CHAPTER 13

BLOCK DIAGRAMS

A block diagram is an interconnection of symbols representing certain basic mathematical operations in such a way that the overall diagram obeys the system's mathematical model. In the diagram, the lines interconnecting the blocks represent the variables describing the system behavior, such as the input and state variables. Inspecting a block diagram of a system may provide new insight into the system's structure and behavior beyond that available from the differential equations themselves.

Throughout most of this chapter we shall restrict the discussion to fixed linear systems that contain no initial stored energy. After we transform the equations describing such a system, the variables that we shall use will be the Laplace transforms of the corresponding functions of time. The parts of the system can then be described by their transfer functions, as in Section 8.2. Recall that transfer functions give only the zero-state response. However, the steady-state response of a stable system does not depend on the initial conditions, so in that case there is no loss of generality in using only the zero-state response.

When we draw diagrams using functions of time as the variables, we generally call them simulation diagrams rather than block diagrams. Such diagrams can be useful in preparing a computer simulation. Unlike block diagrams, they can also be extended to include initial conditions and nonlinear elements. We shall include an example of a simulation diagram for a nonlinear system, even though our emphasis will be on block diagrams and Laplace-transformed quantities.

After defining the components to be used in our diagrams, we first consider systems described by state-variable models. We then look at systems described by input-output equations and by models in nonstandard form. Finally, we develop rules for simplifying block diagrams, emphasizing those that represent feedback systems.

13.1 DIAGRAM BLOCKS

The operations that we generally use in block diagrams are summation, gain, and multiplication by a transfer function. Unless otherwise stated, all variables will be Laplace-transformed quantities.

Summer

The addition and subtraction of variables is represented by a **summer**, or **summing junction**. A summer is represented by a circle that has any number of arrows directed toward it (denoting inputs) and a single arrow directed away from it (denoting the output). Next to each entering arrowhead is a plus or minus symbol indicating the sign associated with the variable that the particular arrow represents. The output variable, appearing as the one arrow leaving the circle, is defined to be the sum of all the incoming variables, with the associated signs taken into account. A summer having three inputs $X_1(s)$, $X_2(s)$, and $X_3(s)$ appears in Figure 13.1.

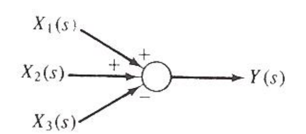


FIGURE 13.1 Summer representing $Y(s) = X_1(s) + X_2(s) - X_3(s)$.

Gain

The multiplication of a single variable by a constant is represented by a **gain** block. We place no restriction on the value of the gain, which may be positive or negative. It may be an algebraic function of other constants and/or system parameters. Several self-explanatory examples are shown in Figure 13.2.

Transfer Function

For a fixed linear system with no initial stored energy, the transformed output $Y(s)$ is given by

$$Y(s) = H(s)U(s)$$

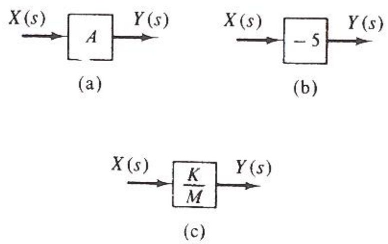


FIGURE 13.2 Gains. (a) $Y(s) = AX(s)$. (b) $Y(s) = -5X(s)$. (c) $Y(s) = (K/M)X(s)$.

where $H(s)$ is the transfer function and $U(s)$ is the transformed input. When dealing with parts of a larger system, we often use $F(s)$ and $X(s)$ for the transfer function and transformed input of an individual part. Then

$$Y(s) = F(s)X(s) \tag{1}$$

Any system or combination of elements can be represented by a block containing its transfer function $F(s)$, as indicated in Figure 13.3(a). For example, the first-order system that obeys the input-output equation

$$\dot{y} + \frac{1}{\tau}y = Ax(t)$$

has as its transfer function

$$F(s) = \frac{A}{s + \dfrac{1}{\tau}}$$

Thus it could be represented by the block diagram shown in Figure 13.3(b). Note that the gain block in Figure 13.2(a) can be considered as a special case of a transfer function block, with $F(s) = A$.

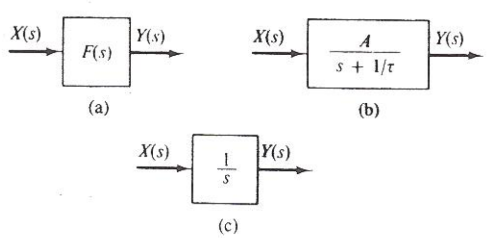


FIGURE 13.3 Basic block diagrams. (a) Arbitrary transfer function. (b) First-order system. (c) Integrator.

Integrator

Another important special case of a general transfer function block—one that will appear frequently in our diagrams—is the **integrator** block. An integrator that has an input $x(t)$ and an output $y(t)$ obeys the relationship

$$y(t) = y(0) + \int_0^t x(\lambda)d\lambda$$

where λ is the dummy variable of integration. Setting $y(0)$ equal to 0 and transforming the equation give

$$Y(s) = \frac{1}{s}X(s)$$

Hence the transfer function of the integrator is $Y(s)/X(s) = 1/s$, as shown in Figure 13.3(c).

The next three sections contain examples of constructing block diagrams using the above components. Because a block diagram is merely a pictorial representation of a set of algebraic Laplace-transformed equations, it is possible to combine blocks by calculating equivalent transfer functions and thereby to simplify the diagram. We shall now present procedures for handling series and parallel combinations of blocks. Methods for simplifying diagrams containing feedback paths will be discussed in Section 13.5.

Series Combination

Two blocks are said to be in **series** when the output of one goes only to the input of the other, as shown in Figure 13.4(a). The transfer functions of the individual blocks in the figure are $F_1(s) = V(s)/X(s)$ and $F_2(s) = Y(s)/V(s)$.

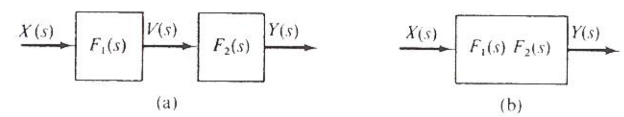


FIGURE 13.4 (a) Two blocks in series. (b) Equivalent diagram.

When we evaluate the individual transfer functions, it is essential that we take any **loading effects** into account. This means that $F_1(s)$ is the ratio $V(s)/X(s)$ when the two subsystems are connected, so any effect the second subsystem has on the first is accounted for in the mathematical model. The same statement holds for calculating $F_2(s)$. For example, the input-output relationship for a linear potentiometer loaded by a resistor connected from its wiper to the ground node was shown in Example 10.1 to differ from that of the unloaded potentiometer.

In Figure 13.4(a), $Y(s) = F_2(s)V(s)$ and $V(s) = F_1(s)X(s)$. It follows that

$$\begin{aligned} Y(s) &= F_2(s)[F_1(s)X(s)] \\ &= [F_1(s)F_2(s)]X(s) \end{aligned}$$

Thus the transfer function relating the input transform $X(s)$ to the output transform $Y(s)$ is $F_1(s)F_2(s)$, the product of the individual transfer functions. The equivalent block diagram is shown in Figure 13.4(b).

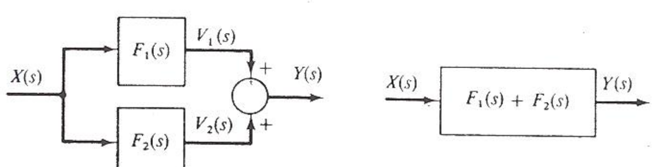
Parallel Combination

Two systems are said to be in **parallel** when they have a common input and their outputs are combined by a summing junction. If, as indicated in Figure 13.5(a), the individual blocks have the transfer functions $F_1(s)$ and $F_2(s)$ and the signs at the summing junction are both positive, the overall transfer function $Y(s)/X(s)$ will be the sum $F_1(s) + F_2(s)$, as shown in Figure 13.5(b). To prove this statement, we note that

$$Y(s) = V_1(s) + V_2(s)$$

where $V_1(s) = F_1(s)X(s)$ and $V_2(s) = F_2(s)X(s)$. Substituting for $V_1(s)$ and $V_2(s)$, we have

$$Y(s) = [F_1(s) + F_2(s)]X(s)$$



(a) (b)

FIGURE 13.5 (a) Two blocks in parallel. (b) Equivalent diagram.

If either of the summing-junction signs associated with $V_1(s)$ or $V_2(s)$ is negative, we must change the sign of the corresponding transfer function in forming the overall transfer function. The following example illustrates the rules for combining blocks that are in parallel or in series.

▶ *EXAMPLE 13.1*

Evaluate the transfer functions $Y(s)/U(s)$ and $Z(s)/U(s)$ for the block diagram shown in Figure 13.6, giving the results as rational functions of s.

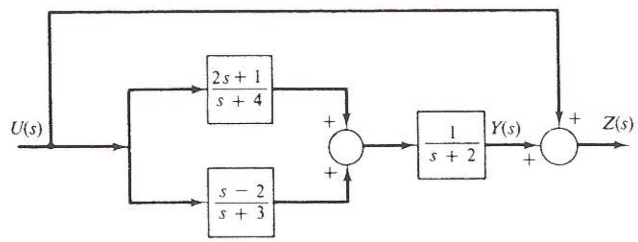


FIGURE 13.6 Block diagram for Example 13.1.

Solution

Because $Z(s)$ can be viewed as the sum of the outputs of two parallel blocks, one of which has $Y(s)$ as its output, we first evaluate the transfer function $Y(s)/U(s)$. To do this, we observe that $Y(s)$ can be considered the output of a series combination of two parts, one of which is a parallel combination of two blocks. Starting with this parallel combination, we write

$$\frac{2s+1}{s+4} + \frac{s-2}{s+3} = \frac{3s^2+9s-5}{s^2+7s+12}$$

and redraw the block diagram as shown in Figure 13.7(a). The series combination in this version has the transfer function

$$\begin{aligned} \frac{Y(s)}{U(s)} &= \frac{3s^2+9s-5}{s^2+7s+12} \cdot \frac{1}{s+2} \\ &= \frac{3s^2+9s-5}{s^3+9s^2+26s+24} \end{aligned}$$

which leads to the diagram shown in Figure 13.7(b). We can reduce the final parallel combination to the single block shown in Figure 13.7(c) by writing

$$\begin{aligned} \frac{Z(s)}{U(s)} &= 1 + \frac{Y(s)}{U(s)} \\ &= 1 + \frac{3s^2+9s-5}{s^3+9s^2+26s+24} \\ &= \frac{s^3+12s^2+35s+19}{s^3+9s^2+26s+24} \end{aligned}$$

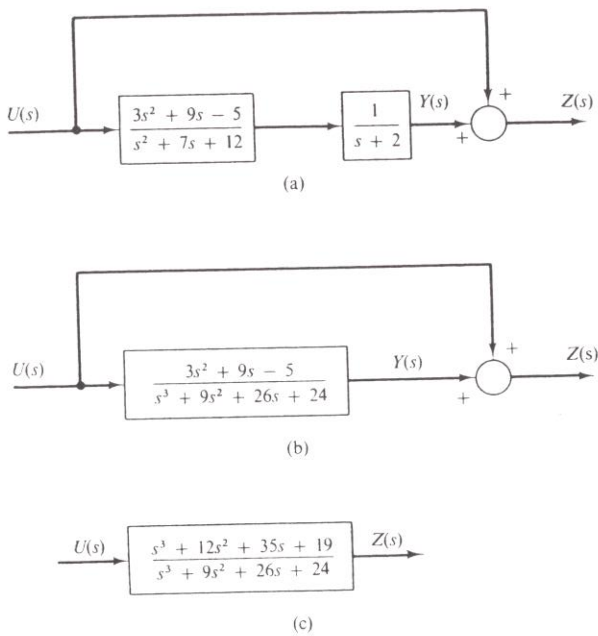


FIGURE 13.7 Equivalent block diagrams for the diagram shown in Figure 13.6.

In general, it is desirable to reduce the transfer functions of combinations of blocks to rational functions of s in order to simplify the subsequent analysis. This will be particularly important in Section 13.5 when we are reducing feedback loops to obtain an overall transfer function.

13.2 DIAGRAMS FOR STATE-VARIABLE MODELS

Having defined the necessary building blocks, we next develop techniques for constructing the block diagram for a fixed linear system whose model is available in state-variable form. We shall transform the equations, with the initial condition terms set equal to zero, and then draw a diagram to fit the transformed equations. We start with a first-order system and then extend the procedure to systems of higher order.

