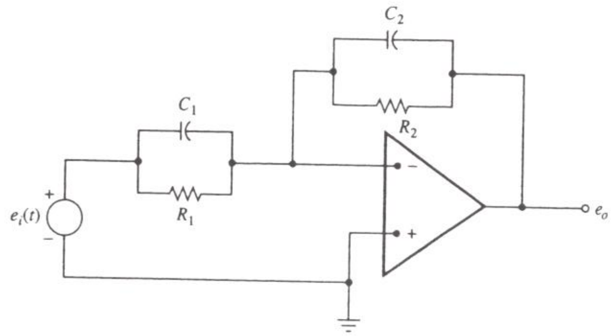


**FIGURE P5.37**

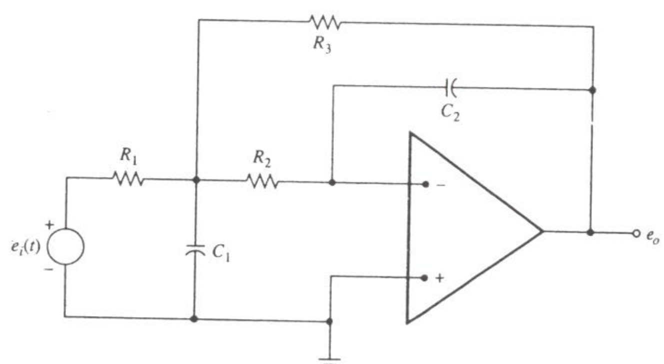


**FIGURE P5.38**

CHAPTER 6

# ANALYTICAL SOLUTION OF LINEAR MODELS

The first several chapters of this book presented methods for formulating the mathematical model for a system and illustrated them with many mechanical and electrical examples. The same techniques will be used for other types of systems in some of the later chapters. The mathematical model can be solved analytically or with the aid of a digital computer. Analytical solutions, which in general are feasible only for fixed linear models, are examined in this chapter.

Section 6.1 summarizes the basic procedures that will be needed in later sections for solving differential equations. For many readers, this will be a review. Throughout the chapter, we assume that we have fixed linear differential equations with real coefficients. Furthermore, the methods we describe are not sufficient for all possible inputs, though they are satisfactory for the inputs that are commonly encountered. Proofs and detailed justifications are omitted, but some general references are included in Appendix D. In the rest of the chapter, we consider in detail the solution of first- and second-order models, examine some specific inputs that are useful in system analysis, and define a number of important terms. Some comments are also made about systems of arbitrary order. In the final section, we present methods of solving models that are in state-variable form, using a number of techniques from linear algebra.

## ■ 6.1 THE COMPLETE SOLUTION OF DIFFERENTIAL EQUATIONS

We assume that the system model has been put into input-output form with all other variables eliminated, as discussed in Section 3.2. The model for an $n$th-order system with input $u(t)$ and output $y(t)$ can be written, as in (3.25), as

$$a_n y^{(n)} + \cdots + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = b_m u^{(m)} + \cdots + b_1 \dot{u} + b_0 u(t) \tag{1}$$

where $y^{(n)} = d^n y/dt^n$, etc. The collection of terms on the right side of (1), which involves the input and its derivatives, is represented by

$$F(t) = b_m u^{(m)} + \cdots + b_1 \dot{u} + b_0 u(t) \tag{2}$$

where $F(t)$ is called the **forcing function**. With this definition, we may rewrite (1) as

$$a_n y^{(n)} + \cdots + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = F(t) \tag{3}$$

The desired solution, $y(t)$ for $t \geq 0$, must satisfy the differential equation for $t \geq 0$ and also $n$ specified initial conditions, which are usually $y(0)$, $\dot{y}(0), \ldots, y^{(n-1)}(0)$.

## Homogeneous Differential Equations

If $F(t)$ is replaced by zero, (3) reduces to the **homogeneous differential equation**

$$a_n y_H^{(n)} + \cdots + a_2 \ddot{y}_H + a_1 \dot{y}_H + a_0 y_H = 0 \tag{4}$$

where the subscript $H$ has been added to emphasize that $y_H(t)$ is the solution to the homogeneous equation. Assume that a solution of (4) has the form

$$y_H(t) = K\epsilon^{rt}$$

where we must determine the constant $r$ such that (4) is satisfied. Multiplying $y_H$ by a constant just multiplies the entire left side of (4) by that constant, so any nonzero value will be satisfactory for the constant $K$. Substituting the assumed solution into (4) gives

$$(a_n r^n + \cdots + a_2 r^2 + a_1 r + a_0)K\epsilon^{rt} = 0$$

Because $K\epsilon^{rt} \neq 0$ for a nontrivial solution, we must have

$$a_n r^n + \cdots + a_2 r^2 + a_1 r + a_0 = 0 \tag{5}$$

which is called the **characteristic equation**. Note that the coefficients in this algebraic equation are identical to those on the left side of the differential equation, so we can write (5) by inspection. An $n$th-order algebraic equation has $n$ roots, which we denote by $r_1, r_2, \ldots, r_n$. Not only are $K_1\epsilon^{r_1 t}$, $K_2\epsilon^{r_2 t}$, $\ldots$, $K_n\epsilon^{r_n t}$ all solutions of (4) for arbitrary $K_i$, but

$$y_H(t) = K_1\epsilon^{r_1 t} + K_2\epsilon^{r_2 t} + \cdots + K_n\epsilon^{r_n t} \tag{6}$$

is a solution for any values of the arbitrary constants $K_1$ through $K_n$. If all the roots of the characteristic equation are different, then (6) is the most general solution.



If two or more of the characteristic roots are identical—that is, if there is a repeated root of (5)—then we must modify the form of (6) to contain $n$ independent terms and to represent the most general solution. If $r_1 = r_2$ with the remaining roots being distinct, the solution is

$$y_H(t) = K_1\epsilon^{r_1 t} + K_2 t\epsilon^{r_1 t} + K_3\epsilon^{r_3 t} + \cdots + K_n\epsilon^{r_n t}$$

If $r_1 = r_2 = r_3$, the most general solution to the homogeneous differential equation is

$$y_H(t) = (K_1 + K_2 t + K_3 t^2)\epsilon^{r_1 t} + K_4\epsilon^{r_4 t} + \cdots + K_n\epsilon^{r_n t}$$

If any of the roots of the characteristic equation are complex numbers, they must occur in complex conjugate pairs because the coefficients in the characteristic equation are real. Suppose that $r_1 = \alpha + j\beta$ and $r_2 = \alpha - j\beta$, where $j = \sqrt{-1}$,[1] and that the characteristic equation has no other roots. Then

$$y_H(t) = K_1\epsilon^{(\alpha+j\beta)t} + K_2\epsilon^{(\alpha-j\beta)t}$$

However, because the solution is a real function of time, we should rewrite this equation by using suitable trigonometric identities. Factoring out $\epsilon^{\alpha t}$ from the right side, then using the first two entries in Table 6.1 with $\theta = \beta t$, and finally collecting like terms, we have

$$\begin{aligned} y_H(t) &= \epsilon^{\alpha t}[K_1\epsilon^{j\beta t} + K_2\epsilon^{-j\beta t}] \\ &= \epsilon^{\alpha t}[K_1(\cos\beta t + j\sin\beta t) + K_2(\cos\beta t - j\sin\beta t)] \\ &= \epsilon^{\alpha t}[(K_1 + K_2)\cos\beta t + j(K_1 - K_2)\sin\beta t] \end{aligned}$$

With $K_3 = K_1 + K_2$ and $K_4 = j(K_1 - K_2)$, the equation becomes

$$y_H(t) = \epsilon^{\alpha t}(K_3\cos\beta t + K_4\sin\beta t) \tag{7}$$

By using the last entry in Table 6.1, we can rewrite (7) in the alternative form

$$y_H(t) = K\epsilon^{\alpha t}\cos(\beta t + \phi) \tag{8}$$

where the constants $K$ and $\phi$ depend on $K_3$ and $K_4$. In these expressions involving complex roots, $K_1$ and $K_2$ are complex numbers, with $K_2$ the complex conjugate of $K_1$; but $K_3$, $K_4$, $K$, and $\phi$ are real constants. Either of the two forms given by (7) and (8) should be used.

We treat repeated complex characteristic roots much like repeated real roots. For a fourth-order characteristic equation with $r_1 = r_3 = \alpha + j\beta$ and $r_2 = r_4 = \alpha - j\beta$, the most general solution to the homogeneous differential equation has the form

$$y_H(t) = K_A\epsilon^{\alpha t}\cos(\beta t + \phi_A) + K_B t\epsilon^{\alpha t}\cos(\beta t + \phi_B)$$

[1] The symbol $i$ is also used for $\sqrt{-1}$, but we reserve $i$ for electrical current.

**TABLE 6.1 Useful Trigonometric Identities**

$$\epsilon^{j\theta} = \cos\theta + j\sin\theta$$
$$\epsilon^{-j\theta} = \cos\theta - j\sin\theta$$
$$\cos\theta = \tfrac{1}{2}(\epsilon^{j\theta} + \epsilon^{-j\theta})$$
$$\sin\theta = \frac{1}{2j}(\epsilon^{j\theta} - \epsilon^{-j\theta})$$
$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$
$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$
$$A\cos\theta + B\sin\theta = \sqrt{A^2+B^2}\sin\left(\theta + \tan^{-1}\frac{A}{B}\right)$$
$$A\cos\theta + B\sin\theta = \sqrt{A^2+B^2}\cos\left(\theta - \tan^{-1}\frac{B}{A}\right)$$

## Nonhomogeneous Differential Equations

We now consider the nonhomogeneous differential equation, where the forcing function $F(t)$ in (3) is nonzero. For mathematical convenience, we express the solution as the sum of two parts, namely

$$y(t) = y_H(t) + y_P(t) \tag{9}$$

where $y_H(t)$ and $y_P(t)$ are known as the complementary and the particular solutions, respectively. The **complementary solution** $y_H(t)$ is the solution to the homogeneous differential equation in (4); examples are given by (6) through (8). The **particular solution** $y_P(t)$ must satisfy the entire original differential equation, so

$$a_n y_P^{(n)} + \cdots + a_2\ddot{y}_P + a_1\dot{y}_P + a_0 y_P = F(t) \tag{10}$$

A general procedure for finding $y_P(t)$ is the variation-of-parameters method. When the forcing function $F(t)$ has only a finite number of different derivatives, however, the method of undetermined coefficients is satisfactory. Some examples of functions possessing only a finite number of independent derivatives are given in the left-hand column of Table 6.2. Others include $\cosh\omega t$ and $t^2\epsilon^{\alpha t}\cos(\omega t + \phi)$. An example of a forcing function that does *not* fall into this category is $1/t$.

In using the **method of undetermined coefficients**, we normally assume that the form of $y_P(t)$ consists of terms similar to those in $F(t)$ and its derivatives, with each term multiplied by a constant to be determined. We find the values of these constants by substituting the assumed $y_P(t)$ into the differential equation and then equating corresponding coefficients. The particular solution to be assumed for some common forcing functions is shown in Table 6.2.



**TABLE 6.2 Usual Form of the Particular Solution**

| $F(t)$ | $y_P(t)$ |
|---|---|
| $\alpha$ | $A$ |
| $\alpha_1 t + \alpha_0$ | $At + B$ |
| $\epsilon^{\alpha t}$ | $A\epsilon^{\alpha t}$ |
| $\cos\omega t$ | $A\cos\omega t + B\sin\omega t$ |
| $\sin\omega t$ | $A\cos\omega t + B\sin\omega t$ |

If $F(t)$ or one of its derivatives contains a term identical to a term in $y_H(t)$, the corresponding terms in the right-hand column of Table 6.2 should be multiplied by $t$. Thus if $y_H(t) = K_1\epsilon^{-t} + K_2\epsilon^{-2t}$ and $F(t) = 3\epsilon^{-t}$, then $y_P(t) = At\epsilon^{-t}$ should be used. If a term in $F(t)$ corresponds to a double root of the characteristic equation, the normal form for $y_P(t)$ is multiplied by $t^2$. If, for example, $y_H(t) = K_1\epsilon^{-t} + K_2 t\epsilon^{-t}$ and $F(t) = \epsilon^{-t}$, then $y_P(t) = At^2\epsilon^{-t}$.

In the **general solution** $y(t) = y_H(t) + y_P(t)$ for an $n$th-order nonhomogeneous differential equation, the complementary solution $y_H(t)$ contains $n$ arbitrary constants, which we earlier denoted by $K_1$ through $K_n$. The original differential equation in (3) will be satisfied regardless of the values of these constants, so their evaluation requires $n$ separately specified initial conditions: $y(0)$, $\dot{y}(0), \ldots, y^{(n-1)}(0)$. Because these initial values are associated with the entire solution and not just with the complementary solution, we cannot evaluate the arbitrary constants until we have found both $y_H(t)$ and $y_P(t)$.

▶ ***EXAMPLE 6.1***

If $\dot{y} + 2y = F(t)$ and $y(0) = 2$, find $y(t)$ for $t \geq 0$ for each of the following forcing functions:

1. $F(t) = 10$
2. $F(t) = 10\cos 2t$
3. $F(t) = 10 + 10\cos 2t$

**Solution**

By inspection of the left-hand side of the differential equation, we find that the characteristic equation is $r + 2 = 0$, which has a single root at $r = -2$. Thus the complementary solution has the form

$$y_H(t) = K\epsilon^{-2t}$$

Because the value of $K$ depends on $F(t)$ as well as $y(0)$, we shall use subscripts on $K$ to distinguish among the three cases.

To find the particular solution for case 1, wherein $F(t) = 10$, we assume that $y_P(t) = A$. Substituting these expressions into the original differential equation gives $0 + 2A = 10$, so $A = 5$, and for $t \geq 0$,

$$\begin{aligned} y(t) &= y_H(t) + y_P(t) \\ &= K_1\epsilon^{-2t} + 5 \end{aligned}$$

We replace $t$ by zero in the last equation and use $y(0) = 2$ to obtain

$$2 = K_1 + 5$$

Thus $K_1 = -3$, and the response for $t \geq 0$ to $F(t) = 10$ is

$$y(t) = -3\epsilon^{-2t} + 5$$

In case 2, the forcing function is $F(t) = 10\cos 2t$, so we assume that $y_P(t) = A\cos 2t + B\sin 2t$. Substituting these expressions into $\dot{y}_P + 2y_P = F(t)$, we have

$$(-2A\sin 2t + 2B\cos 2t) + 2(A\cos 2t + B\sin 2t) = 10\cos 2t$$

or, collecting like terms,

$$(2A + 2B - 10)\cos 2t + (-2A + 2B)\sin 2t = 0$$

Because this must be an identity for all $t \geq 0$, we require that $2A+2B-10 = 0$ and that $-2A + 2B = 0$. Solving these equations gives $A = B = \frac{5}{2}$, so

$$y(t) = K_2\epsilon^{-2t} + \frac{5}{2}(\cos 2t + \sin 2t)$$

Replacing $t$ by zero and $y(t)$ by 2, we have $2 = K_2 + \frac{5}{2}$. Thus $K_2 = -\frac{1}{2}$, and for all $t \geq 0$ the solution is

$$y(t) = -\frac{1}{2}\epsilon^{-2t} + \frac{5}{2}(\cos 2t + \sin 2t)$$

Finally, for case 3, $F(t) = 10 + 10\cos 2t$ and the particular solution is the sum of the particular solutions for the individual terms in $F(t)$. Thus

$$y(t) = K_3\epsilon^{-2t} + 5 + \frac{5}{2}(\cos 2t + \sin 2t)$$

For the right side to reduce to 2 when $t$ is replaced by zero, we require that $K_3 = -\frac{11}{2}$. Hence for $t \geq 0$,

$$y(t) = -\frac{11}{2}\epsilon^{-2t} + 5 + \frac{5}{2}(\cos 2t + \sin 2t)$$

Although $y_P(t)$ for case 3 is the sum of the particular solutions for the first two cases, note that $K_3 \neq K_1 + K_2$. This is because the constants in $y_H(t)$ cannot be evaluated until after the *entire* particular solution has been found.



▶ ***EXAMPLE 6.2***

Find the solution to the differential equation $\ddot{y} + 2\dot{y} + 5y = t$ for $t \geq 0$ if $y(0) = 0$ and $\dot{y}(0) = 2$.

**Solution**

The characteristic equation is $r^2 + 2r + 5 = 0$, which by the quadratic formula has roots at

$$r = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm j2$$

As in (7), we can write the complementary solution as

$$y_H(t) = \epsilon^{-t}[K_1\cos 2t + K_2\sin 2t]$$

The particular solution is assumed to have the form $y_P(t) = At + B$. Substituting this expression into the differential equation gives

$$0 + 2A + 5(At + B) = t$$

or

$$(5A - 1)t + (2A + 5B) = 0$$

which requires that $5A = 1$ and $2A + 5B = 0$. Thus $A = \frac{1}{5}$, $B = -\frac{2}{25}$, and the general solution for $t \geq 0$ is

$$y(t) = \epsilon^{-t}[K_1\cos 2t + K_2\sin 2t] + \frac{1}{5}t - \frac{2}{25}$$

The derivative of this general solution is

$$\dot{y}(t) = \epsilon^{-t}[(2K_2 - K_1)\cos 2t - (K_2 + 2K_1)\sin 2t] + \frac{1}{5}$$

At $t = 0$ and with the given initial conditions, the last two equations reduce to

$$\begin{aligned} 0 &= K_1 - \frac{2}{25} \\ 2 &= 2K_2 - K_1 + \frac{1}{5} \end{aligned}$$

from which $K_1 = \frac{2}{25}$ and $K_2 = \frac{47}{50}$. For $t \geq 0$,

$$y(t) = \epsilon^{-t}\left[\frac{2}{25}\cos 2t + \frac{47}{50}\sin 2t\right] + \frac{1}{5}t - \frac{2}{25}$$

## 6.2 FIRST-ORDER SYSTEMS

In this section, we consider the solution of first-order systems, which usually consist of one energy-storing element and any number of dissipative elements. We also introduce some definitions that will be used throughout the rest of the book. Although they are illustrated in the context of first-order systems, we shall define the terms in a general way to make their extension to systems of arbitrary order obvious.

