CHAPTER 8

TRANSFER-FUNCTION ANALYSIS

We showed in Chapter 7 how to use the Laplace transform to obtain the responses of fixed linear systems. Now we shall consider in some detail two important special cases: the zero-input response, where the system's excitation consists only of some initial stored energy, and the response to a given input when the initial stored energy is zero. From the second case, we shall develop and illustrate the important concept of the transfer function.

Following a discussion of the transform solution for the complete response, we examine in some detail the responses to the unit impulse, the unit step function, and sinusoidal functions. We also show that finding the transfer function for electrical systems can be simplified by introducing the concept of impedances. Finally, we discuss the application of the Laplace transform to any fixed linear system that is described by a state-variable model in matrix form.

8.1 THE ZERO-INPUT RESPONSE

In Section 6.2, we defined the zero-input response to be the output when the input $u(t)$ is zero for all $t > 0$ and when the initial conditions are nonzero. Hence the zero-input response of a fixed linear nth-order system satisfies a homogeneous nth-order differential equation of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 \dot{y} + a_0 y = 0 \tag{1}$$

which is similar to (6.4) and where $y^{(n)}$ denotes $d^n y/dt^n$. The solution to (1) will involve the n initial conditions $y(0)$, $\dot{y}(0)$, ..., $y^{(n-1)}(0)$. Using

the expressions for the transforms of derivatives given in Appendix B, we can transform (1) term by term to obtain

$$\begin{aligned} a_n[s^n Y(s) - s^{n-1}y(0) - \cdots - y^{(n-1)}(0)] & \\ + a_{n-1}[s^{n-1}Y(s) - s^{n-2}y(0) - \cdots - y^{(n-2)}(0)] & \\ + \cdots + a_1[sY(s) - y(0)] + a_0 Y(s) = 0 & \end{aligned} \tag{2}$$

where $Y(s) = \mathcal{L}[y(t)]$. If we retain the terms involving $Y(s)$ on the left-hand side and collect those involving the initial conditions on the right-hand side, (2) becomes

$$\begin{aligned} (a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)Y(s) & \\ = a_n y(0)s^{n-1} + [a_n \dot{y}(0) + a_{n-1}y(0)]s^{n-2} + \cdots & \\ + [a_n y^{(n-1)}(0) + a_{n-1}y^{(n-2)}(0) + \cdots + a_1 y(0)] & \end{aligned}$$

Thus the transform of the zero-input response is

$$Y(s) = \frac{F(s)}{P(s)} \tag{3}$$

where

$$\begin{aligned} F(s) = a_n y(0)s^{n-1} + [a_n \dot{y}(0) + a_{n-1}y(0)]s^{n-2} + \cdots & \\ + [a_n y^{(n-1)}(0) + a_{n-1}y^{(n-2)}(0) + \cdots + a_1 y(0)] & \end{aligned} \tag{4}$$

and

$$P(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \tag{5}$$

Because $P(s)$ is of degree n and $F(s)$ is at most of degree $n-1$, $Y(s)$ is a strictly proper rational function. The numerator polynomial $F(s)$ depends on the initial conditions. The denominator polynomial $P(s)$ is identical to the characteristic polynomial in (6.5), except that it is written in terms of the complex variable s rather than r. When $P(s)$ is factored, it will have the form

$$P(s) = a_n(s - s_1)(s - s_2)\cdots(s - s_n)$$

As we discussed in Section 7.3, the quantities $s_1, s_2, \ldots, s_n$ are the poles of the transformed output; hence they determine the form of the zero-input response. They are also the roots of the characteristic equation given by (6.5). Thus the discussion in Chapter 6 about the roots of the characteristic equation applies equally well to the poles of the transformed output when the input is zero for $t > 0$.

For a first-order system, where $n = 1$,

$$\begin{aligned} P(s) &= a_1 s + a_0 \\ &= a_1(s + 1/\tau) \end{aligned}$$

where τ is the time constant. The zero-input response will have one of the forms shown in Figure 6.1.

