a) Evaluate the resolvent matrix $\Phi(s)$.
b) Calculate the state-transition matrix $\phi(t)$.
c) Express each element of $\mathbf{q}$ in terms of the arbitrary initial state vector $\mathbf{q}(0)$.

8.41 $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$

*** 8.42** $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

8.43 $\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$

8.44 $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}$

8.45 Consider the dynamic system defined by the matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & -2 \end{bmatrix} \quad \mathbf{D} = 0$$

a) Indicate the numbers of state variables, inputs, and outputs.
b) Using $\phi(t)$ as found in Problem 8.41, evaluate the state vector $\mathbf{q}(t)$ and the output vector $\mathbf{y}(t)$ for the initial state vector $\mathbf{q}(0) = [2 \quad 1]^T$ and zero input.
c) Find the transfer-function matrix $\mathbf{H}(s)$.
d) Evaluate and sketch the output $\mathbf{y}$ when $\mathbf{q}(0) = [2 \quad 1]^T$ and $\mathbf{u}(t) = U(t)$.

*** 8.46** Consider the dynamic system defined by the matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

a) Indicate the numbers of state variables, inputs, and outputs.
b) Using $\Phi(s)$ as found in Problem 8.42, determine the transfer-function matrix $\mathbf{H}(s)$.
c) Evaluate the impulse-response matrix $\mathbf{h}(t)$ and sketch each element versus time.

8.47 Consider the dynamic system defined by the matrices

$$\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

a) Indicate the numbers of state variables, inputs, and outputs.
b) Using $\Phi(s)$ as found in Problem 8.43, determine the transfer-function matrix $\mathbf{H}(s)$.
c) Evaluate the zero-input response when $\mathbf{q}(0) = \begin{bmatrix} 4 & -1 \end{bmatrix}^T$ and sketch each element versus time.
d) Evaluate the zero-state response when

$$\mathbf{u}(t) = \begin{cases} 0 & \text{for } t \le 0 \\ 5\epsilon^{-t} & \text{for } t > 0 \end{cases}$$

CHAPTER 9

DEVELOPING A LINEAR MODEL

In nearly all the examples used in earlier chapters, the elements were assumed to be linear. In practice, however, many elements are inherently nonlinear and may be considered linear over only a limited range of operating conditions.

When confronted with a mathematical model that contains nonlinearities, the analyst has essentially three choices: (1) to attempt to solve the differential equations directly, (2) to derive a fixed linear approximation that can be analyzed, or (3) to obtain computer solutions of the response for specific numerical cases. The first alternative is possible only in specialized cases and will not be pursued. We present the linearization approach in this chapter and discuss computer solutions in Chapter 15.

We first develop a method for linearizing an element law where the two variables, such as force and displacement, are not directly proportional. We then show how to incorporate the linearized element law into the system model. We consider mechanical systems with nonlinear stiffness or friction elements, and we then discuss nonlinear electrical systems.

9.1 LINEARIZATION OF AN ELEMENT LAW

The object of linearization is to derive a linear model whose response will agree closely with that of the nonlinear model. Although the responses of the linear and nonlinear models will not agree exactly and may differ significantly under some conditions, there will generally be a set of inputs and initial conditions for which the agreement will be satisfactory. In this section, we consider the linearization of a single element law that is a nonlinear

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function of a single variable. We can express such an element law as an algebraic function $f(x)$. If x represents the total length of a nonlinear spring and $f(x)$ the force on the spring, the function $f(x)$ might appear as shown in Figure 9.1(a), where x_0 denotes the free or unstretched length.

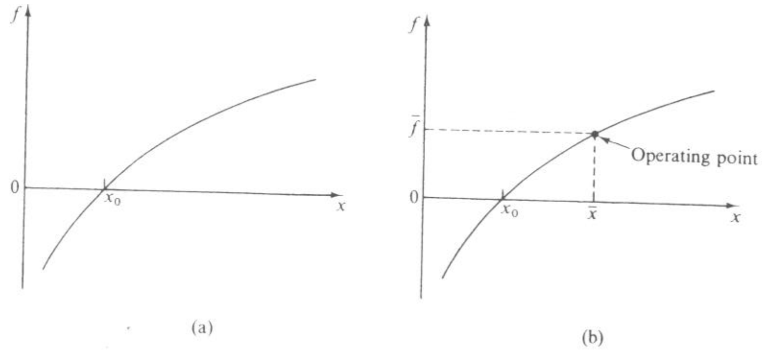

FIGURE 9.1 (a) A nonlinear spring characteristic. (b) Nonlinear spring characteristic with operating point.

We shall carry out the linearization of the element law with respect to an **operating point**, which is a specific point on the nonlinear characteristic denoted by $\bar{x}$ and $\bar{f}$. A sample operating point is shown in Figure 9.1(b). We discuss the procedure for determining the operating point in the following section; for now, we shall assume that the values of $\bar{x}$ and $\bar{f}$ are known.

We can write $x(t)$ as the sum of a constant portion, which is its value at the operating point, and a time-varying portion $\hat{x}(t)$ such that

$$x(t) = \bar{x} + \hat{x}(t) \tag{1}$$

The constant term $\bar{x}$ is called the **nominal value** of x, and the time-varying term $\hat{x}(t)$ is the **incremental variable** corresponding to x. Likewise, we can write $f(t)$ as the sum of its nominal value $\bar{f}$ and the incremental variable $\hat{f}(t)$:

$$f(t) = \bar{f} + \hat{f}(t) \tag{2}$$

where the dependence of $\hat{f}$ on time is shown explicitly. Because $\bar{x}$ and $\bar{f}$ always denote a point that lies on the curve for the nonlinear element law,

$$\bar{f} = f(\bar{x}) \tag{3}$$

Having defined the necessary terms, we shall develop two equivalent methods of linearizing the element law to relate the incremental variables $\hat{x}$ and $\hat{f}$. The first uses a graphical approach; the second is based on a Taylor-series expansion.

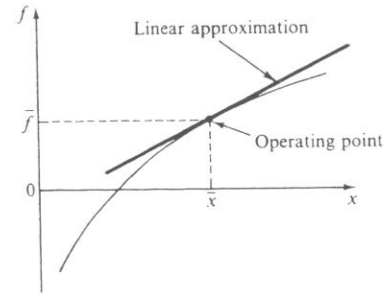

FIGURE 9.2 Nonlinear spring characteristic with linear approximation.

Graphical Approach

Figure 9.2 shows the nonlinear element law $f(x)$ with the tangent to the curve at the operating point appearing as the straight line. For the moment, we note that the tangent line will be a good approximation to the nonlinear curve provided that the independent variable x does not deviate greatly from its nominal value $\bar{x}$ and that the curvature of the curve $f(x)$ is small in the vicinity of the operating point. The slope of the tangent line is

$$k = \left.\frac{df}{dx}\right|_{x=\bar{x}}$$

or, written more concisely,

$$k = \left.\frac{df}{dx}\right|_{\bar{x}} \tag{4}$$

where the subscript $\bar{x}$ after the vertical line indicates that the derivative must be evaluated at $x = \bar{x}$. The tangent passes through the operating point that has the coordinates $(\bar{x}, \bar{f})$ and is described by the equation

$$f = \bar{f} + k(x - \bar{x})$$

which can be written as

$$f - \bar{f} = k(x - \bar{x}) \tag{5}$$

Noting from (1) and (2) that the incremental variables are

$$\hat{x} = x - \bar{x} \tag{6a}$$

$$\hat{f} = f - \bar{f} \tag{6b}$$

we see that (5) reduces to

$$\hat{f} = k\hat{x} \tag{7}$$

where k is given by (4).

We can represent (7) in graphical form by redrawing the nonlinear function $f(x)$ with a coordinate system whose axes are $\hat{x}$ and $\hat{f}$ and whose origin is located at the operating point, as depicted in Figure 9.3. The incremental variables $\hat{x}$ and $\hat{f}$ are linearly related, the constant of proportionality k being the slope of the tangent line at the operating point.

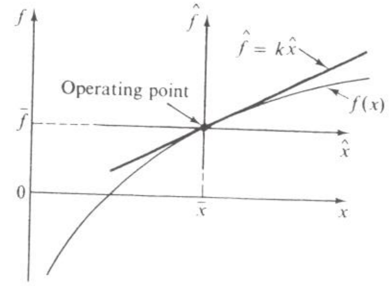

FIGURE 9.3 Nonlinear spring characteristic with incremental-variable coordinates.

Obviously, the accuracy of the linear approximation depends on the curvature of $f(x)$ in the vicinity of the operating point $(\bar{x}, \bar{f})$ and on the extent to which x deviates from $\bar{x}$ as the system responds to its excitations. We expect that the linearized model should be a good approximation for those values of x for which the straight line closely approximates the original curve. If the nonlinear spring is part of a larger system, we are unlikely to know in advance what range of values we will encounter for x. The problem is further complicated by the fact that usually we are primarily interested in how well certain responses of the overall system are approximated by calculations made on the linearized model. For example, we can expect some of the elements in a particular system to play a more critical role than others in determining the response of interest. Generally, the only way to assess with certainty the quality of the approximations is to compare computer solutions for the nonlinear model and the linearized model.

Series-Expansion Approach

As an alternative to the geometric arguments just presented, we can derive the linearized approximation in (7) by expressing $f(x)$ in terms of its **Taylor-series expansion** about the operating point $(\bar{x}, \bar{f})$. This expansion is

$$f(x) = f(\bar{x}) + \left.\frac{df}{dx}\right|_{\bar{x}} (x - \bar{x}) + \frac{1}{2!}\left.\frac{d^2 f}{dx^2}\right|_{\bar{x}} (x - \bar{x})^2 + \cdots$$

where the subscript $\bar{x}$ after the vertical line indicates that the associated derivative is evaluated at $x = \bar{x}$. We can find the first two terms of this expansion provided that f and its first derivative exist for $x = \bar{x}$. We seek a linear approximation to the actual curve, so we shall neglect subsequent terms, which are higher-order in $x - \bar{x}$. The justification for truncating the series after the first two terms is that if x is sufficiently close to $\bar{x}$, then the higher-order terms are negligible compared to the constant and linear terms. Hence we write

$$f(x) \simeq f(\bar{x}) + \left.\frac{df}{dx}\right|_{\bar{x}} (x - \bar{x}) \tag{8}$$

Because $\bar{f} = f(\bar{x})$ and $k = [df/dx]|_{\bar{x}}$, this equation is equivalent to (5), which reduces to (7).

