified sketch within a class of stable systems (1-dimensional systems described by class $\mathcal{K}$ functions). In fact, these functions can be framed categorically, as inspired by the corresponding notions for Zeno stability [8], by noting that a $\mathbb{R}_0^+$-valued function is positive definite if the equalizer of said function and the zero function is the equilibrium point of interest. Thus, in an appropriate category of dynamical systems, we can consider a *Lyapunov morphism* to be a morphism that maps to the subcategory of the simplest stable objects that satisfies the universal property that the point or set we wish to certify the stability of is the equalizer. The categorical notion of Lyapunov morphisms allow us to generalize Lyapunov-like functions; for example, barrier functions can be framed in this setting by changing the target objects. This hints towards the "special class" of Lyapunov-like morphisms we wish to explore in the proposed work. In particular, we plan to study the role of stability and safety in general hybrid systems by considering Lyapunov morphisms in the category of hybrid systems. This will lead to the actual synthesis of controllers enforcing stability and safety in real-world hybrid systems, e.g., walking robots.

An implementation of the categorical viewpoint on the problems of safety and stability can be achieved by reinterpreting the notions of Lyapunov stability[10] as a *no-escape* condition: the totality of trajectories starting in small vicinity of the equilibrium and ending outside a prespecified vicinity is empty. This opens a testable approach to stability for (arbitrary executions of) the *open systems*, via a topological characterization of the requirements on the collection of safety (no-escape) certificates ensuring the coverage of the space of trajectories by the collections of subsets where those safety certificates are true.

2.2.2 Topological Invariants of the Hybrid Spaces

Forgetting about the dynamics in our Definition 2 leaves one with the branched covering $G \to E \to M$, and the reset maps R. This setup allows one to form the *diagram of spaces* (R, i) : $G \rightrightarrows E$ (here R is the reset, and i the embedding maps). We note that the diagram can be quite complicated if the guards and the space E have multiple components.

In the setting we can apply the standard construction of the *homotopy limit*, and derive a *model* $\mathtt{M}_{E,G}(R)$ *of hybrid space*.[11] We claim that the *homotopy type* of this space captures signif-

[10] Which in closed loop dynamics is just a statement of continuous dependence of the trajectory over $\mathbb{R}_+$ on its initial value.

[11] The construction of the homotopy colimits is not complicated but somewhat tedious, and we do not provide here any details; the treatises [151, 157] provide a very readable account. We remark here only that the fundamental

icant information about the hybrid *space*, imposing significant restrictions on any *dynamics*[12] supported by this space, especially if the information about the initial configuration and the task (goal) is taken into account.

Consider the simple example of a pair of vehicles about to merge lanes (see Fig. 1), a cause of many concerns in the industry [150]. The physical configuration space is a plane with a ray on the diagonal removed; the hybrid space is a branched covering. Topologically, the configuration space modulo the task region is homotopy equivalent to a circle; this means, in particular, that that there is *no continuous feedback control taking any initial configuration to its goal* [128].

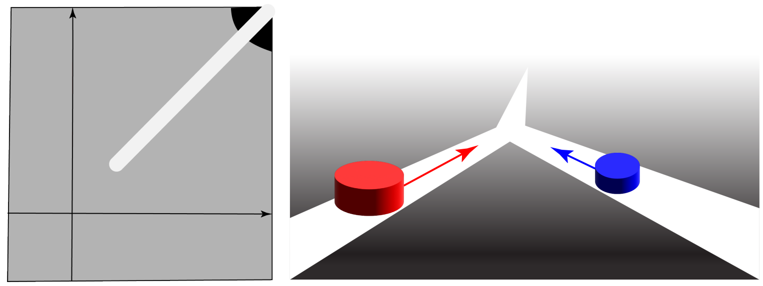

Figure 1: The underlying physical space M is not contractible (modulo the space of initial configurations, depicted as the black region on the left display), hence it cannot be covered by an unramified hybrid space: the internal logics is nontrivial.

The hybrid space is fibered over the configuration space, with the fibers representing the logics of the process. Intuitively (this intuition is easy to support it this situation), the hybrid space should have two components, corresponding to the order in which the vehicle follow each other after joining the single lane. Moreover, it is immediate that there is a nontrivial part of the physical configuration space to which both components project: the *topological perplexity* forces nontrivial collision resolution done through an interaction between the vehicles, and moreover, this resolution process is unavoidable [20].

One important feature of the reasoning above is its full independence from the actual realization of the hybrid dynamics: it is valid for any feedback compatible with the underlying open loop system targeting the indicated regions. The homotopy limit of the resulting hybrid space is that of the unit segment modulo its midpoint: contractible, hence stabilizable [128].

Given the same underlying phase space structure, the semantic dynamics, - which was codified in the example above by the fixation of the target subspace of the phase space, - can be vastly generalized by requesting a particular attractor-repeller structure encoded, traditionally, by Conley decomposition. While the goal of Conley's theory was to create a simple sketch describing an existing dynamics, modeled on the Morse-Smale decomposition, we consider a Conley structure as a *pseudo-code* of the desired behavior [117], *imposing* it on an open hybrid system, and seek a realization through a closed loop, feedback control, compare [93].

Of course such a control might not exist: the quest of looking for the topological invariants of the hybrid space and underlying topological category is one of the key thrusts of this project.

In general, the homotopy type of the homotopy limits of the diagrams of spaces encoding the structure of the hybrid space is a hard to quantify structure[13] which is usually replaced by far easier to compute invariants, - cohomology rings.

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assumptions on the hybric spaces imply that all existential restrictions (such as the defining maps being cofibrations etc) are satified, by default.

[12] Remark that the model $\mathtt{M}(R)$ does not depend on the flow, only on the attachment data specified by the guards G and resets R.

[13] A complete descriptors involves a sequence, in general, infinite, of spaces, intricately attached to each other, - hardly a usable construct.

Relatively efficient computational pipes for deriving the (co)homology groups for various classes of spaces (primarily simplicial and cubical complexes [40, 55, 158]) do exist, but several significant improvements for our settings of branched fibrations in some computationally feasible o-minimal structure will be needed.

We note that the topological formalism outlined above was implicitly surfacing in various hybrid and discontinuous dynamic settings through intrinsic space gluing procedures.

