C H A P T E R 3

STANDARD FORMS FOR SYSTEM MODELS

In Chapter 2, we introduced the element and interconnection laws for translational systems and the procedure for drawing free-body diagrams and applying D'Alembert's law. These are essential steps regardless of the final form of the equations.

Not only should the complete mathematical model contain as many independent equations as unknown variables, but it should also have a form that is convenient for its solution. The two most common forms for the model when a direct solution is contemplated are discussed, illustrated, and compared in this chapter. Although translational mechanical systems are used as examples, the same techniques are applied in subsequent chapters to other types of systems. Other important forms of the model, based on Laplace transform procedures, will be found later in the book.

3.1 STATE-VARIABLE EQUATIONS

One procedure for formulating the system model is to begin by selecting a set of **state variables**. This set of variables must completely describe the effect of the past history of the system on its response in the future. Then, knowing the values of the state variables at a reference time t_0 and the values of the inputs for all $t \geq t_0$ is sufficient for evaluating the state variables and outputs for all $t \geq t_0$. It is further understood that the state variables must be independent; that is, it must be impossible to express any state variable as an algebraic function of the remaining state variables and the inputs.

Although the choice of state variables is not unique, the state variables for mechanical systems are usually related to the energy stored in each of the system's energy-storing elements. Because any energy that is initially

stored in these elements can affect the response of the system at a later time, one state variable is normally associated with each of the energy-storing elements. Generally, this adequately summarizes the effect of the past history of the system. In some systems, the number of state variables is different from the number of energy-storing elements because a particular interconnection of elements causes redundant variables or because there is need for a state variable that is not related to the storage of energy.

Suppose the system has n state variables, m inputs, and p outputs. Let $q_1, q_2, \ldots, q_n$ denote the state variables, let $u_1, u_2, \ldots, u_m$ be the inputs, and let $y_1, y_2, \ldots, y_p$ be the outputs. We must find a set of first-order differential equations having a particular form. Each equation must express the derivative of one of the state variables as an *algebraic* function of the state variables and inputs. For a linear system, these algebraic functions will consist of a sum of terms, each of which is just a state variable multiplied by a coefficient or an input multiplied by a coefficient. If, for example, there are three state variables and two inputs, the state-variable equations have the form

$$\begin{aligned}
\dot{q}_1 &= a_{11}q_1 + a_{12}q_2 + a_{13}q_3 + b_{11}u_1 + b_{12}u_2 \\
\dot{q}_2 &= a_{21}q_1 + a_{22}q_2 + a_{23}q_3 + b_{21}u_1 + b_{22}u_2 \\
\dot{q}_3 &= a_{31}q_1 + a_{32}q_2 + a_{33}q_3 + b_{31}u_1 + b_{32}u_2
\end{aligned} \tag{1}$$

The outputs of interest are then expressed as algebraic functions of the state variables and inputs. If, for example, the linear system described by (1) happens to have two outputs, then the output equations normally have the form

$$\begin{aligned}
y_1 &= c_{11}q_1 + c_{12}q_2 + c_{13}q_3 + d_{11}u_1 + d_{12}u_2 \\
y_2 &= c_{21}q_1 + c_{22}q_2 + c_{23}q_3 + d_{21}u_1 + d_{22}u_2
\end{aligned} \tag{2}$$

For fixed linear systems, all of the coefficients given by a_{ij}, b_{ij}, c_{ij}, and d_{ij} are constants. For linear systems whose parameters vary with time, some of these coefficients are functions of time. Only for nonlinear elements could some of the coefficients themselves become functions of the state variables or inputs. In the most general case of time-varying, nonlinear systems, the right sides of the state-variable and output equations can be quite complicated functions of the state variables, the inputs, and time. Even then, however, the right sides of these equations should still be algebraic functions, without any derivative or integral terms. An uncommon exception wherein derivatives of the input must appear in (2) will be illustrated in one of the examples.

In this chapter we shall deal almost entirely with fixed linear systems. The form of the equations will then be generally similar to (1) and (2), although there may be more state variables, inputs, or outputs. An output variable may be identical to a state variable, in which case (2) may contain one or more equations such as $y_i = q_i$. When all the outputs are identical to some of the state variables, (2) becomes trivial and can be omitted.

Solving (1) for $t \geq t_0$ requires knowledge of the inputs for $t \geq t_0$ and also of the initial values of the state variables, namely $q_1(t_0)$, $q_2(t_0)$, and $q_3(t_0)$. In general, we cannot solve the individual state-variable equations separately but must solve them as a group. For example, the equation $\dot{q}_1 = -q_1 + q_2$ cannot be solved for q_1 unless another equation exists that can be solved for q_2.

The energy stored in translational mechanical systems must be associated with the masses or springs. From (2.6)[1] the kinetic energy of a mass is $\frac{1}{2}Mv^2$, and from (2.10) the potential energy of a spring is $\frac{1}{2}K(\Delta x)^2$. Thus it is logical to consider as possible state variables the velocities of the masses and the elongations of the springs. In most problems, we can express the elongations of the springs in terms of the displacements of the masses.

When drawing free-body diagrams, we try to express the forces in terms of state variables and inputs and in a way that avoids unnecessary derivatives. Thus we normally label inertial and friction forces in the form $M\dot{v}$ and Bv, rather than $M\ddot{x}$ and $B\dot{x}$.

In the remainder of this section, a variety of examples illustrate the technique of deriving the mathematical model in state-variable form. The general approach is as follows:

1. Identify the state variables and write those state-variable equations that do not require a free-body diagram, such as equations of the form $\dot{x} = v$.
2. Draw free-body diagrams for each independent mass and junction point that can move with an unknown motion. Sum the forces on each free-body diagram separately to obtain a set of differential equations.
3. Manipulate the equations into state-variable form. For each of the state variables, there must be an equation that expresses its derivative as an algebraic function of the state variables, the inputs, and possibly time.
4. Express the output variables as algebraic functions of the state variables, the inputs, and possibly time. In some unusual cases, it may be necessary to have derivatives of the input on the right side of the output equations. However, input derivatives should be avoided wherever possible.

▶ *EXAMPLE 3.1*

Find the state-variable model for the system shown in Figure 3.1(a), which is identical to Figure 2.12(a). The outputs of interest are the tensile force for the spring K and the velocity and acceleration of the mass M.

Solution

We choose as state variables the elongation x of the spring, which is related to its potential energy, and the velocity v of the mass, which is related to its

[1]This denotes equation (6) in Chapter 2.

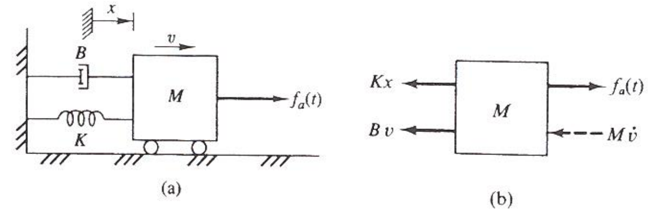

FIGURE 3.1 (a) Translational system for Example 3.1. (b) Free-body diagram.

kinetic energy. Having made this choice, we can write the first of the two state-variable equations by inspection:

$$\dot{x} = v$$

Summing the forces on the free-body diagram, which was drawn for Example 2.1 and which is repeated in Figure 3.1(b), gives

$$M\dot{v} + Bv + Kx = f_a(t)$$

Solving this equation for $\dot{v}$ results in the second state-variable equation, where the right side is a function of only the state variables x and v and the input $f_a(t)$, as required. Thus the state variable equations are

$$\dot{x} = v$$
$$\dot{v} = \frac{1}{M}[-Kx - Bv + f_a(t)]$$

Analytical and computer solutions of such a set of equations are treated in Chapters 6 and 15, respectively. In order to find the state variables x and v as functions of time for all $t \geq 0$, we would need to know the input $f_a(t)$ for $t \geq 0$ and also the initial values of the state variables at $t = 0$. Any outputs that are not also state variables are expressed as algebraic functions of the state variables and inputs, so they can be evaluated as soon as the state variables have been found.

