Online Algorithms with Randomly Infused Advice

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Abstract

We introduce a novel method for the rigorous quantitative evaluation of online algorithms that relaxes the “radical worst-case” perspective of classic competitive analysis. In contrast to prior work, our method, referred to as randomly infused advice (RIA), does not make any probabilistic assumptions about the input sequence and does not rely on the development of designated online algorithms. Rather, it can be applied to existing online randomized algorithms, introducing a means to evaluate their performance in scenarios that lie outside the radical worst-case regime.

More concretely, an online algorithm ALG with RIA benefits from pieces of advice generated by an omniscient but not entirely reliable oracle. The crux of the new method is that the advice is provided to ALG by writing it into the buffer $\mathcal{B}$ from which ALG normally reads its random bits, hence allowing us to augment it through a very simple and non-intrusive interface. The (un)reliability of the oracle is captured via a parameter $0 \leq \alpha \leq 1$ that determines the probability (per round) that the advice is successfully infused by the oracle; if the advice is not infused, which occurs with probability $1 - \alpha$, then the buffer $\mathcal{B}$ contains fresh random bits (as in the classic online setting).

The applicability of the new RIA method is demonstrated by applying it to three extensively studied online problems: paging, uniform metrical task systems, and online set cover. For these problems, we establish new upper bounds on the competitive ratio of classic online algorithms that improve as the infusion parameter α increases. These are complemented with (often tight) lower bounds on the competitive ratio of online algorithms with RIA for the three problems.

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1 Introduction

Motivated by the advent of AI platforms that learn input sequences, in the recent years, many traditional algorithmic models have been extended with a notion of an *oracle* that gives “advice”. Such models typically assume a limit on the amount or quality of advice given, and much research revolved around the design of novel algorithms which can exploit “good advice”, but are never worse than traditional algorithms in case of “non-useful advice”. In order to assess these algorithms, worst-case scenarios are considered in which an adversary chooses the worst-possible inputs for a given algorithm with advice.

This paper provides a novel perspective on online computation with advice, by introducing a method for the rigorous quantitative study of randomized online algorithms. Our method, which we call *randomly infused advice (RIA)*, defines an interface between the algorithm and the oracle, which may serve two purposes: (1) augment (new or existing!) randomized online algorithms with advice; and (2) enable the analysis of randomized online algorithms from a different perspective, beyond the worst-case nature of traditional competitive analysis, without any probabilistic assumptions about the request sequence. Indeed, the RIA method does not require the development of designated online algorithms; rather, it can be applied to existing online algorithms, facilitating the evaluation of their performance in scenarios that lie outside the "radical worst-case" regime, assumed in the classic online computation literature.

Concretely, an online algorithm ALG with RIA has access to pieces of advice generated by an omniscient, though not entirely reliable, oracle. The core of our new approach is the natural and non-intrusive interface between ALG and the oracle: instead of introducing a designated buffer (or tape) into which the oracle writes its advice (as is done in the existing advice-based models), the advice is written into the buffer $\mathcal{B}$ from which ALG normally reads its random bits. The (un)reliability of the oracle is modelled via an *infusion parameter* $0 \leq \alpha \leq 1$, which determines the probability that the advice is successfully infused by the oracle in each round (independently); if the advice is not infused — an event occurring with probability $1 - \alpha$ — then the buffer $\mathcal{B}$ contains fresh random bits (as in the classic online setting). For illustration, see Figure 1.

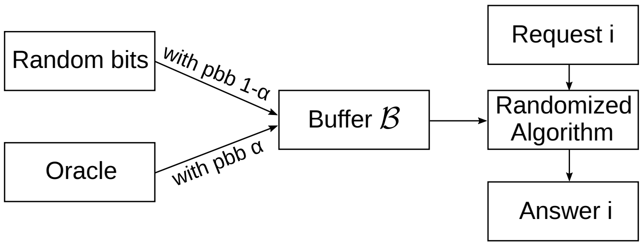

Figure 1 The RIA model. At the beginning of each round, the buffer $\mathcal{B}$ that the randomized algorithm reads its random bits from is either filled with fresh random bits or is infused with advice. The infusion parameter α determines the probability that the advice is successfully infused by the oracle at each round.

Notice that the RIA model does not impose any limitations on the size of the buffer $\mathcal{B}$, and through it, on the advice size (or the number of random bits) provided to ALG in each round. This raises the concern of making the online algorithm "too powerful" as the (successfully) infused advice may hold excessive information regarding the future requests. To overcome this concern, we restrict our attention to randomized online algorithms which are *randomness-oblivious*, namely, in each round, ALG has access to past requests, past answers, the current request, and the current content of the buffer $\mathcal{B}$ (which contains the current advice or random bits), however ALG cannot access the content of $\mathcal{B}$ in previous rounds. Indeed, all algorithms analyzed in this paper are randomness-oblivious.

The motivation for studying the new RIA model is twofold: First, the RIA model introduces a quantitative method (parameterized by the infusion parameter α) for relaxing the worst-case nature of competitive analysis without making explicit assumptions about the request sequence (or the probability distribution thereof). Second, the RIA model provides an abstraction for an unreliable predictor (whose role is assumed by the oracle) whose "mistakes" take a random (rather than worst-case) flavor, where the infusion parameter α indicates the (expected) fraction of rounds in which the predictor is correct.

The non-intrusive interface between the online algorithm and the oracle gives the RIA model a distinctive advantage over existing advice models for online algorithms: in contrast to the latter, which typically require the development of new, model specific, algorithms, the RIA model is applicable (also) to existing randomized algorithms. This allows us to analyze the performance of existing online algorithms in scenarios that include an (unreliable) predictor, while retaining their worst-case guarantees.

1.1 Our Contribution

On top of the conceptual contribution that lies in introducing the RIA model, we make the following technical contribution.

Upper bounds. The applicability of the new RIA model is demonstrated on three extensively studied online problems: the *paging* problem [39], for which we analyze the classic RandomMark algorithm [24], the uniform *metrical task system (MTS)* problem [15], for which we analyze the classic UnifMTS algorithm, and the unweighted *online set cover* problem [6], for which we analyze the influential primal-dual algorithm [17, Ch. 4] with randomized rounding (referred to as RandSC). In all cases, our findings are similar to what is called “robustness” and “consistency” in the literature dedicated to online algorithms with predictions [35]: when augmented with RIA, the competitive ratio of these algorithms is never worse than the original, and improves asymptotically as $\alpha \to 1$. Our results are cast in the following three theorems, where we denote the k-th harmonic number by $H_k \approx \log k$; we emphasize that in all cases, neither the online algorithm nor the oracle are aware of the infusion parameter α.

▶ **Theorem 1.** *The competitive ratio of RandomMark augmented with RIA with infusion parameter $0 \leq \alpha \leq 1$ on instances of cache size k is at most $\min\{2H_k, \frac{2}{\alpha}\}$.*

▶ **Theorem 2.** *The competitive ratio of UnifMTS augmented with RIA with infusion parameter $0 \leq \alpha \leq 1$ on n-state instances is at most $\min\{2H_n, \frac{2}{\alpha} + 2\}$.*

▶ **Theorem 3.** *The competitive ratio of RandSC augmented with RIA with infusion parameter $0 \leq \alpha \leq 1$ on instances with n elements and maximum element degree d is at most $O(\min\{\log d \log n, \frac{\log n}{\alpha}\})$.*

Lower bounds. On the negative side, we prove that the upper bound promised in Theorem 1 is asymptotically tight for the class of *lazy* algorithms, which are not allowed to change their cache configuration unless there is a page miss.