As functoriality is one of our guiding principles, understanding invariants of hubrid systems through their morphisms into a simple(r) hybrid system, one immediately seizes upon the notion of the homotopy type of a dynamical (hybrid) system, a promising yet still inchoate direction of recent work. This viewpoint becomes particularly attractive in the trajectory-based open-loop formalism of RCA1, promising both a data-driven generalization of the Cohen-Jones-Segal (CJS) phase space reconstruction, as well as direct representation of practicable analytical or synthetic expressions (e.g., safe execution, or reachability) as native trajectory space invariants.

2.2.3 Dynamical Invariants of Hybrid Systems

Deriving discrete, symbolic dynamics from continuous, is one of the critical steps in any digital implementation of control systems. Such discrete dynamical systems simplify dramatically the representation of a system's behavior but require significant effort to ensure their properties reflect the relevant patterns. Despite significant recent foundational progress (e.g., in research on the persistence of stability under control quantization), other key properties remain widely open.

Conceptually, the *uncertainty* is quantified by the variety of executions feasible for the given dynamical system, given the past. Various concepts proposed to capture this intuitive notion, including ergodicity, mixing, minimality and others [45]. However, the most universal notion carrying significant uncertainty quantification power is the idea of *entropy*.

In the systems with underlying probability measure preserved by the dynamics, the entropy captures the typical complexity of the execution paths (as introduced by Shannon, who relied on the Gibbs' foundational ideas).

By contrast, *topological entropy* bounds from above any metric entropy and is a purely dynamic concept that describes the growth rates of any informative symbolic representation of the dynamical system.

Symbolic trajectories for continuous dynamical system emerged as critical tool for strongly hyperbolic systems [1, 3], but were quickly recognized as an essential tool in other areas of dynamics. In that approach, a continuous trajectory is encoded by a (potenitall bi-infinite) word listing which of the regions (into which the phase space is partitioned). If the encoding is *faithful*, i.e. identifies the continuous trajectory (up to a shift), then the exponential growth of the size of the dictionary (the number of distinct symbolic words on length n) coincides with the topological entropy of the system.

The uniformly hyperbolic dynamics is the cleanest example of the *closed loop* systems where the topological entropy is well understood; the apparatus of Markov partitions reliably provides for topological conjugations between the smooth and symbolic dynamics. These Markov partitions are intrinsic to the system, but need to be constructed, and do not have a physical semantics.

By contrast, in systems with impacts, and in the hybrid systems, the semantically relevant symbolic trajectories present themselves readily as the sequences of traversals of the guards, see e.g. [12, 17]. The resulting complexities of the symbolic trajectories are often informative even when their entropies vanish.[14]

The mismatch between the *intrinsic* topological entropy of a hybrid dynamical system, and the complexity of the observed symbolic trajectory is a highly nonlinear analogue of non-observability for linear systems.

In most applications, one can augment the intrinsic partition of the trajectory space into the classes of continuous trajectories classified by the sequence of guards they traverse, with additional semantically interpretable data delivered by various sensors. This creates a *filtration of partitions*[15], starting at the *guard trajectories* and being refined; the resulting collection of *sensor entropies*[16] is bounded from above by the topological entropy and serves as an important invariant of the hybrid system.

Returning to our example of the dumbbell, in the system where it bounces off a moving platform, the natural hybrid symbolic sequence encodes which of the balls hits the platform. One can easily construct substantively distinct trajectories resulting in the same symbolic hybrid dynamics (say, when only one ball hits the platform, while the other never does). To distinguish between such trajectories one would need to introduce a semantic sensor ("roll"), registering the instances when the dumbbell is horizontal; that would provide an intrinsic characterization of the trajectory.

The concept of entropy allows one also to address some of the other notions specific to hybrid dynamics, clarifying their nature and offering ways to understand them in a principled way. Thus, the standard example of a bouncing ball dissipating energy is characterized by the vanishing topological entropy of underlying dynamics. Here, despite the infinite length of the symbolic trajectory encoding the continuous dynamics, it can be effectively *finitely recoded*, resolving the paradoxical behavior. One can prove that in the PL category for the hybrid systems, the Zeno phenomena always result in vanishing topological entropy, thus leading to effective resolution of the paradoxical behavior. Generalizations to other o-minimal categories is among the problems this project will address.

An interface between the path spaces and the topological entropy apparatuses can be seen in the hybrid system reinterpretation of the archetypal models for chaotic system, such as the Lorenz system [99]. The template model that was introduced by Birman and Williams [28], and rigorously proved by Tucker to be conjugated to the attractor of the original system [141] can be seen as a simplest hybrid system. Its phase space is the unit square; the flow is given by the horizontal constant vector field $\partial/\partial x$; the right vertical segment of the boundary is the union of the segments, representing the guards, and the reset map sends piece-wise linearly each of the two vertical half-segments to the left boundary of the square.

The resulting branched surface is the archetypal model for our hybrid branched covering E; its stratification fits the paradigm of tameness, and the resulting dynamics can be seen to be easily recovered using the CJS formalism, for appropriately chosen fragments of the observed trajectories.

In the context of topological entropy and complexity, the fact that a version of the Lorenz system, with slightly more complicated branched template (still 2-dimensional) embedded into

[14] In fact, in some situations the low complexity allows one to completely characterize the underlying continuous dynamics [13].

[15] Conceptually similar to the formalization of accumulation of information provided by a filtration of σ-algebras.

[16] Compare the hierarchy of robotic and sensing systems abilities [94].

the Euclidean 3 space, can have any knot (isotopy class of embedding of circle into $\mathbb{R}^3$) as its closed trajectory [67] is especially interesting. This universality exhibits intrinsic complexities hidden in the simplest hybrid systems, and the fact that the tameness of o-minimal structures we take as our fundamental assumption precludes neither chaos, nor intricate topological phenomena. One can see a prototype of the modular hybrid space composition in the implicit, yet highly operational apparatus of templates that this theory developed [156].