A first-order system is described by a single differential equation of the form

$$\dot{y} + \frac{1}{\tau} y = F(t) \tag{11}$$

where $\tau$ is a real, nonzero constant. To find the response for $t \geq 0$, we must know the forcing function $F(t)$ for all positive values of time and the initial condition $y(0)$. Because the characteristic equation $r + (1/\tau) = 0$ has one root at $r = -1/\tau$, the complementary solution is

$$y_H(t) = K\epsilon^{-t/\tau} \tag{12}$$

### Stability

Typical curves for the complementary solution are shown in Figure 6.1. If $y_H(t)$ decays to zero as $t$ approaches infinity, the system is said to be **stable**. If, on the other hand, $y_H(t)$ increases without limit as $t$ becomes large, the system is **unstable**. A first-order system is stable if $\tau > 0$ and unstable if $\tau < 0$. If the magnitude of $\tau$ approaches infinity, $y_H(t)$ becomes constant, as shown in Figure 6.1(c). Such a system is said to be **marginally stable**.

### Parts of the Complete Solution

A variety of names are used to identify the terms in the response of a fixed linear system, and these names can be easily confused. We shall explain the general terminology and relate the new terms to the complementary and particular solutions.

We have seen that we can find the form of the complementary solution by letting $F(t) = 0$ and solving for the roots of the characteristic equation. Thus the form of $y_H(t)$ does not depend on the input, but only on the system elements and their interconnections. The form of $y_H(t)$ represents the natural behavior of the system when the external input is removed and the system is excited by some initial stored energy. Thus $y_H(t)$ is also called the **free response**.

The size of the terms in the free response is given by the arbitrary constants, such as $K$ in (12). In Section 6.1, we saw that the values of such constants depend on $y_P(t)$ and hence on the input. The constants also



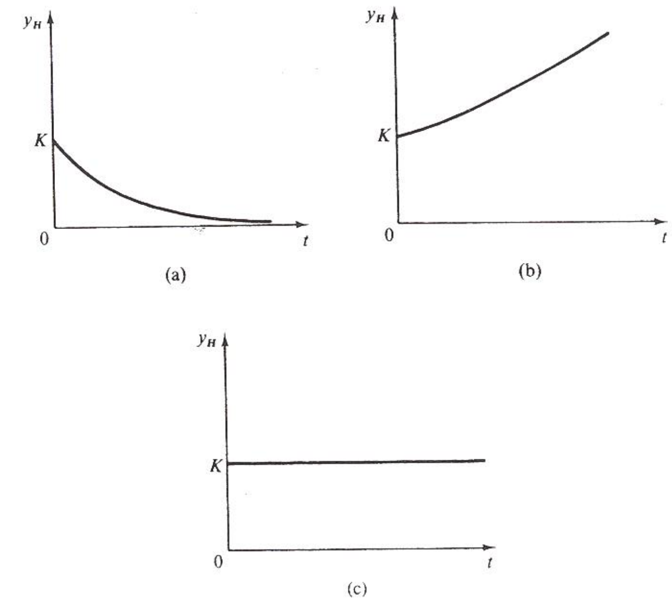


**FIGURE 6.1** The complementary solution for a first-order system. (a) $\tau > 0$. (b) $\tau < 0$. (c) $\tau$ infinite.

depend on the initial conditions, which in turn represent the effect of the history of the system prior to the initial instant.

Studying the method of undetermined coefficients, we saw that the particular solution depends on the form of the forcing function. For this reason, $y_P(t)$ is called the **forced response**. In summary, the free and forced responses are identical to the complementary and particular solutions, respectively.

An alternative way of dividing the total response into two parts is to regard it as the sum of the zero-state response and the zero-input response. The **zero-state response** $y_{zs}(t)$ is the complete response to the input when the initial values of the state variables are zero. It includes both the particular solution and the complementary solution, where the arbitrary constants in the complementary solution are found for initial conditions that correspond to zero initial values for the state variables. For a first-order system, we can write

$$y_{zs}(t) = y_P(t) + K_{zs}\epsilon^{-t/\tau}$$

where $K_{zs}$ is the arbitrary constant evaluated for zero-state conditions.

The **zero-input response** $y_{zi}(t)$ corresponds to specified initial values of the state variables and a forcing function of zero. Hence it consists of the complementary solution but not the particular solution, and for a first-order system we can denote it by

$$y_{zi}(t) = K_{zi}\epsilon^{-t/\tau}$$

where $K_{zi}$ is the arbitrary constant evaluated for zero input. Note that although both $y_{zs}$ and $y_{zi}$ contain a term of the form $\epsilon^{-t/\tau}$, the constants in front of the terms are not the same.

A linear system with a specified input and specified initial values for its state variables has a response that we can find by superposition of the zero-state and zero-input responses. Thus

$$y(t) = y_{zs}(t) + y_{zi}(t)$$

Combining these expressions for a first-order system gives

$$y(t) = y_P(t) + (K_{zs} + K_{zi})\epsilon^{-t/\tau}$$

By comparing this equation to (9), we see that the two terms $K_{zs}\epsilon^{-t/\tau}$ and $K_{zi}\epsilon^{-t/\tau}$ combine to give the free response $y_H(t)$.

It is often convenient to decompose the total response into transient and steady-state components. The **transient response** $y_t(t)$ consists of those terms that decay to zero as $t$ approaches infinity. The **steady-state response** $y_{ss}(t)$ is the part of the solution that remains after the transient terms have disappeared. For a first-order system, $y_H(t) = K\epsilon^{-t/\tau}$ will be part of the transient response if $\tau$ is positive—that is, if the system is stable. If $y_P(t)$ has a form such as $A$, $A\cos\omega t + B\sin\omega t$, or $At + B$, then $y_P(t)$ will constitute the steady-state response. On the other hand, if $y_P(t) = At\epsilon^{-t} + B\epsilon^{-t}$, then $y_P(t)$ will be part of the transient response. For systems that are stable and for which the terms in $y_P$ do not decay to zero, the transient and steady-state components are identical to the free and the forced responses, respectively.

### The Complete Response to a Constant Input

If $F(t)$ has a constant value of $A$, then the forced response to the first-order differential equation

$$\dot{y} + \frac{1}{\tau}y = F(t)$$

is $y_P(t) = \tau A$, as discussed in Example 6.1. Assume that $\tau > 0$, so that the free response

$$y_H(t) = K\epsilon^{-t/\tau}$$

decays to zero. Then $y_P(t) = y_{ss}$ and we can write the complete solution as

$$y(t) = y_{ss} + K\epsilon^{-t/\tau} \tag{13}$$

If $y(0)$ is known, we can evaluate this solution at $t = 0$, giving $y(0) = y_{ss} + K$ or $K = y(0) - y_{ss}$. Thus for $t \geq 0$,

$$y(t) = y_{ss} + [y(0) - y_{ss}]\epsilon^{-t/\tau} \tag{14}$$



which is shown in Figure 6.2(a). Special cases of this result are shown in Figure 6.2(b) and Figure 6.2(c).

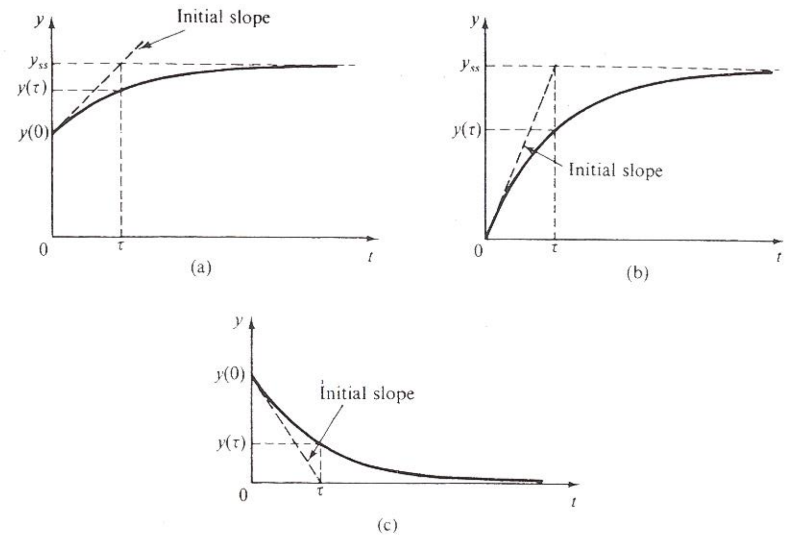


**FIGURE 6.2** Response of a stable first-order system to a constant input. (a) $y_{ss}$ and $y(0)$ both nonzero. (b) $y(0) = 0$. (c) $y_{ss} = 0$.

To give significance to the value of $y(\tau)$ shown in Figure 6.2(a), first note from (14) that, for $t = \tau$, we have

$$y(\tau) = y_{ss} + [y(0) - y_{ss}]\epsilon^{-1}$$

Because $\epsilon^{-1} = 0.3679$,

$$y(\tau) = y(0) + 0.6321[y_{ss} - y(0)]$$

Thus after $\tau$ seconds, the response to a constant input is approximately 63% of the way from the initial value to the steady-state value. Because $\epsilon^{-4} = 0.0183$, the response after $4\tau$ seconds is approximately 98% of the way from the initial value to the steady-state value. We see that the system parameter $\tau$, which is called the **time constant**, is a measure of the system's speed of response. Another interpretation of the time constant results from considering the initial slope of the response. Differentiation of (14) gives

$$\dot{y}(t) = \frac{1}{\tau}[y_{ss} - y(0)]\epsilon^{-t/\tau}$$

and setting $t = 0$ yields

$$y_{ss} = y(0) + \tau\dot{y}(0)$$

Thus if the slope of the response curve were maintained at its initial value of $\dot{y}(0)$, it would take $\tau$ seconds, instead of an infinite time, for the response to reach its steady-state value.

### EXAMPLE 6.3

Find the response of each of the first-order systems shown in Figure 6.3. The systems are at rest with no stored energy at $t = 0$, and for $t > 0$ the respective inputs are $f_a(t) = A$ and $\omega_a(t) = A$. In Figure 6.3(a), the output $x$ is the elongation of the spring from its unstretched position. The output for Figure 6.3(b) is the angular velocity $\omega_1$.

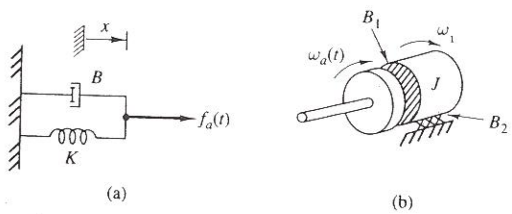


FIGURE 6.3 First-order mechanical systems for Example 6.3. (a) Translational. (b) Rotational.

**Solution**

The translational system shown in Figure 6.3(a) is described by the differential equation

$$B\dot{x} + Kx = f_a(t)$$

or

$$\dot{x} + \frac{K}{B}x = \frac{1}{B}f_a(t)$$

Comparing this equation to (11), we see that the time constant is

$$\tau = \frac{B}{K} \tag{15}$$

Note that if the friction coefficient $B$ is increased, $\tau$ increases and the system takes a longer time to approach its steady-state value. Because $f_a(t) = A$ for all positive values of time, $x_P = x_{ss} = A/K$. Although we can get this result by setting the derivative in the differential equation equal to zero, we can also see this by looking at Figure 6.3(a). When the applied force is a constant, the right ends of $B$ and $K$ will eventually become stationary. Then no force will be exerted by the friction element. The only steady-state forces will be those exerted by the input and the spring, so $A = Kx_{ss}$. Finally, because there is no initial energy stored in the spring, $x(0) = 0$. Thus by



(14), the complete response is

$$x = \frac{A}{K}(1 - \epsilon^{-t/\tau}) \tag{16}$$

where $\tau$ is given by (15).

For the rotational system shown in Figure 6.3(b),

$$J\dot{\omega}_1 + (B_1 + B_2)\omega_1 = B_1\omega_a(t)$$

or

$$\dot{\omega}_1 + \frac{B_1 + B_2}{J}\omega_1 = \frac{B_1}{J}\omega_a(t)$$

The time constant is

$$\tau = \frac{J}{B_1 + B_2} \tag{17}$$

With $\omega_a(t) = A$ for all positive time, $(\omega_1)_{ss} = AB_1/(B_1 + B_2)$. Because $\omega_1(0) = 0$,

$$\omega_1 = \frac{AB_1}{B_1 + B_2}(1 - \epsilon^{-t/\tau}) \tag{18}$$

As should be expected for the rotational system, the time needed to approach steady-state conditions is directly proportional to the inertia of the disk. However, once the disk reaches a constant speed, the only torques on it are those from the friction elements. Thus the steady-state angular velocity is determined by the relative sizes of $B_1$ and $B_2$.

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An electrical circuit consisting of one capacitor or one inductor, together with any number of resistors and sources, can also be described by a first-order differential equation. If the sources are constant for $t > 0$, the response of any output voltage or current will have the form shown in (14). The steady-state response $y_{ss}$ and the time constant $\tau$ can be determined from inspection of the circuit's differential equation. The quantity $y(0)$ will depend on the energy contained in the circuit at $t = 0$ and on the input.

It is possible for the output to undergo, at $t = 0$, an instantaneous change caused by the sudden application of a source or the opening or closing of a switch. An output having a discontinuity at $t = 0$ is relatively rare in mechanical systems but fairly common in electrical ones. Recall that the energy stored in a passive element cannot change instantaneously unless there is an infinitely large power flow. Hence, as long as all voltages and currents remain finite, neither the voltage across a capacitor nor the current through an inductor can change instantaneously.

Other electrical variables, however, are not directly related to the stored energy and thus can have discontinuities. Included in this category are resistor voltages and currents, capacitor currents, and inductor voltages. When we must consider functions that have a discontinuity at $t = 0$, it is customary to use the notations $y(0-)$ and $y(0+)$ for the limiting values of $y(t)$ as $t$ approaches zero through negative and positive values, respectively. The function $y(t)$ shown in Figure 6.4 has a discontinuity at $t = 0$, where it jumps instantaneously from $y(0-)$ to $y(0+)$. This is typical for the responses of first-order circuits. To account for the possibility of discontinuous behavior, we modify (14) slightly to read

$$y(t) = y_{ss} + [y(0+) - y_{ss}]\epsilon^{-t/\tau} \tag{19}$$

This change is consistent with the fact that in computing the response for $t > 0$, we must use $y(0+)$, the value of $y(t)$ immediately following any discontinuity.

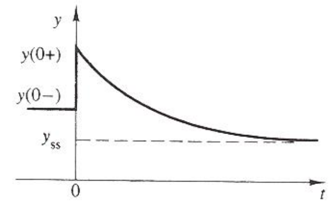


**FIGURE 6.4** Typical response of a first-order circuit.

### ▶ *EXAMPLE 6.4*

The voltage source shown in Figure 6.5(a) is zero for $t < 0$ but has a constant value of $A$ for all $t > 0$. Find and sketch the output voltage $e_o$.

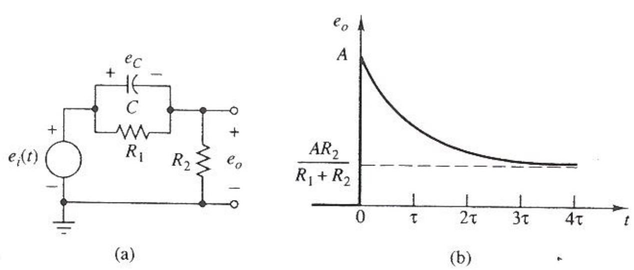


**FIGURE 6.5** First-order circuit for Example 6.4. (a) Circuit diagram. (b) Zero-state response when $e_i(t) = A$ for $t > 0$.

**Solution**

Setting the sum of the currents that are leaving the upper right-hand node equal to zero, we have



$$C(\dot{e}_o - \dot{e}_i) + \frac{1}{R_1}[e_o - e_i(t)] + \frac{1}{R_2}e_o = 0$$

which gives the first-order input-output equation

$$C\dot{e}_o + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)e_o = C\dot{e}_i + \frac{1}{R_1}e_i(t) \tag{20}$$

In order to obtain an expression for $e_o(t)$ for $t > 0$, we first evaluate the terms in the forcing function on the right-hand side of this equation for $t > 0$. The general solution will contain one arbitrary constant. To evaluate this constant, we will need $e_o(0+)$, the value of the output at the beginning of the interval $0 < t < \infty$.

Because $e_i(t) = A$ and $\dot{e}_i = 0$ for all $t > 0$, (20) reduces to

$$C\dot{e}_o + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)e_o = \frac{A}{R_1} \quad \text{for } t > 0$$

or

$$\dot{e}_o + \left(\frac{R_1 + R_2}{R_1 R_2 C}\right)e_o = \frac{A}{R_1 C} \quad \text{for } t > 0 \tag{21}$$

By inspection of this equation, we see that

$$\tau = \frac{R_1 R_2 C}{R_1 + R_2} \tag{22a}$$

$$(e_o)_{ss} = \frac{R_2}{R_1 + R_2}A \tag{22b}$$

Then by (19),

$$e_o(t) = \frac{R_2 A}{R_1 + R_2} + K\epsilon^{-t/\tau} \quad \text{for } t > 0 \tag{23}$$

where

$$K = e_o(0+) - (e_o)_{ss} \tag{24}$$

Because $e_o$ is the voltage across a resistor, it can change instantaneously when $e_i(t)$ changes from 0 to $A$ at $t = 0$. However, the voltage across the capacitor, denoted by $e_C$ in Figure 6.5(a), must be continuous at $t = 0$ because it is related to the energy stored in the capacitor, which cannot change instantaneously. By applying Kirchhoff's voltage law to a loop consisting of the voltage source, the capacitor, and the resistor $R_2$, we have

$$e_i(t) - e_C(t) - e_o(t) = 0 \tag{25}$$

In order to find $e_o(0+)$, we apply (25) at $t = 0+$. We know that $e_i(0+) = A$. Because the capacitor contains no energy at $t = 0-$ and because its energy cannot change instantaneously, $e_C(0+) = e_C(0-) = 0$. Thus for $t = 0+$, (25) gives

$$e_o(0+) = e_i(0+) - e_C(0+) = A \tag{26}$$

Substituting (22b) and (26) into (24), we find that the complete response for $t > 0$ is

$$e_o(t) = \frac{A}{R_1 + R_2}(R_2 + R_1\epsilon^{-t/\tau})$$

where $\tau$ is given by (22a). The response is sketched in Figure 6.5(b).