▶ **Theorem 4.** *There does not exist a lazy (randomness-oblivious) online paging algorithm augmented with RIA with infusion parameter $0 \leq \alpha \leq 1$ whose competitive ratio on instances of cache size k is better than $\min\{H_k, \frac{1}{\alpha}\}$.*

Omitting the restriction to lazy algorithms, we can establish a weaker lower bound.

▶ **Theorem 5.** *There does not exist a (randomness-oblivious) online paging algorithm augmented with RIA with infusion parameter $0 \leq \alpha \leq 1$ whose competitive ratio on instances of cache size k is better than $\min\{H_k, \frac{1}{k \cdot \alpha}\}$.*

The uniform MTS problem generalizes the paging problem on instances that include $n = k + 1$ pages. As Theorems 4 and 5 hold (already) for such instances, their promised lower bounds are transferred to the uniform MTS problem, where laziness translates to online MTS algorithms that may switch state only when the processing cost is positive [23] (an algorithm class that includes UnifMTS).

▶ **Theorem 6.** *There does not exist a lazy (randomness-oblivious) online uniform MTS algorithm augmented with RIA with infusion parameter $0 \leq \alpha \leq 1$ whose competitive ratio on n-state instances is better than $\min\{H_{n-1}, \frac{1}{\alpha}\}$.*

▶ **Theorem 7.** *There does not exist a (randomness-oblivious) online uniform MTS algorithm augmented with RIA with infusion parameter* $0 \leq \alpha \leq 1$ *whose competitive ratio on* n*-state instances is better than* $\min\{H_{n-1}, \frac{1}{(n-1)\cdot\alpha}\}$.

For online set cover, we establish a lower bound for lazy algorithms, namely, online algorithms which are allowed to buy a set only if it contains the current (uncovered) element (an algorithm class that includes RandSC).

▶ **Theorem 8.** *There does not exist a lazy (randomness-oblivious) unweighted online set cover algorithm augmented with RIA with infusion parameter* $0 \leq \alpha \leq 1$ *whose competitive ratio on instances with maximum element degree* d *is better than* $\min\{\frac{1}{2}\log d, \frac{1}{2\cdot\alpha}\}$.

1.2 Related Work

Our contribution touches and builds upon a number of recent works.

Models of Advice. A well-known and suitable advice model for machine-learned predictions is the model of online algorithms with untrusted advice introduced by Purohit, Svitkina and Kumar [35], where the existing literature includes papers on paging [9, 28, 31, 37], metrical task system [7], and online set cover via the primal-dual approach [8]. In this model, the predictor may be faulty, and the competitive ratio depends on its error so that for low error, the algorithm should perform close to the offline optimum (a.k.a. *consistency*), while even for large error, the algorithm should still fallback to guarantees similar to those of non-augmented online algorithms (a.k.a. *robustness*).

Another well-known advice model is the *perfect* advice model [13, 22] under which many online problems have been studied, including paging, metrical task system [16], and online set cover [21]. In this model, the oracle is fully trustworthy, and its power is therefore quantified via the size (i.e., number of bits) of the advice provided to the online algorithm.

Unlike these two models, the RIA model does not require any new algorithmic features (e.g., a designated advice tape) and is therefore applicable to existing (standard) online algorithms. Furthermore, our model does not limit the advice size, unlike the perfect advice model, and still allows to arrive at asymptotically tight lower bounds under natural assumptions, in contrast to the machine-learned prediction model where no general lower bounds are known.

Beyond the Worst-Case Analysis of Online Algorithms. The analysis of existing (randomized) online algorithms through the RIA lens sheds new light on their competitiveness under relaxed worst-case analysis. The RIA model is thus related to other relaxed settings for studying online algorithms. These include both "stochastic relaxations" such as distributional analysis [25, 36], diffused adversary [30], independent sampling [20], and random arrival order [2–4], and "deterministic relaxations" such as the lookahead model [26]. We refer to the book of Roughgarden [38] for further resources on analysis of online algorithms beyond the worst-case regime.

Online algorithms for paging, MTS, and set cover. Two optimally competitive algorithms for paging are known: PARTITION [32] and EQUITABLE [1]. For the uniform MTS problem, a $(2H_n)$-competitive algorithm was presented in [15], later improved to $H_n + O(\sqrt{\log n})$ in [27]; the latter result nearly matches the H_n lower bound of [15].

For online set cover, the state-of-the-art competitive ratio upper bounds are $O(\log m \log n)$ for the weighted case [6] and $O(\log m \log(n/\mathsf{OPT}))$ for the unweighted case [18], where m and

n denote the number of sets (an upper bound on the maximum element degree d) and the number of elements, respectively; interestingly, both bounds can be realized by deterministic online algorithms. On the negative side, no (randomized) online algorithm has a competitive ratio better than $\Omega(\log m)$ [29] and no deterministic online algorithm has a competitive ratio better than $\Omega(\log m \log n / (\log \log m + \log \log n))$ [6]. If the (randomized) online algorithm is required to admit a polynomial time implementation, then the competitiveness lower bound improves to $\Omega(\log m \log n)$ assuming that $NP \nsubseteq BPP$ [29].

2 Online Algorithms with Randomly Infused Advice

We begin by recalling standard definitions of online algorithms as request-answer games [12]. Our model of online algorithms with randomly infused advice is then defined as a generalization of this model.

2.1 Online Algorithms as Request-Answer Games

Consider a finite sequence $\sigma = \langle r_1, \dots, r_{|\sigma|} \rangle$ of *requests*, where each request r_i is taken from a set $\mathcal{R}$. A *solution* for σ is a sequence $\lambda = \langle a_1, \dots, a_{|\sigma|} \rangle$ of *answers*, where each answer a_i is taken from a set $\mathcal{A}$. For a given minimization problem, the quality of a solution λ for a request sequence σ is determined by means of a *cost function* $f : \mathcal{R}^{|\sigma|} \times \mathcal{A}^{|\sigma|} \to \mathbb{R} \cup \{\infty\}$.[1] Let $\mathsf{OPT}(\sigma) = \inf_{\lambda \in \mathcal{A}^{|\sigma|}} f(\sigma, \lambda)$ denote the cost of an *optimal solution* for σ.

In the realm of online algorithms, the requests are revealed one-by-one, in discrete *rounds*, so that upon receiving request r_i in round i, a (randomized) online algorithm ALG outputs the (random) answer a_i irrevocably. That is, the solution $\lambda_{\mathsf{ALG}} = \langle a_1, \dots, a_{|\sigma|} \rangle$ produced by ALG is defined so that each answer a_i is computed as a function of (1) the request subsequence $r_1, \dots, r_i$; (2) the answer subsequence $a_1, \dots, a_{i-1}$; and (3) round i's random bit string $\mathcal{B}_i \in_R \{0,1\}^L$, where the parameter $L \in \mathbb{Z}_{\geq 0}$ is specified by the algorithm's designer (possibly as a function of the parameters of the problem).[2]

The performance of an online algorithm ALG is measured via competitive analysis: we say that ALG is *c-competitive* if there exists a constant b (that may depend on the parameters of the problem) such that $\mathbb{E}[\mathsf{ALG}(\sigma)] \leq c \cdot \mathsf{OPT}(\sigma) + b$ for any request sequence σ, where $\mathsf{ALG}(\sigma)$ is the random variable that takes on the cost of the solution produced by ALG in response to a request sequence σ. The request sequence σ is assumed to be determined by a malicious *adversary*; we stick to the convention of an *oblivious adversary* [14, Ch. 4] which means that the adversary knows ALG's description, but is unaware of the outcome of ALG's random coin tosses.