The accuracy of the linearized approximation in (8) depends on the extent to which the higher-order terms we have omitted from the Taylor-series expansion are truly negligible. This in turn depends on the magnitude of $x - \bar{x}$, which is the incremental independent variable, and on the values of the higher-order derivatives of $f(x)$ at the operating point. Before addressing the linearization of the complete model, we shall consider two numerical examples that illustrate the linearization of a nonlinear element law.

▶ *EXAMPLE 9.1*

A nonlinear translational spring obeys the force-displacement relationship $f(x) = |x|x$, where x is the elongation of the spring from its unstretched length. Determine the linearized element law in numerical form for each of the operating points corresponding to the nominal spring elongations $\bar{x}_1 = -1$, $\bar{x}_2 = 0$, $\bar{x}_3 = 1$, and $\bar{x}_4 = 2$.

Solution

We can rewrite $f(x)$ as

$$f(x) = \begin{cases} -x^2 & \text{for } x < 0 \\ x^2 & \text{for } x \geq 0 \end{cases} \tag{9}$$

The slope of the tangent at the operating point is

$$\begin{aligned} k &= \left.\frac{df}{dx}\right|_{\bar{x}} = \begin{cases} -2\bar{x} & \text{for } \bar{x} < 0 \\ 2\bar{x} & \text{for } \bar{x} \geq 0 \end{cases} \\ &= 2|\bar{x}| \qquad \text{for all } \bar{x} \end{aligned} \tag{10}$$

Thus the linear approximation to the spring characteristic is

$$\hat{f} = 2|\bar{x}|\hat{x}$$

where the coefficient $k = 2|\bar{x}|$ can be thought of as an effective spring constant whose numerical value depends on the nominal value of the spring's elongation $\bar{x}$. Substituting the four specified values of $\bar{x}$ into (9) and (10) gives the values of $\bar{f}$ and k that appear in Table 9.1. Figure 9.4 shows the four linear approximations superimposed on the nonlinear spring characteristic. Note that the value of the effective spring constant k is strongly dependent on the location of the operating point. In fact, k vanishes for $\bar{x} = 0$, which implies that the spring would not appear in the linearized model of a system having $\bar{x} = 0$ as its operating point. It is also interesting to note that k is the same for $\bar{x} = -1$ and $\bar{x} = +1$, although the values of $\bar{f}$ differ.

TABLE 9.1 Nominal Elongations, Nominal Forces, and Effective Spring Constants

i	$\bar{x}_i$	$\bar{f}_i$	k_i
1	−1	−1	2
2	0	0	0
3	1	1	2
4	2	4	4

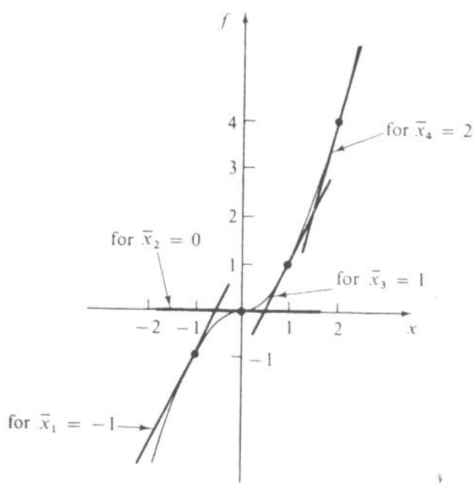

FIGURE 9.4 Nonlinear spring characteristic and linear approximations for four values of $\bar{x}$.

Concerning the accuracy of the approximation, one might say that for deviations in x up to 0.25 from the operating point, the approximation seems to be quite good; for deviations exceeding 1.0 it would be poor. It is difficult

to make a definitive statement, however, without knowing the system in which the element is to appear.

▶ *EXAMPLE 9.2*

A torque τ_M exerted on a body that can rotate with an angular displacement θ is given by the equation $\tau_M = D\sin\theta$. Determine the linearized element law relating $\hat{\tau}_M$ and $\hat{\theta}$. Consider the five operating points corresponding to $\bar{\theta}_1 = 0, \bar{\theta}_2 = \pi/4, \bar{\theta}_3 = \pi/2, \bar{\theta}_4 = 3\pi/4$, and $\bar{\theta}_5 = \pi$.

Solution

The Taylor-series expansion for τ_M is

$$\begin{aligned}\tau_M &= D\sin\bar{\theta} + \frac{d}{d\theta}D\sin\theta\bigg|_{\bar{\theta}}(\theta - \bar{\theta}) + \cdots \\ &= D\sin\bar{\theta} + D(\cos\bar{\theta})\hat{\theta} + \cdots\end{aligned}$$

Using the first two terms in this series, and noting that $\bar{\tau}_M = D\sin\bar{\theta}$, we can write

$$\hat{\tau}_M = D(\cos\bar{\theta})\hat{\theta} = k\hat{\theta}$$

where $k = D\cos\bar{\theta}$ is the linearized stiffness constant. In Table 9.2, the values of $\bar{\theta}$, $\bar{\tau}_M$, and k are listed for each of the five specified operating poin**t**s.

TABLE 9.2 Operating Points and Effective Stiffness Constants for Example 9.2

i	$\bar{\theta}_i$	$\bar{\tau}_{M_i}$	k_i
1	0	0	D
2	$\pi/4$	$D/\sqrt{2}$	$D/\sqrt{2}$
3	$\pi/2$	D	0
4	$3\pi/4$	$D/\sqrt{2}$	$-D/\sqrt{2}$
5	π	0	$-D$

The values of k_i, which were found from $k_i = D\cos\bar{\theta}_i$, are also the slopes of the tangents to the characteristic curve drawn at the operating points. The tangent lines at three of these operating points are shown in Figure 9.5.

In the free-body diagram for the pendulum shown in Figure 4.23, one of the torques was $\tau_M = MgL\sin\theta$. In Examples 4.7 and 6.9, we used the small-angle approximation for θ and a trigonometric identity to obtain

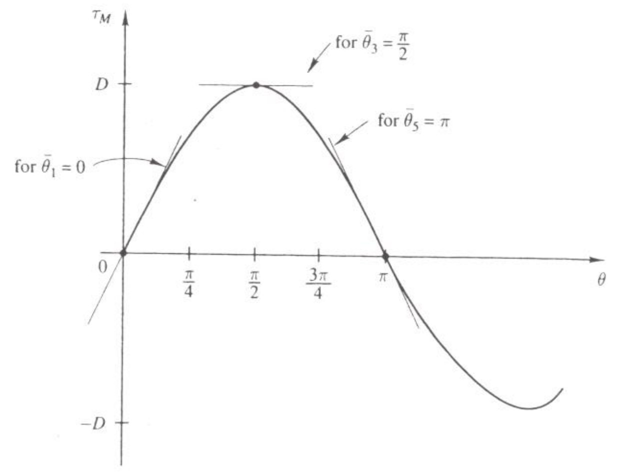

FIGURE 9.5 Nonlinear characteristic curve and linear approximations for Example 9.2.

a linearized element law for small variations about the operating points $\bar{\theta} = 0$ and $\bar{\theta} = \pi$. We found that $\hat{\tau}_M = (MgL)\hat{\theta}$ for $\bar{\theta} = 0$ and that $\hat{\tau}_M = -(MgL)\hat{\theta}$ for $\bar{\theta} = \pi$. In Example 9.2, these same results were obtained by a more general linearization method.

9.2 LINEARIZATION OF THE MODEL

We shall now consider the process of incorporating one or more linearized element laws into a system model. Starting with a given nonlinear model, we need to do the following:

1. Determine the operating point of the model by writing and solving the appropriate nonlinear algebraic equations. Select the proper operating-point value if extraneous solutions also appear.
2. Rewrite all linear terms in the mathematical model as the sum of their nominal and incremental variables, noting that the derivatives of constant terms are zero.
3. Replace all nonlinear terms by the first two terms of their Taylor-series expansions—that is, the constant and linear terms.
4. Using the algebraic equation(s) defining the operating point, cancel the constant terms in the differential equations, leaving only linear terms involving incremental variables.

5. Determine the initial conditions of all incremental variables in terms of the initial conditions of the variables in the nonlinear model.

For all situations we shall consider, the operating point of the system will be a condition of equilibrium in which each variable will be constant and equal to its nominal value and in which all derivatives will be zero. Inputs will take on their nominal values, which are typically selected to be their average values. For example, if a system input were $u(t) = A + B\sin\omega t$, then the nominal value of the input would be taken as $\bar{u} = A$. Under these conditions, the differential equations reduce to algebraic equations that we can solve for the operating point, using a computer if necessary.

Upon completion of step 4, the terms remaining in the model should involve only incremental variables and they should all be linear with constant coefficients. In general, the coefficients involved in those terms that came from the expansion of nonlinear terms depend on the equilibrium conditions. Hence we must find a specific operating point before we can express the linearized model in numerical form. The entire procedure will be illustrated by several examples.

▶ *EXAMPLE 9.3*

Derive a linearized model of the translational mechanical system shown in Figure 9.6(a), where the nonlinear spring characteristic $f_K(x)$ is given in Figure 9.6(b) and where the average value of the applied force $f_a(t)$ is zero.

Solution

First we derive the nonlinear model by drawing the free-body diagram shown in Figure 9.6(c) and summing forces. This yields

$$M\ddot{x} + B\dot{x} + f_K(x) = f_a(t) \tag{11}$$

To find the operating point, we replace $f_a(t)$ by its average value $\bar{f}_a$ and x by $\bar{x}$:

$$M\ddot{\bar{x}} + B\dot{\bar{x}} + f_K(\bar{x}) = \bar{f}_a$$

Noting that $\dot{\bar{x}} = \ddot{\bar{x}} = 0$ because $\bar{x}$ is a constant and that $\bar{f}_a$ was specified to be zero, we see that

$$\bar{f}_K = f_K(\bar{x}) = 0$$

Thus the operating point is at $\bar{x} = 0$, $\bar{f}_K = 0$, which corresponds to the origin of the spring characteristic in Figure 9.6(b).