First-Order Systems

Consider a fixed linear system that has a single state variable q, a single input u, and a single output y. The form of the state-variable model is

$$\dot{q} = aq + bu \tag{2a}$$

$$y = cq + du \tag{2b}$$

where a, b, c, and d are constants. Transforming these equations, with $q(0) = 0$, gives

$$sQ(s) = aQ(s) + bU(s) \tag{3a}$$

$$Y(s) = cQ(s) + dU(s) \tag{3b}$$

To construct the block diagram, we assume that the transformed input $U(s)$ is available for use in forming $sQ(s)$ and $Y(s)$ and that the coefficients $a, b, c,$ and d are known. From (3a) we see that the quantity $sQ(s)$ can be formed by using gain and summing blocks, provided that $Q(s)$ is available. If we make $sQ(s)$ the input to an integrator, whose transfer function is $1/s$, then the output of the integrator will be $Q(s)$, which becomes available for use in implementing (3a). The process is circular in that $sQ(s)$ leads to $Q(s)$ via the integrator, whereas $Q(s)$, along with $U(s)$, is used to form $sQ(s)$ by (3a). We can then finish the diagram by using (3b) to form $Y(s)$ from the variables $Q(s)$ and $U(s)$. The steps in the process are as follows, and the diagram as it appears after each step is shown in Figure 13.8.

1. Draw an integrator block with input $sQ(s)$ and output $Q(s)$.

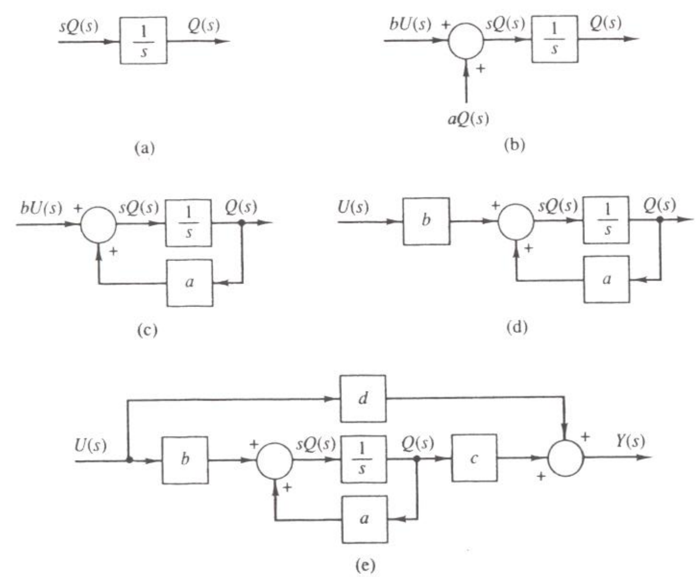


FIGURE 13.8 Block diagram for (3). (a), (b), (c) Partial diagrams for (3a). (d) Complete diagram for (3a). (e) Complete diagram for both equations.

2. Draw a summer to the left of the integrator having $sQ(s)$ as its output and $bU(s)$ and $aQ(s)$ as its inputs. Label both input arrows with plus signs.
3. Draw a block with gain a having $Q(s)$ as its input, and connect its output to the arrow labeled $aQ(s)$ entering the summer.
4. Draw a block with gain b having $U(s)$ as its input, and connect its output to the arrow labeled $bU(s)$ entering the summer. This step completes the simulation of (3a).
5. Append blocks with gains of c and d to the right side of the diagram, and make their inputs be $Q(s)$ and $U(s)$, respectively. Then draw a summer with inputs $cQ(s)$ and $dU(s)$ and output $Y(s)$. The diagram is now complete.

It is interesting to note that Figure 13.8 clearly illustrates the feedback nature of the system—that is, the fact that the rate of change of the output is dependent on the output itself, in addition to the input. The diagrams of most dynamic systems have such feedback paths from the outputs of one or more integrators back to the integrator input by way of gain blocks and summing junctions.

Second- and Higher-Order Systems

An nth-order fixed linear system having m inputs and p outputs is represented in state-variable form by a set of n first-order differential equations, each of which has the form

$$\dot{q}_i = a_{i1}q_1 + a_{i2}q_2 + \cdots + a_{in}q_n + b_{i1}u_1 + \cdots + b_{im}u_m \quad i = 1, 2, \ldots, n \tag{4}$$

The outputs are given by algebraic equations of the form

$$y_j = c_{j1}q_1 + c_{j2}q_2 + \cdots + c_{jn}q_n + d_{j1}u_1 + \cdots + d_{jm}u_m \quad j = 1, 2, \ldots, p \tag{5}$$

Transforming (4) and (5), with $q_i(0) = 0$ for all i, gives

$$sQ_i(s) = a_{i1}Q_1(s) + a_{i2}Q_2(s) + \cdots + a_{in}Q_n(s) + b_{i1}U_1(s) + \cdots + b_{im}U_m(s) \quad i = 1, 2, \ldots, n \tag{6a}$$

$$Y_j(s) = c_{j1}Q_1(s) + c_{j2}Q_2(s) + \cdots + c_{jn}Q_n(s) + d_{j1}U_1(s) + \cdots + d_{jm}U_m(s) \quad j = 1, 2, \ldots, p \tag{6b}$$

Generalizing the method we outlined for a first-order system, we draw n integrator blocks. We label the input and output of each integrator $sQ_i(s)$ and $Q_i(s)$, respectively, where $i = 1, 2, \ldots, n$. Then, for each value of i, we construct $sQ_i(s)$ according to (6a) from the n transformed state variables that appear as the integrator outputs and from the m inputs, using gain and summer blocks. Finally, we form the p outputs according to (6b) with additional gain and summer blocks.

In the following example, we illustrate this procedure for the general second-order fixed linear system having a single input and a single output.

▶ ***EXAMPLE 13.2***

Draw a block diagram for the system described by the state-variable equations

$$\dot{q}_1 = a_{11}q_1 + a_{12}q_2 + b_1u \tag{7a}$$

$$\dot{q}_2 = a_{21}q_1 + a_{22}q_2 + b_2u \tag{7b}$$

and the output equation

$$y = c_1q_1 + c_2q_2 + du \tag{8}$$

Solution

The transformed equations, with $q_1(0) = q_2(0) = 0$, are

$$sQ_1(s) = a_{11}Q_1(s) + a_{12}Q_2(s) + b_1U(s) \tag{9a}$$

$$sQ_2(s) = a_{21}Q_1(s) + a_{22}Q_2(s) + b_2U(s) \tag{9b}$$

and

$$Y(s) = c_1Q_1(s) + c_2Q_2(s) + dU(s) \tag{10}$$

The system has two state variables, so we need two integrator blocks that have outputs of $Q_1(s)$ and $Q_2(s)$ and inputs of $sQ_1(s)$ and $sQ_2(s)$, respectively. Each of the integrator inputs is the output of a summer which itself has three inputs as defined by (9a) and (9b). The output $Y(s)$ is also formed by using a summer with three inputs according to (10). The resulting diagram is shown in Figure 13.9.

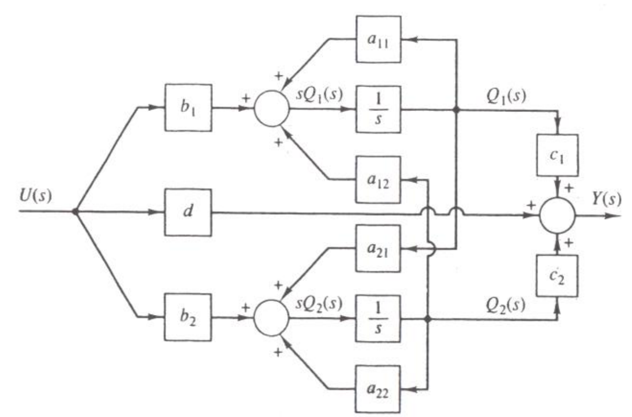


FIGURE 13.9 Block diagram for a general fixed linear second-order system with a single input and a single output in state-variable form.

Simulation Diagrams

For a fixed linear system, the form of the **simulation diagram** is essentially the same as that of the corresponding block diagram, except that all variables are functions of time rather than transformed quantities. The summer and gain blocks are exactly the same except for the labeling of the variables. The integrator, instead of being represented by the transfer function $1/s$ as in Figure 13.10(a), is shown by a block containing an integral sign as in part (b) of the figure. The input-output relationship is

$$y(t) = y(0) + \int_0^t x(\lambda)d\lambda \tag{11}$$

where the initial condition $y(0)$ is generally not explicitly shown. However, $y(0)$, which for transfer function analysis would be assumed to be zero, can be included if desired, as in Figure 13.10(c). Note that (11) can also be written in its differential form as

$$\dot{y}(t) = x(t) \tag{12}$$

leading to the representation in part (d) of the figure.

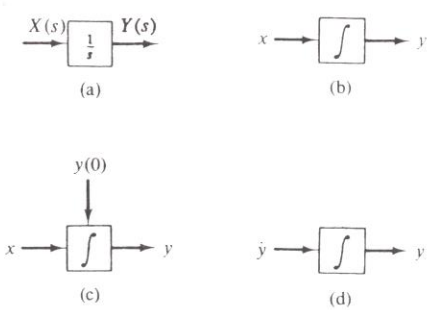


FIGURE 13.10 Integrators. (a) Transfer-function representation. (b) Without initial condition shown. (c) With initial condition shown. (d) Showing differential relationship.

The simulation diagram for Example 13.2 would be the same as that shown in Figure 13.9 except for the following changes. The transformed functions $U(s)$, $Q_1(s)$, $Q_2(s)$, and $Y(s)$ would be changed to the corresponding functions of time, and $sQ_1(s)$ and $sQ_2(s)$ would be replaced by $\dot{q}_1$ and $\dot{q}_2$, respectively. Finally, the integrators would be represented by blocks like the one in Figure 13.10(d) rather than by the transfer function $1/s$.

When considering nonlinear or time-varying systems, we may need additional blocks. A **multiplier** is represented by a block that is labeled MULT and has two inputs and one output. The output variable is the product of the two input variables; thus it is a nonlinear block, unlike the summer, gain,

and integrator. Using multipliers in simulation diagrams, we may implement a variety of nonlinear characteristics and time-varying coefficients. Several examples are shown in Figure 13.11. In Figure 13.11(c), the time-varying coefficient $a(t)$ is assumed to be available as an input to the simulation diagram, as are the inputs to the system.

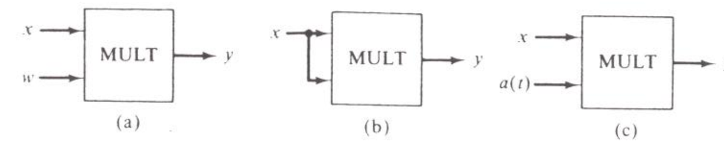


FIGURE 13.11 Multipliers. (a) $y = wx$. (b) $y = x^2$. (c) $y = a(t)x$.

Other nonlinear operations can be represented by a rectangular box that shows the appropriate symbol (if there is one) or a sketch of the function. Three examples are shown in Figure 13.12.

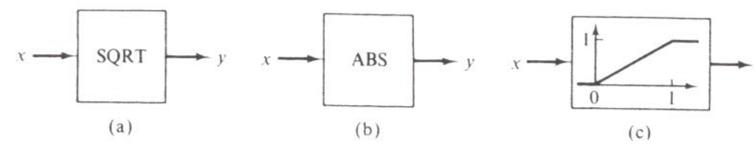


FIGURE 13.12 Nonlinearities. (a) $y = \sqrt{x}$. (b) $y = |x|$. (c) $y = 0$ for $x < 0$, $y = x$ for $0 \le x \le 1$, and $y = 1$ for $x > 1$.

To construct a simulation diagram directly from a state-variable model, we use an integrator for each state variable. The input of the integrator is $\dot{q}_i$ and its output is q_i. Then we combine gains, summers, and any necessary nonlinear blocks to form $\dot{q}_i$ as given by the corresponding state-variable equation. The general procedure is similar to that used for the block diagrams discussed earlier. It is illustrated for a nonlinear system in the following example.

▶ *EXAMPLE 13.3*

Draw a simulation diagram for the high-speed vehicle discussed in Example 9.6 and described by the equation

$$M\dot{v} + Bv + D|v|v = f_a(t)$$

where $f_a(t)$ is the driving force and v is the velocity of the vehicle.

Solution

The model for this vehicle is a first-order nonlinear differential equation with the velocity v as both the state variable and the output. The input is the applied force $f_a(t)$. In state-variable form, the model is

$$\dot{v} = \frac{1}{M}[-Bv - D|v|v + f_a(t)] \tag{13}$$

We initiate the simulation diagram by drawing an integrator with input $\dot{v}$ and output v. The coefficient $1/M$ appears as a gain whose output is $\dot{v}$ and whose input corresponds to the three terms inside the brackets in (13). The bracketed term is the output of a summer with the three inputs Bv, $D|v|v$, and $f_a(t)$, each having the appropriate sign at the corresponding arrowhead. The term $|v|v$ is formed by using a nonlinear block for the absolute value and a multiplier. The completed diagram is shown in Figure 13.13.