2.2 Randomly Infused Advice

In this paper, we introduce an extension of online algorithms, referred to as online algorithms with *randomly infused advice (RIA)*. In the RIA model, an algorithm ALG is assisted by a powerful, yet not entirely reliable, *oracle* that has access to the entire request sequence σ. Formally, for any request sequence $\sigma = \langle r_1, \dots, r_{|\sigma|} \rangle$ and round $1 \leq i \leq |\sigma|$, the oracle $\mathcal{O}$ is defined by an *advice function* $\mathcal{O}_{\sigma,i} : \mathcal{A}^{i-1} \to \{0,1\}^L$ that maps each answer subsequence

[1] We restrict our attention to minimization problems as these are the problems addressed in the current paper. Extending our setting to maximization problems is straightforward.

[2] We use a single parameter L (that is often kept implicit in the online algorithm's description) for simplicity of the exposition; it can be easily generalized to a (not necessarily bounded) sequence $L_1, L_2, \dots$ of round-dependent parameters.

$\langle a_1, \ldots, a_{i-1} \rangle$ to a bit string $\mathcal{O}_{\sigma,i}(a_1, \ldots, a_{i-1}) \in \{0,1\}^L$, referred to as the round i's *advice*. Notice that the length of the advice bit string is equal to the length L of ALG's random bit string.

The RIA model is associated with an *infusion parameter* $0 \leq \alpha \leq 1$ that quantifies the (un)reliability of the oracle $\mathcal{O}$. Specifically, in each round i, the bit string $\mathcal{B}_i$ (provided to the online algorithm in that round) is now determined based on the following random experiment (independently of the other rounds): with probability α, the round i's advice is *infused* into $\mathcal{B}_i$, that is, $\mathcal{B}_i \leftarrow \mathcal{O}_{\sigma,i}(a_1, \ldots, a_{i-1})$; with probability $1 - \alpha$, the bit string $\mathcal{B}_i$ is picked uniformly at random, that is, $\mathcal{B}_i \in_R \{0,1\}^L$.

In other words, in each round i where the infusion is successful (an event occurring with probability α), the oracle's advice "smoothly" substitutes the random bit string $\mathcal{B}_i$ before it is provided to ALG; if the infusion is not successful, then $\mathcal{B}_i$ remains a random bit string. We emphasize that ALG and $\mathcal{O}$ are not aware (at least not directly) of whether the advice is successfully infused in the round i, nor are they aware of the infusion parameter α itself.

The competitive ratio of online algorithms ALG with RIA is typically expressed as a function of the infusion parameter α, where the extreme case of $\alpha = 0$ corresponds to standard online computation (with no advice). The ultimate goal is to provide guarantees on the competitiveness of ALG for any $0 \leq \alpha \leq 1$.

2.3 Randomness-Oblivious Online Algorithms

Recall that the aforementioned definition of online algorithms dictates that when the online algorithm ALG determines the answer a_i associated with round i, it is aware of the requests $r_{i'}$ and answers $a_{i'}$ associated with past rounds $i' < i$, as well as the request r_i and random bit string $\mathcal{B}_i$ associated with the current round i, however it is not aware (at least not directly) of the random bit strings $\mathcal{B}_{i'}$ associated with past rounds $i' < i$. This model choice is made to prevent an online algorithm ALG with RIA from passing information received through the (successfully infused) advice to future rounds, thus over-exploiting the lack of an explicit (model specific) bound on the length of the random / advice bit strings. To distinguish the online algorithms that adhere to this formulation from general online algorithms (that may maintain a persistent memory that encodes past random bits), we refer to the former as *randomness-oblivious* online algorithms.

3 Paging

In the *online paging* problem [39], we manage a two-level memory hierarchy, consisting of a slow memory that stores the set of all n pages, and a fast memory, called the *cache*, that stores any size k subset of pages. We are given a sequence σ of requests to the pages. If a requested page is not in the cache, a *page fault* occurs, and the page must be moved to the cache. Since the cache is limited in size, we must specify which page to evict to make space for the requested page. The goal is to minimize the number of page faults.

In this section, we analyze an elegant randomized online algorithm RandomMark, introduced by Fiat, Karp, Luby, McGeoch, Sleator and Young [24], in the randomly infused advice framework. The algorithm RandomMark maintains a bit associated with each page in the cache. Initially the bits of all pages are set to 0 (the pages are *unmarked*), and after requesting a page, we bring it to the cache if it is not in the cache yet, and we set its bit to 1 (we *mark* the page). To bring a page to the cache, we may need to evict another page to make space for it. In such a case, RandomMark evicts a page uniformly at random chosen from the unmarked pages. If no unmarked page exists, we unmark all pages. This strategy has been shown to be $2H_k$-competitive [24], where H_k is the harmonic number, and no randomized algorithm can be better than H_k-competitive.

3.1 RandomMark with infused advice

With help of randomness, the classic RandomMark decides on the final candidate to evict: a random node among unmarked pages. With infused advice, in some rounds the randomness source used by RandomMark contains advice instead of random bits. The presence of clairvoyent advice brings obvious advantages, but also brings challenges: not all pages can be evicted, only the unmarked ones.

Unmarked Longest-Forward-Distance Oracle. An optimal offline algorithm for paging is to evict the item with the access time furthest in the future [11], also known as *longest forward distance* (LFD) algorithm. However, we cannot directly design an oracle for RandomMark around LFD, as it may advise to evict a marked page, but RandomMark never evicts marked pages. Hence, we propose a variant of this algorithm that can act as an oracle for RandomMark. Such an oracle, denoted O_{ULFD}, advises RandomMark to evict the page with the longest forward distance *among the unmarked items* of RandomMark.

Analysis of RandomMark. How well can RandomMark perform with infused advice? To find out, we consider the RandomMark algorithm assisted with the oracle O_{ULFD}, and we express the algorithm's competitive ratio of in terms of the infusion parameter α (the probability of receiving advice in each round). Later in this paper, we will show that RandomMark with O_{ULFD} is asymptotically optimal (Theorem 18).

▶ **Theorem 9.** *The competitive ratio of RandomMark with the oracle O_{ULFD} with RIA on k-page instances (against the oblivious adversary) is at most $\min\{2H_k, \frac{2}{\alpha}\}$, where H_k is the k-harmonic number, and $0 \leq \alpha \leq 1$ is the infusion parameter.*

Before proving this theorem, we recall the definition of a k-phase partitioning of an input sequence, and we derive sufficient conditions to stop incurring further page faults in a phase.

3.2 Phases and Blame Graphs

We begin by recalling basic definitions from the analysis of RandomMark by [24]. We consider the *k-phase partition* of the input sequence σ, following the notation from [14]: phase 0 is the empty sequence, and each phase $i > 0$ is the maximal sequence following the phase $i-1$ that contains at most k distinct page requests since the start of the ith phase.

To minimize cache misses, our advice aims to terminate each k-phase as soon as possible. To understand how advice helps, we study the sufficient conditions for phase termination, using an auxiliary *blame graph* of [31], which is defined for any k-phase of any marking algorithm (and hence for RandomMark).

We build the definitions on top of the known concepts of clean and stale pages. In a phase of any marking algorithm for paging, a page is *stale* if it is unmarked but was marked in the previous phase, and a page is *clean* if it is neither stale nor marked. To study the cost incurred in a phase, we further partition the stale pages: let the *vanishing pages* be the stale pages not requested in the phase, and let the *returning pages* be the stale pages requested in the phase.

