C H A P T E R 5

ELECTRICAL SYSTEMS

Except at quite high frequencies, electrical circuits can usually be considered an interconnection of lumped elements. In such cases, which include a large and very important portion of the applications of electrical phenomena, we can model a circuit by using ordinary differential equations and apply the solution techniques discussed in this book.

In this chapter, we shall consider fixed linear circuits using the same approach we used for mechanical systems. We shall introduce the element and interconnection laws and then combine them to form procedures for finding the model of a circuit. After developing a general technique for finding the input-output model, we shall present several specialized results for the important case of resistive circuits. We shall then discuss systematic procedures for obtaining the model as a set of state-variable equations. A discussion of controlled sources, with an emphasis on the operational amplifier, concludes the chapter.

5.1 VARIABLES

The variables most commonly used to describe the behavior of electrical circuits are

e, voltage in volts (V)

i, current in amperes (A)

The related variables

q, charge in coulombs (C)

ϕ, flux in webers (Wb)

Λ, flux linkage in weber-turns

may be used on occasion. Current is the time derivative of charge, so i and q are related by the expressions

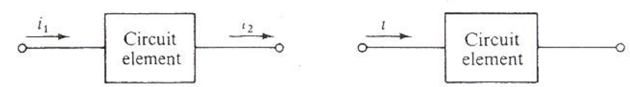
$$i = \frac{dq}{dt} \tag{1}$$

and

$$q(t) = q(t_0) + \int_{t_0}^{t} i(\lambda)d\lambda \tag{2}$$

Flux and flux linkage are related by the number of turns N in a coil of wire, such that if all the turns are linked by all the flux, then $\Lambda = N\phi$.

We represent the current into and out of a circuit element by arrows drawn on the circuit diagram as shown in Figure 5.1. The arrows point in the direction in which positive charge—that is, positive ions—flows when the current has a positive value. Equivalently, a positive current can also correspond to electrons (which have a negative charge) flowing in the opposite direction.



(a) (b)

FIGURE 5.1 Conventions for denoting current. (a) Acceptable. (b) Preferred.

Because a net charge cannot exist within any circuit element, the current entering one end of a two-terminal element must leave the other end. Hence $i_1 = i_2$ in Figure 5.1(a) at all times, so only one current arrow need be shown, as in Figure 5.1(b).

The voltage at a point in a circuit is a measure of the difference between the electrical potential of that point and the potential of an arbitrarily established reference point called the **ground node**, or **ground** for short. The ground associated with a circuit is denoted by the symbol shown in the lower part of Figure 5.2(a). Any point in the circuit that has the same potential as the ground has a voltage of zero, by definition. The voltage e_1 shown in Figure 5.2(a) is positive if the point with which it is associated is at a higher potential than the ground; it is negative if the potential of the point with which it is associated is lower than that of the ground.

We can define the voltages of the two terminals of a circuit element individually with respect to ground by writing appropriate symbols next to the terminals, as shown in Figure 5.2(b). We define the voltage between the terminals of an element by placing a symbol next to the element and plus

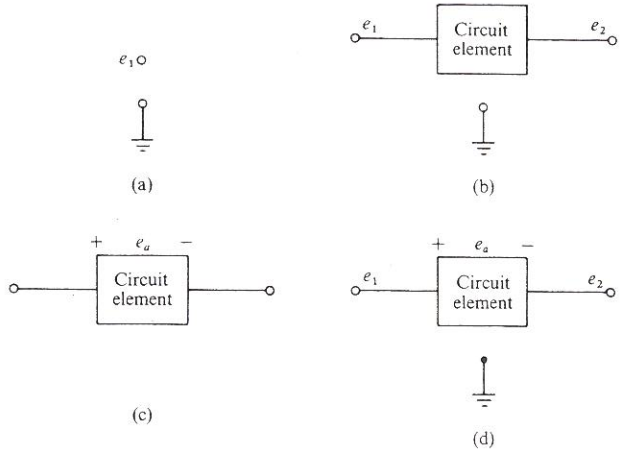


FIGURE 5.2 Conventions for denoting voltages.

and minus signs on either side of the element or at the terminals, as shown in Figure 5.2(c). When the element voltage e_a is positive, the terminal marked with the plus sign is at a higher potential than the other terminal.

In Figure 5.2(d), e_1 and e_2 denote the terminal voltages with respect to ground, and e_a is the voltage across the element, with its positive sense indicated by the plus and minus signs. These three voltages are related by the equation

$$e_a = e_1 - e_2$$

Interchanging the plus and minus signs reverses the sign of the voltage e_a in any equation in which it appears.

When we define the positive senses of the current and voltage associated with a circuit element as shown in Figure 5.3, such that a positive current is assumed to enter the element at the terminal designated by the plus sign, then the power supplied to the element is

$$p = ei \tag{3}$$

which has units of watts. If at some instant p is negative, then the circuit element is supplying power to the rest of the circuit at that instant. Because

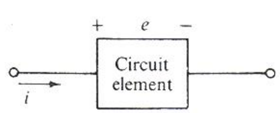


FIGURE 5.3 Positive senses of voltage and current for (3).

power is the time derivative of energy, the energy supplied to the element over the interval t_0 to t_1 is

$$\int_{t_0}^{t_1} p(t)dt$$

which has units of joules, where 1 joule = 1 volt-ampere-second.

5.2 ELEMENT LAWS

The elements in the electrical circuits that we shall consider are resistors, capacitors, inductors, and sources. The first three of these are referred to as **passive elements** because, although they can store or dissipate energy that is present in the circuit, they cannot introduce additional energy. They are analogous to the dashpot, mass, and spring for mechanical systems. In contrast, sources are **active elements** that can introduce energy into the circuit and that serve as the inputs. They are analogous to the force or displacement inputs for mechanical systems.

Resistor

A **resistor** is an element for which there is an algebraic relationship between the voltage across its terminals and the current through it—that is, an element that can be described by a curve of e versus i. A linear resistor is one for which the voltage and current are directly proportional to each other—that is, one described by **Ohm's law**:

$$e = Ri \tag{4}$$

or

$$i = \frac{1}{R}e \tag{5}$$

where R is the **resistance** in ohms (Ω). A resistor and its current and voltage are denoted as shown in Figure 5.4. If we reversed either the current arrow or the voltage polarity (but not both) in the figure, we would introduce a minus sign into (4) and (5). The resistance of a body of length ℓ and constant cross-sectional area A made of a material with resistivity ρ is $R = \rho\ell/A$.

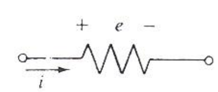


FIGURE 5.4 A resistor and its variables.

A resistor dissipates any energy supplied to it by converting it into heat (in this it is analogous to the frictional element of mechanical systems). We can write the power ei dissipated by a linear resistor as

$$p = Ri^2 = \frac{1}{R}e^2$$

Capacitor

A **capacitor** is an element that obeys an algebraic relationship between the voltage and the charge, where the charge is the integral of the current. We use the symbol shown in Figure 5.5 to represent a capacitor. For a linear capacitor, the charge and voltage are related by

$$q = Ce \tag{6}$$

where C is the **capacitance** in farads (F). For a fixed linear capacitor, the capacitance is a constant. If (6) is differentiated and $\dot{q}$ replaced by i, the element law for a fixed linear capacitor becomes

$$i = C\frac{de}{dt} \tag{7}$$

To express the voltage across the terminals of the capacitor in terms of the current, we solve (7) for de/dt and then integrate, getting

$$e(t) = e(t_0) + \frac{1}{C}\int_{t_0}^{t} i(\lambda)d\lambda \tag{8}$$

where $e(t_0)$ is the voltage corresponding to the initial charge, and where the integral is the charge delivered to the capacitor between the times t_0 and t.

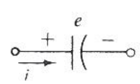


FIGURE 5.5 A capacitor and its variables.

One form of a capacitor consists of two parallel metallic plates, each of area A, separated by a dielectric material of thickness d. Provided that fringing of the electric field is negligible, the capacitance of this element is $C = \varepsilon A/d$, where ε is the permittivity of the dielectric material. The values of practical capacitances are typically expressed in microfarads (μF), where $1\mu\text{F} = 10^{-6}$ F. However, for numerical convenience we may use farads in our examples.

The energy supplied to a capacitor is stored in its electrical field and can affect the response of the circuit at future times. For a fixed linear capacitor, the stored energy is

$$w = \frac{1}{2}Ce^2$$

Because the energy stored is a function of the voltage across its terminals, the initial voltage $e(t_0)$ of a capacitor is one of the conditions we need in order to find the complete response of a circuit for $t \geq t_0$.

Inductor

An **inductor** is an element for which there is an algebraic relationship between the voltage across its terminals and the derivative of the flux linkage. The symbol for an inductor and the convention for defining its current and voltage are shown in Figure 5.6. For a linear inductor,

$$e = \frac{d}{dt}(Li)$$

where L is the **inductance** with units of henries (H). For a fixed linear inductor, L is constant and we can write the element law as

$$e = L\frac{di}{dt} \tag{9}$$

We can find an expression for the current through the inductor by using (9) to integrate di/dt, giving

$$i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} e(\lambda)d\lambda \tag{10}$$

where $i(t_0)$ is the initial current through the inductor.

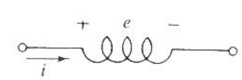


FIGURE 5.6 An inductor and its variables.

For a linear inductor made by winding N turns of wire around a toroidal core of a material having a constant permeability μ, cross-sectional area A, and mean circumference ℓ, the inductance is $L = \mu N^2 A/\ell$. Typical values of inductance are usually less than 1 henry and are often expressed in millihenries (mH).

The energy supplied to an inductor is stored in its magnetic field, and for a fixed linear inductor this energy is given by

$$w = \frac{1}{2}Li^2$$

To find the complete response of a circuit for $t \geq t_0$, we need to know the initial current $i(t_0)$ for each inductor.

Sources

The inputs for electrical circuit models are provided by ideal voltage and current sources. A **voltage source** is any device that causes a specified voltage to exist between two points in a circuit, regardless of the current that may flow. A **current source** causes a specified current to flow through the branch containing the source, regardless of the voltage that may be required. The symbols used to represent general voltage and current sources are shown in Figure 5.7(a) and Figure 5.7(b). We often represent physical sources by the combination of an ideal source and a resistor, as shown in parts (c) and (d) of the figure.

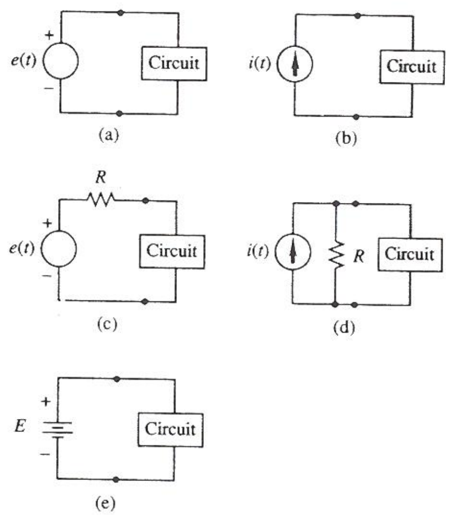


FIGURE 5.7 Sources. (a) Voltage. (b) Current. (c), (d) Possible representations of non-ideal sources. (e) Constant voltage source.

A voltage source that has a constant value for all time is often represented as shown in Figure 5.7(e). The symbol E denotes the value of the voltage, and the terminal connected to the longer line is the positive terminal. A battery is often represented in this fashion.

Open and Short Circuits

An **open circuit** is any element through which current cannot flow. For example, a switch in the open position provides an open circuit, as shown in Figure 5.8(a). Likewise, we can consider a current source that has a value of $i(t) = 0$ over a nonzero time interval an open circuit and can draw it as shown in Figure 5.8(b).

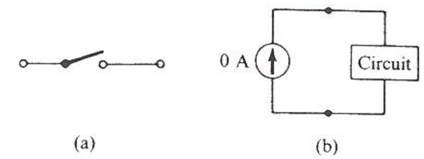


FIGURE 5.8 Examples of open circuits. (a) Open switch. (b) Zero current source.

A **short circuit** is any element across which there is no voltage. A switch in the closed position, as shown in Figure 5.9(a), is an example of a short circuit. Another example is a voltage source with $e(t) = 0$, as indicated in Figure 5.9(b).

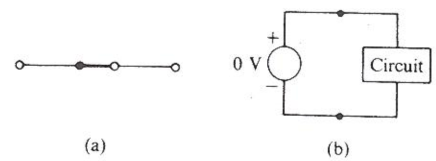


FIGURE 5.9 Examples of short circuits. (a) Closed switch. (b) Zero voltage source.

5.3 INTERCONNECTION LAWS

Two interconnection laws are used in conjunction with the appropriate element laws in modeling electrical circuits. These laws are known as Kirchhoff's voltage law and Kirchhoff's current law.

