CHAPTER 10

ELECTROMECHANICAL SYSTEMS

We can construct a wide variety of very useful devices by combining electrical and mechanical elements. Among the electromechanical devices that we shall consider are potentiometers, the galvanometer, the microphone, and motors and generators.

We first discuss coupling the electrical and mechanical parts of the system by mechanically varying a resistance. In the following section we consider the coupling associated with the movement of a current-carrying conductor through a magnetic field. Two new laws are needed to describe the additional forces and voltages caused by this action. A number of magnetically coupled devices are then considered in detail.

Some of the examples incorporate many of the analytical tools developed in the previous four chapters, such as damping ratio, steady-state response, transfer functions, and linearization. The chapter concludes with a comprehensive example that points out some of the important features that must be considered in the design of an electromechanical measuring device.

10.1 RESISTIVE COUPLING

We can control a variable resistance by mechanical motion either continuously by moving an electrical contact or discretely by opening and closing a switch. Because resistors cannot store energy, this method of coupling electrical and mechanical parts of a system, in contrast to coupling by magnetic and electrical fields, does not involve mechanical forces that depend on electrical variables.

Figure 10.1(a) shows a strip of conducting material having resistivity ρ and cross-sectional area A. One terminal is fixed to the left end of the conductor. The other terminal, known as the **wiper**, is free to slide along the bar while maintaining a good electrical connection at all times—that is, zero resistance at the contact point. The resistance per unit length of the bar is ρ/A, so the resistance between the terminals is

$$R = \left[\frac{\rho}{A}\right] x(t) \tag{1}$$

where $x(t)$, the displacement of the wiper, can vary with time. The rotational equivalent of this device is shown in Figure 10.1(b). Here the resistance between the two terminals is a function of the angle $\theta(t)$, which defines the angular orientation of the wiper with respect to the fixed terminal.

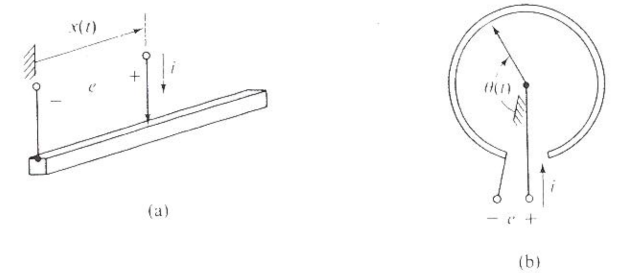

FIGURE 10.1 Variable resistors. (a) Translational. (b) Rotational.

A very useful device known as a **potentiometer** is obtained by adding a third terminal to the right ends of the variable resistors shown in Figure 10.1. For the translational potentiometer shown in Figure 10.2(a), the two end terminals are normally connected across a voltage source, and the voltage at the wiper is considered the output. The resistances R_1 and R_2 depend on the wiper position and are therefore labeled $R_1(x)$ and $R_2(x)$ in the figure. In our mathematical development, however, we shall omit the x in parentheses. We can regard either the position $x(t)$ or the voltage source $e_i(t)$, or both, as inputs.

The circuit diagram for the potentiometer is shown in Figure 10.2(b). Provided that no current flows through the wiper, we can apply the voltage-divider rule of (5.29) to obtain

$$e_o = \left[\frac{R_2}{R_1 + R_2}\right] e_i(t) \tag{2}$$

If the distance and the resistance between the fixed terminals are denoted by $x_{\max}$ and R_T, respectively, then $R_2 = [R_T/x_{\max}]x(t)$ and $R_1 + R_2 = R_T$. Substituting these two expressions into (2), we can write the output voltage

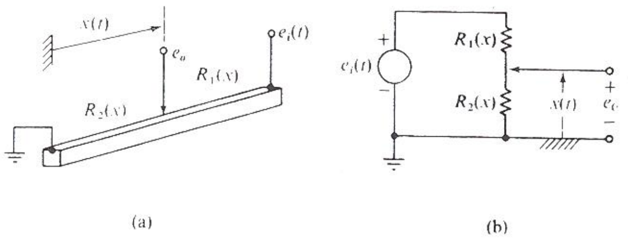

FIGURE 10.2 (a) Translational potentiometer. (b) Equivalent circuit.

as

$$e_o = \left[\frac{1}{x_{\max}}\right] x(t) e_i(t) \tag{3}$$

where the ratio $x(t)/x_{\max}$ must be between 0 and 1, inclusive.

We can interpret (3) as saying that the output voltage is proportional to the product of the input voltage $e_i(t)$ and the mechanical variable $x(t)$. For a constant input voltage, the output voltage is proportional only to the mechanical displacement $x(t)$. Sometimes, however, the electrical circuit includes an additional resistance across part of the potentiometer, which can complicate the desired relationship. This possible problem is illustrated in the following example.

▶ *EXAMPLE 10.1*

The rotary potentiometer shown in Figure 10.3 has a constant voltage E applied across its fixed terminals, and the wiper is connected to a constant resistance R_o, which might represent the resistance of a voltmeter or a recording device. Find the relationship between the output voltage e_o and the angular orientation of the mechanical rotor attached to the wiper. Assume that the resistance per unit length of the potentiometer is constant.

Solution

The wiper contact divides the total potentiometer resistance R_T into R_1 and R_2, such that in the circuit shown in Figure 10.4(a),

$$\begin{aligned} R_2 &= \left[\frac{R_T}{\theta_{\max}}\right]\theta(t) \\ R_1 + R_2 &= R_T \end{aligned} \tag{4}$$

The values of R_1 and R_2 depend on $\theta(t)$, as we emphasize by labeling them $R_1(\theta)$ and $R_2(\theta)$ in the figure. Because R_o and R_2 are connected in parallel, they can be replaced by a single equivalent resistance $R_{eq} = R_oR_2/(R_o+R_2)$, which gives the circuit shown in Figure 10.4(b).

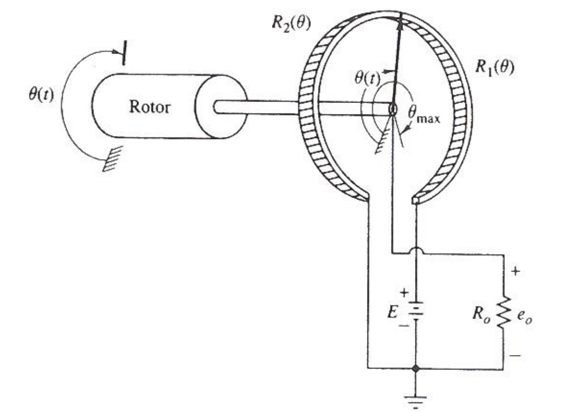

FIGURE 10.3 Potentiometer used to measure the angular orientation of a rotor.

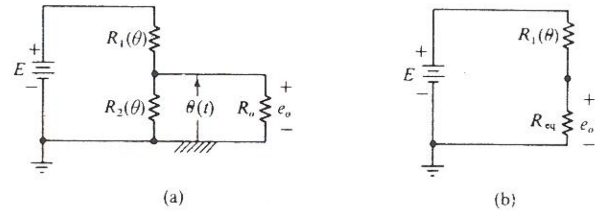

FIGURE 10.4 Equivalent circuits for Example 10.1.

By the voltage-divider rule, the output voltage is

$$\begin{aligned} e_o &= \left(\frac{R_{eq}}{R_{eq}+R_1}\right)E = \left(\frac{R_oR_2}{R_oR_2+R_oR_1+R_1R_2}\right)E \\ &= \left(\frac{R_oR_2}{R_oR_T+R_1R_2}\right)E \\ &= \left[\frac{1}{1+\left(\frac{R_T}{R_o}\right)\left(\frac{R_1}{R_T}\right)\left(\frac{R_2}{R_T}\right)}\right]\left(\frac{R_2}{R_T}\right)E \end{aligned} \tag{5}$$

Substituting (4) into (5) gives

$$e_o = \left(\frac{E}{\theta_{\max}}\right)\left[\frac{1}{1+\left(\dfrac{R_T}{R_o}\right)\left[1-\dfrac{\theta(t)}{\theta_{\max}}\right]\left[\dfrac{\theta(t)}{\theta_{\max}}\right]}\right]\theta(t) \tag{6}$$

which is the desired result.

Unfortunately, (6) is a nonlinear relationship between the output voltage and the rotor angle because of the presence of $\theta(t)$ in the coefficient of $\theta(t)$. Although we could still use the potentiometer to measure $\theta(t)$, the voltmeter or recording device would require a nonlinear scale, which would be inconvenient.

However, if the output resistance R_o is large compared to the total potentiometer resistance R_T, then the coefficient in the brackets is approximately unity because $0 \leq \theta(t)/\theta_{\max} \leq 1$. Then (6) reduces to the linear relationship

$$e_o = \left[\frac{E}{\theta_{\max}}\right]\theta(t) \quad \text{for } R_o \gg R_T \tag{7}$$

Equation (7) indicates that we can treat the potentiometer as a gain of $E/\theta_{\max}$ volts per radian with $\theta(t)$ as its input and e_o as its output. This simplification can be used whenever the parallel combination of R_o and R_2 has a resistance equal to R_2. This is equivalent to saying that the current through R_o is negligible compared to that through R_2. When R_o is not sufficiently large for us to use (7), we say that the resistance R_o **loads** the potentiometer, and we use the nonlinear expression in (6).

10.2 COUPLING BY A MAGNETIC FIELD

A great variety of electromechanical devices contain current-carrying wires that can move within a magnetic field. The physical laws governing this type of electromechanical coupling are given in introductory physics textbooks such as Halliday and Resnick (see Appendix D). These laws state that (1) a wire in a magnetic field that carries a current will have a force exerted on it, and (2) a voltage will be induced in a wire that moves relative to the magnetic field. The variables we need to model such devices are

f_e, the force on the conductor in newtons (N)

v, the velocity of the conductor with respect to the magnetic field in meters per second (m/s)

ℓ, the length of the conductor in the magnetic field in meters (m)

ϕ, the flux in webers (Wb)

$\mathcal{B}$, the flux density of the magnetic field in webers per square meter (Wb/m^2)

i, the current in the conductor in amperes (A)

e_m, the voltage induced in the conductor in volts (V)

In all of our examples the variables will be scalar quantities. However, we shall first introduce the basic laws in a very general way, where the force, velocity, length, and flux density can have any spatial orientation. For the general case we will represent these four quantities as vectors and use the boldface symbols $\mathbf{f}_e$, $\mathbf{v}$, $\boldsymbol{\ell}$, and $\boldsymbol{\mathcal{B}}$.