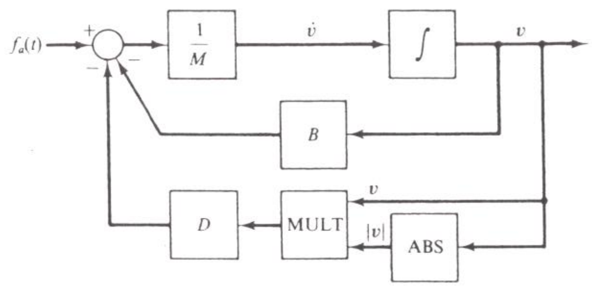


FIGURE 13.13 Simulation diagram for high-speed vehicle.

13.3 DIAGRAMS FOR INPUT-OUTPUT MODELS

The input-output form of the model of an nth-order fixed linear system with input $u(t)$ and output y is the single differential equation

$$a_n y^{(n)} + a_{n-1} y^{n-1} + \cdots + a_0 y = b_m u^{(m)} + \cdots + b_0 u(t) \tag{14}$$

where $y^{(k)}$ denotes $d^k y/dt^k$, and where in practice $m \le n$. Transforming this equation with the initial-condition terms set equal to zero gives

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0)Y(s) = (b_m s^m + \cdots + b_0)U(s)$$

corresponding to the transfer function

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0} \tag{15}$$

In this section we shall present a general method of representing these equations by a block diagram consisting only of summers, gains, and integrators. We also want to be able to write a state-variable model corresponding

to a given input-output equation. Because the choice of state variables is not unique, our method will result in only one of several possible state-variable models.

We shall develop the method in two stages. First we consider systems for which (14) contains no derivatives of the input, and then we extend the method to include input derivatives. For a first-order system, the state-variable and input-output models involve a single first-order differential equation and are essentially identical. Thus we start with second-order systems and generalize the results to apply to higher-order systems.

Second-Order Systems

We shall consider the input-output differential equation

$$a_2 \ddot{y} + a_1 \dot{y} + a_0 y = F(t)$$

for three choices of the forcing function: (1) $F(t) = u(t)$, (2) $F(t) = b_1 \dot{u} + b_0 u(t)$, and (3) $F(t) = b_2 \ddot{u} + b_1 \dot{u} + b_0 u(t)$. We shall construct the corresponding block diagrams in the following three examples, the last one of which will constitute the most general case of a second-order fixed linear system.

▶ ***EXAMPLE 13.4***

Construct the block diagram for the system described by the differential equation

$$a_2 \ddot{y} + a_1 \dot{y} + a_0 y = u(t) \tag{16}$$

Then use the block diagram to find a state-variable model for the system.

Solution

With $y(0) = \dot{y}(0) = 0$, the transformed equation is

$$a_2 s^2 Y(s) + a_1 s Y(s) + a_0 Y(s) = U(s)$$

so

$$s^2 Y(s) = \frac{1}{a_2}[-a_1 s Y(s) - a_0 Y(s) + U(s)] \tag{17}$$

We begin by drawing a series combination of two integrators in Figure 13.14(a), which shows the variables $Y(s)$, $sY(s)$, and $s^2Y(s)$. Using gain blocks and a summer, we then form $s^2Y(s)$ according to (17). The complete diagram is shown in Figure 13.14(b).

In order to write the state-variable equations, it can be helpful to draw or visualize the simulation diagram, where the variables are shown as functions of time. The simulation diagram corresponding to part (b) of Figure 13.14 is given in part (c). We usually take the output of each integrator as a state

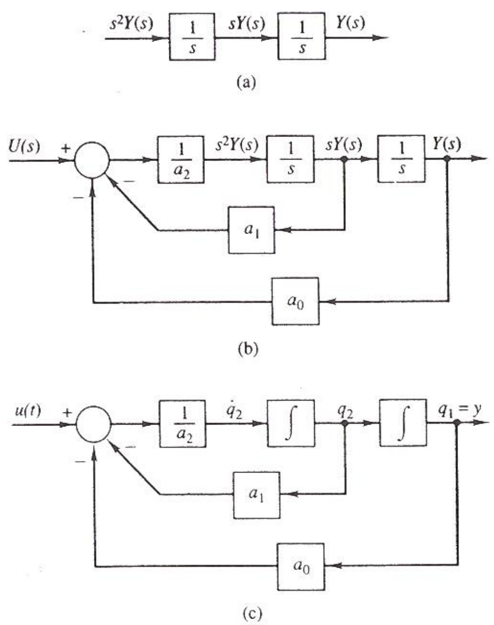


FIGURE 13.14 Diagrams for Example 13.4. (a) Partial diagram. (b) Complete block diagram. (c) Simulation diagram.

variable, and we have labeled the diagram in this way. If the output of an integrator is q_i, then its input is $\dot{q}_i$ and we can get an equation for $\dot{q}_i$ by inspection of the diagram. In our example,

$$\dot{q}_1 = q_2 \tag{18a}$$

$$\dot{q}_2 = \frac{1}{a_2}[-a_0 q_1 - a_1 q_2 + u(t)] \tag{18b}$$

$$y = q_1 \tag{18c}$$

In the last example, note that (18a) and (18b) do have the correct form for the state-variable equations describing a second-order fixed linear system. It is easy to check (18) by showing that these equations satisfy the input-output equation in (16). To do this, we need only substitute (18a) into (18b) and then replace q_1 by y. We also note that the transfer function corresponding to (16) is

$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{a_2 s^2 + a_1 s + a_0} \tag{19}$$

By the methods to be presented in Section 13.5, we can show as a further check that the block diagram in Figure 13.14(b) does indeed have this transfer function.

▶ *EXAMPLE 13.5*

Draw the block diagram and write the state-variable model when

$$a_2\ddot{y} + a_1\dot{y} + a_0 y = b_1\dot{u} + b_0 u(t) \tag{20}$$

Solution

With zero initial conditions, the transformed output can be written as

$$Y(s) = \frac{b_1 s + b_0}{a_2 s^2 + a_1 s + a_0} U(s) \tag{21}$$

If we define $Q(s)$ to be

$$Q(s) = \left[\frac{1}{a_2 s^2 + a_1 s + a_0}\right] U(s) \tag{22}$$

we can rewrite (21) as

$$\begin{aligned} Y(s) &= (b_1 s + b_0) Q(s) \\ &= b_1 s Q(s) + b_0 Q(s) \end{aligned} \tag{23}$$

The quantity inside the brackets in (22) is the transfer function corresponding to (16) in Example 13.4. Thus (22) is described by the block diagram in Figure 13.14(b) if we replace the symbol $Y(s)$ by $Q(s)$. This is done in Figure 13.15(a). Once $Q(s)$ and $sQ(s)$ are available in the block diagram, we use two additional gain blocks and a summer to satisfy (23), which results in the complete diagram in Figure 13.15(b).

The corresponding simulation diagram is drawn in part (c) of the figure, where we have again used the outputs of the integrators as the state variables. We can write the state-variable model as

$$\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= \frac{1}{a_2}[-a_0 q_1 - a_1 q_2 + u(t)] \\ y &= b_0 q_1 + b_1 q_2 \end{aligned} \tag{24}$$

Comparing these equations with the results of Example 13.4, we see that the only effect of the more complicated forcing function has been to change the algebraic output equation to include both state variables.

In order to avoid having to draw a simulation diagram before writing the state-variable equations, we have shown in Figure 13.15(b) the transformed

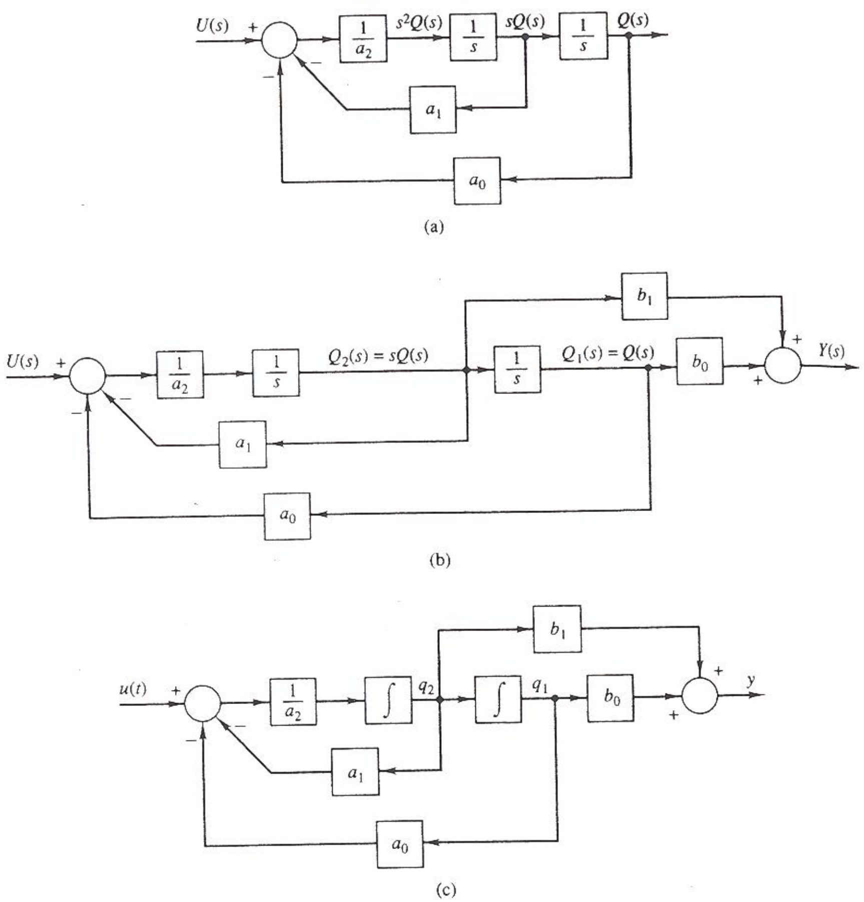


FIGURE 13.15 Diagram for Example 13.5. (a) Diagram for $Q(s)$. (b) Complete block diagram. (c) Simulation diagram.

variables $Q_1(s) = Q(s)$ and $Q_2(s) = sQ(s)$. Then, directly from the figure, we have

$$
\begin{aligned}
sQ_1(s) &= Q_2(s) \\
sQ_2(s) &= \frac{1}{a_2}[-a_0Q_1(s) - a_1Q_2(s) + U(s)] \qquad (25) \\
Y(s) &= b_0Q_1(s) + b_1Q_2(s)
\end{aligned}
$$

Remember that when constructing block diagrams, we always assume the initial conditions to be zero. Because $\mathscr{L}[\dot{q}_1] = sQ_1(s)$ and $\mathscr{L}[\dot{q}_2] = sQ_2(s)$

for zero initial conditions, (24) follows immediately from (25). With a little practice, it is possible to write the state-variable model directly from the block diagram, and we shall do this in subsequent examples.

▶ *EXAMPLE 13.6*

Repeat the previous example when

$$a_2\ddot{y} + a_1\dot{y} + a_0y = b_2\ddot{u} + b_1\dot{u} + b_0u(t) \qquad (26)$$

Solution

We can write the zero-state transformed output as

$$Y(s) = \left[\frac{b_2s^2 + b_1s + b_0}{a_2s^2 + a_1s + a_0}\right]U(s) \qquad (27)$$

or, with $Q(s)$ defined by (22),

$$
\begin{aligned}
Y(s) &= (b_2s^2 + b_1s + b_0)Q(s) \\
&= b_2s^2Q(s) + b_1sQ(s) + b_0Q(s) \qquad (28)
\end{aligned}
$$

The definition of $Q(s)$ is the same as for Example 13.5, so the relationship between $Q(s)$ and $U(s)$ is still described by Figure 13.15(a). To satisfy (28), we can add three gain blocks and a summer to obtain the complete block diagram in Figure 13.16(a). In order to facilitate writing the state-variable equations, we have replaced $Q(s)$ and $sQ(s)$ by $Q_1(s)$ and $Q_2(s)$ when labeling the outputs of the integrators. From this diagram we see that

$$
\begin{aligned}
\dot{q}_1 &= q_2 \qquad (29a) \\
\dot{q}_2 &= \frac{1}{a_2}[-a_0q_1 - a_1q_2 + u(t)] \qquad (29b) \\
y &= b_0q_1 + b_2\dot{q}_2 + b_1q_2 \qquad (29c)
\end{aligned}
$$

Note, however, that (29c) does not have the standard form of the output equation in a state-variable model, because y is given as a function of the *derivative* of one of the state variables, in addition to the state variables themselves. In spite of this feature, (29c) is a useful form of the output equation for many purposes, particularly because each of the six gains appearing in the block diagram is a coefficient of the input-output differential equation. Thus we can draw the diagram without performing any calculations to evaluate the gains.