After evicting the last vanishing page, any marking algorithm resides in a configuration where all the remaining requests in the current phase are free (page hits).

▶ **Lemma 10.** *Fix an input sequence σ, consider its k-phase partition, and fix any phase P that is not the last phase. Then, the following holds:*

- *we have exactly c vanishing pages, where c is the number of the clean pages in the phase, and*
- *after evicting all vanishing pages, no marking algorithm for paging incurs further cost in the phase P.*

Proof. We first claim that we have exactly c vanishing pages: this follows since there are c clean pages and $k - c$ requested stale pages in each phase, and we have k distinct requests in each phase but the last one. Hence, the first claim holds.

Fix any phase P but the last one. We claim that if at any point all c vanishing pages are evicted, the algorithm incurs no further page faults on clean pages because all of them have been requested already.

To show this, we construct a blame graph of [31] for the phase P to reason why a page was evicted. In the construction of the blame graph, there are two cases for the evicted page x: either x was evicted when a clean page y arrived, in which case we add a directed edge from x to y, or it was evicted because a stale element z arrived, but z was evicted before. In the latter case, we add a directed edge from x to z.

The blame graph for any phase is always a set of paths. Each path can contain at most one vanishing page (at the end of it), because the vanishing pages are not requested in this phase, so the path extends no further. Each path contains one clean page (at the beginning of it), hence at least i clean items were requested beforehand for i vanishing items to be evicted.

We claim that after evicting all c vanishing pages, the complementing set of stale pages to be requested remains in the cache. This follows because the contents of the cache can consist of a subset of c clean and k stale pages only, and we evicted c pages that were not requested in the phase P, so the cache must contain the $k - c$ stale pages requested.

After evicting all vanishing pages, the cache consists of returning and clean pages, and those incur the cost, so the second claim holds. ◀

3.3 Bounding the competitive ratio

Finally, we prove our main claim for paging: RandomMark is $\min\{2H_k, \frac{2}{\alpha}\}$-competitive. We repeat the classic arguments of [24] to arrive at the bound $2H_k$, and we analyze the offline algorithm *unmarked longest forward distance*, employed by the oracle that probabilistically interacts with the oracle, to arrive at the bound $\frac{2}{\alpha}$.

Proof of Theorem 9. Fix any input sequence σ and consider its k-phase partition. Consider any phase that is not the first or the last one. Let c be the number of clean pages in the phase.

We claim that the expected number of page faults is upper bounded by c/α. If the algorithm incurs a page fault, and it receives the oracle's advice, and there are still some vanishing pages in the cache, then the algorithm evicts a vanishing page; this follows since the vanishing pages are not requested in the current phase, hence they have larger forward distance than other stale pages, and the vanishing pages are unmarked. By Lemma 10, evicting all vanishing pages means that no further cost is incurred throughout the phase, hence the number of page faults in the phase is upper bounded by the number of page faults until the algorithm receives c rounds of advice from the oracle (not necessarily consecutive).

The expected number of page faults until receiving c rounds of advice is c/α, since this is the expected number of independent tosses of α-biased coin until getting c heads outcomes.

Next, we repeat the classic arguments of [24]: the expected number of page faults of the algorithm is also upper bounded by $c \cdot H_k$. Consider an i-th request to a stale page in the phase for $i = 1, 2, 3, \dots, s$. Let $c(i)$ denote the number of clean pages requested in the phase immediately before the i-th request to a stale page, and let $S(i)$ denote the set stale pages that remain in the cache before the i-th request to a stale page, and let $s(i) = |S(i)|$. For $i = 1, 2, 3, \dots, s$, we compute the expected cost of the i-th request to a stale page. When the algorithm serves the i-th request to a stale page, exactly $s(i) - c(i)$ of the $s(i)$ stale pages are in the cache. The returning pages (recall that those are the stale pages that are requested in this phase) are in the cache with equal probability, say p, since these are never evicted with the help of advice, but are evicted uniformly from unmarked pages when a page fault occurs in rounds without advice. The vanishing pages are in the cache with probability at most p, since they can be evicted both in the rounds with and without the advice. For the all $s(i)$ stale pages the probability of being in the cache sums to 1, hence $p \leq 1/s(i)$. Fix a request to a stale page; note it is a returning page as it is just being requested. The page is in the cache with probability $(s(i) - c(i)) \cdot p$, hence the expected cost of the request is

$$1 - (s(i) - c(i)) \cdot p \leq 1 - \frac{s(i) - c(i)}{s(i)} = \frac{c(i)}{s(i)} \leq \frac{c}{k - i + 1}.$$

Hence, the total cost of the request to the stale pages is $\sum_{i=1}^{s} c/(k-i+1) \leq \sum_{i=2}^{k} c/i = c \cdot (H_k - 1)$. The total cost in the phase includes the cost of serving the clean page and stale pages, in total $c \cdot H_k$.

We conclude that the number of page faults of the algorithm in a phase is upper-bounded by both $c \cdot H_k$ and c/α. By arguments of [24, Theorem 1], the amortized number of faults made by OPT during the phase is at least $c/2$. Summing over all phases but the first and the last one, the competitive ratio is at most $\min\{2H_k, \frac{2}{\alpha}\}$. The first and the last phase incurs cost bounded by $2k$, which we account in the additive in the competitive ratio. ◀

For the special case $n = k + 1$, the competitive ratio of RandomMark with the oracle O_{ULFD} is $\min\{H_k, \frac{1}{\alpha}\}$, since in each phase but the last phase, any offline algorithm pays at least 1, and the number of clean pages is also 1.

The algorithm RandomMark with perfect advice ($\alpha = 1$) is equivalent to an offline algorithm that evicts the unmarked item with the longest forward distance. The Theorem 9 implies that this algorithm is optimal for $n = k + 1$, and a 2-approximation for any n.

4 Uniform Metrical Task System

In the *metrical task system (MTS)* problem, we are given a finite metric space (S, d) consisting of a set $S = \{s_1, \dots, s_n\}$ of n *states* and a distance function $d : S^2 \to \mathbb{R}_{\geq 0}$ assumed to be a metric. A *task* $r \in \mathbb{R}^n_{\geq 0}$ is an n-sized vector of non-negative *processing costs*, where the entry $r(i)$ is defined to be the processing cost of serving r in state s_i. Given a sequence $\sigma = r^1, \dots, r^{|\sigma|}$ of tasks, the cost of a schedule $s^1, \dots, s^{|\sigma|}$ is the sum between the total transition cost and the total processing cost. The goal in the MTS problem is to find a schedule of minimal cost. We focus on algorithms for the MTS problem in the online setting, where the state s^i that serves task r^i is chosen without knowing the subsequence $r^{i+1}, \dots, r^{|\sigma|}$.

In this section, we focus on the MTS problem on a uniform metric, i.e., the metric where $d(s_i, s_j) = 1$ for all $i \neq j$. We shall present a randomized algorithm, henceforth referred to as UnifMTS, with advice. This algorithm is inspired by the classical $2H_n$-competitive algorithm by [15].

Consider a sequence $\sigma = r^1, \ldots, r^{|\sigma|}$ of tasks given at times $t = 1, \ldots, |\sigma|$. For an integer $i \in \{1, \ldots, |\sigma|\}$, and $i \leq \ell < \ell' \leq i+1$, let us define the processing cost $\pi(s_j, \ell, \ell')$ of being in state s_j in the time interval $[\ell, \ell']$ as $\pi(s_j, \ell, \ell') = (\ell' - \ell) \cdot r^i(j)$. We now naturally extend this notion to time intervals $[\ell, \ell']$ such that $i \leq \ell \leq i+1 < \ell' \leq |\sigma|+1$ by defining $\pi(s_j, \ell, \ell') = \pi(s_j, \ell, i+1) + \pi(s_j, \lfloor \ell' \rfloor, \ell') + \sum_{k=i+1}^{\lfloor \ell' \rfloor - 1} \pi(s_j, k, k+1)$.