The force on a conductor of differential length $d\boldsymbol{\ell}$ carrying a current i in a magnetic field of flux density $\boldsymbol{\mathcal{B}}$ is

$$d\mathbf{f}_e = i(d\boldsymbol{\ell} \times \boldsymbol{\mathcal{B}}) \tag{8}$$

The cross in (8) represents the vector cross product. To obtain the total electrically induced force $\mathbf{f}_e$, we must integrate (8) along the length of the conductor.

In our applications, the wires will be either straight conductors that are perpendicular to a unidirectional magnetic field or circular conductors in a radial magnetic field. In either case, the differential length $d\boldsymbol{\ell}$ will be perpendicular to a uniform flux density $\boldsymbol{\mathcal{B}}$. Then (8) simplifies to the scalar relationship

$$f_e = \mathcal{B}\ell i \tag{9}$$

where the direction of the force is perpendicular to both the wire and the magnetic field and can be found by the following rule. If the bent fingers of the right hand are pointed from the positive direction of the current toward the positive direction of the magnetic field (through the right angle), the thumb will point in the positive direction of the force. Figure 10.5(a) shows that for a positive current into the page and a positive flux to the left, the force on a straight conductor is upward. The position of the right hand corresponding to this situation is shown in Figure 10.5(b).

The voltage induced in a conductor of differential length $d\boldsymbol{\ell}$ moving with velocity $\mathbf{v}$ in a field of flux density $\boldsymbol{\mathcal{B}}$ is

$$de_m = (\mathbf{v} \times \boldsymbol{\mathcal{B}}) \cdot d\boldsymbol{\ell} \tag{10}$$

where the dot denotes the scalar product, or dot product, used with vector notation. We obtain the total induced voltage for a conductor by integrating (10) between the ends of the conductor.

In practice, the three vectors in (10) will be mutually perpendicular, so integrating (10) yields the scalar relationship

$$e_m = \mathcal{B}\ell v \tag{11}$$

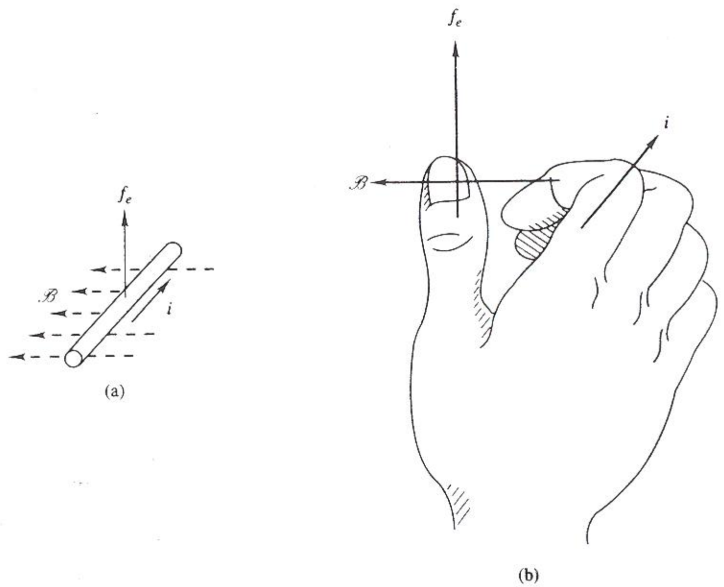

FIGURE 10.5 Right-hand rule for the force on a conductor.

If the bent fingers of the right hand are pointed from the positive direction of the velocity toward the positive direction of the magnetic field (through the right angle), the thumb will point in the direction in which the current caused by the induced voltage tends to flow. In Figure 10.6(a), a straight conductor is shown moving downward in a magnetic field directed to the left. As indicated by the polarity signs and by the sketch shown in Figure 10.6(b), the positive sense of the induced voltage is into the page. If the conductor were part of a complete circuit with no external sources, the current in the conductor would be into the page. According to Figure 10.5, this current would cause there to be exerted on the conductor an upward force that would oppose the downward motion.

Equations (9) and (11), which describe the force and induced voltage associated with a wire moving perpendicularly to a magnetic field, can be incorporated in a schematic representation of a translational electromechanical system as shown in Figure 10.7. The induced voltage is represented by a source in the electrical circuit, whereas the magnetically induced force is shown acting on the mass M to which the conductor is attached. We can determine the proper polarity marks for e_m and the reference direction for f_e by examining the specific device under consideration, but the figure indicates two combinations of reference directions that are consistent with the conservation of energy.

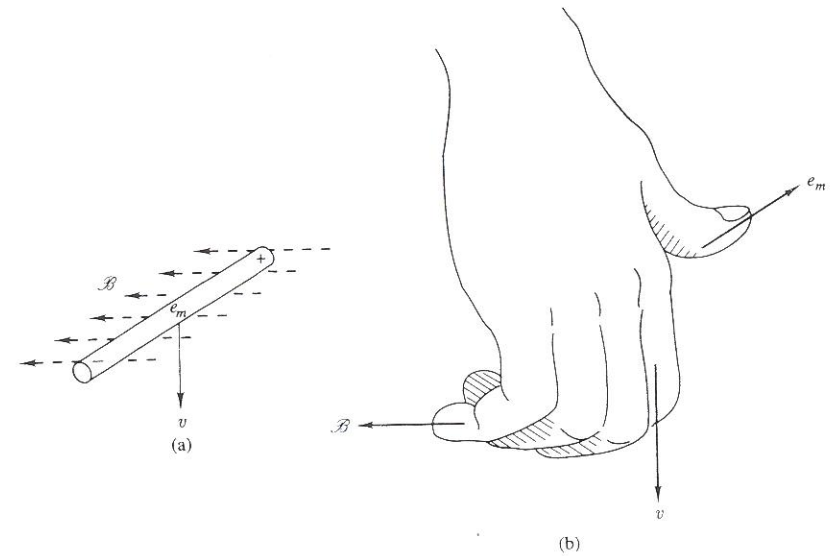

FIGURE 10.6 Right-hand rule for the voltage induced in a moving conductor.

In Figure 10.7(a), the polarity of the electrical source is such that power is absorbed from the remainder of the circuit when both e_m and i are positive. Likewise, when f_e and v are positive, there is a transfer of power into the mechanical part of the system. Conversely, for Figure 10.7(b), power flows from the mechanical side to the electrical side when all four variables are positive.

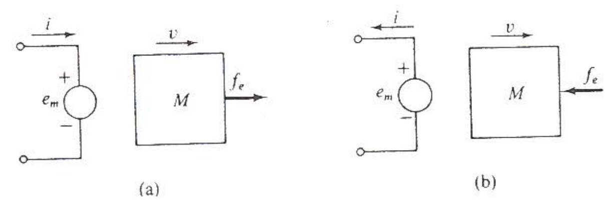

FIGURE 10.7 Representations of a translational electromechanical system. (a) Electrical-to-mechanical power flow. (b) Mechanical-to-electrical power flow.

It is instructive to evaluate the power involved in the coupling mechanism. The external power delivered to the electrical part of Figure 10.7(a) is

$$p_e = e_m i = (\mathscr{B}\ell v)i$$

whereas the power available to whatever mechanical elements are attached to the coil is

$$p_m = f_e v = (\mathscr{B}\ell i)v$$

Hence,

$$p_m = p_e$$

which says that any power delivered to the coupling mechanism in electrical form will be passed on undiminished to the mechanical portion. Of course, any practical system has losses resulting from the resistance of the conductor and the friction between the moving mechanical elements. However, any such dissipative elements can be modeled separately by a resistor in the electrical circuit or a viscous-friction element acting on the mass. In a similar way, you can demonstrate that for the coupling shown in Figure 10.7(b), the mechanical power supplied by the force f_e is transmitted to whatever electrical elements are connected across e_m.

10.3 DEVICES COUPLED BY MAGNETIC FIELDS

Having introduced the basic laws that govern the behavior of a single conductor in a magnetic field, we shall describe several of the most common types of electromechanical systems and derive their mathematical models. In each case, we consider idealized versions of the device that omit certain aspects that may be important from a design standpoint but are not essential to understanding its operation as part of an overall system. We shall examine in turn the galvanometer, a microphone, and a motor.

The Galvanometer

The galvanometer is a device that produces an angular deflection dependent on the current passing through a coil attached to a pointer. It is widely used in electrical measurement devices. As shown in Figure 10.8(a), a permanent magnet supplies a radial magnetic field, and the flux passes through a stationary iron cylinder between the poles of the magnet. A coil of wire whose terminals can be connected to an external circuit is suspended by bearings so that it can rotate about a horizontal axis passing through the center of the cylinder. A torsional spring mounted on its axis restrains the coil.

The magnet provides a uniform flux density $\mathscr{B}$ within the air gaps between it and the iron cylinder directed from the north to the south pole. The moment of inertia of the coil is J, and the combination of bearing friction and damping due to the air is represented by the viscous-damping coefficient B. The torsional spring has the rotational spring constant K. It is assumed that the electrical connection between the external circuit and the

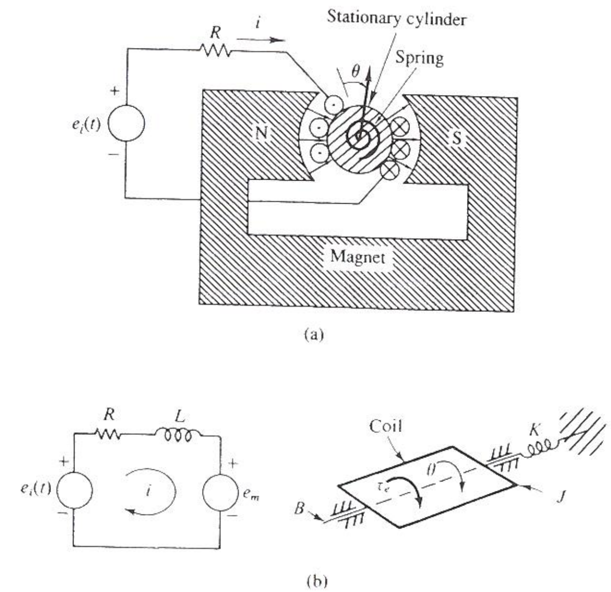

FIGURE 10.8 Galvanometer. (a) Physical device. (b) Diagrams used for analysis.

movable coils is made in such a way that the connection exerts no torque on the coil.

