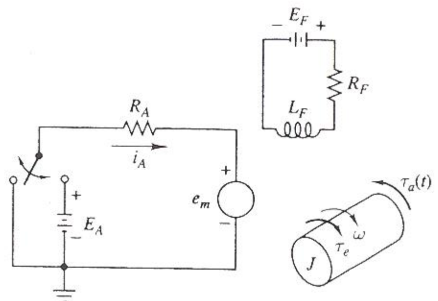


FIGURE P10.20

c) After a new steady-state speed has been reached under the conditions of part (b), the applied torque is removed, with the switch still in the right-hand position. Again find and sketch ω versus t.

* **10.21** The magnetic field of a small motor is supplied by a permanent magnet. The following two sets of measurements are made in the steady state with a 6-V battery connected across the armature. When there is no load and negligible friction, the motor rotates at 100 rad/s. When the motor is mechanically blocked to prevent its movement, the armature current is 2 A. The rotor is then attached to a propeller, which creates a viscous-friction load of 2.0×10^{-3} N·m·s, and the 6-V battery is again connected across the armature. Find the steady-state speed of the propeller.

10.22 Consider the electric motor shown in Figure 10.13 and discussed in Example 10.2. Taking i_A and ω as the state variables, write the state-variable equations in matrix form. Also write a matrix output equation for the output vector $\mathbf{y} = [\tau_e \ e_m]^T$. Identify the matrices **A**, **B**, **C**, and **D**.

CHAPTER 11

THERMAL SYSTEMS

Thermal systems are systems in which the storage and flow of heat are involved. Their mathematical models are based on the fundamental laws of thermodynamics. Examples of thermal systems include a thermometer, an automobile engine's cooling system, an oven, and a refrigerator. Generally thermal systems are distributed, and thus they obey partial rather than ordinary differential equations. We shall restrict our attention to lumped mathematical models by making approximations where necessary. Our purpose is to obtain linear ordinary differential equations that are capable of describing the dynamic response to a good approximation. We shall not examine the steady-state analysis of thermodynamic cycles that might be required in the design of a chemical process. Although systems that involve changes of phase (such as boiling or condensation) can be modeled, such treatment is beyond the scope of this book.

As in the previous chapters on modeling, we first introduce the variables and the element laws used to describe the dynamic behavior of thermal systems. Then we present a number of examples illustrating their application. A comprehensive example of the analysis of a thermal system appears in Section 11.4.

11.1 VARIABLES

The variables used to describe the behavior of a thermal system are

θ, temperature in kelvins (K) [1]

q, heat flow rate in joules per second (J/s) or in watts (W)

where 1 watt = 1 joule per second.

[1] Although the kelvin is the SI temperature unit, degrees Celsius (°C) may be more familiar. A temperature expressed in kelvins can be converted to degrees Celsius by subtracting 273.15 from its value.

The temperatures at various points in a distributed body usually differ from one another. For modeling and analysis, however, it is desirable to assume that all points in the body have the same temperature, presumably the average temperature of the body. If the temperature deviations from the average at various points do affect the validity of the single-temperature model, then the body may be partitioned into segments. Each of the segments can have a different average temperature associated with it, as illustrated in a later example. Unless otherwise noted, we shall use average temperatures for individual bodies. Furthermore, because the temperature is a measure of the energy stored in a body (provided there are no changes of phase), we normally select the temperatures as the state variables of a thermal system.

For most thermal systems, an equilibrium condition exists that defines the nominal operation. Generally, only deviations of the variables from their nominal values are of interest from a dynamic point of view. In these cases, **incremental temperatures** and **incremental heat flow rates** are defined by relationships of the form

$$\hat{\theta}(t) = \theta(t) - \bar{\theta}$$

$$\hat{q}(t) = q(t) - \bar{q}$$

where $\bar{\theta}$ and $\bar{q}$ are the nominal values. The ambient temperature of the environment surrounding the system is considered constant and is denoted by θ_a. In some problems, the nominal values of the temperature variables may be equal to θ_a, in which case we may refer to the incremental temperatures as **relative temperatures**.

11.2 ELEMENT LAWS

A consequence of the laws of thermodynamics is that there are only two types of passive thermal elements: thermal capacitance and thermal resistance. Strictly speaking, thermal capacitance and thermal resistance are characteristics associated with bodies that are distributed in space and are not lumped elements. However, because we seek to describe the dynamic behavior of thermal systems by lumped models, we shall refer to them as elements. These elements are described next, and a brief discussion of thermal sources follows.

Thermal Capacitance

An algebraic relationship exists between the temperature of a physical body and the heat stored within it. Provided that there is no change of phase and that the range of temperatures is not excessive, this relationship can be considered linear.

If $q_{in}(t) - q_{out}(t)$ denotes the net heat flow rate into the body as a function of time, then the net heat supplied between time t_0 and time t is

$$\int_{t_0}^{t} [q_{in}(\lambda) - q_{out}(\lambda)]d\lambda$$

We assume that the heat supplied during this time interval equals a constant C times the change in temperature. If the temperature of the body at the reference time t_0 is denoted by $\theta(t_0)$, then

$$\theta(t) = \theta(t_0) + \frac{1}{C}\int_{t_0}^{t} [q_{in}(\lambda) - q_{out}(\lambda)]d\lambda \qquad (1)$$

The constant C is known as the **thermal capacitance** and has units of joules per kelvin (J/K). For a body having a mass M and specific heat σ, with units of joules per kilogram-kelvin, the thermal capacitance is $C = M\sigma$.

Differentiating (1), we have

$$\dot{\theta} = \frac{1}{C}[q_{in}(t) - q_{out}(t)] \qquad (2)$$

which relates the rate of temperature change to the instantaneous net heat flow rate into the body. Because we generally select the temperatures of the bodies that constitute a thermal system as the state variables, we shall use (2) extensively to write the state-variable equations. As indicated earlier, we can use (1) and (2) only when the temperature of the body is assumed to be uniform. If the thermal gradients within the body are so great that we cannot make this assumption, then the body should be divided into two or more parts with separate thermal capacitances.

Thermal Resistance

Heat can flow between points by three different mechanisms: conduction, convection, and radiation. We shall consider only conduction, whereby heat flows from one body to another through the medium connecting them at a rate proportional to the temperature difference between the points. Specifically, the flow of heat by conduction from a body at temperature θ_1 to a body at temperature θ_2 obeys the relationship

$$q(t) = \frac{1}{R}[\theta_1(t) - \theta_2(t)] \qquad (3)$$

where R is the **thermal resistance** of the path between the bodies, with units of kelvin-seconds per joule (K·s/J) or kelvins per watt (K/W). For a path of cross-sectional area A and length d composed of material having a thermal conductivity α (with units of watts per meter-kelvin), the thermal resistance is

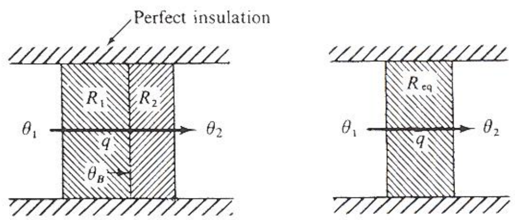
$$R = \frac{d}{A\alpha} \qquad (4)$$

We can use (3) only when the material or body being treated as a thermal resistance does not store any heat. Should it become important to account for the heat stored in the resistance, then we must also include a thermal capacitance in the model.

In developing lumped models of thermal systems, we often find it convenient to combine two or more thermal resistances into a single equivalent resistance. The following two examples illustrate the techniques for doing this for combinations of two resistances.

▶ *EXAMPLE 11.1*

Figure 11.1(a) shows two bodies at temperatures θ_1 and θ_2 separated by two resistances R_1 and R_2. Heat flows through each of the resistances at the rate q but cannot flow through the perfect insulating material above and below the resistances. Find the value of the equivalent thermal resistance R_{eq} in Figure 11.1(b) and solve for the interface temperature θ_B.



(a) (b)

FIGURE 11.1 (a) Two thermal resistances in series. (b) Equivalent resistance.