Kirchhoff's Voltage Law

When a closed path—that is, a loop—is traced through any part of a circuit, the algebraic sum of the voltages across the elements that make up the loop must equal zero. This property is known as **Kirchhoff's voltage law**. It may be written as

$$\sum_j e_j = 0 \qquad \text{around any loop} \tag{11}$$

where e_j denotes the voltage across the jth element in the loop.

It follows that summing the voltages across individual elements in any two different paths from one point to another will give the same result. For instance, in the portion of a circuit sketched in Figure 5.10(a), summing the voltages around the loop, going in a counterclockwise direction, and taking

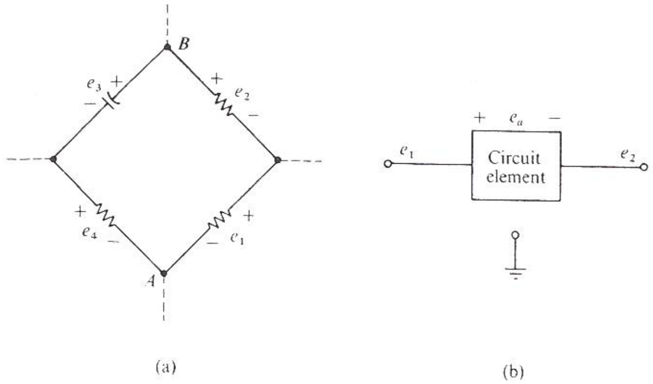


FIGURE 5.10 Partial circuits to illustrate Kirchhoff's voltage law.

into account the polarities indicated on the diagram give

$$e_1 + e_2 - e_3 - e_4 = 0$$

Reversing the direction in which the loop is traversed yields

$$e_4 + e_3 - e_2 - e_1 = 0$$

Likewise, going from point A to point B by each of the two paths shown gives

$$e_1 + e_2 = e_4 + e_3$$

which is, of course, equivalent to both of the foregoing loop equations. In fact, we invoked (11) for the circuit element shown in Figure 5.2(d), which is repeated in Figure 5.10(b), when we stated that $e_a = e_1 - e_2$, because it follows from the voltage law that $e_2 + e_a - e_1 = 0$.

Kirchhoff's Current Law

When the terminals of two or more circuit elements are connected together, the common junction is referred to as a **node**. All the joined terminals are at the same voltage and can be considered part of the node. Because it is not possible to accumulate any net charge at a node, the algebraic sum of the currents at any node must be zero at all times. This property is known as **Kirchhoff's current law**. It may be written as

$$\sum_j i_j = 0 \qquad \text{at any node} \tag{12}$$

where the summation is over the currents through all the elements joined to the node.

In applying (12), we must take into account the directions of the current arrows. We shall use a plus sign in (12) for a current arrow directed away from the node being considered and a minus sign for a current arrow directed toward the node. This is consistent with the fact that the current i entering a node is equivalent to the current $-i$ leaving the node.[1] For the partial circuit shown in Figure 5.11, applying (12) at the node to which the three elements are connected gives $i_1 + i_2 + i_3 = 0$. If we wish, we can also use Kirchhoff's current law in (12) for any closed surface that surrounds part of the circuit.

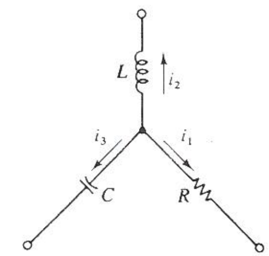


FIGURE 5.11 Partial circuit to illustrate Kirchhoff's current law.

It is a common practice to write the current-law equation directly in terms of the element values and the voltages of the nodes. Consider, for example, the circuit segment shown in Figure 5.12, where e_A, e_B, e_D, and e_F

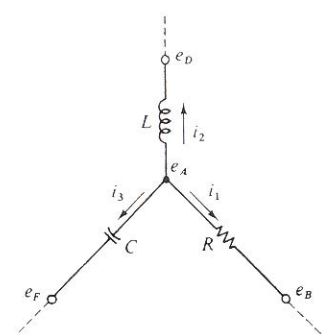


FIGURE 5.12 Partial circuit to illustrate Kirchhoff's current law written in terms of node voltages.

[1]Instead of interpreting the left side of (12) as the algebraic sum of the currents leaving the node, it would also be correct to use the algebraic sum of the currents entering the node.

represent the voltages of the nodes with respect to ground. By Kirchhoff's voltage law, the voltage across the resistor is $e_A - e_B$; and by the element law, the current through the resistor is $i_1 = (e_A - e_B)/R$. Similarly, the current through the inductor is

$$i_2 = i_2(0) + \frac{1}{L}\int_0^t (e_A - e_D)d\lambda$$

and that through the capacitor is $i_3 = C(\dot{e}_A - \dot{e}_F)$. Thus we can write the current law in terms of the node voltages and the initial inductor current as

$$\frac{1}{R}(e_A - e_B) + i_2(0) + \frac{1}{L}\int_0^t (e_A - e_D)d\lambda + C(\dot{e}_A - \dot{e}_F) = 0$$

In the following two examples, Kirchhoff's voltage law and current law are used to derive a circuit model.

EXAMPLE 5.1

Derive the model for the circuit shown in Figure 5.13.

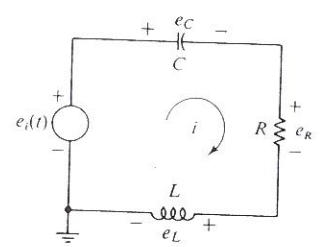


FIGURE 5.13 Series a *RLC* circuit with a voltage source.

Solution

By a trivial application of Kirchhoff's current law, the same current must flow through each of the four elements in the circuit. This current is denoted by i and its positive sense is taken as clockwise, as indicated in Figure 5.13. Because they have the same current flowing through them, the four elements are said to be connected in **series**.

The voltages across the three passive elements are e_L, e_R, and e_C, and we have assigned them the polarities indicated in the diagram. Starting at the ground node and proceeding counterclockwise around the single loop, we have, by Kirchhoff's voltage law,

$$e_L + e_R + e_C - e_i(t) = 0 \tag{13}$$

The element laws (4), (8), and (9) give expressions for e_R, e_C, and e_L:

$$\begin{aligned} e_R &= Ri \\ e_C &= e_C(0) + \frac{1}{C}\int_0^t i(\lambda)d\lambda \\ e_L &= L\frac{di}{dt} \end{aligned} \tag{14}$$

where the initial time has been taken as $t_0 = 0$ in (8). Substituting (14) into (13) and rearranging give the circuit model as the integral-differential equation

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i(\lambda)d\lambda = e_i(t) - e_C(0) \tag{15}$$

To eliminate the constant term and the integral, we differentiate (15) term by term, which yields

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \dot{e}_i$$

a second-order differential equation for the current i with the derivative of the applied voltage acting as the forcing function.

EXAMPLE 5.2

Obtain the input-output differential equation relating the input $i_i(t)$ to the output e_o for the circuit shown in Figure 5.14.

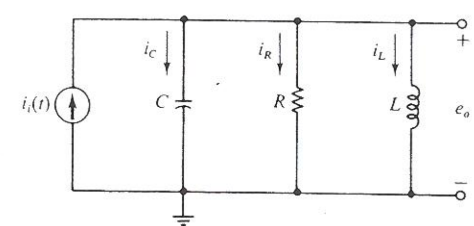


FIGURE 5.14 Parallel *RLC* circuit with a current source.

Solution

Each of the four circuit elements in Figure 5.14 has one terminal connected to the ground node and the other terminal connected to another common node. By a trivial application of Kirchhoff's voltage law, we see that the voltage across each element is e_o. Hence we say that the elements are connected in **parallel**.

Because the circuit has a single node whose voltage is unknown, we shall apply Kirchhoff's current law at that node in order to obtain the circuit model. We could also apply the current law at the ground node, but we would obtain no new information. The currents through the three passive elements are i_C, i_R, and i_L. As indicated by the arrows in Figure 5.14, each of these currents is considered positive when it flows from the upper node to the ground node.

Applying Kirchhoff's current law by summing the currents leaving the upper node, we write

$$i_C + i_R + i_L - i_i(t) = 0 \tag{16}$$

From the element laws given by (5), (7), and (10), we have

$$\begin{aligned} i_R &= \frac{1}{R}e_o \\ i_C &= C\dot{e}_o \\ i_L &= i_L(0) + \frac{1}{L}\int_0^t e_o(\lambda)d\lambda \end{aligned} \tag{17}$$

where the initial time has been taken as $t_0 = 0$.

Substituting (17) into (16) and rearranging the result give the model as

$$C\dot{e}_o + \frac{1}{R}e_o + \frac{1}{L}\int_0^t e_o(\lambda)d\lambda = i_i(t) - i_L(0) \tag{18}$$

Differentiating (18) term by term eliminates the constant term and the integral, resulting in the input-output differential equation

$$C\ddot{e}_o + \frac{1}{R}\dot{e}_o + \frac{1}{L}e_o = \frac{di_i}{dt}$$

5.4 OBTAINING THE INPUT-OUTPUT MODEL

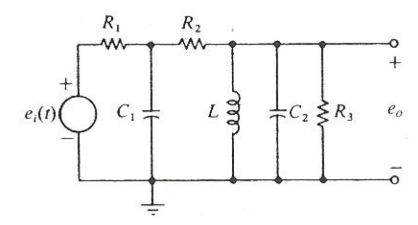
Two general procedures for developing input-output models of electrical circuits are the node-equation method and the loop-equation method. Example 5.1 was actually a simple illustration of the loop-equation method, and Example 5.2 used the node-equation method. In the **loop-equation method**, a rather trivial application of the current law enables us to express the current through every element in terms of one or more loop currents. We then write an appropriate set of simultaneous equations by using the voltage law and the element laws. In the **node-equation method**, we use Kirchhoff's voltage law in a trivial way to express the voltage across every element in terms of node voltages. Then we write a set of simultaneous equations by using Kirchhoff's current law and the element laws.

We shall emphasize the node-equation method, partly because in some circuits the loop-equation method requires us to use fictitious loop currents that do not correspond to measurable currents through individual elements. Furthermore, the node-equation method is well suited to handling the current sources that appear in models of transistor circuits. (References that cover both methods in detail are listed in Appendix D.)

When we use the node-equation method, we start by labeling the voltage of each node with respect to the ground node. If a voltage source is connected between a particular node and ground, the voltage of that node is the known source voltage. Where they are needed, we introduce symbols to define the voltages of the other nodes with respect to ground. Once we have done this, we can express the voltage across each passive element in terms of the node voltages by a trivial application of Kirchhoff's voltage law, as illustrated by the discussion of Figure 5.2(d) and Figure 5.12. We write a current-law equation for each of the nodes whose voltage is unknown, using the element laws to express the currents through the passive elements in terms of the node voltages. We need only combine the resulting set of equations into input-output form to complete the model.

▶ *EXAMPLE 5.3*

Derive the input-output equation for the circuit shown in Figure 5.15(a), using the node-equation method. The input and output voltages are $e_i(t)$ and e_o, respectively.



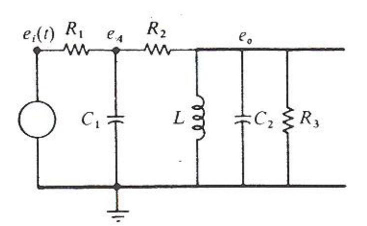


FIGURE 5.15 (a) Circuit for Example 5.3. (b) Circuit with node voltages shown.

Solution

The first step is to define all unknown node voltages and redraw the circuit diagram with all node voltages shown, as in Figure 5.15(b). We use the heavy lines to emphasize that the ground node extends across the bottom of the entire circuit and that the node whose voltage is e_o extends from L to R_3. We show the source voltage $e_i(t)$ at the upper left node and denote the voltage of the remaining node with respect to ground by e_A. Because e_A and e_o are unknown node voltages, we shall write a current-law equation at each of these nodes, using the appropriate element laws.

To assist in writing the equations, we can draw separate sketches for each node, as shown in Figure 5.16 (analogous to the free-body diagrams drawn for mechanical systems). For each element, the voltage across its terminals is shown in terms of the node voltages, with the plus sign placed at the node in question. Then we use the appropriate element law to write an expression for the current leaving the node.

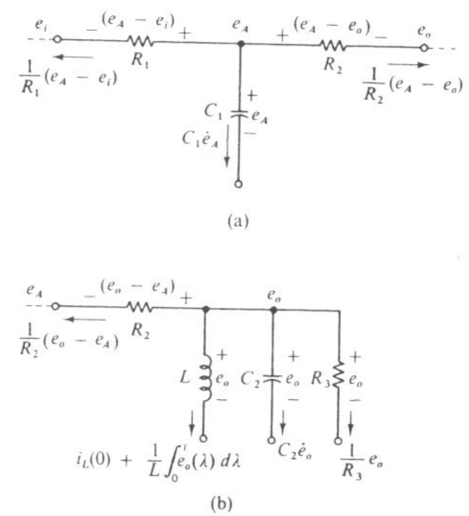


FIGURE 5.16 Nodes with currents expressed by element laws for Example 5.3.