The velocity of the mass is one of the state variables. The output equations for the tensile force f_K in the spring and the acceleration a_M of the mass are

$$f_K = Kx$$
$$a_M = \frac{1}{M}[-Kx - Bv + f_a(t)]$$

The system considered in the following three examples has four energy-storing elements and four state variables. The modeling is carried out for two different choices of state variables in order to illustrate that there are often several satisfactory choices. The examples also show that finding the state-variable equations is not made significantly more difficult when some of the system's elements are nonlinear.

▶ *EXAMPLE 3.2*

Find the state-variable equations for the system shown in Figure 3.2(a), which is identical to Figure 2.13(a). Write output equations for the tensile force f_{K_2} in the spring K_2 and for the total momentum m_T of the masses.

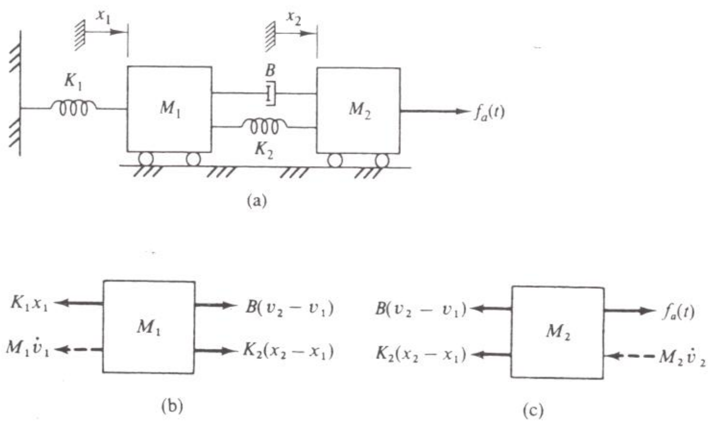

FIGURE 3.2 (a) Translational system for Example 3.2. (b), (c) Free-body diagrams.

Solution

An appropriate choice of state variables is x_1, v_1, x_2 and v_2, because we can express the velocity of each mass and the elongation of each spring in terms of these four variables and because none of these variables can be expressed in terms of the other three. Because $\dot{x}_1 = v_1$ and $\dot{x}_2 = v_2$, two of the four state-variable equations are available immediately.

The free-body diagrams for the two masses are repeated from Example 2.2 in Figure 3.2(b) and Figure 3.2(c), with all forces labeled in terms of the state variables and the input. By D'Alembert's law,

$$M_1\dot{v}_1 + K_1x_1 - K_2(x_2 - x_1) - B(v_2 - v_1) = 0$$

$$M_2\dot{v}_2 + K_2(x_2 - x_1) + B(v_2 - v_1) = f_a(t)$$

which may be solved for $\dot{v}_1$ and $\dot{v}_2$, respectively. The state-variable equations are

$$\dot{x}_1 = v_1 \tag{3a}$$

$$\dot{v}_1 = \frac{1}{M_1}[-(K_1 + K_2)x_1 - Bv_1 + K_2x_2 + Bv_2] \tag{3b}$$

$$\dot{x}_2 = v_2 \tag{3c}$$

$$\dot{v}_2 = \frac{1}{M_2}[K_2x_1 + Bv_1 - K_2x_2 - Bv_2 + f_a(t)] \tag{3d}$$

If we know the element values, the input $f_a(t)$ for $t \geq 0$, and the initial conditions $x_1(0)$, $v_1(0)$, $x_2(0)$, and $v_2(0)$, then we can solve this set of simultaneous first-order differential equations for x_1, v_1, x_2, and v_2 for all $t \geq 0$. The output equations are

$$\begin{aligned} f_{K_2} &= K_2(x_2 - x_1) \\ m_T &= M_1v_1 + M_2v_2 \end{aligned} \tag{4}$$

▶ *EXAMPLE 3.3*

As an alternative choice of state variables for the system in Figure 3.2(a), use the relative displacement x_R and the relative velocity v_R of mass M_2 with respect to M_1, in place of x_2 and v_2. Again find the state-variable and output equations.

Solution

We choose the four state variables x_1, v_1, x_R, and v_R, where

$$x_R = x_2 - x_1$$

$$v_R = v_2 - v_1$$

The free-body diagrams when the forces are labeled in terms of these variables are shown in Figure 3.3. Remember that the inertial forces depend on the absolute accelerations of the respective masses. By D'Alembert's law,

$$M_1\dot{v}_1 + K_1x_1 - K_2x_R - Bv_R = 0$$

$$M_2\dot{v}_1 + M_2\dot{v}_R + Bv_R + K_2x_R = f_a(t)$$

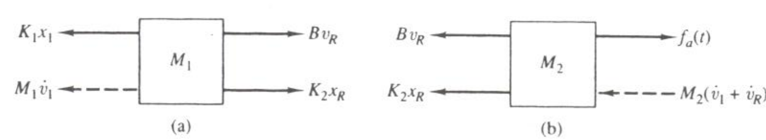

FIGURE 3.3 Free-body diagrams for Example 3.3 using relative displacement and relative velocity.

from which

$$\dot{v}_1 = \frac{1}{M_1}(-K_1x_1 + K_2x_R + Bv_R) \tag{5a}$$

$$\dot{v}_R = \frac{1}{M_2}[-M_2\dot{v}_1 - Bv_R - K_2x_R + f_a(t)] \tag{5b}$$

Although (5a) is a satisfactory state-variable equation, (5b) is not because of the derivative on the right side. Thus we must substitute (5a) into (5b) in order to get an appropriate expression for $\dot{v}_R$. When this is done, we obtain for the complete set of state-variable equations

$$\begin{aligned}
\dot{x}_1 &= v_1 \\
\dot{v}_1 &= \frac{1}{M_1}(-K_1x_1 + K_2x_R + Bv_R) \\
\dot{x}_R &= v_R \\
\dot{v}_R &= \frac{1}{M_1M_2}[K_1M_2x_1 - K_2(M_1 + M_2)x_R \\
&\qquad - B(M_1 + M_2)v_R + M_1f_a(t)]
\end{aligned} \tag{6}$$

To write the output equations in terms of the new state variables, we note that x_R is the elongation of K_2 and that the momentum of a mass depends on its absolute velocity. Thus

$$\begin{aligned}
f_{K_2} &= K_2x_R \\
m_T &= M_1v_1 + M_2(v_1 + v_R) \\
&= (M_1 + M_2)v_1 + M_2v_R
\end{aligned} \tag{7}$$

▶ *EXAMPLE 3.4*

For the system shown in Figure 3.2(a), the forces exerted by the linear spring K_1 and the linear dashpot B were K_1x_1 and Bv_R, respectively. Now assume that these two elements are replaced by nonlinear elements. The expression relating the force f_{K_1} to the displacement x_1 is denoted by $f_{K_1}(x_1)$, whereas the force on the dashpot is denoted by $f_B(v_R)$. Also take as an additional output the energy stored in the linear spring K_2.

Solution

We again choose the state variables to be x_1, v_1, x_R, and v_R. The free-body diagrams are the same as those in Figure 3.3 except that K_1x_1 is replaced by $f_{K_1}(x_1)$, and Bv_R by $f_B(v_R)$, respectively. The resulting state-variable equations are identical to (6) except for the contributions of the two nonlinear

elements:

$$\begin{aligned}
\dot{x}_1 &= v_1 \\
\dot{v}_1 &= \frac{1}{M_1}[-f_{K_1}(x_1) + K_2x_R + f_B(v_R)] \\
\dot{x}_R &= v_R \\
\dot{v}_R &= \frac{1}{M_1M_2}[M_2f_{K_1}(x_1) - K_2(M_1 + M_2)x_R \\
&\qquad - (M_1 + M_2)f_B(v_R) + M_1f_a(t)]
\end{aligned}$$

Note that the procedure for drawing the free-body diagrams and writing the state-variable equations is no more complicated than for the linear case. However, solving the differential equations analytically would be much more difficult, if not impossible.

The output equations for f_{K_2} and m_T remain the same as those in (7). The additional output equation for the energy stored in the linear spring K_2 is

$$w_{K_2} = \frac{1}{2}K_2x_R^2$$

which has a squared variable on the right side. Even in the case of linear components, some possible outputs may require output equations that are not just linear combinations of the state variables and inputs. Even then, however, the output equation should normally be an algebraic function of the state variables and inputs.

