C H A P T E R 2

TRANSLATIONAL MECHANICAL SYSTEMS

The modeling techniques for translational and rotational mechanical systems are discussed in this and the next two chapters. Procedures for solving the mathematical models are developed in later chapters.

After introducing the variables to be used, we discuss the laws for the individual elements, the laws governing the interconnections of the elements, and the use of free-body diagrams as an aid in formulating the equations of the model. Inputs will consist of either the application of a known external force or the moving of some body with a known displacement.

The first examples will be systems that can have only horizontal or only vertical motion. For masses that can move vertically, the gravitational forces must be considered. If the system contains an ideal pulley, then some parts can move horizontally and other parts vertically. Special situations, such as free-body diagrams for massless junctions and rules for the series or parallel combination of similar elements, will be treated later in the chapter.

2.1 VARIABLES

The symbols for the basic variables used to describe the dynamic behavior of translational mechanical systems are

x, displacement in meters (m)

v, velocity in meters per second (m/s)

a, acceleration in meters per second per second (m/s^2)

f, force in newtons (N)

All these variables are functions of time. In general, however, we shall add a t in parentheses immediately after the symbol only when it denotes an input or when we find doing so useful for clarity or emphasis.

Displacements are measured with respect to some reference condition, which is often the equilibrium position of the body or point in question. Velocities are normally expressed as the derivatives of the corresponding displacements. If the reference condition of a displacement is not indicated because it is not of interest, then the reference condition for the velocity needs to be given.

Two conventions used to define displacements are illustrated in Figure 2.1(a) and Figure 2.1(b). In Figure 2.1(a), the variable x represents the displacement of the left side of the body from the fixed vertical wall, whereas in Figure 2.1(b), the reference position corresponding to $x = 0$ is not specifically shown.

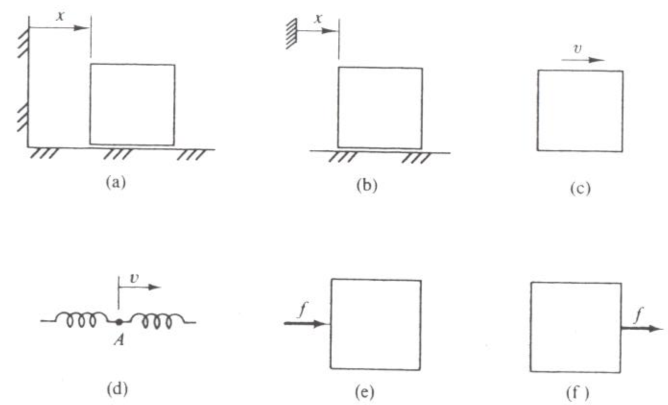


FIGURE 2.1 Conventions for designating variables.

Generally, the reference position will correspond to a condition of equilibrium for which the system inputs are constants and in which the net force on the body being considered is zero. Figure 2.1(c) and Figure 2.1(d) indicate two methods of defining a velocity. All points on the body in Figure 2.1(c) must move with the same velocity, so there is no possible ambiguity about which point has the velocity v. In Figure 2.1(d), the vertical line at the base of the arrow indicates that v is the velocity of the point labeled A. Forces can be represented by arrows pointing either into or away from a body, as depicted in Figure 2.1(e) and Figure 2.1(f), which are equivalent to one another.

Remember that the arrows only indicate an assumed positive sense for the displacement, velocity, or force being considered and by themselves do not imply anything about the actual direction of the motion or of the force at a given instant. If, for example, in Figure 2.1(e) and Figure 2.1(f), the force acting on the body is $f(t) = \sin t$, the force acts to the right for $0 < t < \pi$ and to the left for $\pi < t < 2\pi$, and it continues to change direction every π seconds. Note that an alternative way of describing the identical situation is to draw the arrow pointing to the left and then write $f(t) = -\sin t$. Reversing a reference arrow is equivalent to reversing the sign of the algebraic expression associated with it. There is no unique way of choosing reference directions on a diagram, but the equations must be consistent with whatever choice is made for the arrows.

Reference arrows for the displacement, velocity, and acceleration of a given point are invariably drawn in the same direction so that the equations

$$v = \frac{dx}{dt}$$

and

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

can be used. With this understanding, a reference arrow for acceleration is not shown explicitly on the diagrams, and for the same reason only the reference arrow for either the displacement or the velocity of a point (but not both) is shown in many examples.

Variables in addition to those defined at the beginning of this section include

w, energy in joules (J)

p, power in watts (W)

where 1 joule = 1 newton-meter and 1 watt = 1 joule per second. Because the arrows defining the positive senses of the velocity and the force point in the same direction, the power supplied to the mass in Figure 2.2(a) and a to the resorte spring in Figure 2.2(b) is

$$p = fv \tag{1}$$

Because power is defined to be the rate at which energy is supplied or dissipated, it follows that

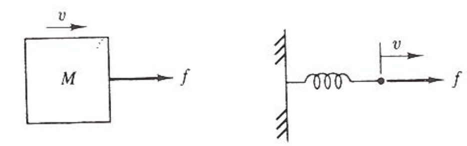
$$p = \frac{dw}{dt} \tag{2}$$

and the energy supplied between time t_0 and t_1 is

$$\int_{t_0}^{t_1} p(t)dt$$

If $w(t_0)$ denotes the energy supplied up to time t_0, then the total energy supplied up to any later time t is

$$w(t) = w(t_0) + \int_{t_0}^{t} p(\lambda)d\lambda \tag{3}$$



(a) (b)

FIGURE 2.2 Reference arrows for (1).

In the last integrand, t has been replaced by the dummy variable λ in order to avoid confusion between the upper limit and the variable of integration.

2.2 ELEMENT LAWS

Physical devices are represented by one or more idealized elements that obey laws involving the variables associated with the elements. As we mentioned in Chapter 1, some degree of approximation is required in selecting the elements to represent a device, and the behavior of the combined elements may not correspond exactly to the behavior of the device. The elements that we include in translational systems are mass, friction, stiffness, and the lever. The element laws for the first three relate the external force to the acceleration, velocity, or displacement associated with the element. The lever is considered in Chapter 4.

Mass

Figure 2.2(a) shows a **mass** M, which has units of kilograms (kg), subjected to a force f. Newton's second law states that the sum of the forces acting on a body is equal to the time rate of change of the momentum:

$$\frac{d}{dt}(Mv) = f \tag{4}$$

which, for a constant mass, can be written as

$$M\frac{dv}{dt} = f \tag{5}$$

For (4) and (5) to hold, the momentum and acceleration must be measured with respect to an inertial reference frame. For ordinary systems at or near the surface of the earth, the earth's surface is a very close approximation to an inertial reference frame, so it is the one we use. The momentum, acceleration, and force are really vector quantities, but in this chapter the mass is constrained to move in a single direction, so we can write scalar equations.

We shall restrict our attention to constant masses and shall neglect relativistic effects so that we can use (5). Hence a mass can be modeled by an algebraic relationship between the acceleration dv/dt and the external force f. For (5) to hold, the positive senses of both dv/dt and f must be the same, because the force will cause the velocity to increase in the direction in which the force is acting.

Energy in a mass is stored as kinetic energy if the mass is in motion and as potential energy if the mass has a vertical displacement relative to its reference position. The kinetic energy is

$$w_k = \frac{1}{2}Mv^2 \tag{6}$$

and the potential energy, assuming a uniform gravitational field, is

$$w_p = Mgh \tag{7}$$

where g is the gravitational constant (approximately 9.807 m/s^2 at the surface of the earth) and h is the height of the mass above its reference position. In order to determine the response for $t \geq t_0$ of a dynamic system containing a mass, we must know its initial velocity $v(t_0)$ and, if vertical motion is possible, its initial height $h(t_0)$.

Friction

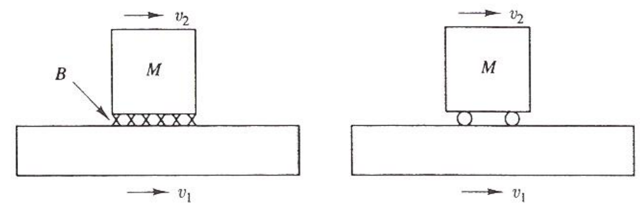
Forces that are algebraic functions of the relative velocity between two bodies are modeled by friction elements. A mass sliding on an oil film that has laminar flow, as depicted in Figure 2.3(a), is subject to **viscous friction** and obeys the linear relationship

$$f = B\Delta v \tag{8}$$

where B has units of newton-seconds per meter (N·s/m) and where $\Delta v = v_2 - v_1$. The direction of a frictional force will be such as to oppose the motion of the mass. For (8) to apply to Figure 2.3(a), the force f exerted on the mass M by the oil film is to the left. (By Newton's third law, the mass exerts an equal force f to the right on the oil film.) The friction coefficient B is proportional to the contact area and to the viscosity of the oil and inversely proportional to the thickness of the film.

Sometimes the frictional forces on adjacent bodies that have relative motion are small enough to be neglected. This might be the situation, for example, if the bodies are separated by bearings. The diagrams for such cases often show small wheels between the two bodies, as illustrated in Figure 2.3(b), in order to emphasize the lack of frictional forces.

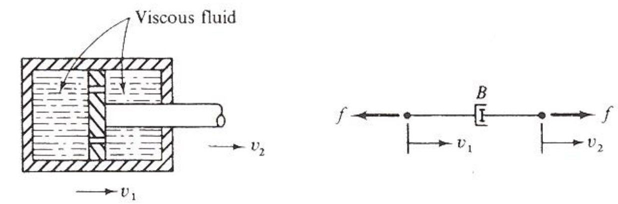
Viscous friction may also be used to model a dashpot, such as the shock absorbers on an automobile. As indicated in Figure 2.4(a), a piston moves through an oil-filled cylinder, and there are small holes in the face of the piston through which the oil passes as the parts move relative to each other. The symbol often used for a dashpot is shown in Figure 2.4(b). Many dashpot devices involve high rates of fluid flow through the orifices and



(a) (b)

FIGURE 2.3 (a) Friction described by (8) with $\Delta v = v_2 - v_1$. (b) Adjacent bodies with negligible friction.

have nonlinear characteristics. If the flow is laminar, then the element is again described by (8). If the lower block in Figure 2.3(a) or the cylinder of the dashpot in Figure 2.4(a) is stationary, then $v_1 = 0$ and the element law reduces to $f = Bv_2$.