The basic reason for the form of (29c) is that one of the signals entering the output summer in Figure 13.16(a) does not come from the output of an integrator. This in turn is caused by the fact that the numerator of the transfer function in (27) is not of lower order than the denominator. When this situation occurs, we can carry out a preliminary step of long division, as

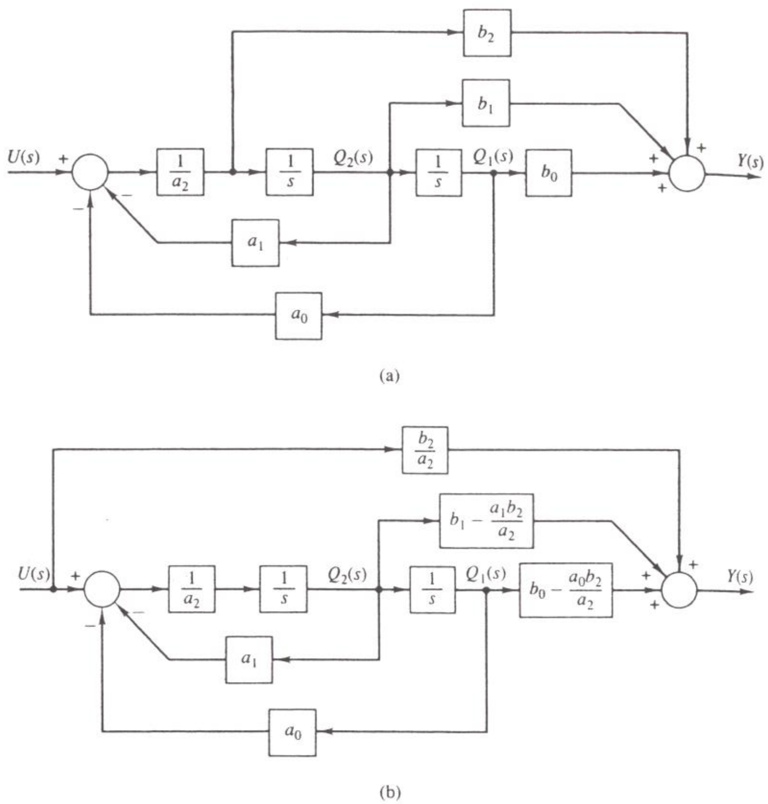


FIGURE 13.16 Diagrams for Example 13.6. (a) Block diagram corresponding to (28). (b) Block diagram corresponding to (30).

was done in Section 7.3 whenever a function of s was not a strictly proper rational function. Then (27) becomes

$$\begin{aligned} Y(s) &= \left[\frac{b_2}{a_2} + \frac{(b_1 - a_1 b_2/a_2)s + (b_0 - a_0 b_2/a_2)}{a_2 s^2 + a_1 s + a_0}\right] U(s) \\ &= \frac{b_2}{a_2} U(s) + [(b_1 - a_1 b_2/a_2)s + (b_0 - a_0 b_2/a_2)]Q(s) \qquad (30) \end{aligned}$$

The quantity $Q(s)$ is still given by (22), and the relationship between $Q(s)$ and $U(s)$ is still shown in Figure 13.15(a). Starting with that figure, adding the extra blocks needed to implement (30), and replacing $Q(s)$ and $sQ(s)$ by $Q_1(s)$ and $Q_2(s)$, we obtain the diagram in Figure 13.16(b). The state-

variable equations corresponding to the diagram are again given by (29a) and (29b), but the output equation becomes

$$y = \left(b_0 - \frac{a_0 b_2}{a_2}\right) q_1 + \left(b_1 - \frac{a_1 b_2}{a_2}\right) q_2 + \frac{b_2}{a_2} u(t) \qquad (31)$$

which includes the input $u(t)$ in addition to the state variables q_1 and q_2.

Higher-Order Systems

Generalizing the three previous examples to systems of higher order than two is straightforward. The general input-output differential equation and the corresponding transfer function are given by (14) and (15), respectively. We first consider the related differential equation

$$a_n q^{(n)} + a_{n-1} q^{(n-1)} + \cdots + a_0 q = u(t) \qquad (32)$$

where the forcing function is assumed to be just the input $u(t)$. Transforming (32) with zero initial conditions, we have

$$a_n s^n Q(s) + a_{n-1} s^{n-1} Q(s) + \cdots + a_0 Q(s) = U(s) \qquad (33)$$

and

$$Q(s) = \left[\frac{1}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}\right] U(s) \qquad (34)$$

We rearrange (33) as

$$s^n Q(s) = \frac{1}{a_n} [-a_{n-1} s^{n-1} Q(s) - \cdots - a_0 Q(s) + U(s)] \qquad (35)$$

which is described by the block diagram shown in Figure 13.17.

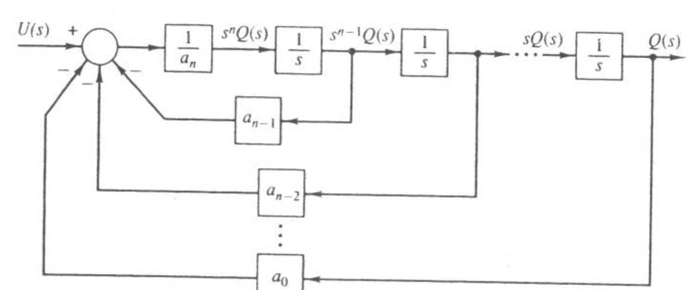


FIGURE 13.17 Block diagram for (32) through (35).

Returning to the more general model given by (14), we rewrite (15) as

$$\begin{aligned} Y(s) &= \left[\frac{b_m s^m + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}\right] U(s) \\ &= (b_m s^m + \cdots + b_0)Q(s) \\ &= b_m s^m Q(s) + \cdots + b_0 Q(s) \end{aligned} \tag{36}$$

where $Q(s)$ is defined by (34) and where the relationship between $Q(s)$ and $U(s)$ is represented by the diagram in Figure 13.17. To form the block diagram corresponding to (14) and (15), we merely add to Figure 13.17 additional gain and summer blocks according to (36).

As long as $m < n$, this procedure will give an entirely satisfactory diagram. For the case of $m = n$, we may choose to carry out a preliminary step of long division on the transfer function $H(s)$ in (15). The following two examples illustrate the method for third-order systems.

▶ *EXAMPLE 13.7*

Draw the block diagram for the system described by

$$\dddot{y} + 5\ddot{y} + 2\dot{y} + y = 3\ddot{u} + 4u(t)$$

Also find a state-variable model.

Solution

The block diagram corresponding to

$$\dddot{q} + 5\ddot{q} + 2\dot{q} + q = u(t)$$

is shown in Figure 13.18(a). Noting that

$$Y(s) = (3s^2 + 4)Q(s) = 3s^2Q(s) + 4Q(s)$$

we add to that figure two gain blocks and a summer in order to obtain the block diagram in Figure 13.18(b).

The outputs of the integrators are labeled $Q_1(s)$, $Q_2(s)$, and $Q_3(s)$ from right to left. Then we can write the state-variable model as

$$\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= q_3 \\ \dot{q}_3 &= -q_1 - 2q_2 - 5q_3 + u(t) \\ y &= 4q_1 + 3q_3 \end{aligned} \tag{37}$$

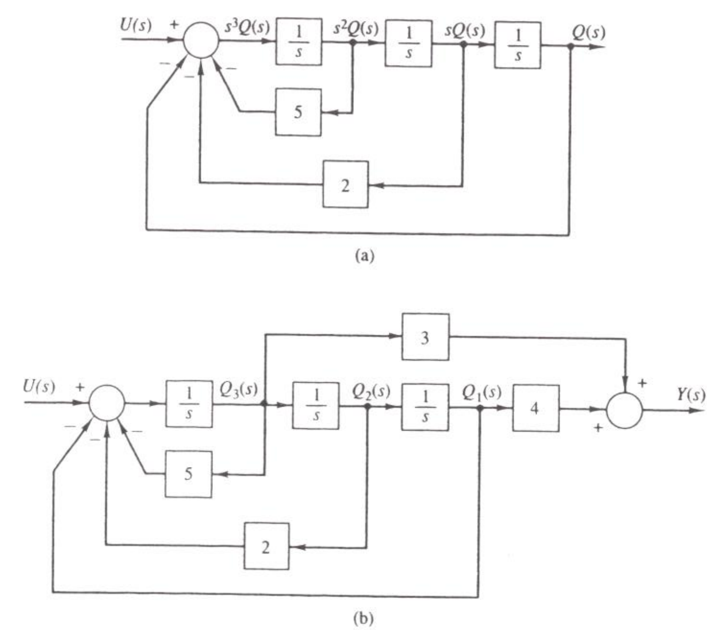


FIGURE 13.18 Diagrams for Example 13.7. (a) Partial block diagram. (b) Complete block diagram.

▶ *EXAMPLE 13.8*

Repeat the previous example for the input-output equation

$$0.5\dddot{y} + 2\ddot{y} + \dot{y} + y = 2\dddot{u} - 3\dot{u} + u(t)$$

Solution

We can write

$$Y(s) = \left[\frac{2s^3 - 3s + 1}{0.5s^3 + 2s^2 + s + 1}\right] U(s) \tag{38}$$

where the quantity inside the bracket is the system's transfer function $H(s)$. Then

$$\begin{aligned} Y(s) &= (2s^3 - 3s + 1)Q(s) \\ &= 2s^3Q(s) - 3sQ(s) + Q(s) \end{aligned} \tag{39}$$

where

$$Q(s) = \left[\frac{1}{0.5s^3 + 2s^2 + s + 1}\right] U(s) \tag{40}$$

The block diagram for (40) is shown in Figure 13.19(a). Adding the gain and summer blocks called for by (39) gives the complete diagram in Figure 13.19(b). Then we have

$$\dot{q}_1 = q_2 \tag{41a}$$

$$\dot{q}_2 = q_3 \tag{41b}$$

$$\dot{q}_3 = 2[-q_1 - q_2 - 2q_3 + u(t)] \tag{41c}$$

$$y = q_1 - 3q_2 + 2\dot{q}_3 \tag{41d}$$

Because of the presence of the term $2\dot{q}_3$, the last equation does not have the standard form for the output equation in a state-variable model. For an alternative block diagram, we perform a step of long division on the transfer function within the brackets in (38). Then

$$Y(s) = \left[4 + \frac{-8s^2 - 7s - 3}{0.5s^3 + 2s^2 + s + 1}\right] U(s)$$

$$= 4U(s) - 8s^2Q(s) - 7sQ(s) - 3Q(s)$$

Again starting with Figure 13.19(a), we add four gain blocks and a summer to obtain the diagram in part (c) of the figure. The corresponding state-variable model consists of (41a), (41b), and (41c), together with the new output equation

$$y = -3q_1 - 7q_2 - 8q_3 + 4u(t) \tag{42}$$

which does not contain any derivatives. This output equation can also be obtained by substituting (41c) into (41d) to eliminate the derivative on the right side of (41d).

13.4 MODELS IN NONSTANDARD FORM

The free-body diagram for a single mass or moment of inertia in a mechanical system will generally result in a second-order differential equation that contains one or more variables not directly associated with the motion of that element. If the system consists of several such elements, each having its own free-body diagram, then the overall model will probably appear first as a set of coupled second-order equations.