For a second-order system, where $n = 2$,

$$P(s) = a_2 s^2 + a_1 s + a_0$$

Possible pole positions and the corresponding typical zero-input responses are shown in Figure 6.17 through Figure 6.20. When a_0 and a_2 have the same sign, we can rewrite $P(s)$ as

$$P(s) = a_2(s^2 + 2\zeta\omega_n s + \omega_n^2)$$

where ζ and ω_n are the damping ratio and the undamped natural frequency, respectively. We pointed out the significance of these two parameters in the discussion associated with Figure 6.22.

Mode Functions

The transformed output $Y(s)$ is a strictly proper rational function, so we can expand it in partial fractions by the methods of Section 7.3. If the characteristic equation $P(s) = 0$ has the distinct roots $s_1, s_2, \ldots, s_n$, then

$$\begin{aligned} Y(s) &= \frac{F(s)}{a_n(s - s_1)(s - s_2)\cdots(s - s_n)} \\ &= \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2} + \cdots + \frac{A_n}{s - s_n} \end{aligned} \tag{6}$$

where the coefficients $A_1, A_2, \ldots, A_n$ depend on the initial conditions $y(0), \dot{y}(0), \ldots, y^{(n-1)}(0)$. Because $A_i/(s - s_i) = \mathscr{L}[A_i\epsilon^{s_i t}]$, the inverse transform of (6) is

$$y(t) = A_1\epsilon^{s_1 t} + A_2\epsilon^{s_2 t} + \cdots + A_n\epsilon^{s_n t} \tag{7}$$

The exponential functions $\epsilon^{s_i t}$ that make up $y(t)$ are referred to as the **mode functions** or **modes** of the system's zero-input response. The s_i are the roots of the characteristic equation, so the mode functions are properties of the system. However, the coefficients A_i in (7) are the weightings of the individual mode functions, and they depend on the particular initial conditions. In fact, we can select specific initial conditions so as to eliminate any of the mode functions in the zero-input response by forcing the corresponding A_i to be zero. The following three examples illustrate the manner in which we can use the Laplace transform to solve for a system's zero-input response and the manner in which the initial conditions affect the weighting of the system's modes.

▶ *EXAMPLE 8.1*

The rotational mechanical system shown in Figure 4.13(a) is redrawn in Figure 8.1. For the parameter values $J = 1$ kg·m², $B = 5$ N·m·s/rad, and

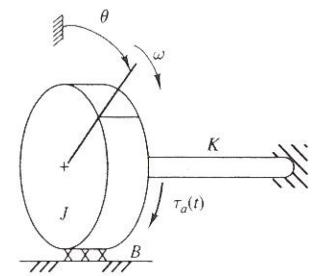

FIGURE 8.1 Rotational system for Example 8.1.

$K = 6$ N·m/rad, and for no applied torque, find and sketch the zero-input responses for the following sets of initial conditions.

1. $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$
2. $\theta(0) = 0$ and $\dot{\theta}(0) = \dot{\theta}_0$

Solution

As derived in Example 4.1, the system model is

$$J\ddot{\theta} + B\dot{\theta} + K\theta = \tau_a(t)$$

which, for the given parameter values and no applied torque, becomes

$$\ddot{\theta} + 5\dot{\theta} + 6\theta = 0$$

Transforming each term in the differential equation and collecting on the right-hand side those terms not involving $\Theta(s)$, we get

$$(s^2 + 5s + 6)\Theta(s) = s\theta_0 + \dot{\theta}_0 + 5\theta_0$$

Dividing through by the characteristic polynomial

$$P(s) = s^2 + 5s + 6$$

gives $\Theta(s)$ for any combination of initial conditions:

$$\Theta(s) = \frac{s\theta_0 + \dot{\theta}_0 + 5\theta_0}{s^2 + 5s + 6} \tag{8}$$