The coil consists of N rectangular turns, each of which has a radius of a and a length of ℓ along the direction of the axis of rotation, and it has a total inductance L. The dots in the wires on the left side of the coil and the crosses on the right side indicate that when i is positive, the current flows out of the page in the left conductors and into the page in the right conductors. The remainder of the circuit consists of a voltage source $e_i(t)$ and a resistance R that accounts for any resistance external to the galvanometer as well as for the resistance of the coils.

For purposes of analysis, we represent the idealized system by the circuit and mechanical diagrams shown in Figure 10.8(b). We have used the rotational equivalent of Figure 10.7(a) rather than Figure 10.7(b) to represent the electromechanical coupling, because the purpose of the device is to convert an electrical variable (current) into a mechanical variable (angular displacement). It remains to use (9) and (11), with the appropriate right-hand rules, to determine the expressions for e_m and τ_e, the electrically induced torque.

Assume that the rotation of the coil is sufficiently small that all the conductors remain in the region of constant flux density. Then we can obtain the torque τ_e acting about the axis of the coil by summing the torques on the $2N$ individual conductors of length ℓ. Because the magnetic field is confined to the iron cylinder as it passes between the air gaps, there is no contribution to the torque from the ends of the coil outside the gaps. If the current is positive, each conductor on the left side of the coil has a force of $f_e = \mathscr{B}\ell i$ acting upward, whereas the conductors on the right side have forces of the same magnitude acting downward. Because the arrow denoting the electrically induced torque τ_e was taken as clockwise in Figure 10.8(b), the expression for the torque is

$$\tau_e = (2N\mathscr{B}\ell a)i \tag{12}$$

Next we express the voltage induced in the coil in terms of the angular velocity $\dot{\theta}$. Because there are $2N$ conductors in series, which move with the velocity $a\dot{\theta}$ with respect to the magnetic field, a voltage of $2N\mathscr{B}\ell a\dot{\theta}$ is induced in the coil. We find the sign of the mechanically induced voltage e_m by applying the right-hand rule illustrated in Figure 10.6(b). The conductors on the right side of the coil in Figure 10.8(a) move downward when $\dot{\theta} > 0$ and the flux-density vector $\mathscr{B}$ points to the right, toward the south pole. Thus the positive sense of e_m is directed toward the viewer, and the corresponding current direction is out of the paper, opposite to the reference arrow for i in the figure. The circuit shown in Figure 10.8(b) was drawn with this assumed polarity for e_m, so we have

$$e_m = (2N\mathscr{B}\ell a)\dot{\theta} \tag{13}$$

Summing torques on the coil and using (12) give

$$J\ddot{\theta} + B\dot{\theta} + K\theta = (2N\mathscr{B}\ell a)i \tag{14}$$

Using (13) in a voltage-law equation for the single loop that makes up the electrical part of the system gives

$$L\frac{di}{dt} + Ri + (2N\mathscr{B}\ell a)\dot{\theta} = e_i(t) \tag{15}$$

Equations (14) and (15) constitute the complete model of the galvanometer, which is a third-order model. From these two equations we can determine the state-variable equations, the transfer function, and the input-output differential equation. In practical situations, the inductance of the coil is often sufficiently small to allow us to neglect the term $L\,di/dt$ in (15). This simplification is particularly helpful in solving for the response to an input voltage, because the model becomes second-order. When this term is dropped, we can solve (15) for the current, obtaining

$$i = \frac{1}{R}\left[e_i(t) - (2N\mathscr{B}\ell a)\dot{\theta}\right]$$

By substituting this expression into (14), we have, for the input-output equation,

$$\ddot{\theta} + \left(\frac{B}{J} + \frac{\alpha^2}{JR}\right)\dot{\theta} + \frac{K}{J}\theta = \frac{\alpha}{JR}e_i(t) \tag{16}$$

where $\alpha = 2N\mathscr{B}\ell a$ is the electromechanical coupling coefficient. Comparing (16) with (6.49), we find that the undamped natural frequency ω_n and the damping ratio ζ of the galvanometer are

$$\omega_n = \sqrt{\frac{K}{J}}$$

$$\zeta = \frac{1}{2\sqrt{KJ}}\left(B + \frac{\alpha^2}{R}\right)$$

Hence the undamped natural frequency depends on the mechanical parameters K and J, and the damping ratio depends on both the mechanical and electrical parameters and the electromechanical coupling coefficient α.

We can find the sensitivity of the galvanometer in radians per volt by solving for the forced response to the constant excitation $e_i(t) = A$. The steady-state value of θ will be the particular solution of (16), namely

$$\theta_{ss} = \left(\frac{\alpha}{KR}\right)A \tag{17}$$

which is a constant. The galvanometer sensitivity is $\theta_{ss}/A = \alpha/KR$. Thus a higher flux density will make the device more sensitive, and increasing either the spring stiffness or the electrical resistance will reduce its sensitivity.

We can arrive at the same result for the sensitivity by the following argument. Because in the steady state the rotor will be stationary, no voltage will be induced in the coils. Thus the steady-state current will be $i_{ss} = A/R$. The steady-state torque exerted on the rotor through the action of the magnetic field will be $(\tau_e)_{ss} = \alpha A/R$, which must be balanced entirely by the torsional spring that exerts the steady-state torque $K\theta_{ss}$. Equating these two torques gives (17).

If we wish to characterize the system by a set of state-variable equations, we can choose θ and ω as the state variables and use (16) to write

$$\dot{\theta} = \omega$$

$$\dot{\omega} = -\frac{K}{J}\theta - \left(\frac{B}{J} + \frac{\alpha^2}{JR}\right)\omega + \frac{\alpha}{JR}e_i(t)$$

Had we retained the inductance of the coil, the current i would have become the third state variable.

A Microphone

The microphone shown in cross section in Figure 10.9 consists of a diaphragm attached to a circular coil of wire that moves back and forth through a magnetic field when sound waves impinge on the diaphragm. The magnetic field is supplied by a cylindrical permanent magnet having concentric north and south poles, which result in radial lines of flux directed inward toward the axis of the magnet. The coil has N turns with a radius of a and is connected in series with an external resistor R, across which the output voltage is measured. The positive direction for the current i is assumed to be counterclockwise from the perspective of viewing the magnet from the diaphragm. In the figure, the resistance of the coil has been neglected, but the inductance of the coil is represented by the inductor L located externally to the coil. If the coil resistance were not negligible, another resistor could be added in series with the external resistor in order to account for it.

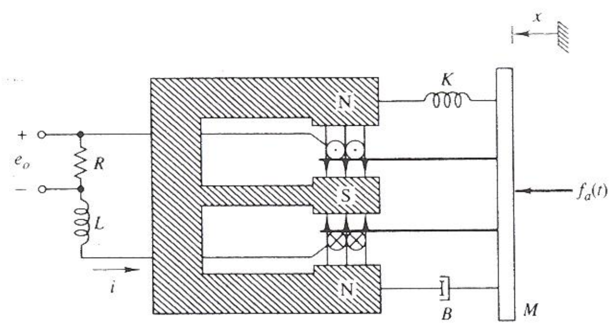

FIGURE 10.9 Representation of a microphone.

A single stiffness element K has been used to represent the stiffness of the entire diaphragm, and the viscous-friction element B has been used to account for the energy dissipation due to air resistance. The net force of the impinging sound waves is represented by $f_a(t)$, which is the input to the system. Although the forces acting on the diaphragm are certainly distributed in nature, it is a justifiable simplification to consider them as point forces associated with the lumped elements K, B, and M.

The first step in modeling the microphone is to construct idealized diagrams of the electrical and mechanical portions. In developing the equations for the galvanometer, we assumed at the outset positive senses for τ_e and e_m. Then we wrote expressions for these quantities, determining their signs by using the right-hand rules. This time, we shall first determine the positive directions of f_e, the electrically induced force on the diaphragm, and the mechanically induced voltage e_m when the other variables are positive. After that, we shall label the diagram with senses that agree with the positive directions of f_e and e_m. When we use this approach, we know in advance that the expressions for f_e and e_m will have plus signs.

Figure 10.10(a) shows the upper portion of a single turn of the coil as viewed from the diaphragm looking toward the magnet. Because the flux arrow points downward from the north to the south pole and the current arrow points to the left, the right-hand rule shown in Figure 10.5(b) indicates that the positive sense of f_e is toward the diaphragm. Because the velocity arrow points away from the diaphragm, the right-hand rule shown in Figure 10.6(b) indicates that the positive sense of the induced voltage is the same as that of the current. We can reach similar conclusions by examining any other part of the coil. Thus we can draw the complete diagram of the system, as shown in Figure 10.10(b).

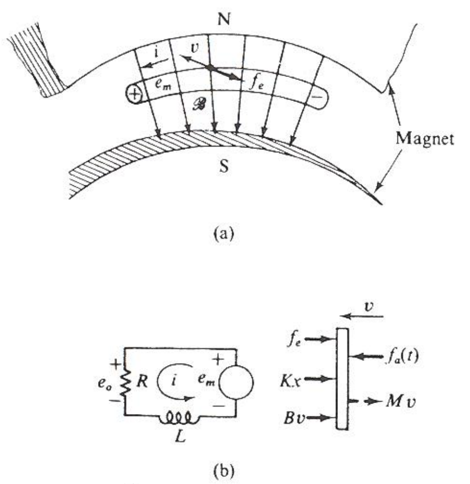

FIGURE 10.10 Microphone. (a) Relationships for a portion of a single coil. (b) Diagram used for analysis.

Because of the radial symmetry of the coil and the flux lines in the air gap, the entire coil of length $2\pi aN$ is perpendicular to the flux. Thus

$$f_e = \alpha i \tag{18a}$$

$$\tau_e = \alpha v \tag{18b}$$

where $\alpha = 2\pi aN\mathscr{B}$ is the electromechanical coupling coefficient for the system.