Next we consider two modifications of Figure 3.2(a) that illustrate circumstances in which the number of state variables might be different from the number of energy-storing elements. In the first of these, the spring K_1 is removed; in the second, an additional spring is added between the mass M_2 and a wall at the right.

▶ *EXAMPLE 3.5*

Find the state-variable model for the linear system shown in Figure 3.2(a) when the spring K_1 is removed. Let the outputs again be the tensile force f_{K_2} in the spring K_2 and the total momentum m_T of the masses. Then reconsider the problem when the displacement x_1 of the mass M_1 is an additional output of interest.

Solution

The free-body diagrams will be the same as those in parts (b) and (c) of Figure 3.2, except that the force K_1x_1 will be missing. Thus (3) and (4) will still be valid if we just let $K_1 = 0$.

Because there are only three energy-storing elements in the modified system, however, we would expect to need only three state variables, rather than the four that appear in (3). Two of the three state variables can be chosen to be v_1 and v_2, which are related to the kinetic energy stored in the masses. We choose the third to be the elongation of the spring K_2, which is related to the potential energy in that element and which is $x_R = x_2 - x_1$. One of the state-variable equations is $\dot{x}_R = v_2 - v_1$. The other two follow from (3b) and (3d), with $K_1 = 0$ and with $x_2 - x_1$ replaced by x_R:

$$\begin{aligned} \dot{x}_R &= v_2 - v_1 \\ \dot{v}_1 &= \frac{1}{M_1}[K_2 x_R - B v_1 + B v_2] \\ \dot{v}_2 &= \frac{1}{M_2}[-K_2 x_R + B v_1 - B v_2 + f_a(t)] \end{aligned} \tag{8}$$

The only variables on the right side of these equations are state variables and the input $f_a(t)$, as is required. The output equations are

$$\begin{aligned} f_{K_2} &= K_2 x_R \\ m_T &= M_1 v_1 + M_2 v_2 \end{aligned} \tag{9}$$

An alternative form of the state-variable model can be found by letting $K_1 = 0$ in (6) and (7). Because the only state variables that are needed are v_1, x_R, and v_R, we can omit the first equation in (6). Then the state-variable and output equations become

$$\begin{aligned} \dot{v}_1 &= \frac{1}{M_1}(K_2 x_R + B v_R) \\ \dot{x}_R &= v_R \\ \dot{v}_R &= \frac{1}{M_1 M_2}[-K_2(M_1 + M_2)x_R - B(M_1 + M_2)v_R + M_1 f_a(t)] \end{aligned} \tag{10}$$

and

$$\begin{aligned} f_{K_2} &= K_2 x_R \\ m_T &= (M_1 + M_2)v_1 + M_2 v_R \end{aligned} \tag{11}$$

Finally, suppose that one of the outputs of interest is x_1. It is not possible to express x_1 as an algebraic function of only x_R, v_1, v_2, and $f_a(t)$. Thus we cannot construct a suitable output equation based on (8). Similarly, because x_1 cannot be expressed as an algebraic function of only v_1, x_R, v_R, and $f_a(t)$, we cannot write a suitable output equation based on (10). Hence, if x_1 is specified as one of the outputs of interest, three state variables are not

sufficient. We would need to consider x_1 as an additional state variable and add the equation $\dot{x}_1 = v_1$ to either (8) or (10).

The previous example shows why an additional state variable that is not related to energy storage is sometimes needed. The following example illustrates how the number of state variables can be less than the number of energy-storing elements when the system's stored energy can be expressed by a reduced number of variables.

▶ *EXAMPLE 3.6*

Find a state-variable model for the linear system shown in Figure 3.4(a).

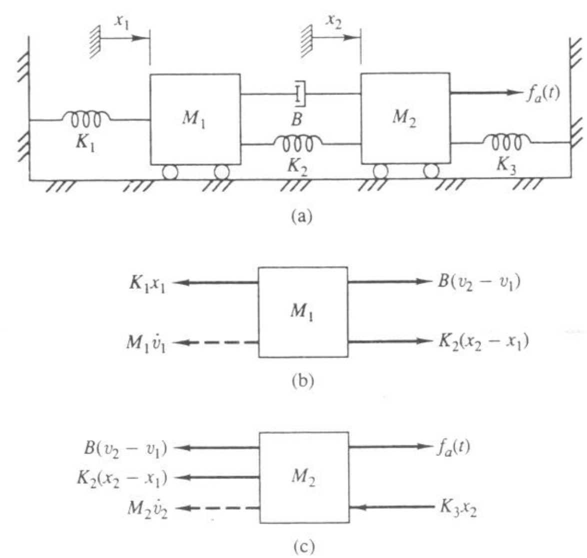

FIGURE 3.4 (a) Translational system with fewer state variables than energy-storing elements. (b), (c) Free-body diagrams.

Solution

Although there are three springs, their elongations are not all independent and can be specified in terms of the two displacement variables x_1 and x_2. The elongations of K_1, K_2, and K_3 are x_1, $x_2 - x_1$, and $-x_2$, respectively. The free-body diagrams for the two masses, which are shown in Figures 3.4(b) and 3.4(c), are the same as those in Figure 3.2 except for the additional force associated with K_3. By D'Alembert's law,

$$\begin{aligned} M_1\dot{v}_1 + (K_1 + K_2)x_1 + Bv_1 - K_2x_2 - Bv_2 &= 0 \\ M_2\dot{v}_2 - K_2x_1 - Bv_1 + (K_2 + K_3)x_2 + Bv_2 &= f_a(t) \end{aligned}$$

By solving these equations for $\dot{v}_1$ and $\dot{v}_2$, we can write the following four state-variable equations:

$$
\begin{aligned}
\dot{x}_1 &= v_1 \\
\dot{v}_1 &= \frac{1}{M_1}[-(K_1+K_2)x_1 - Bv_1 + K_2x_2 + Bv_2] \\
\dot{x}_2 &= v_2 \\
\dot{v}_2 &= \frac{1}{M_2}[K_2x_1 + Bv_1 - (K_2+K_3)x_2 - Bv_2 + f_a(t)]
\end{aligned}
\tag{12}
$$

If the outputs of interest are the tensile force f_{K_2} and the total momentum m_T, then the output equations are again given by (4).

In the following example, two springs and a dashpot are attached to a massless junction. The system contains three energy-storing elements and normally requires three state variables. However, when the dashpot is removed from the massless junction, the number of state variables is reduced.

▶ *EXAMPLE 3.7*

Find the state-variable model for the system shown in Figure 3.5(a). The input is the force $f_a(t)$, and the output is the displacement x_2 of the massless junction A. Repeat the problem when the dashpot B_2 is removed.

Solution

A satisfactory choice of state variables is x_1, v_1, and x_2, because these three variables determine the elongations of the springs and the velocity of the mass. One of the three state-variable equations is $\dot{x}_1 = v_1$.

To obtain the other two equations, we draw free-body diagrams for both the mass and the junction point, as shown in Figure 3.5(b) and Figure 3.5(c). Because the point A is massless, no inertial force is present in its free-body diagram. Summing the forces shown in these diagrams gives

$$M\dot{v}_1 + B_1v_1 + K_1(x_1 - x_2) = f_a(t) \tag{13a}$$

$$B_2\dot{x}_2 + K_2x_2 + K_1(x_2 - x_1) = 0 \tag{13b}$$

Solving (13) for $\dot{v}_1$ and $\dot{x}_2$, we arrive at the state-variable equations

$$\dot{x}_1 = v_1 \tag{14a}$$

$$\dot{v}_1 = \frac{1}{M}[-K_1x_1 - B_1v_1 + K_1x_2 + f_a(t)] \tag{14b}$$

$$\dot{x}_2 = \frac{1}{B_2}[K_1x_1 - (K_1+K_2)x_2] \tag{14c}$$

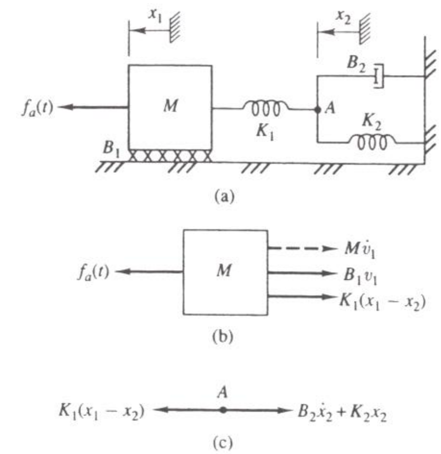

FIGURE 3.5 (a) Translational system containing a massless junction. (b), (c) Free-body diagrams.