Solution

We can use (3) for each of the thermal resistances to express the heat flow rate q in terms of the resistance and the temperature difference. Specifically,

$$q = \frac{1}{R_1}(\theta_1 - \theta_B) \tag{5a}$$

$$q = \frac{1}{R_2}(\theta_B - \theta_2) \tag{5b}$$

Combining the equations to eliminate θ_B and arranging the result in the form of (3), we find that

$$q = \frac{1}{R_1 + R_2}(\theta_1 - \theta_2)$$

Hence the equivalent thermal resistance is

$$R_{eq} = R_1 + R_2 \tag{6}$$

where R_1 and R_2 are said to be in **series** because the heat flow rate is the same through each.

To calculate the interface temperature θ_B, we combine (5a) and (5b) to eliminate q, getting

$$\theta_B = \frac{R_2\theta_1 + R_1\theta_2}{R_1 + R_2} \tag{7}$$

You should verify that equivalent forms of (7) are

$$\theta_B = \theta_1 - \frac{R_1}{R_{eq}}(\theta_1 - \theta_2)$$

$$\theta_B = \theta_2 + \frac{R_2}{R_{eq}}(\theta_1 - \theta_2)$$

▶ *EXAMPLE 11.2*

Figure 11.2 shows a hollow cylindrical vessel whose walls have thickness T and a material whose thermal conductivity is α. Calculate the thermal resistances of the side of the cylinder (R_c) and of each end (R_e). Then find the equivalent resistance of the entire vessel in terms of R_c and R_e.

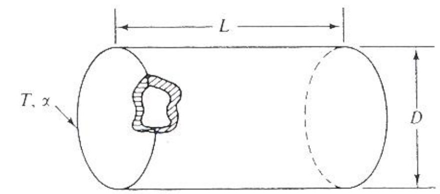


FIGURE 11.2 Cylindrical vessel for Example 11.2.

Solution

From (4), with d replaced by T and A replaced by $\pi D^2/4$, the resistance of each end of the vessel is

$$R_e = \frac{4T}{\pi D^2 \alpha} \tag{8}$$

Similarly, the resistance of the cylindrical portion is

$$R_c = \frac{T}{\pi D L \alpha} \tag{9}$$

The rate at which heat flows through each end is

$$q_e = \frac{\Delta\theta}{R_e}$$

where $\Delta\theta$ denotes the interior temperature minus the exterior temperature. Likewise, the heat flow rate through the cylindrical wall is

$$q_c = \frac{\Delta\theta}{R_c}$$

The total heat flow rate is

$$\begin{aligned} q_T &= 2q_e + q_c \\ &= \left(\frac{2}{R_e} + \frac{1}{R_c}\right)\Delta\theta \end{aligned} \tag{10}$$

Because the equivalent thermal resistance R_{eq} must satisfy the relationship

$$q_T = \frac{\Delta\theta}{R_{\text{eq}}}$$

it follows from (10) that

$$\frac{1}{R_{\text{eq}}} = \frac{2}{R_e} + \frac{1}{R_c}$$

and

$$R_{\text{eq}} = \frac{R_c R_e}{2R_c + R_e} \tag{11}$$

Because the two ends and the cylindrical wall present independent paths for the heat to flow between the interior and exterior of the vessel, with the same temperature difference existing across each path, the three thermal resistances are said to be in **parallel**.

Thermal Sources

There are two types of ideal thermal sources. Figure 11.3 represents a source that adds or removes heat at a specified rate. Heat is added to the system when $q_i(t)$ is positive and removed when $q_i(t)$ is negative. On occasion, we shall consider the temperature of a body to be an input, in which case the temperature is a known function of time regardless of the rate at which heat flows between that body and the rest of the system.

FIGURE 11.3 Representation of an ideal thermal source.

11.3 DYNAMIC MODELS OF THERMAL SYSTEMS

We shall demonstrate how to construct and analyze dynamic models of thermal systems by considering several examples. The general technique is to select the temperature of each thermal capacitance as a state variable and use (2) to obtain the corresponding state-variable equation. The net heat flow rate into a thermal capacitance depends on heat sources and heat flow rates through thermal resistances. By using (3), we can express the heat flow rates through the resistances in terms of the system's state variables, the temperatures of the thermal capacitances.

We first consider systems that have thermal capacitances from which heat can escape to the environment through thermal resistances. We then illustrate approximating a distributed system by a lumped model by analyzing two possible models for heating a bar. In the final example, a thermal capacitance is heated by both a heater and an incoming liquid stream.

▶ *EXAMPLE 11.3*

Figure 11.4 shows a thermal capacitance C enclosed by insulation that has an equivalent thermal resistance R. The temperature within the capacitance is θ and is assumed to be uniform. Heat is added to the interior of the system at the rate $q_i(t)$. The nominal values of $q_i(t)$ and θ are denoted by $\bar{q}_i$ and $\bar{\theta}$, respectively. The ambient temperature surrounding the exterior of the insulation is θ_a, a constant. Find the system model in terms of θ, $q_i(t)$, and θ_a and also in terms of incremental variables. Solve for the transfer function and the unit step response.

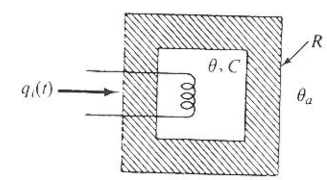


FIGURE 11.4 Thermal system with one capacitance.

Solution

The appropriate state variable is θ. We obtain an expression for its derivative by using (2) with

$$q_{\text{in}}(t) = q_i(t)$$

and, from (3),

$$q_{\text{out}}(t) = \frac{1}{R}(\theta - \theta_a)$$

Thus the state-variable model is

$$\dot{\theta} = \frac{1}{C}\left[q_i(t) - \frac{1}{R}(\theta - \theta_a)\right]$$

where we consider the ambient temperature θ_a an input to the system, along with $q_i(t)$. Rewriting the model, we have

$$\dot{\theta} + \frac{1}{RC}\theta = \frac{1}{C}q_i(t) + \frac{1}{RC}\theta_a \tag{12}$$

which is readily recognized as the differential equation of a linear first-order system with the time constant $\tau = RC$ and the inputs $q_i(t)$ and θ_a. At the operating point, (12) reduces to

$$\frac{1}{RC}\bar{\theta} = \frac{1}{C}\bar{q}_i + \frac{1}{RC}\theta_a \tag{13}$$

so

$$\bar{\theta} = \theta_a + R\bar{q}_i \tag{14}$$

When the system is in equilibrium, the temperature of the thermal capacitance is constant, and the heat flow rate $\bar{q}_i$ supplied by the heater must equal the rate of heat flow through the thermal resistance. Then the temperature difference across the resistance is $R\bar{q}_i$, which agrees with (14).

To obtain a model in terms of incremental variables, we define

$$\hat{\theta}(t) = \theta(t) - \bar{\theta}$$

$$\hat{q}_i(t) = q_i(t) - \bar{q}_i$$

Substituting these expressions into (12) gives

$$\dot{\hat{\theta}} + \frac{1}{RC}(\hat{\theta} + \bar{\theta}) = \frac{1}{C}[\hat{q}_i(t) + \bar{q}_i] + \frac{1}{RC}\theta_a$$

By using (13), we can cancel the constant terms in the last equation, giving

$$\dot{\hat{\theta}} + \frac{1}{RC}\hat{\theta} = \frac{1}{C}\hat{q}_i(t) \tag{15}$$

Examining (14) shows that if $\bar{q}_i > 0$, then $\bar{\theta} > \theta_a$ and the capacitance is being heated. If $\bar{q}_i < 0$, then $\bar{\theta} < \theta_a$ and the capacitance is being cooled. If $\bar{q}_i = 0$, then the nominal value of the temperature is $\bar{\theta} = \theta_a$ and $\hat{q}_i(t) = q_i(t)$. Because the system is linear, the incremental model given by (15) has the same coefficients regardless of the operating point.