Summing forces on the free-body diagram for the diaphragm and using (18a), we obtain

$$M\dot{v} + Bv + Kx = -\alpha i + f_a(t) \tag{19}$$

We find the circuit equation by applying Kirchhoff's voltage law and using (18b), which yields

$$L\frac{di}{dt} + Ri = \alpha v \tag{20}$$

The system has three independent energy-storing elements (L, K, and M), so an appropriate set of state variables is i, x, and v. By rewriting (19) and (20), we obtain the state-variable equations

$$\frac{di}{dt} = \frac{1}{L}(-Ri + \alpha v)$$

$$\dot{x} = v$$

$$\dot{v} = \frac{1}{M}[-\alpha i - Kx - Bv + f_a(t)]$$

The output e_o is not a state variable, but we can find it from

$$e_o = Ri \tag{21}$$

In one of the end-of-chapter problems, you are asked to verify that the corresponding transfer function is

$$H(s) = \frac{E_o(s)}{F_a(s)} = \frac{\alpha Rs}{MLs^3 + (MR + BL)s^2 + (KL + BR + \alpha^2)s + KR} \tag{22}$$

and that the input-output equation is

$$ML\dddot{e}_o + (MR + BL)\ddot{e}_o + (KL + BR + \alpha^2)\dot{e}_o + KRe_o = \alpha R\dot{f}_a \tag{23}$$

It is interesting to note that because the right side of (23) is proportional to $\dot{f}_a$ rather than to $f_a(t)$, the forced response for a constant-force input is zero. Thus a constant force applied to the diaphragm yields no output voltage in the steady state. This same conclusion can be reached by noting from (22) that $H(0) = 0$. In Figure 10.9, the constant applied force will be balanced by the steady-state spring force Kx_{ss}, the diaphragm will become stationary, and no voltage will be induced in the coil.

A Direct-Current Motor

A direct-current (dc) motor is somewhat similar to the galvanometer but differs from it in several significant respects. In all but the smallest motors, the magnetic field is established not by a permanent magnet but by a current in a separate field winding on the iron core that constitutes the stationary part of the motor, the **stator**. Figure 10.11 indicates the manner in which the field winding establishes the flux when the field current i_F flows. Because of the saturation effects of magnetic fields in iron, the flux ϕ is not necessarily proportional to i_F at high currents.

In a dc motor, the iron cylinder between the poles of the magnet is free to rotate and is called the **rotor**. The rotating coils are embedded in the surface

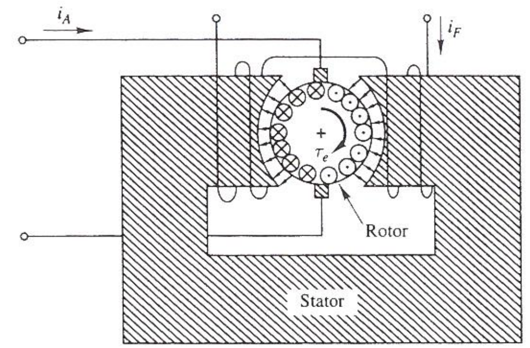

FIGURE 10.11 DC motor showing field and armature windings.

of the rotor and are known as the **armature winding**. There is no restraining torsional spring, and the rotor is free to rotate through an indefinite number of revolutions. However, this fact requires a significant deviation from the galvanometer in the construction of the rotor and the armature winding. First, if the rotor shown in Figure 10.11 rotates through 180° without a change in the direction of the currents in the individual armature conductors, then the torque exerted on it through the magnetic field will undergo a change in direction. Second, if there were a direct connection of the external armature circuit to the rotating armature, the wires would soon become tangled and halt the machine.

To solve both these problems, we use a **commutator**, which consists of a pair of low-resistance carbon brushes that are fixed with respect to the stator and make contact with the ends of the armature windings on the rotor (see the references in Appendix D for details). As indicated in Figure 10.11, when a conductor is located to the right of the commutator brushes, a positive value of i_A implies that the individual conductor current will be directed toward the reader. When the conductor is located to the left of the brushes, its current flows away from the reader when $i_A > 0$. As the ends of a particular conductor pass under the brushes, the direction of the current in that conductor changes sign. Under the arrangement just described, each conductor exerts a unidirectional torque on the rotor as it passes through a complete revolution. Hence the sliding contact at the brushes solves the mechanical problem of connecting the stationary and rotating parts of the armature circuit.

For modeling purposes, it is convenient to represent the important characteristics of the motor as shown in Figure 10.12. Representing the armature by a stationary circuit having resistance R_A, inductance L_A, and induced-voltage source e_m is justified by the presence of the commutator, which makes the armature behave as if it were stationary even though the individ-

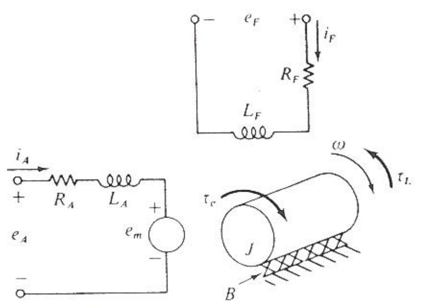

FIGURE 10.12 Diagram of the dc motor used for analysis.

ual conductors are indeed rotating. Likewise, the field circuit has resistance R_F and inductance L_F but no induced voltage. The rotor has moment of inertia J, rotational viscous-damping coefficient B, a driving torque τ_e caused by the forces acting on the individual conductors, and a load torque τ_L. The electrical inputs to the motor may be considered to be the currents i_A and i_F or the applied voltages e_A and e_F. The output is ω, the angular velocity of the rotor.

We begin the modeling process by expressing the voltage e_m and the torque τ_e in terms of the other system variables. We then write voltage-law equations for both the armature and field circuits, unless i_A or i_F is an input, and apply D'Alembert's law to the rotor.

When we are modeling dc motors and generators, it is convenient to express the flux density $\mathscr{B}$ as

$$\mathscr{B} = \frac{1}{A}\phi(i_F) \tag{24}$$

where $\phi(i_F)$ is the total flux established by the field current and A is the effective cross-sectional area of the flux path in the air gap between the rotor and stator. If ℓ denotes the total length of the armature conductors within the magnetic field and a denotes the radius of the armature, the electromechanical torque exerted on the rotor is

$$\tau_e = \left(\frac{\phi}{A}\right)\ell a i_A \tag{25}$$

Because the parameters ℓ, a, and A depend only on the geometry of the motor, we can define the motor parameter

$$\gamma = \frac{\ell a}{A} \tag{26}$$

and rewrite (25) in terms of the flux and armature current as

$$\tau_e = [\gamma\phi(i_F)]i_A \tag{27}$$

Similarly, the voltage induced in the armature is

$$e_m = [\gamma\phi(i_F)]\omega \tag{28}$$

Keep in mind that the flux $\phi(i_F)$ in (27) and (28) is a function of the field current i_F and, for that matter, is generally a nonlinear function.

Having established the model for the internal behavior of a dc motor represented by Figure 10.12 and (26), (27), and (28), we are prepared to develop models of complete electromechanical systems involving dc motors. We shall consider two such systems in the following examples.

▶ *EXAMPLE 10.2*

Derive the state-variable equations for a dc motor that has a constant field voltage E_F, an applied armature voltage $e_i(t)$, and a load torque $\tau_L(t)$. Also obtain the input-output equation with ω as the output, and determine the steady-state angular velocities corresponding to the following sets of inputs: (1) $e_i(t) = E$, $\tau_L(t) = 0$ and (2) $e_i(t) = 0$, $\tau_L(t) = T$.

Solution

The basic motor diagram in Figure 10.12 is repeated in Figure 10.13, with the specified field and armature input voltages added. The field voltage E_F is constant, so the field current will be the constant $i_F = E_F/R_F$. We can write the electromechanical driving torque τ_e and the induced voltage e_m as

$$\begin{aligned} \tau_e &= \alpha i_A \\ e_m &= \alpha\omega \end{aligned} \tag{29}$$

where α is a constant defined by

$$\alpha = \gamma\phi(i_F) \tag{30}$$

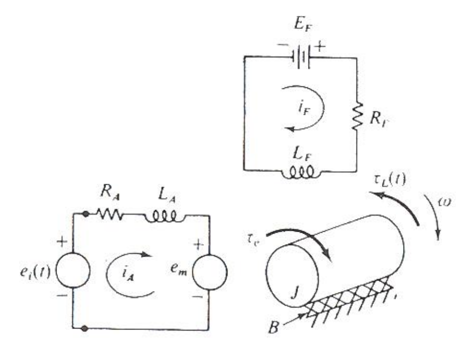

FIGURE 10.13 DC motor with a constant field current.

and γ is given by (26). We select i_A and ω as the state variables and write a voltage equation for the armature circuit and a torque equation for the rotor. Then, using (29) and solving for the derivatives of the state variables, we find the state-variable equations to be

$$\begin{aligned}\frac{di_A}{dt} &= \frac{1}{L_A}[-R_A i_A - \alpha\omega + e_i(t)] \\ \dot{\omega} &= \frac{1}{J}[\alpha i_A - B\omega - \tau_L(t)]\end{aligned} \tag{31}$$

In order to find the system's transfer functions, we apply the Laplace transform to (31) and assume that there is no initial stored energy. With $i_A(0) = 0$ and $\omega(0) = 0$, we have

$$L_A s I_A(s) = -R_A I_A(s) - \alpha\Omega(s) + E_i(s)$$
$$Js\Omega(s) = \alpha I_A(s) - B\Omega(s) - \tau_L(s)$$

Eliminating $I_A(s)$ from this pair of algebraic equations and solving for the transformed output $\Omega(s)$, we find that

$$\Omega(s) = H_1(s)E_i(s) + H_2(s)\tau_L(s)$$

where

$$H_1(s) = \frac{\alpha/JL_A}{P(s)}$$
$$H_2(s) = \frac{-(1/J)s - (R_A/JL_A)}{P(s)}$$
$$P(s) = s^2 + \left(\frac{R_A}{L_A} + \frac{B}{J}\right)s + \left(\frac{R_A B + \alpha^2}{JL_A}\right)$$

The quantity $H_1(s)$ is the transfer function relating the output velocity and input voltage when $\tau_L(t) = 0$. $H_2(s)$ relates $\Omega(s)$ and $\tau_L(s)$ when $e_i(t) = 0$. The general input-output differential equation is

$$\ddot{\omega} + \left(\frac{R_A}{L_A} + \frac{B}{J}\right)\dot{\omega} + \left(\frac{R_A B + \alpha^2}{JL_A}\right)\omega = \frac{\alpha}{JL_A}e_i(t) - \frac{1}{J}\dot{\tau}_L - \frac{R_A}{JL_A}\tau_L(t) \tag{32}$$

As expected, both the electrical and the mechanical parameters contribute to the system's undamped natural frequency ω_n and to the damping ratio ζ.