We define a partition of $[1, |\sigma|+1]$ into time intervals $[t_0, t_1], [t_1, t_2], \ldots, [t_{m-1}, t_m] \subseteq [1, |\sigma|+1]$ called *phases* such that $t_0 = 1$ and $t_m = |\sigma|+1$. The i-th phase starts at time t_{i-1}. We say that a state s_j is *saturated* for phase i at time $t > t_{i-1}$ if the processing cost associated with being in s_j during the entire time interval $[t_{i-1}, t]$ is at least 1. The i-th phase ends in time t_i, defined to be the minimal time in which all states are saturated for the i-th phase. Observe that upon the arrival of a task r^i at time i, an online algorithm can determine which states will become saturated for the current phase by time $i+1$.

The UnifMTS algorithm operates as follows. Consider the task r^i arriving at time i and let φ be the current phase. If the current state does not become saturated for φ at time $i+1$, then UnifMTS stays in the same state. Otherwise, if φ ends by time $i+1$, then UnifMTS moves to a state that minimizes the processing cost in r^i. Otherwise, UnifMTS moves uniformly at random to a state that is unsaturated for φ at time $i+1$ (notice that such state exists since in this case φ does not end by time $i+1$). We note that while phases may end at non-discrete times, the scheduling decisions made by the algorithm all occur at discrete times.

Consider an oracle O_{LTS} that advises UnifMTS to move to a state with the longest time until saturation for the current phase. In the following theorem, we bound the competitive ratio of UnifMTS with O_{LTS}.

▶ **Theorem 11.** *The competitive ratio of UnifMTS with the oracle O_{LTS} against an oblivious adversary is at most $\min\{2H_n, \frac{2}{\alpha}+2\}$, where $0 \leq \alpha \leq 1$ is the infusion parameter.*

Proof. Observe that an optimal offline algorithm OPT must incur a cost of at least 1 during each phase. Indeed, if OPT changed states during a phase, then it pays at least 1 in transition cost. Otherwise, OPT resided in a state that became saturated in this phase, hence it pays a processing cost of 1.

We now bound the expected cost of UnifMTS with O_{LTS} during a phase $\varphi = [t_{start}, t_{end}]$. Observe that if there exists $i \in \{1, \ldots, |\sigma|\}$ such that $i \leq t_{start} < t_{end} \leq i+1$, then by definition, at time i UnifMTS moved to a state that minimizes the processing cost incurred during $[i, i+1]$. This means that UnifMTS pays 1 in processing cost during $[t_{start}, t_{end}]$ and possibly 1 in transition cost at time i. Thus, in this case the expected cost of UnifMTS during φ is at most 2.

Now we consider the case that there exists $i \in \{1, \ldots, |\sigma|\}$ such that $t_{start} < i < t_{end}$. We show that the cost of UnifMTS during φ is at most $\frac{2}{\alpha}+2$. Let s^* be the state given in the advice of O_{LTS} during φ. By definition, by the time s^* is saturated for φ, all other states have also been saturated. Therefore, when UnifMTS receives an advice from the oracle, it transitions into the final state of phase φ. Hence, the additional cost incurred by the UnifMTS in φ following the advice is at most 2 (1 for the transition to s^* and at most 1 for processing cost). Since the algorithm uses randomization only at transition rounds, hence the expected number of transitions before the algorithm receives the advice is $1/\alpha$ (recall that at each transition the advice is given with probability α). For each state that we visit, we pay 1 in transition cost. Since UnifMTS only moves to states that are unsaturated for φ, it pays at most 1 in processing cost at each state. Overall, the expected total cost is at most $\frac{2}{\alpha}+2$.

We now show that the cost of UnifMTS during φ is at most $2H_n$. Notice that for every transition, UnifMTS pays 1 in transition cost and at most 1 in processing cost. Thus, it suffices to show that the expected number of transitions during φ is at most H_n. Let $f(k)$ be the expected number of transitions UnifMTS performs given that there are k unsaturated states left. Clearly, $f(1) = 1$. For $k < 1$, after a single transition we have $k - 1$ unsaturated states with probability at most $1/k$. Thus, $f(k) \leq f(k-1) + 1/k$, which implies that $f(n) \leq H_n$. Summing over the costs of all phases, we get a competitive ratio of $\min\{2H_n, \frac{2}{\alpha} + 2\}$. ◀

5 Set Cover

In the *set cover* problem, we are given a universe $\mathcal{U}$ of n elements and a set $\mathcal{F} = \{S_1, \ldots, S_m\}$ of m subsets $S_1, \ldots, S_m \subseteq \mathcal{U}$ such that $S_1 \cup \cdots \cup S_m = \mathcal{U}$. For each element $e \in \mathcal{U}$, let $\mathcal{F}(e) = \{S \in \mathcal{F} \mid e \in S\}$ be the collection of sets that cover it. In the online setting, a subset $\mathcal{U}' \subseteq \mathcal{U}$ of elements arrive one by one in an arbitrary order.[3] Upon the arrival of an element e, the algorithm is required to cover it (i.e., if e was not previously covered by the algorithm, then the algorithm must select a set from $\mathcal{F}(e)$). We emphasize that the algorithm does not know $\mathcal{U}'$ (or its size) in advance and that any previously selected set cannot be removed from the solution obtained by the online algorithm. The cost of a solution to the set cover problem is the number of sets selected.

In the standard linear program (LP) relaxation for set cover, each set $S \in \mathcal{F}$ is associated with a variable x_S. The objective is to minimize the sum $\sum_{S \in \mathcal{F}} x_S$ subject to the constraints $\sum_{S \in \mathcal{F}(e)} x_S \geq 1$ for each element $e \in \mathcal{U}'$, and $x_S \geq 0$ for all $S \in \mathcal{F}$.

Recall that in the context of set cover in the RIA model, we focus on *lazy* algorithms, i.e., algorithms that adhere to the following restrictions upon the arrival of an elemnt e: (1) if e is already covered by the algorithm, then in the current round the algorithm does not select any additional sets to its solution; and (2) if e is not covered yet, then in the current round the algorithm may only select sets from $\mathcal{F}(e)$. Notice that this restriction prevents the trivial oracle strategy of simply advising to select all the sets of an optimal set cover at each round.

We describe an online algorithm with RIA for set cover in three stages. First, we present an algorithm that obtains a fractional solution $\mathbf{x}$ to the relaxed LP. Then, we present an online randomized rounding scheme that can be incorporated into the fractional set cover algorithm to obtain an integral solution which is feasible with high probability. Finally, we present the oracle's advice.

Fractional set cover algorithm. We use is the basic discrete algorithm presented by Buchbinder and Naor in [17, Chapter 4.2, Algorithm 1].[4] The algorithm operates as follows. Initially, set $x_S = 0$ for all $S \in \mathcal{F}$. Upon arrival of an element e, if $\sum_{S \in \mathcal{F}(e)} x_S < 1$, then update $x_S \leftarrow 2 \cdot x_S + 1/|\mathcal{F}(e)|$ for all $S \in \mathcal{F}(e)$. Observe that at the end of the round, it is guaranteed that the fractional primal solution maintained by the algorithm satisfies the constraint since the algorithm adds at least $1/|\mathcal{F}(e)|$ to the variable x_S for each set $S \in \mathcal{F}(e)$.