(a) (b)

FIGURE 2.4 (a) A dashpot. (b) Its representation.

If the dashpot or oil film is assumed to be massless and if the accelerations are to remain finite, then when a force f is applied to one side, a retarding force of equal magnitude must be exerted on the other side (either by a wall or by some other component) as shown in Figure 2.4(b), again with $f = B(v_2 - v_1)$. This means that in the system shown in Figure 2.5, the force f is transmitted through the dashpot and exerted directly on the mass M.

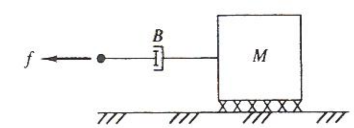


FIGURE 2.5 Force transmitted through a dashpot.

The viscous friction described by (8) is a linear element, for which the plot of f versus Δv is a straight line passing through the origin, as shown in Figure 2.6(a). Examples of friction that obey nonlinear relationships are **dry friction** and **drag friction**. The former is modeled by a force that is independent of the magnitude of the relative velocity, as indicated in Figure 2.6(b), and that can be described by the equation

$$f = \begin{cases} -A & \text{for } \Delta v < 0 \\ A & \text{for } \Delta v > 0 \end{cases}$$

Drag friction is caused by resistance to a body moving through a fluid (such as wind resistance) and can often be described by an equation of the form $f = D|\Delta v|\Delta v$, as depicted in Figure 2.6(c). Various other nonlinearities may be encountered in friction elements.

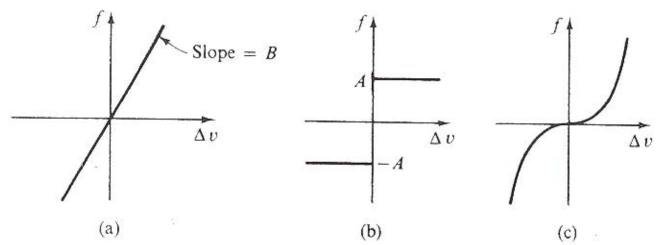


FIGURE 2.6 Friction characteristics. (a) Linear. (b) Dry. (c) Drag.

The power dissipated by friction is the product of the force exerted and the relative velocity of the two ends of the element. This power is immediately converted to heat and thus cannot be returned to the rest of the mechanical system at a later time. Accordingly, we do not usually need to know the initial velocities of the friction elements in order to solve the model of a system.

Stiffness

Any mechanical element that undergoes a change in shape when subjected to a force can be characterized by a **stiffness element**, provided only that an algebraic relationship exists between the elongation and the force. The most common stiffness element is the spring, although most mechanical elements undergo some deflection when stressed. For the spring sketched in Figure 2.7(a), we define d_0 to be the length of the spring when no force is applied and x to be the elongation caused by the force f. Then the total length at any instant is $d(t) = d_0 + x$, and the stiffness property refers to the algebraic relationship between x and f, as depicted in Figure 2.7(b). Because x has been defined as an elongation and the plot shows that f and x always have the same sign, it follows that the positive sense of f must be to the right in Figure 2.7(a); that is, f represents a tensile rather than

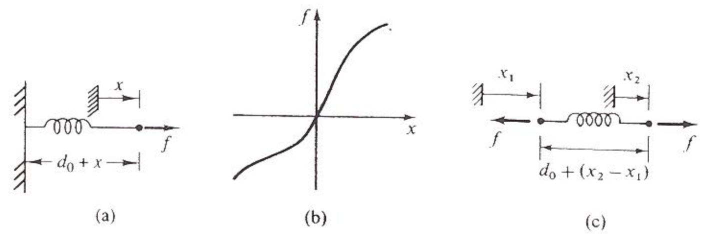


FIGURE 2.7 Characteristics of a spring.

a compressive force. For a linear spring, the curve in Figure 2.7(b) is a straight line and $f = Kx$, where K is a constant with units of newtons per meter (N/m).

Figure 2.7(c) shows a spring whose ends are displaced by the amounts x_1 and x_2 relative to their respective reference positions. If $x_1 = x_2 = 0$ corresponds to a condition when no force is applied to the spring, then the elongation at any instant is $x_2 - x_1$. For a linear spring,

$$f = K\Delta x \tag{9}$$

where $\Delta x = x_2 - x_1$. For small elongations of a structural shaft, K is proportional to the cross-sectional area and to Young's modulus and is inversely proportional to the length.

When a force f is applied to one side of a stiffness element that is assumed to have no mass, a force equal in magnitude but of opposite direction must be exerted on the other side. Thus for the system shown in Figure 2.8, the force f passes through the first spring and is exerted directly on the mass M. Of course, all physical devices have some mass, but to obtain a lumped model we assume either that it is negligible or that it is represented by a separate element.

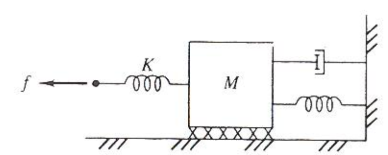


FIGURE 2.8 Force transmitted through a spring.

Potential energy is stored in a spring that has been stretched or compressed, and for a linear spring that energy is given by

$$w_p = \frac{1}{2}K(\Delta x)^2 \tag{10}$$

This energy may be returned to the rest of the mechanical system at some time in the future. Therefore, the initial elongation $\Delta x(t_0)$ is one of the initial conditions we need in order to find the complete response of a system.

2.3 INTERCONNECTION LAWS

Having identified the individual elements in translational systems and having given equations describing their behavior, we next present the laws that describe the manner in which the elements are interconnected. These include D'Alembert's law, the law of reaction forces, and the law for displacement variables.

D'Alembert's Law

D'Alembert's law is just a restatement of Newton's second law governing the rate of change of momentum. For a constant mass, we can write

$$\sum_i (f_{\text{ext}})_i = M\frac{dv}{dt} \tag{11}$$

where the summation over the index i includes all the external forces $(f_{\text{ext}})_i$ acting on the body. The forces and velocity are in general vector quantities, but they can be treated as scalars provided that the motion is constrained to be in a fixed direction. Rewriting (11) as

$$\sum_i (f_{\text{ext}})_i - M\frac{dv}{dt} = 0 \tag{12}$$

suggests that the mass in question can be considered to be in equilibrium—that is, the sum of the forces is zero—provided that the term $-M\,dv/dt$ is thought of as an additional force. This fictitious force is called the **inertial force** or **D'Alembert force**, and including it along with the external forces allows us to write the force equation as one of equilibrium:

$$\sum_i f_i = 0 \tag{13}$$

This equation is known as **D'Alembert's law**. The minus sign associated with the inertial force in (12) indicates that when $dv/dt > 0$, the force acts in the negative direction.

In addition to applying (13) to a mass, we can apply it to any point in the system, such as the junction between components. Because a junction is considered massless, the inertial force is zero in such a case.

The Law of Reaction Forces

In order to relate the forces exerted by the elements of friction and stiffness to the forces acting on a mass or junction point, we need Newton's third law regarding reaction forces. Accompanying any force of one element on another, there is a **reaction force** on the first element of equal magnitude and opposite direction.

In Figure 2.9(a), for example, let f_k denote the force exerted by the mass on the right end of the spring, with the positive sense defined to be to the right. Newton's third law tells us that there acts on the mass a reaction force f_k of equal magnitude with its positive sense to the left, as indicated in Figure 2.9(b). Likewise, at the left end, the fixed surface exerts a force f_k on the spring with the positive sense to the left, while the spring exerts an equal and opposite force on the surface.

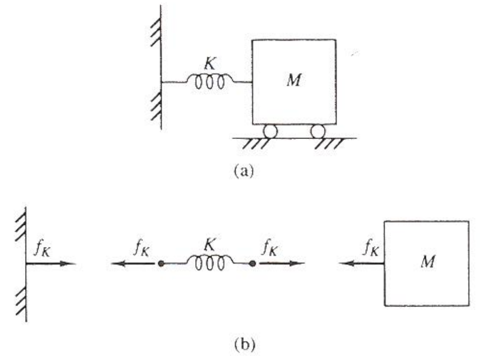


FIGURE 2.9 Example of reaction forces.

The Law for Displacements

If the ends of two elements are connected, those ends are forced to move with the same displacement and velocity. For example, because the dashpot and spring in Figure 2.10(a) are both connected between the wall and the mass, the right ends of both elements have the same displacement x and move with the same velocity v. In Figure 2.10(b), where B_2 and K are connected

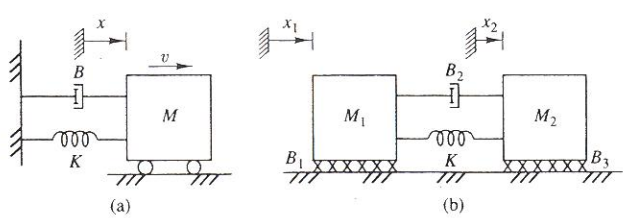


FIGURE 2.10 Two elements connected between the same endpoints. (a) One endpoint fixed. (b) Both endpoints movable.

between two moving masses, the elongation of both elements is $x_2 - x_1$. An equivalent statement is that if we go from M_1 to M_2 and record the elongation of the dashpot B_2, and then return to M_1 and subtract the elongation of the spring K, the result is zero. In effect, we are saying that the difference between the displacements of any two points is the same regardless of which elements we are examining between those points. This statement is really a consequence of our being able to uniquely define points in space.

The discussion in the previous paragraph can be summarized by saying that at any instant the algebraic sum of the elongations around any closed path is zero; that is

$$\sum_i (\Delta x)_i = 0 \quad \text{around any closed path} \tag{14}$$

It is understood that the left side of (14) is the algebraic sum of the elongations with signs that take into account the direction in which the path is being traversed. Furthermore, it is understood that for two elements connected between the same two points, such as B_2 and K in Figure 2.10(b), the elongations of both elements must be measured with respect to the same references. If for some reason the two elongations were measured with respect to different references, then the algebraic sum of the elongations around the closed path would be a constant but not zero.

In Figure 2.11, let x_1 and x_2 denote displacements measured with respect to reference positions that correspond to a single equilibrium condition of the system. Then the respective elongations of B_1, B_2, and K are x_1, $x_2 - x_1$, and x_2. When the elongations are summed going from the fixed surface to the mass by way of the friction elements B_1 and B_2 and going back by way of the spring K, (14) gives

$$x_1 + (x_2 - x_1) - x_2 = 0$$

This equation can be regarded as a justification for the statement that the elongation of B_2 is $x_2 - x_1$ and that an additional symbol for this elongation is not needed.