Recall from Chapter 8 that the transfer function is $H(s) = Y(s)/U(s)$, where $U(s)$ is the transformed input and $Y(s)$ is the transform of the zero-state response. Thus we transform (15) with $\hat{\theta}(0) = 0$ to obtain

$$s\hat{\Theta}(s) + \frac{1}{RC}\hat{\Theta}(s) = \frac{1}{C}\hat{Q}_i(s)$$

We can rearrange this transformed equation to give the transfer function $H(s) = \hat{\Theta}(s)/\hat{Q}_i(s)$ as

$$H(s) = \frac{\dfrac{1}{C}}{s + \dfrac{1}{RC}}$$

which has a single pole at $s = -1/RC$. The response of $\hat{\theta}$ to a unit step function for $\hat{q}_i(t)$ will be

$$\hat{\theta} = R(1 - \epsilon^{-t/RC}) \quad \text{for } t > 0$$

which approaches a steady-state value of R with the time constant RC. Note that we can also find the steady-state value of $\hat{\theta}$ by evaluating $H(s)$ at $s = 0$. To find the response in terms of the actual temperature θ, we merely add $\bar{\theta}$ to $\hat{\theta}$, getting

$$\theta = \bar{\theta} + R(1 - \epsilon^{-t/RC}) \quad \text{for } t > 0$$

which is shown in Figure 11.5, along with the input $q_i(t)$.

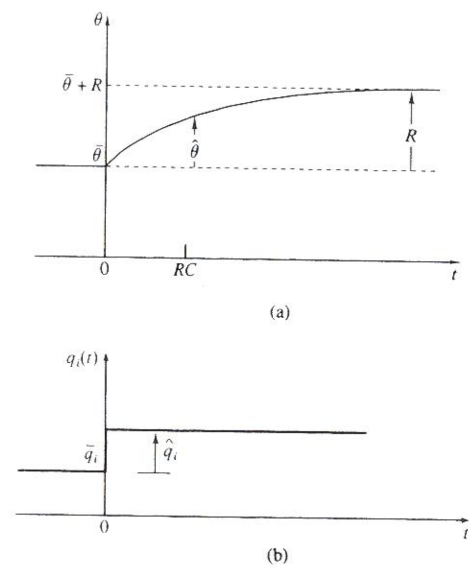


FIGURE 11.5 (a) Temperature response for Example 11.3. (b) Heat input rate.

▶ *EXAMPLE 11.4*

A system with two thermal capacitances is shown in Figure 11.6. Heat is supplied to the left capacitance at the rate $q_i(t)$ by a heater, and it is lost at the right end to the environment, which has the constant ambient temperature θ_a. Except for the thermal resistances R_1 and R_2, the enclosure is assumed to be perfectly insulated. Find the transfer function relating the transforms of the incremental variables $\hat{q}_i(t)$ and $\hat{\theta}_2$.

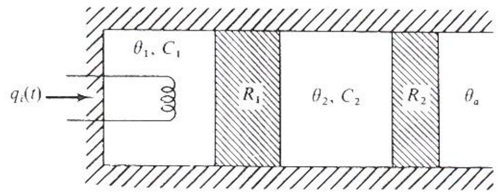


FIGURE 11.6 Thermal system with two capacitances.

Solution

Taking θ_1 and θ_2 as the two state variables and using (2) for each thermal capacitance, we can write the differential equations

$$\begin{aligned} \dot{\theta}_1 &= \frac{1}{C_1}\left[q_i(t) - \frac{1}{R_1}(\theta_1 - \theta_2)\right] \\ \dot{\theta}_2 &= \frac{1}{C_2}\left[\frac{1}{R_1}(\theta_1 - \theta_2) - \frac{1}{R_2}(\theta_2 - \theta_a)\right] \end{aligned} \tag{16}$$

At the operating point corresponding to the nominal input $\bar{q}_i$, (16) reduces to

$$\begin{aligned} \bar{q}_i - \frac{1}{R_1}(\bar{\theta}_1 - \bar{\theta}_2) &= 0 \\ \frac{1}{R_1}(\bar{\theta}_1 - \bar{\theta}_2) - \frac{1}{R_2}(\bar{\theta}_2 - \theta_a) &= 0 \end{aligned} \tag{17}$$

from which

$$\bar{\theta}_2 = \theta_a + R_2\bar{q}_i \tag{18a}$$

$$\bar{\theta}_1 = \theta_a + (R_1 + R_2)\bar{q}_i \tag{18b}$$

At the operating point, where equilibrium conditions exist and the temperatures are constant, the heat flow rates through R_1 and R_2 must both equal $\bar{q}_i$. Because the rates of heat flow are the same, we can regard the two resistances as being in series. The equivalent resistance is $R_{\text{eq}} = R_1 + R_2$,

as in (6), so $\bar{\theta}_1$ must be the temperature difference $(R_1 + R_2)\bar{q}_i$ plus the ambient temperature θ_a, which agrees with (18b).

We define the incremental temperatures $\hat{\theta}_1 = \theta_1 - \bar{\theta}_1$ and $\hat{\theta}_2 = \theta_2 - \bar{\theta}_2$ and substitute these expressions into (16). After canceling the constant terms by using (17), we obtain

$$\begin{aligned} \dot{\hat{\theta}}_1 + \frac{1}{R_1C_1}\hat{\theta}_1 &= \frac{1}{R_1C_1}\hat{\theta}_2 + \frac{1}{C_1}\hat{q}_i(t) \\ \dot{\hat{\theta}}_2 + \left(\frac{1}{R_1C_2} + \frac{1}{R_2C_2}\right)\hat{\theta}_2 &= \frac{1}{R_1C_2}\hat{\theta}_1 \end{aligned} \tag{19}$$

where the ambient temperature θ_a no longer appears. To find the transfer function $H(s) = \hat{\Theta}_2(s)/\hat{Q}_i(s)$, we transform (19) with $\hat{\theta}_1(0) = \hat{\theta}_2(0) = 0$, getting

$$\begin{aligned} \left(s + \frac{1}{R_1C_1}\right)\hat{\Theta}_1(s) &= \frac{1}{R_1C_1}\hat{\Theta}_2(s) + \frac{1}{C_1}\hat{Q}_i(s) \\ \left[s + \left(\frac{1}{R_1C_2} + \frac{1}{R_2C_2}\right)\right]\hat{\Theta}_2(s) &= \frac{1}{R_1C_2}\hat{\Theta}_1(s) \end{aligned}$$

Combining these equations to eliminate $\hat{\Theta}_1(s)$ and rearranging to form the ratio $\hat{\Theta}_2(s)/\hat{Q}_i(s)$, we get

$$H(s) = \frac{\dfrac{1}{R_1C_1C_2}}{s^2 + \left(\dfrac{1}{R_1C_1} + \dfrac{1}{R_1C_2} + \dfrac{1}{R_2C_2}\right)s + \dfrac{1}{R_1C_1R_2C_2}} \tag{20}$$

Note that the denominator of $H(s)$ is a quadratic function of s, which implies that it will have two poles. In fact, for any combination of numerical values for R_1, R_2, C_1, and C_2, these poles will be real, negative, and distinct. As a consequence, the transient response of the system consists of two decaying exponential functions.

▶ *EXAMPLE 11.5*

Consider a bar of length L and cross-sectional area A that is perfectly insulated on all its boundaries except at the left end, as shown in Figure 11.7. The temperature at the left end of the bar is $\theta_i(t)$, a known function of time that is the system's input. The interior of the bar is initially at the ambient temperature θ_a. The specific heat of the material is σ with units of joules per kilogram-kelvin, its density is ρ expressed in kilograms per cubic meter, and its thermal conductivity is α expressed in watts per meter-kelvin. Although the system is distributed and can be modeled exactly only by a partial differential equation, develop a lumped model consisting of a single thermal capacitance and resistance and then find the step response.

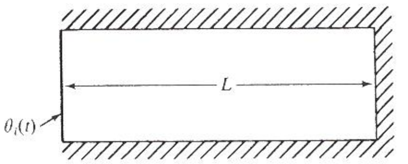


FIGURE 11.7 Insulated bar considered in Example 11.5.