Let $d = \max_{e \in \mathcal{U}'} |\mathcal{F}(e)|$ be the maximum degree of an element. The following assertion on the competitive ratio is established by Buchbinder and Naor in [17].

▶ **Lemma 12** ([17]). *The fractional set cover algorithm is $O(\log d)$-competitive.*

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[3] While our results in the current section are expressed in terms of the size of the universe n, it can be modified to obtain the same asymptotic bounds in terms of the length of the element sequence $|\mathcal{U}'|$.

[4] We note that the algorithm presented in [17] is designed for weighted set cover. The algorithm presented in this paper is its application for the case of unit weights.

Randomized rounding. An online rounding scheme that randomly obtains an integral solution from the fractional set cover algorithm was constructed by Alon et al. in [5]. The solution produced by the rounding scheme of [5] is feasible with high probability while incurring a multiplicative factor of $O(\log n)$ to the expected cost. However, this rounding method does not fit our advice framework. This is because all random coins are tossed in the beginning to compute a threshold for each set. Thus, we present a slightly different rounding method that fits our framework while maintaining similar guarantees.

The rounding procedure operates as follows. Consider an element e and let $\mathbf{x}$ and $\mathbf{x}_{int}$ be the solution maintained by the fractional algorithm and the (integral) solution maintained by the rounding scheme, respectively, at the time of e's arrival. If e is already covered by either the current fractional solution or the current integral solution produced by the rounding, then we do nothing (we will later show that the feasibility of $\mathbf{x}_{int}$ is maintained with high probability in this case). Otherwise (e is not covered by both solutions), we update $\mathbf{x}$ according to the fractional algorithm. For each $S \in \mathcal{F}(e)$, let x_S^{beg} be the value of the variable x_S at the beginning of the round and let $\delta(S) = x_S^{beg} + 1/|\mathcal{F}(e)|$ be the additive increase to x_S that occurs during the round. The rounding is obtained by independently selecting each set $S \in \mathcal{F}(e)$ to the cover with probability $\min\{1, \delta(S) \cdot \Theta(\log n)\}$.

We refer to the randomized algorithm described above (i.e., the fractional set cover algorithm combined with the rounding scheme) as RandSC. The properties of RandSC are described in the following lemma.

▶ **Lemma 13.** *RandSC is $O(\log n \log d)$-competitive and computes a feasible solution with high probability.*[5]

Proof. Let $\mathbf{x}$ be the solution obtained by the fractional algorithm at termination. Recall that in each round, set S is selected with probability at most $\delta(S) \cdot c \log n$ (for a constant $c > 0$). By linearity of expectation, the total expected cost associated with S is $O(\log n) \cdot x_S$. Thus, the expected cost of RandSC is $O(\log n) \cdot \sum_{S \in \mathcal{F}} x_S = O(\log n \log d) \cdot \mathsf{OPT}$.

We now bound the probability that there exists an element that was not covered by the integral solution produced by RandSC when it arrived. Consider an element e' arriving at round r. Notice that by construction, e' must be covered by the fractional solution at the end of round r. We argue that this implies that e' is covered by the integral solution with high probability. Let $\ell = |\mathcal{F}(e')|$ and let $S^1, \ldots S^\ell$ denote the sets in $\mathcal{F}(e')$. Let us denote by $\delta_{i,j}$ the increase to the variable x_{S^i} associated with set S^i in round j and let $p_{i,j}$ the probability that S^i was selected to the integral solution at round j. If $p_{i,j} = 1$ for some $i \leq \ell$ and $j \leq r$, then e' is covered by the end of round r with probability 1. Otherwise, due to the independence of selection events, the probability that e' is not covered by the integral solution at the end of round r is

$$\prod_{i=1}^{\ell} \prod_{j=1}^{r} (1 - p_{i,j}) \leq e^{-\sum_{i=1}^{\ell} \sum_{j=1}^{r} p_{i,j}} = e^{-c \log n \sum_{i=1}^{\ell} \sum_{j=1}^{r} \delta_{i,j}} \leq n^{-c},$$

where the final inequality holds because the fractional algorithm guarantees that e' is covered at round r and thus $\sum_{i=1}^{\ell} \sum_{j=1}^{r} \delta_{i,j} \geq 1$. By a union bound argument, the probability that there exists a set that is not covered by the integral solution is at most n^{1-c}. Thus, RandSC produces a feasible solution with probability at least $1 - 1/n^{c-1}$. ◀

[5] For simplicity, RandSC is described as a Monte Carlo algorithm. It can be easily transformed into a Las Vegas algorithm as follows: whenever an element e is not covered by RandSC upon the end of a round, select an arbitrary set that covers e into the solution. Notice that the added expected cost is negligible.

Oracle's advice. The idea of the oracle's advice is to boost the probability of selecting "good" sets while not losing the probabilistic feasibility guarantee of Lemma 13. For the sake of analysis, let us assume that the oracle is randomized (observe that this assumption does not enhance the oracle's power since the oracle can deterministically compute an optimal realization of the randomized selection). Let $\mathcal{A}^* \subseteq \mathcal{F}$ be an optimal solution for the set cover instance. Consider the arrival of an element e that was not covered yet by both the fractional and integral solutions and let p_S be the probability that set S is selected in the current round of RandSC for each set $S \in \mathcal{F}(e)$. The oracle's advice is as follows: (1) each set $S \in \mathcal{F}(e) \cap \mathcal{A}^*$ is selected to the advice; and (2) each set $S \in \mathcal{F}(e) - \mathcal{A}^*$ is independently selected to the advice with probability p_S. Notice that the argument used in Lemma 13 regarding the feasibility of the solution still holds since the oracle does not decrease the selection probability of any set at a given round.

Analysis. Let us denote the oracle described above by O_{boost}. We establish the following theorem.

▶ **Theorem 14.** *The competitive ratio of RandSC with the oracle O_{boost} against an oblivious adversary is $O(\log n) \cdot \min\{1/\alpha, \log d\}$, where $0 \leq \alpha \leq 1$ is the infusion parameter.*

Proof. We start by showing that RandSC with O_{boost} is $O(\log n \log d)$-competitive. Notice that by Lemma 13, the total expected cost associated with sets $S \in \mathcal{F} - \mathcal{A}^*$ is $O(\log n \log d) \cdot \mathsf{OPT}$. In addition, the total cost of sets in $\mathcal{A}^*$ is bounded by $|A^*| = \mathsf{OPT}$. Therefore, the expected cost of the solution produced by RandSC with O_{boost} is $O(\log n \log d) \cdot \mathsf{OPT}$.

We now show that RandSC with O_{boost} is $O(\frac{\log n}{\alpha})$-competitive. Consider the run of RandSC with O_{boost} on some element sequence. We refer to a round as a selection round if there exists a set that is selected with a positive probability in that round. Notice that we can bound the cost of RandSC with O_{boost} only in selection rounds (for non-selection rounds no cost is incurred). Observe that in each selection round, the probability of selecting a set from $\mathcal{A}^*$ is at least α (the probability of receiving advice). Moreover, if at some point in the execution all sets from $\mathcal{A}^*$ were selected, then there are no selection rounds after that point (since $\mathcal{A}^*$ covers all elements). Hence, the expected number of selection rounds during the execution is at most $|\mathcal{A}^*|/\alpha$.