In the analysis of mechanical systems, (14) is normally used implicitly and automatically in the process of labeling the system diagram. For example, we use the same symbol for two elongations that are forced to be equal

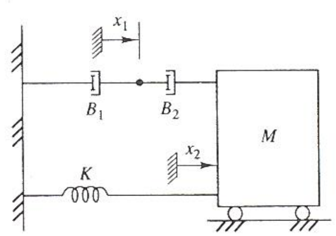


FIGURE 2.11 Illustration for the displacement law.

by the element interconnections, and we avoid using different symbols for displacements that are known to be identical.

Differentiating (14) would lead to a similar equation in terms of relative velocities. However, we shall use only a single symbol for the velocities of two points that are constrained to move together. It will therefore not be necessary to invoke (14) in a formal way.

2.4 OBTAINING THE SYSTEM MODEL

The system model must incorporate both the element laws and the interconnection laws. The element laws involve displacements, velocities, and accelerations. Because the acceleration of a point is the derivative of the velocity, which in turn is the derivative of the displacement, we could write all the element laws in terms of x and its derivatives or in terms of x, v, and dv/dt. It is important to indicate the assumed positive directions for displacements, velocities, and accelerations. We shall always choose the assumed positive directions for a, v, and x to be the same, so it will not be necessary to indicate all three positive directions on the diagram. Throughout this book, dots over the variables are used to denote derivatives with respect to time. For example, $\dot{x} = dx/dt$ and $\ddot{y} = d^2y/dt^2$.

Free-Body Diagrams

We normally need to apply D'Alembert's law, given by (13), to each mass or junction point in the system that moves with a velocity that is unknown beforehand. To do so, it is useful to draw a free-body diagram for each such mass or point, showing all external forces and the inertial force by arrows that define their positive senses. The element laws are used to express all forces except inputs in terms of displacements, velocities, and accelerations. We must be sure that the signs of these expressions are consistent with the directions of the reference arrows. After the free-body diagram is completed, we can apply (13) by summing the forces indicated on the diagram, again taking into account their assumed positive senses. Normally all forces must be added as vectors, but in our examples the forces in the free-body diagram will be collinear and can be summed by scalar equations. The following two examples illustrate the procedure in some detail. The first system contains a single mass; the second has two masses that can move with different velocities.

▶ ***EXAMPLE 2.1***

Draw the free-body diagram and apply D'Alembert's law for the system shown in Figure 2.12(a). The mass is assumed to move horizontally on frictionless bearings, and the spring and dashpot are linear.

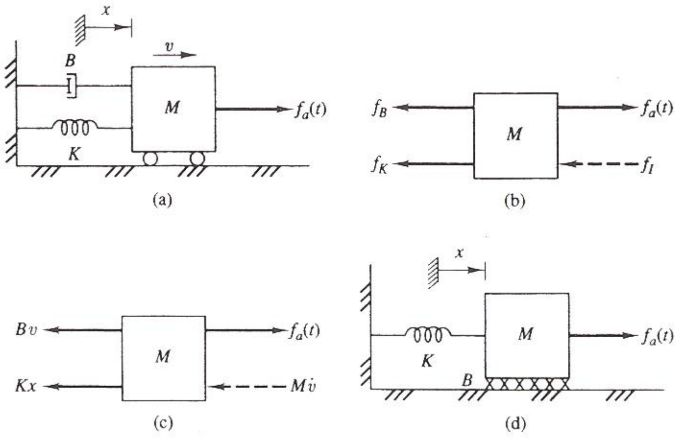


FIGURE 2.12 (a) Translational system for Example 2.1. (b) Free-body diagram. (c) Free-body diagram including element laws. (d) Additional system described by (15).

Solution

The free-body diagram for the mass is shown in Figure 2.12(b). The vertical forces on the mass (the weight Mg and the upward forces exerted by the frictionless bearings) have been omitted because these forces are perpendicular to the direction of motion. The horizontal forces, which are included in the free-body diagram, are

f_K, the force exerted by the spring
f_B, the force exerted by the dashpot
f_I, the inertial force
$f_a(t)$, the applied force

The choice of directions for the arrows representing f_K, f_B, and f_I is arbitrary and does not affect the final result. However, the expressions for these individual forces must agree with the choice of arrows. The use of a dashed arrow for the inertial force f_I emphasizes that it is not an external force like the other three.

We next use the element laws to express the forces f_K, f_B, and f_I in terms of the element values K, B, and M and the system variables x and v. In Figure 2.12(a), the positive direction of x and v is defined to be to the right, so the spring is stretched when x is positive and compressed when x is negative. If the spring undergoes an elongation x, then there must be a tensile force Kx on the right end of the spring directed to the right and a reaction force $f_K = Kx$ on the mass directed to the left. Because the arrow for f_K does point to the left in Figure 2.12(b), we may relabel this force as Kx in Figure 2.12(c). Note that if x is negative at some instant of time, the spring will be compressed and will exert a force to the right on the mass. Under these conditions, Kx will be negative and the free-body diagram will show a negative force on the mass to the left, which is equivalent to a positive force to the right. Although the result is the same either way, it is customary to assume that all displacements are in the assumed positive directions when determining the proper expressions for the forces.

Similarly, when the right end of the dashpot moves to the right with velocity v, a force $f_B = Bv$ is exerted on the mass to the left. Finally, because of (12), the inertial force $f_I = M\dot{v}$ must have its positive direction opposite to that of dv/dt. After trying a few examples, the reader should be able to draw a free-body diagram such as the one in Figure 2.12(c) without first having to show explicitly the diagram in Figure 2.12(b).

D'Alembert's law can now be applied to the free-body diagram in Figure 2.12(c), with due regard for the assumed arrow directions. If forces acting to the right are regarded as positive, the law yields

$$f_a(t) - (M\dot{v} + Bv + Kx) = 0$$

Replacing v by $\dot{x}$ and $\dot{v}$ by $\ddot{x}$, and rearranging the terms, we can rewrite this equation as

$$M\ddot{x} + B\dot{x} + Kx = f_a(t) \tag{15}$$

In place of the dashpot between the mass and the wall in part (a) of Figure 2.12, part (d) shows viscous friction between the mass and the horizontal support. The free-body diagram for this system would remain the same as in Figure 2.12(c), and D'Alembert's law would again yield (15).

▶ *EXAMPLE 2.2*

Draw the free-body diagrams and write D'Alembert's law for the two-mass system shown in Figure 2.13(a).

Solution

Because there are two masses that can move with different unknown velocities, a separate free-body diagram should be drawn for each one. This is done in Figure 2.13(b) and Figure 2.13(c). In Figure 2.13(b), the forces K_1x_1 and $M_1\ddot{x}_1$ are similar to those in Example 2.1. As indicated in our earlier discussion of displacements, the net elongation of the spring and dashpot connecting the two masses is $x_2 - x_1$. Hence a positive value of $x_2 - x_1$ results in a reaction force by the spring to the right on M_1 and to the left on M_2, as indicated in the figure. Of course, the force on either free-body diagram could be labeled $K_2(x_1 - x_2)$, provided that the corresponding reference arrow was reversed. For a positive value of $\dot{x}_2 - \dot{x}_1$, the reaction force of the middle dashpot is to the right on M_1 and to the left on M_2.

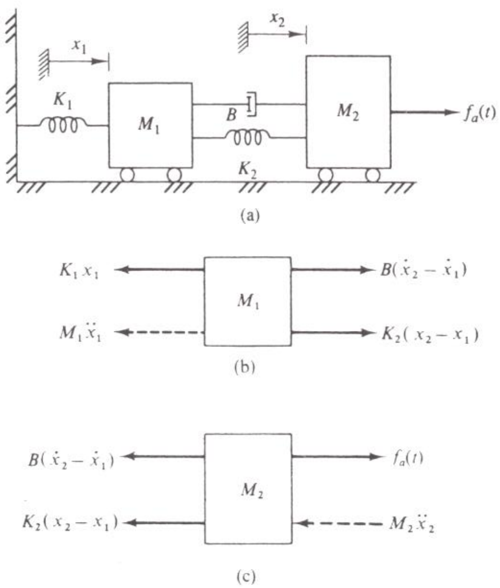


FIGURE 2.13 (a) Translational system for Example 2.2. (b), (c) Free-body diagrams.

As always, the inertial forces $M_1\ddot{x}_1$ and $M_2\ddot{x}_2$ are opposite to the positive directions of the accelerations.

Summing the forces on each free-body diagram separately and taking into account the directions of the reference arrows give the following pair of differential equations:

$$B(\dot{x}_2 - \dot{x}_1) + K_2(x_2 - x_1) - M_1\ddot{x}_1 - K_1x_1 = 0$$

$$f_a(t) - M_2\ddot{x}_2 - B(\dot{x}_2 - \dot{x}_1) - K_2(x_2 - x_1) = 0$$

Rearranging, we have

$$M_1\ddot{x}_1 + B\dot{x}_1 + (K_1 + K_2)x_1 - B\dot{x}_2 - K_2x_2 = 0 \tag{16a}$$

$$-B\dot{x}_1 - K_2x_1 + M_2\ddot{x}_2 + B\dot{x}_2 + K_2x_2 = f_a(t) \tag{16b}$$

Equations (16a) and (16b) constitute a pair of coupled second-order differential equations. In the next chapter, we shall discuss two alternative methods of presenting the information contained in such a set of equations.

In the force equation (16a) for the mass M_1 in the last example, note that all the terms involving the displacement x_1 and its derivatives have the

same sign. Similarly in (16b) for the mass M_2, all the terms with x_2 and its derivatives have the same sign. This is generally true for systems where the only permanent energy sources are associated with the external inputs. The reason for this will become apparent from the discussion of stability in Chapter 6, but the reader may wish to use this fact now as a check on the work.

Suppose that D'Alembert's law is applied to any mass M_i and that terms involving corresponding variables are collected together. Then all the terms involving the displacement x_i of the mass M_i should be expected to have the same sign. If they do not, the engineer should suspect that an error has been made and should check the steps leading to that equation. In the simplified equation for M_i no general statement can be made about the signs of terms involving displacements other than x_i and its derivatives, because the other signs depend on the reference directions used for the definition of the variables.