We can apply Kirchhoff's current law to each of the nodes shown in Figure 5.16 by setting the algebraic sum of the currents leaving each node equal to zero. The result is the pair of equations

$$\frac{e_A - e_i(t)}{R_1} + C_1\dot{e}_A + \frac{e_A - e_o}{R_2} = 0 \tag{19a}$$

$$\frac{e_o - e_A}{R_2} + i_L(0) + \frac{1}{L}\int_0^t e_o(\lambda)d\lambda + C_2\dot{e}_o + \frac{e_o}{R_3} = 0 \tag{19b}$$

With a bit of experience, the reader should be able to write these equations directly from the circuit diagram, without drawing the sketches shown in Figure 5.16. It is worthwhile to note that the current through R_2 is labeled $(e_A - e_o)/R_2$ in the sketch for node A and is labeled $(e_o - e_A)/R_2$ in Figure 5.16(b). However, the reference arrows for this current are also reversed on the two parts of the figure, so there is no inconsistency in the expressions.

We can now differentiate (19b) to eliminate the constant term and the integral. Doing this and rearranging terms give the circuit model as the following pair of coupled differential equations for the node voltages e_A and e_o:

$$C_1\dot{e}_A + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)e_A - \frac{1}{R_2}e_o = \frac{1}{R_1}e_i(t) \tag{20a}$$

$$-\frac{1}{R_2}\dot{e}_A + C_2\ddot{e}_o + \left(\frac{1}{R_2} + \frac{1}{R_3}\right)\dot{e}_o + \frac{1}{L}e_o = 0 \tag{20b}$$

By combining these equations to eliminate e_A, we can obtain the input-output differential equation relating $e_i(t)$ and e_o. To simplify the calculations, assume that the passive elements have the numerical values $R_1 = R_2 = 2\ \Omega$, $R_3 = 4\ \Omega$, $C_1 = 1$ F, $C_2 = 4$ F, and $L = \frac{1}{2}$ H. With these parameter values, (20) becomes

$$\dot{e}_A + e_A - \frac{1}{2}e_o = \frac{1}{2}e_i(t)$$

$$-\frac{1}{2}\dot{e}_A + 4\ddot{e}_o + \frac{3}{4}\dot{e}_o + 2e_o = 0$$

By using the p-operator method described in Section 3.2 or by using a combination of substitution and differentiation, we can show that the circuit obeys the third-order equation

$$16\dddot{e}_o + 19\ddot{e}_o + 10\dot{e}_o + 8e_o = \dot{e}_i \tag{21}$$

for the prescribed element values. In order to solve (21) for e_o for all $t \geq 0$, we would need to know the three initial conditions $e_o(0)$, $\dot{e}_o(0)$, and $\ddot{e}_o(0)$, as well as the input $e_i(t)$ for $t \geq 0$.

Several results in the last example should be specifically noted. The order of an input-output equation is generally the same as the number of energy-storing elements—that is, it is the number of capacitors plus the number of inductors. Thus we could have anticipated that (21) would be third-order. In unusual cases, the order of the input-output equation might be less than the number of energy-storing elements. This can happen when the specified output does not depend on the values of some of the passive elements or when the capacitor voltages or inductor currents are not all independent. An illustration of this will be considered in Example 5.10.

In order to avoid drawing partial circuits like those in Figure 5.16, we can use the following rule for the current leaving a node through a passive element. The voltage that appears in the basic element law is replaced by the voltage of the node being considered minus the voltage at the other end of the passive element. The reader should examine (19a) for node A and (19b) for node O to see how the terms can be written down directly from Figure 5.15(b). If a current source is attached to the node, we can easily include the known source when writing Kirchhoff's current law. If, however, a voltage source is connected directly to the node, its current will not be known until after the circuit has been completely solved. To prevent introducing an additional unknown, we try to avoid summing currents at nodes to which voltage sources are attached. Thus, in the last example, we would not write a current-law equation at the junction of R_1 and $e_i(t)$.

Finally, consider the simplified node equations that remain after any integral signs have been eliminated by differentiation and after like terms have been collected together. In (20a) for node A, all the terms involving e_A and its derivatives have the same sign. Similarly in (20b) for node O, all the terms with e_o and its derivatives have the same sign. Some insight into the reason for this can be found in Chapter 6, but the reader may wish to use this general property now as a check on the work.

The next example involves a voltage source neither end of which is connected to ground. In such cases, we must take special care when labeling the node voltages and writing the current-law equations.

▶ *EXAMPLE 5.4*

Find the input-output differential equation for the circuit shown in Figure 5.17(a) where the inputs are the two voltage sources $e_1(t)$ and $e_2(t)$ and the output is the voltage e_o.

Solution

The voltage of each node with respect to the ground is shown in Figure 5.17(b). Two of the nodes are labeled $e_1(t)$ and e_o to correspond to the left voltage source and the output voltage, respectively. In labeling the node to the right of L and R_1, we do not introduce a new symbol (such as e_A) but take advantage of the fact that $e_2(t)$ is a source voltage. By Kirchhoff's voltage law, this remaining node voltage is $e_o + e_2(t)$, as shown on the diagram. This approach avoids the introduction of unnecessary variables whenever there is a voltage source not connected to ground.

The directions of the current reference arrows included in Figure 5.17(b) are arbitrary, but our equations must be consistent with the directions selected. Applying Kirchhoff's current law to the upper right node, we have

$$i_C + i_{R_2} + i_2 = 0 \tag{22}$$

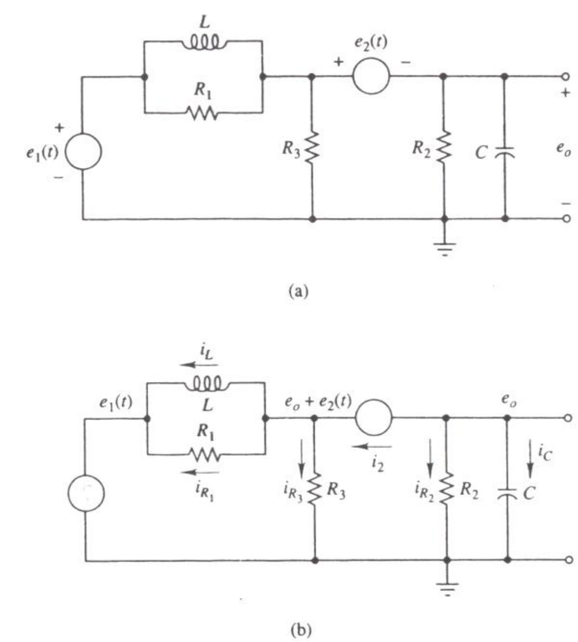


FIGURE 5.17 Circuit for Example 5.4. (a) As specified by the example statement. (b) With currents and node voltages defined.

Although we can express i_C and i_{R_2} in terms of the node voltage e_o by the element laws, the current i_2 through the voltage source cannot be directly related to $e_2(t)$. However, by applying the current law to the node labeled $e_o + e_2(t)$, we see that

$$i_2 = i_{R_3} + i_{R_1} + i_L$$

which, when inserted into (22), gives

$$i_C + i_{R_2} + i_{R_3} + i_{R_1} + i_L = 0 \tag{23}$$

Using the element laws in the forms given by (5), (7), and (10), we write

$$i_{R_1} = \frac{1}{R_1}[e_o + e_2(t) - e_1(t)] \tag{24a}$$

$$i_{R_2} = \frac{1}{R_2}e_o \tag{24b}$$

$$i_{R_3} = \frac{1}{R_3}[e_o + e_2(t)] \tag{24c}$$

$$i_C = C\dot{e}_o \tag{24d}$$

$$i_L = i_L(0) + \frac{1}{L}\int_0^t (e_o + e_2 - e_1)d\lambda \tag{24e}$$

Substituting (24) into (23) and rearranging terms give the integral-differential equation

$$C\dot{e}_o + \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)e_o + \frac{1}{L}\int_0^t e_o d\lambda$$
$$= \frac{1}{R_1}e_1(t) - \left(\frac{1}{R_1} + \frac{1}{R_3}\right)e_2(t) + \frac{1}{L}\int_0^t (e_1 - e_2)d\lambda - i_L(0)$$

By differentiating this expression term-by-term, we obtain the desired input-output model:

$$C\ddot{e}_o + \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)\dot{e}_o + \frac{1}{L}e_o$$
$$= \frac{1}{R_1}\dot{e}_1 - \left(\frac{1}{R_1} + \frac{1}{R_3}\right)\dot{e}_2 + \frac{1}{L}[e_1(t) - e_2(t)] \tag{25}$$

which is a second-order differential equation. To solve it, we must know the two initial conditions $e_o(0)$ and $\dot{e}_o(0)$, in addition to the source voltages $e_1(t)$ and $e_2(t)$.

The solution of a circuit model, such as the one in (25), is discussed in the next chapter. We may sometimes wish to see how the nature of the response changes when a source branch is disconnected. Even with all the sources disconnected, there will still be some output due to any energy that has been previously stored in the capacitors and inductors. The circuit in the following example contains only one energy storing element, so we should expect that it will be described by a first-order input-output equation.

▶ *EXAMPLE 5.5*

Find the differential equation for the output voltage e_o in Figure 5.18(a) when the switch is closed. Numerical values are given for the resistors but not for the capacitor. Repeat the problem when the left branch is disconnected by opening the switch.

Solution

The node voltages with respect to ground, with the switch closed, are labeled in Figure 5.18(b). Summing currents at nodes A and O gives the following pair of equations:

$$C(\dot{e}_A - \dot{e}_o) + \frac{1}{3}(e_A - e_o) + \frac{1}{4}e_A + \frac{3}{4}[e_A - e_i(t)] = 0 \tag{26a}$$

$$C(\dot{e}_o - \dot{e}_A) + \frac{1}{3}(e_o - e_A) + \frac{1}{2}e_o = 0 \tag{26b}$$

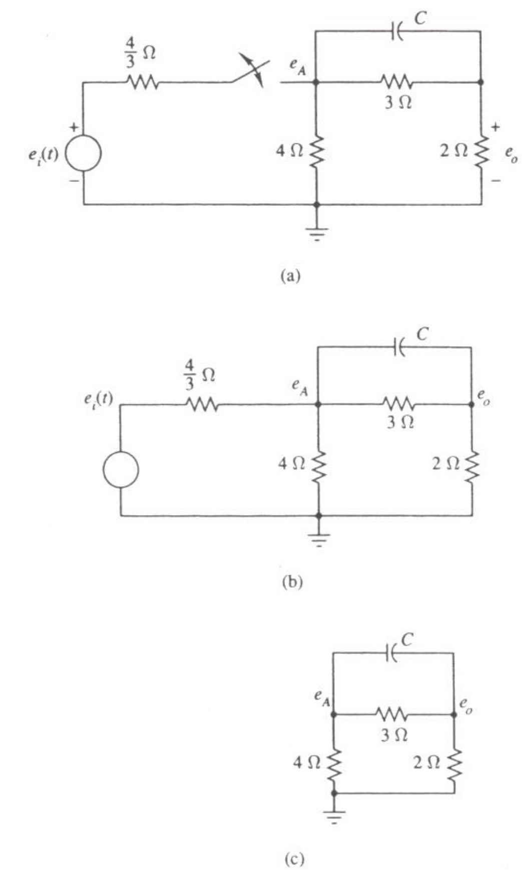


FIGURE 5.18 (a) Circuit for Example 5.5. (b) With the switch closed. (c) With the switch open.

We could collect like terms in each of these equations and then use the p-operator method to eliminate e_A and to obtain an equation in e_o and its derivatives. However, by adding (26a) and (26b) we see that

$$\frac{1}{2}e_o + e_A - \frac{3}{4}e_i(t) = 0$$

Replacing e_A in (26b) by $-\frac{1}{2}e_o + \frac{3}{4}e_i(t)$, we have

$$C\left[\dot{e}_o + \frac{1}{2}\dot{e}_o - \frac{3}{4}\dot{e}_i\right] + \frac{1}{3}\left[e_o + \frac{1}{2}e_o - \frac{3}{4}e_i(t)\right] + \frac{1}{2}e_o = 0$$

from which

$$\frac{3}{2}C\dot{e}_o + e_o = \frac{3}{4}C\dot{e}_i + \frac{1}{4}e_i(t)$$

With the switch open, the circuit reduces to the one shown in Figure 5.18(c). Once again summing currents at nodes A and O, we obtain

$$C(\dot{e}_A - \dot{e}_o) + \frac{1}{3}(e_A - e_o) + \frac{1}{4}e_A = 0$$

$$C(\dot{e}_o - \dot{e}_A) + \frac{1}{3}(e_o - e_A) + \frac{1}{2}e_o = 0$$

Following the same procedure as before, we find that for part (c) of the figure,

$$2C\dot{e}_o + e_o = 0$$

Note that, as is the case in most examples, disconnecting the source has not only caused the input terms to disappear but has also changed the coefficients on the left side of the differential equation.