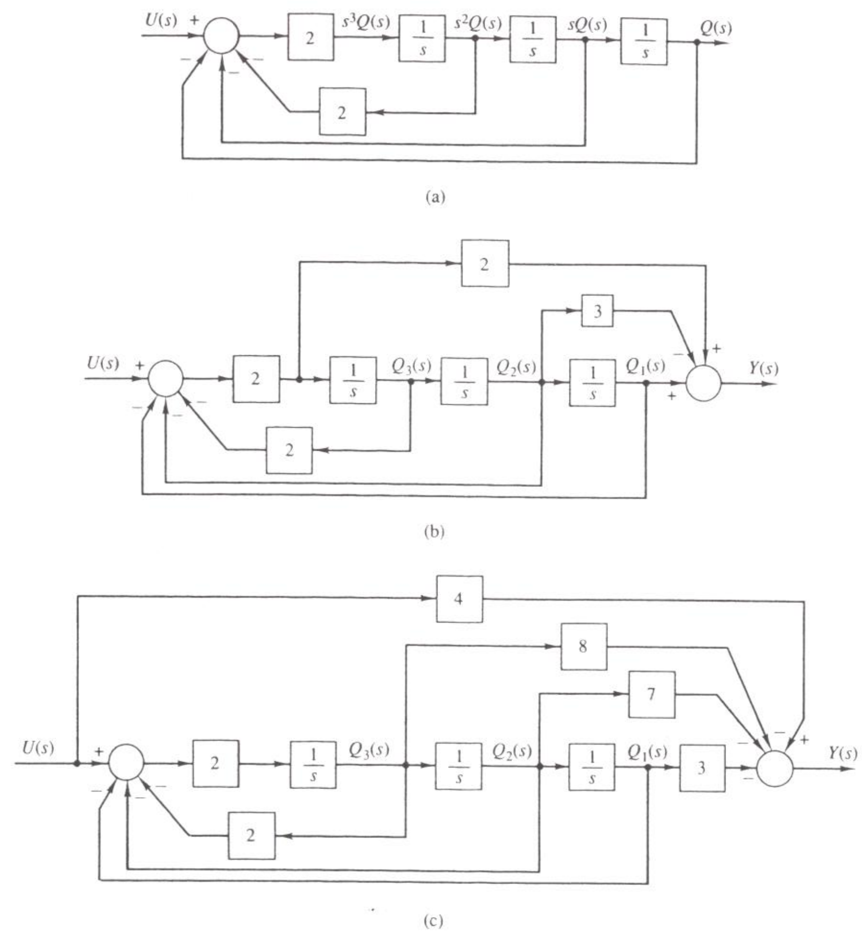


FIGURE 13.19 Diagrams for Example 13.8. (a) Partial block diagram. (b) Complete block diagram. (c) Alternative block diagram.

In such situations, we may wish to draw a block diagram directly from the nonstandard form of the model, which is in neither state-variable nor input-output form. Because the variables associated with a single mass are the position, velocity, and acceleration, the corresponding portion of the diagram generally consists of a chain of two integrators with the variables being fed back to a summer at the input of the first integrator. Hence the complete diagram is an interconnection of subdiagrams that contain two-integrator chains. These ideas are illustrated in the following example.

▶ *EXAMPLE 13.9*

Draw a block diagram for the mechanical system shown in Figure 13.20 and modeled in Example 4.11.

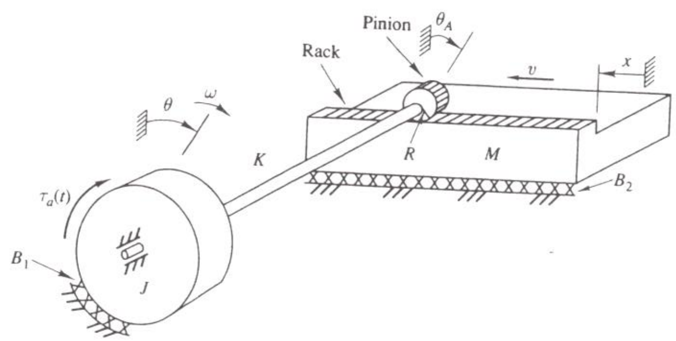


FIGURE 13.20 System for Example 4.11 with rack and pinion gear.

Solution

In the solution of Example 4.11, we drew free-body diagrams for the rotor, the pinion gear, and the mass. These diagrams lead to the three equations

$$J\ddot{\theta} + B_1\dot{\theta} + K(\theta - \theta_A) = \tau_a(t) \tag{43a}$$

$$M\ddot{x} + B_2\dot{x} = f_c \tag{43b}$$

$$Rf_c = K(\theta - \theta_A) \tag{43c}$$

which are identical to (4.61) except for a minor rearrangement of the terms and the use of the identities $\dot{\theta} = \omega$ and $\dot{x} = v$. In addition to (43), we have the geometric relationship

$$R\theta_A = x \tag{44}$$

Transforming (43) and (44) with zero initial conditions, we obtain

$$s^2\Theta(s) = \frac{1}{J}\{-B_1 s\Theta(s) - K[\Theta(s) - \Theta_A(s)] + \tau_a(s)\} \tag{45a}$$

$$s^2X(s) = \frac{1}{M}[-B_2 sX(s) + F_c(s)] \tag{45b}$$

$$F_c(s) = \frac{1}{R}K[\Theta(s) - \Theta_A(s)] \tag{45c}$$

$$\Theta_A(s) = \frac{1}{R}X(s) \tag{45d}$$

We start the block diagram with a pair of two-integrator chains, whose outputs are $\Theta(s)$ and $X(s)$ and whose inputs are $s^2\Theta(s)$ and $s^2X(s)$. From (45c) and (45d) we construct $F_c(s)$ and $\Theta_A(s)$ in terms of $X(s)$ and $\Theta(s)$. Finally, we use (45a) and (45b) to determine the inputs to the summers that precede the integrator chains. The completed diagram is shown in Figure 13.21(a).

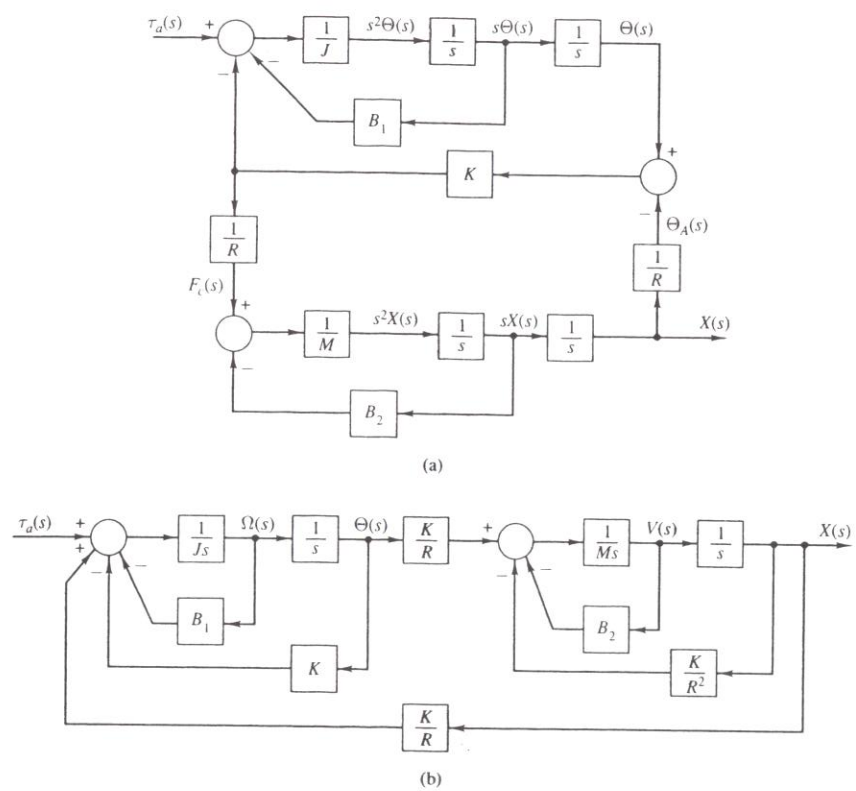


FIGURE 13.21 Block diagrams for the mechanical system of Example 13.9. (a) Nonstandard form. (b) State-variable form.

When discussing block diagrams for specific physical systems, engineers often refer to transformed variables by their original time-domain names. For example, the quantity $K[\Theta(s) - \Theta_A(s)]$ is the *transform* of the torque transmitted by the shaft connecting the pinion gear to the rotational inertia. However, it would commonly be referred to as simply the torque transmitted by the shaft. Note that the two parallel two-integrator chains in Figure 13.21(a) are coupled by this torque.

We can also draw a block diagram to correspond to the state-variable form of the model given by (4.63), which is repeated here:

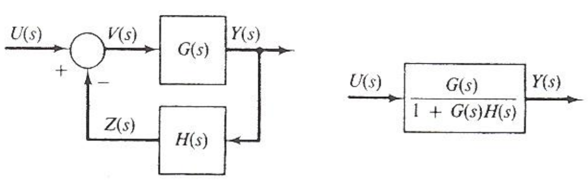
$$
\begin{aligned}
\dot{\theta} &= \omega \\
\dot{\omega} &= \frac{1}{J}\left[-K\theta - B_1\omega + \frac{K}{R}x + \tau_a(t)\right] \\
\dot{x} &= v \\
\dot{v} &= \frac{1}{M}\left(\frac{K}{R}\theta - \frac{K}{R^2}x - B_2 v\right)
\end{aligned}
$$

The reader should verify that these equations lead to the diagram shown in Figure 13.21(b). As drawn, the diagram has the two-integrator chains arranged in series rather than in parallel. Although the contact force $F_c(s)$ is not explicitly shown in this version, it can be generated from $\Theta(s)$ and $X(s)$ by using gains and a summer. In comparing the two diagrams, remember that in order to draw the state-variable version, we had to use the two algebraic equations (43c) and (44) to eliminate $F_c(s)$ and $\Theta_A(s)$. In contrast, the diagram in Figure 13.21(a) can be drawn directly from the free-body diagrams and (44).

13.5 BLOCK DIAGRAMS OF FEEDBACK SYSTEMS

The block diagrams in the previous three sections contained only integrators, gain blocks, and summers. Feedback paths were formed around the integrators by multiplying the transformed state variables by constants and then feeding these signals back to an input summer. In this section and also in the next chapter, we shall consider the more general situation where the individual blocks can have arbitrary transfer functions.

Figure 13.22(a) shows the block diagram of a general feedback system that has a forward path from the summing junction to the output and a feed-



(a) (b)

FIGURE 13.22 (a) Block diagram of a feedback system. (b) Equivalent diagram.

back path from the output back to the summing junction. The transforms of the system's input and output are $U(s)$ and $Y(s)$, respectively. The transfer function $G(s) = Y(s)/V(s)$ is known as the **forward transfer function**, and $H(s) = Z(s)/Y(s)$ is called the **feedback transfer function**. We must evaluate both of these transfer functions with the system elements connected in order properly to account for the loading effects of the interconnections. The product $G(s)H(s)$ is referred to as the **open-loop transfer function**. The sign associated with the feedback signal from the block $H(s)$ at the summing junction is shown as minus because a minus sign naturally occurs in the majority of feedback systems, particularly in control systems.

Given the model of a feedback system in terms of its forward and feedback transfer functions $G(s)$ and $H(s)$, it is often necessary to determine the **closed-loop transfer function** $T(s) = Y(s)/U(s)$. We do this by writing the algebraic transform equations corresponding to the block diagram shown in Figure 13.22(a) and solving them for the ratio $Y(s)/U(s)$. We can write the following transform equations directly from the block diagram.

$$
\begin{aligned}
V(s) &= U(s) - Z(s) \\
Y(s) &= G(s)V(s) \\
Z(s) &= H(s)Y(s)
\end{aligned}
$$

If we combine these equations in such a way as to eliminate $V(s)$ and $Z(s)$, we find that

$$Y(s) = G(s)[U(s) - H(s)Y(s)]$$

which can be rearranged to give

$$[1 + G(s)H(s)]Y(s) = G(s)U(s)$$

Hence the closed-loop transfer function $T(s) = Y(s)/U(s)$ is

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} \tag{46}$$

where it is implicit that the sign of the feedback signal at the summing junction is negative. It is readily shown that when a plus sign is used at the summing junction for the feedback signal, the closed-loop transfer function becomes

$$T(s) = \frac{G(s)}{1 - G(s)H(s)} \tag{47}$$

A commonly used simplification occurs when the feedback transfer function is unity—this is, when $H(s) = 1$. Such a system is referred to as a

unity-feedback system, and (46) reduces to

$$T(s) = \frac{G(s)}{1 + G(s)} \tag{48}$$

We now consider three examples that use (46) and (47). The first two illustrate determining the closed-loop transfer function by reducing the block diagram. They also show the effects of feedback gains on the closed-loop poles, time constant, damping ratio, and undamped natural frequency. In the third example, a block diagram is drawn directly from the system's state-variable equations and then reduced to give the system's transfer functions.

▶ *EXAMPLE 13.10*

Find the closed-loop transfer function for the feedback system shown in Figure 13.23(a), and compare the locations of the poles of the open-loop and closed-loop transfer functions in the s-plane.