Because the only output is one of the state variables, we shall not write a separate output equation.

Removing the dashpot corresponds to setting $B_2 = 0$. We cannot make this substitution in (14c), because division by zero is an invalid mathematical operation. However, replacing B_2 by zero in (13) gives

$$M\dot{v}_1 + B_1v_1 + K_1(x_1 - x_2) = f_a(t) \tag{15a}$$

$$K_2x_2 + K_1(x_2 - x_1) = 0 \tag{15b}$$

The second of these equations is purely algebraic and can be solved for x_2 in terms of x_1:

$$x_2 = \left(\frac{K_1}{K_1+K_2}\right)x_1 \tag{16}$$

The displacements x_1 and x_2 are now proportional to each other, so they cannot both be state variables. If we choose x_1 and v_1 as the state variables, we still have $\dot{x}_1 = v_1$. To find the second state-variable equation, we substitute (16) into (15a) and solve for $\dot{v}_1$. Thus

$$
\begin{aligned}
\dot{x}_1 &= v_1 \\
\dot{v}_1 &= \frac{1}{M}\left[-\left(\frac{K_1K_2}{K_1+K_2}\right)x_1 - B_1v_1 + f_a(t)\right]
\end{aligned}
\tag{17}
$$

As required, the variables appearing on the right sides of these equations are either state variables or the input.

The system for $B_2 = 0$ is shown in Figure 3.6. From the discussion associated with Example 2.10 and Figure 2.23(a), we see that the two springs

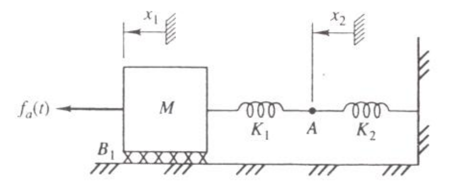

FIGURE 3.6 Translational system with springs in series.

are now in series and can be replaced by a single equivalent spring for which $K_{eq} = K_1K_2/(K_1 + K_2)$. This is reflected by the corresponding term in (17) and by the fact that only two state variables are needed. Because the output x_2 is no longer a state variable, we need a separate output equation to accompany (17). It is given by (16).

Sometimes the proposed state-variable equations that are initially found contain the derivative of an input on the right side. It might be argued that because the input is presumably known completely, its derivative could be found. However, it is important for the standard methods of solution (whether the solution is to be carried out analytically or with a computer) to eliminate any derivatives on the right side of the state-variable equations.

It is also desirable to avoid input derivatives on the right side of the output equations. For some possible outputs, however, such derivatives may be unavoidable.

A general method for eliminating derivatives of the input whenever possible is discussed in Chapter 13, which uses Laplace-transform techniques. However, a basic approach is illustrated in the following example, which also has a displacement rather than an applied force as the specified input. An input derivative on the right side of the state-variable equations is removed by redefining one of the state variables. One of the output equations, however, has unavoidable derivatives on the right side.

▶ *EXAMPLE 3.8*

Find the state-variable model for the system shown in Figure 3.7(a), which is identical to Figure 2.14. The displacement $x_2(t)$ is a prescribed function of time that is the input of the system. The primary outputs of interest are the displacement and velocity of M_1. Consider as an additional output the force f_2 that must be applied to M_2 in order to move it with the prescribed displacement.

Solution

We know the motion of M_2 in advance, so there is no need to draw a free-body diagram for it unless it is needed for one of the output equations. The

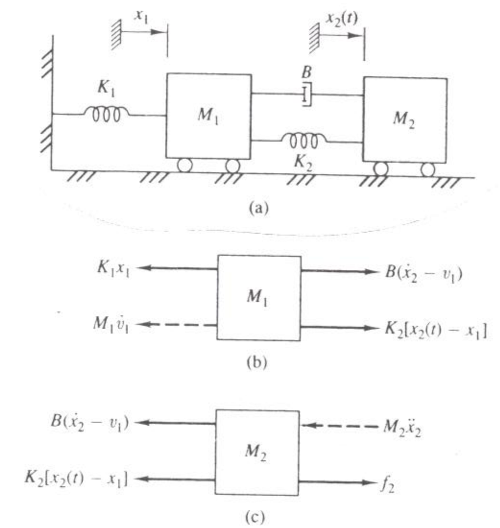

FIGURE 3.7 (a) Translational system with displacement input. (b), (c) Free-body diagrams.

free-body diagram for M_1 is shown in Figure 3.7(b), and the corresponding force equation is

$$M_1\dot{v}_1 + Bv_1 + (K_1 + K_2)x_1 = B_2\dot{x}_2 + K_2x_2(t)$$

We may choose x_1 and v_1 as the state variables, write $\dot{x}_1 = v_1$, and solve the last equation for $\dot{v}_1$, which yields

$$\dot{x}_1 = v_1 \tag{18a}$$

$$\dot{v}_1 = \frac{1}{M_1}[-(K_1 + K_2)x_1 - Bv_1 + B\dot{x}_2 + K_2x_2(t)] \tag{18b}$$

Equation (18b) does not fit the required form for state-variable equations given by (1) because its right side contains $\dot{x}_2$, the derivative of the input. In order to eliminate $\dot{x}_2$ from the initial state-variable equations, we introduce a new state variable q to be used in place of v_1. To determine how q must be defined, we rewrite (18b) with the derivative of x_2 moved to the left side:

$$\dot{v}_1 - \frac{B}{M_1}\dot{x}_2 = \frac{1}{M_1}[-(K_1 + K_2)x_1 - Bv_1 + K_2x_2(t)] \tag{19}$$

If we select the new state variable q as

$$q = v_1 - \frac{B}{M_1}x_2(t) \tag{20}$$

then the left side of (19) can be written as $\dot{q}$. Rearranging (20), we see that the old state variable v_1 is given by

$$v_1 = q + \frac{B}{M_1} x_2(t) \tag{21}$$

Finally, we insert (21) into the right sides of (18a) and (19) to obtain

$$\begin{aligned} \dot{x}_1 &= q + \frac{B}{M_1} x_2(t) \\ \dot{q} &= \frac{1}{M_1}\left[-(K_1 + K_2)x_1 - Bq + \left(K_2 - \frac{B^2}{M_1} \right) x_2(t) \right] \end{aligned} \tag{22}$$

which is in state-variable form. The output x_1 is a state variable, and the output v_1 is given by (21), which fits the standard form for an output equation.

To obtain an expression for the force f_2 applied to M_2, we draw the free-body diagram shown in Figure 3.7(c). By D'Alembert's law,

$$f_2 = -K_2x_1 - Bv_1 + M_2\ddot{x}_2 + B\dot{x}_2 + K_2x_2(t)$$

Using (21) for v_1, we rewrite this equation as

$$f_2 = -K_2x_1 - Bq + M_2\ddot{x}_2 + B\dot{x}_2 + \left(K_2 - \frac{B^2}{M_1} \right) x_2(t)$$

which contains derivatives of the input in addition to the input $x_2(t)$ and the state variables x_1 and q. For this particular output, however, there is no way to avoid input derivatives on the right side.

Systems may have several inputs. In the final state-variable example, a time-varying force $f_a(t)$ is applied to M_1, and there is a gravitational force on M_2.

▶ ***EXAMPLE 3.9***

Find a state-variable model for the system that was shown in Figure 2.19(a) and considered in Example 2.7. The two inputs are $f_a(t)$ and the gravitational constant g. The outputs are x_{K_1} and f_{B_2}, defined as the elongation of K_1 and the upward force on the top of B_2, respectively.