The next step is to rewrite the linear terms in (11) in terms of the incremental variables $\hat{x} = x - \bar{x}$ and $\hat{f}_a(t) = f_a(t) - \bar{f}_a$. This yields

$$M(\ddot{\bar{x}} + \ddot{\hat{x}}) + B(\dot{\bar{x}} + \dot{\hat{x}}) + f_K(x) = \bar{f}_a + \hat{f}_a(t)$$

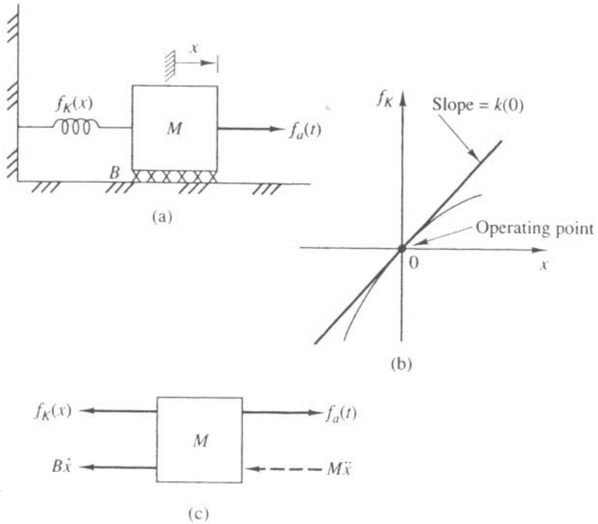

FIGURE 9.6 (a) Nonlinear system for Example 9.3. (b) Nonlinear spring characteristic. (c) Free-body diagram.

Because $\dot{\bar{x}} = \ddot{\bar{x}} = 0$, we can rewrite the equation as

$$M\ddot{\hat{x}} + B\dot{\hat{x}} + f_K(x) = \bar{f}_a + \hat{f}_a(t) \tag{12}$$

Expanding the spring force $f_K(x)$ about $\bar{x} = 0$ gives

$$f_K(x) = f_K(0) + \left.\frac{df_K}{dx}\right|_{x=0} \hat{x} + \cdots$$

Substituting the first two terms into (12) yields the approximate equation

$$M\ddot{\hat{x}} + B\dot{\hat{x}} + f_K(0) + k(0)\hat{x} = \bar{f}_a + \hat{f}_a(t)$$

The constant $k(0)$ denotes the derivative df_K/dx evaluated at $x = 0$ and is the slope of the tangent to the spring characteristic at the operating point, as indicated in Figure 9.6(b).

The spring force at the operating point is $f_K(0) = \bar{f}_a = 0$, so the linearized model is

$$M\ddot{\hat{x}} + B\dot{\hat{x}} + k(0)\hat{x} = \hat{f}_a(t) \tag{13}$$

which is a fixed linear differential equation in the incremental variable $\hat{x}$ with the incremental input $\hat{f}_a(t)$. The coefficients are the constants M, B, and $k(0)$. To solve (13), we must know the initial values $\hat{x}(0)$ and $\dot{\hat{x}}(0)$, which we find from the initial values $x(0)$ and $\dot{x}(0)$. Because $\hat{x}(t) = x(t) - \bar{x}$

and $\dot{\hat{x}}(t) = \dot{x}(t) - \dot{\bar{x}}$ for all values of t,

$$\hat{x}(0) = x(0) - \bar{x}$$
$$\dot{\hat{x}}(0) = \dot{x}(0) - \dot{\bar{x}}$$

In this example, $\bar{x} = \dot{\bar{x}} = 0$, so $\hat{x}(0) = x(0)$ and $\dot{\hat{x}}(0) = \dot{x}(0)$. Once we have solved the linearized model, we find the approximate solution of the nonlinear model by adding the nominal value $\bar{x}$ to the incremental solution $\hat{x}(t)$. Remember that the sum of the terms, $\bar{x}+\hat{x}(t)$, is only an approximation to the actual solution of the nonlinear model.

Should we want to put the linearized model given by (13) into state-variable form, we need only define the incremental velocity $\hat{v} = v - \bar{v}$, where $\bar{v} = 0$. Then $\hat{v} = \dot{\hat{x}}$, and we can write the pair of first-order equations

$$\dot{\hat{x}} = \hat{v}$$
$$\dot{\hat{v}} = \frac{1}{M}[-k(0)\hat{x} - B\hat{v} + \hat{f}_a(t)]$$

Because $\bar{x} = \bar{v} = 0$, the appropriate initial conditions are $\hat{x}(0) = x(0)$ and $\hat{v}(0) = v(0)$.

▶ *EXAMPLE 9.4*

Repeat Example 9.3 with the applied force $f_a(t)$ having a nonzero average value of $\bar{f}_a$ for positive time, as shown in Figure 9.7(a).

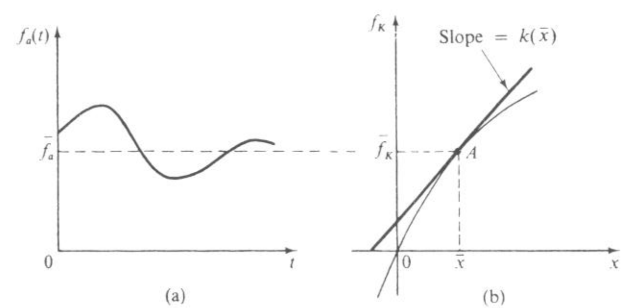

FIGURE 9.7 (a) Applied force for Example 9.4. (b) Nonlinear spring characteristic with new operating point.

Solution

The form of the nonlinear model given by (11) is unaffected by the value of $\bar{f}_a$. However, a new operating point will exist that is defined by the equation

$$f_K(\bar{x}) = \bar{f}_K = \bar{f}_a \tag{14}$$

For the spring characteristic shown in Figure 9.7(b) and the value of $\bar{f}_a$ shown in part (a) of the figure, the operating point is point A. In graphical terms, the straight line $\bar{f}_a$ in Figure 9.7(a) is projected horizontally onto the curve of the spring characteristic in part (b) of the figure, intersecting it at the operating point, the coordinates of which are $x = \bar{x}$, $f_K = \bar{f}_K$.

Upon substituting $x = \bar{x} + \hat{x}$ and $f_a(t) = \bar{f}_a + \hat{f}_a(t)$ into (11) and using $\dot{\bar{x}} = \ddot{\bar{x}} = 0$, we obtain

$$M\ddot{\hat{x}} + B\dot{\hat{x}} + f_K(x) = \bar{f}_a + \hat{f}_a(t) \tag{15}$$

which is identical to (12). The first two terms in the Taylor series for the spring force are

$$f_K(\bar{x}) + \left.\frac{df_K}{dx}\right|_{\bar{x}} \hat{x} \tag{16}$$

where $f_K(\bar{x}) = \bar{f}_K$ and where $\bar{x}$ must satisfy (14). Substituting (16) for $f_K(x)$ into (15) and invoking (14) yield the desired linear model:

$$M\ddot{\hat{x}} + B\dot{\hat{x}} + k(\bar{x})\hat{x} = \hat{f}_a(t) \tag{17}$$

where $k(\bar{x}) = [df_K/dx]|_{\bar{x}}$ and is the slope of the straight line in Figure 9.7(b). Note that the form of the model given by (17) with $\bar{f}_a \neq 0$ is the same as that with $\bar{f}_a = 0$, which is given by (13). The only difference between the two equations is the value of the effective spring constant. The value of $k(\bar{x})$ depends on the value of $\bar{x}$ at which the slope of $f_K(x)$ is measured. Hence the responses of the two linearized models could be rather different, even for the same incremental applied force $\hat{f}_a(t)$.

▶ *EXAMPLE 9.5*

Derive a linear model for the mechanical system and spring characteristic shown in Figure 9.8, where $x = 0$ corresponds to an unstretched spring.

Solution

We obtain the nonlinear model of the system by drawing the free-body diagram shown in Figure 9.9(a) and setting the sum of the vertical forces equal to zero. Because the mass is constrained to move vertically, we must include its weight Mg in the free-body diagram. The resulting nonlinear model is

$$M\ddot{x} + B\dot{x} + f_K(x) = f_a(t) + Mg$$

Note that its form is similar to that given by (11), the nonlinear model for the two preceding examples. By setting $x = \bar{x}$ and $f_a(t) = \bar{f}_a$ and by noting that $\dot{\bar{x}} = \ddot{\bar{x}} = 0$, we find the algebraic equation for the operating point to be

$$f_K(\bar{x}) = \bar{f}_a + Mg$$

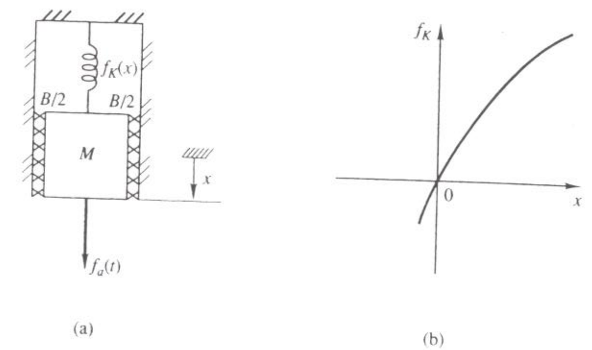

FIGURE 9.8 (a) Mechanical system for Example 9.5. (b) Nonlinear spring characteristic.

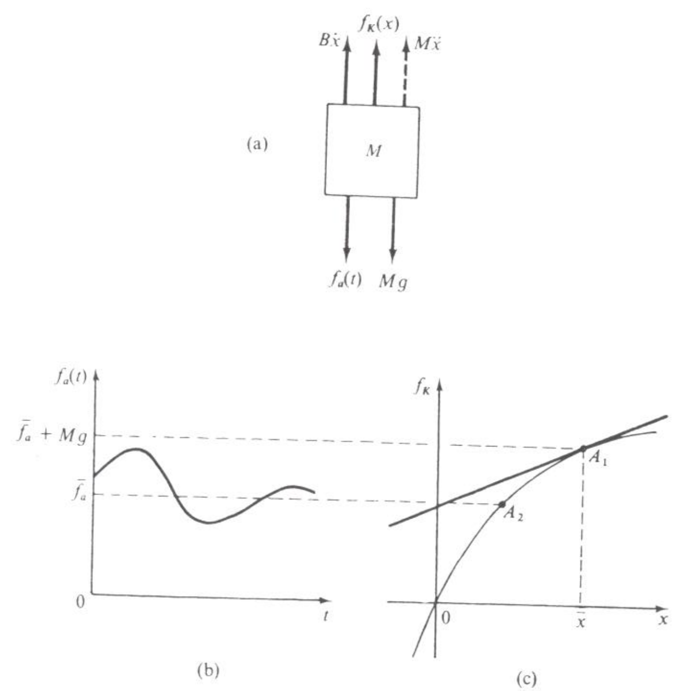

FIGURE 9.9 (a) Free-body diagram for Example 9.5. (b) Input. (c) Nonlinear spring characteristic with two operating points.

which is the same as (14) except for inclusion of the force Mg on the right side.