The characteristic polynomial is quadratic and has a damping ratio $\zeta > 1$, so $\Theta(s)$ has two real distinct poles. To carry out the partial-fraction expansion in a general form before substituting specific values for θ_0 and $\dot{\theta}_0$, we write

$$\begin{aligned} \Theta(s) &= \frac{s\theta_0 + \dot{\theta}_0 + 5\theta_0}{(s+2)(s+3)} \\ &= \frac{A_1}{s+2} + \frac{A_2}{s+3} \end{aligned}$$

where

$$A_1 = (s+2)\Theta(s)|_{s=-2} = 3\theta_0 + \dot{\theta}_0$$
$$A_2 = (s+3)\Theta(s)|_{s=-3} = -(2\theta_0 + \dot{\theta}_0)$$

Hence the transform of the response is

$$\Theta(s) = \frac{3\theta_0 + \dot{\theta}_0}{s+2} - \frac{2\theta_0 + \dot{\theta}_0}{s+3}$$

and the corresponding time function is

$$\theta(t) = (3\theta_0 + \dot{\theta}_0)\epsilon^{-2t} - (2\theta_0 + \dot{\theta}_0)\epsilon^{-3t} \quad \text{for } t > 0 \; . \tag{9}$$

From an examination of (9), it is evident that the zero-input response is composed of the two exponentially decaying mode functions ϵ^{-2t} and ϵ^{-3t}, with the weighting of each mode function dependent on the initial angle θ_0 and the initial angular velocity $\dot{\theta}_0$. Specifically, the two desired responses are

1. $\theta(t) = \theta_0(3\epsilon^{-2t} - 2\epsilon^{-3t})$ for $\theta_0 \neq 0$ and $\dot{\theta}_0 = 0$
2. $\theta(t) = \dot{\theta}_0(\epsilon^{-2t} - \epsilon^{-3t})$ for $\theta_0 = 0$ and $\dot{\theta}_0 \neq 0$

The responses are shown in Figure 8.2, along with the respective mode functions.

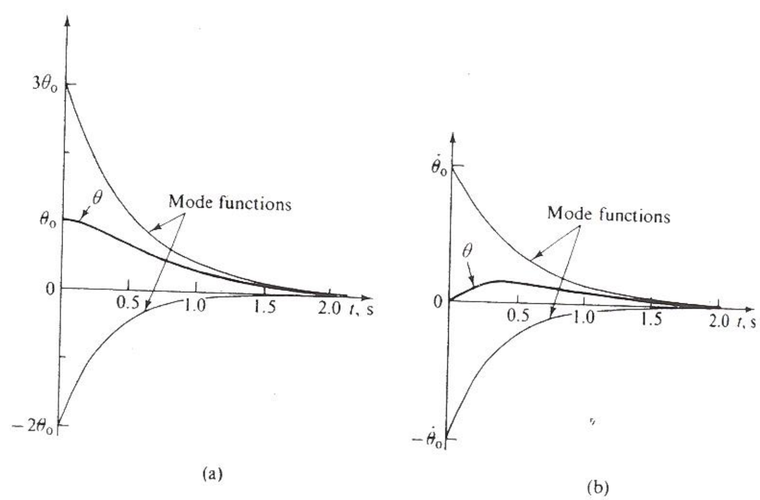

FIGURE 8.2 Responses of the rotational system in Example 8.1 to specified initial conditions. (a) $\theta_0 \neq 0$ and $\dot{\theta}_0 = 0$. (b) $\theta_0 = 0$ and $\dot{\theta}_0 \neq 0$.

From inspection of (9), we can determine for the last example the relative values of θ_0 and $\dot{\theta}_0$ that will suppress either of the two modes in the zero-input response. For example, if $2\theta_0 + \dot{\theta}_0 = 0$, the response given by (9) reduces to

$$\begin{aligned}\theta(t) &= (3\theta_0 + \dot{\theta}_0)\epsilon^{-2t} \\ &= \theta_0\epsilon^{-2t}\end{aligned}$$

which would appear as the response of a first-order system having a time constant of $\tau = \frac{1}{2}$ s. Likewise, initial conditions satisfying $3\theta_0 + \dot{\theta}_0 = 0$ result in

$$\theta(t) = \theta_0\epsilon^{-3t}$$

which is the response of a first-order system with $\tau = \frac{1}{3}$ s. Hence one should not attempt to deduce the order of a system or the character of its mode functions solely on the basis of a single sample of the zero-input response. Rather, one should be certain that each of the mode functions appears in the observed responses.