Solution

The free-body diagrams for the two masses were shown in parts (b) and (c) of Figure 2.19. The four state variables can be chosen to be the displacements x_1 and x_2 and the velocities v_1 and v_2. In Example 2.7, the force equations were shown to be

$$\begin{aligned} M_1\dot{v}_1 + B_1v_1 + K_1(x_1 - x_2) &= f_a(t) \\ M_2\dot{v}_2 + B_2v_2 + K_2x_2 + M_2g &= K_1(x_1 - x_2) \end{aligned}$$

We solve the first equation for $\dot{v}_1$, solve the second equation for $\dot{v}_2$, and then write the following four state-variable equations:

$$\begin{aligned} \dot{x}_1 &= v_1 \\ \dot{v}_1 &= \frac{1}{M_1}[-K_1x_1 - B_1v_1 + K_1x_2 + f_a(t)] \\ \dot{x}_2 &= v_2 \\ \dot{v}_2 &= \frac{1}{M_2}[K_1x_1 - (K_1 + K_2)x_2 - B_2v_2 - M_2g] \end{aligned} \tag{23}$$

The output equations are

$$\begin{aligned} x_{K_1} &= x_1 - x_2 \\ f_{B_2} &= B_2v_2 \end{aligned} \tag{24}$$

The right sides of both sets of equations are algebraic functions of the four state variables and the two inputs, as required.

■ 3.2 INPUT-OUTPUT EQUATIONS

In this section, we develop the system models in the form of input-output differential equations by eliminating all variables except the inputs and outputs and their derivatives. For a system with one input $u(t)$ and one output y, the input-output equation has the general form

$$a_n y^{(n)} + \cdots + a_2\ddot{y} + a_1\dot{y} + a_0 y = b_m u^{(m)} + \cdots + b_1\dot{u} + b_0 u(t) \tag{25}$$

where $y^{(n)} = d^n y/dt^n$ and $u^{(m)} = d^m u/dt^m$, and where for systems of practical interest $m \le n$. For fixed linear systems, all the coefficients in (25) are constants. In order to solve such an equation for y for all $t \ge t_0$, we need to know not only the input $u(t)$ for $t \ge t_0$ but also the n initial conditions $y(t_0)$, $\dot{y}(t_0)$, $\ddot{y}(t_0)$, ..., $y^{(n-1)}(t_0)$. Finding these initial conditions may be a difficult task.

For systems with more than one input, the right side of (25) will include additional input terms. If there are several outputs, we need a separate equation similar to (25) for each output. For example, the pair of equations

$$\begin{aligned} \ddot{y}_1 + 2\dot{y}_1 + 2y_1 &= 3\dot{u}_1 + 2u_1(t) + u_2(t) + 3\dot{u}_3 \\ \ddot{y}_2 + 2\dot{y}_2 + 2y_2 &= u_1(t) + 2u_2(t) + u_3(t) \end{aligned}$$

corresponds to a system with two outputs and three inputs. In the general case, each of the input-output equations involves only one unknown variable and its derivatives. Thus, unlike state-variable equations, each equation can be solved independently of the others.

An input-output equation can be constructed by combining the equations in a state-variable model, if such a model has already been found. A more direct method is to label the forces in the free-body diagrams in terms of the output variables and the minimum number of additional variables. For a translational system in which the outputs are displacements, for example, we would normally write the inertial and friction forces in the form $M\ddot{x}$ and $B\dot{x}$, rather than as $M\dot{v}$ and Bv.

▶ *EXAMPLE 3.10*

Write the input-output equation for the system shown in Figure 3.1(a) when the output is the displacement x.

Solution

In Example 3.1, the state-variable equations were found to be

$$\dot{x} = v$$

$$\dot{v} = \frac{1}{M}[-Kx - Bv + f_a(t)]$$

We can combine these equations by replacing v and $\dot{v}$ in the second equation by $\dot{x}$ and $\ddot{x}$, respectively, to obtain

$$M\ddot{x} + B\dot{x} + Kx = f_a(t)$$

The direct approach, which does not make use of the state-variable model, would be to label all the forces on the free-body diagram except $f_a(t)$ in terms of x and its derivatives. Then the foregoing input-output equation follows directly from D'Alembert's law.

▶ *EXAMPLE 3.11*

Find the input-output equations that relate the outputs x_1 and x_2 to the input $f_a(t)$ for the system shown in Figure 3.5(a).

Solution

The free-body diagrams for the mass M and the junction point A were shown in parts (b) and (c) of Figure 3.5, and the corresponding force equations were written in (13). When all forces except the input are expressed in terms of x_1 and x_2, these equations become

$$M\ddot{x}_1 + B_1\dot{x}_1 + K_1x_1 - K_1x_2 = f_a(t) \tag{26a}$$

$$B_2\dot{x}_2 + (K_1 + K_2)x_2 - K_1x_1 = 0 \tag{26b}$$

To obtain a single differential equation relating x_1 to $f_a(t)$ from this pair of coupled equations, we must eliminate x_2 and $\dot{x}_2$. If one of the equations

contains an unwanted variable but none of its derivatives, we can solve for it in terms of the remaining variables and their derivatives. Then we can eliminate the unwanted variable from the model by substitution. Thus we rewrite (26a) as

$$x_2 = \frac{1}{K_1}[M\ddot{x}_1 + B_1\dot{x}_1 + K_1x_1 - f_a(t)]$$

and, by differentiating once, we obtain

$$\dot{x}_2 = \frac{1}{K_1}(M\dddot{x}_1 + B_1\ddot{x}_1 + K_1\dot{x}_1 - \dot{f}_a)$$

Substituting these expressions for x_2 and $\dot{x}_2$ into (26b) gives

$$\frac{B_2}{K_1}(M\dddot{x}_1 + B_1\ddot{x}_1 + K_1\dot{x}_1 - \dot{f}_a) + \frac{K_1 + K_2}{K_1}[M\ddot{x}_1 + B_1\dot{x}_1 + K_1x_1 - f_a(t)] - K_1x_1 = 0$$

or

$$\begin{aligned} MB_2\dddot{x}_1 &+ (B_1B_2 + K_1M + K_2M)\ddot{x}_1 \\ &+ (B_2K_1 + B_1K_1 + B_1K_2)\dot{x}_1 + K_1K_2x_1 \\ &= B_2\dot{f}_a + (K_1 + K_2)f_a(t) \end{aligned} \tag{27}$$

which is the input-output equation for x_1. To obtain a differential equation with x_2 as the only unknown variable, we use (26b) to get an expression for x_1, which we then substitute into (26a). The result is

$$\begin{aligned} MB_2\dddot{x}_2 &+ (B_1B_2 + K_1M + K_2M)\ddot{x}_2 \\ &+ (B_2K_1 + B_1K_1 + B_1K_2)\dot{x}_2 + K_1K_2x_2 = K_1f_a(t) \end{aligned} \tag{28}$$

Note that the coefficients on the left sides of (27) and (28) are the same. Note also that the input-output equations are third-order and that the system has three state variables.

For the special case where $B_2 = 0$, which corresponds to removing the dashpot, (27) and (28) reduce to

$$M\ddot{x}_1 + B_1\dot{x}_1 + \left(\frac{K_1K_2}{K_1 + K_2}\right)x_1 = f_a(t)$$

$$M\ddot{x}_2 + B_1\dot{x}_2 + \left(\frac{K_1K_2}{K_1 + K_2}\right)x_2 = \left(\frac{K_1}{K_1 + K_2}\right)f_a(t)$$

The input-output equation for x_1 is now similar to that in Example 3.10, with K replaced by an equivalent constant $K_{eq} = K_1K_2/(K_1 + K_2)$ for the two springs that are now in series. We also see that $x_2 = [K_1/(K_1 + K_2)]x_1$

and that our results are consistent with those in Example 3.7. Because the two springs can be replaced by a single equivalent spring, the input-output equations are only second-order.

Reduction of Simultaneous Differential Equations

As we have seen, it is often necessary to combine a set of differential equations involving more than one dependent variable into a single differential equation with a single dependent variable. We did this in Example 3.11 by straightforward substitution. However, when it is not obvious how to eliminate the unwanted variable easily, the following procedure is recommended.

Let p denote the differentiation operator d/dt such that $py = \dot{y}$, $p^2 y = \ddot{y}$, etc. Then, for example,

$$(p+2)y = \dot{y} + 2y$$
$$[(p+1)(p+2)]\,y = (p^2+3p+2)y$$
$$= \ddot{y} + 3\dot{y} + 2y$$
$$(a_n p^n + \cdots + a_2 p^2 + a_1 p + a_0)y = a_n y^{(n)} + \cdots + a_2\ddot{y} + a_1\dot{y} + a_0 y$$

where $y^{(k)} = d^k y/dt^k$ for any positive integer value of k. Remember that p must operate on the variable or expression that follows it and that it is not a variable or algebraic quantity itself.