Although the steady-state response to a constant input can always be obtained from the circuit's differential equation, it can also be found directly from the circuit diagram. For a constant input, the steady-state values of all currents and voltages are constants. Under these conditions, $di_L/dt = 0$ and $de_C/dt = 0$. Then from the respective element laws, we see that the voltage across any inductor and the current through any capacitor must be zero. Hence in the steady state, an inductor acts like a short circuit, and a capacitor becomes an open circuit.

When finding the steady-state response to a constant input, we may redraw the circuit with any inductor replaced by a short circuit and any capacitor replaced by an open circuit. The original circuit then reduces to a purely resistive one. In the case of Figure 6.5(a), we can draw the steady-state equivalent circuit shown in Figure 6.6. The resistances $R_1$ and $R_2$ now form a series combination directly across the source, so we can use the voltage-divider rule to write

$$(e_o)_{ss} = \frac{R_2}{R_1 + R_2}A$$

which agrees with (22b).

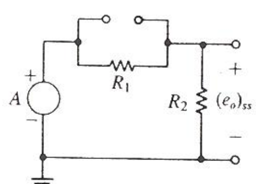


**FIGURE 6.6** Steady-state equivalent circuit for Figure 6.5(a).

In the final first-order example, a source branch is disconnected from the rest of the circuit. We shall note that the time constant changes when the branch is removed. We shall replace the original circuit by a resistive one in order to calculate the voltage across the capacitor just before the source is disconnected. Because the capacitor voltage cannot change instantaneously, we know that $e_C(0+) = e_C(0-)$. Once again, however, the output voltage has a discontinuity at $t = 0$.



### ▶ *EXAMPLE 6.5*

Find the time constant for the circuit shown in Figure 6.7(a) when the switch is closed and when the switch is open. Let the source voltage have a constant value of 24 volts. Assume that the switch has been closed for a long time for $t < 0$, so that the circuit is in the steady state at $t = 0-$. If the switch opens at $t = 0$, find and sketch $e_o$ versus $t$ for $t > 0$.

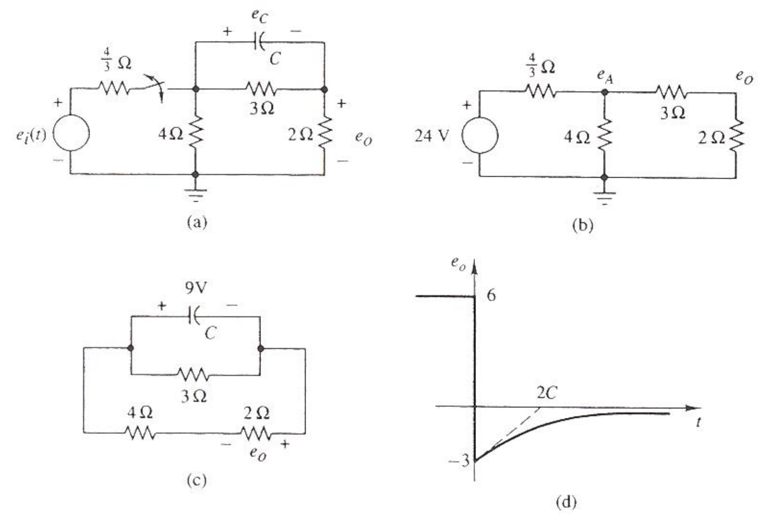


**FIGURE 6.7** (a) Circuit for Example 6.5. (b) Circuit at $t = 0-$. (c) Circuit at $t = 0+$. (d) Sketch of $e_o$ versus $t$.

**Solution**

From Example 5.5, the input-output equation with the switch closed is

$$\dot{e}_o + \frac{2}{3C}e_o = \frac{1}{2}\dot{e}_i + \frac{1}{6C}e_i(t)$$

so the time constant is $\tau = 3C/2$. Again from Example 5.5, when the switch is open,

$$\dot{e}_o + \frac{1}{2C}e_o = 0 \qquad (27)$$

which corresponds to a time constant $\tau = 2C$.

To find the response for $t > 0$ with the switch open, we first note that the general solution to (27) is

$$e_o(t) = K\epsilon^{-t/2C} \qquad (28)$$

As the first step in determining the initial condition $e_o(0+)$, we shall find the capacitor voltage at $t = 0-$, when the circuit is in the steady state with the switch closed. Replacing the capacitor by an open circuit, we obtain Figure 6.7(b). The parallel combination of 4 Ω and 5 Ω is 20/9 Ω. By the voltage-divider rule,

$$e_A(0-) = \frac{20/9}{4/3 + 20/9}(24) = 15 \text{ V}$$

$$e_o(0-) = \frac{2}{3+2} e_A(0-) = 6 \text{ V}$$

so

$$e_C(0-) = e_A(0-) - e_o(0-) = 9 \text{ V}$$

which must also be the value of $e_C(0+)$.

The circuit at $t = 0+$ is shown in Figure 6.7(c), with $e_C(0+) = 9$V, with the source branch removed, and with the 4-Ω and 2-Ω resistors positioned slightly differently for increased clarity. Again using the voltage-divider rule, we see that

$$e_o(0+) = \frac{2}{4+2}(-9) = -3 \text{ V}$$

With this initial condition, we can rewrite (28) as

$$e_o(t) = -3\epsilon^{-t/2C} \quad \text{for } t > 0$$

which is sketched in part (d) of the figure. Note the discontinuity in the output voltage at $t = 0$.

## 6.3 THE STEP FUNCTION AND IMPULSE

Two of the most important inputs we encounter in the analysis of dynamic systems are the unit step function and the unit impulse.

### The Unit Step Function

One frequently encounters inputs that are zero before some reference time and that have a nonzero constant value thereafter. To treat such inputs mathematically, we define the **unit step function**, which is denoted by $U(t)$. This function is defined to be zero for $t \leq 0$ and unity for $t > 0$; it is shown in Figure 6.8(a).[2] If the step discontinuity occurs at some later time $t_1$, as shown in Figure 6.8(b), the function is defined by

$$U(t - t_1) = \begin{cases} 0 \text{ for } t \leq t_1 \\ 1 \text{ for } t > t_1 \end{cases} \tag{29}$$

This notation is consistent with the fact that when any function $f(t)$ is plotted versus $t$, replacing every $t$ in $f(t)$ by $t - t_1$ shifts the curve $t_1$ units to the right. If the height of the step is $A$ rather than unity, we simply have $A$ times the unit step function, as shown in Figure 6.8(c).

From the definition of $U(t - t_1)$, we note that

$$f(t)U(t - t_1) = \begin{cases} 0 & \text{for } t \leq t_1 \\ f(t) & \text{for } t > t_1 \end{cases} \tag{30}$$

Thus the output of a system that was at rest for $t \leq 0$ is often written in the form $f(t)U(t)$, where the multiplying factor $U(t)$ is used in place of the phrase "for $t > 0$."

We define the **unit step response** of a system, denoted by $y_U(t)$, as the output that occurs when the input is the unit step function $U(t)$ and when the system contains no initial stored energy—that is, $y_U(t)$ is the zero-state response to the input $U(t)$. For the translational system shown in Figure 6.3(a),

$$y_U(t) = \frac{1}{K}[1 - \epsilon^{-Kt/B}] \quad \text{for } t \geq 0$$

whereas for the rotational system shown in Figure 6.3(b),

$$y_U(t) = \frac{B_1}{B_1 + B_2}[1 - \epsilon^{-(B_1+B_2)t/J}] \quad \text{for } t \geq 0$$

We can represent any function that consists of horizontal and vertical lines as the sum of step functions. Consider, for example, the pulse shown in Figure 6.9(a). This function is the sum of the two functions shown in Figure 6.9(b) and Figure 6.9(c), so

$$f_1(t) = AU(t) - AU(t - t_1) \tag{31}$$

[2]Note that a capital $U$ is used for the unit step function $U(t)$, in contrast to the lowercase letter in the symbol $u(t)$ for a general input. The value of the unit step function at time zero could be defined to be unity, or its value could be left undefined at this instant. Defining $U(0) = 0$ will be convenient in Chapter 7.



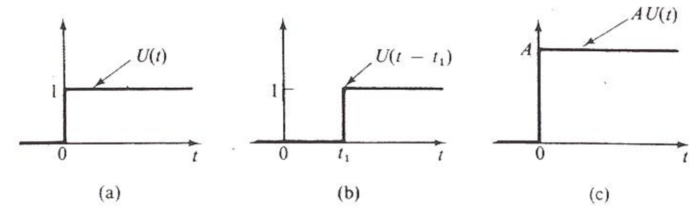


FIGURE 6.8 Step functions. (a) $U(t)$. (b) $U(t - t_1)$. (c) $AU(t)$.

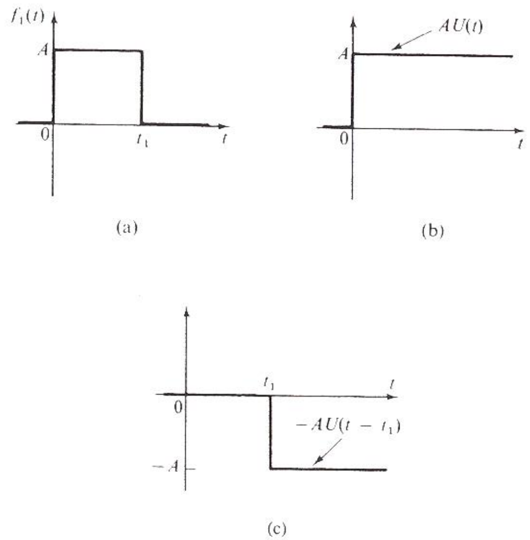


**FIGURE 6.9** (a) Rectangular pulse. (b), (c) Formation of the rectangular pulse by the sum of two step functions.

Suppose that the pulse in (31) is the input to a linear first-order system described by the differential equation

$$\dot{y} + \frac{1}{\tau}y = u(t) \tag{32}$$

and for which $y(0) = 0$. From Section 6.2, we know that the response to $AU(t)$ is $A\tau(1 - \epsilon^{-t/\tau})$ for $t \geq 0$, and this is also the response for $0 \leq t \leq t_1$ to the input in (31). For $t > t_1$, we may use superposition and sum the responses to the components $AU(t)$ and $-AU(t - t_1)$ to obtain

$$\begin{aligned} y(t) &= A\tau[1 - \epsilon^{-t/\tau}] - A\tau[1 - \epsilon^{-(t-t_1)/\tau}] \\ &= A\tau(-1 + \epsilon^{t_1/\tau})\epsilon^{-t/\tau} \end{aligned}$$

Hence we may write

$$y(t) = \begin{cases} A\tau(1 - \epsilon^{-t/\tau}) & \text{for } 0 \leq t \leq t_1 \\ A\tau(\epsilon^{t_1/\tau} - 1)\epsilon^{-t/\tau} & \text{for } t > t_1 \end{cases} \tag{33}$$

which is shown in Figure 6.10(a). As expected, the response for the first $t_1$ seconds is the same as in Figure 6.2(b), whereas for $t > t_1$ the output decays exponentially to zero with a time constant $\tau$. The output is the superposition of the two functions shown in Figure 6.10(b) and Figure 6.10(c). It can also



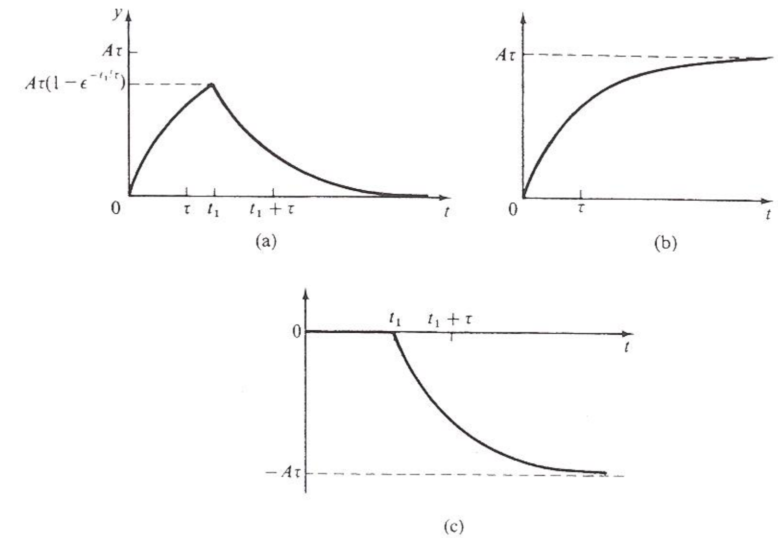


**FIGURE 6.10** Responses of a first-order system to the inputs shown in Figure 6.9.

be written in the alternative form

$$y(t) = A\tau(1 - \epsilon^{-t/\tau})U(t) - A\tau(1 - \epsilon^{-(t-t_1)/\tau})U(t - t_1)$$

It is instructive to rewrite the pulse response for $t > t_1$ in (33) with $\epsilon^{t_1/\tau}$ replaced by its Taylor-series expansion

$$\epsilon^{t_1/\tau} = 1 + \frac{t_1}{\tau} + \frac{1}{2!}\left(\frac{t_1}{\tau}\right)^2 + \cdots$$

Then, for $t > t_1$,

$$y(t) = A\tau\left[\frac{t_1}{\tau} + \frac{1}{2!}\left(\frac{t_1}{\tau}\right)^2 + \cdots\right]\epsilon^{-t/\tau}$$

Suppose that the pulse width is small compared to the time constant of the system, so that $t_1 \ll \tau$. Then we can neglect all the terms inside the brackets except the first and write

$$y(t) \simeq At_1\epsilon^{-t/\tau} \quad \text{for } t > t_1 \tag{34}$$

where $At_1$ is the area underneath the input pulse. We could also consider input pulses that have somewhat different shapes but have the same area underneath the curve. We would find that as long as the width $t_1$ of any pulse that is the input to a first-order system is small compared to the system's time constant, the response for $t > t_1$ depends on the area underneath the pulse but not on its shape.

As background for another important property, note that the response of the system modeled by (32) to a step function of height $At_1$, with $y(0) = 0$, is $At_1$ times the unit step response:

$$At_1 y_U(t) = At_1 \tau(1 - \epsilon^{-t/\tau}) \quad \text{for } t \geq 0$$

Note that

$$\frac{d}{dt}[At_1 y_U(t)] = At_1 \epsilon^{-t/\tau} \tag{35}$$

for all positive values of $t$. Because the right side of (35) is identical to (34), we see that if $t_1 \ll \tau$, the response for $t > t_1$ to a pulse of area $At_1$ is the derivative of the response to a step function of height $At_1$.

## The Unit Impulse

The response of a first-order system to a pulse of given area appears to be independent of the pulse shape as long as the pulse width $t_1$ is small compared to the time constant $\tau$, so it is reasonable to try to define an idealized pulse function whose width is small compared to the time constant of all first-order systems. However, in order to have $t_1 \ll \tau$ for all nonzero values of $\tau$, $t_1$ must be infinitesimally small; and to have a nonzero pulse area, the height of the pulse must become infinitely large. Although such an idealized pulse creates conceptual and mathematical difficulties, let us reconsider the rectangular input pulse that is shown in Figure 6.11(a). Note that $f_1(t)$ in part (a) of the figure is the derivative of the function $f_2(t)$ shown in part (b).

Let us specify $A = 1/t_1$ in Figure 6.11 such that the area underneath $f_1(t)$ is unity and such that the value of $f_2(t)$ for $t > t_1$ is also unity. If we continually decrease the value of $t_1$, then the height of the pulse $f_1(t)$ increases in order to maintain unit area, while $f_2(t)$ rises to its final value more rapidly. The dashed lines in Figure 6.12 show the changes in $f_1(t)$ and $f_2(t)$ when $t_1$ is halved.

As $t_1$ approaches zero, $f_1(t)$ approaches a pulse of infinitesimal width, infinite height, and unit area. The limit of this process is called the **unit impulse**, denoted by the symbol $\delta(t)$. It is represented graphically as shown in Figure 6.13(a). The number 1 next to the head of the arrow indicates the area underneath the function that approached the impulse. In the limit, as $t_1$ approaches zero, $f_2(t)$ becomes the unit step function $U(t)$, shown in Figure 6.13(b). Because $f_1(t)$ in Figure 6.11 is the time derivative of $f_2(t)$, it appears that in the limit,

$$\delta(t) = \frac{d}{dt}U(t) \tag{36}$$

The arguments we used in the preceding paragraph are heuristic and not mathematically rigorous. Questions can be raised about (36) because differentiation is a limiting process and we introduced $\delta(t)$ as the result



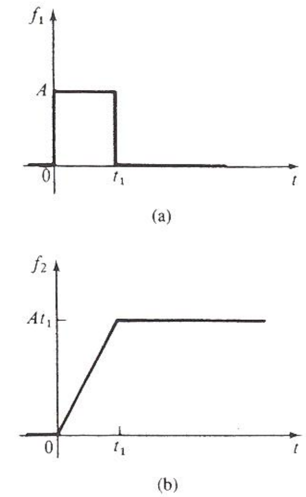


**FIGURE 6.11** (a) Rectangular pulse with area $At_1$. (b) $f_2(t) = \int_0^t f_1(\lambda)d\lambda$.

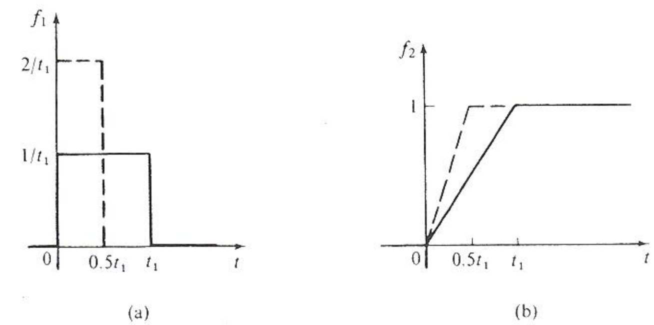


**FIGURE 6.12** The functions $f_1(t)$ and $f_2(t)$ shown in Figure 6.11 when $A = 1/t_1$.

of another limiting process. Then, without mathematical justification, we interchanged the order of the two limiting processes. Furthermore, if we were to put our description of the unit impulse into equations, we might write

$$\delta(t) = 0 \quad \text{for } t \neq 0 \tag{37a}$$

$$\int_{-\varepsilon}^{\varepsilon} \delta(t)dt = 1 \quad \text{for } \varepsilon > 0 \tag{37b}$$

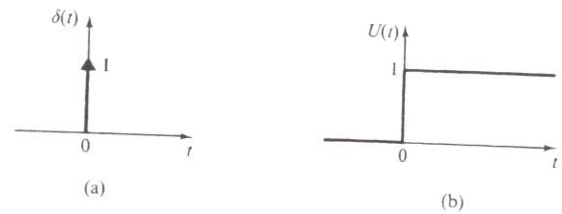


**FIGURE 6.13** (a) The unit impulse. (b) The unit step function.

where (37b) suggests that the unit impulse has unit area. Although we shall not discuss the matter in detail here, it happens that (37) violates the axioms of real-function theory and is an impermissible way of defining a function. In fact, $\delta(t)$ is not a function in the usual sense, and we have purposely avoided calling it the unit impulse "function." However, results obtained in this section can be justified by rigorous mathematical arguments.[3]

We can define the unit impulse formally in terms of an integral and in a way that is consistent with distribution theory. For any function $f(t)$ that is continuous at $t = 0$, the unit impulse $\delta(t)$ must satisfy the integral expression

$$\int_{-a}^{b} f(t)\delta(t)dt = f(0) \quad \text{for } a > 0, b > 0 \tag{38}$$

Note that (37b) is a special case of this equation, with $f(t) = 1$ for all values of $t$.