▶ *EXAMPLE 8.2*

Find and sketch the zero-input response of the rotational system discussed in Example 8.1 for the parameter values $J = 1$ kg·m², $B = 2$ N·m·s/rad, and $K = 5$ N·m/rad and the initial conditions $\theta(0) = \theta_0$, $\dot{\theta}(0) = \dot{\theta}_0$.

Solution

Substituting the given parameter values into the model and transforming the resulting equation in terms of the arbitrary initial conditions θ_0 and $\dot{\theta}_0$ give

$$\Theta(s) = \frac{s\theta_0 + \dot{\theta}_0 + 2\theta_0}{s^2 + 2s + 5} \tag{10}$$

which is similar to (8) but has different coefficients in both numerator and denominator. A check of the damping ratio indicates that $\zeta = 1/\sqrt{5}$, which is less than unity, implying that the two roots of the characteristic equation are complex. Because the mode functions $\epsilon^{s_i t}$ are themselves complex in such a case, it is preferable to leave the denominator in its quadratic form, rather than to expand $\Theta(s)$ into partial fractions with first-order denominators. Recalling that

$$\begin{aligned}\mathcal{L}[\epsilon^{-at}\cos\omega t] &= \frac{s+a}{(s+a)^2 + \omega^2} \\ \mathcal{L}[\epsilon^{-at}\sin\omega t] &= \frac{\omega}{(s+a)^2 + \omega^2}\end{aligned} \tag{11}$$

we rewrite (10) as

$$\begin{aligned}\Theta(s) &= \frac{\theta_0(s+1)+\dot{\theta}_0+\theta_0}{(s+1)^2+2^2} \\ &= \theta_0\left[\frac{s+1}{(s+1)^2+2^2}\right]+\frac{\dot{\theta}_0+\theta_0}{2}\left[\frac{2}{(s+1)^2+2^2}\right] \qquad (12)\end{aligned}$$

From a comparison of (11) and (12), it is apparent that the zero-input response is

$$\theta(t) = \theta_0\epsilon^{-t}\cos 2t + \frac{1}{2}(\dot{\theta}_0+\theta_0)\epsilon^{-t}\sin 2t \quad \text{for } t>0 \qquad (13)$$

The responses for the two sets of initial conditions $\theta_0 \neq 0, \dot{\theta}_0 = 0$ and $\theta_0 = 0, \dot{\theta}_0 \neq 0$ are shown in Figure 8.3.

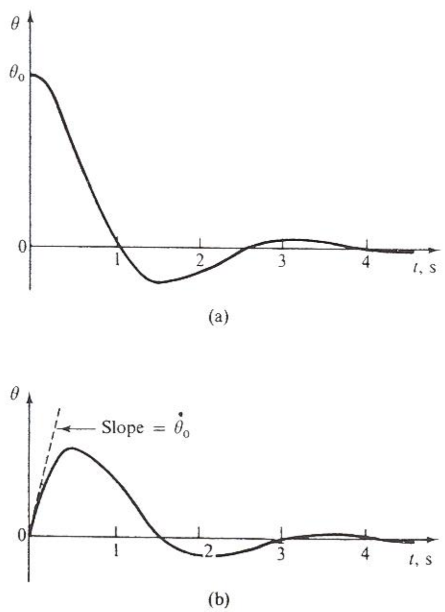

FIGURE 8.3 Responses of the rotational system in Example 8.2 to specified initial conditions. (a) $\theta_0 \neq 0$ and $\dot{\theta}_0 = 0$. (b) $\theta_0 = 0$ and $\dot{\theta}_0 \neq 0$.