To solve for the steady-state motor speed when the voltage source has the constant value $e_i(t) = E$ and when $\tau_L(t) = 0$, we omit all derivative terms in (32) and substitute these values for $e_i(t)$ and $\tau_L(t)$, obtaining

$$\omega_{ss} = \frac{\alpha E}{R_A B + \alpha^2} \tag{33}$$

In physical terms, the motor will run at a constant speed such that the driving torque $\tau_e = \alpha i_A$ exactly balances the viscous-frictional torque $B\omega_{ss}$. However, the steady-state armature current is $i_A = (E - e_m)/R_A$, where $e_m = \alpha\omega_{ss}$. Making the appropriate substitutions, we again obtain (33). We get the same expression by examining $H_1(0)$.

When $e_i(t) = 0$ and $\tau_L(t)$ has the constant value T, the steady-state solution to (32) is

$$\omega_{ss} = -\frac{R_A T}{R_A B + \alpha^2} \tag{34}$$

which indicates that the motor will be driven backward at a constant angular velocity. We can also obtain (34) by noting that $\omega_{ss} = TH_2(0)$. To understand the behavior of the motor under this condition, we observe that the electromechanical torque, $\tau_e = \alpha i_A$, must balance the sum of the load torque T and the viscous-frictional torque $B\omega_{ss}$. Thus an armature current must flow that will make $\alpha i_A = T + B\omega_{ss}$. Furthermore, because the applied armature voltage is zero and $e_m = \alpha\omega_{ss}$, it follows that $i_A R_A = -\alpha\omega_{ss}$. As anticipated, solving these two equations for ω_{ss} results in (34).

In this situation, the motor is acting as a **generator** connected to a load of zero resistance. Part of the mechanical power supplied by the load torque is being converted to electrical form and dissipated in the armature resistance R_A. Basically, we may think of a generator as a motor that is being driven mechanically and that delivers a portion of the power to an electrical load connected across the armature terminals.

You can verify that when the constant applied voltage is $e_i(t) = E$ and the constant load torque is $\tau_L(t) = \alpha E/R_A$, the steady-state motor speed is $\omega_{ss} = 0$. In this condition, the electromechanical driving torque τ_e exactly matches the load torque and the motor is stalled.

▶ *EXAMPLE 10.3*

A dc motor has a constant armature current $\bar{i}_A$ but a variable current source $i_F(t)$ supplying the field winding. The relationship between the flux ϕ and the field current $i_F(t)$ is nonlinear and is shown in Figure 10.14(a). A load having moment of inertia J_L and viscous-damping coefficient B_L is connected to the rotor by a rigid shaft. Find a linear model suitable for analyzing small perturbations about the operating point A indicated on the flux-versus-current curve.

Solution

In Figure 10.14(b), we have repeated the basic motor diagram in Figure 10.12 and have also added the specified current sources and the mechanical load. From (27), the electromechanical torque is

$$\tau_e = \gamma\phi\bar{i}_A \tag{35}$$

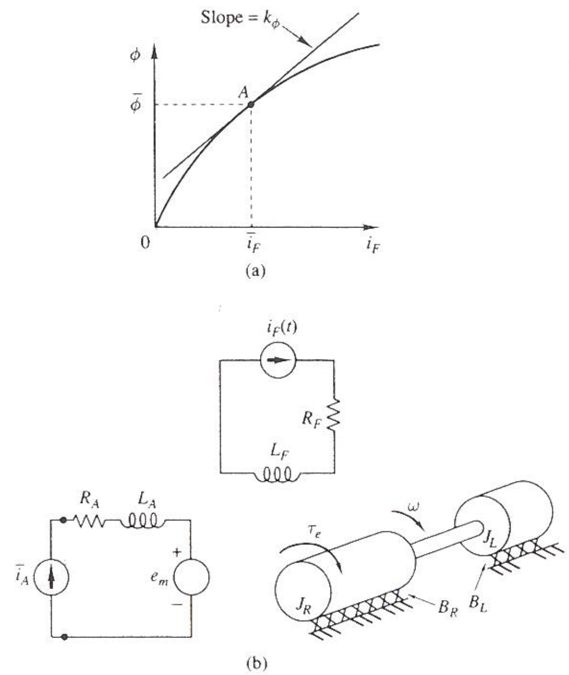

FIGURE 10.14 DC motor for Example 10.3. (a) Nonlinear field characteristic. (b) Diagram used for analysis.

where, in this case, γ and $\bar{i}_A$ are constants and $\phi = \phi(i_F)$. Because the armature current is constant and the field current is specified, there is no need to write an equation for either of the electrical circuits. Rather, we obtain the system model by summing torques on the rotor and load and using (35). This yields

$$(J_R + J_L)\dot{\omega} + (B_R + B_L)\omega = \gamma \bar{i}_A \phi(i_F) \tag{36}$$

To obtain the linearized model, we write the field current and angular velocity as $i_F(t) = \bar{i}_F + \hat{i}_F(t)$ and $\omega = \bar{\omega} + \hat{\omega}$, respectively. The operating-point values $\bar{i}_F$ and $\bar{\omega}$ must satisfy (36), which reduces to

$$\bar{\omega} = \frac{\gamma \bar{i}_A \phi(\bar{i}_F)}{B_R + B_L} \tag{37}$$

The flux is approximated by the first two terms in the Taylor-series expansion for $\phi(i_F)$ about the operating point, namely

$$\phi(\bar{i}_F) + k_\phi \hat{i}_F \tag{38}$$

where

$$k_\phi = \left.\frac{d\phi}{di_F}\right|_{\bar{i}_F} \tag{39}$$

Substituting (38) into (36), writing $\omega = \bar{\omega} + \hat{\omega}$, and using (37) to cancel the constant terms, we find the linearized model to be

$$(J_R + J_L)\dot{\hat{\omega}} + (B_R + B_L)\hat{\omega} = \gamma \bar{i}_A k_\phi \hat{i}_F(t) \tag{40}$$

where k_ϕ is the slope of the curve of ϕ versus i_F evaluated at the operating point.

In the last example note that (40) is a first-order equation, which implies that the system is only first-order. Although there are two inductors that can store energy, their currents are those of the current sources $\bar{i}_A$ and $i_F(t)$ and thus cannot be state variables.

Furthermore, it is apparent that the moments of inertia J_R and J_L and the friction coefficients B_R and B_L are merely added to obtain equivalent parameters for the system as a whole. If the motor had been connected to its load through a gear train (as would usually be the case), the equivalent parameters, when reflected to the motor, would be

$$J_{eq} = J_R + \frac{1}{N^2} J_L$$

$$B_{eq} = B_R + \frac{1}{N^2} B_L$$

The symbol N denotes the motor-to-load gear ratio; that is, the motor rotates at N times the angular velocity of the load.

10.4 A DEVICE FOR MEASURING ACCELERATION

In many areas of technology, it is important to be able to measure and record the acceleration of a moving body as a function of time. For example, deceleration measurements are required in the testing of automobiles for crash resistance. Accelerometers are important components in ship, aircraft, and rocket navigation and guidance systems.

Figure 10.15 shows an electromechanical device whose response depends on the acceleration of its case relative to an inertially fixed reference frame. The device is not intended to be representative of commonly used accelerometers, but modeling and analyzing its response will provide practice in using many of the techniques that we have discussed in this chapter and in Chapters 6 and 8.

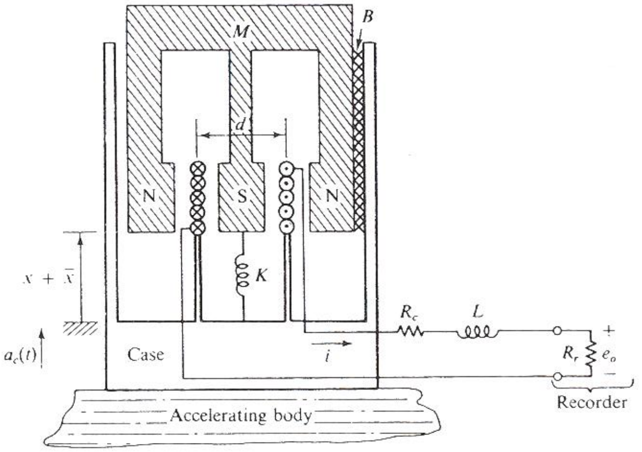

FIGURE 10.15 Acceleration-measuring device.

System Description

Basically, the device shown in Figure 10.15 consists of a case attached to the body or material whose motion is to be measured, a circular coil fixed to the case, and a permanent magnet supported on the case by a spring, with viscous damping between the magnet and the case. As the case moves vertically as a result of motion of the body supporting it, a voltage is induced in the coil because of the relative motion of the coil and the magnetic field. A recorder is attached to the terminals of the coil and draws a graph of the coil voltage e_o as a function of time.

The magnet has mass M and provides a flux density $\mathscr{B}$ in the annular space between its north and south poles. Although several springs would be used to support the mass, the single spring shown with spring constant K may be considered equivalent to whatever springs are present. The viscous-damping coefficient B accounts for all viscous effects between the magnet and the case.

The coil has N turns of diameter d, and the parameter $\ell = \pi d N$ denotes the length of the coil. The total coil resistance and inductance are modeled by lumped elements having values R_c and L, respectively. The dots and crosses associated with the coil in Figure 10.15 indicate that the assumed positive direction of current is clockwise as viewed from above. The recorder is attached directly to the terminals of the coil and is assumed to provide a resistance R_r in the coil circuit.

The acceleration of the case relative to a fixed inertial reference frame is denoted by $a_c(t)$ and is the input to the system. The vertical distance between the magnet and the case is $x + \bar{x}$, so the variable x denotes the

incremental displacement of the magnet relative to the case, with the value $x = 0$ corresponding to the equilibrium condition. Likewise, the relative velocity of the magnet with respect to the case is $\dot{x}$.

System Model

We derive the differential equations describing the behavior of the system by drawing a free-body diagram for the magnet, drawing a circuit diagram for the coil and recorder, and expressing the electromechanically induced force and voltage in terms of the appropriate system variables. Three aspects of this task deserve specific mention.