Solution

Assume that all points within the bar have the same temperature θ except the left end, which has the prescribed temperature $\theta_i(t)$. Then the thermal capacitance is

$$C = \sigma \rho A L \tag{21}$$

with units of joules per kelvin. To complete the single-capacitance approximation, we assume that the thermal resistance of the entire bar, which from (4) is

$$R = \frac{L}{A\alpha} \tag{22}$$

with units of kelvins per watt, separates the left end from the remainder of the bar. The lumped approximation is shown in Figure 11.8, with C and R given by (21) and (22), respectively. From (2), with $q_{\text{out}} = 0$ because of the perfect insulation, and with

$$q_{\text{in}} = \frac{1}{R}[\theta_i(t) - \theta]$$

it follows that the single-capacitance model is

$$\dot{\theta} = \frac{1}{RC}[\theta_i(t) - \theta]$$

or

$$\dot{\theta} + \frac{1}{RC}\theta = \frac{1}{RC}\theta_i(t) \tag{23}$$

where $\theta(0) = \theta_a$.

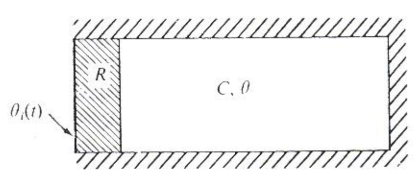


FIGURE 11.8 Single-capacitance approximation to the insulated bar shown in Figure 11.7.

If the input $\theta_i(t)$ is the sum of the constant ambient temperature θ_a and a step function of height B, we can write

$$\theta_i(t) = \theta_a + BU(t)$$

In this example, it is convenient to define the nominal values of the temperatures to be the ambient temperature θ_a and to let the incremental variables be the temperatures relative to θ_a. The relative input temperature is

$$\begin{aligned}\hat{\theta}_i(t) &= \theta_i(t) - \theta_a \\ &= BU(t)\end{aligned}$$

and the relative bar temperature is

$$\hat{\theta} = \theta - \theta_a$$

With these definitions, (23) becomes

$$\dot{\hat{\theta}} + \frac{1}{RC}(\hat{\theta} + \theta_a) = \frac{1}{RC}[\theta_a + BU(t)]$$

which reduces to

$$\dot{\hat{\theta}} + \frac{1}{RC}\hat{\theta} = \frac{B}{RC}U(t) \tag{24}$$

with the initial condition $\hat{\theta}(0) = 0$. The response of the relative temperature is

$$\hat{\theta} = B(1 - \epsilon^{-t/RC}) \qquad \text{for } t > 0$$

which indicates that the temperature of the bar will rise from the ambient temperature to that of its left end with the time constant

$$RC = \frac{\sigma \rho L^2}{\alpha}$$

▶ ***EXAMPLE 11.6***

Analyze the step response of the insulated bar shown in Figure 11.7, using the two-capacitance approximation shown in Figure 11.9.

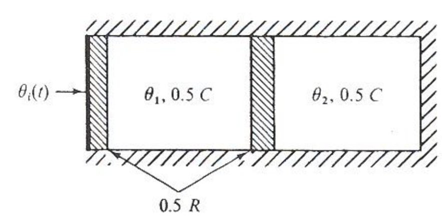


FIGURE 11.9 A two-capacitance approximation to the insulated bar shown in Figure 11.7.

Solution

As indicated in Figure 11.9, we take the value of each thermal capacitance as $0.5C$, where C is given by (21). Likewise, we take the value of each thermal resistance as $0.5R$, where R is given by (22). Applying (2) to the left capacitance with

$$q_{in} = \frac{1}{0.5R}[\theta_i(t) - \theta_1]$$

and

$$q_{out} = \frac{1}{0.5R}(\theta_1 - \theta_2)$$

gives the state-variable equation

$$\dot{\theta}_1 = \frac{1}{0.25RC}[\theta_i(t) - 2\theta_1 + \theta_2] \tag{25}$$

Applying (2) to the right capacitance with

$$q_{in} = \frac{1}{0.5R}(\theta_1 - \theta_2)$$

and $q_{out} = 0$ gives the second state-variable equation

$$\dot{\theta}_2 = \frac{1}{0.25RC}(\theta_1 - \theta_2) \tag{26}$$

To evaluate the responses of θ_1 and θ_2 to the input $\theta_i(t) = \theta_a + BU(t)$, we can define the relative temperatures $\hat{\theta}_1 = \theta_1 - \theta_a$, $\hat{\theta}_2 = \theta_2 - \theta_a$, and $\hat{\theta}_i(t) = \theta_i(t) - \theta_a$ and then derive the transfer functions $H_1(s) = \hat{\Theta}_1(s)/\hat{\Theta}_i(s)$ and $H_2(s) = \hat{\Theta}_2(s)/\hat{\Theta}_i(s)$. Knowing $H_1(s)$ and $H_2(s)$, we can write, for the step responses,

$$\hat{\theta}_1(t) = \mathcal{L}^{-1}\left[H_1(s)\frac{B}{s}\right]$$
$$\hat{\theta}_2(t) = \mathcal{L}^{-1}\left[H_2(s)\frac{B}{s}\right]$$

Rewriting (25) and (26) in terms of the relative temperatures, we have

$$\begin{aligned}\dot{\hat{\theta}}_1 + \frac{8}{RC}\hat{\theta}_1 &= \frac{4}{RC}\hat{\theta}_2 + \frac{4}{RC}\hat{\theta}_i(t)\\ \dot{\hat{\theta}}_2 + \frac{4}{RC}\hat{\theta}_2 &= \frac{4}{RC}\hat{\theta}_1\end{aligned} \tag{27}$$

Transforming (27) with $\hat{\theta}_1(0) = \hat{\theta}_2(0) = 0$ gives the pair of algebraic equations

$$\left(s + \frac{8}{RC}\right)\hat{\Theta}_1(s) = \frac{4}{RC}[\hat{\Theta}_2(s) + \hat{\Theta}_i(s)] \tag{28a}$$

$$\left(s + \frac{4}{RC}\right)\hat{\Theta}_2(s) = \frac{4}{RC}\hat{\Theta}_1(s) \tag{28b}$$

Solving (28b) for $\hat{\Theta}_2(s)$, substituting the result into (28a), and rearranging terms, we find that

$$\left[\left(s + \frac{8}{RC}\right)\left(s + \frac{4}{RC}\right) - \left(\frac{4}{RC}\right)^2\right]\hat{\Theta}_1(s) = \left[\frac{4}{RC}s + \left(\frac{4}{RC}\right)^2\right]\hat{\Theta}_i(s)$$

Simplifying the quadratic factor on the left side and solving for $H_1(s) = \hat{\Theta}_1(s)/\hat{\Theta}_i(s)$, we obtain

$$\begin{aligned}H_1(s) &= \frac{4}{RC}\left[\frac{s + \frac{4}{RC}}{s^2 + \frac{12}{RC}s + \frac{16}{(RC)^2}}\right]\\ &= \frac{4}{RC}\left[\frac{s + \frac{4}{RC}}{\left(s + \frac{1.528}{RC}\right)\left(s + \frac{10.472}{RC}\right)}\right]\end{aligned} \tag{29}$$

The transfer function has poles at $s_1 = -1.528/RC$ and $s_2 = -10.472/RC$, and the free response has two terms that decay exponentially with the time constants

$$\tau_1 = \frac{RC}{1.528} = 0.6545RC$$
$$\tau_2 = \frac{RC}{10.472} = 0.0955RC$$

Because $H_1(0) = 1$ from (29), the steady-state value of $\hat{\theta}_1$ equals B when $\hat{\theta}_i(t) = BU(t)$. For this input,

$$\begin{aligned}\hat{\Theta}_1(s) &= \frac{4}{RC}\left[\frac{s + \frac{4}{RC}}{\left(s + \frac{1.528}{RC}\right)\left(s + \frac{10.472}{RC}\right)}\right]\cdot\frac{B}{s}\\ &= \frac{A_1}{s} + \frac{A_2}{s + \frac{1.528}{RC}} + \frac{A_3}{s + \frac{10.472}{RC}}\end{aligned} \tag{30}$$