Examination of (13) suggests that in place of the complex mode functions $\epsilon^{(-1+j2)t}$ and $\epsilon^{(-1-j2)t}$ corresponding to the poles of $\Theta(s)$ at $s_1 = -1+j2$ and $s_2 = -1-j2$, it is appropriate to consider the functions

$\epsilon^{-t}\cos 2t$ and $\epsilon^{-t}\sin 2t$. In more general terms, if a system has a pair of complex poles at $s_1 = \alpha + j\beta$ and $s_2 = \alpha - j\beta$, we can take the mode functions as either the complex functions $\epsilon^{(\alpha+j\beta)t}$ and $\epsilon^{(\alpha-j\beta)t}$ or the real functions $\epsilon^{\alpha t}\cos\beta t$ and $\epsilon^{\alpha t}\sin\beta t$.

▶ *EXAMPLE 8.3*

The two-inertia rotational system modeled in Example 4.2 is shown in Figure 8.4, with the viscous damping in the original version omitted. Find and sketch the zero-input response $\theta_1(t)$ for the parameter values $J_1 = J_2 = 1$ kg·m^2, $K_1 = 1$ N·m/rad, and $K_2 = 2$ N·m/rad, with the initial conditions $\theta_1(0) = \theta_2(0) = 0.5$ rad and $\dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$.

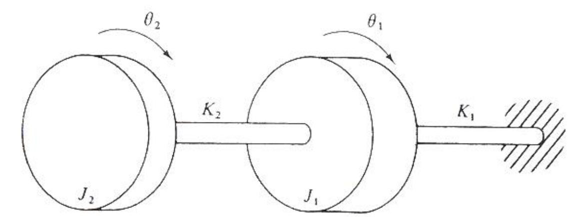

FIGURE 8.4 Rotational system for Example 8.3.

Solution

By substituting numerical values for J_1, J_2, K_1, and K_2 into (4.26) and setting $B_1 = B_2 = \tau_a(t) = 0$, we obtain the system model

$$\begin{aligned}\ddot{\theta}_1 + 3\theta_1 - 2\theta_2 &= 0 \\ \ddot{\theta}_2 + 2\theta_2 - 2\theta_1 &= 0\end{aligned} \qquad (14)$$

At this point, we could combine the pair of equations into a single homogeneous fourth-order differential equation for θ_1 and then transform the result to obtain $\Theta_1(s)$. However, that approach would require knowledge of $\ddot{\theta}_1(0)$ and $\dddot{\theta}_1(0)$. When we are working with coupled equations, including state-variable equations, it is more convenient to transform the differential equations immediately and then to solve the transformed equations for the transform of the desired output.

When transformed with the specified initial conditions, (14) becomes

$$\begin{aligned}(s^2+3)\Theta_1(s) - 2\Theta_2(s) &= 0.5s \\ -2\Theta_1(s) + (s^2+2)\Theta_2(s) &= 0.5s\end{aligned}$$

Solving these two simultaneous algebraic equations for $\Theta_1(s)$ yields

$$\Theta_1(s) = \frac{0.5s^3+2s}{s^4+5s^2+2} \qquad (15)$$

Although the denominator of $\Theta_1(s)$ is a polynomial of degree four in s, it contains only even powers of s and can be factored into the product $(s^2+0.4384)(s^2+4.562)$ by means of the quadratic formula. Thus we can decompose $\Theta_1(s)$ into the sum of two terms having quadratic denominators. The expansion of $\Theta_1(s)$ is

$$\Theta_1(s) = \frac{A_1 s + B_1}{s^2+0.4384} + \frac{A_2 s + B_2}{s^2+4.562} \tag{16}$$

To evaluate the coefficients A_1, A_2, B_1, and B_2, we put the right-hand side of (16) over a common denominator and compare the numerator coefficients to those of (15). Doing this, we obtain the following four equations:

$$A_1 + A_2 = 0.5$$

$$B_1 + B_2 = 0$$

$$4.562A_1 + 0.4384A_2 = 2$$

$$4.562B_1 + 0.4384B_2 = 0$$

Solving these equations and substituting the results into (16), we find that

$$\Theta_1(s) = \frac{0.4319s}{s^2+0.6622^2} + \frac{0.0681s}{s^2+2.136^2}$$

Referring to Appendix B, we see that the response is

$$\theta_1(t) = 0.4319\cos 0.6622t + 0.0681\cos 2.136t \quad \text{for } t > 0$$

which is plotted in Figure 8.5. The figure indicates that the disk in question responds in a rather complicated fashion, which is the superposition of undamped oscillations at the two frequencies $\omega = 0.6622$ and 2.136 rad/s. These cosine functions and the corresponding sine functions having the same frequencies can be considered the mode functions of the system. Any other combination of initial conditions would result in a response that is a weighted sum of these four mode functions.