First, the inertial force shown on the free-body diagram must use the acceleration of the mass relative to the inertial reference frame. Hence this force on the diagram shown in Figure 10.16(a) is $M[a_c(t) + \ddot{x}]$ in the

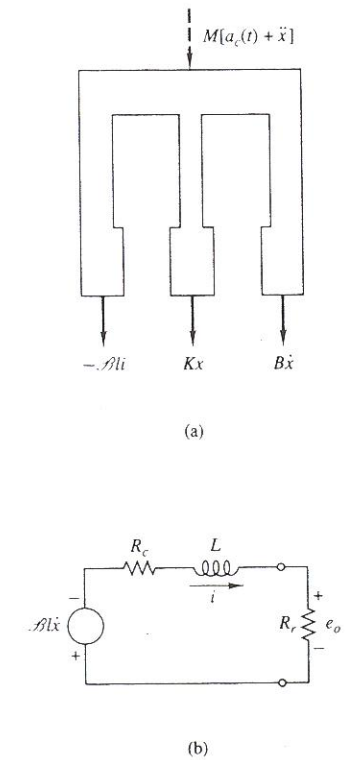

FIGURE 10.16 (a) Free-body diagram for the magnet. (b) Circuit diagram.

downward direction. Second, recall that the electrically induced force on the coil is given by $f_e = \mathscr{B}\ell i$. In this instance, however, we must show on the free-body diagram the force on the magnet, which, by the law of reaction forces, is $-\mathscr{B}\ell i$ in the downward direction. Finally, in order to determine the proper sign for the voltage induced in the coil, we note that when $\dot{x} > 0$, the coil is moving downward relative to the magnetic field. This is because the magnet is moving upward relative to the case, and the coil is attached to the case.

Taking these points into consideration, we can draw the free-body and circuit diagrams shown in Figure 10.16. Summing forces on the magnet and writing a voltage equation for the circuit, we obtain

$$L\frac{di}{dt} + (R_c + R_r)i = -\mathscr{B}\ell\dot{x} \tag{41a}$$

$$M\ddot{x} + B\dot{x} + Kx = \mathscr{B}\ell i - Ma_c(t) \tag{41b}$$

$$e_o = R_r i \tag{41c}$$

Transfer Function

To determine a single overall transfer function relating the input transform $A_c(s)$ to the output transform $E_o(s)$, we shall transform (41) with zero initial conditions. Doing this, we obtain the following set of three algebraic transform equations:

$$(Ls + R_c + R_r)I(s) = -\mathscr{B}\ell s X(s) \tag{42a}$$

$$(Ms^2 + Bs + K)X(s) = \mathscr{B}\ell I(s) - MA_c(s) \tag{42b}$$

$$E_o(s) = R_r I(s) \tag{42c}$$

We can combine these equations to eliminate the variables $X(s)$ and $I(s)$, obtaining a single equation relating the input transform $A_c(s)$ and the output transform $E_o(s)$.

First, if we combine (42a) and (42b) to eliminate $X(s)$, the result is

$$(Ms^2 + Bs + K)\left(\frac{Ls + R}{-\mathscr{B}\ell s}\right)I(s) - \mathscr{B}\ell I(s) = -MA_c(s)$$

where $R = R_c + R_r$. Combining the two terms involving $I(s)$ in this equation and then using (42c) to express $I(s)$ in terms of $E_o(s)$, we obtain the single transformed equation

$$\frac{1}{R_r}[(Ms^2 + Bs + K)(Ls + R) + \mathscr{B}^2\ell^2 s]E_o(s) = \mathscr{B}\ell M s A_c(s)$$

Solving for the ratio $E_o(s)/A_c(s)$ yields the overall transfer function

$$\frac{E_o(s)}{A_c(s)} = \frac{R_r\mathscr{B}\ell M s}{P(s)} \tag{43}$$

where $P(s)$ is the characteristic polynomial of the device and is

$$P(s) = MLs^3 + (BL + MR)s^2 + (BR + KL + \mathscr{B}^2\ell^2)s + KR \tag{44}$$

Because the acceleration, velocity, and displacement of the case are related by $a_c(t) = \dot{v}_c = \ddot{x}_c$, and because for zero initial conditions $A_c(s) = sV_c(s) = s^2X_c(s)$, we can also write the transfer functions corresponding to a velocity or displacement input:

$$\frac{E_o(s)}{V_c(s)} = \frac{R_r\mathscr{B}\ell M s^2}{P(s)}$$

$$\frac{E_o(s)}{X_c(s)} = \frac{R_r\mathscr{B}\ell M s^3}{P(s)}$$

The characteristic polynomial $P(s)$ is a cubic, so it is difficult to be very specific about the behavior of the system as a measuring device without substituting numerical values for the parameters and calculating the frequency response or simulating the response to specific inputs, such as the impulse and step function.

Rather than doing this, we shall make the approximation that the coil inductance can be neglected. In terms of the frequency response, this approximation should be well justified for low frequencies but not for high frequencies. Because our interest is principally in the response at low frequencies, we are justified in setting L equal to zero in (44). With this change $P(s)$ becomes quadratic. In essence, we have eliminated the current as a state variable, and (43) reduces to

$$\frac{E_o(s)}{A_c(s)} = \frac{R_r\mathscr{B}\ell M s}{MRs^2 + (BR + \mathscr{B}^2\ell^2)s + KR} = \frac{(\mathscr{B}\ell R_r/R)s}{s^2 + \left(\frac{B}{M} + \frac{\mathscr{B}^2\ell^2}{MR}\right)s + \frac{K}{M}} \tag{45}$$

Inspection of (45) indicates that the transfer function $E_o(s)/A_c(s)$ has a zero at $s = 0$ and a pair of poles that may be real or complex. To determine the undamped natural frequency ω_n and the damping ratio ζ associated with these poles, we compare (45) to

$$H(s) = \frac{Cs}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{46}$$

which can be viewed as the standard form for a transfer function having a zero at $s = 0$ and two poles. Comparing the coefficients in (45) and (46), we obtain

$$C = \frac{\mathscr{B}\ell R_r}{R}$$

$$\omega_n = \sqrt{\frac{K}{M}} \tag{47}$$

$$\zeta = \frac{1}{2}\sqrt{\frac{M}{K}}\left(\frac{B}{M} + \frac{\mathscr{B}^2\ell^2}{MR}\right)$$

If it were known what values of ζ and ω_n would result in a device that would perform well as a measuring instrument, a designer could attempt to select the physical parameters in order to achieve these values of ζ and ω_n. Frequency-response plots can be used to determine suitable values for ζ and ω_n.

Frequency Response

The steady-state response to a sinusoidal input is formed by examining $H(j\omega)$, as discussed in Section 8.5. A frequency-response analysis of (45) would be cumbersome unless specific numerical values were given for all but one or two of the physical parameters, so we shall work with (46). When s is replaced by $j\omega$, (46) becomes

$$H(j\omega) = \frac{jC\omega}{\omega_n^2 - \omega^2 + j2\zeta\omega_n\omega}$$

$$= \frac{j(C/\omega_n)(\omega/\omega_n)}{1-(\omega/\omega_n)^2 + j2\zeta(\omega/\omega_n)} \tag{48}$$

where the second form of the equation results from dividing both the numerator and the denominator by ω_n^2. The quantity ω/ω_n can be thought of as a normalized frequency. Because the factor C/ω_n in the numerator of (48) is a multiplying constant that does not affect the variation of $H(j\omega)$ with ω, we shall also normalize the magnitude of the transfer function by defining

$$H_N(j\omega) = \frac{\omega_n}{C}H(j\omega)$$

$$= \frac{j(\omega/\omega_n)}{1-(\omega/\omega_n)^2 + j2\zeta(\omega/\omega_n)} \tag{49}$$

We obtain the magnitude of $H_N(j\omega)$ by dividing the magnitude of its numerator by that of its denominator. This results in

$$|H_N(j\omega)| = \frac{\omega/\omega_n}{\sqrt{(\omega/\omega_n)^4 + (4\zeta^2-2)(\omega/\omega_n)^2 + 1}} \tag{50}$$

Comparing $|H_N(j\omega)|$ for different values of ζ indicates the relative shapes of the frequency-response magnitudes corresponding to different values of the damping ratio. If the device is to be of value in measuring the acceleration of the case, there should be a range of frequencies for which $|H_N(j\omega)|$ is fairly flat—that is, independent of frequency.

From (50), we see that $|H_N(j\omega)| \simeq \omega/\omega_n$ for small values of ω/ω_n and that it approaches $1/(\omega/\omega_n)$ for large values of ω/ω_n. For $\omega/\omega_n = 1$, $|H_N(j\omega)| = 1/2\zeta$. Using this information and calculating a few additional points, we can draw the plots shown in Figure 10.17. Logarithmic scales are commonly used for frequency-response plots, and they enable us to include a wide range of values of $|H_N(j\omega)|$ and ω/ω_n.

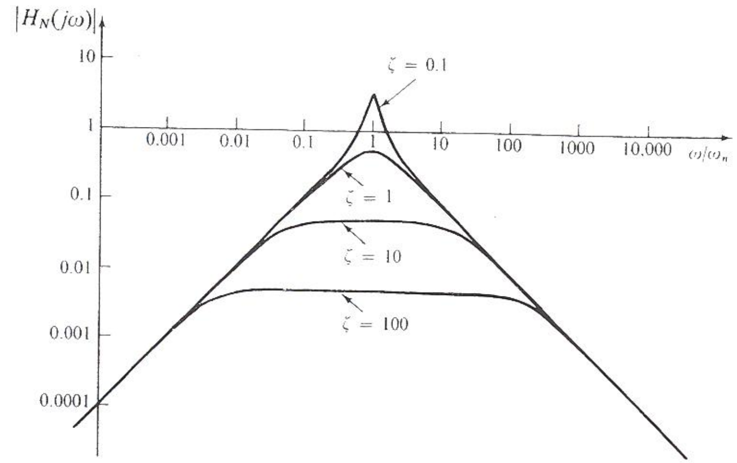

FIGURE 10.17 Frequency response of the acceleration-measuring device for several damping ratios.

It is apparent that the device must be heavily damped ($\zeta \gg 1$) if a range of frequencies is to be achieved for which $|H_N(j\omega)|$ is essentially constant. If $\zeta = 100$, $|H_N(j\omega)|$ will remain between 0.0045 and 0.0050 for $0.01 < \omega/\omega_n < 100$, which is a four-decade range of frequencies. In contrast, if $\zeta \le 1$, there is no range of frequencies over which $|H_N(j\omega)|$ is essentially constant.