The nonlinear spring characteristic in Figure 9.8(b) is repeated in Figure 9.9(c). Figure 9.9(b) shows $f_a(t)$ and $\bar{f}_a$ as in Example 9.4, but it also shows the total nominal force $\bar{f}_a + Mg$ that must be projected onto the characteristic curve in Figure 9.9(c) to establish the operating point A_1. The point A_2, which is obtained by projecting the force $\bar{f}_a$ from Figure 9.9(b), would be the operating point if the motion of the mass were horizontal, as it is in Example 9.4. For applied forces having the same average values, the two systems will have different linearized spring characteristics if the curve of $f_K(x)$ does not have the same slope at points A_1 and A_2. If the spring were linear, however, the slope of the characteristic in Figure 9.9(c) would be constant and the presence of the weight would have no influence on the effective spring constant, as was observed in Example 2.5. With the provision that $k(\bar{x})$ is the slope of $f_K(x)$ measured at the point A_1 rather than at point A_2, the resulting linearized model is again given by (17).

EXAMPLE 9.6

A high-speed vehicle of mass M moves along a horizontal track and is subject to a linear retarding force Bv caused by viscous friction associated with the bearings and a nonlinear retarding force $D|v|v$ caused by air drag. Obtain a linear model that is valid when the driving force $f_a(t)$ undergoes variations about a positive nominal value $\bar{f}_a$.

Solution

The nonlinear differential equation governing the vehicle's velocity is

$$M\dot{v} + Bv + D|v|v = f_a(t) \tag{18}$$

Setting $v = \bar{v}$ and $f_a(t) = \bar{f}_a$ and noting that $\dot{\bar{v}} = 0$, we have

$$B\bar{v} + D|\bar{v}|\bar{v} = \bar{f}_a$$

for the operating-point equation. Because $\bar{f}_a$ is positive, we know that $\bar{v}$ is positive and we can replace $D|\bar{v}|\bar{v}$ by $D(\bar{v})^2$. Then

$$D(\bar{v})^2 + B\bar{v} - \bar{f}_a = 0 \tag{19}$$

By inspection we see that (19) will have two real roots, one positive and the other negative. However, we are interested only in the positive root, which is

$$\bar{v} = \frac{-B + \sqrt{B^2 + 4\bar{f}_a D}}{2D} \tag{20}$$

The negative root was introduced when we replaced $|\bar{v}|\bar{v}$ by $(\bar{v})^2$ and is not a root of the original operating-point equation.

Provided that v always remains positive, we can replace $D|\bar{v}|\bar{v}$ in (18) by Dv^2. Then, using $v = \bar{v} + \hat{v}$ and $f_a(t) = \bar{f}_a + \hat{f}_a(t)$ and noting that $\dot{\bar{v}} = 0$, we can rewrite (18) as

$$M\dot{\hat{v}} + B(\bar{v} + \hat{v}) + Dv^2 = \bar{f}_a + \hat{f}_a(t) \tag{21}$$

To linearize the term v^2, we replace it by the constant and linear terms in its Taylor series:

$$(\bar{v})^2 + \frac{d}{dv}(v^2)\bigg|_{\bar{v}}(v - \bar{v}) = (\bar{v})^2 + 2\bar{v}\hat{v} \tag{22}$$

Substituting (22) for v^2 into (21) and regrouping, we have

$$M\dot{\hat{v}} + (B + 2D\bar{v})\hat{v} + B\bar{v} + D(\bar{v})^2 = \bar{f}_a + \hat{f}_a(t)$$

The constant terms cancel because of (19), the operating-point equation. Thus the desired linearized model, which holds for $\bar{f}_a > 0$ and $v > 0$, is

$$M\dot{\hat{v}} + b\hat{v} = \hat{f}_a(t)$$

where b denotes the effective damping coefficient

$$b = B + 2D\bar{v}$$

with $\bar{v}$ given by (20) as a function of the average driving force $\bar{f}_a$.

It is worthwhile to observe that in this particular case the Taylor series of the nonlinearity has only three terms, and we could have obtained it without differentiation by writing

$$v^2 = (\bar{v} + \hat{v})^2 = (\bar{v})^2 + 2\bar{v}\hat{v} + (\hat{v})^2$$

In this example, we can see that $(\hat{v})^2$ is the error introduced by replacing v^2 by $(\bar{v})^2 + 2\bar{v}\hat{v}$. Provided that $\bar{v} \gg |\hat{v}|$, the error will be small compared to the two terms that are retained.

EXAMPLE 9.7

A nonlinear system obeys the state-variable equations

$$\dot{x} = y \tag{23a}$$

$$\dot{y} = -|x|x - 2x - 2y^3 - 3 + 0.2\cos t \tag{23b}$$

Find the operating point and develop the linearized model in numerical form.

Solution

The operating point, described by $\bar{x}$ and $\bar{y}$, must satisfy the conditions $\dot{\bar{x}} = \dot{\bar{y}} = 0$ with the incremental portion of the input set to zero. Hence the operating-point equations reduce to

$$\begin{aligned} \bar{y} &= 0 \\ |\bar{x}|\bar{x} + 2\bar{x} + 3 &= 0 \end{aligned} \tag{24}$$

which have the solution $\bar{x} = -1$, $\bar{y} = 0$. We replace the two nonlinear elements in (23b) by the first two terms in their respective Taylor-series expansions. For $|x|x$, we write

$$|\bar{x}|\bar{x} + 2|\bar{x}|\hat{x}$$

and we replace y^3 by

$$(\bar{y})^3 + 3(\bar{y})^2\hat{y}$$

By substituting these approximations into (23) and using $x = \bar{x} + \hat{x}$ and $y = \bar{y} + \hat{y}$, we obtain

$$\begin{aligned} \dot{\hat{x}} &= \bar{y} + \hat{y} \\ \dot{\hat{y}} &= -(|\bar{x}|\bar{x} + 2|\bar{x}|\hat{x}) - 2(\bar{x} + \hat{x}) - 2[(\bar{y})^3 + 3(\bar{y})^2\hat{y}] - 3 + 0.2\cos t \end{aligned} \tag{25}$$

With $\bar{x} = -1$ and $\bar{y} = 0$, (25) reduces to

$$\begin{aligned} \dot{\hat{x}} &= \hat{y} \\ \dot{\hat{y}} &= -4\hat{x} + 0.2\cos t \end{aligned} \tag{26}$$

which is the linearized model in state-variable form. Comparing (26) with (25), we note that (1) all the constant terms have been canceled, (2) the coefficient of $\hat{y}$ in the second equation is zero because $\bar{y} = 0$, and (3) the coefficient of $\hat{x}$ reflects the combined effects of the linear and nonlinear terms.

9.3 CIRCUITS WITH NONLINEAR RESISTORS

We have defined resistors, capacitors, and inductors as elements for which there is an algebraic relationship between the voltage and current, voltage and charge, and current and flux linkage, respectively. If the two variables involved in the algebraic relationship are directly proportional to one another, then the element is linear, as was the case in all the examples in Chapter 5.

In this section, we consider circuits with nonlinear resistors. The general procedure for obtaining a linearized model is the same as that used for the mechanical examples. As in Section 9.1, we express the variables as the sum of a constant portion and a time-varying portion. For example, we write a voltage e_o as

$$e_o = \bar{e}_o + \hat{e}_o$$

where the constant term $\bar{e}_o$ is the nominal value, corresponding to a particular operating point, and where $\hat{e}_o$ is the incremental time-varying component. In the circuit diagrams, we indicate the fact that a resistor is nonlinear by drawing a curved line through its symbol. The procedure is illustrated by three examples of increasing complexity.

▶ *EXAMPLE 9.8*

The circuit shown in Figure 9.10(a) contains a nonlinear resistor that obeys the element law $i_o = 2e_o^3$. Write the differential equation relating e_o and $e_i(t)$. If $e_i(t) = 18 + A\cos\omega t$, find the operating point and derive the linearized input-output differential equation. Also determine the time constant of the linearized model.

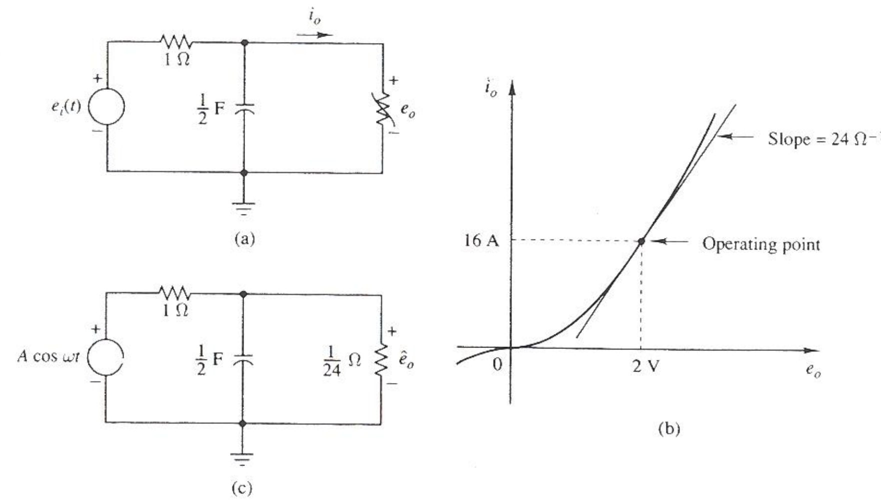

FIGURE 9.10 (a) Circuit for Example 9.8. (b) Characteristic curve for the nonlinear resistor. (c) Linearized equivalent circuit.

Solution

The right-hand resistor is nonlinear because the current i_o is not directly proportional to the voltage e_o. Summing the currents leaving the node at the upper right gives

$$\frac{1}{2}\dot{e}_o + [e_o - e_i(t)] + 2e_o^3 = 0$$

Thus the input-output equation is

$$\frac{1}{2}\dot{e}_o + 2e_o^3 + e_o = e_i(t) \tag{27}$$

To determine the operating point, we replace $e_i(t)$ by $\bar{e}_i = 18$ V, e_o by $\bar{e}_o$, and $\dot{e}_o$ by zero to obtain

$$2\bar{e}_o^3 + \bar{e}_o = 18$$

The only real value of $\bar{e}_o$ that satisfies this algebraic equation is

$$\bar{e}_o = 2 \text{ V} \tag{28}$$

From the nonlinear element law, we see that $\bar{i}_o = 16$ A, which gives the operating point shown in Figure 9.10(b).