2.3 Rules of and obstacles to composition (RCA3)

Defining categories intricate enough to capture real physical semantics of assemblies of objects (that in the simplest situations are described by the fibered products/coequalizers) is highly nontrivial as evidenced by category theoretic hybrid systems models developed by different members of our team: they all still miss various key desirable features. Whatever the formalism, merely casting subsystem interactions in terms of some high-level language does not yield a usable, scalable computational pipeline. Turning, for example, Hamiltonian descriptions, even the most basic model of dynamics at the interface of two media with different Hamiltonians yields a hybrid (Hamiltonian!) system whose dynamics is still not properly understood.

2.3.1 Networks of Hybrid Differential Spaces

One of us introduced in [96] an approach to hybrid *open* systems, their networks and maps between networks creating a blueprint for building a larger system out of smaller subsystems by specifying a pattern of interactions between subsystems—an interconnection map. Maps between networks allow one to produce maps between complex hybrid dynamical systems by specifying maps between their simpler subsystems. To actually implement this program, however, we need to redesign the techniques of [96] using the differential spaces setup described above: the approach thus far was assuming that the configuration spaces are manifolds with corners.

Once this is done we intend to revisit [46] and replace its somewhat *ad hoc* concept of a "smooth sets" developed there with differential spaces.

A broader conceptualization of the notion of networks of open systems formalized as a *double category* was introduced in [95], building on earlier work with DeVille on coupled cell networks [50] and on the work of Lerman with Vagner and Spivak on the operad of wiring diagrams aimed to bridge the notion of a network of dynamical systems as a decorated graph and as a morphism in a colored operad [143]. That work relied heavily on Brockett's definition of an open system as a surjective submersion $p : E \to B$ and a morphism of bundles $F : E \to TB$ [33]. Local triviality of the fibration p allows one, again, to deploy categorical pullback constructions defining compositions (products) and interconnections. However, when one attempts to apply the same circle of ideas to Hamiltonian systems (in which case we assume that the spaces B, which are smooth to start, and differential in full generality, carry a symplectic structure), the situation becomes trickier.

How does one *interconnect* these open systems in a generally contravariant, functorial way? A naive attempt to use functions on one of the spaces as parameters of the other quite easily leads to the necessity to adjust symplectic structure in some ad hoc way. Similar problems arise with Lagrangian formalism.

The issue is well-known in the control theory community, see, e.g. [39] and is usually attacked using the popular formalism of *port-Hamiltonian systems*, see, for example, [16]. There is also a categorical approach to port-Hamiltonian systems; see [103]. However, the heavy dependence of port-Hamiltonian systems on a fixed $q - p$ split of coordinate system (or, more generally, on a Lagrangian polarization) makes port-Hamiltonian systems ill-adapted to our desired level of generality. While one may be able to think of the tangent "bundle" of a differential space as a sheaf, this sheaf is not locally free and not even quasi-coherent [81].

Returning to our model system of multi-agent systems, one can introduce a collection of links, some rigid, some having less than full degree of freedom, some soft (which one can interpret as a software enabled Control Barrier Function precluding collisions). While the unconstrained bodies collectively move in the configuration space $\mathtt{Eu}(3)^n$, their composition, as we mentioned above, can have highly nontrivial topological and local differential-geometric structures. The individual coordinates of the bodies become highly interdependent, illuminating the necessity of functorial structures enabling Hamiltonian or Lagrangian dynamics, going far beyond the existing approaches.

Therefore, understanding the category-theoretic content of such a composition operation would be critical for this RCA. We expect that a colored operad (and a symmetric monoidal category of underlying the operad) which would give us the *pattern* of these operations and a language for reasoning about the operations. We will know that our category theory is on the right track if we can see that our construction applies to some nontrivial examples of complex Lagrangian (or Hamiltonian) systems.

Other obvious complications include compositions of stable systems that can be unstable (a standard example is the emergence of biological cycles); or, conversely, compositions can stabilize unstable subsystems. Analysis of such transgressing properties of composites through the category theory is key to our project's success.

2.3.2 Limbs, Bodies, Swarms as Networks of Hybrid Systems

From the control-theoretical perspective, the caveats permeating compositions of the dynamical system has been known for decades. Already [142] showed that carefully arranged logical switching compositions of abstract stable dynamical systems can result in overall instability (as well as the opposite – overall stabilization from logically composed unstable constituents).

Hamiltonian dynamics, with its conservation laws, can engender only rather weak versions of stability, dependent, in classical mechanics, on rather subtle settings of KAM theory (unless some integrability can be ensured). However, within the last two decades it has been discovered that very natural lossless physical models [119] can yield asymptotically stable hybrid dynamical systems when composed via guards and resets whose implementation on working robots [118] requires purely mechanical feedback [4].

Notably, these analytical results and their empirical corroboration in robots were motivated by careful biomechanics models [92] and ingenious animal experiments [77] that demonstrated the reality of "preflexes" [34] – animal feedback strategies implemented by the arrangement of body parts and materials with no need to recruit neural pathways. But such fascinating and technologically valuable examples of "mechanical intelligence" [129, 130, 144] presently arise from inspired biological observation or specific metamaterial design domains [133, 134] and continue to resist general algorithmic integration of form with function. We are convinced that the abstractions of category theory offer the promise of such unification whereby, for instance, categorification of robot co-design problems [38] might intersect category theoretic representations of hybrid behavior [46].

The topological dimension of the composition problem can be best seen in situations of the iterative interactions of subsystem exceptions, such as exclusion constraints in multi-agent systems or protein dimerization. Back-propagation of the trajectories leading to the reset conflicts (e.g., encoding which of two vehicles clears a narrow passage first) leads to a combinatorial explosion, precluding efficient verification of implementations. A practical, context-aware way to address these patterns of complex interactions of subsystems is to create a toolbox of application domain-aware algebraic-topological abstractions. For example, the basic high-level abstraction for the systems subject to exclusion is the small-cubes operad (an abstraction at the overlap of the topological and categorical foundational domains; heavily used the team's prior work in the area [14, 15]).

2.4 Compositions and Invariants (RCA4)

In an ideal world, invariants of the components should translate into the invariants of the composites, if the formalism is chosen correctly. The (heuristic, at this stage) functoriality of the composition process poses strong constraints on the approaches to be taken, biasing the candidate constructions towards natural, intrinsic formulations.