To complete our analysis, we argue that the expected cost associated with sets that are not in $\mathcal{A}^*$ at each selection round is $O(\log n)$. Consider a selection round in which an elements e arrived. Recall that for each set $S \in \mathcal{F}(e) - \mathcal{A}^*$, we define $\delta(S) = x_S^{beg} + 1/|\mathcal{F}(e)|$, where x_S^{beg} is the value of variable x_S at the beginning of the round, and select each set $S \in \mathcal{F}(e) - \mathcal{A}^*$ to the cover with probability $\min\{1, \delta(S) \cdot \Theta(\log n)\}$. Thus, the total expected cost that comes from the sets $S \in \mathcal{F}(e) - \mathcal{A}^*$ in the round is bounded by $O(\log n) \cdot \sum_{S \in \mathcal{F}(e) - \mathcal{A}^*} x_S^{beg} + \frac{1}{|F(e)|} \leq O(\log n) \cdot 2 = O(\log n)$. Since the total cost associated with sets from $\mathcal{A}^*$ is at most $|\mathcal{A}^*|$, we get that the total expected cost of RandSC with O_{boost} is $O(\log n) \cdot |\mathcal{A}^*|/\alpha = O(\frac{\log n}{\alpha}) \cdot \mathsf{OPT}$. ◀

6 Lower Bounds

In this section we show fundamental limitations of online algorithms with RIA. First, we give a lower bound for competitiveness with RIA for online set cover, under the assumption that the algorithm is lazy (buys sets only when they are needed to cover the current element). Second, we give a lower bound for competitiveness with RIA for paging, that we improve to an asymptotically tight lower bound for the case of lazy algorithms. The lower bound for paging implies the lower bound for the uniform metrical task system.

6.1 Online set cover

We give a lower bound for the competitive ratio of any online randomized algorithm with RIA for online set cover. The construction of the input sequence is similar to the lower bounds given in [29, Theorem 2.2.1] and [17, Lemma 4.6]. The bound is given for randomness-oblivious (defined in Section 2.3) and lazy algorithms (lazy algorithms are allowed to buy a set only if it contains the current element).

▶ **Theorem 15.** *Assume that an online randomized algorithm with RIA for online set-cover is lazy, randomness-oblivious and strictly c-competitive against the oblivious adversary. Then* $c \geq \min\{\frac{1}{2}\log n, \frac{1}{2\alpha}\}$*, where n is the size of the universe of element, and α is the infusion parameter.*

Proof. Fix any lazy, randomness-oblivious online randomized algorithm ALG with RIA, its oracle O and the infusion parameter α. The adversary is oblivious to random choices of the algorithm, but it has access to the description of the algorithm, the oracle and the infusion parameter, hence can maintain the probability distribution of ALG's cache configurations.

Consider a complete binary tree with d leaves. The items to be covered are the nodes of the tree, and the sets are the d root-leaf paths. Our sequence σ will be the items on one root-leaf path, starting from the root and going downward.

We chose the sequence of items to request corresponding to a path in the complete binary tree as follows. Let $F(e)$ be the family of sets that cover the item e, and let p_S be the probability that ALG currently has the set S in the solution. The first request is to the root of the tree. For the i-th request, we choose one of the children, x or y of the item requested in the $(i-1)$-th request, depending on the probability distribution of the sets that cover these items. To decide between x and y, we choose the item $r \in \{x, y\}$ with no smaller sum of the probability mass $\sum_{F(x)} p_S$.

We consider two cases depending on whether or not the algorithm received advice for σ.

(1) Assume the algorithm did not receive advice for σ. In such case, the algorithm acts as an online algorithm without advice. Notice that the total probability mass of sets that do not appear in subsequent iterations add up to at least 1/2. Each path has length $\log n$, and the algorithm pays at least $\frac{1}{2}$ for each such round, hence overall the algorithm pays $\frac{1}{2}\log d$.

(2) Assume the algorithm received advice for σ. In expectation, the number of rounds before getting advice is $\frac{1}{\alpha}$, and the algorithm pays at least $\frac{1}{2}$ for each such round, hence in total the algorithm pays at least $\frac{1}{2} \cdot \frac{1}{\alpha} = \frac{1}{2\alpha}$.

Note that σ can be covered by a single set, namely the one that corresponds to the leaf where the path ends, hence $\mathsf{OPT}(\sigma) = 1$. The online algorithm pays at least $\min\{\frac{1}{2}\log d, \frac{1}{\alpha}\}$ for any sequence σ of the form described above, hence ALG is at least strictly $\min\{\frac{1}{2}\log d, \frac{1}{2\alpha}\}$-competitive. ◀

For lazy algorithms, we can obtain a lower bound in terms of the number of d. We say that an online algorithm for online set cover is *lazy* if it buys a set only if the current element is not yet covered, and then it may buy only sets that cover the current element. The next bound is stronger than the previous one, as it the bound is on the competitive ratio in the classic sense, with the possible additive constant, as opposed to the previous bound on the strict competitiveness.

▶ **Theorem 16.** *Assume that an online randomized algorithm with RIA for online set-cover is lazy, randomness-oblivious and c-competitive against the oblivious adversary. Then*

$c \geq \min\{\frac{1}{2}\log d, \frac{1}{2\alpha}\}$*, where* d *is the maximum element degree, and* α *is the infusion parameter.*

Proof. We repeat the construction from the previous proof of Theorem 15 in phases, in each phase using a binary tree of $2d$ items.

As the algorithm is lazy, it cannot buy sets from future phases, and the sets used in different phases are disjoint, hence advice received in any phase cannot decrease the cost of the algorithm in any future phase.

Fix any phase. We consider two cases depending on whether or not the algorithm received advice in this phase. If the algorithm received advice, then it pays at least $\frac{1}{2} \cdot \frac{1}{\alpha} = \frac{1}{2\alpha}$, as the expected number of rounds in this phase before receiving advice concerning sets in this phase is $\frac{1}{\alpha}$. Otherwise, if the algorithm did not receive advice, then it pays at least $\frac{1}{2} \cdot \log d$, following the arguments from the previous proof.

In total, the algorithm pays at least $\min\{\frac{1}{2}\log d, \frac{1}{2\alpha}\}$ in each phase, and an optimal algorithm can cover the items in each phase using a single set, hence the algorithm is at least $\min\{\frac{1}{2}\log d, \frac{1}{2\alpha}\}$-competitive.

Note that we can repeat this construction arbitrary number of iterations to obtain a lower bound on the competitive ratio, as opposed to a lower bound on strict competitive ratio. In each iteration, we use a new set of items and sets corresponding to a binary balanced tree, and the maximum number of sets that cover any item d does not increase by repeating the construction. Hence, no randomized algorithm with infused advice can be better than $\min\{\frac{1}{2}\log d, \frac{1}{2\alpha}\}$-competitive. ◀

6.2 Paging and metrical task systems

In this section we give a lower bound for competitiveness of randomized online algorithms with RIA for paging. The uniform metrical task system problem generalizes the paging problem on instances that include $n = k + 1$ pages, hence the lower bound for paging is a common lower bound for paging and uniform metrical task system. We restrict our attention to randomness-oblivious algorithms, as defined in Section 2.3. Our lower bound for any randomness-oblivious algorithm is loose by a factor of $1/k$; but with the natural assumption that the algorithm is *lazy*, we get rid of the $1/k$ factor, and for lazy algorithms the upper bounds for paging (Theorem 9) and uniform metrical task system (Theorem 11) are asymptotically optimal.