Inputs are variables that are specified functions of time, which are completely known at the beginning of a problem rather than depending on the values of the system's components. Inputs for translational mechanical systems may be either forces or displacements. A displacement input exists when one part of a system is moved in a predetermined way. We assume that the mechanism that provides the specified displacement has a sufficient source of energy to carry out the motion regardless of any retarding forces that might come from the system components.

▶ *EXAMPLE 2.3*

The system shown in Figure 2.14 is the same as that in Figure 2.13(a), except that the excitation is the displacement input $x_2(t)$ instead of an applied force. Write the equation governing the motion.

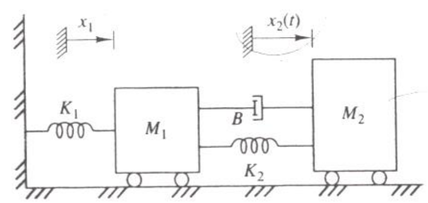


FIGURE 2.14 Translational system with displacement input for Example 2.3.

Solution

We know the motion of mass M_2 in advance, so there is no need to draw a free-body diagram for it. The free-body diagram for M_1 is still the one shown in Figure 2.13(b), and D'Alembert's law again gives

$$M_1\ddot{x}_1 + B\dot{x}_1 + (K_1 + K_2)x_1 - B\dot{x}_2 - K_2x_2(t) = 0 \tag{17}$$

Because $x_2(t)$ and $\dot{x}_2$ are known functions of time, x_1 is the only unknown variable in (17) and a second equation is not needed.

If we wished to determine the force f_2 that needs to be applied in order to move M_2 with the prescribed displacement, we could also draw a free-body diagram for M_2. This would be the same as the one in Figure 2.13(c), except that $f_a(t)$ would be replaced by the unknown force f_2, with its positive sense to the right. The corresponding equation is

$$f_2 = -B\dot{x}_1 - K_2x_1 + M_2\ddot{x}_2 + B\dot{x}_2 + K_2x_2(t) \tag{18}$$

The quantities $x_2(t)$, $\dot{x}_2$, and $\ddot{x}_2$ are known. Once we have solved (17) for the unknown displacement $x_1(t)$ as a function of time, we can insert that result into (18) in order to calculate f_2.

A D'Alembert equation is not usually written for a body whose motion is known in advance. If, however, a free-body diagram is drawn for such a body, it is important to show the force associated with the displacement input. The quantity f_2 in (18) would then be regarded as one of the outputs in the system model.

Relative Displacements

In the previous examples, the displacement variable associated with each mass was expressed with respect to its own fixed reference position. However, the position of a mass is sometimes measured with respect to some other moving body, rather than from a fixed reference. Such a **relative displacement** is used as one of the variables in the following example.

▶ *EXAMPLE 2.4*

For the two-mass system shown in Figure 2.15(a), x denotes the position of mass M_1 with respect to a fixed reference, and z denotes the relative displacement of mass M_2 with respect to M_1. The positive direction for both displacements is to the right. Assume that the two springs are neither stretched nor compressed when $x = z = 0$. Find the equations describing the system.

Solution

The free-body diagrams for the two masses are shown in parts (b) and (c) of the figure. The force on mass M_1 through the viscous friction element B_2 is proportional to the relative velocity $\dot{z}$ of the two masses. If M_2 is moving to the right faster than M_1, so that $\dot{z}$ is positive, then the force on M_1 through B_2 tends to pull the mass to the right, as indicated in Figure 2.15(b).

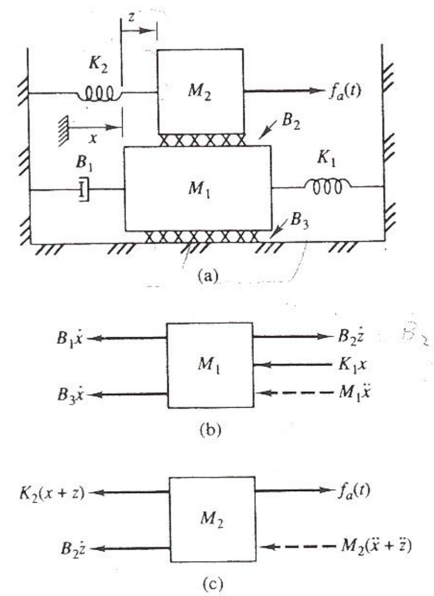


FIGURE 2.15 (a) Translational system for Example 2.4. (b), (c) Free-body diagrams.

To draw the free-body diagram for M_2, we note that the elongation of the spring K_2 is $x + z$. Furthermore, the inertial force is always proportional to the *absolute* acceleration $\ddot{x} + \ddot{z}$, not to the *relative* acceleration with respect to some other moving body. Thus only the force exerted through the friction element B_2 is expressed in terms of the *relative* motion of the two masses. Summing the forces on each of the free-body diagrams gives

$$B_2\dot{z} - K_1x - M_1\ddot{x} - B_1\dot{x} - B_3\dot{x} = 0$$

$$f_a(t) - M_2(\ddot{x} + \ddot{z}) - K_2(x + z) - B_2\dot{z} = 0$$

or, after we rearrange the terms,

$$M_1\ddot{x} + (B_1 + B_3)\dot{x} + K_1x - B_2\dot{z} = 0 \tag{19a}$$

$$M_2\ddot{x} + K_2x + M_2\ddot{z} + B_2\dot{z} + K_2z = f_a(t) \tag{19b}$$

The comments made after Example 2.2 about the signs in the force equations for M_1 and M_2 still apply when a relative displacement variable is used. In the force equation (19a) for M_1, all the terms with the displacement x of the mass M_1 and its derivatives have the same sign. Similarly in (19b), all the terms involving the relative displacement z of M_2 with respect to M_1 have the same sign.

The reader is encouraged to repeat this example when the displacement of each mass is expressed with respect to its own fixed reference position. If x_1 and x_2 denote the displacements of M_1 and M_2, respectively, with the positive senses to the right, we find that

$$M_1\ddot{x}_1 + (B_1 + B_2 + B_3)\dot{x}_1 + K_1 x_1 - B_2\dot{x}_2 = 0$$
$$-B_2\dot{x}_1 + M_2\ddot{x}_2 + B_2\dot{x}_2 + K_2 x_2 = f_a(t) \qquad (20)$$

When x_1 is replaced by x, and x_2 is replaced by $x + z$, this pair of equations reduces to those in (19). However, it is useful to be able to obtain (19) directly from the free-body diagrams in Figure 2.15.

Displacement variables must be expressed with respect to some reference position. These references are commonly chosen such that any springs are neither stretched nor compressed when the values of the displacement variables are zero. The following two examples show why this may not necessarily be the case for systems with vertical motion.

▶ EXAMPLE 2.5

Draw the free-body diagram, including the effect of gravity, and find the differential equation describing the motion of the mass shown in Figure 2.16(a).

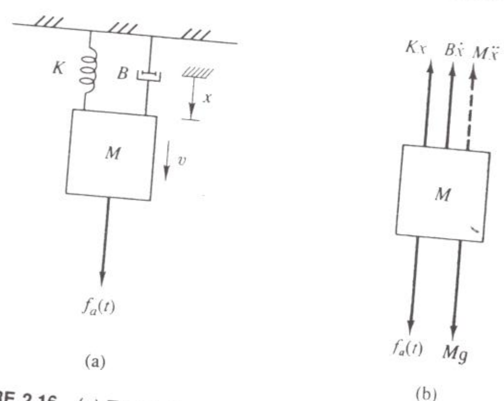


FIGURE 2.16 (a) Translational system with vertical motion. (b) Free-body diagram.

Solution

Assume that x is the displacement from the position corresponding to a spring that is neither stretched nor compressed. The gravitational force on the mass is Mg, and we include it in the free-body diagram shown in

Figure 2.16(b) because the mass moves vertically. By summing the forces on the free-body diagram, we obtain

$$M\ddot{x} + B\dot{x} + Kx = f_a(t) + Mg \qquad (21)$$

Suppose that the applied force $f_a(t)$ is zero and that the mass is not moving. Then $x = x_0$, where x_0 is the constant displacement caused by the gravitational force. Because $\dot{x}_0 = \ddot{x}_0 = 0$, the foregoing differential equation reduces to the algebraic equation

$$Kx_0 = Mg \qquad (22)$$

We can also see this directly from the free-body diagram by noting that all but two of the five forces vanish under these conditions.

We now reconsider the case where $f_a(t)$ is nonzero and where the mass is moving. Let

$$x = x_0 + z \qquad (23)$$

This equation defines z as the displacement caused by the input $f_a(t)$, namely the additional displacement beyond that resulting from the constant weight Mg. Substituting (23) into (21) and again noting that $\dot{x}_0 = \ddot{x}_0 = 0$, we have

$$M\ddot{z} + B\dot{z} + K(x_0 + z) = f_a(t) + Mg$$

or, by using (22),

$$M\ddot{z} + B\dot{z} + Kz = f_a(t) \qquad (24)$$

Comparison of (21) and (24) indicates that we can ignore the gravitational force Mg when drawing the free-body diagram and when writing the system equation, provided that the displacement is defined to be the displacement from the static position corresponding to no inputs except gravity.

This conclusion is valid for all cases where masses are suspended vertically by one or more linear springs. Under static-equilibrium conditions, there are no inertial or friction forces, and the force exerted by the spring has the form

$$\begin{aligned} f_K(t) &= K(x_0 + z) \\ &= Kx_0 + Kz \end{aligned}$$

which is just the superposition of the static force caused by gravity and the force caused by additional inputs. For nonlinear springs, however, this conclusion is not valid because superposition does not hold.

Normally, a new symbol such as z is not introduced in problems involving gravitational forces. Instead, the symbol x can be redefined to be the additional displacement from the static position.

▶ *EXAMPLE 2.6*

For the system shown in Figure 2.17(a), x_1 and x_2 denote the elongations of K_1 and K_2, respectively. Note that x_1 is the displacement of mass M_1 with respect to a fixed reference but that x_2 is the relative displacement of M_2 with respect to M_1. When $x_1 = x_2 = 0$, all three springs shown in the figure are neither stretched nor compressed. Draw the free-body diagram for each mass, including the effect of gravity, and find the differential equations describing the system's behavior. Determine the values of x_1 and x_2 that correspond to the static-equilibrium position, when $f_a(t) = 0$ and when the masses are motionless.