5.5 RESISTIVE CIRCUITS

There are many useful circuits that contain only resistors and sources, with no energy-storing elements. Such circuits are known as **resistive circuits** and are modeled by algebraic rather than differential equations. In this section we shall develop rules for finding the voltages and currents in such circuits and for replacing certain combinations of resistors by a single equivalent resistor.

The analysis of resistive circuits is important for other reasons as well. Even for circuits with energy-storing elements, we are often interested primarily in the steady-state response to a constant input, after any initial transients have died away. In the next chapter, we shall see how any circuit reduces to a resistive one under these circumstances.

Even when we need to find the complete response of a general electrical system, a combination of two or more resistors will frequently be connected to the remainder of the circuit by a single pair of terminals, as shown in Figure 5.19(a). In such situations, it is possible to replace the entire combination of resistors by a single equivalent resistor R_{eq}, as shown in Figure 5.19(b). Provided that R_{eq} is selected such that $e = R_{eq}i$ is satisfied, the response of the remainder of the circuit is identical in both cases. Once we have found R_{eq}, it is easier to analyze the complete circuit because there are fewer nodes and thus fewer equations. The two most important cases of combinations of resistors are the series and parallel connections.

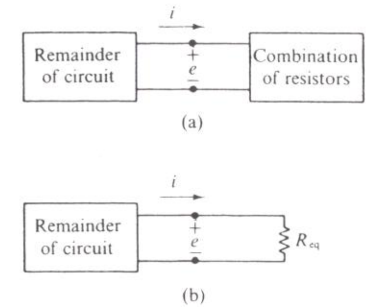


FIGURE 5.19 Replacement of a combination of resistors by an equivalent resistance.

Resistors in Series

Two resistors are in series when a single terminal of each resistor is connected to a single terminal of the other with no other element connected to the common node, as shown in Figure 5.20(a). Obviously, two resistors in series must have the same current flowing through them.

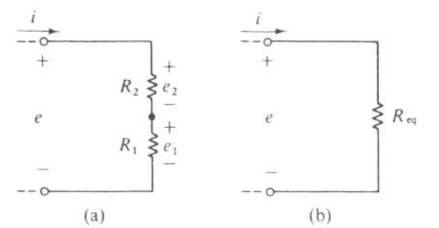


FIGURE 5.20 (a) Two resistors in series. (b) Equivalent resistance.

It follows from Ohm's law that $e_1 = R_1 i$ and $e_2 = R_2 i$ and from Kirchhoff's voltage law that $e = e_1 + e_2$. Thus

$$e = (R_1 + R_2)i \tag{27}$$

Because $e = R_{eq}i$, we know from (27) that the equivalent resistance shown in Figure 5.20(b) for the series combination shown in Figure 5.20(a) is

$$R_{eq} = R_1 + R_2 \tag{28}$$

Using (27) with the expressions for e_1 and e_2, we see that

$$e_1 = \left(\frac{R_1}{R_1 + R_2}\right) e \tag{29a}$$

$$e_2 = \left(\frac{R_2}{R_1 + R_2}\right) e \tag{29b}$$

which is known as the **voltage-divider rule**. From (29), the ratio of the individual resistor voltages is

$$\frac{e_1}{e_2} = \frac{R_1}{R_2} \tag{30}$$

Resistors in Parallel

Two resistors are in parallel when each terminal of one resistor is connected to a separate terminal of the other resistor, as shown in Figure 5.21(a). It is apparent that two resistors in parallel must have the same voltage across their terminals.

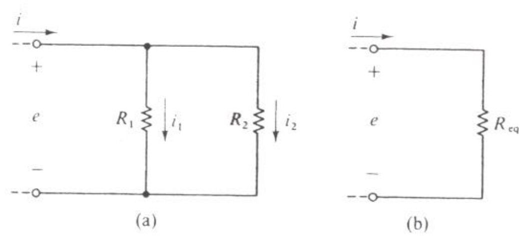


FIGURE 5.21 (a) Two resistors in parallel. (b) Equivalent resistance.

From Ohm's law, the individual currents are $i_1 = (1/R_1)e$ and $i_2 = (1/R_2)e$. From Kirchhoff's current law, $i = i_1 + i_2$. Thus

$$i = \left(\frac{1}{R_1} + \frac{1}{R_2}\right) e \tag{31}$$

For the equivalent resistance shown in Figure 5.21(b), we have $i = (1/R_{eq})e$. From (31), we see that for the parallel combination in Figure 5.21(a),

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

or

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2} \tag{32}$$

To relate i_1 to the total current, we can write $i_1 = (1/R_1)e = (R_{eq}/R_1)i$. Doing this for i_2 and then using (32) to express R_{eq} in terms of R_1 and R_2, we obtain

$$i_1 = \left(\frac{R_2}{R_1 + R_2}\right) i \tag{33a}$$

$$i_2 = \left(\frac{R_1}{R_1 + R_2}\right) i \tag{33b}$$

which is known as the **current-divider rule**. From (33), the ratio of the individual resistor currents is $i_1/i_2 = R_2/R_1$.

Calculating equivalent resistances and solving for the currents and voltages in most types of resistive networks can be simplified by using these rules for series and parallel combinations, as demonstrated in the following example.

▶ ***EXAMPLE 5.6***

The resistive circuit shown in Figure 5.22(a) consists of a voltage source connected to a combination of seven resistors. The output is the voltage e_o. Find the equivalent resistance R_{eq} of the seven-resistor combination and evaluate e_o.

Solution

To obtain R_{eq}, we use (28) and (32) repeatedly to combine series or parallel combinations of resistors into single equivalent resistors. Starting with the original circuit in Figure 5.22(a), we replace the 6-Ω and 3-Ω resistors that are in parallel by a single 2-Ω resistor. We also combine the 10-Ω and 2-Ω resistors in series into a 12-Ω resistor, which yields the intermediate circuit diagram shown in Figure 5.22(b). Note that the output voltage e_o does not appear on this diagram. Next we replace the parallel combination of the 4-Ω and 12-Ω resistors by a 3-Ω resistor, and obtain Figure 5.22(c). The series combination of 2 Ω and 3 Ω gives a 5-Ω resistor, which is in parallel with another 5-Ω branch, yielding the resistor of $\frac{5}{2}$ Ω that is shown in Figure 5.22(d). Thus the equivalent resistance connected across the voltage source is

$$R_{eq} = \frac{1}{2} + \frac{5}{2} = 3\ \Omega$$

To find the output voltage e_o, we make repeated use of the voltage-divider rule given by (29a) to obtain, in turn, e_A, e_B, and finally e_o. From Figure 5.22(d),

$$e_A = \left(\frac{5/2}{1/2 + 5/2}\right) e_i(t) = \frac{5}{6} e_i(t)$$

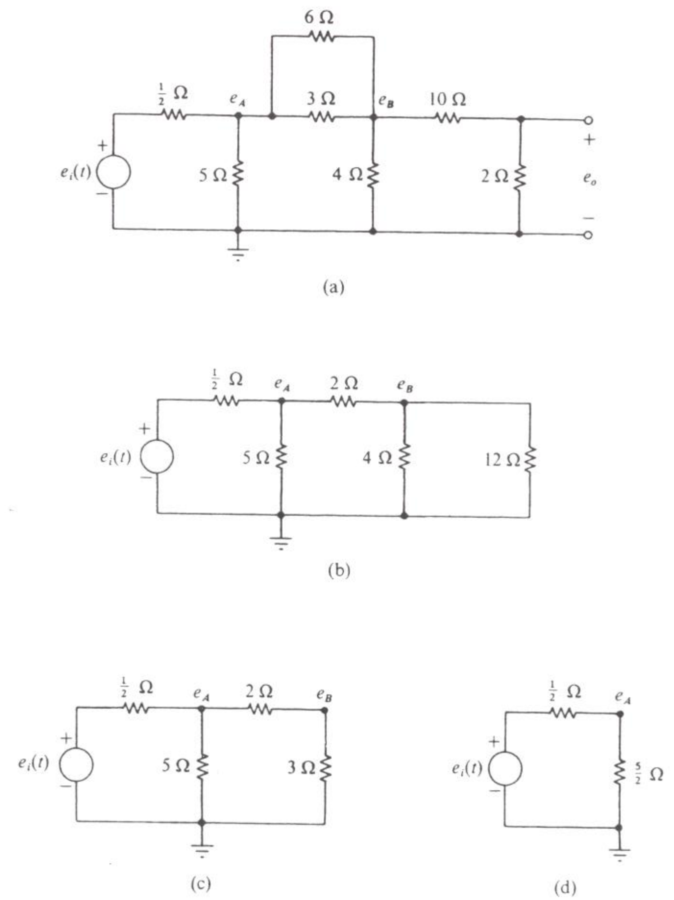


FIGURE 5.22 Circuits for Example 5.6. (a) Original circuit. (b), (c), (d) Equivalent circuits.

and from Figure 5.22(c),

$$e_B = \left(\frac{3}{2+3}\right) e_A = \frac{1}{2} e_i(t)$$

Then, from the original circuit diagram,

$$e_o = \left(\frac{2}{2+10}\right) e_B = \frac{1}{12} e_i(t)$$

Although the rules for combining series and parallel resistors often simplify the process of modeling a circuit, there are circuits in which the resistances do not occur in series or parallel combinations. In such situations, we can find an equivalent resistance by writing and solving the appropriate node equations, which will be strictly algebraic when only resistors and sources are involved.

5.6 OBTAINING THE STATE-VARIABLE MODEL

To obtain the model of a circuit in state-variable form, we define an appropriate set of state variables and then derive an equation for the derivative of each state variable in terms of only the state variables and inputs. The choice of state variables is not unique, but they are normally related to the energy in each of the circuit's energy-storing elements. Recalling that the energy stored in a capacitor is $\frac{1}{2}Ce^2$ and for an inductor is $\frac{1}{2}Li^2$, we generally select the capacitor voltages and inductor currents as the state variables. For fixed linear circuits, exceptions occur only when there are capacitor voltages or inductor currents that are not independent of one another. This unusual situation will be illustrated in Example 5.10.

For each capacitor or inductor, we want to express $\dot{e}_c$ or di_L/dt as an algebraic function of state variables and inputs. We do this by writing the capacitor and inductor element laws in their derivative forms as

$$\begin{aligned} \dot{e}_C &= \frac{1}{C} i_C \\ \frac{di_L}{dt} &= \frac{1}{L} e_L \end{aligned} \tag{34}$$

and then obtaining algebraic expressions for i_C and e_L in terms of the state variables and inputs. To find these expressions, we use the resistor element laws and Kirchhoff's voltage and current laws.

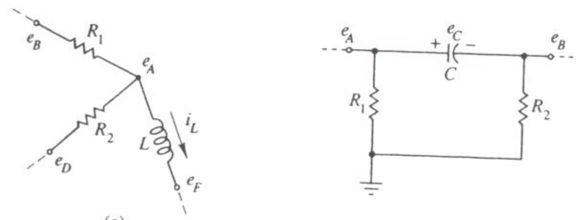
All the techniques we have discussed can still be used. The only basic difference is that we want to retain the variables e_C and i_L wherever they appear in our equations and to express other variables in terms of them. For example, in applying Kirchhoff's current law to node A in the partial circuit shown in Figure 5.23(a), we would write

$$\frac{1}{R_1}(e_A - e_B) + \frac{1}{R_2}(e_A - e_D) + i_L = 0$$

instead of

$$\frac{1}{R_1}(e_A - e_B) + \frac{1}{R_2}(e_A - e_D) + i_L(0) + \frac{1}{L}\int_0^t (e_A - e_F)d\lambda = 0$$

In the partial circuit of Figure 5.23(b), we do not use both e_A and e_B when summing the currents at nodes A and B. If the symbol e_B is used,



(a) (b)

FIGURE 5.23 Partial circuits to illustrate writing Kirchhoff's current law in terms of state variables.

the voltage at node A is written as $e_B + e_C$. If e_A is used, the voltage at node B is expressed as $e_A - e_C$.

▶ *EXAMPLE 5.7*

Write the state-variable equations for the circuit shown in Figure 5.24, for which we found the input-output equation in Example 5.2.

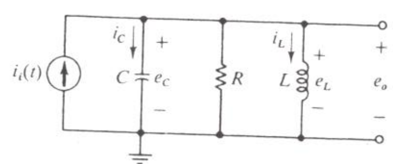


FIGURE 5.24 Parallel *RLC* circuit for Example 5.7.