Because $|H_N(j\omega)| \simeq \omega/\omega_n$, as ω approaches zero, a constant acceleration will result in a zero steady-state output. We can also see this from (46) by noting that the steady-state response to a unit step-function input is $H(0) = 0$. When the acceleration $a_c(t)$ is constant, the velocity of the magnet will become equal to the velocity of the coil. There will then be no voltage induced in the coil, and the output of the recorder will be zero.

In order to read the relative displacement of the magnet with respect to the case, we could attach a pointer to the mass and put scale markings on the case. There would then be a steady-state output reading even for a constant acceleration of the case, and the device could sense acceleration of arbitrarily low frequencies. To obtain an electrical output for recording purposes, we can replace the pointer by the wiper of a potentiometer and use the voltage at the wiper arm as the output.

SUMMARY

In this chapter, we discussed two mechanisms for coupling the electrical and mechanical parts of a system. The potentiometer, which is an example of resistive coupling, can produce an output voltage that is proportional to a mechanical displacement, provided that the loading effect described in Example 10.1 is avoided.

The coupling for most of the electromechanical devices we considered was through a magnetic field. The applications used a current-carrying wire that moved perpendicularly to the current and to the mangetic field. When summing forces, we included the force on the wire, $f_e = \mathscr{B}\ell i$. When summing voltages, we included the induced voltage, $e_m = \mathscr{B}\ell v$. The characteristics of the overall systems depended on the electrical parameters, the mechanical parameters, and the strength of the magnetic coupling.

We presented several examples to illustrate the basic steps in modeling electromechanical devices. The references in Appendix D contain further information about nonlinear effects and construction details, including the commutator needed for a direct-current motor. They also discuss other possible types of electromechanical coupling, such as mechanically varying the characteristics of the flux path.

PROBLEMS

10.1 Use (6) to plot curves of e_o/E versus $\theta/\theta_{\max}$ for a rotary potentiometer with the load resistance R_o when (1) $R_o = 0.5R_T$, (2) $R_o = R_T$, and (3) $R_o = 10R_T$.

* **10.2** The potentiometer model shown in Figure 10.2(b) includes the resistances R_1 and R_2, both of which depend on $x(t)$. However, many potentiometers contain some inductance because they are constructed by winding many turns of wire about a core. The circuit shown in Figure P10.2 represents this inductance by the lumped elements whose values also depend on $x(t)$.

a) Find the differential equation relating e_o to $x(t)$ and $e_i(t)$.

b) Assume that R_2 and L_2 are proportional to the displacement $x(t)$. Let $L_2 = L_T[x(t)/x_{\max}]$ and $R_2 = R_T[x(t)/x_{\max}]$, where $L_1 + L_2 = L_T$ and $R_1 + R_2 = R_T$. Find the differential equation relating e_o to $x(t)$ and $e_i(t)$. Compare your answer to (3).

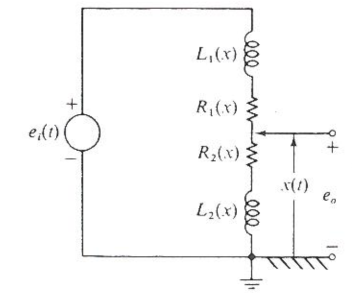

FIGURE P10.2

10.3 Two identical rotary potentiometers similar to the one shown in Figure 10.3 are connected across a constant voltage source E as indicated in Figure P10.3. The output wiper positions are denoted by $\theta_a(t)$ and $\theta_b(t)$. The various resistances satisfy the relationships

$$R_T = R_1 + R_2 = R_3 + R_4$$
$$R_2 = [\theta_a(t)/\theta_{\max}]R_T$$
$$R_4 = [\theta_b(t)/\theta_{\max}]R_T$$

a) Verify that when R_o is infinite (an open circuit),

$$e_o = \left(\frac{\theta_a - \theta_b}{\theta_{\max}}\right)E$$

b) Verify that when no restrictions are placed on R_o other than $R_o > 0$,

$$e_o = \left[\frac{\theta_a - \theta_b}{\theta_{\max} + \dfrac{R_T}{R_o}\left(1 - \dfrac{\theta_a}{\theta_{\max}}\right)\theta_a + \dfrac{R_T}{R_o}\left(1 - \dfrac{\theta_b}{\theta_{\max}}\right)\theta_b}\right]E$$

c) Show that when $R_T/R_o = 0$, the expression in part (b) for finite R_o reduces to the expression given in part (a).

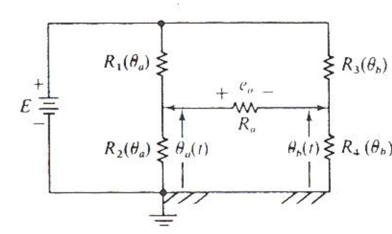

FIGURE P10.3

10.4 a) Find the state-variable equations, the transfer function, and the input-output differential equation for the galvanometer shown in Figure 10.8 when the inductance L of the coil is not neglected.
b) Find the steady-state response to a constant input by examining $H(0)$, and compare the result to (17).

10.5 Figure P10.5 shows a galvanometer whose flux is obtained from the same current i that passes through the movable coil, rather than from a permanent magnet as shown in Figure 10.8. The flux density in the air gaps is $\mathscr{B} = k_{\mathscr{B}} i$. The coil has moment of inertia J and viscous-damping coefficient B, and it is restrained by a torsional spring with spring constant K. The total length of the coil in the magnetic field is d, and its radius is a. Let $\alpha = adk_{\mathscr{B}}$.
a) Verify that a valid state-variable model is

$$\dot{\theta} = \omega$$
$$\dot{\omega} = \frac{1}{J}(-K\theta - B\omega + \alpha i^2)$$
$$\frac{di}{dt} = \frac{1}{L}[-\alpha i\omega - Ri + e_i(t)]$$

b) Find $\bar{\theta}$ corresponding to the operating point $\bar{e}_i$, where $e_i(t) = \bar{e}_i + \hat{e}_i(t)$.
c) Find a set of linearized state-variable equations valid in the vicinity of the operating point you found in part (b). To do this, first let $\theta = \bar{\theta} + \hat{\theta}$, $\omega = \bar{\omega} + \hat{\omega}$, and $i = \bar{i} + \hat{i}$. Then assume that the incremental variables are small, so that $(\hat{i})^2$ and the product $\hat{i}\hat{\omega}$ can be neglected.

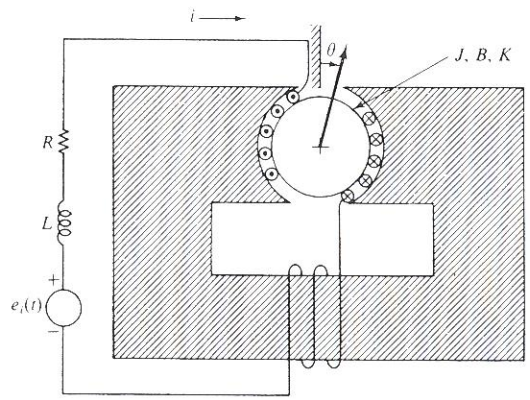

FIGURE P10.5

10.6 a) Verify (22) and (23) for the microphone shown in Figure 10.9 by transforming (19), (20), and (21) with zero initial conditions.
b) Write the input-output equation for the case $L = 0$. Find expressions for the damping ratio ζ and the undamped natural frequency ω_n in terms of the physical parameters M, K, B, R, and α.

*** 10.7** A loudspeaker produces sound waves by the movement of a diaphragm in response to an electrical input. In the cross-sectional view shown in Figure P10.7, $e_i(t)$ is the input and the output is the displacement x. A coil of wire with N turns and radius a is attached to the diaphragm. Let $\alpha = 2\pi a N \mathscr{B}$, where $\mathscr{B}$ is the flux density in the air gap of the permanent magnet.
a) Verify that the equations

$$\dot{x} = v$$
$$\dot{v} = \frac{1}{M}(-Kx - Bv + \alpha i)$$
$$\frac{di}{dt} = \frac{1}{L}[-\alpha v - Ri + e_i(t)]$$

represent a valid state-variable model.
b) Find the transfer function and the input-output differential equation.
c) Rewrite the input-output equation for the case $L = 0$, and find expressions for the damping ratio ζ and the undamped natural frequency ω_n.

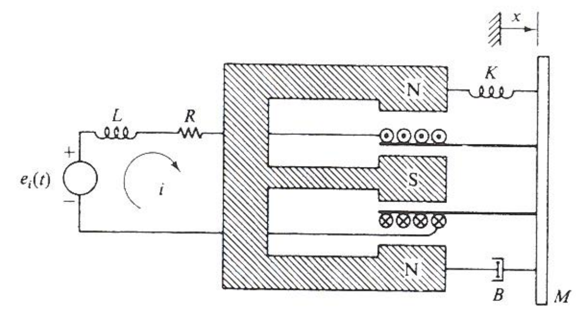

FIGURE P10.7

10.8 A transducer to measure translational motion is shown in Figure P10.8. The permanent magnet produces a uniform magnetic field in the air gap with flux density $\mathscr{B}$ and can move with a displacement x and velocity v. A wire that is fixed in space has a length d within the magnetic field. The inductance and resistance of the wire are included in the lumped elements L and R, and the output of the system is the voltage e_o across the resistor R_o.
a) Verify that the transfer function is

$$H(s) = \frac{E_o(s)}{X(s)} = \frac{d\mathscr{B}R_o s}{sL + (R + R_o)}$$

b) If $i(0) = 0$, find the response for all $t > 0$ when $x(t) = U(t)$. What is the steady-state response? What is the time constant?

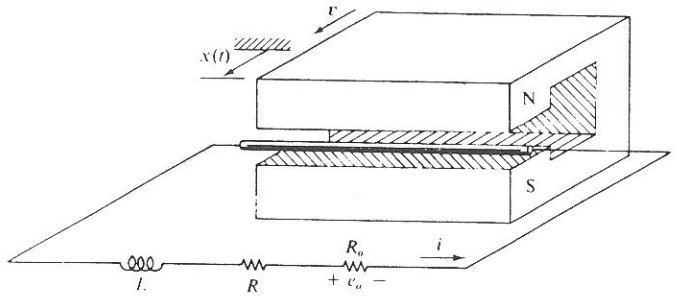

FIGURE P10.8

* **10.9** The magnet shown in Figure P10.8 and considered in Problem 10.8 has mass M and is separated from a fixed horizontal surface by an oil film with viscous friction coefficient B. Instead of a displacement input, a force $f_a(t)$ is applied to the magnet in the positive x direction. The wire of length d remains fixed in space. Find the transfer function $H(s) = E_o(s)/F_a(s)$.