A unit impulse that occurs at time $t_a$ rather than at $t = 0$ is denoted by $\delta(t - t_a)$ and is shown in Figure 6.14(a). Furthermore, for any function $f(t)$ that is continuous at $t = t_a$, we can replace (38) by

$$\int_{b}^{c} f(t)\delta(t - t_a)dt = f(t_a) \quad \text{for } b < t_a < c \tag{39}$$

which is referred to as the **sampling property** of the impulse. The product $A\delta(t)$, which is shown in Figure 6.14(b), is called an impulse of weight $A$. We can visualize it as the limit of a high, narrow pulse of area $A$. Equation (36) may be replaced by the more general relationship

$$A\delta(t - t_a) = \frac{d}{dt}[AU(t - t_a)] \tag{40}$$

In other words, differentiating a step function of height $A$ occurring at $t_a$ gives rise to an impulse of weight $A$ at $t = t_a$, which is indicated in Figure 6.14(c).

[3]The unit impulse is also called the **Dirac delta function** and is used in many areas of science and engineering. Both the unit impulse and ordinary functions can be regarded as special cases of generalized functions or distributions.



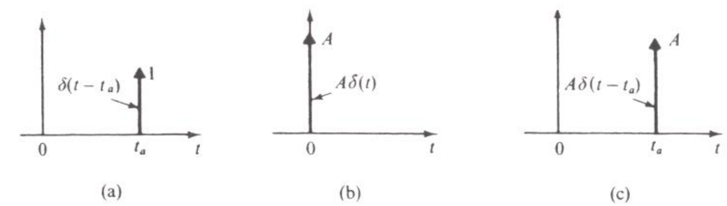


**FIGURE 6.14** Impulses. (a) $\delta(t - t_a)$. (b) $A\delta(t)$. (c) $A\delta(t - t_a)$.

The **unit impulse response** $h(t)$ is defined as the output that occurs when the input is $\delta(t)$ and when the system contains no stored energy before the impulse is applied. Thus $h(t)$ is the zero-state response to $\delta(t)$. The zero-state response for $t > t_1$ to a pulse of any shape is approximately equal to $h(t)$ times the area underneath the original pulse, as long as the pulse width $t_1$ is small compared to the time constant of the system.

Because a unit impulse applied to a system can cause an infinite power flow that instantaneously changes the energy stored in the system, the initial conditions we would need in order to calculate $h(t)$ directly are often hard to find. For this reason, the property described in the following paragraph often proves useful.

Assume that we know the response to a certain input for a linear system that contains no initial stored energy. Then, if we substitute a new input that is the derivative of the original input, the new response will be the derivative of the old response. A formal proof of this property will be found in Section 8.2. Because $\delta(t)$ is the derivative of $U(t)$, the unit impulse response and the unit step response for a linear system are related by

$$h(t) = \frac{d}{dt}y_U(t) \tag{41}$$

Thus once the unit step response $y_U(t)$ is known, we can obtain $h(t)$ from it by using (41).

### ▶ *EXAMPLE 6.6*

For the translational system shown in Figure 6.15, find $y_U(t)$ and $h(t)$ when the output is the velocity $v(t)$.

**Solution**

The system is described by the differential equation

$$M\dot{v} + Bv = f_a(t)$$

or

$$\dot{v} + \frac{B}{M}v = \frac{1}{M}f_a(t)$$

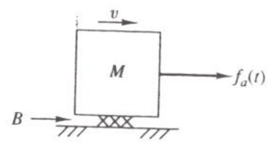


FIGURE 6.15 Translational system for Example 6.6.

Because the system's time constant is $\tau = M/B$, and $v_{ss} = 1/B$ if $f_a(t) = 1$ for all $t > 0$, the unit step response is

$$y_U(t) = \frac{1}{B}(1 - \epsilon^{-Bt/M}) \quad \text{for } t > 0 \tag{42}$$

From (41), the unit impulse response can be found by differentiating $y_U(t)$. It is

$$h(t) = \frac{1}{M}\epsilon^{-Bt/M} \quad \text{for } t > 0 \tag{43}$$

These responses are shown in Figure 6.16.

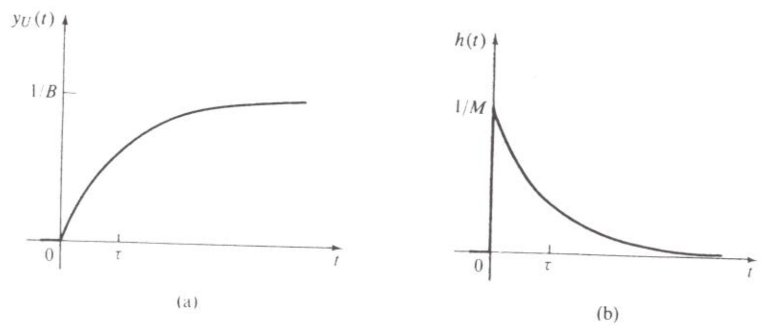


FIGURE 6.16 Responses for the system shown in Figure 6.15 when the output is the velocity. (a) Unit step response. (b) Unit impulse response.

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Note in Figure 6.16(a) that when $f_a(t)$ is a step function, the velocity of the mass does not change instantaneously at $t = 0$. When $f_a(t)$ is an impulse, however, the velocity of the mass does undergo an instantaneous change. In the first case, there is no change in the kinetic energy of the mass at $t = 0$. In the latter case, the impulse causes an instantaneous increase in the energy of the mass.



## 6.4 SECOND-ORDER SYSTEMS

We can write the differential equation for a fixed linear second-order system as

$$\ddot{y} + a_1\dot{y} + a_0 y = F(t) \tag{44}$$

where, without loss of generality, the coefficient of $\ddot{y}$ has been made unity and where $a_0$ and $a_1$ are real constants. To find the response, as in Example 6.2, we need to know $F(t)$ for $t \geq 0$ and also the initial conditions $y(0)$ and $\dot{y}(0)$.

If the forcing function $F(t)$ has a finite number of independent derivatives, we can find the forced response by the method of undetermined coefficients. The free response $y_H(t)$ contains two terms instead of the one that is present for first-order systems. If the roots $r_1$ and $r_2$ of the characteristic equation

$$r^2 + a_1 r + a_0 = 0$$

are real and distinct, then

$$y_H(t) = K_1\epsilon^{r_1 t} + K_2\epsilon^{r_2 t} \tag{45}$$

For $r_1 = r_2$, the two roots are not distinct, and we must replace (45) by

$$y_H(t) = K_1\epsilon^{r_1 t} + K_2 t\epsilon^{r_1 t} \tag{46}$$

If the roots are complex, they must have the form $r_1 = \alpha + j\beta$ and $r_2 = \alpha - j\beta$, and we write the free response as

$$y_H(t) = \epsilon^{\alpha t}[K_1 \cos\beta t + K_2 \sin\beta t]$$

or, equivalently,

$$y_H(t) = K\epsilon^{\alpha t}\cos(\beta t + \phi) \tag{47}$$

Examples of $y_H(t)$ are shown in Figure 6.17 for the three cases represented by (45) through (47) when $r_1, r_2$, and $\alpha$ are negative numbers. The light lines in Figure 6.17(a) and Figure 6.17(b) indicate the two individual functions that are added to give $y_H(t)$, which is the heavy curve. In Figure 6.17(c), dashed lines labeled $K\epsilon^{\alpha t}$ and $-K\epsilon^{\alpha t}$ form the envelope of the damped oscillations. The values of the arbitrary constants $K_1$, $K_2$, $K$, and $\phi$ in (45) through (47) depend on $y_P(t)$ and the initial conditions $y(0)$ and $\dot{y}(0)$.

### The Complex Plane

When the roots of the characteristic equation are plotted in a complex plane, inspection of the plot reveals the nature of the system's free response. The root locations corresponding to the typical free responses shown in Figure 6.17 are indicated by the crosses in the respective parts of Figure 6.18.

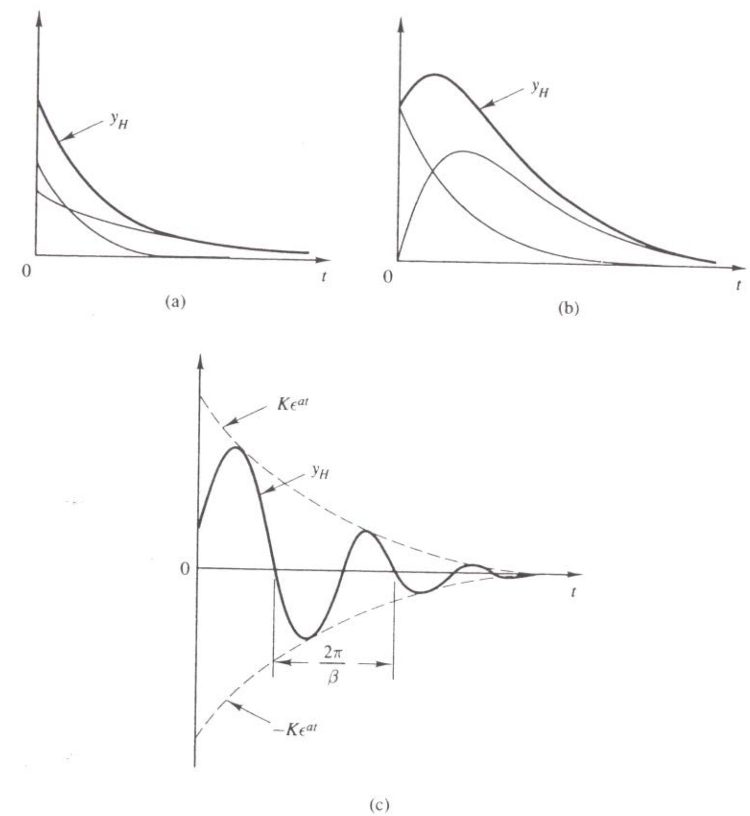


**FIGURE 6.17** Typical curves for the free response of a second-order system. (a) Real distinct negative roots of the characteristic equation. (b) Identical negative roots. (c) Complex roots with $\alpha < 0$.

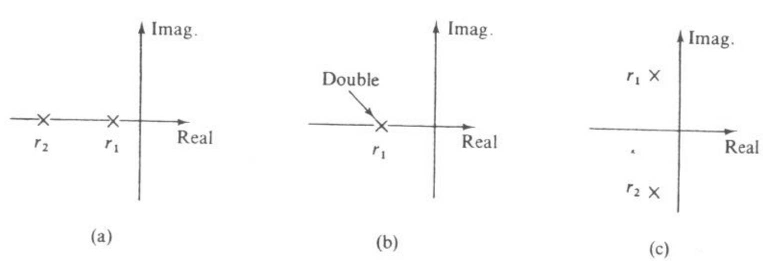


**FIGURE 6.18** Roots of the characteristic equation corresponding to Figure 6.17. (a) Real distinct negative roots. (b) Identical negative roots. (c) Complex roots with $\alpha < 0$.



Recall from Section 6.2 that a stable system is one for which the free response $y_H(t)$ decays to zero as $t$ approaches infinity. All the examples in Figure 6.17 fall into this category. Examples of unstable systems, where $y_H(t)$ increases without bound, are shown in Figure 6.19. The characteristic-root locations appear directly under the corresponding sketches of the free response. Figure 6.19(a) and Figure 6.19(b) correspond to a positive value of $r_1$ in (45), Figure 6.19(c) and Figure 6.19(d) correspond to a positive value of $\alpha$ in (47), and Figure 6.19(e) and Figure 6.19(f) correspond to $r_1 = 0$ in (46).

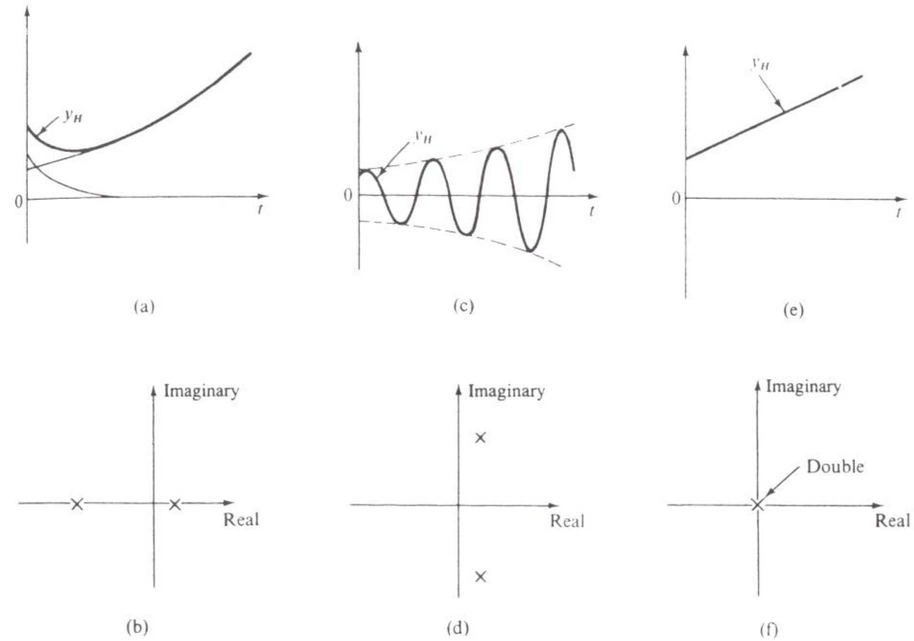


**FIGURE 6.19** Examples of unstable second-order systems. (a), (b) One real root in the right half-plane. (c), (d) Complex roots in the right half-plane. (e), (f) Double root at the origin of the complex plane.

In addition to these stable and unstable classes of systems, it is possible (at least in an idealized case) for a linear system to be marginally stable and have a free response that neither decays to zero nor grows without bound. For second-order systems, such a response occurs when the characteristic equation has either a single root at $r_1 = 0$ with the remaining root in the left half-plane or a pair of imaginary roots at $r_1 = j\beta$ and $r_2 = -j\beta$. The first case corresponds to the characteristic equation $r^2 + a_1 r = 0$ and to the free response

$$y_H(t) = K_1 + K_2\epsilon^{-a_1 t}$$

where $a_1 > 0$. The second case corresponds to the characteristic equation $r^2 + \beta^2 = 0$ and to the free response

$$y_H(t) = K\cos(\beta t + \phi)$$

A system whose free response is a constant-amplitude sine or cosine function is often called a **simple harmonic oscillator**. Sample root locations and typical free-response curves for the two types of marginally stable second-order systems are shown in Figure 6.20.

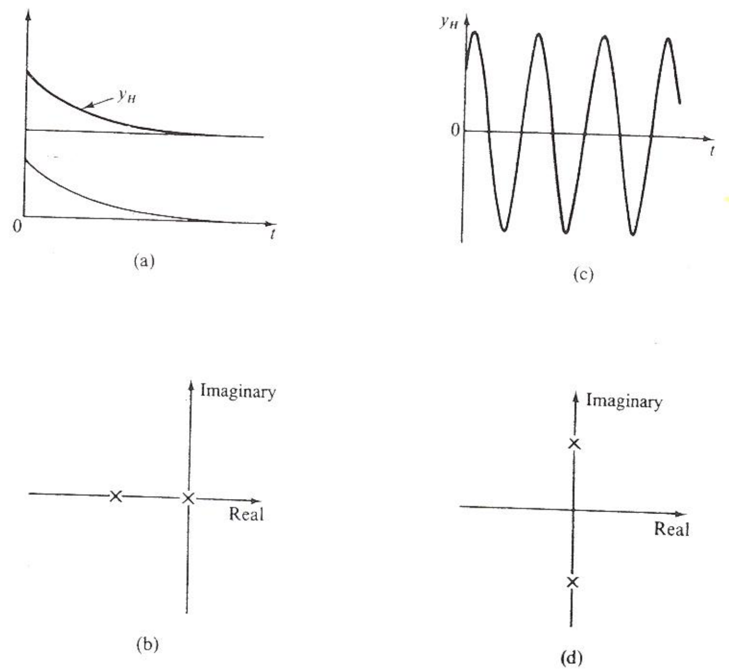


**FIGURE 6.20** Examples of marginally stable second-order systems. (a), (b) One root at the origin of the complex plane. (c) (d) A pair of complex roots on the imaginary axis.

We can summarize the discussion about stability in a way that is applicable to fixed linear systems of any order as follows. If all the roots of the characteristic equation are inside the left half-plane, the system is stable. If there are any roots inside the right half-plane or repeated roots on the imaginary axis, the system is unstable. If all the roots are inside the left



half-plane except for one or more distinct roots on the imaginary axis, the system is marginally stable.