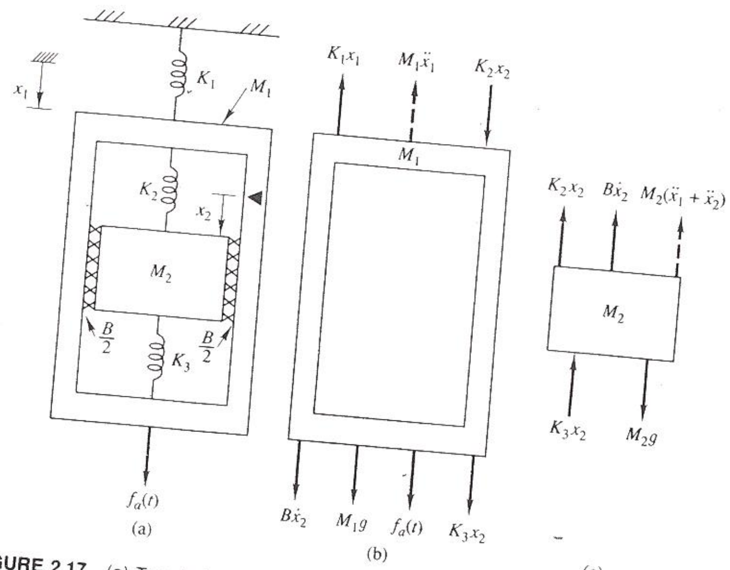


FIGURE 2.17 (a) Translational system for Example 2.6. (b), (c) Free-body diagrams.

Solution

The free-body diagrams are shown in parts (b) and (c) of the figure. Many of the comments made in Example 2.4 also apply to this problem. If x_1 and x_2 are positive, then K_1 and K_2 are stretched and K_3 is compressed. Under these circumstances, K_1 exerts an upward force on M_1, and K_2 and K_3 exert downward forces on M_1. The relative velocity of M_2 with respect to M_1 is $\dot{x}_2$, so frictional forces of $B\dot{x}_2$ are exerted downward on M_1 and upward on M_2. The inertial force on M_2 is proportional to its absolute acceleration, which is $\ddot{x}_1 + \ddot{x}_2$. Summing the forces on each of the free-body diagrams

gives

$$
\begin{aligned}
M_1\ddot{x}_1 + K_1x_1 - B\dot{x}_2 - (K_2 + K_3)x_2 &= M_1g + f_a(t) \\
M_2\ddot{x}_1 + M_2\ddot{x}_2 + B\dot{x}_2 + (K_2 + K_3)x_2 &= M_2g
\end{aligned}
\tag{25}
$$

This pair of coupled equations has two unknown variables. If the element values and $f_a(t)$ are known, and if the necessary initial conditions are given, then (25) can be solved for x_1 and x_2 as functions of time by the methods discussed in later chapters. Because x_2 is a relative displacement, the total displacement of M_2 is $x_1 + x_2$.

To find the displacements x_{1_0} and x_{2_0} that correspond to the static-equilibrium position, we replace $f_a(t)$ and all the displacement derivatives by zero. Then (25) reduces to

$$
\begin{aligned}
K_1x_{1_0} - (K_2 + K_3)x_{2_0} &= M_1g \\
(K_2 + K_3)x_{2_0} &= M_2g
\end{aligned}
\tag{26}
$$

from which

$$
\begin{aligned}
x_{1_0} &= \frac{(M_1 + M_2)g}{K_1} \\
x_{2_0} &= \frac{M_2g}{K_2 + K_3}
\end{aligned}
\tag{27}
$$

If we want the differential equations in terms of displacements z_1 and z_2 measured with respect to the equilibrium conditions given by (27), then we can write $x_1 = x_{1_0} + z_1$ and $x_2 = x_{2_0} + z_2$. Substituting these expressions into (25) and using (26), we find that

$$
\begin{aligned}
M_1\ddot{z}_1 + K_1z_1 - B\dot{z}_2 - (K_2 + K_3)z_2 &= f_a(t) \\
M_2\ddot{z}_1 + M_2\ddot{z}_2 + B\dot{z}_2 + (K_2 + K_3)z_2 &= 0
\end{aligned}
$$

As expected, these equations are similar to (25) except for the absence of the gravitational forces.

The Ideal Pulley

A pulley can be used to change the direction of motion in a translational mechanical system. Frequently, part of the system then moves horizontally and the rest vertically. The basic pulley consists of a cylinder that can rotate about its center and that has a cable resting on its surface. An ideal pulley has no mass and no friction associated with it. We assume that there is no slippage between the cable and the surface of the cylinder—that is, that they both move with the same velocity. We also assume that the cable is always in tension but that it cannot stretch. If we need to consider a cable that can stretch, we can approximate that effect by showing a separate spring leading to an ideal cable. If the pulley is not ideal, then its mass and any frictional effects must be considered, as will be discussed in Chapter 4. The action of an ideal pulley is illustrated in the following example.

▶ *EXAMPLE 2.7*

Find and compare the equations describing the systems shown in Figure 2.18(a) and Figure 2.19(a). Let $x_1 = x_2 = 0$ correspond to the condition when the springs are neither stretched nor compressed.

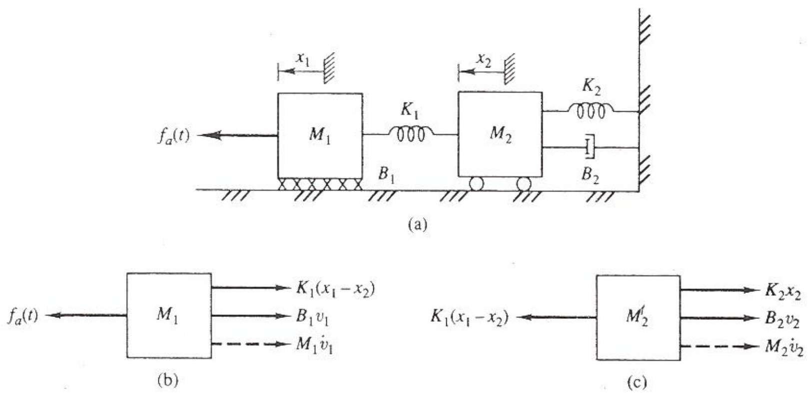


FIGURE 2.18 (a) Translational system for Example 2.7. (b), (c) Free-body diagrams.

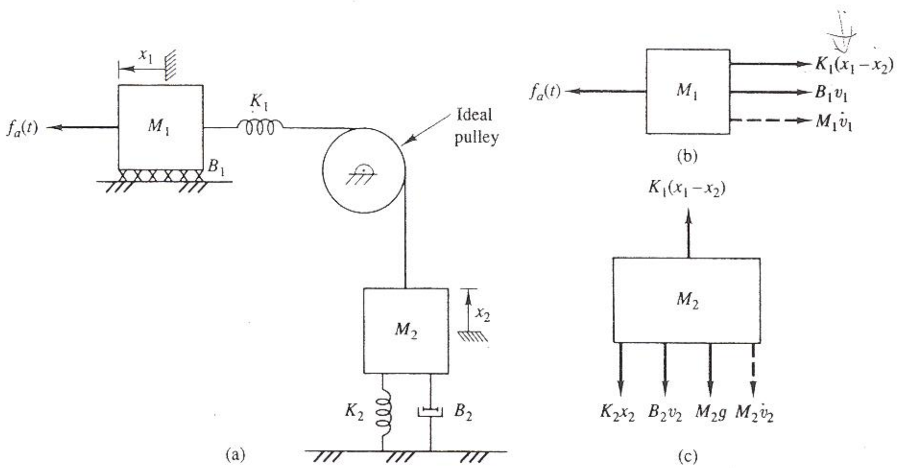


FIGURE 2.19 (a) Translational system with an ideal pulley added. (b), (c) Free-body diagrams.

Solution

Free-body diagrams for the system in part (a) of Figure 2.18 are shown in parts (b) and (c). Note that the spring K_1 exerts equal but opposite forces on M_1 and M_2. Summing the forces shown in the diagrams gives

$$\begin{aligned} M_1\dot{v}_1 + B_1v_1 + K_1(x_1 - x_2) &= f_a(t) \\ M_2\dot{v}_2 + B_2v_2 + K_2x_2 &= K_1(x_1 - x_2) \end{aligned} \tag{28}$$

When an ideal pulley is added to give the system in Figure 2.19(a), the free-body diagram for M_1, which is repeated in part (b) of the figure, remains unchanged. To draw the diagram for M_2, we note that both ends of the cable move with the same motion because it cannot stretch. The force exerted by K_1 passes through the cable and is exerted directly on M_2. Hence the cable does not affect the magnitude of the forces exerted by K_1 but merely changes the direction of the force on M_2 to be upward. Because M_2 moves vertically, we also include the gravitational force in its free-body diagram, which is shown in Figure 2.19(c). Summing the forces shown in parts (b) and (c) of the figure gives

$$\begin{aligned} M_1\dot{v}_1 + B_1v_1 + K_1(x_1 - x_2) &= f_a(t) \\ M_2\dot{v}_2 + B_2v_2 + K_2x_2 + M_2g &= K_1(x_1 - x_2) \end{aligned} \tag{29}$$

As expected, (28) and (29) are identical except for the presence of the gravitational force M_2g.

It is important to be aware of a significant difference between the models given by (28) and (29) for the last example. The difference involves their validity. Positive or negative values of the quantity $x_1 - x_2$ correspond to the elongation or compression, respectively, of the spring K_1. Equations (28) are valid for Figure 2.18(a) for all values of x_1 and x_2. For the system with the pulley, however, the free-body diagram in Figure 2.19(c) indicates that the cable would exert a downward force on M_2 for a negative value of $x_1 - x_2$. This implies that the cable would tend to push apart the bodies attached to its ends, which is not physically possible. In such a case the cable would buckle, become detached from the pulley, and exert no force at all on M_2. Thus (29) can be used only when $x_1 - x_2 \geq 0$. Although this condition is not normally written next to the corresponding equations, the engineer should always examine the analytical or computer solution to be sure that the results correspond to conditions for which the modeling equations are valid.

Parallel Combinations

In some cases, two or more springs or dashpots can be replaced by a single equivalent element. Two springs or dashpots are said to be in **parallel** if the first end of each is attached to the same body and if the remaining ends are also attached to a common body. We shall consider a specific example before formulating a general rule.