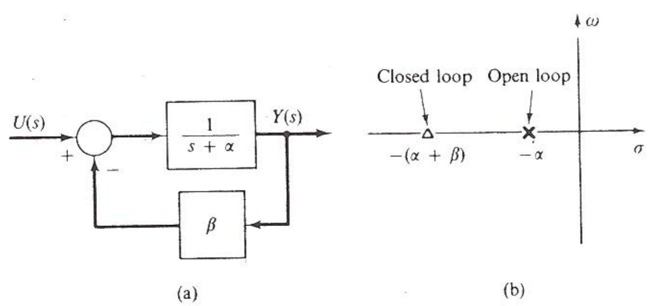


FIGURE 13.23 Single-loop feedback system for Example 13.10.

Solution

By comparing the block diagram shown in Figure 13.23(a) with that shown in Figure 13.22(a), we see that $G(s) = 1/(s+\alpha)$ and $H(s) = \beta$. Substituting these expressions into (46) gives

$$T(s) = \frac{\dfrac{1}{s+\alpha}}{1 + \left(\dfrac{1}{s+\alpha}\right)\beta}$$

which we can write as a rational function of s by multiplying the numerator and denominator by $s + \alpha$. Doing this, we obtain the closed-loop transfer function

$$T(s) = \frac{1}{s + \alpha + \beta}$$

This result illustrates an interesting and useful property of feedback systems: the fact that the poles of the closed-loop transfer function differ from the poles of the open-loop transfer function $G(s)H(s)$. In this case, the single open-loop pole is at $s = -\alpha$, whereas the single closed-loop pole is at $s = -(\alpha + \beta)$. These pole locations are indicated in Figure 13.23(b) for positive α and β. Hence, in the absence of feedback, the pole of the transfer function $Y(s)/U(s)$ is at $s = -\alpha$, and the free response will be of the form $\epsilon^{-\alpha t}$. With feedback, however, the free response will be $\epsilon^{-(\alpha+\beta)t}$. Thus the time constant of the open-loop system is $1/\alpha$, whereas that of the closed-loop system is $1/(\alpha + \beta)$.

▶ *EXAMPLE 13.11*

Find the closed-loop transfer function of the two-loop feedback system shown in Figure 13.24. Also express the damping ratio and the undamped natural frequency of the closed-loop system in terms of the gains a_0 and a_1.

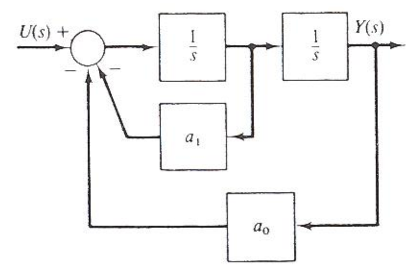


FIGURE 13.24 System with two feedback loops for Example 13.11.

Solution

Because the system's block diagram contains one feedback path inside another, we cannot use (46) directly to evaluate $Y(s)/U(s)$. However, we can redraw the block diagram such that the summing junction is split into two summing junctions, as shown in Figure 13.25(a). Then it is possible to use (46) to eliminate the inner loop by calculating the transfer function $W(s)/V(s)$. Taking $G(s) = 1/s$ and $H(s) = a_1$ in (46), we obtain

$$\frac{W(s)}{V(s)} = \frac{\dfrac{1}{s}}{1 + \dfrac{a_1}{s}} = \frac{1}{s + a_1}$$

Redrawing Figure 13.25(a) with the inner loop replaced by a block having $1/(s + a_1)$ as its transfer function gives Figure 13.25(b). The two

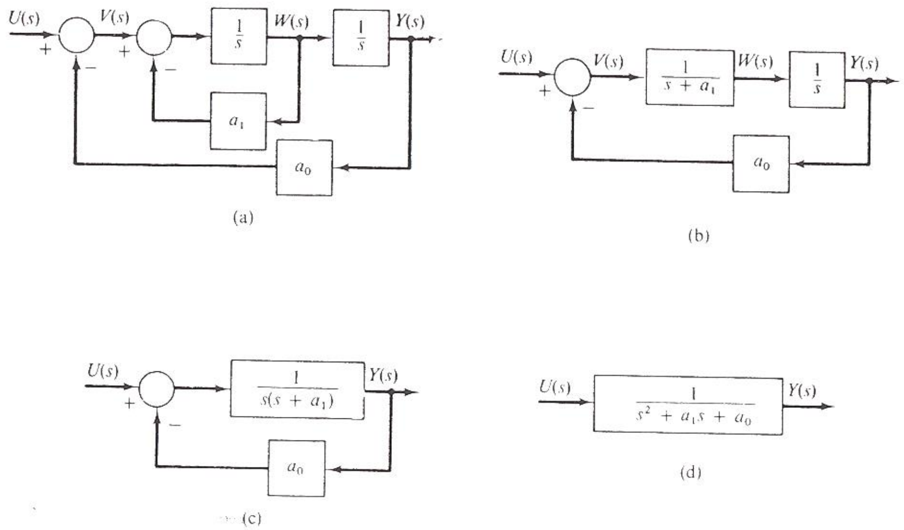


FIGURE 13.25 Equivalent block diagrams for the system shown in Figure 13.24.

blocks in the forward path of this version are in series and can be combined by multiplying their transfer functions, which gives the block diagram shown in Figure 13.25(c). Then we can apply (46) again to find the overall closed-loop transfer function $T(s) = Y(s)/U(s)$ as

$$T(s) = \frac{\dfrac{1}{s(s+a_1)}}{1 + \dfrac{1}{s(s+a_1)} \cdot a_0} = \frac{1}{s^2 + a_1 s + a_0} \tag{49}$$

The block-diagram representation of the feedback system corresponding to (49) is shown in Figure 13.25(d).

The poles of the closed-loop transfer function are the roots of the equation

$$s^2 + a_1 s + a_0 = 0 \tag{50}$$

which we obtain by setting the denominator of $T(s)$ equal to zero and which is the characteristic equation of the closed-loop system. Equation (50) has two roots, which may be real or complex, depending on the sign of the quantity $a_1^2 - 4a_0$. However, the roots of (50) will have negative real parts and the closed-loop system will be stable provided that a_0 and a_1 are both positive.

If the poles are complex, it is convenient to rewrite the denominator of $T(s)$ in terms of the damping ratio ζ and the undamped natural frequency

ω_n, which were introduced in Section 6.4. When the parameter r is replaced by s, the characteristic equation in (6.50) can be compared to (50). Doing this, we see that

$$a_0 = \omega_n^2 \tag{51a}$$

$$a_1 = 2\zeta\omega_n \tag{51b}$$

Solving (51a) for ω_n and substituting it into (51b) give the damping ratio and the undamped natural frequency of the closed-loop system as

$$\zeta = \frac{a_1}{2\sqrt{a_0}}$$

$$\omega_n = \sqrt{a_0}$$

We see from these expressions that a_0, the gain of the outer feedback path in Figure 13.24, determines the undamped natural frequency ω_n and that a_1, the gain of the inner feedback path, affects only the damping ratio. If we can specify both a_0 and a_1 at will, then we can attain any desired values of ζ and ω_n for the closed-loop transfer function.

EXAMPLE 13.12

Draw a block diagram for the translational mechanical system studied in Example 3.7, whose state-variable equations are given by (3.14). Reduce the block diagram to determine the transfer functions $X_1(s)/F_a(s)$ and $X_2(s)/F_a(s)$ as rational functions of s.

Solution

Transforming (3.14) with zero initial conditions, we have

$$sX_1(s) = V_1(s) \tag{52a}$$

$$MsV_1(s) = -K_1X_1(s) - B_1V_1(s) + K_1X_2(s) + F_a(s) \tag{52b}$$

$$B_2sX_2(s) = K_1X_1(s) - (K_1 + K_2)X_2(s) \tag{52c}$$

We use (52b) to draw a summing junction that has $MsV_1(s)$ as its output. After the summing junction, we insert the transfer function $1/Ms$ to get $V_1(s)$, which, from (52a), equals $sX_1(s)$. Thus an integrator whose input is $V_1(s)$ has $X_1(s)$ as its output. Using (52c), we form a second summing junction that has $B_2sX_2(s)$ as its output. Following this summing junction by the transfer function $1/B_2s$, we get $X_2(s)$ and can complete the four feedback paths required by the summing junctions. The result of these steps is the block diagram shown in Figure 13.26(a).

To simplify the block diagram, we use (46) to reduce each of the three inner feedback loops, obtaining the version shown in Figure 13.26(b). To evaluate the transfer function $X_1(s)/F_a(s)$, we can apply (47) to this single-

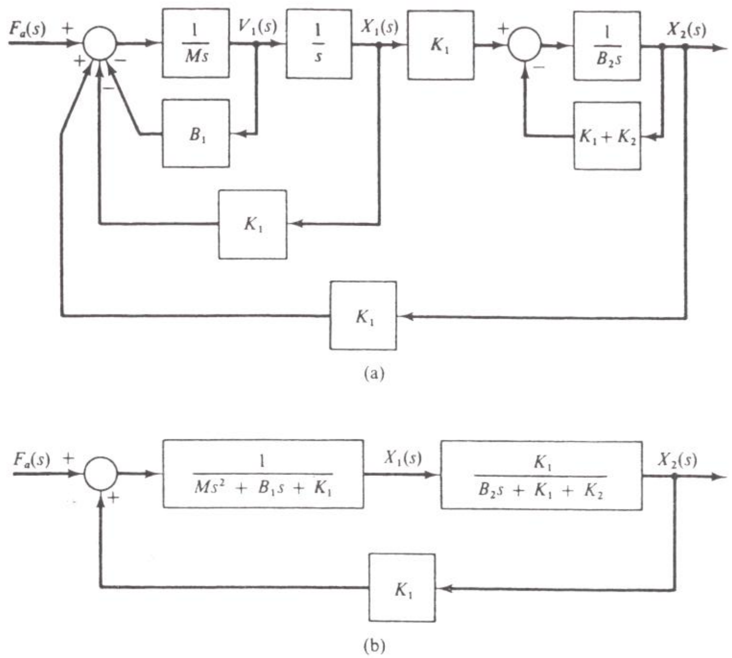


FIGURE 13.26 Block diagrams for the system in Example 13.12. (a) As drawn from (52). (b) With the three inner feedback loops eliminated.

loop diagram because the sign associated with the feedback signal at the summing junction is positive rather than negative. Doing this with

$$G(s) = \frac{1}{Ms^2 + B_1 s + K_1}$$

and

$$H(s) = \frac{K_1^2}{B_2 s + K_1 + K_2}$$

we find

$$\begin{aligned}\frac{X_1(s)}{F_a(s)} &= \frac{\dfrac{1}{Ms^2 + B_1 s + K_1}}{1 - \dfrac{1}{Ms^2 + B_1 s + K_1} \cdot \dfrac{K_1^2}{B_2 s + K_1 + K_2}} \\ &= \frac{B_2 s + K_1 + K_2}{(Ms^2 + B_1 s + K_1)(B_2 s + K_1 + K_2) - K_1^2} \\ &= \frac{B_2 s + K_1 + K_2}{P(s)} \qquad (53)\end{aligned}$$

where

$$\begin{aligned}P(s) = {} & MB_2 s^3 + [(K_1 + K_2)M + B_1 B_2]s^2 \\ & + [B_1(K_1 + K_2) + B_2 K_1]s + K_1 K_2\end{aligned}$$

To obtain $X_2(s)/F_a(s)$, we can write

$$\frac{X_2(s)}{F_a(s)} = \frac{X_1(s)}{F_a(s)} \cdot \frac{X_2(s)}{X_1(s)}$$

where $X_1(s)/F_a(s)$ is given by (53) and, from Figure 13.26(b),

$$\frac{X_2(s)}{X_1(s)} = \frac{K_1}{B_2 s + K_1 + K_2} \qquad (54)$$

The result of multiplying (53) and (54) is a transfer function with the same denominator as (53) but with a numerator of K_1. Note that the two transfer functions are consistent with the corresponding input-output differential equations found in (3.27) and (3.28) for this system.

In the previous examples, we used the rules for combining blocks that are in series or in parallel, as shown in Figures 13.4 and 13.5. We also repeatedly used the rule for simplifying the basic feedback configuration given in Figure 13.22(a). A number of other operations can be derived to help simplify block diagrams. To conclude this chapter, we present and illustrate two of these additional operations.

Keep in mind that a block diagram is just a means of representing the algebraic Laplace-transformed equations that describe a system. Simplifying or reducing the diagram is equivalent to manipulating the equations. In order to prove that a particular operation on the block diagram is valid, we need only show that the relationships among the transformed variables of interest are left unchanged.