2.4.1 Topology of the Spaces of Trajectories

The topology of the spaces of trajectories in the composites is the proper, structural way to address many problems of engineering design, that are often solved in an ad hoc fashion, relying on the analytic structures ultimately ill-fitting the task.

One prominent example of such a problem deals with *stability* of control systems. While traditional approaches focus on state-space based certificates, i.e., Lyapunov functions, it is often advantageous to abandon the state space perspective in favor of a trajectory-based perspective.

In the state space view, the fact that a composition of unstable dynamical systems can be stable seems magical. From the space of trajectories perspective, the stability of the composition can be addressed through the following observation: the *lack* of stability (of, say an equilibrium) is the existence of trajectories that connect points arbitrarily close to the equilibrium to an exterior domain. Therefore, the stability is equivalent to the *absence* of such undesirable, divergent trajectories in the composed systems. Interaction of a system A (with unstable trajectories) with a similarly unstable system B can render the set of them divergent trajecties in the composite empty [155].

Compare this situation to topological approaches to concurrency theory where the trajectories in question are execution paths (and their disappearances are the dreaded *deadlock* phenomenon, now resolved by means of homotopy theory, see [65, 72] after decades of CS research in the area [120]). From our perspective, the disappearance of the escaping trajectories violating stability is *desirable*, but the same arsenal of tools is applicable (similarly to the viewpoint of the Section 2.2.1). The focus in this paradigm shifts to the analysis of the invariants of the spaces of trajectories in the composed (open of closed) systems, addressing the topological properties of the spaces of trajectories by homotopy theory tools.

2.4.2 Homotopy Type of the Spaces of Trajectories

Trajectories hitting a particular unwelcome part of the phase space are a frequent menace of the hybrid systems. Besides the usual zoo of the excluded regions (dictated by the safety protocols, collision avoidance and other causes dictated by the nature of the system), the analysis of hybrid systems, especially those resulting from the compositions, needs to deal with the interactions of the guards inherited from the components in the composite system. Indeed, if the trajectory approaches the point of the intersection of several guards, the potential ambiguity of which of the guards will be hit first, can lead to difficult problems in the modeling of the systems [35, 111]. Similar difficulties arise when the trajectory is near the boundary of the guard: as an example, this is how the Zeno-type phenomena emerge.

Understanding the impact of the excluded trajectories (those that hit the obstacles[17]) is one of the pervasive problems in the hybrid systems theory.

Topological Perplexity It is important to emphasize that obstacles can be forced on the system not just by the explicit constraints present in the model, but also for topological, or category-theoretical reasons. Consider the standard task of designing a feedback control in an open system that would steer the system to, and stabilize it on a flow-invariant set, an *attractor* $\mathscr{A}$. (In applications one encounters most frequently the attractors that are an equilibrium point, or a limit cycle.) One typically requests that the attractor is globally asymptotically stable in the usual sense: for any open vicinity of the attractor, trajectories starting close enough to the attractor never leave that vicinity, and every trajectory of the flow approaches $\mathscr{A}$ asymptotically.

Even in the simple situation, when the configuration space is a smooth manifold with boundary $M \supset \mathscr{A}$, preserved by the flow, some global constraints might force the discontinuities of the steering algorithms. Thus, it is well known that in general a *continuous* feedback control stabilizing on $\mathscr{A}$ might fail to exist [90, 114, 128]. Indeed, under some mild regularity conditions, the existence of such a continuous feedback would imply that the resulting vector field retracts M onto $\mathscr{A}$, and in particular, that the configuration space M and the attractor have the same homotopy type, a strong topological constraint.

We note that the constraints already mentioned, both derived from the nature of the system (such as collision avoidance) or by the algorithmic implementation (guard overlaps) often lead to the situation when the topology of the feasible part of the configuration space is quite complex (say, in the simplest example of the classical configuration spaces describing spacial exclusion of n convex bodies, the total Betti number is $n!$, a superexponential growth), and the assumption of the matching topologies of the configuration space and the attractor are routinely violated.

Thus, a consistent, verifiable non-stalling stabilizing feedback control should have some regions where the feedback is forced to be *discontinuous* or, equivalently, where some logical conditions dictate which way to behave, so that different branches of the hybrid fibration result in distinct execution paths. (A typical everyday situation involves cars arriving at an all-way stop intersection nearly simultaneously; in this case one relies on the embedded human interaction logic, which is far from universal. An engineering design addressing this type of conflicts in a swarm of unmanned vehicles needs a far more efficient non-deadlocking resolution protocol.)

To be precise, let us refer to a closed subset $\boldsymbol{C} \subset M$ as the admissible *cut* if for some the feedback control function k defined on the excised configuration space $M - \boldsymbol{C}$ and preserving it, the resulting flow stabilizes on the attractor $\mathscr{A} \subset M$. We will be concerned with the *topology of cuts*, not their measure which can be vanishing. However, the very existence of the cuts forces the designer into deploying discrete decision flows, whose complexity can be directly related to the total Betti number of the cut sets [29, 62].

It turns out that the mismatch between the topologies of the cut attractor and the configura-

[17] We will be somewhat indiscriminantly referring to the forbidden regions as obstacles, despite the quite different semantics those regions can represent

tion space M very directly impacts the topological *perplexity* (measuring how complicated the topology of the cut is) [20] of the problem.

The results of [20] allow to derive quite powerful constraints on the total Betti numbers of the cuts, in terms of the mismatch between the topologies of the configuration space and the attractor: thus, if the attractor is a limit cycle, one has $\beta_k(C, \partial C) \geq \beta_k(M, \partial M)$ in all dimensions up to $\dim M - 1$ (all homologies are singular over a field).

Needless to say, the situation in a hybrid setting will have additional intricacies: one will need to take into account the guards and the potentially complicated combinatorics of their interactions. Proper formulation of the resulting constraints is one of the important problems in this thread.

2.4.3 Obstacle Avoidance

Among the more natural constructions arising in this context is the category theoretic construct of homotopy colimits of diagrams of spaces, allowing one to reconstruct the invariants of the composites in terms of the topology of the components and their reset maps. At the level of trajectory spaces, analogous constructions, motivated by the problems arising in concurrency via the theory of the directed path spaces also appeared recently in the control-theoretic setting.