▶ *EXAMPLE 2.8*

The system shown in Figure 2.20(a) includes two linear springs between the wall and the mass M. Write the differential equation describing the motion of the mass. Find the spring constant K_{eq} for a single spring that could replace K_1 and K_2. Assume first that the springs have the same unstretched length. Then repeat the problem when the unstretched lengths of K_1 and K_2 are d_1 and d_2, respectively.

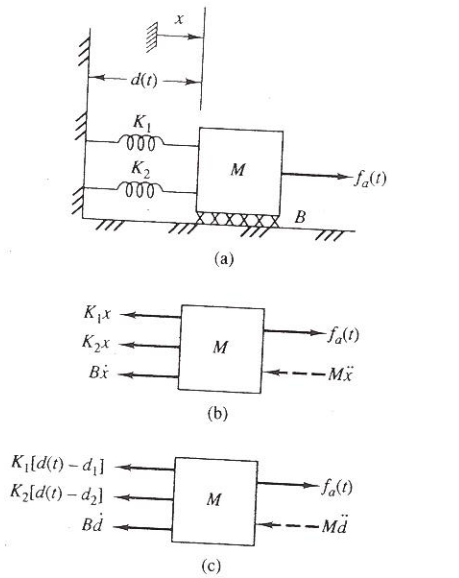


FIGURE 2.20 (a) Translational system for Example 2.8. (b) Free-body diagram when the springs have the same unstretched lengths. (c) Free-body diagram when the springs have different unstretched lengths.

Solution

If the unstretched lengths of the two springs are identical, then they will have the same elongation, denoted by x, when the mass is in motion. The free-body diagram is shown in Figure 2.20(b). Summing the forces gives

$$M\ddot{x} + B\dot{x} + (K_1 + K_2)x = f_a(t) \tag{30}$$

If the combination of K_1 and K_2 is replaced by a single equivalent spring, then the system reduces to that shown in Figure 2.12(d), which is described by (15). Comparing (30) and (15) reveals that

$$K_{eq} = K_1 + K_2 \tag{31}$$

Next suppose that the springs have different unstretched lengths, and let $d(t)$ denote the distance from the wall to the left side of the mass. The elongations of K_1 and K_2, which are no longer the same, are $d(t) - d_1$ and $d(t) - d_2$, respectively. The mass has velocity $\dot{d}$ and acceleration $\ddot{d}$. The new free-body diagram, with all the retarding forces expressed in terms of $d(t)$, is shown in Figure 2.20(c). By D'Alembert's law,

$$M\ddot{d} + B\dot{d} + K_1[d(t) - d_1] + K_2[d(t) - d_2] = f_a(t) \tag{32}$$

Let d_0 be the distance from the wall to the mass when $f_a(t) = 0$ and when the mass is motionless in a position of static equilibrium. The only forces on the mass will be those exerted by the two springs, which will be equal and opposite. From (32), or directly from Figure 2.20(c),

$$K_1(d_0 - d_1) + K_2(d_0 - d_2) = 0 \tag{33}$$

from which

$$d_0 = \frac{K_1 d_1 + K_2 d_2}{K_1 + K_2} \tag{34}$$

The respective static elongations of K_1 and K_2 are

$$d_0 - d_1 = \frac{K_2(d_2 - d_1)}{K_1 + K_2}$$

$$d_0 - d_2 = \frac{K_1(d_1 - d_2)}{K_1 + K_2}$$

If $d_1 \neq d_2$, one of these elongations will be negative. This means that one spring will be stretched and the other compressed, so that the net static force exerted by the two springs on the mass can be zero.

It is now reasonable to define the variable z as the distance of the mass beyond d_0 when a force input is applied—that is, as the displacement with respect to the reference position given by (34). Then

$$d(t) = d_0 + z$$

Substituting this expression into (32) and noting that $\dot{d} = \dot{z}$ and $\ddot{d} = \ddot{z}$, we have

$$M\ddot{z} + B\dot{z} + K_1(d_0 + z - d_1) + K_2(d_0 + z - d_2) = f_a(t)$$

Using (33) to cancel some of the terms in this equation, we obtain

$$M\ddot{z} + B\dot{z} + (K_1 + K_2)z = f_a(t)$$

which is identical to (30) except for the use of z in place of x. Thus (31) is valid for the equivalent spring constant even if the unstretched lengths of

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or dashpots have their respective ends joined, as m the last example, we see that for the parallel

$$K_{eq} = K_1 + K_2 \tag{35}$$

be shown that for two dashpots in parallel, as in part (b) of

$$B_{eq} = B_1 + B_2 \tag{36}$$

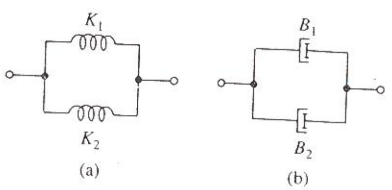

FIGURE 2.21 Parallel combinations. (a) $K_{eq} = K_1 + K_2$. (b) $B_{eq} = B_1 + B_2$.

The formulas for parallel stiffness or friction elements can be extended to situations that may seem somewhat different than those in Figure 2.21. The key requirement for parallel elements is that respective ends move with the same displacement. The individual ends need not be tied directly together in the figure depicting the system. Although one pair of ends may sometimes be connected to a fixed surface, in other cases both pairs of ends may be free to move.

▶ ***EXAMPLE 2.9***

Find the equation describing the motion of the mass in the translational system shown in Figure 2.22(a). Show that the two springs can be replaced by a single equivalent spring, and the three friction elements by an equivalent element.

Solution

Note that each of the springs has one side attached to the mass and the other attached to the fixed surface on the perimeter of the diagram. When x is positive, K_1 is stretched and K_2 compressed. Thus the springs will exert forces of K_1x and K_2x on the mass to the left, as shown on the free-body diagram in Figure 2.22(b).

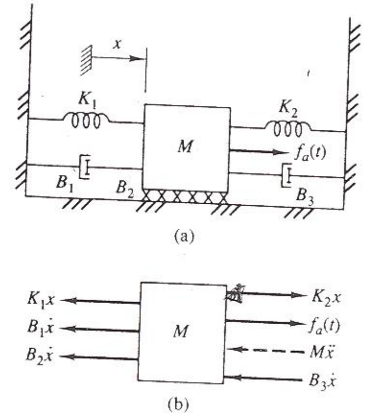

FIGURE 2.22 (a) Translational system with parallel stiffness and friction elements. (b) Free-body diagram.

For each of the three friction elements, one side moves with the velocity of the mass and the other side is stationary. When $\dot{x}$ is positive, each of these elements exerts a retarding force on the mass to the left. Summing the forces shown on the free-body diagram, we have

$$M\ddot{x} + (B_1 + B_2 + B_3)\dot{x} + (K_1 + K_2)x = f_a(t)$$

With $K_{eq} = K_1 + K_2$ and $B_{eq} = B_1 + B_2 + B_3$, this equation becomes

$$M\ddot{x} + B_{eq}\dot{x} + K_{eq}x = f_a(t)$$

When the parallel combinations of stiffness and friction elements in Figure 2.22(a) are replaced by equivalent elements, the system reduces to that shown in Figure 2.12(d), which is described by (15).

Series Combinations

Two springs or dashpots are said to be in **series** if they are joined at only one end of each element and if there is no other element connected to their common junction. The following example has a series combination of two springs and also illustrates the application of D'Alembert's law to a massless junction.

▶ ***EXAMPLE 2.10***

When $x_1 = x_2 = 0$, the two springs shown in Figure 2.23(a) are neither stretched nor compressed. Draw free-body diagrams for the mass M and for the massless junction A, and then write the equations describing the system. Show that the motion of point A is not independent of that of the

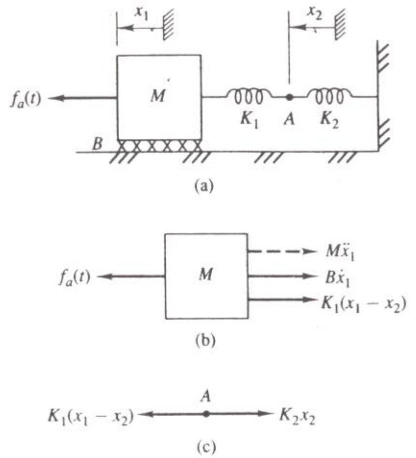

FIGURE 2.23 (a) Translational system with a massless junction. (b), (c) Free-body diagrams.

mass M and that x_1 and x_2 are directly proportional to one another. Finally, find K_{eq} for a single spring that could replace the combination of K_1 and K_2.

Solution

The free-body diagrams are shown in parts (b) and (c) of the figure. Because there is no mass at point A, there is no inertial force in its free-body diagram. Summing the forces for each diagram gives

$$M\ddot{x}_1 + B\dot{x}_1 + K_1(x_1 - x_2) = f_a(t)$$
$$K_2x_2 = K_1(x_1 - x_2)$$

Solving the second equation for x_2 in terms of x_1 gives

$$x_2 = \left(\frac{K_1}{K_1 + K_2}\right)x_1$$

which shows that the two displacements are proportional to one another. Substituting this expression back into the first equation, we have

$$M\ddot{x}_1 + B\dot{x}_1 + K_1\left[1 - \frac{K_1}{K_1 + K_2}\right]x_1 = f_a(t)$$

from which

$$M\ddot{x}_1 + B\dot{x}_1 + \frac{K_1K_2}{K_1 + K_2}x_1 = f_a(t)$$

This equation describes the system formed when the two springs in Figure 2.23(a) are replaced by a single spring for which $K_{eq} = K_1K_2/(K_1 + K_2)$.

Series combinations of stiffness and friction elements are shown in Figure 2.24. It is assumed that no other element is connected to the common junctions. For the two springs in part (a) of the figure, as in the last example,

$$K_{eq} = \frac{K_1K_2}{K_1 + K_2} \tag{37}$$

For two dashpots in series, as in part (b) of the figure, it can be shown that

$$B_{eq} = \frac{B_1B_2}{B_1 + B_2} \tag{38}$$

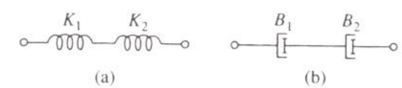

FIGURE 2.24 Series combinations. (a) $K_{eq} = K_1K_2/(K_1 + K_2)$. (b) $B_{eq} = B_1B_2/(B_1 + B_2)$.