Solution

Because the circuit contains an inductor and a capacitor, we select i_L and e_C as the state variables, with the positive senses indicated on the circuit diagram. Starting with the inductor element law in the form $di_L/dt = (1/L)e_L$, we note that e_L is the same as the capacitor voltage e_C because the two elements are in parallel. Thus we obtain the first state-variable equation by replacing e_L by e_C, getting

$$\frac{di_L}{dt} = \frac{1}{L}e_C \tag{35}$$

For the second equation, we write the capacitor element law as $\dot{e}_C = (1/C)i_C$. To express the capacitor current i_C in terms of the state variables and input, we apply Kirchhoff's current law at the upper node, getting

$$i_C + \frac{1}{R}e_C + i_L - i_i(t) = 0 \tag{36}$$

where we have written the resistor current in terms of the state variable e_C. Solving (36) for i_C gives

$$i_C = -i_L - \frac{1}{R}e_C + i_i(t) \tag{37}$$

the right side of which is written entirely in terms of the state variables and input. Substituting (37) into the capacitor element law, we find the second state-variable equation to be

$$\dot{e}_C = \frac{1}{C}\left[-i_L - \frac{1}{R}e_C + i_i(t)\right] \tag{38}$$

Finally, we find the voltage e_o from the algebraic output equation

$$e_o = e_C \tag{39}$$

To solve the state-variable equations in (35) and (38), we must know the input and the initial values of the state variables. Note also that once we have the state-variable and output equations, we can always combine them into an input-output differential equation. In this example, we could differentiate (38), substitute (35) into it, and finally use (39) to replace e_C by e_o. The result would be the answer to Example 5.2.

There are several ways of summarizing a general procedure for constructing a state-variable model. We assume here that each capacitor voltage and inductor current is chosen to be a state variable. The unusual case where the number of state variables is less than the number of energy-storing elements will be treated later in this section.

1. We show the positive senses for each e_C and i_L on the circuit diagram, and then label i_C and e_L so that each current reference arrow enters the capacitor or inductor at the positive end of the voltage reference. Insofar as possible, we label the voltages of the nodes with respect to ground in terms of the state variables and inputs. We can then use additional symbols for any remaining node voltages, but we try to minimize the use of new variables.
2. We need to find algebraic expressions for each capacitor current i_C and each inductor voltage e_L. We make use of Kirchhoff's laws and Ohm's law, but at this time do not use the element laws for the capacitors and inductors. We require algebraic equations in this step, and the laws for the energy-storing elements will be used in the next step. In general, we may have to solve a set of simultaneous algebraic equations in order to get individual equations for each i_C and e_L in terms of the state variables and inputs.
3. For the state-variable equations, we substitute the expressions for i_C and e_L into the capacitor and inductor element laws, as given by (34). Finally, for each output that is not a state variable, we write an algebraic expression in terms of state variables and inputs.

The next two examples illustrate this general procedure for circuits of moderate complexity.

EXAMPLE 5.8

Derive the state-variable model for the circuit shown in Figure 5.25. The outputs of interest are e_B, i_{C_2}, and i_1.

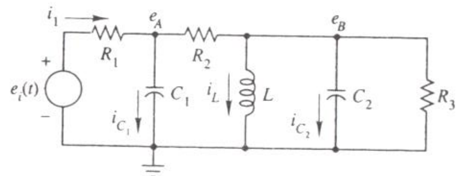


FIGURE 5.25 Circuit for Example 5.8.

Solution

We choose as state variables the inductor current i_L and the capacitor voltages e_A and e_B. We need algebraic equations for the voltage across the inductor and the current through each capacitor. The inductor voltage is identical to the state variable e_B. Thus, by the element law for the inductor, one of the state-variable equations is

$$\frac{di_L}{dt} = \frac{1}{L}e_B$$

The current i_{C_1} will appear in a Kirchhoff current-law equation for node A, namely

$$i_{C_1} = \frac{1}{R_1}[e_i(t) - e_A] - \frac{1}{R_2}(e_A - e_B) \tag{40}$$

For i_{C_2} we consider node B, getting

$$i_{C_2} = \frac{1}{R_2}(e_A - e_B) - i_L - \frac{1}{R_3}e_B \tag{41}$$

Substituting (40) and (41) into the respective element-law equations gives the final two state-variable equations. The complete set of three equations is

$$\begin{aligned}
\frac{di_L}{dt} &= \frac{1}{L}e_B \\
\dot{e}_A &= \frac{1}{C_1}\left[-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)e_A + \frac{1}{R_2}e_B + \frac{1}{R_1}e_i(t)\right] \\
\dot{e}_B &= \frac{1}{C_2}\left[-i_L + \frac{1}{R_2}e_A - \left(\frac{1}{R_2} + \frac{1}{R_3}\right)e_B\right]
\end{aligned} \tag{42}$$

As required, we have expressed the derivative of each of the state variables as an algebraic function of the state variables and the input $e_i(t)$. The output voltage e_B is the same as one of the state variables, and the output current i_{C_2} is given by (41). The output equation for i_1 is

$$i_1 = \frac{1}{R_1}[e_i(t) - e_A] \tag{43}$$

The next example has only two energy-storing elements and hence only two state variables. However, not all of the node voltages can be immediately expressed in terms of state variables and inputs.

EXAMPLE 5.9

Find the state-variable model for the circuit shown in Figure 5.26(a), when e_o is the output.

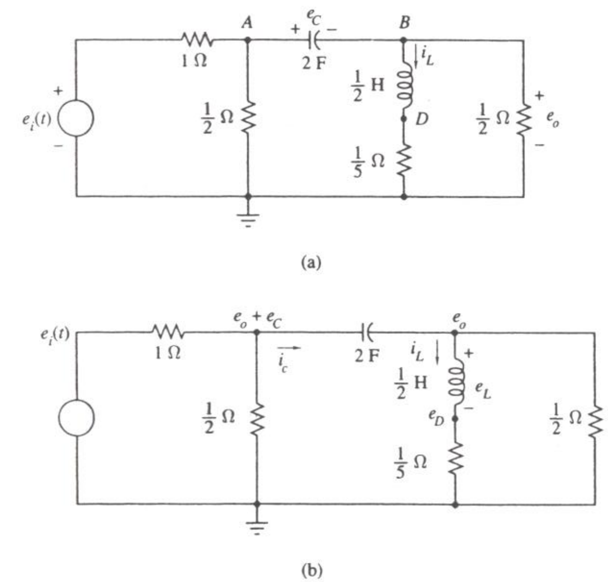


FIGURE 5.26 (a) Circuit for Example 5.9. (b) With additional variables added.

Solution

We make the usual choice of e_C and i_L as state variables, with the positive senses shown on the diagram. The voltage at node D is $e_D = \frac{1}{5}i_L$, and node B corresponds to the output voltage e_o. Because we want to retain e_C in our equations, we use Kirchhoff's voltage law to express the voltage at node A as $e_o + e_C$. These symbols, as well as the reference directions for i_C and e_L, are added to the diagram in Figure 5.26(b). Applying Kirchhoff's current law to nodes B and A, we have

$$\begin{aligned} 2e_o + i_L - i_C &= 0 \\ [e_o + e_C - e_i(t)] + 2(e_o + e_C) + i_C &= 0 \end{aligned} \tag{44}$$

The quantity inside the brackets is the current through the 1-Ω resistor—that is, it is the voltage at node A minus the source voltage all divided by 1 Ω. We now solve (44) simultaneously for e_o and i_C in terms of the state variables and the input. Doing this gives the algebraic equations

$$e_o = \frac{1}{5}[-i_L - 3e_C + e_i(t)] \tag{45a}$$

$$i_C = \frac{1}{5}[3i_L - 6e_C + 2e_i(t)] \tag{45b}$$

We also note that

$$e_L = e_o - e_D = \frac{1}{5}[-2i_L - 3e_C + e_i(t)] \tag{46}$$

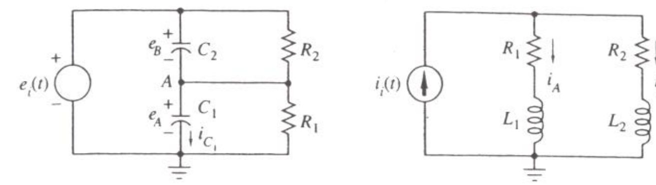
To obtain the state-variable equations, we substitute (45b) and (46) into the element laws, as given by (34) and with $C = 2$ F and $L = \frac{1}{2}$ H. We also repeat the output equation (45a) to obtain the complete state-variable model:

$$\begin{aligned} \dot{e}_C &= \frac{1}{10}[3i_L - 6e_C + 2e_i(t)] \\ \frac{di_L}{dt} &= \frac{2}{5}[-2i_L - 3e_C + e_i(t)] \\ e_o &= \frac{1}{5}[-i_L - 3e_C + e_i(t)] \end{aligned} \tag{47}$$

In the previous three examples, we took as state variables the voltage across each capacitor and the current through each inductor. Figure 5.27 illustrates two exceptions to this procedure. For part (a) of the figure, we might first try to choose both e_A and e_B as state variables. However, by applying Kirchhoff's voltage law to the left loop, we see that

$$e_A + e_B - e_i(t) = 0 \tag{48}$$

Equation (48) is an algebraic relationship between the proposed state variables and the input, which is not allowed. In other words, the capacitor voltages e_A and e_B are not independent and thus cannot both be chosen as state variables. The basic problem is that the circuit contains a loop composed of only capacitors and voltage sources.



(a) (b)

FIGURE 5.27 Circuits having fewer state variables than energy-storing elements.

An analogous situation occurs when a circuit has a node to which only inductors and current sources are connected. For Figure 5.27(b), suppose that we try to choose both i_A and i_B as state variables. By Kirchhoff's current law,

$$-i_i(t) + i_A + i_B = 0$$

Again we have an algebraic relationship between the proposed state variables and the input. Because i_A and i_B are not independent, only one of them should be chosen as a state variable.

The situations illustrated in Figure 5.27 are relatively rare and may even be caused by starting with a circuit diagram that does not correspond closely enough to the physical devices. In part (a), for example, a better model of a physical source might be an ideal voltage source in series with a resistor, which could represent the "internal resistance" of the device. If the diagram were changed in this way, then there would no longer be a loop composed of only capacitors and voltage sources, and both e_A and e_B would be suitable state variables.

A state-variable model for Figure 5.27(a) is developed in Example 5.10. The treatment of Figure 5.27(b) is left for a problem at the end of the chapter. In such cases, we may need to redefine a state variable in order to avoid having the derivative of the input appear on the right-hand side of the state-variable equation. For certain outputs, however, it may not always be possible to avoid an input derivative on the right-hand side of the output equation.

▶ *EXAMPLE 5.10*

Find the state-variable model for the circuit shown in Figure 5.27(a). Take the outputs to be e_A, e_B, and i_{C_1}.

Solution

Because both e_A and e_B cannot be chosen as state variables, suppose we select e_A as the single state variable. Then we need an equation for $\dot{e}_A$ in terms of e_A and $e_i(t)$. First we write the element law for C_1 as

$$\dot{e}_A = \frac{1}{C_1} i_{C_1} \tag{49}$$

where the positive sense of i_{C_1} is downward. Then we apply Kirchhoff's current law at node A, obtaining

$$C_2(\dot{e}_A - \dot{e}_i) + \frac{1}{R_2}[e_A - e_i(t)] + \frac{1}{R_1} e_A + i_{C_1} = 0 \tag{50}$$

Solving (50) for i_{C_1} and substituting the result into (49), we find

$$\dot{e}_A = \frac{1}{C_1}\left[-C_2\dot{e}_A - \left(\frac{1}{R_1} + \frac{1}{R_2}\right) e_A + C_2\dot{e}_i + \frac{1}{R_2} e_i(t)\right]$$

which can be rearranged to yield

$$\dot{e}_A = \left(\frac{1}{C_1 + C_2}\right)\left[- \left(\frac{1}{R_1} + \frac{1}{R_2}\right) e_A + C_2\dot{e}_i + \frac{1}{R_2} e_i(t)\right] \tag{51}$$

Equation (51) would be in state-variable form were it not for the term involving $\dot{e}_i$ on the right side. The derivative of the input should not appear in the final equation, so we define a new state variable, denoted by x, using the same procedure as for the mechanical system in Example 3.8. Transferring the term involving $\dot{e}_i$ to the left side of (51), we have

$$\frac{d}{dt}\left[e_A - \left(\frac{C_2}{C_1 + C_2}\right) e_i(t)\right] = \left(\frac{1}{C_1 + C_2}\right)\left[- \left(\frac{1}{R_1} + \frac{1}{R_2}\right) e_A + \frac{1}{R_2} e_i(t)\right] \tag{52}$$

We define the bracketed term on the left to be the new state variable

$$x = e_A - \left(\frac{C_2}{C_1 + C_2}\right) e_i(t) \tag{53}$$

Then e_A is given by the output equation

$$e_A = x + \left(\frac{C_2}{C_1 + C_2}\right) e_i(t) \tag{54}$$

and, when we substitute (53) and (54) into (52), the state-variable equation becomes

$$\dot{x} = \left(\frac{1}{C_1 + C_2}\right)\left\{ - \left(\frac{1}{R_1} + \frac{1}{R_2}\right) x + \left[\frac{1}{R_2} - \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\left(\frac{C_2}{C_1 + C_2}\right)\right] e_i(t)\right\} \tag{55}$$

Note that the circuit is first-order and can be modeled by a single state-variable equation. However, we did find that the state variable had to be a linear combination of the voltage across C_1 and the input. We can obtain the

capacitor voltage e_A from the algebraic output equation (54) after solving the state-variable equation. For the output equation for e_B, we combine (48) and (54) to obtain

$$e_B = \left(\frac{C_1}{C_1 + C_2}\right) e_i(t) - x$$

For the final output, we substitute (54) into the element law for C_1:

$$i_{C_1} = C_1\dot{e}_A = C_1\left[\dot{x} + \left(\frac{C_2}{C_1 + C_2}\right)\dot{e}_i\right]$$

and then, by (55), we write

$$i_{C_1} = \left(\frac{C_1}{C_1 + C_2}\right)\left\{ - \left(\frac{1}{R_1} + \frac{1}{R_2}\right) x + \left[\frac{1}{R_2} - \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\left(\frac{C_2}{C_1 + C_2}\right)\right] e_i(t) + C_2\dot{e}_i\right\}$$

The output equations in a state-variable model should be purely algebraic whenever possible. This is an example of the unusual case where we cannot avoid a derivative of the input on the right-hand side.