The following property of polynominals can be useful when we are considering stability. If the signs of the coefficients in the characteristic equation are not all the same, then there is at least one root inside the right half-plane and the system is unstable. However, if all the signs are the same, we can be sure that there are no right half-plane roots only in the case of first- and second-order systems; for higher-order systems, all the signs being the same is not sufficient to guarantee the absence of right half-plane roots.

In Chapters 2, 4, and 5 we noted a property that can be used as a check on the equations obtained by examining the free-body diagrams for a mechanical system or the nodes of an electrical circuit. Suppose we sum the forces at a mass $M_i$ and collect like terms. Then all the terms involving the displacement of $M_i$ and its derivatives must have the same sign. We can make a similar statement when summing torques for a rotational system. For an electrical circuit, suppose we sum the currents that leave a node whose voltage is $e_A$ and then collect like terms. All the terms involving $e_A$ and its derivatives must have the same sign.

Although we shall not prove this property, we do now provide an explanation for it. All the systems that we have considered, except for those in Section 5.7, were excited by external independent inputs and consisted only of passive elements (such as $M$, $J$, $B$, $K$, $C$, $R$, and $L$). Systems composed of passive elements cannot be unstable, because there are no permanent sources of energy inside them. Thus all the coefficients in the characteristic equation, which are identical to those on the left-hand side of the input-output differential equation, must have the same sign. Let us first write and combine the system equations in literal form, with the element values represented by letters rather than numbers. Then the coefficients on the left-hand side of the input-output equation must have the same sign for all combinations of positive numbers that might be substituted for the element values. If, however, the property described in the previous paragraph were not satisfied, then we would find that we can make the signs of the coefficients different merely by choosing some of the element values sufficiently large.

▶ ***EXAMPLE 6.7***

Consider the following pair of coupled first-order differential equations:

$$a_1\dot{v}_1 + a_0 v_1 - b v_2 = f_1(t) \tag{48a}$$

$$-c v_1 + d_1\dot{v}_2 + d_0 v_2 = f_2(t) \tag{48b}$$

which describe several simple systems. Two such systems will be presented immediately after this example. Assume, without loss of generality, that $a_1$ and $d_1$ are positive. If this were not the case for either of the original equations, we could make it so by multiplying that equation by $-1$. The magnitudes of $a_0$ and $d_0$ can be made arbitrarily large by an appropriate choice of element values.

Find the input-output equation relating $v_2$ to the inputs $f_1(t)$ and $f_2(t)$. Show that $a_0$ and $d_0$ must be positive in order for the system described by (48) to be stable.

**Solution**

From (48b),

$$v_1 = \frac{1}{c}[d_1\dot{v}_2 + d_0 v_2 - f_1(t)]$$

Substituting this expression into (48a) and collecting terms, we obtain

$$a_1 d_1 \ddot{v}_2 + (a_1 d_0 + a_0 d_1)\dot{v}_2 + (a_0 d_0 - bc)v_2 = cf_1(t) + a_1\dot{f}_2 + a_0 f_2(t)$$

For a stable system, the coefficient of $\dot{v}_2$ must be positive. Because this must be true regardless of the magnitudes of $a_0$ and $d_0$, we require that $a_0$ and $d_0$ be positive. We also require that $a_0 d_0 > bc$, which will be satisfied for passive systems, as illustrated by the two cases that are considered next.

---

For the mechanical system shown in Figure 6.21(a), it is easy to show that (48a) and (48b) are the D'Alembert-law equations written for $M_1$ and $M_2$, respectively, where

$$a_1 = M_1, \quad a_0 = B_1 + B_3, \quad b = c = B_3$$
$$d_1 = M_2, \quad d_0 = B_2 + B_3$$

Thus we should be able to anticipate that $a_1$ and $a_0$ must have the same sign, as must $d_1$ and $d_0$. Similarly, for the electrical circuit in Figure 6.21(b), we

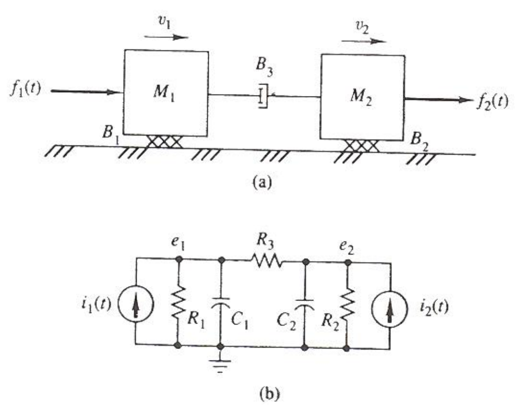


**FIGURE 6.21** Two systems that can be described by (48).



can show that (48a) and (48b) are the current-law equations at nodes 1 and 2, with the following change of symbols:

$$v_1 = e_1, \quad v_2 = e_2, \quad f_1(t) = i_1(t), \quad f_2(t) = i_2(t)$$

$$a_1 = C_1, \quad a_0 = \frac{1}{R_1} + \frac{1}{R_3}, \quad b = c = \frac{1}{R_3}$$
$$d_1 = C_2, \quad d_0 = \frac{1}{R_2} + \frac{1}{R_3}$$

Even though we have not justified the sign rule for the general case, it is helpful to use it as a partial check on the equations for any passive system.

## Damping Ratio and Undamped Natural Frequency

When the roots of the characteristic equation are complex, the parameter $a_0$ in (44) is positive, and it is useful to rewrite the differential equation in the standard form

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = F(t) \tag{49}$$

The parameter $\omega_n$ is called the **undamped natural frequency** and has units of radians per second. The parameter $\zeta$ is dimensionless and is known as the **damping ratio**. The characteristic equation is

$$r^2 + 2\zeta\omega_n r + \omega_n^2 = 0 \tag{50}$$

For $\zeta > 1$, the roots are distinct negative numbers, and $y_H(t)$ consists of two decaying exponentials. For $\zeta = 1$, there is a repeated root at $r = -\omega_n$, and $y_H(t)$ consists of terms having the form $\epsilon^{-\omega_n t}$ and $t\epsilon^{-\omega_n t}$. For $0 \le \zeta < 1$, the roots are complex and are

$$r_1 = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$$
$$r_2 = -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}$$

Then by (47), the free response is

$$y_H(t) = K\epsilon^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}\, t + \phi) \tag{51}$$

For $\zeta < 0$, the system is unstable.

The advantage of introducing the parameters $\zeta$ and $\omega_n$ becomes apparent when the characteristic roots are complex and are plotted in the complex plane, as indicated in Figure 6.22(a). Their distances from the origin are the same and are denoted by $d$. The distance $d$ is the square root of the sum of the squares of the real and imaginary parts of the root. For the upper root,

$$d = \sqrt{(-\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} = \omega_n$$

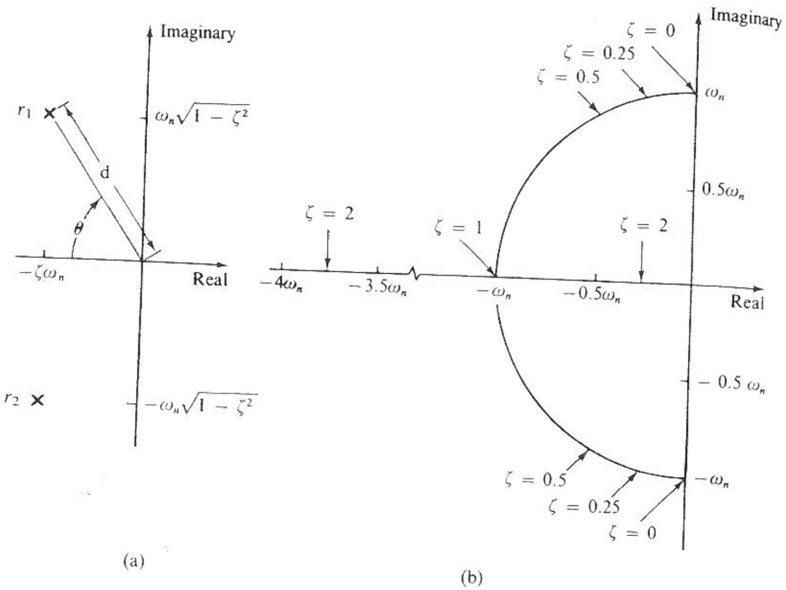


**FIGURE 6.22** Characteristic-root locations in terms of $\zeta$ and $\omega_n$. (a) General complex roots. (b) Locations for constant $\omega_n$ and varying $\zeta$.

Hence, when $0 \le \zeta < 1$, the complex characteristic roots lie on a circle of radius $\omega_n$ centered at the origin. It is easy to show that their locations on the circle depend only on the damping ratio $\zeta$. Specifically, the angle $\theta$ between the negative real axis and the line from the origin to $r_1$ in Figure 6.22(a) satisfies the relationship $\cos\theta = \zeta\omega_n/\omega_n = \zeta$. Thus

$$\theta = \cos^{-1}\zeta \tag{52}$$

The geometric relationships between $\zeta$, $\omega_n$, and the roots of (50) are summarized in Figure 6.22(b) for $\zeta \ge 0$.

It is instructive to observe the effect of the damping ratio $\zeta$ on the responses of a system described by the equation

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \omega_n^2 u(t) \tag{53}$$

The unit step response is shown in Figure 6.23 for several values of $\zeta$, with $\omega_n$ held constant. For $\zeta > 0$, the steady-state response is unity. Note that the value of $\zeta$ determines to what extent, if any, the response will overshoot its steady-state value. The overshoot is 100% for $\zeta = 0$ and decreases to zero when the damping ratio is unity. For $\zeta > 1$, the response approaches its steady-state value monotonically.



The case $\zeta = 1$ is the boundary between responses that oscillate and those that do not, and the system is said to have **critical damping**. Second-order systems for which $\zeta > 1$ have more than critical damping; those for which $0 < \zeta < 1$ are less than critically damped. When $\zeta = 0$ the system is **undamped**. Figure 6.24 shows the unit impulse response of the general second-order system described by (53) for the values of $\zeta$ used in Figure 6.23. Note that the steady-state response is zero for $\zeta > 0$ and that the value of the damping ratio establishes the character of the response.

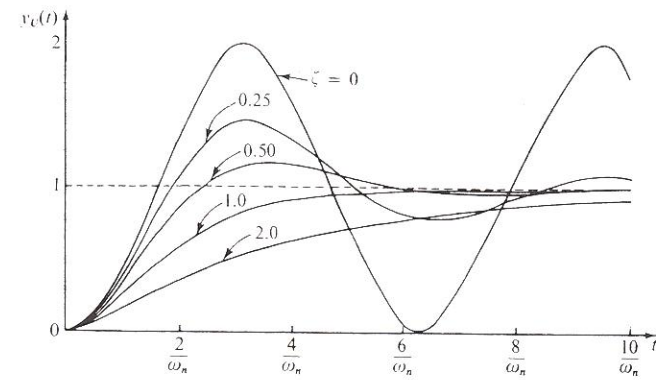


**FIGURE 6.23** The unit step response for a second-order system described by (53).

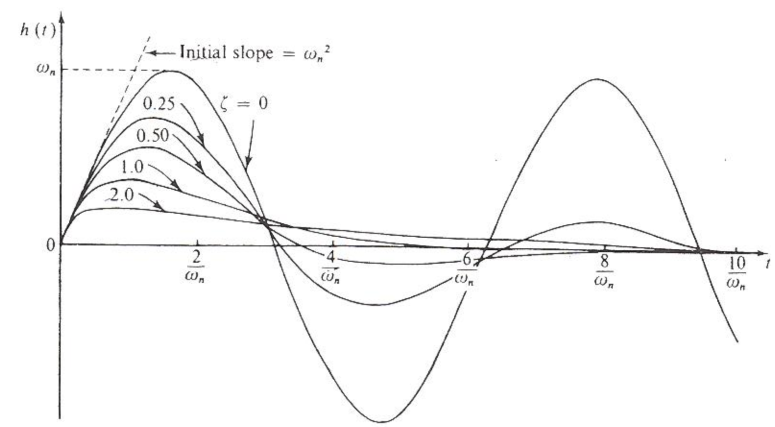


**FIGURE 6.24** The unit impulse response for a second-order system described by (53).

We consider next an example in numerical form. Then we discus briefly the role of friction and resistance in determining the damping ratio We conclude by examining two approximations of a nonlinear second-order system that exhibit quite different behaviors.

### ▶ *EXAMPLE 6.8*

The differential equation describing a certain fixed linear system is

$$2\ddot{y} + \alpha\dot{y} + 50y = F(t)$$

Determine $\omega_n$ and $\zeta$ for $\alpha = 12$ and for $\alpha = 52$. Write the form of the free response for each case. For what values of $\alpha$ does the free response consist of decaying oscillations?

**Solution**

Dividing by 2 to put the left-hand side of the differential equation in the form of (49) gives

$$\ddot{y} + \frac{\alpha}{2}\dot{y} + 25y = \frac{1}{2}F(t)$$

Thus $\omega_n^2 = 25$ and $2\zeta\omega_n = \alpha/2$, from which $\omega_n = 5$ rad/s and $\zeta = \alpha/20$. When $\alpha = 12$, $\zeta = 3/5$, which corresponds to less than critical damping. The roots of the characteristic equation $r^2 + 6r + 25 = 0$ are at $-3 \pm j4$, so $y_H(t) = K\epsilon^{-3t}\cos(4t + \phi)$.

When $\alpha = 52$, $\zeta = 13/5$, showing that the system is more than critically damped. The characteristic equation for this case is $r^2 + 26r + 25 = 0$, which has roots at $r_1 = -1$ and $r_2 = -25$, so $y_H(t) = K_1\epsilon^{-t} + K_2\epsilon^{-25t}$. The free response will contain decaying oscillations when $0 < \zeta < 1$, which requires $0 < \alpha < 20$.

---

Although the damping ratio normally depends on the values of all the passive elements, it is instructive to consider the role of friction and resistance for the four systems shown in Figure 6.25. For the translational system in part (a), which was examined in Example 2.1, the differential equation was shown to be

$$\ddot{x} + \frac{B}{M}\dot{x} + \frac{K}{M}x = \frac{1}{M}f_a(t)$$

and the damping ratio is

$$\zeta = \frac{B}{2\sqrt{MK}} \tag{54}$$

For the rotational system in part (b), which was modeled in Example 4.1, the differential equation is

$$\ddot{\theta} + \frac{B}{J}\dot{\theta} + \frac{K}{J}\theta = \frac{1}{J}\tau_a(t)$$



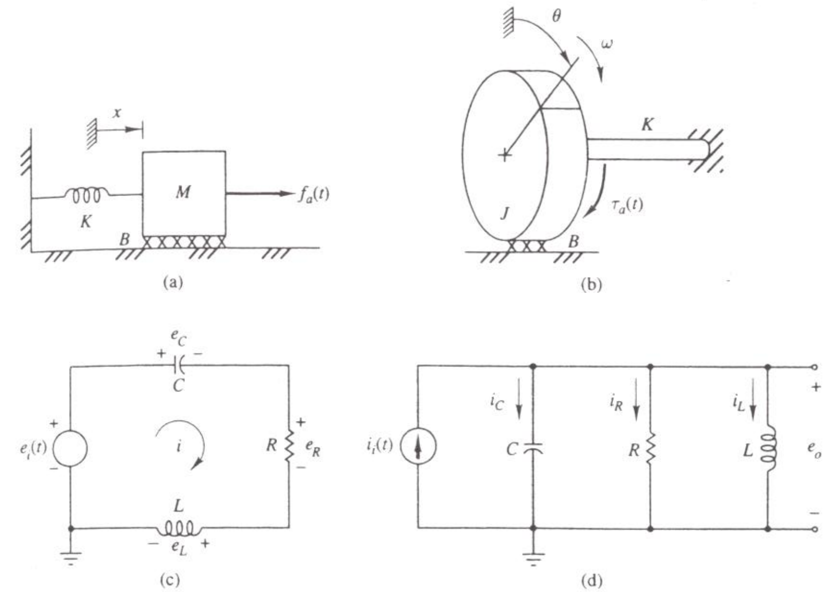


**FIGURE 6.25** Second-order systems illustrating the role of friction and resistance in determining $\zeta$.

and the damping ratio is

$$\zeta = \frac{B}{2\sqrt{JK}} \tag{55}$$

The series circuit in part (c) of the figure was treated in Example 5.1 and is described by

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = \frac{1}{L}\dot{e}_i$$

The damping ratio is

$$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}} \tag{56}$$

Finally, for the parallel circuit in part (d), we have from Example 5.2,

$$\ddot{e}_o + \frac{1}{RC}\dot{e}_o + \frac{1}{LC}e_o = \frac{1}{C}\frac{di_i}{dt}$$

and the damping ratio is

$$\zeta = \frac{1}{2R}\sqrt{\frac{L}{C}} \tag{57}$$

If there is no friction in the two mechanical cases, then we see from (54) and (55) that $\zeta = 0$. The free response will consist of a constant-amplitude oscillation. Because there is no element that can dissipate energy in the form of heat, any initial stored energy will be interchanged undiminished between potential energy in $K$ and kinetic energy in $M$ or $J$. For positive values of $B$ that do not exceed $2\sqrt{MK}$ or $2\sqrt{JK}$, $y_H(t)$ will contain decaying oscillations. If $B$ is increased still further, $y_H(t)$ will have two decaying exponential terms.

When $R = 0$ in part (c) of the figure—that is, when the resistor is replaced by a short circuit—we see from (56) that $\zeta = 0$. Then any initial stored energy is continually swapped back and forth between the capacitor and inductor, and the free response does not die out. When $R$ takes on positive values, energy is lost from the circuit in the form of heat, and $y_H(t)$ decays to zero.

For the circuit in part (d), the undamped case of no energy loss in the form of heat occurs when $R$ is replaced by an open circuit—that is, when $R \to \infty$. Then $y_H(t)$ will again be a constant-amplitude sinusoidal oscillation. As $R$ takes on smaller positive values, $\zeta$ increases.