To develop a linearized model, we let

$$\begin{aligned} e_i(t) &= 18 + A\cos\omega t \\ e_o &= 2 + \hat{e}_o \end{aligned} \tag{29}$$

As in (8), we write the first two terms in the Taylor series for the nonlinear term $2e_o^3$, which are

$$\begin{aligned} \bar{i}_o + \left.\frac{di_o}{de_o}\right|_{\bar{e}_o}(e_o - \bar{e}_o) &= \bar{i}_o + (6\bar{e}_o^2)\hat{e}_o \\ &= 16 + 24\hat{e}_o \end{aligned} \tag{30}$$

This approximation describes the tangent to the characteristic curve at the operating point, as shown in Figure 9.10(b). Substituting (29) and (30) into (27) gives

$$\tfrac{1}{2}(\dot{\bar{e}}_o + \dot{\hat{e}}_o) + (16 + 24\hat{e}_o) + (2 + \hat{e}_o) = 18 + A\cos\omega t$$

Because $\dot{\bar{e}}_o = 0$ and because the constant terms cancel (as is always the case), we have for the linearized model

$$\tfrac{1}{2}\dot{\hat{e}}_o + 25\hat{e}_o = A\cos\omega t$$

or

$$\dot{\hat{e}}_o + 50\hat{e}_o = 2A\cos\omega t \tag{31}$$

By inspecting this equation, we see that the time constant of the linearized model is 0.02 s.

In the last example, we can see from (30), as well as from Figure 9.10(b), that $\hat{i}_o = 24\hat{e}_o$. This equation has the form of Ohm's law for a linear resistor: $\hat{i}_o = (1/r)\hat{e}_o$, where $r = 1/24\ \Omega$. Figure 9.10(c) shows the linearized equivalent circuit that relates the time-varying incremental variables. It has the same form as Figure 9.10(a), except that the nonlinear resistor has been replaced by a linearized element. It is straightforward to show that (31) can be obtained directly from part (c) of the figure. Keep in mind that the value of the linearized element depends on the operating point.

Computer simulations are discussed in Chapter 15, which can be studied at any time. In the final two examples, we not only develop linearized models but also use such simulations to compare the nonlinear and linearized models. This is usually the only way to determine how large the time-varying component of the input can be and still have the linearized model give good results.

▶ *EXAMPLE 9.9*

Write the state-variable equations for the circuit shown in Figure 9.11(a), which contains a nonlinear resistor that obeys the element law $i_2 = \frac{1}{8}e_C^3$. Find the operating point when $e_i(t) = 2 + \hat{e}_i(t)$, and derive the linearized state-variable equations in terms of the incremental variables. Plot and compare i_L versus t for the nonlinear and linearized models when $\hat{e}_i(t) = [A\sin t]U(t)$ for (1) $A = 0.1$ V, (2) $A = 1.0$ V, and (3) $A = 10.0$ V.

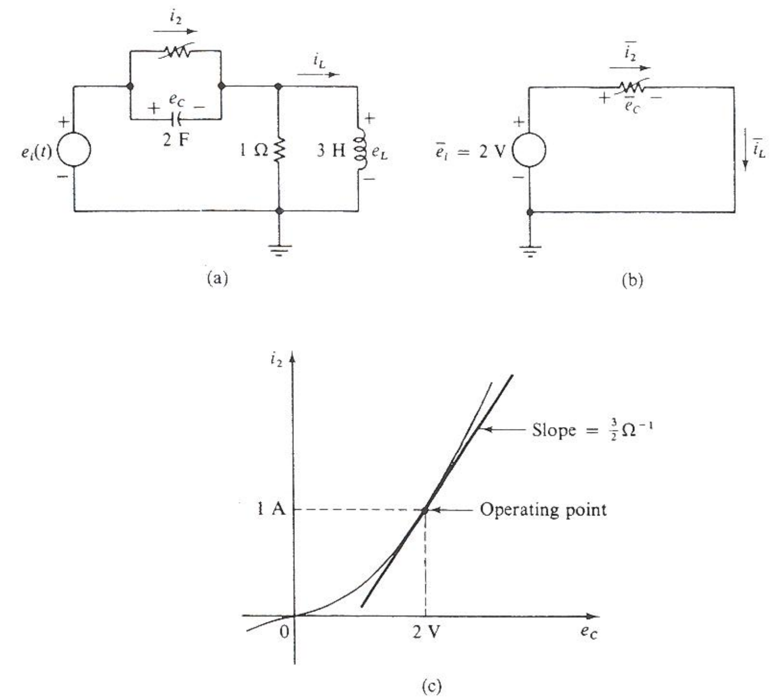

FIGURE 9.11 (a) Circuit for Example 9.9. (b) Circuit used for determining the operating point. (c) Characteristic curve for the nonlinear resistor.

Solution

We choose e_C and i_L as state variables and note that

$$e_L = e_i(t) - e_C \tag{32}$$

Applying Kirchhoff's current law to the upper right node gives

$$i_L + e_L - \tfrac{1}{8}e_C^3 - 2\dot{e}_C = 0$$

Inserting (32) into this equation and into the element law $di_L/dt = \frac{1}{3}e_L$ gives

$$\begin{aligned} \dot{e}_C &= \frac{1}{2}\left[e_i(t) + i_L - \frac{1}{8}e_C^3 - e_C\right] \\ \frac{di_L}{dt} &= \frac{1}{3}[e_i(t) - e_C] \end{aligned} \tag{33}$$

which constitute the nonlinear state-variable equations.

At the operating point, the derivatives of the state variables are zero, and (33) reduces to the algebraic equations

$$\bar{e}_i + \bar{i}_L - \frac{1}{8}\bar{e}_C^3 - \bar{e}_C = 0$$

$$\bar{e}_i - \bar{e}_C = 0$$

With $\bar{e}_i = 2$ V, we find that $\bar{e}_C = 2$ V, $\bar{i}_L = 1$ A, and $\bar{i}_2 = 2^3/8 = 1$ A. An alternative way of determining the operating point is to recall from Section 6.2 that when all the voltages and currents are constant, we can replace the capacitors and inductors by open and short circuits, respectively. We do this in Figure 9.11(b), from which we again see that $\bar{e}_C = 2$ V and $\bar{i}_L = \bar{i}_2 = 1$ A.

Next we define the incremental variables $\hat{e}_C, \hat{i}_L$, and $\hat{e}_i(t)$ by the equations

$$\begin{aligned} e_C &= 2 + \hat{e}_C \\ i_L &= 1 + \hat{i}_L \\ e_i(t) &= 2 + \hat{e}_i(t) \end{aligned} \tag{34}$$

The Taylor-series expansion for the nonlinear resistor is

$$i_2 = \frac{1}{8}e_C^3 = \frac{1}{8}\bar{e}_C^3 + \frac{3}{8}\bar{e}_C^2(e_C - \bar{e}_C) + \cdots$$

Using the first two terms in the series, with $\bar{e}_C = 2$ V, we have

$$i_2 \simeq 1 + \frac{3}{2}\hat{e}_C \tag{35}$$

which is shown graphically by the straight line in Figure 9.11(c). Substituting (34) and (35) into (33) and canceling the constant terms, we obtain

$$\dot{\hat{e}}_C = \frac{1}{2}\left[\hat{e}_i(t) + \hat{i}_L - \frac{5}{2}\hat{e}_C\right] \tag{36a}$$

$$\frac{d\hat{i}_L}{dt} = \frac{1}{3}[\hat{e}_i(t) - \hat{e}_C] \tag{36b}$$

as the linearized state-variable equations.

The results of a computer solution for i_L of the nonlinear model given by (33) with $e_i(t) = 2 + [A \sin t]U(t)$ are shown in Figure 9.12 for three different values of the amplitude A. Plotted on the same axes are curves obtained by calculating $\hat{i}_L$ from the linearized model in (36) with $\hat{e}_i(t) = [A \sin t]U(t)$ and then forming the quantity $\hat{i}_L + \bar{i}_L$. Because $e_i(t) = \bar{e}_i$ for all $t < 0$, we used the initial conditions

$$e_C(0) = \bar{e}_C = 2 \text{ V}, \quad \hat{e}_C(0) = 0$$

$$i_L(0) = \bar{i}_L = 1 \text{ A}, \quad \hat{i}_L(0) = 0$$

Note that the responses of the nonlinear and linearized models are almost identical when $A = 0.1$ V, are in close agreement when $A = 1.0$ V, but differ significantly when $A = 10.0$ V. The steady-state response of the linearized model is always a sinusoidal oscillation about the operating point. For large values of A, however, the steady-state response of the nonlinear model is not symmetrical about the operating point.

We have plotted the three sets of curves with different vertical scales and different origins in order to get a good comparison of the responses of the nonlinear and linearized models for each of the three values of A. If we repeat the example when $\bar{e}_i = 0$ and $\hat{e}_i(t) = [A \sin t]U(t)$, then in the steady state we would expect the response of the nonlinear model to be symmetrical about the operating point. When we carry out a computer run for this case, we obtain not only the expected symmetry but also good agreement of the nonlinear and linearized responses for all three values of A.

▶ *EXAMPLE 9.10*

The voltage source $e_i(t)$ for the circuit shown in Figure 9.13(a) is

$$e_i(t) = \begin{cases} -2 & \text{for } t \le 0 \\ 2 + A\cos 4t & \text{for } t > 0 \end{cases}$$

The element law for the nonlinear resistor is $e_2 = 2|i_L|i_L$.

1. Find a set of state-variable equations and an algebraic output equation for i_o. Give the initial conditions needed to solve the equations.
2. Find a linearized set of state-variable equations about the operating point corresponding to $\bar{e}_i = 2$ V. Write the necessary initial conditions and an algebraic output equation for $\hat{i}_o$. Also draw the linearized equivalent circuit.
3. Simulate the responses of the nonlinear and linearized models, and plot i_o versus time when $A = 1$ V, 5 V, and 20 V.