Suppose that we have the pair of equations

$$\begin{aligned} \dot{y}_1 + 2y_1 + y_2 &= 3u(t) \\ 2\dot{y}_1 + 5y_1 - 2\dot{y}_2 + 2y_2 &= 0 \end{aligned} \tag{29}$$

and want to find a single differential equation involving only the variables y_2 and $u(t)$. In terms of the p operator, we can rewrite (29) as

$$(p+2)y_1 + y_2 = 3u(t)$$
$$(2p+5)y_1 + (-2p+2)y_2 = 0$$

We now premultiply the first equation by $(2p+5)$, premultiply the second equation by $(p+2)$, and then subtract the new second equation from the first. The result is

$$[(2p+5) - (p+2)(-2p+2)]y_2 = (2p+5)3u(t)$$

from which

$$(2p^2 + 4p + 1)y_2 = (6p + 15)u(t) \tag{30}$$

To return to a differential equation relating y_2 and $u(t)$, we observe that (30) is the operator form of

$$2\ddot{y}_2 + 4\dot{y}_2 + y_2 = 6\dot{u} + 15u(t)$$

which is indeed the correct input-output equation. This algebraic procedure provides a useful means of manipulating sets of differential equations with constant coefficients.

▶ *EXAMPLE 3.12*

For the system in Figure 3.2(a), which is identical to Figure 2.13(a), find the input-output equation relating x_1 and $f_a(t)$.

Solution

The free-body diagrams for the two masses were shown in Figure 2.13 (with all forces labeled in terms of displacements and the input) and in Figure 3.2 (with forces labeled in terms of state variables and the input). Repeating (2.16), where the force equations were written in terms of displacements, we have the pair of simultaneous second-order differential equations

$$M_1\ddot{x}_1 + B\dot{x}_1 + (K_1+K_2)x_1 - B\dot{x}_2 - K_2x_2 = 0$$
$$-B\dot{x}_1 - K_2x_1 + M_2\ddot{x}_2 + B\dot{x}_2 + K_2x_2 = f_a(t)$$

In terms of the p operator, these equations become

$$[p^2M_1 + pB + (K_1+K_2)]x_1 - (pB+K_2)x_2 = 0$$
$$-(pB+K_2)x_1 + (p^2M_2 + pB + K_2)x_2 = f_a(t)$$

When we combine this pair of operator equations algebraically to eliminate x_2, we find that

$$\{M_1M_2p^4 + (M_1+M_2)Bp^3 + [M_1K_2 + M_2(K_1+K_2)]p^2 + BK_1p + K_1K_2\}x_1 = (pB+K_2)f_a(t)$$

which is the operator form of the differential equation

$$M_1M_2x_1^{(\text{iv})} + (M_1+M_2)Bx_1^{(\text{iii})} + [M_1K_2 + M_2(K_1+K_2)]\ddot{x}_1 + BK_1\dot{x}_1 + K_1K_2x_1 = B\dot{f}_a + K_2f_a(t)$$

The symbols $x^{(\text{iv})}$ and $x^{(\text{iii})}$ are defined as $x^{(\text{iv})} = d^4x/dt^4$ and $x^{(\text{iii})} = d^3x/dt^3$. As expected for a system that has four state variables, the input-output differential equation is of order four.

Comparison with the State-Variable Method

The order of the input-output differential equation describing a system is usually the same as the number of state variables. Occasionally one or more of the state variables may have no effect on the output, in which case the order of the input-output equation is less than the number of state variables.

For a first-order system, both forms of the system model involve a single first-order differential equation and are essentially identical. For higher-order systems they are quite different. We must solve a set of n first-order differential equations in state-variable form as a group, and we must know the initial value of each state variable to solve the set of n equations. An input-output equation of order n contains only one dependent variable, but we need to know the initial values of that variable and its first $n-1$ derivatives. In practice, finding the input-output equation and the associated initial conditions may require more effort than finding the information needed for a state-variable solution.

Using the state-variable equations has significant computational advantages when a computer solution is to be found, as is discussed in Chapter 15. In fact, standard methods for solving a high-order, possibly nonlinear input-output differential equation on a computer usually require decomposition into a set of simultaneous first-order equations anyway. The analytical solution of input-output equations and of sets of state-variable equations is considered in Chapter 6.

State-variable equations are particularly convenient for complex multi-input, multi-output systems. They are often written in matrix form, and, in addition to their computational advantages, they can be used to obtain considerable insight into system behavior. The state-variable concept has formed the basis for many of the recent theoretical developments in system analysis.

3.3 MATRIX FORMULATION OF STATE-VARIABLE EQUATIONS

One of the characteristics of state-variable models is that they are suitable for matrix notation and for the techniques of linear algebra. For example, we can represent any number of first-order state-variable equations by a single matrix differential equation merely by making the appropriate definitions. Furthermore, when we are dealing with complex multi-input, multi-output systems, using matrix concepts and properties leads to an understanding of system behavior that would be difficult to achieve otherwise. An additional advantage of using matrices is that we can apply many of the theoretical properties of matrices that are taught in introductory linear algebra courses to the study of dynamic systems once we have put their models into matrix form.

We assume that the reader is somewhat familiar with matrix methods through an introductory course in linear algebra. Such basic matrix operations as multiplication, evaluation of determinants, and inversion are summarized in Appendix C.

The required form for the state-variable model was explained in Section 3.1. For a fixed linear system in which $q_1, q_2, \ldots, q_n$ are the state

variables, where $u_1, u_2, \ldots, u_m$ are the inputs, and where $y_1, y_2, \ldots, y_p$ are the outputs, the state-variables equations are

$$
\begin{aligned}
\dot{q}_1 &= a_{11}q_1 + a_{12}q_2 + \cdots + a_{1n}q_n + b_{11}u_1 + \cdots + b_{1m}u_m \\
\dot{q}_2 &= a_{21}q_1 + a_{22}q_2 + \cdots + a_{2n}q_n + b_{21}u_1 + \cdots + b_{2m}u_m \\
&\vdots \\
\dot{q}_n &= a_{n1}q_1 + a_{n2}q_2 + \cdots + a_{nn}q_n + b_{n1}u_1 + \cdots + b_{nm}u_m
\end{aligned} \tag{31}
$$

and the output equations have the form

$$
\begin{aligned}
y_1 &= c_{11}q_1 + c_{12}q_2 + \cdots + c_{1n}q_n + d_{11}u_1 + \cdots + d_{1m}u_m \\
y_2 &= c_{21}q_1 + c_{22}q_2 + \cdots + c_{2n}q_n + d_{21}u_1 + \cdots + d_{2m}u_m \\
&\vdots \\
y_p &= c_{p1}q_1 + c_{p2}q_2 + \cdots + c_{pn}q_n + d_{p1}u_1 + \cdots + d_{pm}u_m
\end{aligned} \tag{32}
$$

where the coefficients a_{ij}, b_{ij}, c_{ij}, and d_{ij} are constants. The initial conditions associated with (31) are the initial values of the state variables, $q_1(0), q_2(0), \ldots, q_n(0)$.

When dealing with the set of n state variables $q_1, q_2, \ldots, q_n$, we shall use the symbol $\mathbf{q}$ to denote the entire set.[2] Thus

$$
\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}
$$

which is a matrix having n rows and a single column, with each of its elements being one of the state variables. As such, its elements are functions of time. Matrices having a single column are commonly referred to as **column vectors** or, for short, **vectors**. Hence the symbol $\mathbf{q}$ will be called the **state vector**, and it is understood that its ith element is the state variable q_i. The initial value of the state vector $\mathbf{q}$ is the vector

$$
\mathbf{q}(0) = \begin{bmatrix} q_1(0) \\ q_2(0) \\ \vdots \\ q_n(0) \end{bmatrix}
$$

[2]Boldface symbols are used to denote matrices. The symbols are generally capitals, but lowercase letters may be used for vectors. Variables written in italic type are scalars.