▶ *EXAMPLE 6.9*

The pendulum shown in Figure 6.26(a) was considered in Example 4.7. From (4.51), its input-output differential equation is

$$\ddot{\theta} + \frac{B}{ML^2}\dot{\theta} + \frac{g}{L}\sin\theta = \frac{1}{ML^2}\tau_a(t) \tag{58}$$

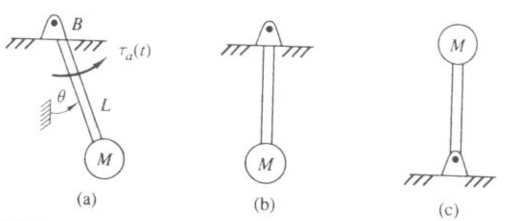


**FIGURE 6.26** (a) Pendulum for Example 6.9. (b) Stable equilibrium position. (c) Unstable equilibrium position.

Discuss the motion of the pendulum for small variations about each of the two equilibrium positions shown in Figure 6.26(b) and Figure 6.26(c).

**Solution**

For the positions shown in parts (b) and (c) of the figure, $\theta = 0$ and $\theta = \pi$ rad, respectively. Note that (58) is satisfied if $\tau_a(t) = 0$ and if $\theta$ has a constant value of either zero or $\pi$ rad. This indicates that if the pendulum



is stationary in either of these two positions, it will not move unless there is an applied torque.

Equation (58) is nonlinear because of the factor $\sin\theta$. In Example 4.7, we noted that $\sin\theta \simeq \theta$ for small values of $\theta$. Thus for small deviations about the vertical position shown in Figure 6.26(b), we can approximate (58) by the linear equation

$$\ddot{\theta} + \frac{B}{ML^2}\dot{\theta} + \frac{g}{L}\theta = \frac{1}{ML^2}\tau_a(t) \tag{59}$$

The free response $\theta_H(t)$ will then have the form of (51) with

$$\omega_n = \sqrt{\frac{g}{L}}$$

$$\zeta = \frac{B}{2Mg^{1/2}L^{3/2}}$$

As expected, the system is stable as long as the pendulum remains close to the position shown in Figure 6.26(b). Decreasing the friction coefficient $B$ decreases the damping ratio $\zeta$. If $B = 0$, then (59) becomes

$$\ddot{\theta} + \frac{g}{L}\theta = \frac{1}{ML^2}\tau_a(t)$$

for which

$$\theta_H(t) = K\cos\left(\sqrt{\frac{g}{L}}\,t + \psi\right)$$

where $K$ and $\psi$ are arbitrary constants that depend on the initial conditions. Such a system is marginally stable. If $\tau_a(t) = 0$ for $t > 0$ but if the pendulum is given some initial angular displacement $\theta_0$, then it will have a sinusoidal oscillation with an angular frequency of $\sqrt{g/L}$ radians per second.

Finally, we consider small displacements about the position shown in Figure 6.26(c). If $\phi(t)$ represents the deviation from that position, then $\theta(t) = \phi(t) + \pi$. We note that $\dot{\theta} = \dot{\phi}$ and $\ddot{\theta} = \ddot{\phi}$, and we also use the mathematical identity $\sin(\phi + \pi) = -\sin\phi \simeq -\phi$. Thus for motion close to the position shown in Figure 6.26(c), (58) reduces to

$$\ddot{\phi} + \frac{B}{ML^2}\dot{\phi} - \frac{g}{L}\phi = \frac{1}{ML^2}\tau_a(t) \tag{60}$$

This linearized model is unstable, because one root of the characteristic equation is in the right half-plane. For the case where $B = 0$,

$$\ddot{\phi} - \frac{g}{L}\phi = \frac{1}{ML^2}\tau_a(t)$$

and

$$\phi_H(t) = K_1\epsilon^{\sqrt{g/L}\,t} + K_2\epsilon^{-\sqrt{g/L}\,t}$$

This agrees with our expectation that the equilibrium position in Figure 6.26(c) would be unstable. Even a very small initial displacement away from the upright position would cause the pendulum to fall.

## 6.5 SYSTEMS OF ORDER THREE AND HIGHER

Except for situations involving repeated characteristic roots, the free response of third- and higher-order systems is composed of those functions that comprise the free response of first- and second-order systems. The basis for this property is the fact that all the characteristic roots must either be real or occur in complex-conjugate pairs. For example, the third-order differential equation

$$\dddot{y} + a_2\ddot{y} + a_1\dot{y} + a_0 y = F(t)$$

has as its characteristic equation

$$r^3 + a_2 r^2 + a_1 r + a_0 = 0 \tag{61}$$

If two of its roots are complex, we may write this equation in factored form as

$$(r - \alpha - j\beta)(r - \alpha + j\beta)(r - \gamma) = 0$$

where the roots are $r_1 = \alpha + j\beta$, $r_2 = \alpha - j\beta$, and $r_3 = \gamma$. The general form of the free response is

$$y_H(t) = K_1\epsilon^{r_1 t} + K_2\epsilon^{r_2 t} + K_3\epsilon^{r_3 t}$$

which, by using (47) for the terms corresponding to the complex roots, we can write as

$$y_H(t) = K_4\epsilon^{\alpha t}\cos(\beta t + \phi) + K_3\epsilon^{\gamma t}$$

Figure 6.27 shows a typical set of characteristic roots in the complex plane and a sample free response for such a third-order system. Figure 6.27(b) shows the individual response curves, with the response associated with the roots $r_1$ and $r_2$ designated as $y_{12}$. Note that the term $y_3 = K_3\epsilon^{\gamma t}$ decays more slowly than the envelope of $y_{12}$, because $|\alpha| > |\gamma|$ in Figure 6.27(a). In graphical terms, $r_3$ is further to the right in the complex plane than $r_1$ and $r_2$. Figure 6.27(c) shows the free response, which is the sum of the two individual responses.

We find the forced response $y_P(t)$ of a third- or higher-order system just as we found it for first- and second-order systems. Although it requires no



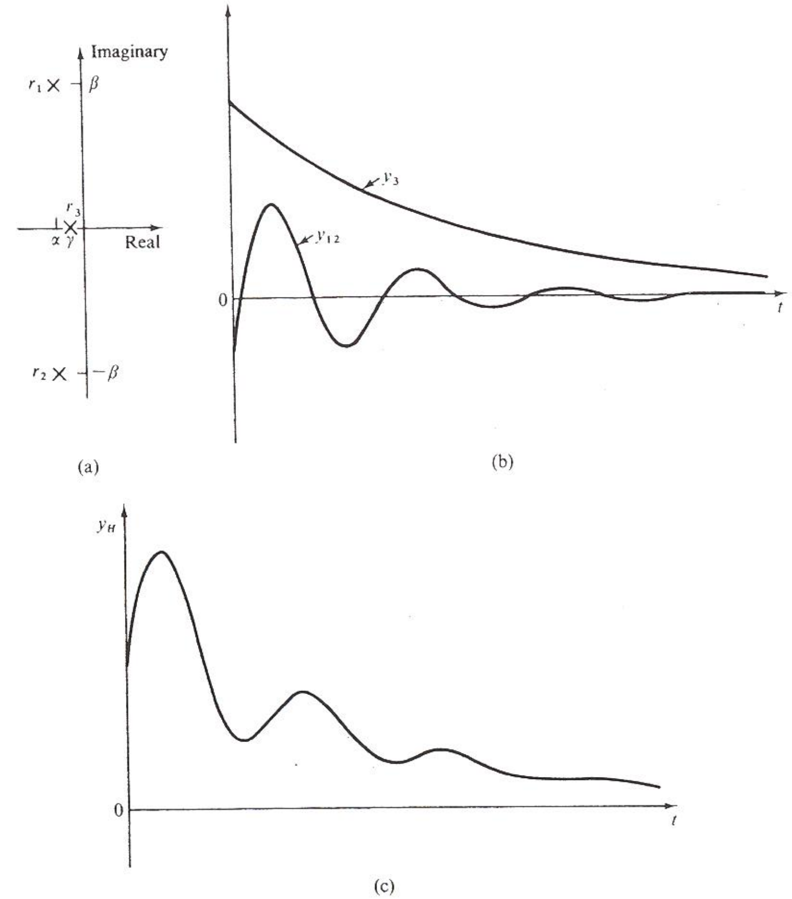


**FIGURE 6.27** (a) Characteristic roots for a third-order system with $\alpha = -2$, $\beta = 10$, and $\gamma = -1$. (b) Components of the free response. (c) The free response.

new concepts or techniques, the solution is more cumbersome than that for the lower-order cases.

## 6.6 TIME-DOMAIN SOLUTION OF MATRIX STATE-VARIABLE EQUATIONS

For fixed linear systems, the state-variable and output equations have the forms shown in (3.31) and (3.32). The matrix form of these equations, which was given in (3.33), is

$$\dot{\mathbf{q}} = \mathbf{Aq} + \mathbf{Bu} \tag{62a}$$

$$\mathbf{y} = \mathbf{Cq} + \mathbf{Du} \tag{62b}$$

where the elements in the coefficient matrices **A**, **B**, **C**, and **D** are constants. The matrix **A** is always square, but this is not necessarily the case for the other coefficient matrices. To solve (62a) for $\mathbf{q}(t)$ for $t \geq t_0$, we must know the input vector $\mathbf{u}(t)$ for $t \geq t_0$ and the initial state vector $\mathbf{q}(t_0)$. We shall normally take $t_0$ to be zero. Once $\mathbf{q}(t)$ has been found, it is relatively easy to find the output vector $\mathbf{y}(t)$ from (62b).

Although (62a) is a set of first-order differential equations, the individual equations are generally coupled together; that is, more than one state variable appears in each of the equations. Thus the individual differential equations that make up (62a) cannot be solved independently of one another. One approach is to try to find a change of variables that will uncouple the individual equations, so that each of the new equations contains only one unknown variable. This is equivalent to converting **A** into a diagonal matrix. Ways of finding what change of variables is needed are discussed in books on linear algebra, often under the topic of similarity transformations or diagonalizing a matrix. One limitation of this approach is that it is sometimes impossible to diagonalize the matrix **A**.

In this section, we shall develop procedures for solving the state-variable equations as a group without resorting to a substitution of variables. We shall draw upon a number of linear algebra results, some of which may be unfamiliar. For proofs and further details, consult the references in Appendix D.

## The Zero-Input Response

We consider first the case where the inputs are zero for $t > 0$, in which case (62) reduces to

$$\dot{\mathbf{q}} = \mathbf{Aq} \tag{63a}$$

$$\mathbf{y} = \mathbf{Cq} \tag{63b}$$

Recall that the solution to the first-order scalar equation $\dot{q} = aq$ is $q(t) = \epsilon^{at}q(0)$ and that the infinite series for the exponential factor is

$$\epsilon^{at} = 1 + at + \frac{1}{2!}a^2t^2 + \frac{1}{3!}a^3t^3 + \cdots$$

Thus it is reasonable to try, as a solution to (63a),

$$\mathbf{q}(t) = \epsilon^{\mathbf{A}t}\mathbf{q}(0) \tag{64}$$

Because **A** is a matrix, the proper interpretation of $\epsilon^{\mathbf{A}t}$ is not obvious. However, most functions of a matrix can be represented by an infinite series in



powers of **A**. As might be expected from the series for the scalar exponential function,

$$\epsilon^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \frac{1}{3!}\mathbf{A}^3t^3 + \cdots \tag{65}$$

If **A** is a square matrix of order $n$, each of the terms in this series, and hence $\epsilon^{\mathbf{A}t}$ itself, is also a square matrix of order $n$.

It can be proved by direct substitution that (64), with $\epsilon^{\mathbf{A}t}$ defined by (65), does indeed satisfy the matrix differential equation in (63a). However, we want solutions in closed form whenever possible, rather than as an infinite series. By using the Cayley–Hamilton theorem, we can also express most matrix functions as a series of finite length. If **A** is a square matrix of order $n$, then

$$\epsilon^{\mathbf{A}t} = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} + \alpha_2\mathbf{A}^2 + \cdots + \alpha_{n-1}\mathbf{A}^{n-1} \tag{66}$$

The standard method for finding the coefficients $\alpha_0$ through $\alpha_{n-1}$ for a particular matrix **A** starts by finding the characteristic values of **A**. As illustrated in Appendix C, $|s\mathbf{I} - \mathbf{A}|$ is a polynomial of degree $n$ in the scalar quantity $s$. The **characteristic values** (or **eigenvalues**) of **A** are the values of $s$ for which

$$|s\mathbf{I} - \mathbf{A}| = 0 \tag{67}$$

Many mathematics books use the symbol $\lambda$ for the scalar quantity in (67). However, we have reserved $\lambda$ for a dummy variable of integration. Furthermore, using the symbol $s$ is consistent with the alternative treatment of matrix state-variable equations in Section 8.7.

If we replace the matrix **A** in (66) by the scalar quantity $s$, then the corresponding scalar equation

$$\epsilon^{st} = \alpha_0 + \alpha_1 s + \alpha_2 s^2 + \cdots + \alpha_{n-1}s^{n-1} \tag{68}$$

can be shown to be valid for each of the characteristic values of **A**. We first assume that all $n$ of the characteristic values are distinct, and we denote them by $s_1, s_2, \ldots, s_n$. Then

$$\begin{aligned}
\epsilon^{s_1t} &= \alpha_0 + \alpha_1 s_1 + \alpha_2 s_1^2 + \cdots + \alpha_{n-1}s_1^{n-1} \\
\epsilon^{s_2t} &= \alpha_0 + \alpha_1 s_2 + \alpha_2 s_2^2 + \cdots + \alpha_{n-1}s_2^{n-1} \\
&\vdots \\
\epsilon^{s_nt} &= \alpha_0 + \alpha_1 s_n + \alpha_2 s_n^2 + \cdots + \alpha_{n-1}s_n^{n-1}
\end{aligned} \tag{69}$$

For distinct characteristic values, these $n$ equations are linearly independent and can be solved simultaneously for the coefficients $\alpha_o$ through $\alpha_{n-1}$. The functions $\epsilon^{s_1t}, \epsilon^{s_2t}, \ldots, \epsilon^{s_nt}$ are called the **mode functions** for the matrix **A**. The $\alpha$'s will be linear combinations of these mode functions.

### EXAMPLE 6.10

Find $\epsilon^{\mathbf{A}t}$ in the form of (66) when

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

**Solution**

We first evaluate the determinant of $s\mathbf{I} - \mathbf{A}$ and factor the resulting characteristic polynomial.

$$\begin{aligned} |s\mathbf{I} - \mathbf{A}| &= \begin{vmatrix} s & -1 \\ 6 & s+5 \end{vmatrix} \\ &= s^2 + 5s + 6 \\ &= (s+2)(s+3) \end{aligned} \tag{70}$$

By (67), we see that the characteristic values of $\mathbf{A}$ are $s_1 = -2$ and $s_2 = -3$. Because the order of $\mathbf{A}$ is $n = 2$, the power series for $\epsilon^{\mathbf{A}t}$ in (66) contains only two terms:

$$\epsilon^{\mathbf{A}t} = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} \tag{71}$$

Replacing $\mathbf{A}$ by $s_1$ and then by $s_2$, we have the two scalar equations

$$\begin{aligned} \epsilon^{-2t} &= \alpha_0 - 2\alpha_1 \\ \epsilon^{-3t} &= \alpha_0 - 3\alpha_1 \end{aligned}$$

Solving these simultaneously yields

$$\begin{aligned} \alpha_0 &= 3\epsilon^{-2t} - 2\epsilon^{-3t} \\ \alpha_1 &= \epsilon^{-2t} - \epsilon^{-3t} \end{aligned} \tag{72}$$

Inserting (72) into (71) gives

$$\begin{aligned} \epsilon^{\mathbf{A}t} &= \begin{bmatrix} \alpha_0 & \alpha_1 \\ -6\alpha_1 & \alpha_0 - 5\alpha_1 \end{bmatrix} \\ &= \begin{bmatrix} 3\epsilon^{-2t} - 2\epsilon^{-3t} & \epsilon^{-2t} - \epsilon^{-3t} \\ -6\epsilon^{-2t} + 6\epsilon^{-3t} & -2\epsilon^{-2t} + 3\epsilon^{-3t} \end{bmatrix} \end{aligned} \tag{73}$$

---

If the characteristic values of $\mathbf{A}$ are not all distinct, then not all the equations in (69) are independent. Suppose that the characteristic value $s_1$ has multiplicity $p$. Then we can use not only (68) but also the first $p-1$ derivatives of this equation with respect to $s$, if $s$ is then replaced by $s_1$. For example, suppose the order of $\mathbf{A}$ is four, such that

$$\epsilon^{\mathbf{A}t} = \alpha_0\mathbf{I} + \alpha_1\mathbf{A} + \alpha_2\mathbf{A}^2 + \alpha_3\mathbf{A}^3$$



If $s_1$ is a characteristic value of multiplicity three, and $s_2$ is a characteristic value of multiplicity one, we may write

$$\begin{aligned} \epsilon^{s_1t} &= \alpha_0 + \alpha_1 s_1 + \alpha_2 s_1^2 + \alpha_3 s_1^3 \\ t\epsilon^{s_1t} &= \alpha_1 + 2\alpha_2 s_1 + 3\alpha_3 s_1^2 \\ t^2\epsilon^{s_1t} &= 2\alpha_2 + 6\alpha_3 s_1 \\ \epsilon^{s_2t} &= \alpha_0 + \alpha_1 s_2 + \alpha_2 s_2^2 + \alpha_3 s_2^3 \end{aligned}$$

which can be solved simultaneously for the coefficients $\alpha_0$ through $\alpha_3$ in terms of the four mode functions $\epsilon^{s_1t}, t\epsilon^{s_1t}, t^2\epsilon^{s_1t}$, and $\epsilon^{s_2t}$.