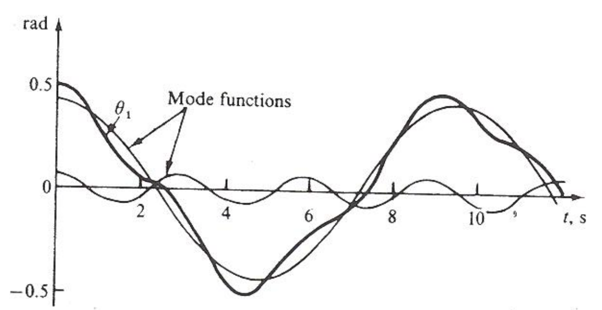

FIGURE 8.5 Response of the rotational system shown in Figure 8.4 to the initial conditions $\theta_1(0) = \theta_2(0) = 0.5$ rad and $\dot\theta_1(0) = \dot\theta_2(0) = 0$.

8.2 THE ZERO-STATE RESPONSE

As defined in Section 6.2, the zero-state response is the response to a nonzero input when the initial stored energy is zero. We assume that the input starts at $t = 0+$ and that $y(0)$, $\dot y(0)$, ..., $y^{(n-1)}(0)$, $u(0)$, $\dot u(0)$, ..., $u^{(m-1)}(0)$ are zero. The zero-state response will consist of both the forced response $y_P(t)$ and the free response $y_H(t)$. We shall consider the system model to be in the form of the general nth-order input-output equation:

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot y + a_0 y = b_m u^{(m)} + \cdots + b_0 u(t) \tag{17}$$

The Transfer Function

Transforming both sides of (17) and collecting terms, we obtain the algebraic equation

$$(a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)Y(s) = (b_m s^m + \cdots + b_0)U(s)$$

which can be rearranged to give the transform of the output as

$$Y(s) = \left(\frac{b_m s^m + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}\right)U(s) \tag{18}$$

Hence the output transform $Y(s)$ is the product of the input transform $U(s)$ and a rational function of the complex variable s whose coefficients are the coefficients in the input-output differential equation. This rational function of s is known as the system's **transfer function**. It plays a key role in the analysis of linear systems. Denoting the transfer function by $H(s)$, we rewrite (18) as

$$Y(s) = H(s)U(s) \tag{19}$$

Note that when we know a system's input-output differential equation, we can write its transfer function directly as

$$H(s) = \frac{b_m s^m + \cdots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \tag{20}$$

Even if the system model is in a form other than the input-output differential equation, the transfer function can be found from (19) or from its equivalent,

$$H(s) = \frac{Y(s)}{U(s)} \tag{21}$$

We denote the input by a general symbol, rather than using a specific function of time. We can then transform any version of the modeling equations, replace all of the initial-condition terms by zero, and solve for the transformed output. This was done in Chapter 7 for the mechanical system shown in Figure 7.10(a). In Example 7.10 we transformed the equations that were written directly from the free-body diagrams. The transfer function was the quantity inside the brackets in (7.69). In Example 7.11, we obtained the same result by transforming the state-variable model. Finally, in Example 7.12, we started with the input-output equation. An example of finding the transfer function for an electrical system will be given in Section 8.5.

Poles and Zeros

When the two polynomials in s that constitute $H(s)$ are factored, the transfer function will have the form

$$H(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \tag{22}$$

The quantities $z_1, z_2, \ldots, z_m$ are those values of s for which the numerator of $H(s)$ is zero, and they are called the **zeros** of the transfer function.[1] The quantities $p_1, p_2, \ldots, p_n$ are those values of s for which the denominator of $H(s)$ vanishes and for which $H(s)$ becomes infinite. They are the **poles** of the transfer function. From a comparison of (20) and (22), it follows that $K = b_m/a_n$, where we have assumed that neither b_m nor a_n may be zero. In the following discussion, we shall assume that none of the zeros coincides with any of the poles—that is, that $z_i \neq p_j$ for all $1 \leq i \leq m$ and all $1 \leq j \leq n$. If a pole and a zero of $H(s)$ were coincident, some of the factors in (22) could be canceled, in which case we could not reconstruct the input-output differential equation from $H(s)$. Then it would not be possible to find the response of all the system's modes from the transfer function.