Solution

We make our usual choice of e_C and i_L as the state variables. The voltage at the top of the 1-Ω resistor is $e_C + 6$, so $i_1 = e_C + 6$. By Kirchhoff's

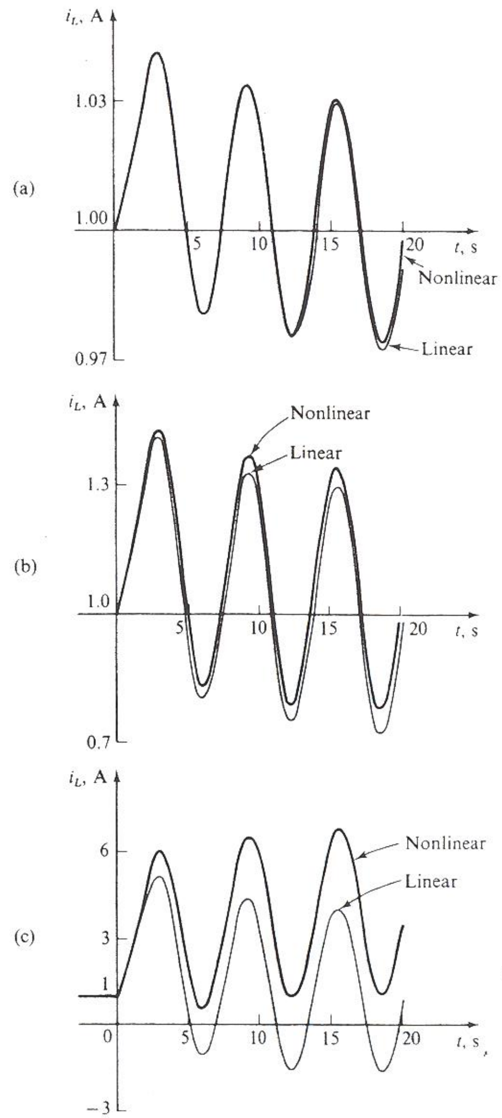

FIGURE 9.12 Results of computer simulation for Example 9.9 with $e_i(t) = 2 + [A \cos t]U(t)$. (a) $A = 0.1$ V. (b) $A = 1.0$ V. (c) $A = 10.0$ V.

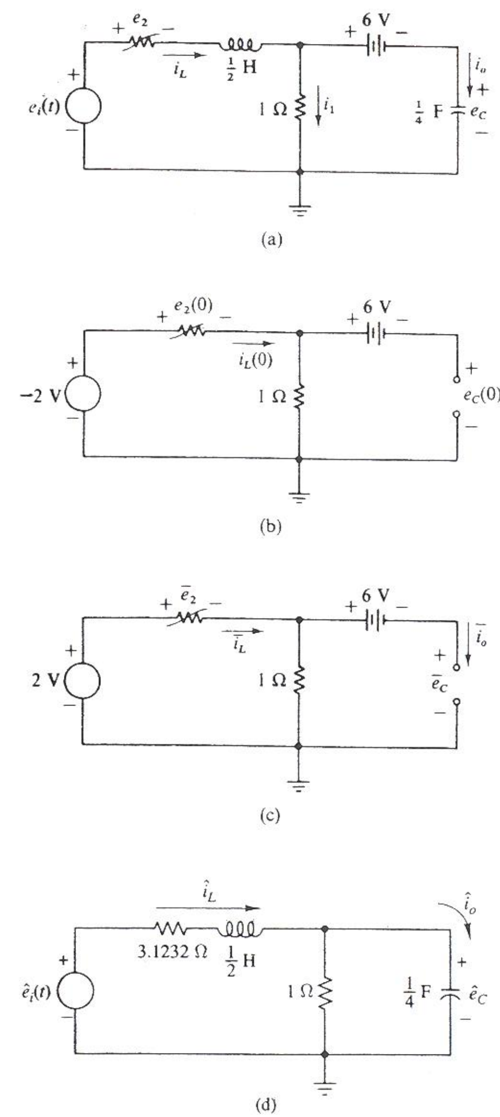

FIGURE 9.13 (a) Circuit for Example 9.10. (b) Circuit for determining the initial conditions. (c) Circuit for determining the operating point. (d) Linearized equivalent circuit.

current law,

$$\begin{aligned} i_o &= i_L - i_1 \\ &= i_L - e_C - 6 \end{aligned} \tag{37}$$

which is the desired output equation. Because $i_o = \frac{1}{4}\dot{e}_C$, one state-variable equation is

$$\dot{e}_C = 4(i_L - e_C - 6) \tag{38}$$

Summing voltages around the outside loop gives

$$e_C + 6 + \frac{1}{2}\frac{di_L}{dt} + 2|i_L|i_L - e_i(t) = 0$$

from which we have, for the second state-variable equation,

$$\frac{di_L}{dt} = 2[e_i(t) - 2|i_L|i_L - e_C - 6] \tag{39}$$

To find the values of $e_C(0)$ and $i_L(0)$, we can use (38) and (39) with the derivatives replaced by zero and $e_i(t)$ replaced by -2 V. Equivalently, we can redraw the circuit as shown in Figure 9.13(b), replacing the capacitor and inductor by open and short circuits, respectively, and using the voltage source -2 V. Then

$$2|i_L(0)|i_L(0) + i_L(0) + 2 = 0$$

Only a negative value of $i_L(0)$ can satisfy this equation, so we rewrite it as

$$-2[i_L(0)]^2 + i_L(0) + 2 = 0 \tag{40}$$

which has roots at -0.7808 and 1.2808. The latter root results from rewriting the equation in quadratic form and is extraneous, so

$$\begin{aligned} i_L(0) &= -0.7808 \text{ A} \\ e_C(0) &= -0.7808 - 6 = -6.7808 \text{ V} \end{aligned} \tag{41}$$

We can find the operating point corresponding to $\bar{e}_i = 2$ V from (38) and (39) or from the simplified circuit shown in Figure 9.13(c). We see that

$$2|\bar{i}_L|\bar{i}_L + \bar{i}_L - 2 = 0$$

This time only a positive value of $\bar{i}_L$ can satisfy the equation, so we rewrite it as

$$2\bar{i}_L^2 + \bar{i}_L - 2 = 0 \tag{42}$$

Because (42) has the same form as (40) except that $i_L(0)$ is replaced by $-\bar{i}_L$, we know that the desired root is at 0.7808. Thus

$$\begin{aligned} \bar{i}_L &= 0.7808 \text{ A} \\ \bar{e}_C &= 0.7808 - 6 = -5.2192 \text{ V} \\ \bar{i}_o &= 0 \end{aligned} \tag{43}$$

We now introduce the incremental variables

$$\begin{aligned} \hat{e}_i(t) &= e_i(t) - 2 \\ \hat{i}_L &= i_L - 0.7808 \\ \hat{e}_C &= e_C + 5.2192 \\ \hat{i}_o &= i_o \end{aligned} \tag{44}$$

The Taylor-series expansion for e_2, the voltage across the nonlinear resistor, is

$$e_2 = 2|i_L|i_L = 2|\bar{i}_L|\bar{i}_L + \left.\frac{de_2}{di_L}\right|_{\bar{i}_L}(i_L - \bar{i}_L) + \cdots$$

Because $\bar{i}_L$ is positive, $|\bar{i}_L| = \bar{i}_L$ and the approximate expression for e_2 becomes

$$e_2 \simeq 2\bar{i}_L^2 + (4\bar{i}_L)\hat{i}_L = 1.2193 + 3.1232\hat{i}_L \tag{45}$$

Substituting (44) and (45) into (37), (38), and (39) and canceling the constant terms give the two state-variable equations

$$\begin{aligned} \dot{\hat{e}}_C &= 4(\hat{i}_L - \hat{e}_C) \\ \frac{d\hat{i}_L}{dt} &= 2[\hat{e}_i(t) - 3.1232\hat{i}_L - \hat{e}_C] \end{aligned} \tag{46}$$

and the output equation

$$\hat{i}_o = \hat{i}_L - \hat{e}_C \tag{47}$$

The required initial conditions are

$$\begin{aligned} \hat{i}_L(0) &= i_L(0) - 0.7808 = -1.5616 \text{ A} \\ \hat{e}_C(0) &= e_C(0) + 5.2192 = -1.5616 \text{ V} \end{aligned} \tag{48}$$

To draw the linearized equivalent circuit, we first note from (45) that $\hat{e}_2 = 3.1232\hat{i}_L$. Thus, by a comparison with Ohm's law, the nonlinear resistor is replaced by a linear element whose resistance is 3.1232 Ω. We also replace all the voltages and currents in Figure 9.13(a) by their incremental components. Because the 6-V source has no incremental component, it becomes a short circuit. The complete linearized model is shown in Figure 9.13(d). If we so desire, we can obtain (46) and (47) directly from this figure.

The results of a computer solution of the nonlinear model when $e_i(t) = 2 + A\cos 4t$ for $t > 0$ and for the initial conditions in (41) are shown in Figure 9.14. A solution of the linearized model when $\hat{e}_i(t) = A\cos 4t$ and for comparable initial conditions is shown on the same set of axes. Because $\bar{i}_o = 0$, we can compare i_o from the nonlinear model to $\hat{i}_o$ from the linearized

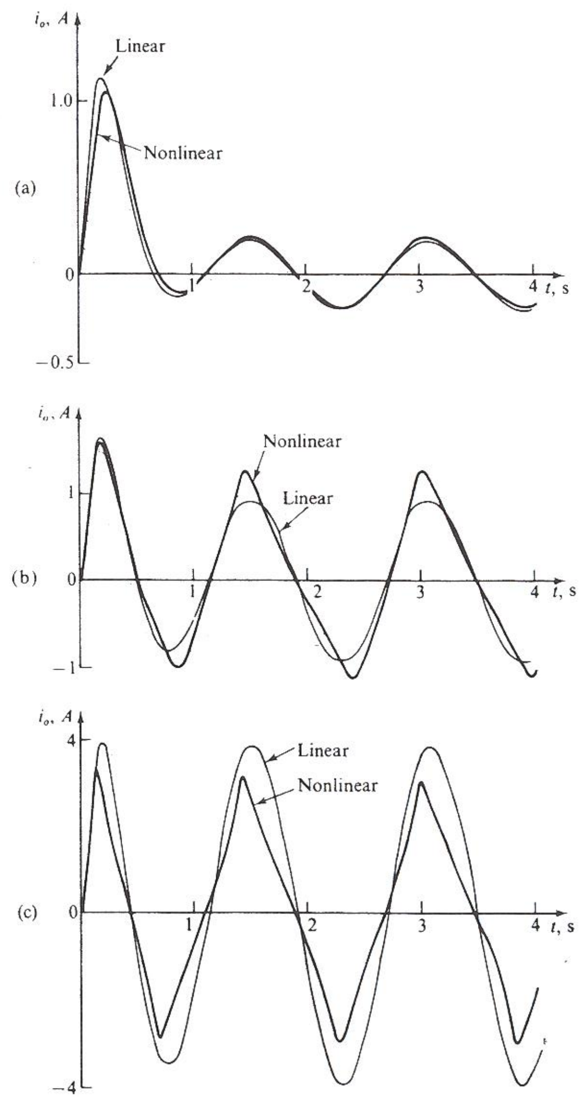

FIGURE 9.14 Results of computer simulation for Example 9.10 with $e_i(t) = 2 + A\cos 4t$ for $t > 0$. (a) $A = 1$ V. (b) $A = 5$ V. (c) $A = 20$ V.

model directly. The responses agree closely for $A = 1$ V. Note, however, that when $A = 5$ V and 20 V, the output waveshape for the nonlinear model begins to look triangular in the steady state. For the linearized model, the steady-state response to a sinusoidal input must remain sinusoidal regardless of the size of the input.