Moving a Pick-Off Point

A **pick-off point** is a point where an incoming variable in the diagram is directed into more than one block. In the partial diagram of Figure 13.27(a),

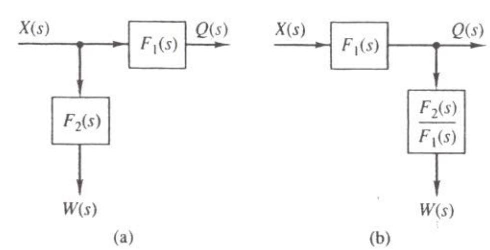


FIGURE 13.27 Moving a pick-off point.

the incoming signal $X(s)$ is used not only to provide the output $Q(s)$ but also to form the signal $W(s)$, which in practice might be fed back to a summer that appears earlier in the complete diagram. The pick-off point can be moved to the right of $F_1(s)$ if the transfer function of the block leading to $W(s)$ is modified as shown in Figure 13.27(b). Both parts of the figure give the same equations:

$$Q(s) = F_1(s)X(s)$$
$$W(s) = F_2(s)X(s)$$

EXAMPLE 13.13

Use Figure 13.27 to find the closed-loop transfer function for the system shown in Figure 13.24.

Solution

The pick-off point leading to the gain block a_1 can be moved to the output $Y(s)$ by replacing a_1 by a_1s, as shown in Figure 13.28(a). Then the two integrator blocks, which are now in series, can be combined to give the transfer function $G(s) = 1/s^2$. The two feedback blocks are now in parallel and can be combined into the single transfer function $a_1s + a_0$, as shown in Figure 13.28(b). Finally, by (46),

$$T(s) = \frac{Y(s)}{U(s)} = \frac{1/s^2}{1 + (a_1s + a_0)/s^2} = \frac{1}{s^2 + a_1s + a_0}$$

which agrees with (49), as found in Example 13.11.

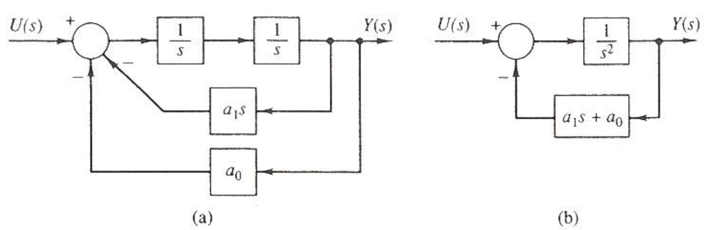


FIGURE 13.28 Equivalent block diagrams for the system shown in Figure 13.24.

Moving a Summing Junction

Suppose that, in the partial diagram of Figure 13.29(a), we wish to move the summing junction to the left of the block that has the transfer function

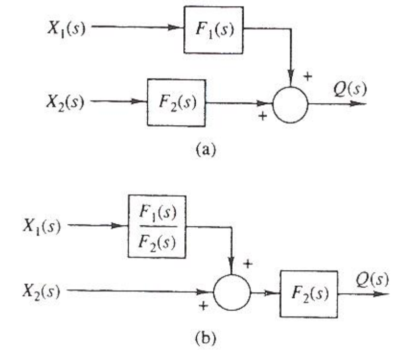


FIGURE 13.29 Moving a summing junction.

$F_2(s)$. We can do this by modifying the transfer function of the block whose input is $X_1(s)$, as shown in part (b) of the figure. For each part of the figure,

$$Q(s) = F_1(s)X_1(s) + F_2(s)X_2(s)$$

EXAMPLE 13.14

Find the closed-loop transfer function $T(s) = Y(s)/U(s)$ for the feedback system shown in Figure 13.30(a).

Solution

We cannot immediately apply (46) to the inner feedback loop consisting of the first integrator and the gain block a_1, because the output of block b_1 enters a summer within that loop. We therefore use Figure 13.29 to move this summer to the left of the first integrator block, where it can be combined with the first summer. The resulting diagram is given in Figure 13.30(b).

Now (46) can be applied to the inner feedback loop to give the transfer function

$$G_1(s) = \frac{1/s}{1 + a_1/s} = \frac{1}{s + a_1}$$

The equivalent block with the transfer function $G_1(s)$ is then in series with the remaining integrator, which results in a combined transfer function of $1/[s(s + a_1)]$. Also, the two blocks with gains of sb_1 and 1 are in parallel and can be combined into a single block. These simplifications are shown in Figure 13.30(c).

We can now repeat the procedure and move the right summer to the left of the block labeled $1/[s(s + a_1)]$, where it can again be combined with

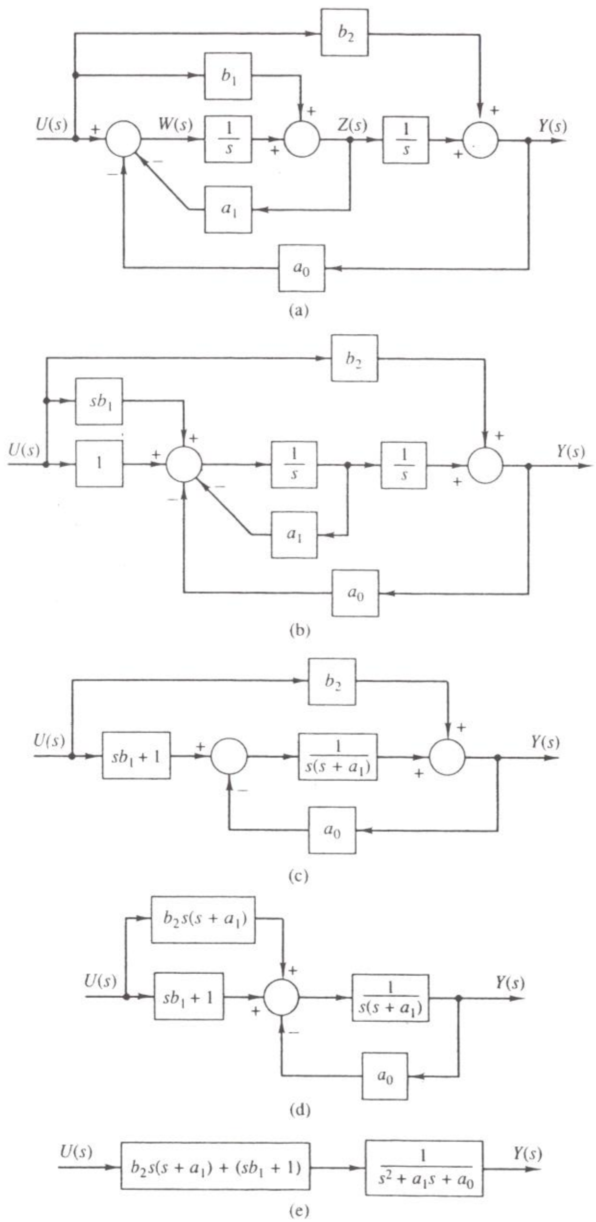


FIGURE 13.30 (a) Block diagram for Example 13.14. (b), (c), (d), (e) Equivalent block diagrams.

the first summer. This is done in part (d) of the figure. The two blocks in parallel at the left can now be combined by adding their transfer functions, and (46) can be applied to the right part of the diagram to give

$$\frac{\dfrac{1}{s(s+a_1)}}{1+\dfrac{a_0}{s(s+a_1)}}=\frac{1}{s^2+a_1s+a_0}$$

These steps yield Figure 13.30(e), from which we see that

$$T(s)=\frac{b_2s^2+(a_1b_2+b_1)s+1}{s^2+a_1s+a_0} \tag{55}$$

Because performing operations on a given block diagram is equivalent to manipulating the algebraic equations that describe the system, it may sometimes be easier to work with the equations themselves. As an alternative solution to the last example, suppose that we start by writing the equations for each of the three summers in Figure 13.30(a):

$$W(s)=U(s)-a_0Y(s)-a_1Z(s) \tag{56a}$$

$$Z(s)=\frac{1}{s}W(s)+b_1U(s) \tag{56b}$$

$$Y(s)=\frac{1}{s}Z(s)+b_2U(s) \tag{56c}$$

Substituting (56a) into (56b), we see that

$$Z(s)=\frac{1}{s}[U(s)-a_0Y(s)-a_1Z(s)]+b_1U(s)$$

from which

$$Z(s)=\frac{1}{s+a_1}[-a_0Y(s)+(b_1s+1)U(s)] \tag{57}$$

Substituting (57) into (56c) gives

$$Y(s)=\frac{1}{s(s+a_1)}[-a_0Y(s)+(b_1s+1)U(s)]+b_2U(s)$$

Rearranging this equation, we find that

$$\frac{Y(s)}{U(s)}=\frac{b_2s^2+(a_1b_2+b_1)s+1}{s^2+a_1s+a_0}$$

which agrees with (55), as found in Example 13.14.

SUMMARY

Block diagrams are an important way of representing the structure and properties of fixed linear systems. We start the construction of a block diagram by transforming the system equations, assuming zero initial conditions. When the system is described by a state-variable model, by an input-output model, or by coupled equations in nonstandard form, we can construct a diagram using integrators, gain blocks, and summers. For a system of order n we need n integrators, each with a transfer function of $1/s$. We feed the signals represented by the integrator outputs through individual gain blocks and then back to an input summer, in accordance with the transformed equations. For general state-variable models or input-output equations containing derivatives of the input, we also feed signals forward to an output summer.

If a block diagram is drawn for a given input-output differential equation or for a given transfer function, we can use the diagram to obtain a state-variable model. The integrator outputs are usually taken to be the Laplace-transformed state variables.

In simulation diagrams, the variables are functions of time rather than transformed quantities. For fixed linear systems, they have a form that is essentially the same as that of the block diagrams. The use of simulation diagrams, however, can be extended to systems where the initial stored energy is important or where there are time-varying or nonlinear elements.

The blocks used in diagrams for feedback systems may contain transfer functions of any degree of complexity. We developed a number of rules, including those for series and parallel combinations and for the basic feedback configuration in Figure 13.22(a), for simplifying block diagrams. Reducing block diagrams and adding components to improve the performance of feedback systems will be important in the next chapter.

PROBLEMS

13.1 Find the transfer functions of the blocks shown in parts (b) and (c) of Figure P13.1 such that the transfer functions $Y(s)/X_1(s)$ and $Y(s)/X_2(s)$ are identical to those for part (a) of the figure.

13.2 Find the transfer functions of the blocks shown in parts (b) and (c) of Figure P13.2 such that the transfer functions $Y_1(s)/X(s)$ and $Y_2(s)/X(s)$ are identical to those for part (a) of the figure.

* **13.3** Evaluate the transfer functions $T_1(s) = Y(s)/U(s)$ and $T_2(s) = Z(s)/U(s)$ as rational functions for the block diagram shown in Figure P13.3.

13.4 Draw block diagrams for each of the following sets of state-variable equations.

a) $\dot{x} = -4x + 6y + 2u(t)$
$\dot{y} = -2x - 3y$

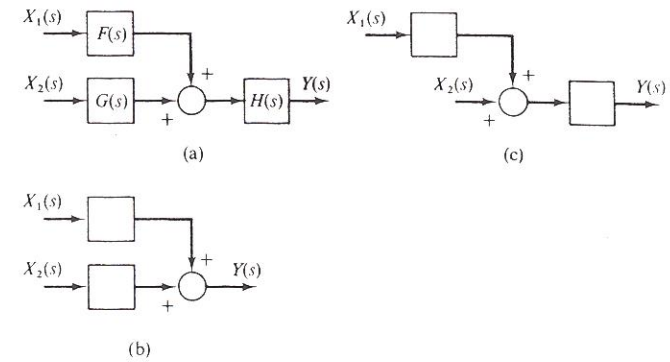


FIGURE P13.1

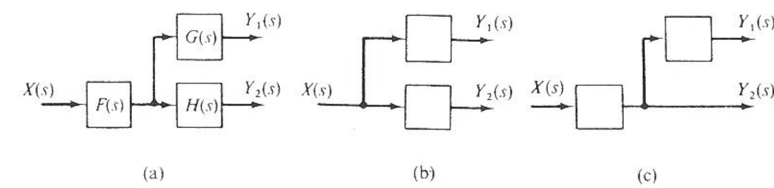


FIGURE P13.2

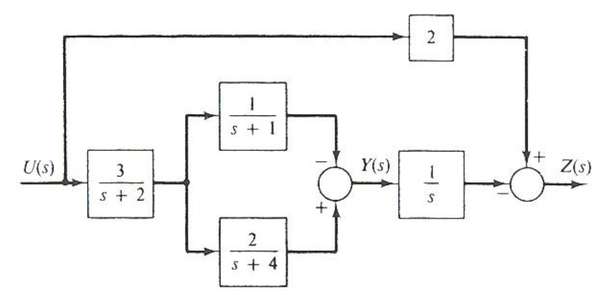


FIGURE P13.3

b) $\dot{x}_1 = -3x_1 + 5x_2 + 3u(t)$
$\dot{x}_2 = 4x_1 - 6x_2 - u(t)$

c) $\dot{\theta} = \omega$
$\dot{\omega} = -8\theta - 4\omega + 2x$
$\dot{x} = v$
$\dot{v} = 6\theta - 3x + u(t)$