From the perspective of the behavioral approach to the dynamical systems, one of the key questions is to characterize the topology of the space of *trajectories* feasible in a particular hybrid setting. One frequent pattern that arises in such systems is the temporally limited, often instantaneous nature of the obstacle the trajectory needs to avoid: thus, the flows typically are transversal to guards, and therefore one does not expect the trajectories to linger on them. Hence, the obstacles arising this way are experienced instantaneously. How does this impact the structure of the space of such trajectories?

Consider the archetypal example of the obstacle avoidance in the case of simple integrator: the trajectories of the system $\dot{x} = u, |u| \leq 1$ need to avoid instantaneous obstacles (see Fig. 2; the horizontal axis represents time; the vertical x).

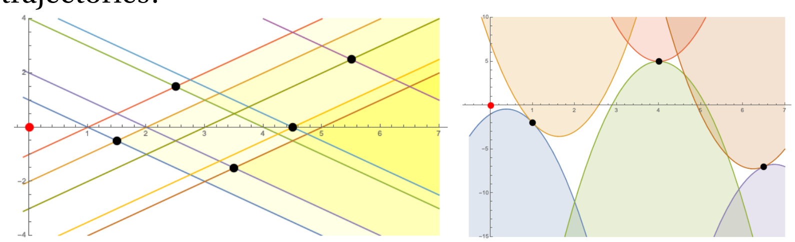

Figure 2: On the left, the space of trajectories of the simple integrator avoiding pointlike obstacles consists of 11 contractible components. On the right: the space of trajectories of the double integrator avoiding pointlike obstacles is homotopy equivalent to the product of a circle and the wedge of two circles.

The theory of the obstacle-avoiding spaces of control trajectories is emerging [21, 107], but the basic contours of it are becoming quite clear. The key observations, through approximability of the space of solutions of control problems through some exhausting sequence of finite-dimensional approximations, is that Poincare duality enables one to represent the topology in question (at least at the level of stable homotopy type, and, at the very minimum, at the level of the homology groups) via the data of the trajectories *hitting* certain sequences of obstacles (*chains*): the homology group of the space of trajectories *avoiding* the obstacles can be rendered in terms of the (Borel-Moore) cohomologies of the *union* of the trajectories *hitting* the obstacles.

Given that, one can construct a diagram of spaces [151, 157] corresponding to the obstacles. Recall that this is a contravariant morphism of a poset (or small category) $\mathscr{P}$ into the category of topological spaces. A general result implies that under quite general conditions (which are satisfied in many examples of controlled systems), the classifying space of the resulting diagram is homotopy equivalent to the union of the spaces. This, either via Segal's formalism, or through the explicit constructions of [157], implies a computationally efficient description of the homologies of the space of obstacle avoiding trajectories.

As an example, for the trajectories of the *double integrator* shown on the right display of Fig. 2, the space of the trajectories avoiding the (codimension 2 obstacles) has Poincare polynomial $1+3t+2t^2$.

2.5 Formal Verification and Analysis (RCA5)

To render continuous systems suitable for formal handling requires passage from signals to symbols (by discretization, homotopy sketches, factoring out symmetries, simplifications etc). The research directions targeted by the thrusts described above are naturally aligned with this goal. Moreover, the approaches we outlined would be amenable to enriching their respective logics with temporal and modal operators.

To achieve this, one would need to consider a significant range of potential verification tools: from (relatively) efficient satisfiability solvers, built on top of the model theories reflecting the model semantics, to automated theorem provers (the latter could be critical in addressing expansions of Hamiltonian and Legendrian formalisms to hybrid systems). Integration of the extremely rich and versatile tools from category theory, such as polynomial functors, is an important route to explore. Incorporation of the trajectory-based CJS paradigm would both expand the expressive power (allowing, inter alia, delocalized constraints), and introduction of sheaf-theoretic tools of integration of local solutions. The key desiderata also include Domain Specific Languages (DSLs) for grounding the verification techniques. The DSL expressiveness should be sufficient to describe executions and their predicates, topological type annotations and a new type-inference system based on these annotations.

In this thrust, we will investigate topological invariants for hybrid systems, develop algorithms for checking whether or not a given invariant is satisfied by a given hybrid model, and also the theory and algorithms for synthesizing inputs that make the model meet a target invariant. For example, path-connectedness of the space of solutions, gives a basic feasibility check which can be used prior to finding optimal solutions of robotic planning problems; compactness of the same space can be used for systematically finding counterexamples to safety requirements, through simulations and sensitivity analysis.

2.5.1 Computing Topological Invariants

For concreteness, consider a controlled system $\dot{x} = f(x,u), x \in X := \mathbb{R}^n; u \in U$, where $U \subset \mathbb{R}^m$ is a convex and open, and the initial state $x_o \in X$. Among all the trajectories of the system one seeks to exclude those running through a set of unwanted, *bad* states or instantaneous obstacles specified as $\mathscr{O} = \{O_k \subset X, k = 1,\ldots,K\}$, generalizing the setup of Section 2.4.2.

Our verification and synthesis questions will be related to the set of trajectories $\mathscr{T}_{\mathscr{O}}$ avoiding the obstacles and reaching the goal $x_f \in X$. For an autonomous system without control inputs and obstacles, computing the set $\mathscr{T}_{\mathscr{O}}$ is equivalent to the well-studied reachability question: does there exist any trajectory from that reaches the goal? Algorithms for answering this question for linear and nonlinear models, and more general switched and hybrid models, have been studied for over three decades, and they are the bedrock of successful verification software tools.[18]

Intertwining with this setup is the class of problems introduced in the last part of the Section 2.2.2. There, the Conley decomposition into a poset of attractors/repellors was used to create a sketch of a closed loop-system, which can then be considered as a specific topological *design request* for the feedback control (vastly generalizing the conventional requirement of a stable equilibrium, or a stable periodic trajectory). The verification tools created within this project would evaluate the feasibility of such a design specification, and its robustness with respect to imprecision of the system specifications.

Tasks and Topological Obstacles Besides reachability, many other verification questions that are critically important in applications of hybrid systems (such as stability or temporal consistency) are strongly geometric. This implies that one needs strong certificates for sets like $\mathscr{T}_{\mathscr{O}}$ (which often cannot be computed exactly), or at least for their under or over-approximated, to answer the verification question.