### EXAMPLE 6.11

Find $\epsilon^{\mathbf{A}t}$ when

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix}$$

**Solution**

To find the characteristic values of $\mathbf{A}$, we examine

$$\begin{aligned} |s\mathbf{I} - \mathbf{A}| &= \begin{vmatrix} s+1 & -1 \\ 1 & s+3 \end{vmatrix} \\ &= s^2 + 4s + 4 \\ &= (s+2)^2 \end{aligned}$$

from which $s_1 = s_2 = -2$. We know that

$$\begin{aligned} \epsilon^{\mathbf{A}t} &= \alpha_0\mathbf{I} + \alpha_1\mathbf{A} \\ &= \begin{bmatrix} \alpha_0 - \alpha_1 & \alpha_1 \\ -\alpha_1 & \alpha_0 - 3\alpha_1 \end{bmatrix} \end{aligned} \tag{74}$$

The scalar equations

$$\begin{aligned} \epsilon^{st} &= \alpha_0 + \alpha_1 s \\ t\epsilon^{st} &= \alpha_1 \end{aligned}$$

are valid when $s$ is replaced by $-2$. Thus

$$\begin{aligned} \epsilon^{-2t} &= \alpha_0 - 2\alpha_1 \\ t\epsilon^{-2t} &= \alpha_1 \end{aligned}$$

from which $\alpha_0 = \epsilon^{-2t} + 2t\epsilon^{-2t}$. Substituting the expressions for $\alpha_0$ and $\alpha_1$ into (74) gives

$$\epsilon^{\mathbf{A}t} = \begin{bmatrix} (1+t)\epsilon^{-2t} & t\epsilon^{-2t} \\ -t\epsilon^{-2t} & (1-t)\epsilon^{-2t} \end{bmatrix}$$

---

The quantity $\epsilon^{\mathbf{A}t}$, which plays a key role in many aspects of system analysis, is called the **state-transition matrix** and is usually given the special symbol $\phi(t)$. With $\epsilon^{\mathbf{A}t}$ replaced by the new symbol $\phi(t)$ in (64), the solution to $\dot{\mathbf{q}} = \mathbf{A}\mathbf{q}$ can be written as

$$\mathbf{q}(t) = \phi(t)\mathbf{q}(0) \tag{75}$$

Once $\phi(t)$ has been found, the state vector $\mathbf{q}(t)$ and the output vector $\mathbf{y}$ can be found in terms of the initial state vector $\mathbf{q}(0)$ by straightforward substitution into (75) and (63b). Because $\mathbf{q}$ is the product of an $n \times n$ matrix of time functions, $\phi(t)$, and an $n \times 1$ vector of constants, $\mathbf{q}(0)$, it follows that the elements of $\mathbf{q}$ are given by

$$q_i = \phi_{i1}(t)q_1(0) + \phi_{i2}(t)q_2(0) + \cdots + \phi_{in}(t)q_n(0) \quad i = 1, 2, \ldots, n$$

From this expression, we see that $q_i$ is a linear combination of the $n$ functions of time $\phi_{ij}(t)$, $j = 1, 2, \ldots, n$, where the weightings of these functions are the corresponding initial values of the state variables, $q_1(0), q_2(0), \ldots, q_n(0)$. Furthermore, we have previously seen that the functions $\phi_{ij}(t)$ are themselves linear combinations of the $n$ mode functions. Hence it follows that each of the state variables $q_i(t)$ is merely a linear combination of the system's mode functions, with the relative weightings dependent on the initial state vector $\mathbf{q}(0)$ and the coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ in (66).

Once we have found the state vector $\mathbf{q}$, it is a straightforward matter to determine the response of the output variables. Because $\mathbf{y} = \mathbf{C}\mathbf{q}$ when the input is zero, the $k$th output is a linear combination of all the state variables, the weightings being given by the elements in the $k$th row of the $\mathbf{C}$ matrix. Specifically,

$$y_k = c_{k1}q_1 + c_{k2}q_2 + \cdots + c_{kn}q_n \quad k = 1, 2, \ldots, p$$

Before discussing some of the properties of the state-transition matrix, we shall illustrate its use in evaluating both the state and output vectors for a second-order example in numerical form

▶ *EXAMPLE 6.12*

Determine the zero-input responses of the state and output vectors for the system described by the matrix equations

$$\dot{\mathbf{q}} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}\mathbf{q} \tag{76a}$$

$$\mathbf{y} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\mathbf{q} \tag{76b}$$

Give the responses in terms of the elements of the initial state vector $\mathbf{q}(0)$.



**Solution**

From inspection of (76a) and (76b), we see that the matrices $\mathbf{A}$ and $\mathbf{C}$ characterizing the system are

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

and

$$\mathbf{C} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$$

The state-transition matrix corresponding to $\mathbf{A}$ was evaluated in Example 6.10 and is given by (73). Hence to find $\mathbf{q}$ in terms of the initial states $q_1(0)$ and $q_2(0)$, we need only substitute (73) into (75) and carry out the matrix multiplication. For the first state variable, we find that

$$q_1 = (3\epsilon^{-2t} - 2\epsilon^{-3t})q_1(0) + (\epsilon^{-2t} - \epsilon^{-3t})q_2(0)$$

which can be rewritten in terms of the mode functions $\epsilon^{-2t}$ and $\epsilon^{-3t}$ as

$$q_1 = [3q_1(0) + q_2(0)]\epsilon^{-2t} + [-2q_1(0) - q_2(0)]\epsilon^{-3t}$$

Similarly, you may verify that the response of the second state variable is

$$q_2 = [-6q_1(0) - 2q_2(0)]\epsilon^{-2t} + [6q_1(0) + 3q_2(0)]\epsilon^{-3t}$$

Finally, we can obtain the response of the output variable $\mathbf{y}$ by substituting these expressions for $q_1$ and $q_2$ into the output equation given by (76b). For $y_1$, we get

$$y_1 = 2q_1 - q_2$$

which simplifies to

$$y_1 = [12q_1(0) + 4q_2(0)]\epsilon^{-2t} - [10q_1(0) + 5q_2(0)]\epsilon^{-3t} \tag{77}$$

The second element of the output vector is just $q_2$, so

$$y_2 = [-6q_1(0) - 2q_2(0)]\epsilon^{-2t} + [6q_1(0) + 3q_2(0)]\epsilon^{-3t} \tag{78}$$

## Properties of the State-Transition Matrix

Starting with (75), which states that

$$\mathbf{q}(t) = \phi(t)\mathbf{q}(0) \tag{79}$$

we can develop several properties of $\phi(t)$ that are useful in the analysis of dynamic systems. First, by setting $t$ equal to zero in (79), we have

$$\mathbf{q}(0) = \phi(0)\mathbf{q}(0)$$

which implies that

$$\phi(0) = \mathbf{I} \tag{80}$$

Thus the state-transition matrix reduces to the $n \times n$ identity matrix when its argument equals zero.

The initial time in (79) need not be restricted to $t = 0$. More generally, we can rewrite (79) in terms of the initial time $t_0$ as

$$\mathbf{q}(t) = \phi(t - t_0)\mathbf{q}(t_0) \tag{81}$$

which reduces to (79) if $t_0 = 0$. Using (81), we can express the state vector at some time $t_2$ in terms of $\mathbf{q}(t_0)$ as

$$\mathbf{q}(t_2) = \phi(t_2 - t_0)\mathbf{q}(t_0)$$

or in terms of $\mathbf{q}(t_1)$ as

$$\mathbf{q}(t_2) = \phi(t_2 - t_1)\mathbf{q}(t_1)$$

By using (81) again to express $\mathbf{q}(t_1)$ in terms of $\mathbf{q}(t_0)$, we can write the second expression for $\mathbf{q}(t_2)$ as

$$\mathbf{q}(t_2) = \phi(t_2 - t_1)\phi(t_1 - t_0)\mathbf{q}(t_0)$$

Now, comparing the two expressions for $\mathbf{q}(t_2)$, we see that

$$\phi(t_2 - t_1)\phi(t_1 - t_0) = \phi(t_2 - t_0) \tag{82}$$

Hence the product of two state-transition matrices (for the same $\mathbf{A}$ matrix, of course) is merely the state-transition matrix with its argument set equal to the sum of the arguments of the two matrices being multiplied.

If we write (82) with $t_0 = t_2 = 0$, we see that

$$\phi(-t_1)\phi(t_1) = \phi(0)$$

But we know from (80) that $\phi(0)$ is the identity matrix, so

$$\phi(-t_1)\phi(t_1) = \mathbf{I}$$

which implies that

$$\phi^{-1}(t) = \phi(-t) \tag{83}$$

Thus the state-transition matrix is nonsingular for all values of its argument, and we can obtain its inverse by merely changing the sign of its argument.

One way of gaining some insight into the physical significance of the state-transition matrix is to consider (79) for some very specific values of $\mathbf{q}(0)$. For instance, suppose $q_1(0) = 1$ and all the other elements of $\mathbf{q}(0)$ are zero. It follows from the definition of matrix multiplication that the resulting response for the state vector $\mathbf{q}$ will be the first column of $\phi(t)$. In similar fashion, setting the $i$th initial state to unity and the rest to zero will result in a response for $\mathbf{q}$ that consists of the $i$th column of $\phi(t)$. Hence we could use a computer simulation of a system repetitively to compute and plot each element of the state-transition matrix, one column at a time, until we had determined all $n^2$ time functions.



### ▶ *EXAMPLE 6.13*

For the state-transition matrix found in Example 6.10 and given by (73), verify that (80) and (83) are true in general and show that (82) holds for $t_0 = 0$, $t_1 = 1$, and $t_2 = 2$.

**Solution**

Setting $t$ equal to zero in (73) gives

$$\phi(0) = \begin{bmatrix} 3\epsilon^0 - 2\epsilon^0 & \epsilon^0 - \epsilon^0 \\ -6\epsilon^0 + 6\epsilon^0 & -2\epsilon^0 + 3\epsilon^0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which agrees with (80).

To check (83), we shall show that $\phi(-t)\phi(t) = \mathbf{I}$, where $\mathbf{I}$ is the $2 \times 2$ identity matrix. From (73), we have

$$\phi(t) = \begin{bmatrix} 3\epsilon^{-2t} - 2\epsilon^{-3t} & \epsilon^{-2t} - \epsilon^{-3t} \\ -6\epsilon^{-2t} + 6\epsilon^{-3t} & -2\epsilon^{-2t} + 3\epsilon^{-3t} \end{bmatrix}$$

so

$$\phi(-t) = \begin{bmatrix} 3\epsilon^{2t} - 2\epsilon^{3t} & \epsilon^{2t} - \epsilon^{3t} \\ -6\epsilon^{2t} + 6\epsilon^{3t} & -2\epsilon^{2t} + 3\epsilon^{3t} \end{bmatrix}$$

For example, the 2,2 element[4] of $\phi(-t)\phi(t)$ is the product of the second row of $\phi(-t)$ and the second column of $\phi(t)$:

$$\begin{aligned} &[-6\epsilon^{2t} + 6\epsilon^{3t} \quad -2\epsilon^{2t} + 3\epsilon^{3t}]\begin{bmatrix} \epsilon^{-2t} - \epsilon^{-3t} \\ -2\epsilon^{-2t} + 3\epsilon^{-3t} \end{bmatrix} \\ &\quad = (-6\epsilon^{2t} + 6\epsilon^{3t})(\epsilon^{-2t} - \epsilon^{-3t}) + (-2\epsilon^{2t} + 3\epsilon^{3t})(-2\epsilon^{-2t} + 3\epsilon^{-3t}) \\ &\quad = 1 \end{aligned}$$

In similar fashion, you can verify that the 1,1 element is also unity and that the 1,2 and 2,1 elements are zero, giving the $2 \times 2$ identity matrix.

To verify (82) for the specified values of $t_0$, $t_1$, and $t_2$, we must show that

$$\phi(2 - 1)\phi(1 - 0) = \phi(2 - 0)$$

[4]The $i, j$ element of a matrix is the one in the $i$th row and the $j$th column.

which is equivalent to

$$\phi(1)\phi(1) = \phi(2) \tag{84}$$

For example, we can write the 1,1 element of the left-hand side of (84) as the matrix product

$$\begin{aligned}
&[3\epsilon^{-2} - 2\epsilon^{-3} \quad \epsilon^{-2} - \epsilon^{-3}]\begin{bmatrix} 3\epsilon^{-2} - 2\epsilon^{-3} \\ -6\epsilon^{-2} + 6\epsilon^{-3} \end{bmatrix} \\
&= (3\epsilon^{-2} - 2\epsilon^{-3})^2 + (\epsilon^{-2} - \epsilon^{-3})(-6\epsilon^{-2} + 6\epsilon^{-3}) \\
&= 3\epsilon^{-4} - 2\epsilon^{-6}
\end{aligned}$$

which is indeed equal to the 1,1 element of $\phi(2)$. By evaluating the remaining three elements that make up the left-hand and right-hand sides of (84), you can show that (82) does hold for the values of $t_0$, $t_1$, and $t_2$ selected. It would be possible to verify (82) for arbitrary $t_0$, $t_1$, and $t_2$, but the process would be more tedious than it is when specific values are selected.

## The Complete Response

Finally, we consider the response for the case where the input vector is nonzero. The special case of the zero-state response corresponds to setting $\mathbf{q}(0) = \mathbf{0}$ in the following presentation.

In the solution of scalar differential equations earlier in this chapter, we regarded the response as the sum of a complementary and particular solution. Moreover, we had to know the form of the input before starting to find the particular solution. A different approach would have been necessary in order to obtain an explicit formula that would be valid for all inputs. It can be shown, however, that the solution to the scalar equation $\dot{q} = aq + bu$ can be written as

$$q(t) = \underbrace{\epsilon^{at}q(0)}_{q_{zi}(t)} + \underbrace{\int_0^t \epsilon^{a(t-\lambda)}bu(\lambda)d\lambda}_{q_{zs}(t)} \tag{85}$$

for $t > 0$. The symbols $q_{zi}(t)$ and $q_{zs}(t)$ denote the zero-input and zero-state portions, respectively, of the general expression. The development of such equations as (85) can be found not only in books on differential equations but also in books on circuits and systems, often under the heading of convolution. An even more general result for the zero-state response, which is valid for a scalar differential equation of any order, is

$$q_{zs}(t) = \int_0^t h(t-\lambda)bu(\lambda)d\lambda$$

where $h(t)$ is the response of $q(t)$ to a unit impulse in $u(t)$ when the system has no stored energy for $t < 0$.



We return now to the general matrix equations in (62), which are repeated here:

$$\dot{\mathbf{q}} = \mathbf{Aq} + \mathbf{Bu} \tag{86a}$$

$$\mathbf{y} = \mathbf{Cq} + \mathbf{Du} \tag{86b}$$

As expected by analogy to (85), the solution to (86a) is

$$\mathbf{q}(t) = \underbrace{\phi(t)\mathbf{q}(0)}_{\mathbf{q}_{zi}(t)} + \underbrace{\int_0^t \phi(t-\lambda)\mathbf{Bu}(\lambda)d\lambda}_{\mathbf{q}_{zs}(t)} \tag{87}$$

where $\phi(t)$ is the state-transition matrix, and where $\phi(t-\lambda)$ is $\phi(t)$ with every $t$ replaced by $t - \lambda$. The symbols $\mathbf{q}_{zi}(t)$ and $\mathbf{q}_{zs}(t)$ denote the zero-input and zero-state parts of the state vector. Once (87) has been evaluated, the output vector can be easily obtained via the algebraic equation (86b). To illustrate the evaluation of the integral in (87) and to show the general procedure, the following example examines the zero-state response for a system that has the same **A** and **C** matrices as in Example 6.12.

► *EXAMPLE 6.14*

Find the state and output vectors for the system described by (86) when

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}, \quad \text{and } \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and when $\mathbf{q}(0) = \mathbf{0}$. Let the input vector be

$$\mathbf{u} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ U(t) \end{bmatrix}$$

That is, let $u_1(t)$ be zero, and let $u_2(t)$ be the unit step function.

**Solution**

The state-transition matrix $\phi(t)$ for the given matrix **A** was found in (73). Thus

$$\begin{aligned}
\phi(t)\mathbf{B} &= \begin{bmatrix} 3\epsilon^{-2t} - 2\epsilon^{-3t} & \epsilon^{-2t} - \epsilon^{-3t} \\ -6\epsilon^{-2t} + 6\epsilon^{-3t} & -2\epsilon^{-2t} + 3\epsilon^{-3t} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 5\epsilon^{-2t} - 3\epsilon^{-3t} & 3\epsilon^{-2t} - 2\epsilon^{-3t} \\ -10\epsilon^{-2t} + 9\epsilon^{-3t} & -6\epsilon^{-2t} + 6\epsilon^{-3t} \end{bmatrix}
\end{aligned}$$

Replacing $t$ by $t - \lambda$ and postmultiplying by the column vector $\mathbf{u}(\lambda)$ give

$$\begin{aligned}
&\phi(t-\lambda)\mathbf{Bu}(\lambda) \\
&= \begin{bmatrix} \left(5\epsilon^{-2(t-\lambda)} - 3\epsilon^{-3(t-\lambda)}\right)u_1(\lambda) + \left(3\epsilon^{-2(t-\lambda)} - 2\epsilon^{-3(t-\lambda)}\right)u_2(\lambda) \\ \left(-10\epsilon^{-2(t-\lambda)} + 9\epsilon^{-3(t-\lambda)}\right)u_1(\lambda) + \left(-6\epsilon^{-2(t-\lambda)} + 6\epsilon^{-3(t-\lambda)}\right)u_2(\lambda) \end{bmatrix}
\end{aligned}$$

Substituting $u_1(\lambda) = 0$ and $u_2(\lambda) = 1$ for all $\lambda > 0$ yields

$$\phi(t-\lambda)\mathbf{B}\mathbf{u}(\lambda) = \begin{bmatrix} 3\epsilon^{-2t}\epsilon^{2\lambda} - 2\epsilon^{-3t}\epsilon^{3\lambda} \\ -6\epsilon^{-2t}\epsilon^{2\lambda} + 6\epsilon^{-3t}\epsilon^{3\lambda} \end{bmatrix}$$

Integrating each row of this matrix and inserting the limits of 0 and $t$, we see that (86a) reduces to

$$\mathbf{q}(t) = \begin{bmatrix} \frac{3}{2}\epsilon^{-2t}\epsilon^{2\lambda} - \frac{2}{3}\epsilon^{-3t}\epsilon^{3\lambda} \\ -3\epsilon^{-2t}\epsilon^{2\lambda} + 2\epsilon^{-3t}\epsilon^{3\lambda} \end{bmatrix}_0^t = \begin{bmatrix} \frac{5}{6} - \frac{3}{2}\epsilon^{-2t} + \frac{2}{3}\epsilon^{-3t} \\ -1 + 3\epsilon^{-2t} - 2\epsilon^{-3t} \end{bmatrix}$$

The output matrix $\mathbf{y}$ is, according to (86b),

$$\mathbf{y}(t) = \begin{bmatrix} 2q_1 - q_2 \\ q_2 \end{bmatrix} = \begin{bmatrix} \frac{8}{3} - 6\epsilon^{-2t} + \frac{10}{3}\epsilon^{-3t} \\ -1 + 3\epsilon^{-2t} - 2\epsilon^{-3t} \end{bmatrix}$$

Note that the two outputs become constants in the steady state, because the inputs have constant values for $t > 0$. As expected, the transient terms are linear combinations of the two mode functions $\epsilon^{-2t}$ and $\epsilon^{-3t}$.