13.5 Draw simulation diagrams for each of the following nonlinear models.

a) $\dot{v}_1 = -3v_1 + 2|v_2|v_2 + 3\cos 2t$

$\dot{v}_2 = v_1 - v_2$

b) $\ddot{y} + 2|\dot{y}|\dot{y} + y^3 = 5u(t)$

13.6 Draw block diagrams for each of the following input-output models.

a) $\dot{y} + 3y = 2u(t)$

b) $\dot{y} + 3y = 2\dot{u} + u(t)$

c) $\ddot{z} + 4\dot{z} + 2z = u(t)$

d) $\ddot{z} + 4\dot{z} + 2z = 3\dot{u} + 2u(t)$

* **13.7** Draw block diagrams for each of the following input-output models. Also write a set of state-variable equations for each of the models.

a) $\ddot{y} + 5\dot{y} + 6y = 12u(t)$

b) $\ddot{y} + 5\dot{y} + 6y = 2\ddot{u} + \dot{u} + 2u(t)$

c) $\dddot{y} + 2\ddot{y} + 3\dot{y} + 4y = 2\ddot{u} - \dot{u} + 4u(t)$

13.8 Repeat Problem 13.7 for each of the following models.

a) $2\ddot{y} + 3\dot{y} + 6y = 4u(t)$

b) $2\ddot{y} + 3\dot{y} + 6y = -2\dot{u} + 3u(t)$

c) $\ddot{z} + 4\dot{z} + 3z = \ddot{u} + 2u(t)$

13.9 Draw a block diagram for the rotational mechanical system described by the following two equations, where $\theta_a(t)$ is the input. Let $z = 2\dot{\theta}_1 - \dot{\theta}_a$.

$$2\ddot{\theta}_1 + \dot{\theta}_1 + 3(\dot{\theta}_1 - \dot{\theta}_2) + \theta_1 = \dot{\theta}_a + \theta_a(t)$$

$$2\ddot{\theta}_2 + \dot{\theta}_2 = 3(\dot{\theta}_1 - \dot{\theta}_2) + \theta_2$$

In Problems 13.10 through 13.19, draw block diagrams for the systems that were modeled in the examples indicated, using the equations cited.

13.10 (2.19) in Example 2.4.

13.11 (2.25) in Example 2.6.

13.12 (3.3) and (3.4) in Example 3.2.

13.13 (3.8) and (3.9) in Example 3.5.

13.14 (3.21) and (3.22) in Example 3.8.

13.15 (3.23) and (3.24) in Example 3.9.

13.16 a) (4.27) in Example 4.2.
b) (4.29) in Example 4.2.

13.17 (4.64) and (4.66) in Example 4.12.

13.18 a) (5.21) in Example 5.3.
b) (5.41), (5.42), and (5.43) in Example 5.8.

13.19 (5.47) in Example 5.9.

In Problems 13.20 through 13.23, draw a simulation diagram for the nonlinear system that was modeled in the example indicated, using the equation(s) cited. Also draw a simulation diagram for the linearized model.

13.20 (9.23) in Example 9.7.

13.21 (9.27) in Example 9.8.

13.22 (9.33) in Example 9.9.

13.23 (9.37), (9.38), and (9.39) in Example 9.10.

In Problems 13.24 through 13.26, determine the closed-loop transfer function $Y(s)/U(s)$ as a rational function of s for the block diagram shown in the figure cited.

* **13.24** Figure P13.24.

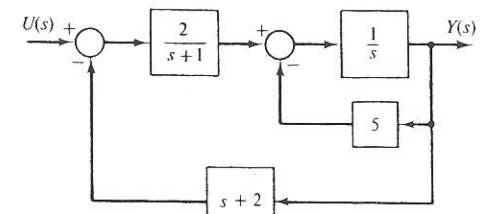


FIGURE P13.24

13.25 Figure P13.25.

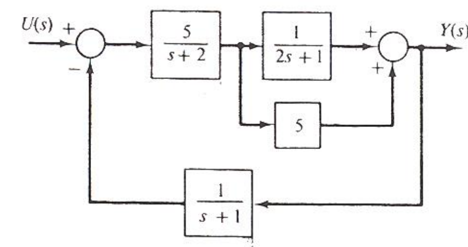


FIGURE P13.25

* **13.26** Figure 13.21(b), where $U(s) = \tau_a(s)$ and $Y(s) = X(s)$.

13.27 a) Draw a block diagram for the rotational mechanical system modeled in Example 4.5 by transforming (4.42) and drawing the corresponding diagram.
b) Set $\tau_L(s)$ equal to zero and reduce the block diagram to find the transfer function $T_1(s) = \Omega_2(s)/\tau_a(s)$.
c) Set $\tau_a(s)$ equal to zero and reduce the block diagram you found in part (a) to obtain the transfer function $T_2(s) = \Omega_2(s)/\tau_L(s)$.
d) Transform (4.43) and use the result to verify your answers to parts (b) and (c).

13.28 a) Draw a block diagram for the electromechanical system modeled in Example 10.2 by (10.31).
b) Obtain the transfer functions $T_1(s) = \Omega(s)/E_i(s)$ and $T_2(s) = \Omega(s)/\tau_L(s)$ by reducing the block diagram, first with $\tau_L(s) = 0$ and then with $E_i(s) = 0$. Compare your answers with the transfer functions $H_1(s)$ and $H_2(s)$ derived in Example 10.2.

13.29 a) Draw a block diagram for the thermal system modeled in Example 11.4 by (11.19).

b) Obtain the transfer function $T(s) = \hat{Q}_2(s)/\hat{Q}_i(s)$ by reducing the block diagram. Compare your answer with (11.20).

* **13.30** **a)** For the block diagram shown in Figure P13.30, find the closed-loop transfer function $T(s) = Y(s)/U(s)$ as a ratio of polynomials.
b) Determine the steady-state response to a unit step-function input in terms of K.
c) Write the damping ratio ζ and the undamped natural frequency ω_n in terms of K. Solve for the value of K for which $\zeta = 1/\sqrt{2}$, and find the corresponding value of ω_n.

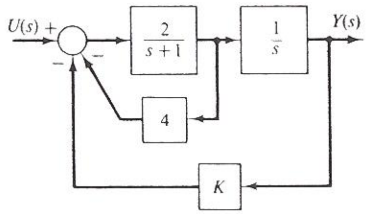


FIGURE P13.30

13.31 **a)** Find the closed-loop transfer function $T(s) = Y(s)/U(s)$ as a ratio of polynomials for the block diagram shown in Figure P13.31.
b) Determine the undamped natural frequency ω_n and the damping ratio ζ of the closed-loop system. Solve for the value of K for which $\omega_n = 5$ rad/s, and find the corresponding value of ζ.
c) Find the steady-state value of the response when the input is a step function of height 5 and when K has the value determined in part (b).

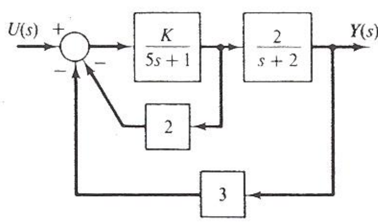


FIGURE P13.31

* **13.32** Consider the feedback system shown in Figure P13.32, which has inputs $U(s)$ and $V(s)$ and output $Y(s)$.
a) Find the transfer function $T_1(s) = Y(s)/U(s)$ as a ratio of polynomials. *Note:* $V(s)$ should be set equal to zero for this calculation.
b) Find the transfer function $T_2(s) = Y(s)/V(s)$ as a ratio of polynomials, taking $U(s) = 0$. *Hint:* Define the output of the right summing junction as

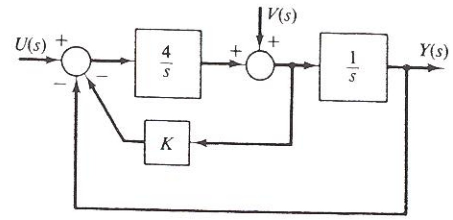


FIGURE P13.32

$Z(s)$, and write the algebraic transform equations relating $V(s)$, $Z(s)$, and $Y(s)$.
c) Determine the damping ratio ζ and the undamped natural frequency ω_n of the closed-loop system. Determine the value of K for which $\zeta = 1$.
d) Find the steady-state response when $u(t)$ is the unit step function and $v(t) = 0$. Repeat the solution when $u(t) = 0$ and $v(t)$ is the unit step function, and comment on the difference in the results.

13.33 **a)** Find the closed-loop transfer function $T(s) = Y(s)/U(s)$ as a ratio of polynomials for the block diagram shown in Figure P13.33.
b) Express the damping ratio ζ and the undamped natural frequency ω_n in terms of K. Show that both poles of $T(s)$ are on the negative real axis for $0 \le K \le 1/4$.

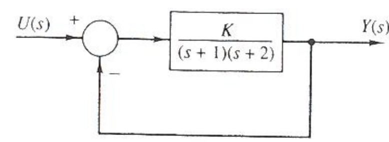


FIGURE P13.33

* **13.34** **a)** Find the closed-loop transfer function $T(s) = Y(s)/U(s)$ as a ratio of polynomials for the block diagram shown in Figure P13.34.

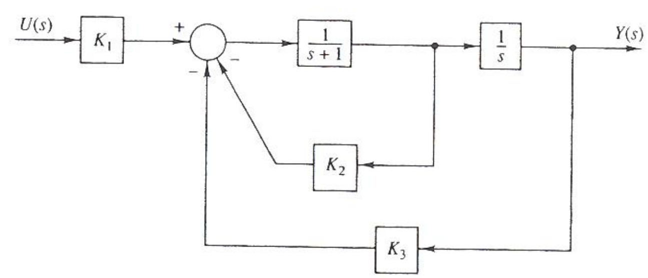


FIGURE P13.34

b) Express the damping ratio ζ and the undamped natural frequency ω_n in terms of the gains K_2 and K_3. Assume that K_3 is positive but that K_2 can be either positive or negative. Explain why ζ and ω_n are not affected by K_1.
c) For what values of K_2 is the system stable?
d) For what values of K_2 will the zero-input response contain decaying oscillations?
e) Determine the values of K_2 and K_3 such that $\omega_n = 2$ rad/s and $\zeta = 1/2$.

13.35 a) Write a set of state-variable equations for the system represented by the block diagram in Figure P13.35.
b) Determine the closed-loop transfer function $T(s) = \Theta(s)/F_a(s)$ as a ratio of polynomials.

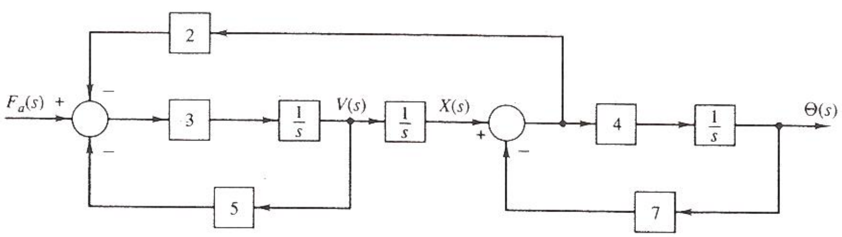


FIGURE P13.35

* **13.36** Find the closed-loop transfer functions $T_1(s) = Y(s)/U(s)$ and $T_2(s) = Z(s)/U(s)$ in terms of the individual transfer functions $A(s)$, ..., $E(s)$ for the block diagram shown in Figure P13.36. Give your answer as a ratio of terms that involve only sums, differences, and products of the individual transfer functions.

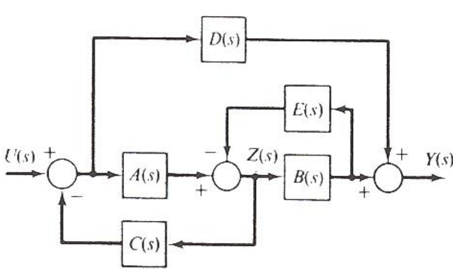


FIGURE P13.36

13.37 Repeat Problem 13.36 for the block diagram shown in Figure P13.37.

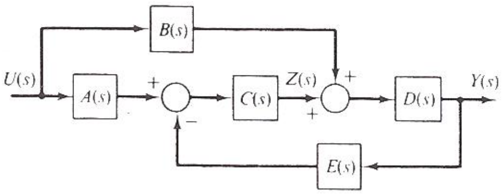


FIGURE P13.37