In this project, we radically reshape this geometric path, and target topological invariants of the space of trajectories. Correspondingly, our verification algorithms will exploit the inherent computational bootstrapping of homotopy invariants.

Here is a simple example (see Fig. 3). Consider two positions of the dumbbell: is the final (red) configuration reachable from the original (blue) one? For each of the balls in isolation, the answer is yes, but together, the configurations are separated: the obstacles form a *topological* barrier. One can show [19, 116] that this unreachibility can be expressed in terms of nontriviality of a certain homology group, resulting from a long exact sequence reflecting the interaction between the obstacles (the composition of invariants). This approach is generalizable to more complicated systems comprised of several bodies and to dynamic setting; the appropriate algebraic apparatus involves a version of Mayer-Vietoris spectral sequence [31] describing the interactions between the collections of guards. The spectral sequence can be explicitly computed, but suffers from the combinatorial explosion in terms of the interactions of the collection of guards; however in many practical situations one can prove this explosion is checked by the dimensional considerations.

Moving on, once the reachability is assured, one can recover a trajectory connecting the initial and the target states (say by configuration space exploration). However, this trajectory can be far from optimal. The (now, closed) system of optimal trajectories is again a hybrid system, with the guards corresponding to the states where one of the subcomponents hits the obstacle, resets switching between the different Hamiltonians.

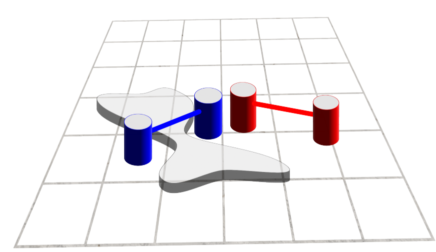

Figure 3: Each of the objects can be steered around the obstacle from the initial (blue) to the final (red) position. Under composition, however, this task becomes infeasible. This is a topological phenomenon, having implications for categorical view on safety and stability.

As this example indicates, this thrust will deal with the interaction of two aspects: algorithms for computing invariants of the configuration space, detecting the topological feasibility of the task, or, more generally, the topological structure of the space of the trajectories executing the task, and the synthesis of the controls realizing an execution optimizing a criterion of optimality.

[18]Among these tools are such as Flow*, CORA, dReal, C2E2, HyLAA and DryVR [5, 42, 59, 64]

One can think of this dual task as reflecting the Janus-like synthesis of the topological and algebraic sides of our program.

Interfaces to Computing Algorithmically, the approach we outline presents a clean-cut interface to the DSL whose semantics reflects on the underlying topology of the spaces of trajectories. Explicitly, the standard tasks or queries necessitated by the relevant applications (such as modeling and control of multi-agent swarms, or of complex polymorphic transformations in biochemical systems) can be quite universally expressed as compatibilities under the composition of the bundles of the strategies (trajectories) that are feasible over the components.

One of the key innovations of our program is the introduction of topological predicates that express properties of the configuration and phase spaces and spaces of trajectories for components, that allow one to reason about the relevant aspects of the structure of constraints and solutions in the composite. Consider the standard class of examples, where the positions of agents is detected by localized presence sensors (say, infrared interruption ones), and their trajectories are therefore evaluated. A long line of work (see, e.g. [26, 27, 139]) describe how to interpret these discrete signals in terms of the basic topological characteristics of the trajectories, such as the homotopy classes of paths (also known as the elements of the fundamental group when the starting and final points of the trajectory coincide). A DSL incorporating the predicates semantically codifying the topological information is one of the desired products of this project.

The necessity for the verification tools becomes especially transparent in general problem of the integration of topological data.

In some cases a topological predicate of a composition of systems can be simply expressed in terms of the predicates over the components. On the other hand, quite frequently the machinery of verification of feasibility requires intricate recursive constructs, like the spectral sequences we alluded to above (for example in the cases where a certificate for the *impossibility* of a section is sought). The DSLs we will be developing, relying on past research [**?**, **?**] should deliver a rich yet usable apparatus for expressing such algebraic and combinatorial constructions.

2.5.2 Topological Entropy

Topological entropy, a quintessential invariant of a closed loop dynamics, helps in the intrinsically discontinuous setting of hybrid systems for several reasons.

First, as entropy describes the rate of growth of state uncertainty (without new measurements), it also is related to the rate of measurements *necessary* to accurately estimate, contain, or control the state or its output. Different requirements give rise to different versions of topological entropy for hybrid systems. For example, in [97, 98] introduces the notion of *estimation entropy*, i.e., the minimal bit rate necessary for sending measurements of a plant's state $\dot{x} = f(x)$ to a decoder, so that the latter can construct accurate estimates of $x(t)$. These lower bounds hold across all algorithms and codes, and therefore, can take the guesswork out of communication network design. Modern fly-by-wire vehicles have hundreds of sensors, actuators, and embedded computing units distributed across its body. The software in these end-points implement the perception and control algorithms while the protocols (such as CAN, Flex-Ray, ethernet, USB, etc.) connecting the endpoints are tied to hardware and are fixed. An emerging challenge is the problem of designing the *intra-vehicle communication protocols* that will assure the system-level requirements, while the lower-level perception, state estimation, and control algorithms are updated. The topological entropy lower bounds and the accompanying encoder-decoder algorithms can serve as building blocks for this design problem.

Secondly, encoder-decoder algorithms for state estimation is a building block for functionalities like monitoring, failure detection, runtime assurance. The entropy-based bandwidth requirements transfer to those functions as well.

Finally, bounding the uncertainty in the system's state is also a core issue in verification. Entropy can also be used to get lower bounds on to the number of tests or simulation samples needed for constructing such proofs based on simulation data. This in turn can inform budgeting for simulation and testing resources needed for different modules in a vehicle system.

Therefore, we need to develop compositional data rate theorems. For example, suppose we have a chain of subsystems connected in a feed-forward fashion: $\dot{x}_{i+1} = f_{i+1}(x_{i+1}, x_i)$, for $i = 0, \dots, n-1$, and $\dot{x}_0 = f_0(x_0)$. We would like to be able to infer the topological entropy of the whole system from the appropriate entropies of the individual subsystems, and to expand the results to far more general interconnection networks. That would require a careful analysis of how the entropy behaves compositionally, especially int he open loop systems. To add to complexity, it has been recently observed [123], that a simple system with input has an infinite entropy when using the standard notion of entropy; but in a way dependent on the input signal decreases, we can recover the usual estimation entropy bound for closed systems given in [97]. How to address this kind of examples, is one of the problems we will attack.