The vector $\dot{\mathbf{q}}$ is an n-element vector whose elements are the derivatives of the corresponding elements in $\mathbf{q}$:

$$\dot{\mathbf{q}} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

The m inputs $u_1, u_2, \ldots, u_m$ will be represented by $\mathbf{u}$, the **input vector**, which is defined as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

The p outputs $y_1, y_2, \ldots, y_p$ will be elements in $\mathbf{y}$, the **output vector**, which is given by

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}$$

The coefficients are put into the following four matrices. We define the $n \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

the $n \times m$ matrix

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}$$

the $p \times n$ matrix

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix}$$

and the $p \times m$ matrix

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots & \vdots & & \vdots \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix}$$

Then we can write the state-variable and output equations for the fixed linear system described by (31) and (32) as

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{Aq} + \mathbf{Bu} \\ \mathbf{y} &= \mathbf{Cq} + \mathbf{Du} \end{aligned} \tag{33}$$

Reflecting for a moment, we note that $\dot{q}_i$ is the ith element of $\dot{\mathbf{q}}$ and is obtained by multiplying the ith row of $\mathbf{A}$ by the corresponding elements in the column vector $\mathbf{q}$ and adding to that product the product of the ith row of $\mathbf{B}$ and the column vector $\mathbf{u}$. Carrying out these operations gives

$$\dot{q}_i = a_{i1}q_1 + a_{i2}q_2 + \cdots + a_{in}q_n + b_{i1}u_1 + \cdots + b_{im}u_m$$

Likewise, we find the kth element of $\mathbf{y}$ by multiplying the kth row of $\mathbf{C}$ by $\mathbf{q}$ and adding to that product the product of the kth row of $\mathbf{D}$ and the vector $\mathbf{u}$, obtaining

$$y_k = c_{k1}q_1 + c_{k2}q_2 + \cdots + c_{kn}q_n + d_{k1}u_1 + \cdots + d_{km}u_m$$

▶ ***EXAMPLE 3.13***

For the system shown in Figure 3.1(a) and modeled in Example 3.1, write the state-variable model in the form of (33) and identify the coefficient matrices **A**, **B**, **C**, and **D**.

Solution

From Example 3.1, the state-variable equations are

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= \frac{1}{M}[-Kx - Bv + f_a(t)] \end{aligned} \tag{34}$$

and the output equations are

$$\begin{aligned} f_K &= Kx \\ v &= v \\ a_M &= \frac{1}{M}[-Kx - Bv + f_a(t)] \end{aligned} \tag{35}$$

Note that v is both a specified output and a state variable. Although the corresponding output equation may seem trivial, we should include it when writing the model in matrix form.

Because there are two state variables, one input, and three outputs, the state vector $\mathbf{q}$, the input vector $\mathbf{u}$, and the output vector $\mathbf{y}$ will have two, one, and three elements respectively. Let

$$\mathbf{q} = \begin{bmatrix} x_1 \\ v_1 \end{bmatrix}, \quad \mathbf{u} = [\, f_a(t) \,], \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} f_K \\ v \\ a_M \end{bmatrix}$$

When a matrix contains only a single element, as is the case with $\mathbf{u}$, the brackets around the matrix are usually omitted. With the above definitions of $\mathbf{q}$, $\mathbf{u}$, and $\mathbf{y}$, we can write (34) and (35) as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -K/M & -B/M \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1/M \end{bmatrix}}_{\mathbf{B}} f_a(t)$$

and

$$\begin{bmatrix} f_K \\ v \\ a_M \end{bmatrix} = \underbrace{\begin{bmatrix} K & 0 \\ 0 & 1 \\ -K/M & -B/M \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} x \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1/M \end{bmatrix}}_{\mathbf{D}} f_a(t)$$

which have the form of (33) and where the coefficient matrices **A**, **B**, **C**, and **D** are specifically identified. Because there is only one input, **B** and **D** have only one column. The reader should verify that carrying out the matrix operations in these equations gives (34) and (35).

▶ *EXAMPLE 3.14*

Write the state-variable model in matrix form for the system shown in Figure 3.5(a) and considered in Example 3.7. Let the only output be the tensile force f_{K_1} in the spring K_1.

Solution

From (14), the state-variable equations are

$$\begin{aligned} \dot{x}_1 &= v_1 \\ \dot{v}_1 &= \frac{1}{M}[-K_1x_1 - B_1v_1 + K_1x_2 + f_a(t)] \\ \dot{x}_2 &= \frac{1}{B_2}[K_1x_1 - (K_1 + K_2)x_2] \end{aligned}$$

and the output equation is $f_{K_1} = K_1(x_1 - x_2)$. We take the state vector $\mathbf{q}$, the input vector $\mathbf{u}$, and the output vector $\mathbf{y}$ to be

$$\mathbf{q} = \begin{bmatrix} x_1 \\ v_1 \\ x_2 \end{bmatrix}, \quad \mathbf{u} = [\, f_a(t) \,], \quad \text{and} \quad \mathbf{y} = [\, f_{K_1} \,]$$

Then

$$\dot{\mathbf{q}} = \begin{bmatrix} 0 & 1 & 0 \\ -\dfrac{K_1}{M} & -\dfrac{B_1}{M} & \dfrac{K_1}{M} \\ \dfrac{K_1}{B_2} & 0 & -\left(\dfrac{K_1 + K_2}{B_2}\right) \end{bmatrix} \mathbf{q} + \begin{bmatrix} 0 \\ \dfrac{1}{M} \\ 0 \end{bmatrix} \mathbf{u}$$

and

$$\mathbf{y} = [\, K_1 \;\; 0 \;\; -K_1 \,]\mathbf{q} + [\, 0 \,]\mathbf{u}$$

which have the form of (33). Because there is only one input, **B** and **D** have only one column. Because there is only one output, **C** and **D** have only one row. Thus **B** is a column vector, **C** a row vector, and **D** a scalar. Such systems are called single-input, single-output systems to distinguish them from the more general class of multi-input, multi-output systems.

▶ *EXAMPLE 3.15*

Rewrite in matrix form the state-variable and output equations for the system shown in Figure 2.19(a) and modeled in Example 3.9.

Solution

From Example 3.9, the state-variable equations are

$$\begin{aligned} \dot{x}_1 &= v_1 \\ \dot{v}_1 &= \frac{1}{M_1}[-K_1x_1 - B_1v_1 + K_1x_2 + f_a(t)] \\ \dot{x}_2 &= v_2 \\ \dot{v}_2 &= \frac{1}{M_2}[K_1x_1 - (K_1 + K_2)x_2 - B_2v_2 - M_2g] \end{aligned}$$

and the output equations are

$$\begin{aligned} x_{K_1} &= x_1 - x_2 \\ f_{B_2} &= B_2v_2 \end{aligned}$$

The state vector has four elements and is taken to be

$$\mathbf{q} = \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix}$$

We take the input vector $\mathbf{u}$ and the output vector $\mathbf{y}$ to be

$$\mathbf{u} = \begin{bmatrix} f_a(t) \\ g \end{bmatrix}$$

and

$$\mathbf{y} = \begin{bmatrix} x_{K_1} \\ f_{B_2} \end{bmatrix}$$

With these definitions, we can write the model in the form of (33) as

$$\dot{\mathbf{q}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{K_1}{M_1} & -\dfrac{B_1}{M_1} & \dfrac{K_1}{M_1} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{K_1}{M_2} & 0 & -\dfrac{(K_1+K_2)}{M_2} & -\dfrac{B_2}{M_2} \end{bmatrix} \mathbf{q} + \begin{bmatrix} 0 & 0 \\ \dfrac{1}{M_1} & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{u}$$

and

$$\mathbf{y} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & B_2 \end{bmatrix} \mathbf{q} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u}$$

For any fixed linear system, the elements in the coefficients matrices **A**, **B**, **C**, and **D** will all be constants. Equations (33) would also be valid for a time-varying linear system, with the understanding that some of the elements in **A**, **B**, **C**, and **D** would then be functions of time.