## SUMMARY

After a review of procedures for solving differential equations, this chapter emphasized first- and second-order systems. Among the important general concepts were the definitions of the free response (usually the same as the transient response), the forced response (usually the same as the steady-state response), the zero-input and zero-state responses, and system stability. The nature of the free response is characterized by the time constant $\tau$ for a first-order system and by the damping ratio $\zeta$ and the undamped natural frequency $\omega_n$ for a second-order system.

Inputs of special importance include the unit step function and the unit impulse. The steady-state response to a constant input can always be determined from the differential equations, but it can also be found directly from the figures representing the systems.

The state-variable model of a first-order system is not essentially different from an input-output model. However, for systems of order two and higher, the state-variable equations are usually coupled together and cannot be solved individually. A systematic way of solving a state-variable model was presented in Section 6.6. The key step is finding the state-transition matrix $\phi(t)$, which is the same as the matrix function $\epsilon^{\mathbf{A}t}$.

## PROBLEMS

**6.1** Find and sketch the response of the system described by the equation

$$\dot{y} + 4y = F(t)$$

**a)** When the input is $F(t) = 3$ for $t \geq 0$ and $y(0) = 2$.
**b)** When $F(t) = t$ for $t \geq 0$ and $y(0) = 0$.



* **6.2** Find the response $y(t)$ for $t \geq 0$ for a linear first-order system described by $\dot{y} + 0.5y = F(t)$ for each of the following conditions.
**a)** $F(t) = \sin t + \cos t$ and $y(0) = 0$
**b)** $F(t) = \epsilon^{-t/2}$ and $y(0) = 1$

**6.3** Find and sketch the response of the system described by the equation

$$3\dot{y} + y = F(t)$$

with the initial condition $y(0) = 4$ when the input is
**a)** $F(t) = 5$
**b)** $F(t) = 4\epsilon^{-t/2}$
**c)** $F(t) = 5 + 4\epsilon^{-t/2}$

**6.4** Find the response $y(t)$ for $t \geq 0$ for a linear first-order system described by $2\dot{y} + 6y = F(t)$ for each of the following conditions.
**a)** $F(t) = 4\epsilon^{-t}$ and $y(0) = 8$
**b)** $F(t) = \sin 3t$ and $y(0) = 0$

* **6.5** Find the response of the system described by

$$\ddot{y} + 3\dot{y} + 2y = \dot{u} + 2u(t)$$

when $u(t) = t$ for $t \geq 0$, $y(0) = 0$, and $\dot{y}(0) = 1$.

**6.6** Consider the differential equation $\ddot{y} + 7\dot{y} + 6y = F(t)$.
**a)** Find the complete response for $t \geq 0$ when $F(t) = 3$, $y(0) = 0$, and $\dot{y}(0) = -2$.
**b)** Repeat part (a) when the initial conditions are changed to $y(0) = 2$, $\dot{y}(0) = 0$.

**6.7** Repeat Problem 6.5 when $u(t) = t\epsilon^{-t}$ for $t \geq 0$ and $y(0) = \dot{y}(0) = 0$.

* **6.8** Find the response of the first-order system described by

$$3\dot{y} + 2y = A$$

when $y(0) = 0$ and $A$ is a constant. Give the values of the time constant and the steady-state response. Identify the transient response and sketch the complete response.

**6.9** Repeat Problem 6.8 for the equation

$$\frac{1}{2}\dot{y} + 4y = A$$

**6.10** Repeat Problem 6.8 for the equation

$$2\dot{y} + 5y = A$$

**6.11** A velocity input $v_a(t)$ is applied to point $A$ in the mechanical system shown in Figure P6.11.
**a)** Write the system's differential equation in terms of the velocity $v_1$.
**b)** What is the time constant $\tau$ for the system?
**c)** Sketch the response when $v_a(t) = 0$ for $t \geq 0$ and $v_1(0) = 10$.

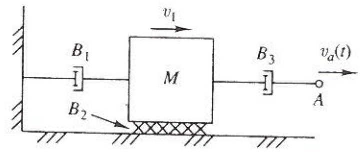


FIGURE P6.11

**d)** Repeat parts (a), (b), and (c) when the velocity input is replaced by a force $f_a(t)$ applied at point $A$, with the positive sense to the right. Explain why the expression for $\tau$ differs from the answer to part (b).

* **6.12** Assume that the circuit shown in Figure P5.1 contains no stored energy for $t < 0$ and that the input is the unit step function. The input-output differential equation, found in Problem 5.1, is

$$\frac{3}{4}\dot{e}_o + \frac{1}{2}e_o = \frac{1}{2}\frac{di_i}{dt} + i_i(t)$$

**a)** Verify the differential equation and determine the time constant.
**b)** For $t > 0$, let $i_i(t)$ have a constant value of unity. Then write the general solution for $t > 0$.
**c)** Find $e_o(0+)$ and evaluate the arbitrary constant in the answer to part (b).
**d)** Find and sketch $e_o$ versus $t$ for all $t > 0$.
**e)** Check the steady-state response by replacing the capacitor by an open circuit.

**6.13** Repeat Problem 6.12 for the circuit shown in Figure P5.4, for which the input-output differential equation, found in Problem 5.4, is

$$C(R_1 + R_2)\dot{e}_o + e_o = CR_1R_2\frac{di_i}{dt}$$

**6.14** Assume that the circuit shown in Figure P5.2 contains no stored energy for $t < 0$ and that the input is the unit step function. The input-output differential equation, found in Problem 5.2, is

$$\dot{e}_o + 9e_o = 3e_i(t)$$

**a)** Verify the differential equation and determine the time constant.
**b)** Find $e_o(0+)$.
**c)** Find and sketch $e_o$ versus $t$ for all $t > 0$.
**d)** Check the steady-state response by replacing the inductor by a short circuit.

**6.15** The switch shown in Figure P6.15 has been closed for a long time, so the circuit is in the steady state at $t = 0-$. If the switch opens at $t = 0$, find and sketch $e_o$ versus $t$. Show on the same scale the curves for $R_o = 5\ \Omega$ and for $R_o = 10\ \Omega$. What would be the circuit's response if $R_o$ were to approach infinity?

* **6.16** Consider a first-order system with input $u(t)$ and output $y$ that is described by $\dot{y} + 0.5y = u(t)$.
**a)** Find and sketch the unit step response $y_U(t)$.
**b)** Find and sketch the response to $u(t) = 2$ for $t > 0$ and with $y(0) = -1$.
**c)** Find and sketch the response to $u(t) = U(t) - U(t-2)$ with $y(0) = 0$.



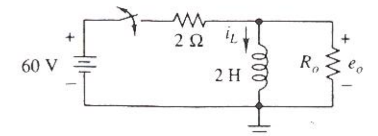


FIGURE P6.15

**d)** Find and sketch the unit impulse response $h(t)$.

**6.17** Figure P6.17 shows the unit step responses of three first-order systems that have the same steady-state value but different time constants. The characteristic-equation root of system $A$ is $r_A = -0.2$, that of system $B$ is $r_B = -1.0$, and that of system $C$ is $r_C = -0.5$. Determine which system generated each response, give the time constant, and state your reasoning.

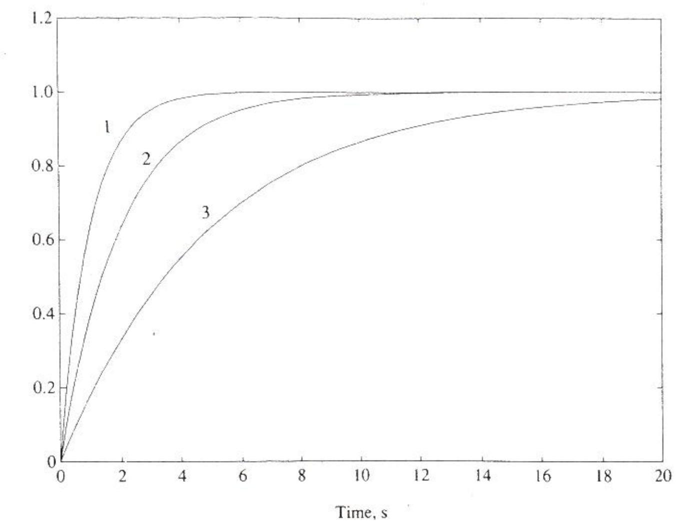


FIGURE P6.17

**6.18** The unit step responses of four second-order systems are shown in Figure P6.18, and four pairs of characteristic roots are listed below. Match each pair of roots with the corresponding step-response curve.

| | |
|---|---|
| A | $r = -0.5 \pm j2$ |
| B | $r = -0.5 \pm j4$ |
| C | $r = -1 \pm j4$ |
| D | $r = -2 \pm j4$ |

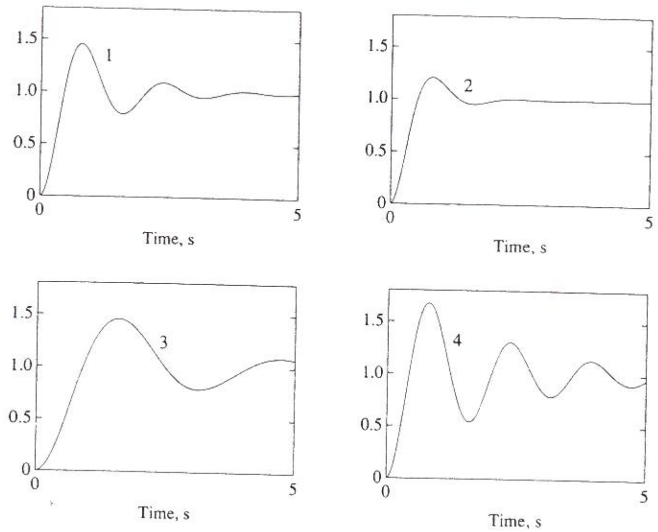


FIGURE P6.18

**6.19** The unit impulse responses of four second-order systems are shown in Figure P6.19, and four pairs of characteristic roots are listed below. Match each pair of roots with the corresponding impulse-response curve.

| | |
|---|---|
| A | $r = -0.2 \pm j1$ |
| B | $r = -0.2 \pm j3$ |
| C | $r = -0.2, -0.3$ |
| D | $r = -0.5 \pm j1$ |

* **6.20** If the spring $K_1$ in Figure 2.19 is replaced by a cable that does not stretch, the system can be shown to obey the differential equation

$$(M_1 + M_2)\ddot{x} + (B_1 + B_2)\dot{x} + K_2 x = f_a(t) - M_2 g$$

where $x_1 = x_2 = x$.

**a)** Verify that the differential equation is correct.
**b)** Find expressions for the damping ratio $\zeta$ and the undamped natural frequency $\omega_n$.
**c)** Find the steady-state response when the applied force is the unit step function.

**6.21** The input-output differential equation for the mechanical system shown in Figure P4.22 is

$$(M + J/R^2)\ddot{x} + B\dot{x} + (K_2 + K_1/R^2)x = -f_a(t)$$

**a)** Verify that the differential equation is correct.



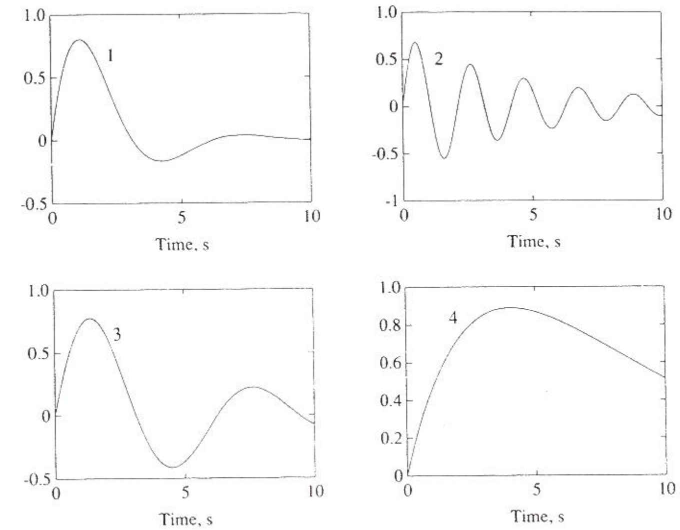


FIGURE P6.19

**b)** Find expressions for the damping ratio $\zeta$ and the undamped natural frequency $\omega_n$.
**c)** Find the steady-state response when the applied force is the unit step function. Check your answer by examining Figure P4.22 directly in the steady state.

**6.22** The input-output differential equation for the circuit shown in Figure P6.22 is

$$C\ddot{e}_o + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\dot{e}_o + \frac{1}{L}e_o = \frac{1}{R_1}\dot{e}_i$$

**a)** Verify that the differential equation is correct.
**b)** Find expressions for the damping ratio $\zeta$ and the undamped natural frequency $\omega_n$.

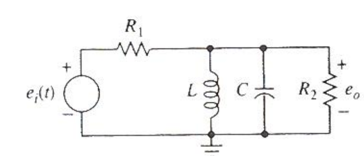


FIGURE P6.22

**6.23** For the differential equation $\ddot{y} + 5\dot{y} + 4y = u(t)$, find and sketch the unit step response $y_U(t)$ and the unit impulse response $h(t)$.

**6.24** A second-order system is described by the equation $\ddot{y} + 4\dot{y} + 25y = 50u(t)$.
**a)** Find the damping ratio $\zeta$ and the undamped natural frequency $\omega_n$.
**b)** Find and sketch the unit step response $y_U(t)$ and the unit impulse response $h(t)$.

* **6.25** Repeat Problem 6.24 for the differential equation $\ddot{y} + 2\dot{y} + 2y = u(t)$.

**6.26** A second-order system is described by $\ddot{y} + 2\dot{y} + 4y = 4u(t)$.
**a)** Find the damping ratio $\zeta$ and the undamped natural frequency $\omega_n$.
**b)** Find and sketch the response when $u(t) = 2$ for $t > 0$ and when the initial conditions are $y(0) = 0$ and $\dot{y}(0) = 1$.

* **6.27** The unit step response of a certain linear system is

$$y_U(t) = [2 + 2\epsilon^{-t}\cos(2t + \pi/4)]U(t)$$

**a)** Find the numerical values of the damping ratio $\zeta$ and the undamped natural frequency $\omega_n$.
**b)** Find the unit impulse response $h(t)$. Sketch $y_U(t)$ and $h(t)$.

* **6.28** A certain third-order system is described by the differential equation

$$\dddot{y} + 2\ddot{y} + \dot{y} + 2y = 4.$$

Find $y(t)$ when $y(0) = 0$, $\dot{y}(0) = 5$, and $\ddot{y}(0) = -3$. The roots of the system's characteristic equation are $r_1 = -2$, $r_2 = j1$, and $r_3 = -j1$.

**6.29** The input-output equation for the translational mechanical system shown in Figure 3.5(a) was found in (3.27). When, in a consistent set of units, the parameter values $M = B_1 = B_2 = K_1 = K_2 = 1$ are used and the input $f_a(t)$ is the unit step function, the equation becomes

$$\dddot{x}_1 + 3\ddot{x}_1 + 3\dot{x}_1 + x_1 = 2 \quad \text{for } t > 0$$

**a)** Verify that the differential equation is correct.
**b)** By examining the figure, show that $x_1(0+) = \dot{x}_1(0+) = 0$ and $\ddot{x}_1(0+) = 1$.
**c)** Find $x_1(t)$ for all $t > 0$.

**6.30** A linear fourth-order system is described by the input-output equation

$$a_4 y^{(iv)} + a_3 y^{(iii)} + a_2\ddot{y} + a_1\dot{y} + a_0 y = F(t)$$

Write the form of the complete response when $F(t) = A$ for each of the following sets of characteristic-root locations. Do not evaluate the multiplying constants in front of the individual terms. Identify the transient and steady-state responses.
**a)** $0,\ -2,\ -2,\ -5$
**b)** $-1,\ -2,\ \pm j3$
**c)** $-2,\ -2,\ -2 \pm j3$
**d)** $-2 \pm j3,\ -2 \pm j3$

**6.31** Repeat Problem 6.30 when $F(t) = \cos 3t$.

**6.32** A system obeying the state-variable equations $\dot{\mathbf{q}} = \mathbf{A}\mathbf{q}$ and $\mathbf{y} = \mathbf{C}\mathbf{q}$ has

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 2 & -1 \end{bmatrix}$$

**a)** Calculate the state-transition matrix $\phi(t)$.
**b)** Write $\mathbf{y}_{zi}(t)$ in terms of the initial states $q_i(0)$.



**c)** Verify that
i. $\phi(0) = \mathbf{I}$
ii. $\phi^{-1}(t) = \phi(-t)$
iii. $\phi(1)\,\phi(2) = \phi(3)$

**6.33** Repeat Problem 6.32 when

$$\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 4 & 1 \end{bmatrix}$$

* **6.34** Repeat parts (a) and (b) of Problem 6.32 when

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} -1 & -2 \end{bmatrix}$$

* **6.35** Find the zero-state response of the output vector $\mathbf{y}(t)$ for the multi-input multi-output system discussed in Example 6.14 when the inputs are $u_1(t) = \epsilon^{-t}U(t)$ and $u_2(t) = 0$.

**6.36** Using the results of Examples 6.12 and 6.14, find the complete response of the system output vector $\mathbf{y}(t)$ when the initial states are $q_1(0) = -1$ and $q_2(0) = 1$ and the inputs are $u_1(t) = 0$ and $u_2(t) = 1$ for $t > 0$.