Reflecting on the results of this example and the preceding one, we see that even in the steady state, a nonlinearity can alter the average value of the output and also its waveshape. Its effect may be different for different operating points, and it is difficult to predict in advance the range of incremental inputs for which a linearized model will give a satisfactory approximation. Because it is difficult to establish general guidelines for approximating nonlinear systems, we must usually rely heavily on computer simulations.

SUMMARY

To linearize an element law about a particular operating point, we can use the first two terms in its Taylor-series expansion. This is equivalent to approximating its characteristic curve by a straight line tangent to the curve at the operating point.

For a system containing nonlinear elements, we first find the operating point corresponding to a specified constant value of the input. Then, in the nonlinear differential equations, we express the variables as the sum of constant and time-varying components, with nonlinear terms replaced by the first two terms in their Taylor-series expansions. After canceling the constant terms, we obtain linear equations that involve only the time-varying incremental variables.

In subsequent chapters, we shall use the same technique to obtain linearized approximations to some electromechanical, thermal, and hydraulic systems. The basic technique can also be extended to systems whose parameters vary with time and to nonlinearities that depend on two or more independent variables. Such extensions, along with nonlinear inductors and capacitors, are treated in some of the references in Appendix D.

PROBLEMS

In Problems 9.1 through 9.6, use a Taylor-series expansion to derive the linearized model for the element law and operating point(s) specified. In each case, show the linearized characteristic on a sketch of the nonlinear element law.

9.1 $f(x) = 0.5x^3$ where $\bar{x} = -2, 0,$ and 2

* **9.2** $f(x) = \begin{cases} -A(1-\epsilon^{x}) & \text{for } x < 0 \\ A(1-\epsilon^{-x}) & \text{for } x \geq 0 \end{cases}$ where $\bar{x} = -1$, 0, and 1

9.3 $f(\theta) = \begin{cases} -\sin^2\theta & \text{for } \theta < 0 \\ \sin^2\theta & \text{for } \theta \geq 0 \end{cases}$ where $\bar{\theta} = 0$, $\pi/4$, $\pi/2$, and $3\pi/4$

9.4 $f(\theta) = \begin{cases} -\sin(\theta^2) & \text{for } \theta < 0 \\ \sin(\theta^2) & \text{for } \theta \geq 0 \end{cases}$ where $\bar{\theta} = 0$, $\sqrt{\pi}/2$, and $\sqrt{\pi/2}$

* **9.5** $f(y) = 1/y$ where $y > 0$ and $\bar{y} = 0.5$

9.6 $f(z) = \begin{cases} -\sqrt{|z|} & \text{for } z < 0 \\ \sqrt{z} & \text{for } z \geq 0 \end{cases}$ where $\bar{z} = -2$ and 2

9.7 A nonlinear spring characteristic $f_K(x)$ is shown in Figure P9.7, where x denotes the total length. For each of the operating points specified, determine graphically the force exerted by the spring at the operating point, and evaluate graphically the linearized spring constant.

a) $\bar{x} = 0.1$ m
b) $\bar{x} = 0.2$ m
c) $\bar{x} = 0.3$ m
d) $\bar{x} = 0.4$ m

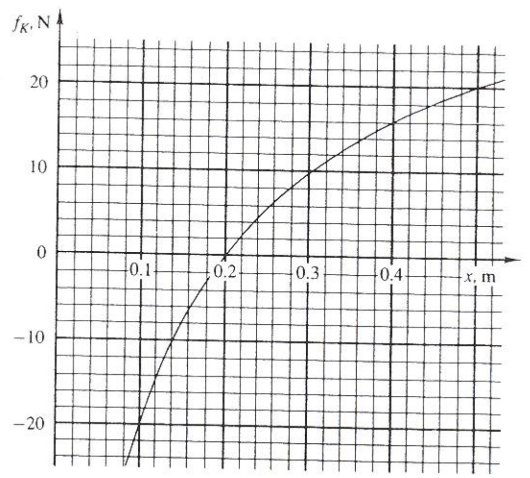

FIGURE P9.7

* **9.8** The nonlinear mechanical system shown in Figure P9.8 has $M = 1.5$ kg, $B = 0.5$ N·s/m, and the spring characteristic $f_K(x)$ plotted in Figure P9.7. The gravitational constant is 9.807 m/s^2. The variable x denotes the total length of the spring.

a) Verify that the nonlinear input-output differential equation is $1.5\ddot{x} + 0.5\dot{x} + f_K(x) = 14.71$.
b) Solve for the operating point $\bar{x}$.

c) Derive the linearized differential equation that is valid in the vicinity of the operating point.
d) Give the approximate range of x for which the linearized spring force is within 25% of the nonlinear spring force.

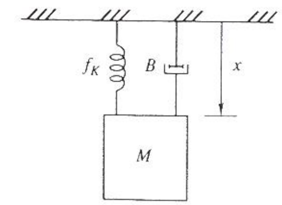

FIGURE P9.8

9.9 For the system shown in Figure P9.9, the mass is attached to the wall by a series combination of a linear spring and a nonlinear spring. The parameter values are $M = 4.0$ kg, $B = 0.3$ N·s/m, and $K = 25$ N/m. The applied force is $f_a(t) = 10 + 2\sin 3t$, and the nonlinear spring characteristic $f_K(x)$ is plotted in Figure P9.7. The position $z = 0$ corresponds to $f_a(t) = 0$ with both springs undeflected.

a) Find the nonlinear input-output differential equation relating z to $f_a(t)$.
b) Solve for the operating point $\bar{z}$.
c) Derive the linearized differential equation that is valid in the vicinity of the operating point.
d) Give the approximate range of z for which the linearized spring force is within 25% of the nonlinear spring force.

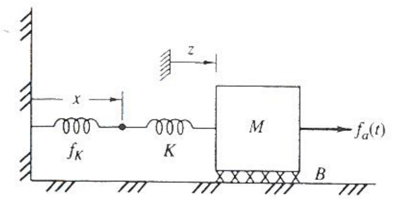

FIGURE P9.9

* **9.10** A mechanical system containing a nonlinear spring obeys the differential equation

$$\ddot{x} + 2\dot{x} + f(x) = A + B\sin 3t$$

where

$$f(x) = \begin{cases} -4\sqrt{|x|} & \text{for } x < 0 \\ 4\sqrt{x} & \text{for } x \geq 0 \end{cases}$$

a) Find the operating point $\bar{x}$ and derive the linearized model in numerical form for $A = 8$.
b) Find the operating point $\bar{x}$ and the linearized spring constant for $A = 4$ and -4.
c) Find $\hat{x}(0)$ and $\dot{\hat{x}}(0)$ when $A = 4$, $x(0) = 1.5$, and $\dot{x}(0) = 0.5$.

9.11 A nonlinear system obeys the equation

$$\ddot{x} + 2\dot{x} + \dot{x}^3 + \frac{4}{x} = A + B\cos t$$

a) Solve for the operating-point conditions on $\bar{x}$ and $\dot{\bar{x}}$. What restriction must be placed on the value of A?
b) Derive the linearized model, evaluating the coefficients in terms of A or numbers.

* **9.12** A system is described by the nonlinear equation

$$\ddot{y} + 2(\dot{y} + \dot{y}^3) + 2y + |y|y = A + B\cos t$$

a) For $A = -3$, find the operating point and derive the linearized model, expressing all coefficients in numerical form.
b) Repeat part (a) for $A = 15$.

9.13 The model of a nonlinear system is described by the equation

$$\ddot{x} + \dot{x} + 3x\sqrt{|x|} = A + B\sin\omega t$$

The equilibrium position is known to be $\bar{x} = 4$. Determine the linearized incremental model, and evaluate $\hat{x}(0)$ and $\dot{\hat{x}}(0)$ when $x(0) = 5$ and $\dot{x}(0) = 0$.

9.14 A nonlinear system obeys the equation

$$\dot{x} + 0.5x^2 = 2 + A\sin t$$

a) Sketch the nonlinear term $0.5x^2$ and indicate all possible operating points.
b) For each operating point you found in part (a), derive the linearized model for the system and indicate whether the linear model is stable or unstable.

9.15 A nonlinear system obeys the differential equation

$$\dot{y} + 4\sqrt{y} = 8 + B\cos 2t$$

for $y \geq 0$ and has the initial condition $y(0) = 5$.
a) Derive the linearized incremental model corresponding to the specified input.
b) Sketch the nonlinear element law for $y \geq 0$, and indicate the operating point and the linear approximation.
c) Evaluate the appropriate initial condition for the incremental variable.

9.16 A system obeys the differential equation

$$\ddot{x} + 3|\dot{x}|\dot{x} + 4x^3 = A + B\sin 2t$$

and has the initial conditions $x(0) = 2$ and $\dot{x}(0) = 1$.
a) Find the operating point and derive the linearized model for $A = 4$. Also, find the initial values of $\hat{x}$ and $\dot{\hat{x}}$.
b) Repeat part (a) for $A = 32$.

9.17 In developing the linearized model for a high-speed vehicle in Example 9.6, we assumed that $\bar{f}_a$, the nominal value of the driving force, was positive. Repeat the development for $\bar{f}_a < 0$, and find an expression for b, the effective damping coefficient of the linearized model, that is valid for both cases.