For the case of systems that can be time-varying and nonlinear, a more comprehensive form for the equations is necessary. For a general nth-order system having m inputs, the individual state-variable equations would have the form

$$\begin{aligned} \dot{q}_1 &= f_1(q_1, q_2, \ldots, q_n, u_1, \ldots, u_m, t) \\ \dot{q}_2 &= f_2(q_1, q_2, \ldots, q_n, u_1, \ldots, u_m, t) \\ &\vdots \\ \dot{q}_n &= f_n(q_1, q_2, \ldots, q_n, u_1, \ldots, u_m, t) \end{aligned} \tag{36}$$

where $q_1, q_2, \ldots, q_n$ are the state variables and $u_1, u_2, \ldots, u_m$ are the inputs. The algebraic functions $f_1, f_2, \ldots, f_n$ express the derivatives $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$ in terms of the state variables, the inputs, and possibly the time t.[3] The initial conditions associated with (36) are the initial values of the state variables, $q_1(0), q_2(0), \ldots, q_n(0)$. If the system

[3]Note that the use of the symbol f in this very general context is not related to the use of f to denote forces in mechanical systems.

has p outputs, the individual output equations have the general form

$$\begin{aligned} y_1 &= g_1(q_1, q_2, \ldots, q_n, u_1, u_2, \ldots, u_m, t) \\ y_2 &= g_2(q_1, q_2, \ldots, q_n, u_1, u_2, \ldots, u_m, t) \\ &\vdots \\ y_p &= g_p(q_1, q_2, \ldots, q_n, u_1, u_2, \ldots, u_m, t) \end{aligned} \tag{37}$$

where $g_1, g_2, \ldots, g_p$ are again algebraic functions of the state variables, the inputs, and possibly time. The sets of equations in (36) and (37) can be represented by the two vector equations

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{f}(\mathbf{q}, \mathbf{u}, t) \\ \mathbf{y} &= \mathbf{g}(\mathbf{q}, \mathbf{u}, t) \end{aligned} \tag{38}$$

where **f** and **g** are vector functions having n and p elements, respectively.

SUMMARY

In this chapter we introduced the concept of state variables and showed how to write the mathematical model of a dynamic system in terms of these variables. Such a model consists of a set of coupled first-order differential equations, whose right-hand sides are algebraic functions of the state variables and inputs. Once the state variables have been found as known functions of time, the system outputs can be found easily as algebraic functions of the state variables and inputs.

In Section 3.2 we used a very different approach, eliminating from our equations all variables except the inputs and outputs and their derivatives. For an nth-order system, we obtain an nth-order differential equation for each output. These equations, which are not coupled together and which can be solved independently of each other, have only the inputs and their derivatives as variables on the right-hand side.

The final section showed how to write the state-variable equations in matrix form. The analytical solution of input-output differential equations and also of matrix state-variable equations is considered in Chapter 6. Computer solutions are discussed in Chapter 15.

PROBLEMS

3.1 Find the state-variable model for the system shown in Figure 2.16(a) and considered in Example 2.5:

a) When x and $v = \dot{x}$ are state variables.

b) When z and $w = \dot{z}$ are state variables.

The inputs are $f_a(t)$ and g. Take the energy stored in the spring as the output.

3.2 The mathematical model of a linear system with input $f_a(t)$ and output y is

$$\dddot{x} + \alpha\ddot{x} + \beta\dot{x} + \gamma x = f_a(t)$$

$$y = \ddot{x} + \dot{x}$$

Write the model in state-variable form, where the state variables are x, $v = \dot{x}$, and $a = \ddot{x}$.

* **3.3** The following pair of equations corresponds to a linear dynamic system having $f_a(t)$ as its input and y as its output.

$$\ddot{y} + 4\dot{y} + 2y = x$$

$$\dot{x} + x + y = f_a(t)$$

Derive the state-variable form of the model. Define any new symbols.

3.4 A mechanical system having input $f_a(t)$ and output $y = \dot{x}_1 - \dot{x}_2$ obeys the pair of differential equations

$$M_1\ddot{x}_1 + B(\dot{x}_1 - \dot{x}_2) + K_1x_1 = 0$$

$$M_2\ddot{x}_2 + B(\dot{x}_2 - \dot{x}_1) + K_2x_2 = f_a(t)$$

Write the model in state-variable form when the state variables are x_1, $v_1 = \dot{x}_1$, x_2, and $v_2 = \dot{x}_2$.

* **3.5** Write state-variable equations for the mechanical system shown in Figure 2.15(a) and modeled in Example 2.4 by (2.20). Take the state variables to be the absolute displacements x_1 and x_2 and their derivatives v_1 and v_2. Include algebraic output equations for y_1, the viscous force of M_2 on M_1 (positive to the right), and y_2, the tensile force in spring K_2.

3.6 Repeat Problem 3.5 using (2.19). Take the state variables to be the absolute displacement x, its derivative $v_x = \dot{x}$, the relative displacement z, and its derivative $v_z = \dot{z}$.

3.7 Write state-variable equations for the mechanical system shown in Figure 2.17(a) and modeled in Example 2.6 by (2.25). The inputs are the applied force $f_a(t)$ and the gravitational constant g. Include algebraic output equations for the tensile force in each of the three springs.

3.8 Find a set of state-variable equations for the system shown in Figure P2.5. The output is the total momentum of the system.

* **3.9** Find the state-variable model for the system shown in Figure P2.15 when the outputs are the elongation y_1 of the spring K_1 and the acceleration a_1 of the mass M_1. The inputs are $f_a(t)$ and the gravitational constant g.

3.10 Repeat Problem 3.5 when the spring K_1 in Figure 2.15(a) is removed. Keep the same outputs, but use x_2, v_1, and v_2 as the state variables. Explain why it is not necessary to have x_1 as a state variable.

* **3.11** Find the state-variable model for the system shown in Figure P2.16. The output is the elongation of the lower spring.

3.12 Find the state-variable model for the system shown in Figure P2.21. The output is the increase in the separation between M_1 and M_2.

* **3.13** Write the state-variable equations for the system shown in Figure P2.27, where the input is the displacement $x_3(t)$. The output is the force exerted by the dashpot B_2 on point A, with the positive sense to the left.

3.14 Repeat Problem 3.13 when $B_2 = 0$. Let the output be the displacement x_2 instead of the force exerted by the dashpot.

* **3.15** The model for a certain dynamic system is

$$\dot{x}_1 = -3x_1 + 2x_2 + u_1(t) + 2\dot{u}_2$$

$$\dot{x}_2 = 2x_1 + x_2 + \dot{u}_1$$

$$y = x_1 - x_2 + u_2(t)$$

where x_1 and x_2 are state variables, $u_1(t)$ and $u_2(t)$ the inputs, and y the output. Rewrite the model in state-variable form by avoiding derivatives on the right side of the equations. Define any new symbols and identify the state variables.

3.16 The input to the system shown in Figure P3.16 is the displacement $x_2(t)$ of M_2. Write a set of state-variable equations. The springs are relaxed when $x_1 = x_2 = 0$. Include an output equation for the downward force f_2 that must be applied to M_2 to achieve the specified motion.

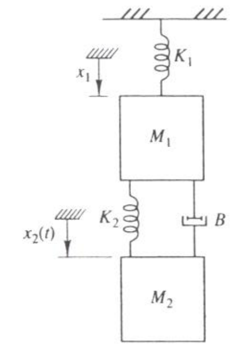

FIGURE P3.16

3.17 For the system shown in Figure P2.2, replace the force input $f_a(t)$ by the displacement $x_2(t)$ of M_2. The springs are relaxed when $x_1 = x_2 = 0$. Write a set of state-variable equations. Include an output equation for the force f_2 that must be applied to M_2 to achieve the specified motion, with the positive sense to the left.

* **3.18** Find a set of state-variable equations for the system shown in Figure P2.6 when the inputs are $x_3(t)$ and $f_a(t)$. The outputs are the velocities v_1 and v_2.

3.19 A linear dynamic system with input $u(t)$, output y, and state variables x_1 and x_2 is characterized by the equations

$$\dot{x}_1 + 2\dot{x}_2 = 3x_1 + 4x_2 - 5u(t)$$

$$\dot{x}_1 - \dot{x}_2 = 2x_1 + 2x_2$$

$$y = \dot{x}_1 + 2x_2$$

Find the state-variable and output equations.