3 Future DoD Relevance

3.1 Application domains

The implementation avenues outlined below engage various team members' research track records and current research thrusts, largely in the DoD context:

Multi-agent systems: this includes the problem of safe coordination of swarms of manned and unmanned vehicles (on which we have a record of cooperation with AFRL), and many adjacent problems (say, of assured coverage), and encompasses more rigorous approaches to verification of cyberphysical systems.

Robotic systems are a rich source of models; the interaction of the robotic limbs with the environment and between themselves produce frustratingly complex models that cry for an efficient formalism, or for innovative data-driven approaches bypassing explicit modeling.

Material science and biological systems deliver an unusual but important source of models. This class of dynamical systems is an intellectually compelling and technologically rich domain heretofore barely touched upon by hybrid systems research.

3.2 Impacts

We expect this project to result in following outcomes, aimed, in synergy, to transform the research paradigm of hybrid, and, more generally, control system, leading to a new generation of tools that engineers across DoD relevant industries would adopt. Our ultimate goal is to create a suite of theoretical tools that would percolate beyond engineering academe into the actually used toolboxes (algorithmic pipes and libraries) for systems designers. In the outline below, [x.x.x] is the pointer to the corresponding subsections of the research narrative.

Differential spaces, We expect to fully execute on the program outlined in [2.1.1] and introduce a workable, versatile, formally defined class of singular spaces on which all the components of the *Cartan package*, including Lie brackets and exterior calculus working across strata, are well-defined. This would enable development of consistent, flexible *Lagrangian and Hamiltonian formalisms* for singular spaces [2.1.3, 2.3.1]. Once this is done, not only the modeling of mechanical systems under constraints would become streamlined, but the applicability of the tools of optimal control would expand drastically.

Path spaces. The implementation and testing of the approach of [2.1.2] allowing the recovery of the phase space structure would in itself have a significant impact on how practitioners analyze empirically presented dynamical systems. While the initial focus would be on the computational implementations and statistical certificates [2.4.2, 2.5.1], the thrust to expanding this set of tools to multiconnected hybrid systems and to open systems, has potential for significantly larger impact.

Homotopy and Category Theoretical Tools. The significance of homotopy invariants such as (co)homology groups, and structures on them has been recognized in data analysis for decades. We aim at creating a framework allowing one to efficiently derive the key topological invariants of hybrid systems [2.2.2], and their path spaces [2.4.2]. The categorically based approaches as promoted in [2.2.1, 2.3.1] would provide provably reliable topological estimates, allowing one to express and to reason about in topological terms the notions of safety and stability, non-dedlocking and reachibility, expanding dramatically ideas of [2.3.2]. We hope that the diagrams of spaces will be in 21st century what the Nyquist plots were in 20th.

Application Focussed Deliverables. Among the more specialized tools, we plan to effectivize usage of operads for multi-agent systems as hierarchical composites [2.3.1], to create software tools for evaluation of topological invariants of path spaces in spaces with obstacles [2.4.3], and to develop apparatuses of computing complexity invariants for hybrid spaces [2.2.3, 2.5.2].

Several deliverables would have rather immediate applicability within the general AFOSR spectrum of sought capabilities. These include formally defined framework for categoric compositions of open systems providing tools for safety, obstacle avoidance, stability verification [2.5.1]; computational tools for obstacle avoiding path planning in the context of multi-agent systems; persistence based data-analytic tools for data-driven phase spaces discovery [2.1.2]; compositional language for limbed robotic operations in congested environments [2.1.3].

4 Project Schedule and Milestones

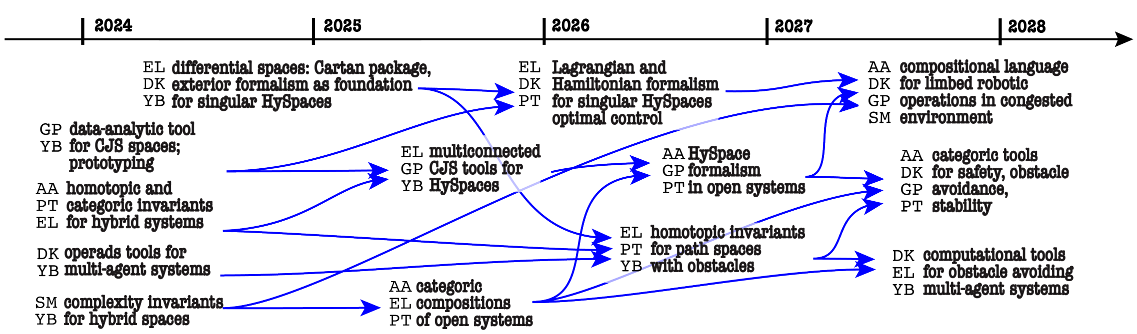

Figure 4: Schematic representation of the subprojects, their interdependencies, and involved PIs. Right edges of the project names are aligned with the expected times when the key results of the theory or main components of the toolset are realized.

The chart on Figure 4 exhibits interdependences between the key deliverables of this proposal. While predicting the rate of progress in a project heavily relying on development of highly abstract mathematical concepts and their alignment with engineering practices is nearly impossible, we made some educated guesses and assigned the expected completion times to those subprojects. As an example of the temporal-topical dependences, the study by Baryshnikov and Mitra of the complexity invariants, entropy being the focus,[19] would provide a natural setting for thread on understanding the intrinsic combinatorial complexities of the categorical compositional constructs for open hybrid systems run by Ames, Lerman, and Tabudada [20] which will inform in turn the search for proper formulation of the CJS formalism for open hybrid systems, that Ames, Pappas and Tabuada are to undertake with around Y3 of the project. All these theoretical developments will feed into the creating a template of a compositional language (which will be to URDF what Julia is to `BASIC`), allowing for the direct implementation of the category-based tools.