From (22), it is apparent that a transfer function can be completely specified by its poles, its zeros, and the multiplying constant K. The poles and zeros may be complex numbers and can be represented graphically by points in a complex plane. This plane is called the **s-plane** or the **complex-frequency plane,** and because $s = \sigma + j\omega$, its real and imaginary axes are labeled σ and ω, respectively. The poles of $H(s)$ are indicated by crosses and the zeros by circles placed at the appropriate points.

Referring to (5), we see that the denominator of $H(s)$ in (20) is the characteristic polynomial of the system. Thus the poles $p_1, p_2, \ldots, p_n$ of $H(s)$ are identical to the characteristic roots that appear in the zero-input response. As a consequence, we can write down the form of the zero-input response as soon as we know the poles of $H(s)$.

[1] If $m < n$, which is often the case, $H(s)$ will have a zero of multiplicity $n - m$ at infinity. However, one is usually interested in only the numerator zeros $z_1, z_2, \ldots, z_m$.

To lend further significance to the transfer-function concept, recall that the stability of a system is determined by the roots of its characteristic equation. Hence the stability can also be characterized by the locations of the poles of the transfer function. If all the poles are inside the left half of the s-plane ($\sigma < 0$), the system is stable; if at least one pole is in the right half of the s-plane ($\sigma > 0$), the system is unstable. If all the poles of $H(s)$ are in the left half of the s-plane except for distinct poles on the imaginary axis ($\sigma = 0$), the system is marginally stable. In addition, a system is unstable if its transfer function has repeated poles on the imaginary axis. Before discussing the zero-state response in more detail, we shall illustrate some of these notions in the following example.

▶ *EXAMPLE 8.4*

Find the transfer function and draw the corresponding pole-zero plot in the s-plane for the system described by the input-output equation

$$\dddot{y} + 7\ddot{y} + 15\dot{y} + 25y = 2\ddot{u} + 6\dot{u}$$

Also comment on the stability of the system and give the form of its zero-input response.

Solution

Transforming the input-output equation with zero initial conditions and with $u(0) = \dot{u}(0) = 0$, we have

$$(s^3 + 7s^2 + 15s + 25)Y(s) = (2s^2 + 6s)U(s)$$

which, when we solve for the ratio $Y(s)/U(s)$, gives

$$H(s) = \frac{2s^2 + 6s}{s^3 + 7s^2 + 15s + 25}$$

Alternatively, we could have used (20) to write $H(s)$ by inspection of the input-output equation.

In order to draw the pole-zero plot representing $H(s)$, we must factor its numerator and denominator. Although the numerator is readily factored, the denominator is a cubic in s. It turns out that

$$H(s) = \frac{2s(s+3)}{(s+5)(s^2+2s+5)} \tag{23a}$$

$$= 2\left[\frac{s(s+3)}{(s+5)(s+1-j2)(s+1+j2)}\right] \tag{23b}$$

Thus $H(s)$ has two real zeros (at $s = 0$ and $s = -3$) and three poles (a real one at $s = -5$ and a complex pair at $s = -1 + j2$ and $s = -1 - j2$), all of which can be represented by the pole-zero plot shown in Figure 8.6. In addition, the multiplying constant is $K = 2$.

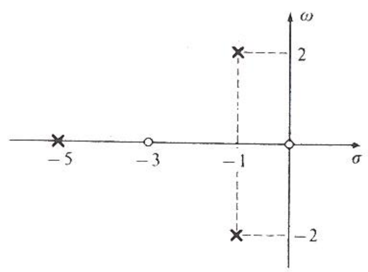

FIGURE 8.6 Pole-zero plot for $H(s)$ in Example 8.4.