You can verify that the numerical values of the coefficients in the p
fraction expansion are

$$A_1 = B$$
$$A_2 = -0.7235B$$
$$A_3 = -0.2764B$$

Substituting these values into (30) and taking the inverse transform, we
that for $t > 0$,

$$\hat{\theta}_1(t) = B(1 - 0.7235\epsilon^{-1.528t/RC} - 0.2764\epsilon^{-10.472t/RC})$$

From (28b),

$$\hat{\Theta}_2(s) = \frac{4}{RCs + 4}\hat{\Theta}_1(s)$$
$$= \frac{16}{(RC)^2}\left[\frac{B}{s\left(s + \dfrac{1.528}{RC}\right)\left(s + \dfrac{10.472}{RC}\right)}\right]$$

whose inverse transform can be shown to be

$$\hat{\theta}_2(t) = B(1 - 1.1708\epsilon^{-1.528t/RC} + 0.1708\epsilon^{-10.472t/RC})$$

The ratios $\hat{\theta}_1(t)/B$ and $\hat{\theta}_2(t)/B$ are shown in Figure 11.10 versus the nor
malized time variable t/RC. Other lumped-element approximations tha
lead to more accurate results for the insulated bar are investigated in some
of the problems at the end of the chapter.

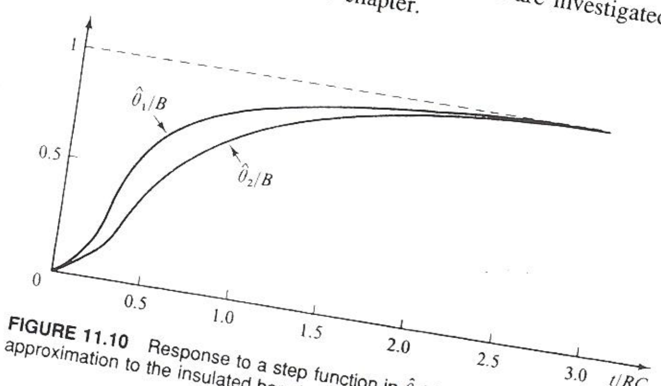


FIGURE 11.10 Response to a step function in $\hat{\theta}_i(t)$ for the two-capacitance approximation to the insulated bar shown in Figure 11.7.

▶ *EXAMPLE 11.7*

The insulated vessel shown in Figure 11.11 is filled with liquid at a temperature θ, which is kept uniform throughout the vessel by perfect mixing. Liquid enters at a constant volumetric flow rate of $\bar{w}$, expressed in units of cubic meters per second, and at a temperature $\theta_i(t)$. It leaves at the same rate and at the temperature θ_o. Because of the perfect mixing, the exit temperature θ_o is the same as the liquid temperature θ. The thermal resistance of the vessel and its insulation is R, and the ambient temperature is θ_a, a constant.

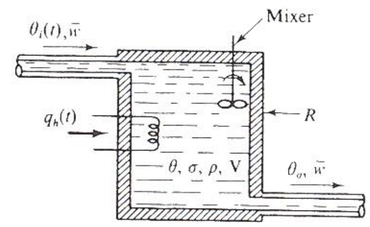


FIGURE 11.11 Insulated vessel with liquid flowing through it.

Heat is added to the liquid in the vessel by a heater at a rate $q_h(t)$. The volume of the vessel is V, and the liquid has a density of ρ (with units of kilograms per cubic meter) and a specific heat of σ (with units of joules per kilogram-kelvin). Derive the system model and find the appropriate transfer functions.

Solution

The thermal capacitance of the liquid is the product of the liquid's volume, density, and specific heat. The heat entering the vessel is the sum of that due to the heater and that contained in the incoming stream. The heat leaving the vessel is the sum of that taken out by the outgoing stream and that lost to the ambient through the vessel walls and insulation.

From (2),

$$\dot{\theta} = \frac{1}{C}(q_{\text{in}} - q_{\text{out}}) \tag{31}$$

where

$$q_{\text{in}} = q_h(t) + \bar{w}\rho\sigma\theta_i(t)$$
$$q_{\text{out}} = \bar{w}\rho\sigma\theta + \frac{1}{R}(\theta - \theta_a)$$
$$C = \rho\sigma V$$

Substituting these expressions into (31) and rearranging, we obtain the first-order differential equation

$$\dot{\theta} + \left(\frac{\bar{w}}{V} + \frac{1}{RC}\right)\theta = \frac{\bar{w}}{V}\theta_i(t) + \frac{1}{C}q_h(t) + \frac{1}{RC}\theta_a \tag{32}$$

The time constant of the system is

$$\tau = \frac{1}{\dfrac{\bar{w}}{V} + \dfrac{1}{RC}} \tag{33}$$

which approaches RC as $\bar{w}$ approaches zero (no liquid flow). As R approaches infinity (perfect insulation), the time constant becomes $V/\bar{w}$, which is the time required to replace the total tank volume at the volumetric flow rate $\bar{w}$.

Because the initial values of the state variables must be zero when we calculate the transfer functions, we first rewrite the model in terms of incremental variables defined with respect to the operating point. Setting $\dot{\theta}$ equal to zero in (32) yields the relationship between the nominal values $\bar{q}_h$, $\bar{\theta}_i$, $\bar{\theta}$, and the ambient temperature θ_a. It is

$$\bar{q}_h + \frac{C\bar{w}}{V}(\bar{\theta}_i - \bar{\theta}) = \frac{1}{R}(\bar{\theta} - \theta_a) \tag{34}$$

Rewriting (32) in terms of the incremental variables $\hat{\theta} = \theta - \bar{\theta}$, $\hat{\theta}_i(t) = \theta_i(t) - \bar{\theta}_i$, and $\hat{q}_h(t) = q_h(t) - \bar{q}_h$ and using (34), we obtain for the incremental model

$$\dot{\hat{\theta}} + \frac{1}{\tau}\hat{\theta} = \frac{\bar{w}}{V}\hat{\theta}_i(t) + \frac{1}{C}\hat{q}_h(t) \tag{35}$$

The equilibrium condition, or operating point, corresponds to the initial condition $\hat{\theta}(0) = 0$ and to incremental inputs $\hat{q}_h(t) = \hat{\theta}_i(t) = 0$. To find the transfer function $H_1(s) = \hat{\Theta}(s)/\hat{Q}_h(s)$, we transform (35) with $\hat{\theta}(0) = 0$ and $\hat{\theta}_i(t) = 0$, obtaining

$$\left(s + \frac{1}{\tau}\right)\hat{\Theta}(s) = \frac{1}{C}\hat{Q}_h(s)$$

Solving for the ratio $\hat{\Theta}(s)/\hat{Q}_h(s)$ yields

$$H_1(s) = \frac{\dfrac{1}{C}}{s + \dfrac{1}{\tau}}$$

which has a single pole at $s = -1/\tau$. The steady-state value of $\hat{\theta}$ in response to a unit step change in the heater input is

$$H_1(0) = \frac{\tau}{C} = \frac{1}{\bar{w}\rho\sigma + \dfrac{1}{R}}$$

Hence, either a high flow rate $\bar{w}$ or a low thermal resistance R tends to reduce the steady-state effect of a change in the heater input.

In similar fashion, you can verify that the transfer function $H_2(s) = \hat{\Theta}(s)/\hat{\Theta}_i(s)$ is

$$H_2(s) = \frac{\dfrac{\bar{w}}{V}}{s + \dfrac{1}{\tau}}$$

The steady-state response of $\hat{\theta}$ to a unit step change in the inlet temperature is

$$H_2(0) = \frac{\bar{w}\tau}{V} = \frac{\bar{w}\rho\sigma}{\bar{w}\rho\sigma + \dfrac{1}{R}} = \frac{1}{1 + \dfrac{1}{\bar{w}\rho\sigma R}}$$

Thus a step change in the inlet temperature with constant heater input affects the steady-state value of the liquid temperature by only some fraction of the change. Either a high flow rate or a high thermal resistance will tend to make that fraction approach unity.