5.7 CONTROLLED SOURCES AND OPERATIONAL AMPLIFIERS

Some important types of electrical elements, unlike those in earlier sections, have more than two terminals to which external connections can be made. Controlled sources are considered in this section, and the frequently used operational amplifier receives special attention.

Controlled sources arise in the models of transistors and other electronic devices. Rather than being independently specified, the values of such sources are proportional to the voltage or current somewhere else in the circuit. One purpose for which such devices are used is to amplify electrical signals, giving them sufficient power, for example, to drive loudspeakers, instrumentation, or various electromechanical systems. Ideal voltage and current amplifiers are shown in parts (a) and (b), respectively, of Figure 5.28.

The models for many common devices have the two bottom terminals connected together. They may also include the added resistors shown in parts (c) and (d) of the figure in order to represent some of the imperfections of the device. For part (c) of the figure to approach part (a), R_n must be very large and R_o very small. In order for part (d) of the figure to approach part (b), R_n must be very small and R_o very large.

As is the case with any source in a circuit diagram, we assume that a controlled source can supply as much power as is required by the passive elements connected to it. Although we do not discuss in detail the internal

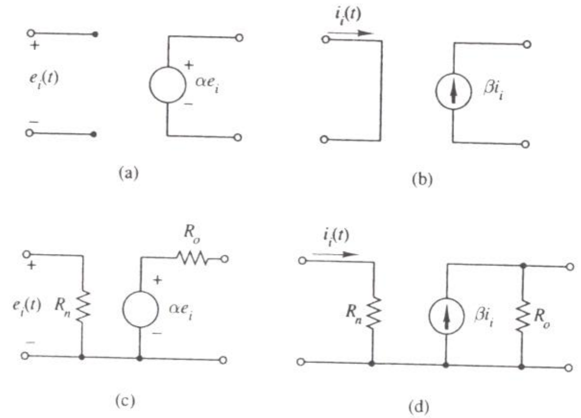


FIGURE 5.28 (a), (b) Ideal voltage and current amplifiers. (c), (d) Amplifiers with internal resistances causing non-ideal behavior.

mechanism responsible for the behavior of any element, a brief comment here should be helpful for those who encounter devices that are represented by controlled sources. The drawings in Figure 5.28 do not show all of the external connections to the physical device. In addition to the time-varying input $e_i(t)$ or $i_i(t)$, there is a constant voltage source that normally is not explicitly shown. This additional voltage, which is usually a battery called the bias supply, has two primary purposes. One is to supply any needed power for the time-varying output signal. The other is to make the electronic device, which usually is inherently nonlinear, operate in its linear region so that the models in Figure 5.28 can be used. Chapter 9 contains a general discussion of how to make a device operate about a particular point in its linear region. The bias voltages that are needed are usually given in the specifications for the device to be used. Also specified are the maximum values of input voltage or current for which the operation of the device can be expected to remain in its linear region.

We shall assume that the values of our controlled sources are directly proportional to the signals controlling them. In addition to the sources represented in Figure 5.28, we can also have a voltage source controlled by a *current* somewhere else in the circuit, as well as a current source controlled by a *voltage*. The problems given at the end of the chapter include these various types of controlled sources. However, we shall emphasize the voltage-controlled voltage source, because that will lead to the concept of the operational amplifier.

Electronic devices can do much more than simply amplify an input signal. In order to accomplish other objectives, additional passive elements are connected around the controlled source. In the following two examples, the controlled source is modeled by the simple circuit in Figure 5.28(a). In the first example, two external resistors are added; in the second, a resistor and a capacitor.

▶ *EXAMPLE 5.11*

Find e_o for the circuit shown in Figure 5.29(a).

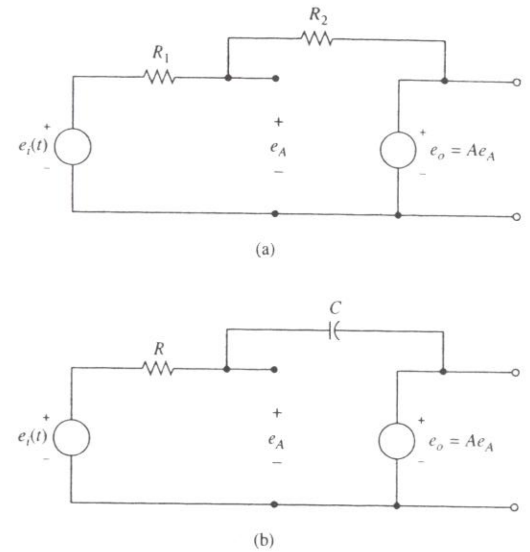


FIGURE 5.29 (a) Circuit for Example 5.11. (b) Circuit for Example 5.12.

Solution

Summing the currents at node A gives

$$\frac{1}{R_1}[e_A - e_i(t)] + \frac{1}{R_2}(e_A - e_o) = 0$$

Because $e_o = Ae_A$,

$$\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{A}{R_2}\right)e_A = \frac{1}{R_1}e_i(t)$$

Multiplying both sides of this equation by R_1R_2, solving for e_A, and then setting $e_o = Ae_A$, we obtain

$$e_o = Ae_A = \left[\frac{AR_2}{R_2 + (1-A)R_1}\right]e_i(t) = \left[\frac{R_2}{-R_1 + \dfrac{1}{A}(R_1 + R_2)}\right]e_i(t)$$

Note that for very large values of A, $e_o = -(R_2/R_1)e_i(t)$. Under these conditions, the size of the voltage gain is determined solely by the ratio of the two resistors.

▶ *EXAMPLE 5.12*

Find an expression for e_o for the circuit shown in Figure 5.29(b).

Solution

Summing the currents at node A gives

$$\frac{1}{R}[e_A - e_i(t)] + C(\dot{e}_A - \dot{e}_o) = 0$$

Because $e_o = Ae_A$, we replace e_A by e_o/A to obtain

$$\frac{1}{AR}e_o + \frac{C}{A}\dot{e}_o - C\dot{e}_o = \frac{1}{R}e_i(t)$$

Dividing both sides of this equation by C and rearranging terms, we obtain the input-output differential equation

$$\left(1 - \frac{1}{A}\right)\dot{e}_o - \frac{1}{ARC}e_o = \frac{-1}{RC}e_i(t)$$

For very large values of A, this reduces to

$$\dot{e}_o = -\left(\frac{1}{RC}\right)e_i(t)$$

If there is no initial stored energy, then $e_o(0) = 0$, and

$$e_o = \frac{-1}{RC}\int_0^t e_i(\lambda)d\lambda$$

The circuit is then called an integrator, because its output is proportional to the integral of the input.

The **operational amplifier** (often called an **op-amp**) is a particularly important building block in the electrical part of many modern systems. The device typically contains over twenty transistors plus a number of resistors and capacitors, and it may have ten or more external terminals. However, its basic behavior is reasonably simple. There are two input terminals for time-varying signals and one output terminal. The symbol for the device is shown in Figure 5.30(a), and a basic circuit model is given in part (b) of the figure.

Complete physical descriptions can be found in undergraduate electronics books. However, before considering any applications, we shall list

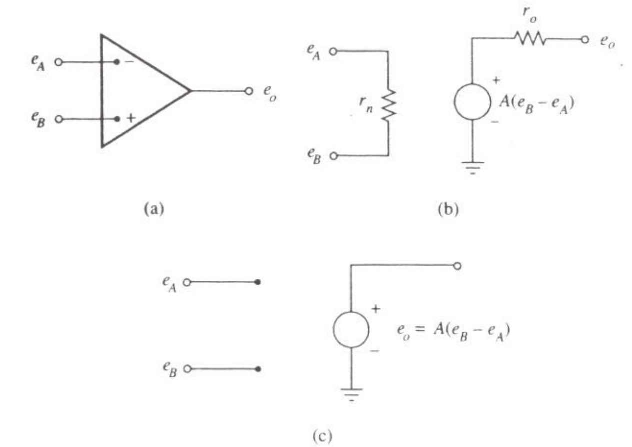


FIGURE 5.30 Operational amplifier. (a) Schematic representation. (b) Equivalent circuit. (c) Idealized equivalent circuit.

without detailed explanations some practical features that users of op-amps should know about.

The input terminals marked with the minus and plus signs are called the inverting and noninverting terminals, respectively. We have denoted their voltages with respect to the ground point (which is the zero-volt reference) by e_A and e_B, although e_1 and e_2 are frequently used. One of the input terminals is often connected to the ground point, but this is not necessary.

Typical values for r_n exceed $10^6\ \Omega$, and r_o is normally less than $100\ \Omega$. In most applications, the resistance r_n can be replaced by an open circuit, and r_o by a short circuit, leading to the simplified model in Figure 5.30(c). Then no current can flow into the device from the left, and the output voltage is $e_o = A(e_B - e_A)$. The voltage amplification A is extremely large, typically exceeding 10^5.

Note that the symbol in Figure 5.30(a) does not show the ground point. In fact, the device itself does not have an external terminal that can be connected directly to ground. There are, however, terminals for the attachment of positive and negative bias voltages. The other ends of these constant bias voltages are connected to a common junction, which is the ground point that appears in parts (b) and (c) of the figure. Circuit diagrams involving op-amps must always show which of the other elements are connected to this external ground. Sometimes ground symbols appear in several different places on the diagram, in which case they can all be connected together. However, the diagrams for our examples will already have had this done.

Because our interest is in the time-varying signals, the constant bias voltages are not normally shown on the circuit diagram. In addition to establishing an external ground point for the op-amp, they provide whatever power is needed for the output signal. Although there may also be other terminals on a practical op-amp, we need be concerned only with those shown in Figure 5.30.

For the examples in this section, we shall use Figure 5.30(c) as the equivalent circuit for the op-amp. When doing so, we shall avoid summing the currents at the output of the op-amp, because the current coming out of that terminal from the controlled voltage source would be an additional unknown. After a little practice, many people prefer not to bother to formally replace the original op-amp symbol in Figure 5.30(a) by the equivalent circuit. In this case, the reader must remember that there is no current into terminals A and B but that an unknown current leaves the output terminal. When we are dealing directly with the op-amp symbol, which does not show all the terminals, it might seem at first glance that the sum of the currents leaving the device is not zero, an apparent violation of Kirchhoff's current law. However, when the symbol in part (a) of Figure 5.30 is replaced by an equivalent circuit that includes the ground point, as in part (c), this apparent contradiction disappears.

▶ *EXAMPLE 5.13*

Find the input-output equation for the two circuits shown in Figure 5.31.

Solution

With the op-amp replaced by the ideal model in Figure 5.30(c), the circuits are equivalent to those in Figure 5.29, except that $e_o = -Ae_A$ rather than $e_o = Ae_A$. Therefore, the results are the same as those in Examples 5.11 and 5.12, except that A is replaced by $-A$. For part (a) of the figure,

$$e_o = \left(\frac{-R_2}{R_1 + \dfrac{1}{A}(R_1 + R_2)} \right) e_i(t)$$

which for very large values of A becomes $e_o = -(R_2/R_1)e_i(t)$. For part (b),

$$\left(1 + \frac{1}{A}\right)\dot{e}_o + \frac{1}{ARC}e_o = \frac{-1}{RC}e_i(t)$$

which for very large values of A becomes

$$\dot{e}_o = -\frac{1}{RC}e_i(t)$$

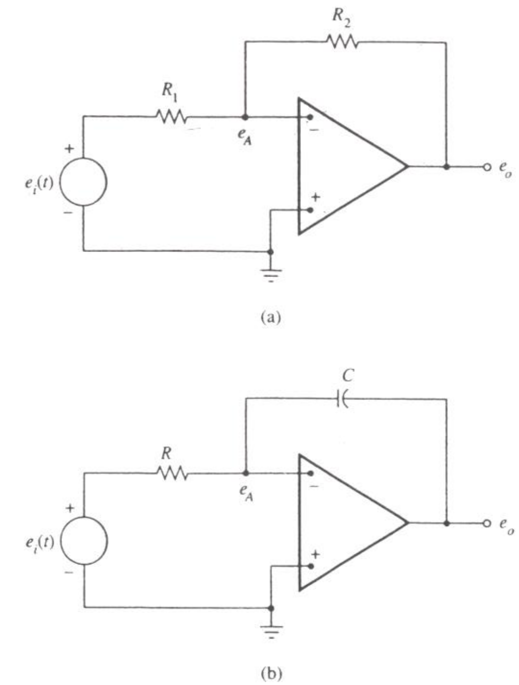


FIGURE 5.31 Circuits for Example 5.13.