To show the lower bounds in this section, we apply Yao's Minimax Principle [41] to competitiveness of randomized online algorithms. In the case of classic online algorithms, the lower bound for the competitiveness of the best *deterministic* online algorithm on a distribution of inputs implies a lower bound on the competitiveness of any randomized online algorithm on any input sequence. To apply Yao's priciple to competitiveness of randomized online algorithms with RIA, we need to define a deterministic equivalent of algorithms with RIA. To this end, we add to each request the information whether the request is served by a deterministic online algorithm or by the oracle. We will analyze performance of such an algorithm on a distribution of requests, where each round is served by the algorithm with probability $1 - \alpha$, and by the oracle with probability α. To give a lower bound for randomness-oblivious algorithms (as defined in Section 2.3), we need to define a deterministic equivalent of such algorithms that we refer to as *deterministic advice-oblivious* algorithms: the answer for each request not served by the oracle is determined by the current request, previous requests and previous answers.

▶ **Theorem 17.** *Assume that an online randomized algorithm with RIA for online paging is randomness-oblivious and c-competitive against the oblivious adversary. Then* $c \geq \min\{H_k, \frac{1}{k \cdot \alpha}\}$, *where* α *is the infusion parameter.*

Proof. To prove the theorem, we apply Yao's Minimax Principle [41] to competitiveness of randomized algorithms. Consider any deterministic advice-oblivious algorithm A for paging, and construct the following distribution over input sequences. Each round is served by A with probability $1 - \alpha$, and by the oracle with probability α. The distribution over requests to pages is constructed as follows. Let $S = \{p_1, p_2, p_3, \ldots, p_{k+1}\}$ be a set of $k + 1$ pages. We construct a probability distribution for choosing a request sequence. The first request $\sigma(1)$ is chosen uniformly at random from S. Every other request $\sigma(t)$, $t > 1$, is made to a page that is chosen uniformly at random from $S \setminus \{\sigma(t-1)\}$. A phase starting with $\sigma(i)$ ends with $\sigma(j)$, where $j, j > i$ is the smallest integer such that $\{\sigma(i), \sigma(i+1), \ldots, \sigma(j)\}$ contains $k + 1$ distinct pages.

We claim that for any advice-oblivious algorithm, the advice received in past phases cannot reduce the cost of the algorithm in future phases. We argue as follows. First, the advice-oblivious algorithm is forbidden to store past advice in its internal memory for future use. Second, no algorithm can store meaningful advice for the future in its cache configuration: each phase contains requests to $k + 1$ different items, so for any cache configuration at the start of the phase, there is always at least 1 *clean page*: a page that is requested in the phase that the algorithm does not have in the cache at the start of the phase.

In our bounds, we use that the average cost of the algorithm for each request is $1/k$; this follows because the requested page is random and each of its pages is outside the cache with equal probability.

We lower-bound the cost of the algorithm in each phase in two ways, depending on whether or not the algorithm receives advice in any round of the phase.

(1) Assume that the algorithm does not receive advice in any round of the phase. In such case, the algorithm acts as an online algorithm without advice throughout the phase, and the expected cost of the algorithm in the phase is at least H_k, following the standard arguments [33]: the expected length of the phase is $k \cdot H_k$, the average cost of the algorithm for each request is $1/k$, therefore the cost of the algorithm within a phase is at least H_k.
(2) Assume that the algorithm receives advice in some round of the phase. To receive advice, we need in expectation $1/\alpha$ rounds prior to the advice round. The average cost of the algorithm for each request is $1/k$, hence the expected cost is at least $\frac{1}{k \cdot \alpha}$.

An optimal offline algorithm OPT incurs 1 page fault during each phase, the algorithm pays at least $\min\{H_k, \frac{1}{k \cdot \alpha}\}$, hence by summing over all phases of σ, we arrive at the desired competitive ratio. ◀

Next, we give an improved lower bound for lazy algorithms for paging. Recall that lazy algorithms for paging are the algorithms that are never allowed to change its cache configuration unless there is a page miss. This class includes RandomMark as well as most other known online paging algorithms. Note that this definition is slightly more general than the usual definition of lazy algorithms, where the algorithm is only allowed to fetch one page per request [14]; the intention of this definition is that the lower bound holds for metrical task systems as well. In the classic setting without infused advice, any algorithm can be turned to a lazy algorithm without increasing its cost; note, however, that the transformed algorithm may not be randomness-oblivious. If we restrict our attention to randomness-oblivious algorithms, the non-lazy algorithms may have an advantage over the lazy algorithms due to non-lazy algorithm's potentially frequent interaction with the oracle, which could be used by the oracle to give advice to prefetch some items even before the first cache miss occurs.

▶ **Theorem 18.** *Assume that an online randomized algorithm with RIA for online paging is lazy, randomness-oblivious and c-competitive against the oblivious adversary. Then* $c \geq \min\{H_k, \frac{1}{\alpha}\}$*, where* α *is the infusion parameter.*

Proof. To prove the theorem, we apply Yao's Minimax Principle [41] to competitiveness of randomized algorithms. Consider any deterministic advice-oblivious online algorithm and the probability distribution for choosing a request sequence as in the proof of Theorem 17.

We claim that for any advice-oblivious algorithm, the advice received in past phases cannot reduce the cost of the algorithm in future phases. We argue as follows. First, the advice-oblivious algorithm is forbidden to store past advice in its internal memory for future use. Second, no algorithm can store meaningful advice for the future in its cache configuration: each phase contains requests to $k+1$ different items, so for any cache configuration at the start of the phase, there is always at least 1 *clean page*: a page that is requested in the phase that the algorithm does not have in the cache at the start of the phase.

We will show that the expected cost of the algorithm is at least $\min\{H_k, \frac{1}{\alpha}\}$ in any phase. We lower-bound the cost of the algorithm in each phase in two ways, depending on whether in this phase the algorithm receives advice in some round with a cache miss or not.

(1) Assume that the algorithm does not receive advice in any round with a cache miss. Since the algorithm is lazy, advice received in rounds without cache misses does not influence the algorithm's cache configuration, and since the algorithm is advice-oblivious, it cannot store such advice either. In such case, the algorithm acts as an online algorithm without advice throughout the phase, and the expected cost of the algorithm in the phase is at least H_k, following the standard arguments [33]: the expected length of the phase is $k \cdot H_k$, the average cost of the algorithm for each request is $1/k$ because the requested page is random and each of its pages is outside the cache with equal probability, therefore the cost of the algorithm within a phase is H_k.

(2) Assume that the algorithm receives advice in a round with a cache miss. To receive advice at a round with a cache miss, we need in expectation $1/\alpha$ rounds with cache misses. Each round with a cache miss costs 1, hence the expected cost of the algorithm is at least $1/\alpha$.

An optimal offline algorithm OPT incurs a single page fault during each phase, and the algorithm pays at least $\min\{H_k, \frac{1}{\alpha}\}$, hence by summing over all phases of σ, we arrive at the desired competitive ratio. ◀

The bound given in Theorem 18 is asymptotically tight for lazy algorithms. However, a gap of a constant factor of 2 remains. To address this gap, an optimal randomized algorithm for paging [32] may be a possible direction for future studies.

7 Conclusions and Future Work

We introduced a novel method for the rigorous quantitative evaluation of online algorithms that relaxes the worst-case perspective of classic competitive analysis. The infused advice model allows the seamless integration of machine-learned predictors with existing randomized online algorithms.

We leave several avenues for future research, in particular to explore the utility of our method applied to other randomized online algorithms. Randomness-oblivious online algorithms are known for many online problems, e.g., all randomized memoryless algorithms [19] such as the COINFLIP algorithm for file migration [40] or the HARMONIC algorithm for k-server [10, 34] are randomness-oblivious.

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