In order to reduce certain combinations of springs or dashpots to a single equivalent element, we may have to use the rules for both parallel and series combinations. The following example illustrates the procedure for two different combinations of dashpots.

▶ *EXAMPLE 2.11*

Find B_{eq} for the single friction element that can replace the three dashpots in each part of Figure 2.25.

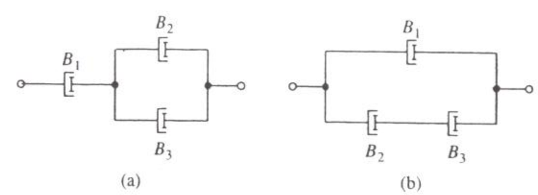

FIGURE 2.25 Parallel-series combinations.

Solution

In part (a) of the figure, the parallel combination of B_2 and B_3 can be replaced by a single element whose viscous friction coefficient is $B_2 + B_3$.

This is then in series with B_1, so that for the overall combination,

$$B_{eq} = \frac{B_1(B_2 + B_3)}{B_1 + B_2 + B_3}$$

In Figure 2.25(b), the series combination of B_2 and B_3 is in parallel with B_1. Thus

$$B_{eq} = B_1 + \frac{B_2 B_3}{B_2 + B_3}$$

SUMMARY

In this chapter we have introduced the variables, element laws, and interconnection laws for linear, lumped-element translational mechanical systems. Either force or displacement inputs can be applied to any part of the system. An applied force is a known function of time, but the motion of the body to which it is applied is not known at the outset of a problem. Conversely, a displacement input moves some part of the system with a specified motion, but the force exerted by the external mechanism moving that part is normally not known.

Displacements may be measured with respect to fixed reference positions or with respect to some other moving body. When relative displacements are used, it is important to keep in mind that the inertial force of a mass is always proportional to its absolute acceleration, not to its relative acceleration.

To model a system, we draw a free-body diagram and sum the forces for every mass or other junction point whose motion is unknown. The free-body diagram for a massless junction is drawn in the usual way, except that there is no inertial force. The modeling process can sometimes be simplified by replacing a series-parallel combination of stiffness or friction elements by a single equivalent element.

Special attention was given to linear systems that involve vertical motion. If displacements are measured from positions where the springs are neither stretched nor compressed, the gravitational forces must be included in the free-body diagrams for any masses that can move vertically. If, however, the displacements are measured with respect to the static-equilibrium positions when the system is motionless and when no other external inputs are applied, then the gravitational forces do not appear in the final equations of motion. Systems containing pulleys can have some parts of the system moving horizontally and other parts moving vertically.

Our basic task in this chapter was to draw free-body diagrams and to write the corresponding equations describing the system. In the next chapter, we shall discuss how to present the information contained in these equations in ways that will facilitate the development of both computer and analytical solutions.

PROBLEMS

Throughout this book, answers to problems marked with an asterisk are given in Appendix E.

* **2.1** For the system shown in Figure P2.1, the springs are undeflected when $x_1 = x_2 = 0$. The input is $f_a(t)$. Draw free-body diagrams and write the modeling equations.

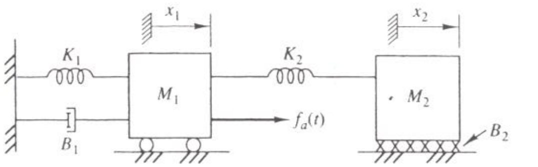

FIGURE P2.1

2.2 Repeat Problem 2.1 for the system shown in Figure P2.2.

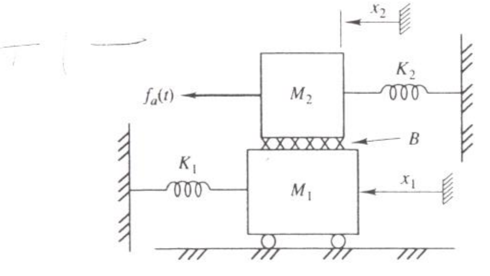

FIGURE P2.2

2.3 Repeat Problem 2.1 for the system shown in Figure P2.3.

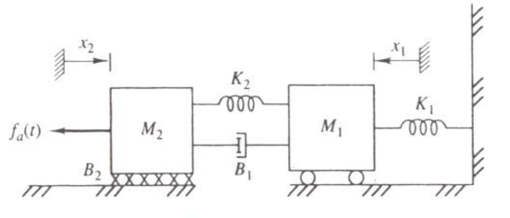

FIGURE P2.3

* **2.4** For the system shown in Figure P2.4, draw the free-body diagram for each mass and write the differential equations describing the system.

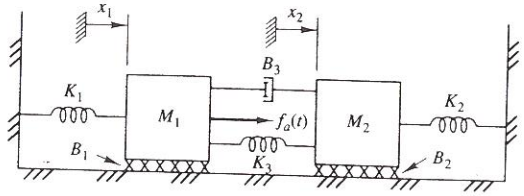

FIGURE P2.4

2.5 Repeat Problem 2.4 for the system shown in Figure P2.5.

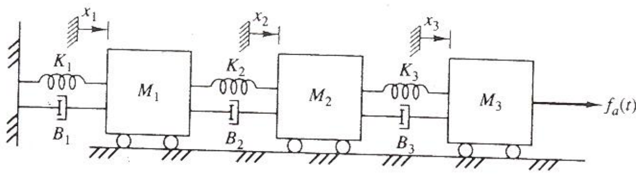

FIGURE P2.5

2.6 In the mechanical system shown in Figure P2.6, the spring forces are zero when $x_1 = x_2 = x_3 = 0$. Let the base be stationary so that $x_3(t) = 0$ for all values of t. Draw free-body diagrams and write a pair of coupled differential equations that govern the motion when the only input is $f_a(t)$.

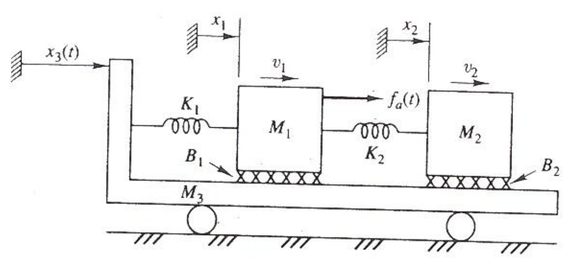

FIGURE P2.6

* **2.7** For the system shown in Figure P2.7, the springs are undeflected when $x_1 = x_2 = 0$. The input is $x_2(t)$, the displacement of the left edge of M_2.

a) Write the equation governing the motion of M_1.

b) Write an expression for the force f_2, positive sense to the right, that must be applied to M_2 in order to achieve the displacement $x_2(t)$.

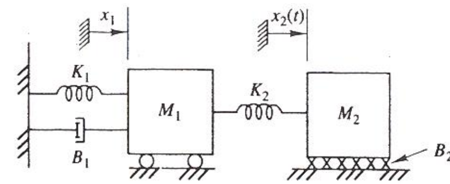

FIGURE P2.7

2.8 Repeat Problem 2.7 for the system shown in Figure P2.8.

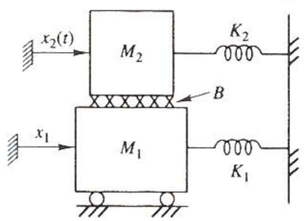

FIGURE P2.8

2.9 **a)** Repeat Problem 2.6 when the displacement $x_3(t)$ is the input and the applied force $f_a(t)$ is zero for all time.

b) Write an expression for the force f_3, positive sense to the right, that must be applied to M_3 in order to move M_3 with the specified displacement $x_3(t)$.

* **2.10** For the system shown in Figure P2.10, the distance between masses M_1 and M_2 is $A + x_2$, where A is a constant. The springs are undeflected when $x_1 = x_2 = 0$. Draw free-body diagrams and write the differential equations for the system.

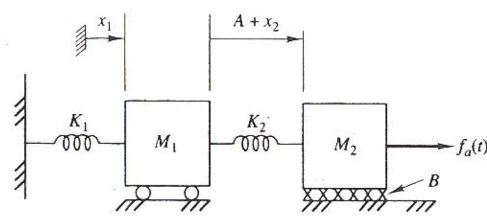

FIGURE P2.10

2.11 For the system shown in Figure P2.11, the input is the displacement $x_1(t)$. The springs are undeflected when $x_1 = x_2 = x_3 = 0$. The variable x_2 represents the displacement of M_2 with respect to M_1. Write the mathematical model to describe the

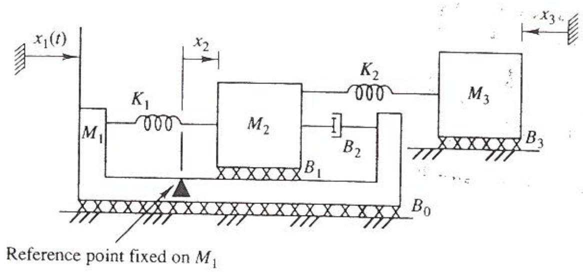

FIGURE P2.11

motion of the masses as a set of coupled differential equations. Include appropriate free-body diagrams.

* **2.12** The input to the translational mechanical system shown in Figure P2.12 is the displacement $x_3(t)$ of the right end of the spring K_1. The displacement of M_2 relative to M_1 is x_2. The forces exerted by the springs are zero when $x_1 = x_2 = x_3 = 0$. Draw the free-body diagrams and write the modeling equations.

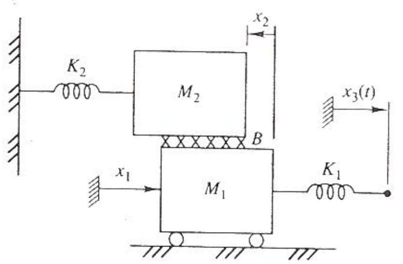

FIGURE P2.12

2.13 For the system shown in Figure P2.13, the displacement of M_1 relative to M_2 is x_1. The spring force is zero when $x_1 = x_2 = 0$. Draw free-body diagrams and write the modeling equations.

2.14 For the system shown in Figure P2.14, the displacements x_1 and x_2 are measured relative to M_3. Both springs are undeflected when $x_1 = x_2 = 0$. Draw free-body diagrams and write the modeling equations.

* **2.15** The mechanical system shown in Figure P2.15 is driven by the applied force $f_a(t)$. When $x_1 = x_2 = 0$, the springs are neither stretched nor compressed.

a) Draw the free-body diagrams and write the differential equations of motion for the two masses in terms of x_1 and x_2.

b) Find x_{1_0} and x_{2_0}, the constant displacements of the masses caused by the gravitational forces when $f_a(t) = 0$ and when the system is in static equilibrium.