In an actual process for which θ must be maintained constant in spite of variations in $\theta_i(t)$, a feedback control system may be used to sense changes in θ and make corresponding adjustments in q_h. The type of mathematical modeling and transfer-function analysis that we have demonstrated here plays an important part in the design of such control systems.

11.4 A THERMAL SYSTEM

Producing chemicals almost always requires control of the temperature of liquids contained in vessels. In a continuous process, a vessel within which a reaction is taking place typically has liquid flowing into and out of it continuously, and a control system is needed to maintain the liquid at a constant temperature. Such a system was modeled in Example 11.7. In a batch process, a vessel would typically be filled with liquid, sealed, and then heated to a prescribed temperature. In the design and operation of batch processes, it is important to be able to calculate in advance the time required for the liquid to reach the desired temperature. Such a batch system is modeled and analyzed in this case study.

System Description

Figure 11.12 shows a closed, insulated vessel that is filled with liquid and contains an electrical heater immersed in the liquid. The heating element is contained within a metal jacket that has a thermal resistance of R_{HL}. The thermal resistance of the vessel and its insulation is R_{La}. The heater has a thermal capacitance of C_H, and the liquid has a thermal capacitance of C_L. The heater temperature is θ_H and that of the liquid is θ_L, which is assumed to be uniform because of the mixer in the vessel. The rate at which energy is supplied to the heating element is $q_i(t)$.

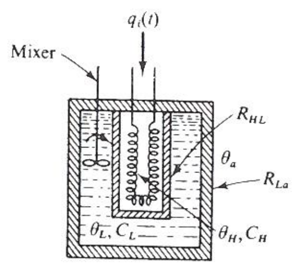


FIGURE 11.12 Vessel with heater.

The heater and the liquid are initially at the ambient temperature θ_a, with the heater turned off. At time $t = 0$, the heater is connected to an electrical source that supplies energy at a constant rate. We wish to determine the response of the liquid temperature θ_L and to calculate the time required for the liquid to reach a desired temperature, denoted by θ_d. The numerical values of the system parameters are

Heater capacitance: $C_H = 20.0 \times 10^3$ J/K
Liquid capacitance: $C_L = 1.0 \times 10^6$ J/K
Heater-liquid resistance: $R_{HL} = 1.0 \times 10^{-3}$ s·K/J
Liquid-ambient resistance: $R_{La} = 5.0 \times 10^{-3}$ s·K/J
Ambient temperature: $\theta_a = 300$ K
Desired temperature: $\theta_d = 365$ K

We will derive the system model for an arbitrary input $q_i(t)$ and for an arbitrary ambient temperature θ_a, using θ_L and θ_H as the state variables.

Then we will define a set of variables relative to the ambient conditions and find the transfer function relating the transforms of the relative liquid temperature and the input. Finally, we will calculate the time required for the liquid to reach the desired temperature.

System Model

Because θ_H and θ_L represent the energy stored in the system, we can write the state-variable model as

$$\begin{aligned} \dot{\theta}_H &= \frac{1}{C_H}[q_i(t) - q_{HL}] \\ \dot{\theta}_L &= \frac{1}{C_L}[q_{HL} - q_{La}] \end{aligned} \tag{36}$$

where $q_{HL} = (\theta_H - \theta_L)/R_{HL}$ and $q_{La} = (\theta_L - \theta_a)/R_{La}$. Substituting these expressions for q_{HL} and q_{La} and the appropriate numerical parameter values into (36) leads to the pair of state-variable equations

$$\begin{aligned} \dot{\theta}_H &= -0.050\theta_H + 0.050\theta_L + [0.50 \times 10^{-4}]q_i(t) \\ \dot{\theta}_L &= -[1.20 \times 10^{-3}]\theta_L + 10^{-3}\theta_H + [0.20 \times 10^{-3}]\theta_a \end{aligned} \tag{37}$$

where the initial conditions are $\theta_H(0) = \theta_L(0) = \theta_a$.

At this point, we could transform (37) for a specific ambient temperature and a specific input $q_i(t)$. Then we could solve for $\Theta_L(s)$ and take its inverse transform to find $\theta_L(t)$. Instead, we define the relative variables

$$\hat{\theta}_H = \theta_H - \theta_a \tag{38a}$$

$$\hat{\theta}_L = \theta_L - \theta_a \tag{38b}$$

$$\hat{q}_i(t) = q_i(t) \tag{38c}$$

where the last equation implies that $\bar{q}_i = 0$. Using (38) to rewrite (37), we find that the system model in terms of the relative variables is

$$\begin{aligned} \dot{\hat{\theta}}_H &= -0.050\hat{\theta}_H + 0.050\hat{\theta}_L + [0.50 \times 10^{-4}]\hat{q}_i(t) \\ \dot{\hat{\theta}}_L &= -[1.20 \times 10^{-3}]\hat{\theta}_L + 10^{-3}\hat{\theta}_H \end{aligned} \tag{39}$$

where the initial conditions are $\hat{\theta}_H = \hat{\theta}_L(0) = 0$.

When transformed and rearranged, (39) becomes the pair of algebraic equations

$$\begin{aligned} (s + 0.050)\hat{\Theta}_H(s) &= 0.050\hat{\Theta}_L(s) + (0.50 \times 10^{-4})\hat{Q}_i(s) \\ (s + 1.20 \times 10^{-3})\hat{\Theta}_L(s) &= 10^{-3}\hat{\Theta}_H(s) \end{aligned} \tag{40}$$

Combining the two equations in (40) to eliminate $\hat{\Theta}_H(s)$, we find that the transfer function $H(s) = \hat{\Theta}_L(s)/\hat{Q}_i(s)$ is

$$H(s) = \frac{0.50 \times 10^{-4}}{1000s^2 + 51.20s + 0.010}$$
$$= \frac{0.50 \times 10^{-7}}{s^2 + 0.05120s + 10^{-5}}$$
$$= \frac{0.50 \times 10^{-7}}{(s + 0.0510)(s + 0.000196)} \tag{41}$$

System Response

Having found the transfer function $\hat{\Theta}_L(s)/\hat{Q}_i(s)$ given by (41), we can now solve for the response to various inputs. Specifically, we shall find $\theta_L(t)$ for the ambient temperature $\theta_a = 300$ K (approximately 27°C or 80°F) and for a step-function input $\hat{q}_i(t) = 1.50 \times 10^4$ W for $t > 0$.

To find $\hat{\Theta}_L(s)$, we multiply $\hat{Q}_i(s) = [1.50 \times 10^4](1/s)$ by $H(s)$ as given by (41), getting

$$\hat{\Theta}_L(s) = \frac{0.750 \times 10^{-3}}{s(s + 0.0510)(s + 0.000196)} \tag{42}$$

Next we expand $\hat{\Theta}_L(s)$ in partial fractions to obtain

$$\hat{\Theta}_L(s) = \frac{75.0}{s} + \frac{0.289}{s + 0.0510} - \frac{75.29}{s + 0.000196}$$

Thus for $t > 0$, the relative liquid temperature is

$$\hat{\theta}_L = 75.0 + 0.289\epsilon^{-0.0510t} - 75.29\epsilon^{-0.000196t} \tag{43}$$

We obtain the actual liquid temperature by adding the ambient temperature of 300 K to $\hat{\theta}_L$, getting

$$\theta_L = 375.0 + 0.289\epsilon^{-0.0510t} - 75.29\epsilon^{-0.000196t} \tag{44}$$

which has a steady-state value of 375 K and is shown in Figure 11.13.

From inspection of either (43) or (44), we see that the transient response of the system has two exponentially decaying modes with time constants of

$$\tau_1 = \frac{1}{0.0510} = 19.61 \text{ s}$$

$$\tau_2 = \frac{1}{0.000196} = 5102 \text{ s}$$

Because τ_2 exceeds τ_1 by several orders of magnitude, the transient having the longer time constant dominates the response. The principal effect of the shorter time constant is to give θ_L a zero slope at $t = 0+$, which it would not have if the transient response were a single decaying exponential.