* **9.18** The disk shown in Figure P9.18 is supported by a nonlinear torsional spring and is subject to both linear and nonlinear frictional torques. The applied torque is $\tau_a(t) = 8 + \hat{\tau}_a(t)$.
a) Find the operating point $\bar{\theta}$.
b) Derive the linearized input-output equation in terms of $\hat{\theta}(t) = \theta - \bar{\theta}$.
c) Find the initial values of the incremental variables $\hat{\theta}$ and $\dot{\hat{\theta}}$ if $\theta(0) = 0.5$ rad and $\dot{\theta}(0) = -0.5$ rad/s.

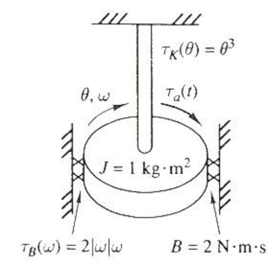

FIGURE P9.18

9.19 The translational system shown in Figure P9.19 has a linear and a nonlinear spring and is subjected to the applied force $f_a(t)$.
a) Show that the nonlinear model is $2\ddot{x} + 6\dot{x} + 3x + |x|x = f_a(t)$.
b) Solve for the operating point $\bar{x}$ when $\bar{f}_a = 10$ N.
c) Derive the linearized model when $f_a(t) = 10 + \hat{f}_a(t)$.
d) Find the initial values $\hat{x}(0)$ and $\dot{\hat{x}}(0)$ if $x(0) = 3$ m and $\dot{x}(0) = 1$ m/s.

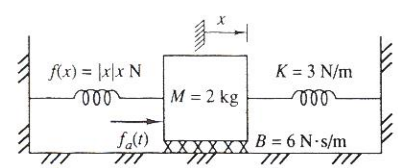

FIGURE P9.19

9.20 The rotating cylinder shown in Figure P9.20 has damping vanes and a linear frictional torque such that the motion is described by the nonlinear equation

$$\dot{\omega} + 2|\omega|\omega + 2\omega = \tau_a(t)$$

where the applied torque is $\tau_a(t) = 12 + \hat{\tau}_a(t)$. Find the operating point and derive a linearized model in terms of the incremental angular velocity $\hat{\omega}$.

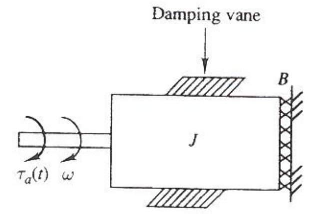

FIGURE P9.20

* **9.21** **a)** Verify that the input-output differential equation for the mechanical system modeled in Problem 2.19 is

$$M\ddot{x} + B\dot{x} + x^3 = f_a(t) + Mg$$

b) Find the operating point and derive the linearized input-output equation for the system for the case when $\bar{f}_a = 0$.
c) Repeat part (b) for the case when $\bar{f}_a = Mg$.

9.22 A nonlinear system obeys the state-variable equations

$$\dot{x} = -x + y$$
$$\dot{y} = \frac{1}{y} + 4 + B\sin t$$

Find the operating point and derive the linearized equations in state-variable form. Also, evaluate the initial conditions on the incremental variables when $x(0) = 0$ and $y(0) = -0.5$.

* **9.23** A second-order nonlinear system having state variables x and y obeys the equations

$$\dot{x} = -2x + y^3$$
$$\dot{y} = x + 4 + \cos t$$

a) Find the operating-point values $\bar{x}$ and $\bar{y}$.
b) Find the linearized state-variable equations in numerical form.
c) Find the linearized model as an input-output equation relating $\hat{x}$ and its derivatives to the incremental input.

9.24 A nonlinear system with state variables x and y and input $u(t)$ obeys the equations

$$\dot{x} = -2|x|x - y + u(t) - 6$$
$$\dot{y} = x - y - 6$$

a) Verify that when $u(t) = 2 + B\cos 2t$, the operating point is $\bar{x} = 0.7808$, $\bar{y} = -5.2192$.
b) Evaluate the linearized model about this operating point.
c) Evaluate the initial conditions for the incremental variables when $x(0) = 1$ and $y(0) = -6$.

9.25 The nonlinear resistor in the circuit shown in Figure P9.25 obeys the element law $e_o = 2i^3$.
a) Verify that the nonlinear input-output equation is

$$0.5\frac{di}{dt} + 3i + 2i^3 = 22 + \hat{e}_i(t)$$

b) Find the operating-point values $\bar{i}$ and $\bar{e}_o$.
c) Derive the linearized model that is valid in the vicinity of this operating point.
d) Find the time constant of the linearized model.

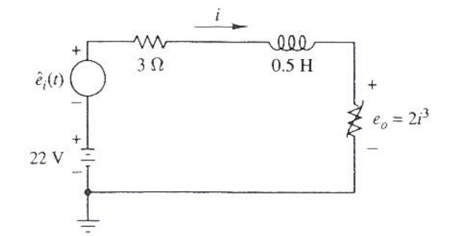

FIGURE P9.25

9.26 For the circuit shown in Figure 9.10(a), the nonlinear resistor is described by the equation $i_o = e_o/(1 + |e_o|)$.
a) Verify that the nonlinear state-variable equation is

$$\dot{e}_o = 2\left[-e_o - \frac{e_o}{1 + |e_o|} + e_i(t)\right]$$

b) If the operating point is defined by $\bar{e}_o = 2$ V, what must be $\bar{e}_i$, the nominal value of the input?
c) Find the linearized state-variable equation for the operating point defined in part (b).
d) Draw the linearized equivalent circuit.

* **9.27** The nonlinear resistor in the circuit shown in Figure P9.27 obeys the element law $e_o = 3|i_L|i_L$.

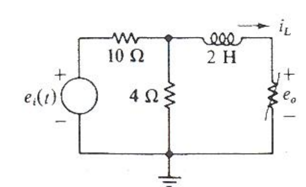

FIGURE P9.27

a) Verify that the nonlinear state-variable model is

$$\frac{di_L}{dt} = -\frac{10}{7}i_L - \frac{3}{2}|i_L|i_L + \frac{1}{7}e_i(t)$$

$$e_o = 3|i_L|i_L$$

b) Find the operating point and derive the linearized model when $e_i(t) = 5 + 0.4\cos t$ for $t > 0$.
c) Find the time constant of the linearized model.

*** 9.28** For the circuit shown in Figure P9.28, $i_o = |e_o|e_o$ and $e_i(t) = 4 + \hat{e}_i(t)$.
a) Show that the nonlinear input-output differential equation is

$$2\ddot{e}_o + 6\dot{e}_o + 2e_o + \frac{d}{dt}(|e_o|e_o) = 4\dot{e}_i + 2e_i(t)$$

b) Find the operating-point values $\bar{e}_o$ and $\bar{i}_o$.
c) Find the linearized input-output equation relating $\hat{e}_o$ and $\hat{e}_i(t)$.
d) Draw the linearized equivalent circuit.

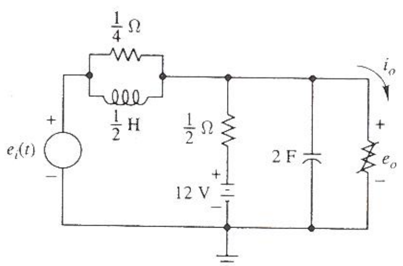

FIGURE P9.28

9.29 For the circuit shown in Figure P9.29, the element law for the nonlinear resistor is $i_2 = 2|e_C|e_C$.

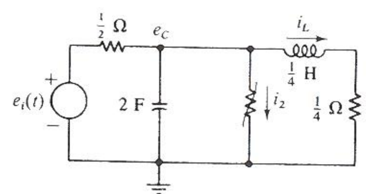

FIGURE P9.29

a) Verify that the circuit obeys the nonlinear state-variable equations

$$\dot{e}_C = -e_C - |e_C|e_C - \frac{1}{2}i_L + e_i(t)$$

$$\frac{di_L}{dt} = 4e_C - i_L$$

b) Determine the values of $\bar{e}_C$ and $\bar{i}_L$ when $e_i(t) = 10 + \hat{e}_i(t)$.
c) Derive the linearized state-variable equations that are valid in the vicinity of the operating point found in part (b).
d) Find $\hat{e}_C(0)$ and $\hat{i}_L(0)$ when $e_C(0) = 3$ V and $i_L(0) = 1$ A.

9.30 For the circuit shown in Figure P9.30, the element law for the nonlinear resistor is $e_2 = 0.5i_L^3$. The input voltage $e_i(t)$ is 4 V for $t \le 0$ and is $4 + \hat{e}_i(t)$ for $t > 0$.
a) Verify that the nonlinear state-variable model is

$$\dot{e}_C = -e_C + \frac{1}{2}i_L + \frac{1}{2}e_i(t)$$

$$\frac{di_L}{dt} = -2e_C - i_L^3 + 2e_i(t)$$

$$e_o = -e_C - \frac{1}{2}i_L^3 + e_i(t)$$

b) Find $e_C(0)$ and $i_L(0)$. Assume that steady-state conditions exist at $t = 0-$.
c) Find the values of the state variables at the operating point, and derive a set of linearized state-variable equations.
d) Find $\hat{e}_C(0)$ and $\hat{i}_L(0)$.

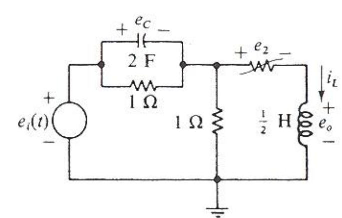

FIGURE P9.30

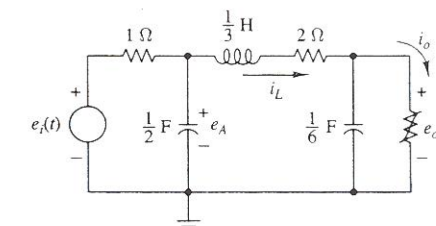

FIGURE P9.31

9.31 For the circuit shown in Figure P9.31, $i_o = \frac{1}{4}e_o^3$ and $e_i(t) = 8 + \hat{e}_i(t)$.
a) Verify that the state-variable equations are

$$\dot{e}_A = 2[-e_A - i_L + e_i(t)]$$

$$\dot{e}_o = 6\left[-\frac{1}{4}e_o^3 + i_L\right]$$

$$\frac{di_L}{dt} = 3[e_A - e_o - 2i_L]$$

b) Verify that the operating point is defined by $\bar{e}_o = 2$ V.
c) Find the linearized state-variable equations for the operating point in part (b) and draw the linearized equivalent circuit.