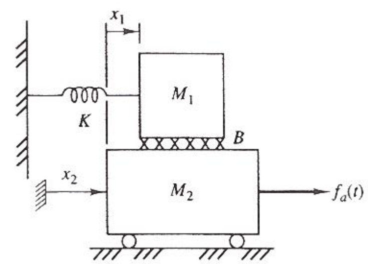

FIGURE P2.13

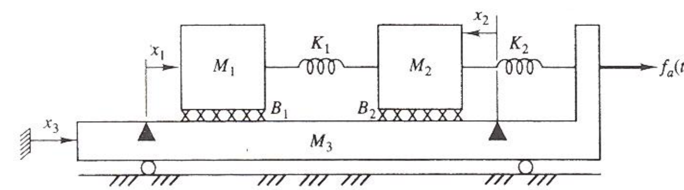

FIGURE P2.14

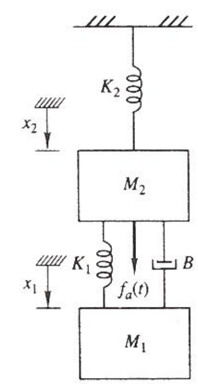

FIGURE P2.15

c) Rewrite the system equations in terms of z_1 and z_2, the relative displacements of the masses with respect to the static-equilibrium positions found in part (b).

2.16 Repeat all three parts of Problem 2.15 for the system shown in Figure P2.16. Each of the three springs has the same spring constant K.

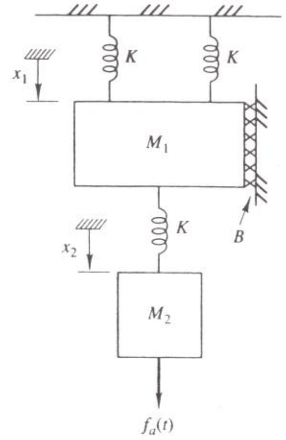

FIGURE P2.16

2.17 All the springs in Figure P2.17 are identical, each with spring constant K. The spring forces are zero when $x_1 = x_2 = x_3 = 0$.

a) Draw the free body diagrams, including the gravitational forces, and write the differential equations describing the system.

b) Determine the constant elongation of each spring caused by the gravitational forces when the masses are stationary in a position of static equilibrium and when $f_a(t) = 0$.

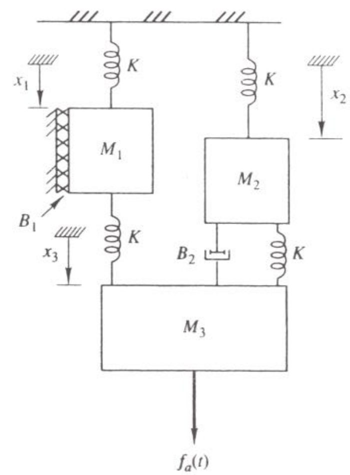

FIGURE P2.17

*** 2.18** Repeat Problem 2.17 for the system shown in Figure P2.18.

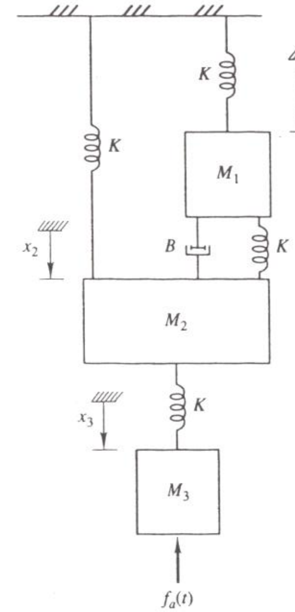

FIGURE P2.18

2.19 The system shown in Figure P2.19 has a nonlinear spring that obeys the expression $f_K = x^3$.

a) Write the differential equation describing the system in terms of the displacement x.

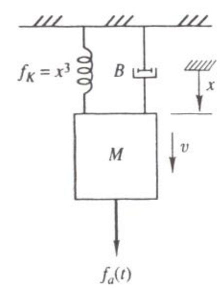

FIGURE P2.19

b) Let $x = x_0 + z$, where x_0 is the constant displacement caused by the gravitational force when the system is in static equilibrium. Rewrite the differential equation in terms of the variable z, canceling the gravitational term.
c) Comparison of (21) and (24) in Example 2.5 for a linear spring showed that the differential equation in z was the same as the one in x except for the deletion of the Mg term. Compare the results of parts (a) and (b) to show that such is not the case in this problem.

2.20 For the system shown in Figure P2.20, x_1 and x_2 are displacements relative to the undeflected spring lengths. The inputs are $f_a(t)$, $f_b(t)$, and gravity. Draw free-body diagrams and write the modeling equations.

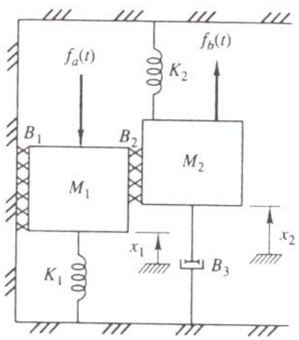

FIGURE P2.20

2.21 Repeat Problem 2.20 for the system shown in Figure P2.21.

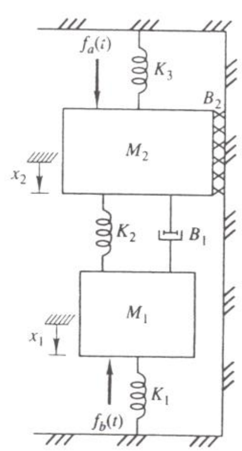

FIGURE P2.21

* **2.22** The pulley shown in Figure P2.22 is ideal. Draw free-body diagrams and write the modeling equations.

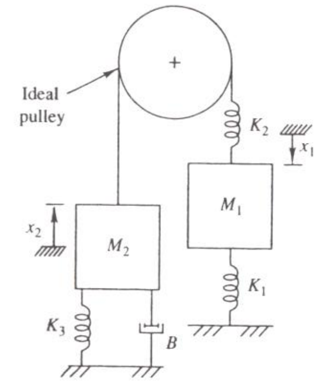

FIGURE P2.22

2.23 Repeat Problem 2.22 for the system shown in Figure P2.23.

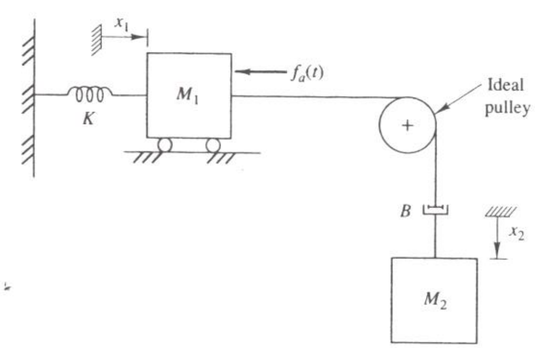

FIGURE P2.23

2.24 Repeat Problem 2.22 for the system shown in Figure P2.24.

* **2.25** For the system shown in Figure 2.11, draw the free-body diagrams for M and the junction of the dashpots. Apply D'Alembert's law and show that $\dot{x}_1$ is proportional to $\dot{x}_2$. Determine an equivalent coefficient B_{eq} for the combination of B_1 and B_2.

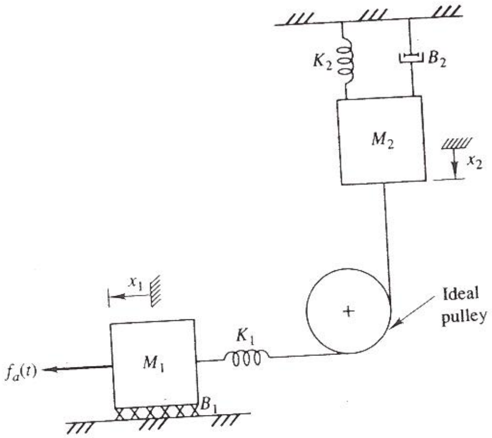

FIGURE P2.24

2.26 For the system shown in Figure P2.26, draw the free-body diagram and write a single differential equation. Determine an equivalent coefficient B_{eq} for the combination of B_1 and B_3.

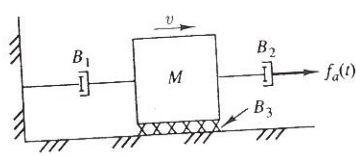

FIGURE P2.26

* **2.27** The input for the system shown in Figure P2.27 is the displacement $x_3(t)$. Draw the free-body diagrams for the mass M and for the massless point A. Write the differential equations describing the system.

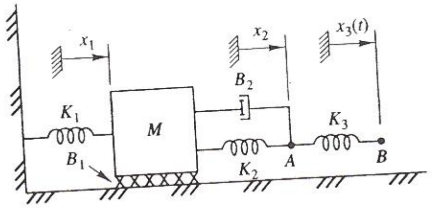

FIGURE P2.27

2.28 Repeat Problem 2.27 when the input is the force $f_a(t)$ applied to point B, with its positive sense to the right.

2.29 For the system shown in Figure P2.29, $x_1 = 0$ when the spring is undeflected. Draw free-body diagrams and write the modeling equations.

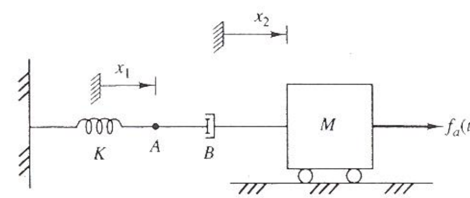

FIGURE P2.29

2.30 **a)** Write the equations describing the series combination of elements shown in Figure P2.30(a).
b) Find expressions for K_{eq} and B_{eq} in Figure P2.30(b) such that the motions of the ends of the combination are the same as those in part (a) of the figure.

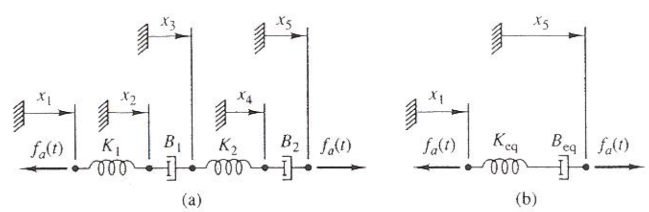

FIGURE P2.30