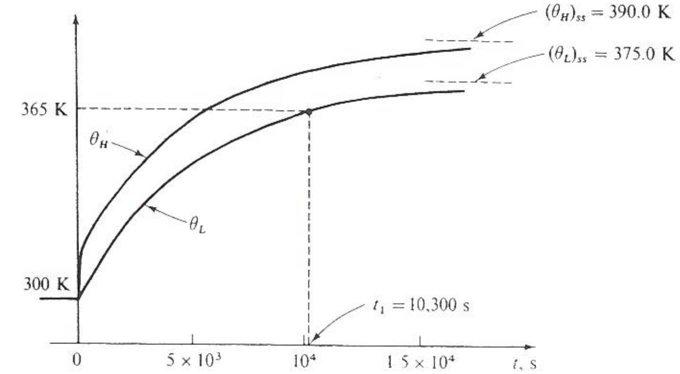


FIGURE 11.13 Responses of liquid and heater temperatures.

Finally, we must calculate the value of t_1, the time required for the liquid to reach the desired temperature $\theta_d = 365$ K. Because two exponential terms are present, an explicit solution of (44) is not possible. However by writing (44) in terms of the time constants τ_1 and τ_2 as

$$\theta_L = 375.0 + 0.289\epsilon^{-t/\tau_1} - 75.29\epsilon^{-t/\tau_2}$$

and noting that the solution for t_1 must be greater that τ_1, we can see that the term $0.289\epsilon^{-t/\tau_1}$ is negligible when $t = t_1$. Hence we can use the simpler approximate expression

$$\theta_L = 375.0 - 75.29\epsilon^{-t/5102} \tag{45}$$

where the value of 5102 s has been substituted for τ_2. Because (45) contains only one exponential function, we can solve it explicitly for t_1. Replacing the left side of (45) by the value 365.0 and replacing t by t_1 on the right side, we have

$$365.0 = 375.0 - 75.29\epsilon^{-t_1/5102}$$

which leads to

$$t_1 = 5102 \ln\left(\frac{75.29}{375.0 - 365.0}\right)$$
$$= 10{,}300 \text{ s}$$

which is approximately 2.86 hours.

Though we have completed our original task, we can use the preceding modeling and analysis results to carry out a variety of additional tasks. For instance, we can find the response of the heater temperature θ_H for the specified $q_i(t)$ by eliminating $\Theta_L(s)$ from (40). The heater temperature θ_H is shown in Figure 11.13, and you are encouraged to evaluate the analytical expression for it. In practice, it might be important to evaluate the time t_1 for constant values of $\hat{q}_i(t)$ other than the value 1.50×10^4 W that we used. For example, if $\hat{q}_i(t)$ is doubled to 3.0×10^4 W, the desired liquid temperature $\theta_d = 365$ K will be reached in only 2916 s (48.6 minutes). On the other hand, if the constant energy-input rate is less than 1.30×10^4 W, the liquid will never reach 365 K. If we want to investigate the effects of replacing the constant power source by a variable source, we can multiply the transfer function $H(s)$ by $\hat{Q}_i(s)$ to give the transform of the relative temperature. After we find the inverse transform, we can add it to the ambient temperature to get θ_L.

SUMMARY

We first introduced the basic variables—temperature and heat flow rate—and then presented the element laws. In contrast to the three types of passive elements in mechanical and electrical systems, there are only two types of passive elements in thermal systems: thermal capacitance and thermal resistance. Also, a greater degree of approximation is often necessary in order to represent a thermal system by a lumped-element model.

The temperature of each body that can store heat is usually taken to be a state variable. Because there is only one type of energy-storing passive element, the poles of the transfer functions are real numbers, and the mode functions that characterize the free response are exponential rather than sinusoidal terms. Of course, adding mechanical or other elements to the thermal part of a system can change the nature of this response.

We investigated many examples, which culminated in the case study presented in Section 11.4. The procedures for finding a state-variable model, an input–output differential equation, or a transfer function are similar to those used for mechanical and electrical examples. In fact, the general methods of modeling and analysis introduced in earlier chapters are applicable to a wide variety of systems. In the next chapter, we shall apply them to hydraulic systems.

PROBLEMS

11.1 Find the equivalent thermal resistance for the three resistances shown in Figure P11.1. Also express the temperatures θ_A and θ_B at the interfaces in terms of θ_1 and θ_2.

* **11.2** The hollow enclosure shown in Figure P11.2 has four sides, each with resistance R_a, and two ends, each with resistance R_b. Find the equivalent thermal

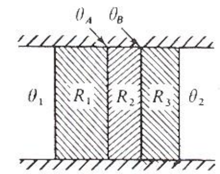


FIGURE P11.1

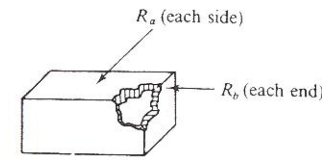


FIGURE P11.2

resistance between the interior and the exterior of the enclosure. Also calculate the total heat flow rate from the interior to the exterior when the interior of the enclosure is at a constant temperature θ_1 and the ambient temperature is θ_a.

11.3 A perfectly insulated enclosure containing a heater is filled with a liquid having thermal capacitance C. The temperature of the liquid is assumed to be uniform and is denoted by θ. The heat supplied by the heater is $q_i(t)$.

a) Write the differential equation obeyed by the liquid temperature θ.

b) Solve the equation you found in part (a) for θ in terms of the arbitrary initial temperature $\theta(0)$.

c) Sketch θ versus t when $q_i(t) = 1$ for $10 < t \leq 20$ and zero otherwise.

11.4 Figure P11.4 shows a volume for which the temperature is θ_1 and the thermal capacitance is C. The volume is perfectly insulated from the environment except for the thermal resistances R_1 and R_2. Heat is supplied at the rate $q_i(t)$. The ambient temperature is θ_a.

a) Write the system model.

b) Find and sketch θ_1 versus t when $q_i(t) = AU(t)$ and $\theta_1(0) = \theta_a$. (*Hint:* First solve for the relative temperature $\hat{\theta}_1 = \theta_1 - \theta_a$.)

c) Evaluate the transfer function $\hat{\Theta}_1(s)/Q_i(s)$ and sketch the magnitude of the frequency response versus ω.

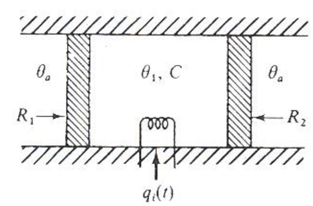


FIGURE P11.4

* **11.5** **a)** Repeat part (a) of Problem 11.4 when the temperature to the right of R_2 is $\theta_2(t)$ rather than the constant ambient temperature θ_a.

b) Using the relative temperatures $\hat{\theta}_1 = \theta_1 - \theta_a$ and $\hat{\theta}_2(t) = \theta_2(t) - \theta_a$, rewrite the model and evaluate the transfer functions $H_1(s) = \hat{\Theta}_1(s)/\hat{\Theta}_2(s)$ and $H_2(s) = \hat{\Theta}_1(s)/Q_i(s)$.

11.6 The system shown in Figure P11.6 is composed of two thermal capacitances, three thermal resistances, and two heat sources.

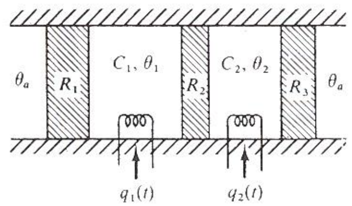


FIGURE P11.6

a) Verify that the model can be written in state-variable form as

$$\dot{\theta}_1 = \frac{1}{C_1}\left[-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\theta_1 + \frac{1}{R_2}\theta_2 + \frac{1}{R_1}\theta_a + q_1(t)\right]$$

$$\dot{\theta}_2 = \frac{1}{C_2}\left[\frac{1}{R_2}\theta_1 - \left(\frac{1}{R_2}+\frac{1}{R_3}\right)\theta_2 + \frac{1}{R_3}\theta_a + q_2(t)\right]$$

b) Derive the input–output differential equation relating θ_1, the inputs $q_1(t)$ and $q_2(t)$, and the ambient temperature θ_a.

c) Defining the relative temperature $\hat{\theta}_1$ as $\hat{\theta}_1 = \theta_1 - \theta_a$, write the transfer function $\hat{\Theta}_1(s)/\hat{Q}_1(s)$ and evaluate the steady-state response to a unit step-function input. Identify the three other transfer functions associated with the system, but do not solve for them.