5 Management Approach

The collaborations will be realized through frequent seminars, and, above all, through shared supervision of postdocs and graduate students. Subteams (initially aligned along the intersectional matrix of RCAs and impact goals, but expected to adjust as organic interactions emerge) will hold weekly seminars (telecons), as well as monthly check-ins with PI Baryshnikov who will oversee the overall project. In-person annual meetings, potentially aligned but not reduced to the reporting events will be held. A project-wide cloud-based computational and storage platforms will be created to coordinate implementations and integrate the results.

The team intends to leverage a track record of collaborations with Dr. Spivak (TOPOS Institute) and Prof. Liberzon (ECE, UIUC) in the areas of category theory for networks, and entropic invariants, as joint supervisors of the postdoctoral scientists, based at UIUC.

6 Principal Investigator Qualifications

The expertise of the PIs forms a thin (low overlap) coverage of the both mathematical and engineering areas we see as critical for our vision of the hybrid systems. (This sparsity justifies the overshot of our team size by one above the suggested limit: it is hard to remove a PI without leaving a quintessential patch of research uncovered.) Even more importantly, their existing research covered essentially all of the *frontier* between mathematical and engineering sides of the general field of hybrid dynamics, ensuring an effective collaboration from the get go.

Aaron Ames received his PhD in hybrid systems at UC Berkeley under the guidance of Shankar Sastry, dealing with a categorical theory of hybrid systems and homotopical algebra of hybrid systems.

He has studied the use of certificates of stability and safety in nonlinear control (of hybrid systems), extending CLFs to a hybrid setting leading to provable stabilization of hybrid periodic orbits (which correspond to walking gaits on bipedal robots). More recently, he invented control barrier functions which give necessary and sufficient conditions for the safety (expressed as forward set invariance) of nonlinear systems.

Yuliy Baryshnikov was trained as an applied mathematician and worked for a decade at Mathematical Center at Bell Laboratories, NJ. His research addresses, among other areas, singularity theory (he worked with V. Arnold and his schools), geometric control, dynamical systems

[19] This branch of the project we hope to collaborate with Prof. Liberzon of UIUC.

[20] In interaction with Dr. Spivak of the Topos Institute.

and applied topology, including topological data analysis, topological robotics, topological integral transforms.

He is is productively collaborating with roboticists and material scientists, neuroscientists and economists.

Dan Koditschek has spent four decades working in the field of robotics with emphasis on achieving embodied intelligence through applications of dynamical systems theory. This entails the systematic design and physical implementation of attractor basins and, consequently, necessitates the introduction of topological tools and thinking toward the specification and control of hybrid Lagrangian dynamics. In recent years, Koditschek, with collaborators, has begun to develop a category theoretic formalization of these synthesis methods with the aim of using the associated type theory as the foundation for a physically grounded language for programming work.

Eugene Lerman learned symplectic geometry and geometric mechanics from Victor Guillemin in 1980s and made major contributions to these two closely related subjects.

Starting in 2010 Lerman started working on dynamics on networks using applied category theory. It started with two papers with DeVille on coupled-cell networks, proceeded to the work on the operad of wiring diagrams (with Vagner and Spivak) and culminated in a paper on networks of open hybrid dynamical systems with Schmidt.

Sayan Mitra was trained in the area of hybrid systems during his PhD with Nancy Lynch at MIT. His point of view is shaped by the saliency of compositional analysis, abstraction relations, and nondeterministic models of computation in automata theory and the theory of distributed computing. He made significant contributions in data-driven analysis of hybrid systems, culminating in several papers, software tools, and ongoing commercialization efforts.

Starting in 2016 Mitra and collaborator Daniel Liberzon initiated an investigation on topological entropy for hybrid systems, as well as data rate theorems for hybrid, switched and open systems.

George Pappas has been one of the original researchers in the topic of hybrid dynamical systems and control. His PhD thesis under the supervision of Shankar Sastry discovered a very important link between computability of hybrid systems verification and order-minimal structures in model theory. PIs Pappas and Tabuada co-developed a categorical foundation for dynamical, control, and hybrid systems. He proposed the first robust semantics for temporal logics when they are interpreted over continuous signals, and developed powerful tools, such as satisfiability-modulo convex optimization, which is one of the most powerful approaches for computing invariant of hybrid systems. Recently, he is focusing on using operator learning approaches to compute invariants of systems when systems are being considered as collections of trajectories.

Paolu Tabuada worked on category theoretic models for dynamical and control systems since 2005 showing that methods and construction of controllers can be unified across several domains including discrete-event systems, nonlinear control systems, behavioral systems, hybrid systems, and many others.

He further used category theory to provide insights into well established nonlinear control concepts and techniques such as backstepping and differential flatness in. He also has an interest in formal methods having recently developed a new temporal logic, and corresponding verification algorithms, to reason about robustness of software systems by drawing inspiration from control theoretic notions of robustness such as input-to-state stability.

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Data Management Plan

This project is primarily theoretical and algorithmic. For evaluation purposes some simulated and experimentally acquired data will be accumulated, but the datasets themselves will not be the primary outcomes of the project. Some outreach materials, aimed at general public will be generated.

The ways each of these categories of our data will be handled is discussed below.

- The technical reports and papers produced by the proposed project will be made available on the PIs' institutional repositories (such as personal research web sites), as well as on public repositories, such as *arXiv*.
- Algorithms and computational pipes will be stored using *git* repositories, either public (such as *github*), or located at the PIs institutions. We will also maintain a centralized git repositories using UIUC github shared platform. We will maintain regular interactions with the funder regarding potential adoption of the computational pipes we generate by the service branches, to ensure that any potential requests for the IP restrictions are resolved immediately.
- The project's web site will be created, containing the links to the generated materials, publications, algorithms and datasets. The outreach materials, as well as any educational materials (lecture series or minicourses) related to the research of this project will be made available to the public on this web site as well. This web site will be hosted by the UIUC.
- We expect the storage and curation of the datasets generated within this project to be performed by the participating PIs' institutions. The project web site will maintain a regularly updated directory of those datasets. We do not expect any privacy-related information to be collected or generated. In case real data are received from any industry collaborators, proprietary and IP-related information will be respected.
- The information generated by the proposed project will be maintained, preserved, and made available for the duration required by the DoD.