▶ *EXAMPLE 5.14*

Find an expression for the output voltage e_o for the circuit in Figure 5.32.

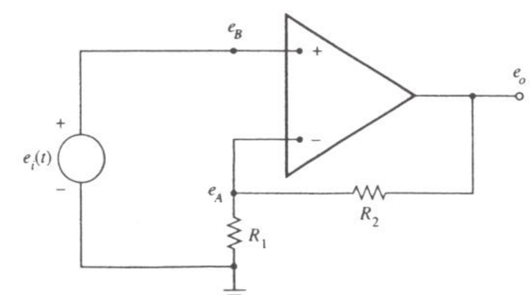


FIGURE 5.32 Circuit for Example 5.14.

Solution

Because no current can flow into the input terminals of the op-amp, we can use the voltage-divider rule to write

$$e_A = \frac{R_1}{R_1 + R_2} e_o$$

Then

$$e_o = A[e_i(t) - e_A] = Ae_i(t) - \left(\frac{AR_1}{R_1 + R_2}\right) e_o$$

from which

$$e_o = \left(\frac{R_1 + R_2}{R_1 + \frac{1}{A}(R_1 + R_2)}\right) e_i(t)$$

which for very large values of A becomes

$$e_o = \left(1 + \frac{R_2}{R_1}\right) e_i(t)$$

In the foregoing examples, note how the elements connected around the op-amp completely determine the behavior when we use the simple model with $A \to \infty$. We first found a general input-output equation in terms of A and then let $A \to \infty$ in order to get a simpler expression. Because A is so very large in practice, people often use an easier method to get the simpler expression directly. The output voltage of the device, given by $e_o = A(e_B - e_A)$, must be finite, so the voltage $e_B - e_A$ between the input terminals must approach zero when A is very large. In practice, this voltage really is only a tiny fraction of one volt, such as 0.1 mV.

Assuming that the voltage difference $e_B - e_A$ is virtually zero is sometimes called the **virtual-short** concept, because the voltage across a short circuit is zero. However, unlike a physical short circuit represented by an ideal wire (through which current could flow), we must still assume that no current flows into either of the input terminals.

▶ *EXAMPLE 5.15*

Use the virtual-short concept to determine directly the input-output behavior for the circuit shown in Figure 5.33 when A is very large.

Solution

Because $e_A = 0$, $i_1 = e_i(t)/R$ and $i_C = -C\dot{e}_o$. No current can flow into the input terminals of the op-amp, so $i_1 = i_C$ and $\dot{e}_o = -(1/RC)e_i(t)$, which agrees with the answer to Example 5.13.

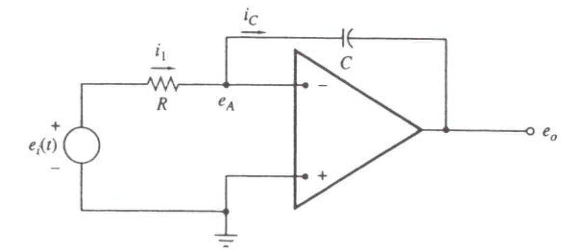


FIGURE 5.33 Circuit for Example 5.15.

▶ *EXAMPLE 5.16*

Find the input-output differential equation describing the circuit shown in Figure 5.34.

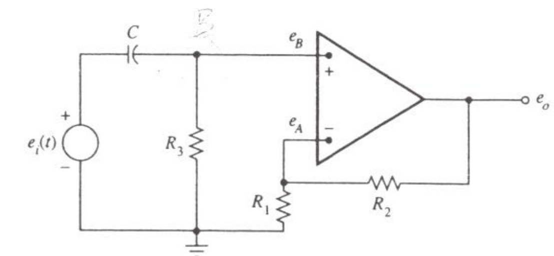


FIGURE 5.34 Circuit for Example 5.16.

Solution

Summing the currents at node B gives

$$C(\dot{e}_B - \dot{e}_i) + \frac{1}{R_3} e_B = 0$$

By the virtual-short concept, $e_B = e_A$, which according to the voltage-divider rule can be written as $e_B = [R_1/(R_1 + R_2)]e_o$. Substituting this into the previous equation gives

$$\frac{R_1 C}{R_1 + R_2}\dot{e}_o - C\dot{e}_i + \frac{R_1}{(R_1 + R_2)R_3} e_o = 0$$

from which

$$R_1 R_3 C \dot{e}_o + R_1 e_o = (R_1 + R_2) R_3 C \dot{e}_i$$

or

$$\dot{e}_o + \frac{1}{R_3 C} e_o = \left(1 + \frac{R_2}{R_1}\right) \dot{e}_i$$

The final example illustrates finding the state-variable model for circuits containing an op-amp. The choice of state variables and the procedure for deriving the model are essentially the same as for the examples in Section 5.6.

▶ *EXAMPLE 5.17*

Find a state-variable model for the circuit shown in Figure 5.35. Assume that the op-amp is ideal, with a gain large enough to allow use of the virtual-short concept. Let the output be e_o, and take as state variables the capacitor voltages e_{C_1} and e_{C_2}.

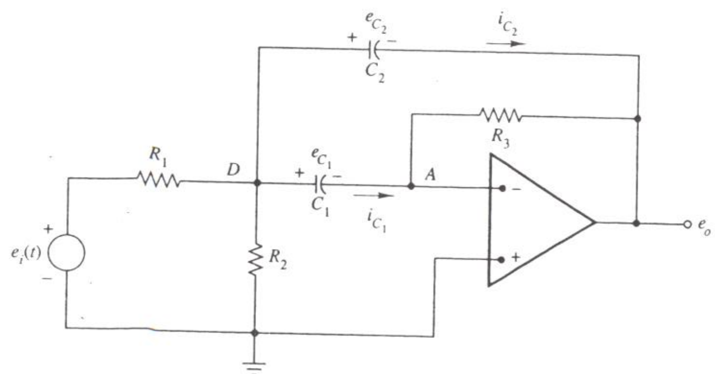


FIGURE 5.35 Circuit for Example 5.17.

Solution

By the virtual-short concept, $e_A = 0$. Then the voltage at point D is e_{C_1}, and

$$e_o = e_{C_1} - e_{C_2} \tag{56}$$

which is the output equation. Because there is no current flowing into the input terminals of the op-amp, and because the voltage at node A is zero, $i_{C_1} = -(1/R_3)e_o$. Using (56), we see that

$$i_{C_1} = -\frac{1}{R_3}(e_{C_1} - e_{C_2}) \tag{57}$$

Summing currents at node D gives

$$i_{C_1} + i_{C_2} + \frac{1}{R_2}e_{C_1} + \frac{1}{R_1}\left[e_{C_1} - e_i(t)\right] = 0$$

Inserting (57) into this equation yields

$$i_{C_2} = \left(\frac{1}{R_3} - \frac{1}{R_1} - \frac{1}{R_2}\right)e_{C_1} - \frac{1}{R_3}e_{C_2} + \frac{1}{R_1}e_i(t) \tag{58}$$

Finally, substituting (57) and (58) into the element law for a capacitor, we have the state-variable equations

$$\dot{e}_{C_1} = -\frac{1}{R_3C_1}(e_{C_1} - e_{C_2})$$

$$\dot{e}_{C_2} = \frac{1}{C_2}\left[\left(\frac{1}{R_3} - \frac{1}{R_1} - \frac{1}{R_2}\right)e_{C_1} - \frac{1}{R_3}e_{C_2} + \frac{1}{R_1}e_i(t)\right]$$

Other op-amp circuits appear in the problems at the end of this chapter and also in Chapters 8 and 14. They include applications involving isolation, summing, inverting, integrating, and filtering of signals.

SUMMARY

After introducing the element and interconnection laws for electrical circuits, we developed systematic procedures for obtaining both the input-output differential equation and the set of state-variable equations. For the general node-equation method, we select a ground node and label the voltages of the other nodes with respect to ground. We then write current-law equations at the nodes whose voltages are unknown, using the element laws to express the currents through the passive elements in terms of the node voltages. If it becomes necessary to sum currents at a node to which a voltage source is connected, keep in mind that the current through such a source is another unknown variable.

For a state-variable model, we normally choose as state variables the voltage across each capacitor and the current through each inductor. Two types of exceptions to this choice were illustrated in Figure 5.27. As far as possible, unknown variables are labeled on the diagram in terms of state variables and inputs. By using Ohm's law and Kirchhoff's laws, we then express the capacitor currents, the inductor voltages, and any other outputs as algebraic functions of the state variables and inputs. Inserting these expressions into the element laws $\dot{e}_C = (1/C)i_C$ and $di_L/dt = (1/L)e_L$ yields the state-variable model.

The important special case of resistive circuits, including the rules for series-parallel combinations, was treated in Section 5.5. Controlled sources, with an emphasis on op-amp applications, were explained in Section 5.7. For an ideal op-amp, no current flows into the input terminals, but the output current is unknown. The op-amp gain is usually large enough so that the voltage between the two input terminals can be assumed to be zero.

PROBLEMS

5.1 Find the input-output differential equation relating e_o and $i_i(t)$ for the circuit shown in Figure P5.1.

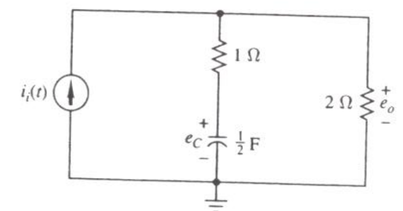


FIGURE P5.1

5.2 Find the input-output differential equation relating e_o and $e_i(t)$ for the circuit shown in Figure P5.2.

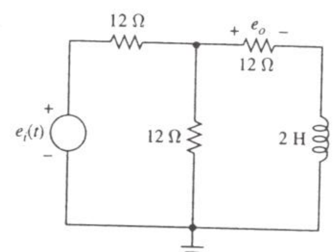


FIGURE P5.2

* **5.3** Repeat Problem 5.2 for the circuit shown in Figure P5.3.

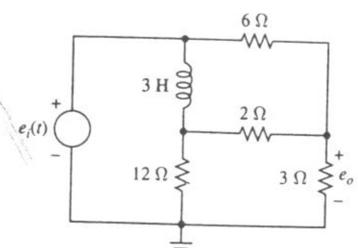


FIGURE P5.3

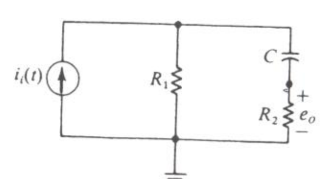


FIGURE P5.4

5.4 Find the input-output differential equation for the circuit shown in Figure P5.4.

5.5 For the circuit shown in Figure P5.5, use the node-equation method to find the input-output differential equation.

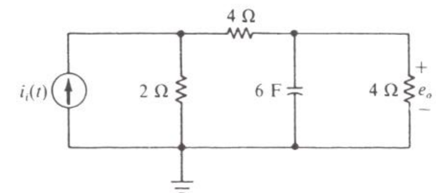


FIGURE P5.5

5.6 Repeat Problem 5.5 for the circuit shown in Figure P5.6.

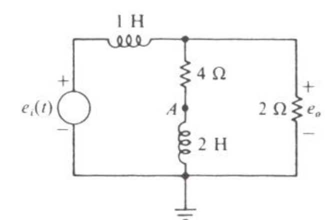


FIGURE P5.6

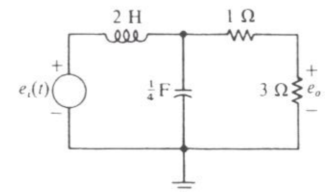


FIGURE P5.7

* **5.7** Repeat Problem 5.5 for the circuit shown in Figure P5.7.

5.8 Repeat Problem 5.5 for the circuit shown in Figure P5.8.

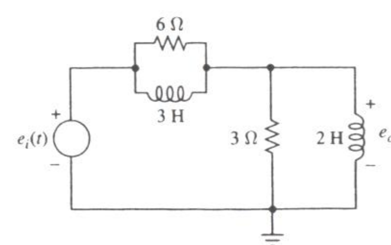


FIGURE P5.8

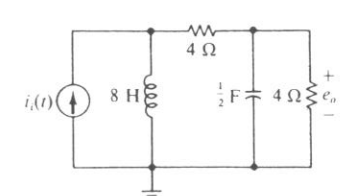


FIGURE P5.9

* **5.9** Repeat Problem 5.5 for the circuit shown in Figure P5.9.