* **11.7** **a)** Find the state-variable model for Figure P11.6 when the temperature to the right of R_3 is $\theta_3(t)$ rather than the constant ambient temperature θ_a.
b) Rewrite the model in terms of the relative temperatures $\hat{\theta}_1 = \theta_1 - \theta_a$, $\hat{\theta}_2 = \theta_2 - \theta_a$, and $\hat{\theta}_3(t) = \theta_3(t) - \theta_a$, and derive the transfer functions $H_1(s) = \hat{\Theta}_1(s)/\hat{\Theta}_3(s)$ and $H_2(s) = \hat{\Theta}_2(s)/\hat{\Theta}_3(s)$.

11.8 Figure P11.8 shows an electronic amplifier and a fan that can be turned on to cool the amplifier. The electronic equipment has a thermal capacitance C and generates heat at the constant rate $\bar{q}$ when it is operating. The amplifier's thermal conductivity to the ambient due to convection is K with the fan off and $2K$ with it on. The ambient temperature is θ_a, and the temperature of the amplifier is θ_1.

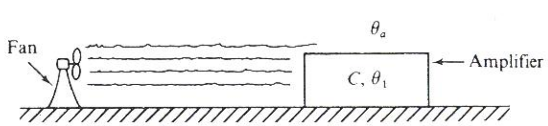


FIGURE P11.8

a) Verify that, with the fan off, the differential equation obeyed by θ_1 is

$$\dot{\theta}_1 + \frac{K}{C}\theta_1 = \frac{1}{C}(K\theta_a + \bar{q})$$

and that when the fan is on, the equation is

$$\dot{\theta}_1 + \frac{2K}{C}\theta_1 = \frac{1}{C}(2K\theta_a + \bar{q})$$

b) Solve for and sketch θ_1 when the system undergoes the following sequence of operations. In each case, indicate the steady-state temperature and the time constant.
(i) With $\theta_1(t_1) = \theta_a$ and the fan off, the amplifier is turned on at $t = t_1$.
(ii) After the system reaches steady-state conditions, the fan is turned on at $t = t_2$.
(iii) After the system reaches steady-state conditions, the amplifier is turned off at $t = t_3$ with the fan running.

c) Show how the response for $t > t_3$ in part (b) is affected when the fan also is turned off at $t = t_3$.

* **11.9** Figure P11.9 shows a piece of hot metal that has been immersed in a water bath to cool it. We can develop a simplified model by assuming that the temperatures of the metal and water, denoted by θ_m and θ_w, respectively, are uniform and that the rate of heat loss from the metal is proportional to the temperature difference $\theta_m - \theta_w$. The thermal capacitances are C_m and C_w, and the thermal resistance is R. We can neglect initially any heat lost to the environment at the surface.

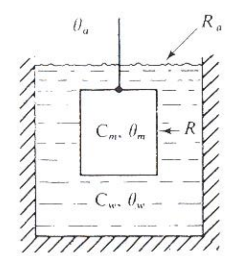


FIGURE P11.9

a) Verify that the mathematical model of the system subject to these assumptions can be written as

$$\dot{\theta}_m = \frac{1}{RC_m}(-\theta_m + \theta_w)$$

$$\dot{\theta}_w = \frac{1}{RC_w}(\theta_m - \theta_w)$$

b) Taking the initial metal temperature and water temperature as $\theta_m(0)$ and $\theta_w(0)$, respectively, solve for and sketch θ_m versus t and θ_w versus t.

c) Modify your answer to part (a) to allow for heat flow from the water to the environment. Denote the ambient temperature by θ_a and the thermal resistance between the water and the environment by R_a.

11.10 A heat exchanger in a chemical process uses steam to heat a liquid flowing in a pipe. We can apply an approximate linear model to relate changes in the temperature of the liquid leaving the heat exchanger to changes in the rate of steam flow. This model is given by the transfer function

$$\frac{\hat{\Theta}(s)}{\hat{W}(s)} = \frac{A\epsilon^{-sT_d}}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

where $\hat{\Theta}(s)$ and $\hat{W}(s)$ are the Laplace transforms of the incremental outlet temperature and the incremental steam flow rate, respectively. We can determine the coefficient A, the time delay T_d, and the time constants τ_1 and τ_2 experimentally by recording the response to a step change in the steam flow rate with all other process conditions held constant. For parameter values $A = 0.5$ K·s/kg, $T_d = 20$ s, $\tau_1 = 15$ s, and $\tau_2 = 150$ s, evaluate and sketch the following:

a) The response to a step in $\hat{w}(t)$ of 20 kg/s

b) The unit impulse response $h(t)$

c) The magnitude of the frequency response $H(j\omega)$ versus ω

* **11.11** Write the state-variable equations, using relative temperatures with respect to the ambient temperature, for a three-capacitance model of the insulated bar considered in Examples 11.5 and 11.6. Use three equal capacitances of value $C/3$ and three equal resistances of value $R/3$. Find the characteristic polynomial in terms of the parameter RC.

11.12 Repeat Problem 11.11, using nonuniform lengths for the three lumped segments of the bar. Specifically, take segments of length $0.2L$, $0.3L$, and $0.5L$, from left to right, where L is the length of the bar. Explain why such an approximation should yield greater accuracy than the equal-segment approximation of Problem 11.11.

11.13 **a)** Solve part (a) and part (b) of Problem 11.9 in numerical form for the following parameter values and initial conditions. The parameter values are

Metal mass = 50 kg
Metal specific heat = 460 J/(kg·K)
Liquid volume = 0.15 m^3
Liquid density = 1000 kg/m^3
Liquid specific heat = 4186 J/(kg·K)
Thermal resistance between the metal and the liquid = 10^{-3} s·K/J
Thermal resistance between the liquid and the environment = 10^{-2} s·K/J

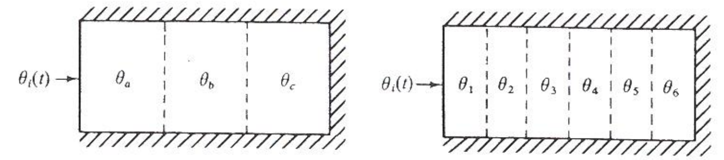
The initial conditions are

Initial metal temperature = 750 K
Initial liquid temperature = 320 K
Ambient temperature = 300 K

b) Do part (c) of Problem 11.9 and solve for θ_m. Sketch θ_m versus t and show qualitatively the shape of the curve for θ_w.

11.14 Using a digital computer, simulate the response of the insulated bar considered in Examples 11.5 and 11.6 by using N elements of equal length, as shown in Figure P11.14. Normalize the model by taking $RC = 1$ and finding the response to $\hat{\theta}_i(t) = U(t)$, where the initial relative temperature of each element is taken as zero.

a) Use $N = 3$ and plot θ_a, θ_b, and θ_c versus time.



(a) (b)

FIGURE P11.14

b) Use $N = 6$ and plot the temperatures

$$\theta_a^* = 0.5(\theta_1 + \theta_2)$$
$$\theta_b^* = 0.5(\theta_3 + \theta_4)$$
$$\theta_c^* = 0.5(\theta_5 + \theta_6)$$

Compare the results for the two cases and comment on the difference.

11.15 The temperature of a uniform bar is analyzed in Example 11.5 and Example 11.6, and extensions of the analysis are proposed in Problems 11.11 and 11.12. For each of the following lumped models, using relative temperatures with respect to the ambient temperature, write the state-variable equations in matrix form and write an output equation for the average temperature of the bar.

a) The two-capacitance model with uniform segment lengths analyzed in Example 11.6

b) The three-capacitance model with uniform segment lengths described in Problem 11.11

c) The three-capacitance model with nonuniform segment lengths described in Problem 11.12