10.10 A wire of length ℓ, rigidly attached to a mass M, is in a magnetic field with flux density $\mathscr{B}$. In the cross-sectional view shown in Figure P10.10, the wire is perpendicular to the page, with point a connected to the front of the wire and point b to the back. The input to the system is a 12-V battery. With no energy initially stored within the system, the switch closes at $t = 0$.

a) Which way will the wire—and hence the mass—move when the switch is closed?

b) What will be the steady-state displacement of the mass from its original position?

c) Define a set of state variables and write the state-variable equations describing the behavior of the system for $t > 0$.

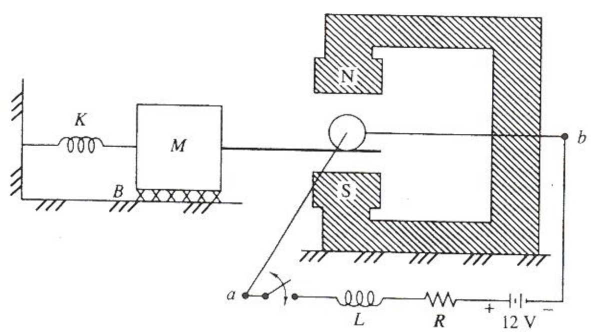

FIGURE P10.10

* **10.11** The conductor of mass M shown in Figure P10.11 can move vertically through a uniform magnetic field of flux density $\mathscr{B}$ whose positive sense is into

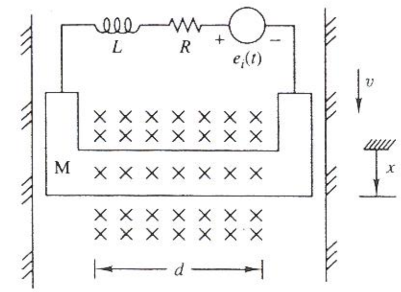

FIGURE P10.11

the page. There is no friction, the effective length of the conductor in the field is d, and $e_i(t)$ is a voltage source. The lumped elements $e_i(t)$, L, and R are outside the magnetic field.

a) Write a set of state-variable equations.

b) For what constant voltage $\bar{e}_i$ will the conductor remain stationary?

c) What will be the steady-state velocity of the conductor if $e_i(t)$ is always zero?

10.12 A plunger is made to move horizontally through the center of a fixed cylindrical permanent magnet by the application of a voltage source $e_i(t)$. Attached to the plunger is a coil having N turns and radius a. Figure P10.12 shows the system, including a cross-sectional view of the magnet and plunger, and indicates typical paths for the magnetic flux by dashed lines. The magnetic field between the plunger and the south pole is assumed to have a constant flux density $\mathscr{B}$. The resistance and inductance of the coil are represented by the lumped elements R and L, respectively, and the plunger has mass M.

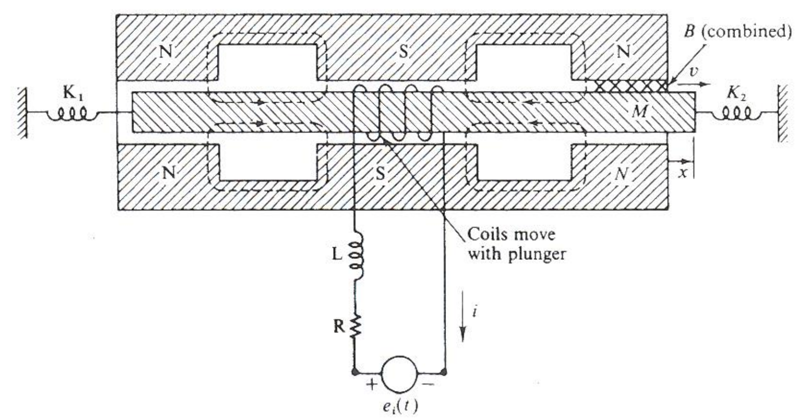

FIGURE P10.12

a) Verify that the following is a valid set of state-variable equations:

$$\dot{x} = v$$
$$\dot{v} = \frac{1}{M}[-(K_1 + K_2)x - Bv + \alpha i]$$
$$\frac{di}{dt} = \frac{1}{L}[-\alpha v - Ri + e_i(t)]$$

where $\alpha = 2\pi a N \mathscr{B}$.

b) Determine the direction in which the plunger will move if there is no initial stored energy and if $e_i(t) = U(t)$.

c) Find the steady-state displacement of the plunger for the input in part (b).

10.13 For the electric motor shown in Figure 10.14(b), let $i_F(t)$ have the constant value $\bar{i}_F$.

a) Write the differential equation describing the system and identify the time constant when $i_A(t)$ is the input and ω the output.

b) If $i_A(t) = i_{A_1}$ for all $t < 0$ and $i_A(t) = i_{A_2}$ for all $t > 0$, sketch ω versus t. Assume that steady-state conditions exist at $t = 0-$.

c) Repeat part (b) when i_{A_2} is replaced by $-i_{A_1}$. Find the value of t for which $\omega = 0$.

* **10.14** a) Write the differential equation describing the motor shown in Figure 10.14(b) when $i_A(t)$ and $i_F(t)$ are separate inputs and when $\phi = k_\phi i_F(t)$.

b) Find an expression for $\bar{\omega}$ at the operating point corresponding to $\bar{i}_A$ and $\bar{i}_F$.

c) If $i_A(t) = \bar{i}_A$ and $i_F(t) = \bar{i}_F + \hat{i}_F(t)$, find a linearized model that is valid in the vicinity of the operating point you found in part (b).

10.15 Let the shaft connecting J_R and J_L in Figure 10.14(b) have a stiffness constant K rather than being rigid. Also replace the current source $\bar{i}_A$ by a voltage source $e_i(t)$, with its positive sense upward. Assume that $\phi = k_\phi i_F$, and denote the angular displacements of the rotor and the load by θ_R and θ_L, respectively. Choose as state variables i_A, ω_R, ω_L, and $\theta = \theta_R - \theta_L$. Write the state-variable equations.

10.16 Assume that $L_A = 0$ for the motor shown in Figure 10.14(b). Replace the current source $\bar{i}_A$ by a voltage source that has a constant value of E_A volts.

a) Write the differential equation describing the system if the input is $i_F(t)$ and if $\phi = k_\phi i_F(t)$.

b) Find the operating point corresponding to $\bar{i}_F$.

c) Find a linearized model that is valid about the operating point you found in part (b). Identify its time constant. Let $\omega = \bar{\omega} + \hat{\omega}$ and $i_F = \bar{i}_F + \hat{i}_F$, and assume that terms involving $(\hat{i}_F)^2$ and the product $\hat{i}_F\hat{\omega}$ can be neglected.

* **10.17** For the motor depicted in Figure 10.13, replace $e_i(t)$ by a constant voltage source E_A, and replace the source in the field winding by a time-varying voltage $e_F(t)$. Assume that $\phi = k_\phi i_F$.

a) Using i_A, i_F, and ω as state variables, write the state-variable equations.

b) Find $\bar{\omega}$ for the operating point corresponding to $e_F(t) = \bar{e}_F$ and $\tau_L(t) = 0$.

c) Derive a linearized model valid about the operating point you found in part (b). Let $\omega = \bar{\omega} + \hat{\omega}$, $i_A = \bar{i}_A + \hat{i}_A$, and $i_F = \bar{i}_F + \hat{i}_F$. Assume that terms involving the products $\hat{i}_F\hat{\omega}$ and $\hat{i}_A\hat{\omega}$ can be neglected.

10.18 The field and armature windings of an electric motor are connected in parallel directly across a voltage source $e_i(t)$, as shown in Figure P10.18. The resistances of the field and armature windings are R_F and R_A, respectively, and the inductances of both windings are negligible.

a) Verify that the differential equation relating ω to $e_i(t)$ and $\tau_L(t)$ is

$$J\dot{\omega} + B\omega = \frac{\gamma k_\phi}{R_F}\left(\frac{R_F - \gamma k_\phi \omega}{R_A R_F}\right) e_i^2(t) - \tau_L(t)$$

b) Find an expression for $\bar{\omega}$ at the operating point corresponding to $e_i(t) = \bar{e}_i$ and $\tau_L(t) = 0$.

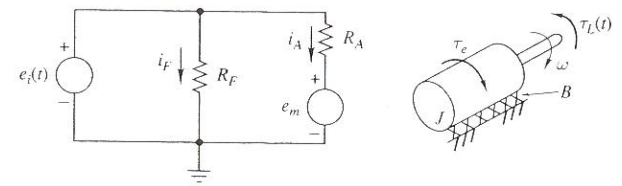

FIGURE P10.18

10.19 For the motor described in Problem 10.18, add an armature inductance L_A and a field inductance L_F in series with R_A and R_F, respectively.

a) Verify that

$$\frac{di_A}{dt} = \frac{1}{L_A}\left[-R_A i_A - \gamma k_\phi i_F \omega + e_i(t)\right]$$
$$\frac{di_F}{dt} = \frac{1}{L_F}[-R_F i_F + e_i(t)]$$
$$\dot{\omega} = \frac{1}{J}\left[\gamma k_\phi i_F i_A - B\omega - \tau_L(t)\right]$$

constitute a suitable set of state-variable equations.

b) Find an expression for $\bar{\omega}$ at the operating point corresponding to $e_i(t) = \bar{e}_i$ and $\tau_L(t) = 0$.

10.20 The rotor shown in Figure P10.20 is driven by a torque $\tau_a(t)$. The rotor has a moment of inertia J, but the friction is negligible. The field winding is excited by a constant voltage source, resulting in a constant magnetic flux. The armature resistance is denoted by R_A, and the armature inductance is negligible. Use (29) to obtain expressions for e_m and τ_e.

a) With the rotor initially at rest and with the switch in the left-hand position connecting the armature to a short circuit, the applied torque is $\tau_a(t) = U(t)$. Find and sketch ω as a function of time.

b) After the rotor has reached a steady-state speed under the conditions of part (a), the switch is thrown to the right, thereby connecting the armature to the battery. Find and sketch ω versus t if a constant unit torque continues to be applied to the rotor.