5.10 Find the input-output differential equation relating e_o to $e_i(t)$ and $i_a(t)$ for the circuit shown in Figure P5.10.

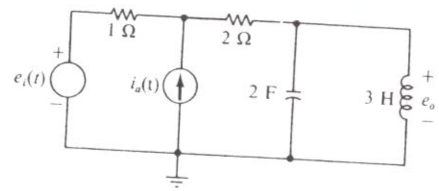


FIGURE P5.10

5.11 **a)** For the circuit shown in Figure P5.11, write current-law equations at nodes A and B to obtain a pair of coupled differential equations in the variables e_A, e_o, and $e_i(t)$.
b) Find the input-output differential equation relating e_o and $e_i(t)$.

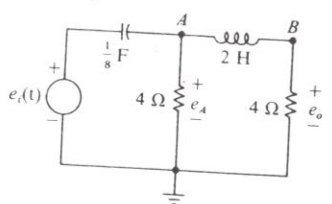


FIGURE P5.11

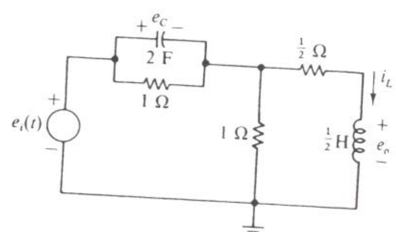


FIGURE P5.12

* **5.12** For the circuit shown in Figure P5.12, use the node-equation method to find the input-output differential equation relating e_o and $e_i(t)$.

5.13 Repeat Example 5.5 with the capacitor C replaced by an inductor L.

5.14 For the circuit shown in Figure P5.14, use the rules for series and parallel combinations of resistors to find i_o and the equivalent resistance connected across the source.

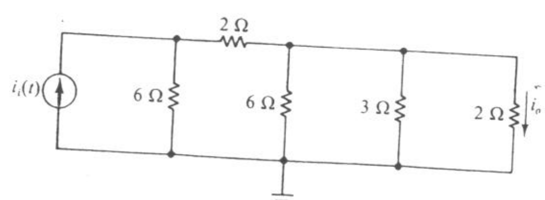


FIGURE P5.14

5.15 For the circuit shown in Figure P5.15, use the rules for series and parallel resistors to find e_o and the equivalent resistance connected across the source.

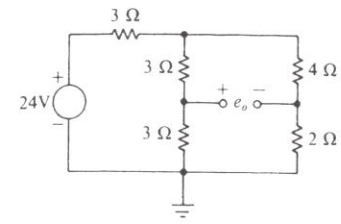


FIGURE P5.15

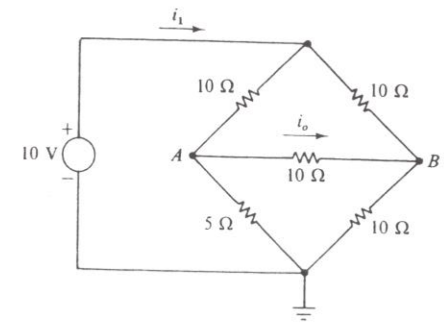


FIGURE P5.16

5.16 **a)** Explain why the rules for series and parallel resistors cannot be used for the circuit shown in Figure P5.16.
b) Use the node-equation method to find the voltages of nodes A and B with respect to the ground node.
c) Find the currents i_o and i_1 and the equivalent resistance connected across the source.

* **5.17** Find e_o for the circuit shown in Figure P5.17.

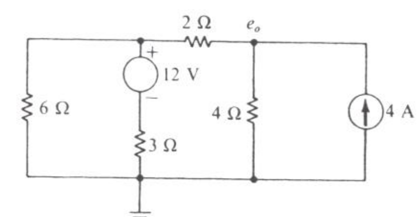


FIGURE P5.17

* **5.18** **a)** Find a set of state-variable equations describing the circuit shown in Figure P5.18. Define the variables and show their positive senses on the diagram.
b) Write an algebraic output equation for i_o, which is the current through the 6-V source.

5.19 For the circuit shown in Figure P5.6, find a set of state-variable equations and an algebraic output equation for e_o.

5.20 Repeat Problem 5.19 for the circuit shown in Figure P5.7.

* **5.21** Repeat Problem 5.19 for the circuit shown in Figure P5.9.

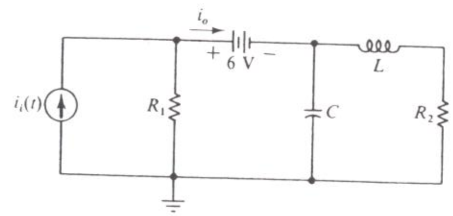


FIGURE P5.18

5.22 Repeat Problem 5.19 for the circuit shown in Figure P5.22.

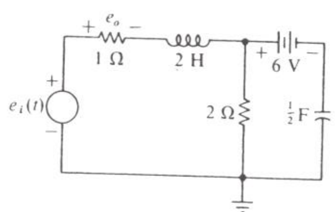


FIGURE P5.22

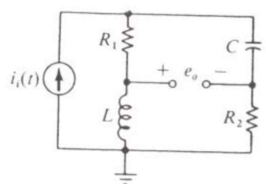


FIGURE P5.23

5.23 Repeat Problem 5.19 for the circuit shown in Figure P5.23.

* **5.24** Repeat Problem 5.19 for the circuit shown in Figure P5.24.

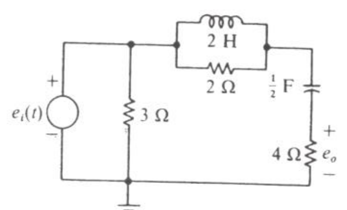


FIGURE P5.24

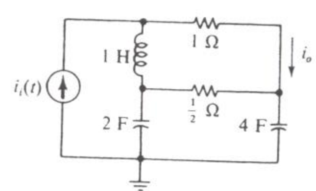


FIGURE P5.25

5.25 For the circuit shown in Figure P5.25, find a set of state-variable equations and write an algebraic output equation for i_o. Define the variables and show their positive senses on the diagram.

5.26 Repeat Problem 5.25 for the circuit shown in Figure P5.26.

* **5.27** Find a set of state-variable equations for the circuit shown in Figure P5.27. Write the algebraic output equation for i_o.

5.28 Find the state-variable equation for the circuit shown in Figure 5.27(a) when the initial choice of the state variable is e_B rather than e_A. Write an algebraic output equation for e_A.

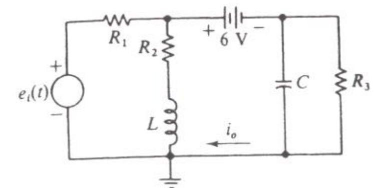


FIGURE P5.26

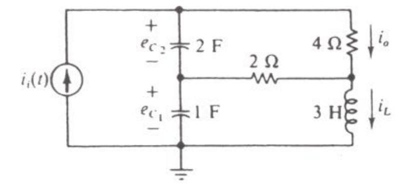


FIGURE P5.27

5.29 Find a set of state-variable equations for the circuit shown in Figure 5.27(a) when the voltage source is replaced by the current source $i_i(t)$ with its reference arrow directed upward. Write an algebraic output equation for i_{C_1}.

5.30 a) Find the state-variable model for the circuit shown in Figure 5.27(b).
b) Write an algebraic output equation for the voltage across the current source.

* **5.31** The circuit shown in Figure P5.31 contains two current sources, one of which is voltage-controlled. Find the input-output differential equation relating e_o and $i_i(t)$.

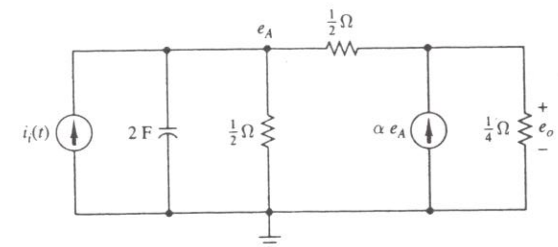


FIGURE P5.31

5.32 The circuit shown in Figure P5.32 contains an independent current source and a current-controlled voltage source. Find the input-output differential equation relating e_o and $i_i(t)$.

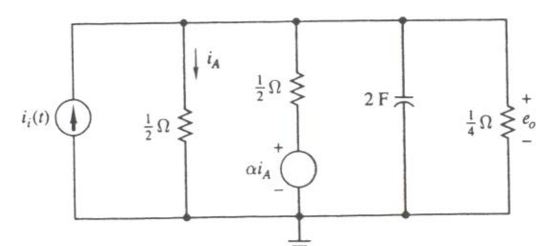


FIGURE P5.32

5.33 The circuit shown in Figure P5.33 contains an independent voltage source and a current-controlled current source. Find the input-output differential equation relating e_o and $e_i(t)$.

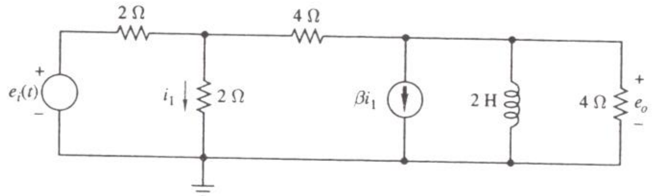


FIGURE P5.33

* **5.34** For the op-amp circuit shown in Figure P5.34, derive the algebraic expression for the output voltage e_o in terms of the two input voltages, $e_1(t)$ and $e_2(t)$. Indicate what mathematical operation the circuit performs.

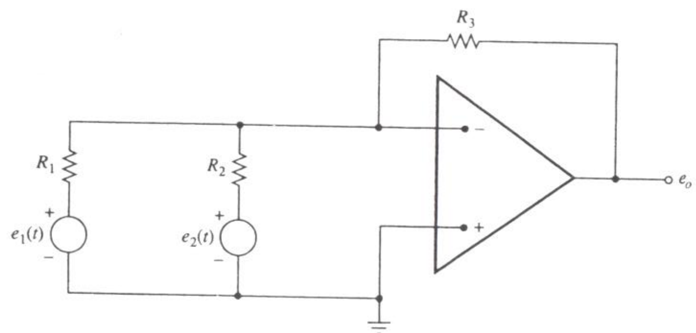


FIGURE P5.34

5.35 Repeat Problem 5.34 for the circuit shown in Figure P5.35.

5.36 For the op-amp circuit shown in Figure P5.36, derive the input-output differential equation relating the output voltage e_o and the input voltage $e_i(t)$.

* **5.37** For the op-amp circuit shown in Figure P5.37, derive the input-output differential equation relating the output voltage e_o and the input voltage $e_i(t)$.

5.38 **a)** For the op-amp circuit shown in Figure P5.38, derive the input-output differential equation relating the output voltage e_o and the input voltage $e_i(t)$. **b)** Derive the state-variable model, taking the output to be the current through R_3, with the positive sense to the right.

5.39 Write in matrix form the circuit model developed in Example 5.7. Identify the matrices **A**, **B**, **C**, and **D**.

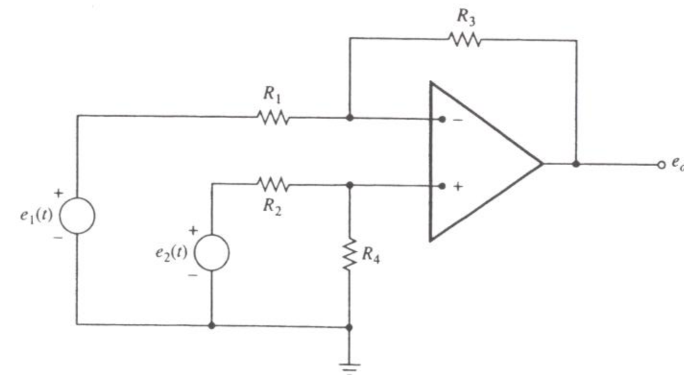


FIGURE P5.35

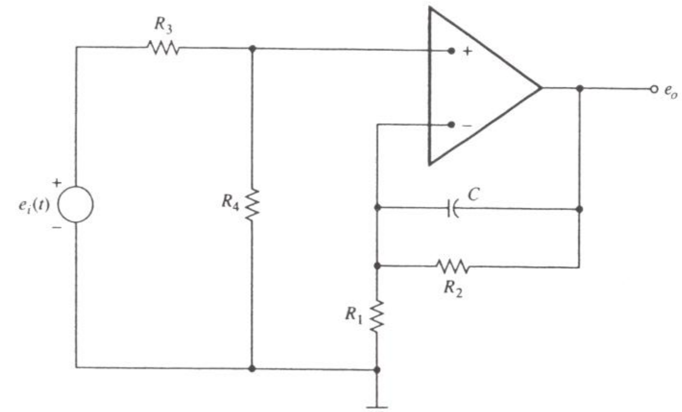


FIGURE P5.36

5.40 Repeat Problem 5.39 for the state-variable model derived in Example 5.8 for the circuit shown in Figure 5.25.

* **5.41** Repeat Problem 5.39 for the state-variable model derived in Example 5.9 for the circuit shown in Figure 5.26(a).