Because the transfer function has all three of its poles in the left half of the s-plane, the system is stable. The fact that one of the zeros (at $s = 0$) is not inside the left half-plane has no bearing on the system's stability. For that matter, $H(s)$ can have zeros in the right half-plane and still correspond to a stable system. Knowing the three poles of the transfer function, we can immediately write down the form of the zero-input response in either of the following two equivalent forms:

$$\begin{aligned} y_{zi}(t) &= K_1\epsilon^{-5t} + \epsilon^{-t}(K_2\cos 2t + K_3\sin 2t) \\ y_{zi}(t) &= K_1\epsilon^{-5t} + K_4\epsilon^{-t}\cos(2t + \phi) \end{aligned} \tag{24}$$

Transient and Steady-State Components

Equation (19) is an expression for the transform of the system's zero-state response—that is, the output when there is no initial stored energy. Recall from Chapter 6 that the transient response consists of those terms that decay to zero as t becomes large, whereas the remaining terms constitute the steady-state response. For a stable system and for an input that does not decay to zero, the steady-state response is the forced response[2] $y_P(t)$. We now show how this relationship is manifested in (19) and how to interpret it in terms of poles and zeros.

Assume that the transform of the input $u(t)$ can be written as a rational function of s:

$$U(s) = \frac{N(s)}{D(s)} \tag{25}$$

where $N(s)$ and $D(s)$ are polynomials. For cases where the input contains a term delayed by a units of time, $N(s)$ will contain a factor ϵ^{-as}, which

[2]If a system is marginally stable, it can exhibit a steady-state response due to initial stored energy, even without an input. An unstable system can yield an unbounded response without an input. Hence the designations *transient* and *steady-state* are most useful for stable systems.

can be treated as in Example 7.14. Other inputs, such as $u(t) = 1/t$, that have Laplace transforms that do not fit (25) are beyond the scope of our consideration.

Rewriting (19) using (22) for $H(s)$ and (25) for $U(s)$, we have

$$Y(s) = K\frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)} \cdot \frac{N(s)}{D(s)} \tag{26}$$

Recognizing that $U(s)$ has its own poles and zeros, we see that the poles and zeros of $Y(s)$ are the combination of those of $H(s)$ and those of $U(s)$. If we expand $Y(s)$ in a partial-fraction expansion and all of its poles are distinct, the expansion will have the form

$$Y(s) = \frac{A_1}{s - p_1} + \frac{A_2}{s - p_2} + \cdots + \frac{A_n}{s - p_n} + \frac{A_{n+1}}{s - p_{n+1}} + \cdots + \frac{A_q}{s - p_q} \tag{27}$$

where the poles $p_1, p_2, \ldots, p_n$ are the poles of $H(s)$ and the poles $p_{n+1}, \ldots, p_q$ are the poles of $U(s)$. Taking the inverse transform of each term, we have

$$y(t) = A_1\epsilon^{p_1 t} + A_2\epsilon^{p_2 t} + \cdots + A_n\epsilon^{p_n t} + A_{n+1}\epsilon^{p_{n+1} t} + \cdots + A_q\epsilon^{p_q t} \tag{28}$$

where the first n terms are the free response $y_H(t)$ and the last $q - n$ terms are the forced response $y_P(t)$. Hence

$$y(t) = y_H(t) + y_P(t)$$

where

$$\begin{aligned} y_H(t) &= A_1\epsilon^{p_1 t} + \cdots + A_n\epsilon^{p_n t} \\ y_P(t) &= A_{n+1}\epsilon^{p_{n+1} t} + \cdots + A_q\epsilon^{qt} \end{aligned}$$

When the system is stable, the term $y_H(t)$ consists of the transients induced in having $y(t)$ start from zero initial conditions and reach the steady-state solution represented by $y_P(t)$. Keep in mind that when complex poles are present, it may be more efficient to include the corresponding second-order term (quadratic denominator and linear numerator) in the partial-fraction expansion, as in (7.57), rather than to factor it.

▶ ***EXAMPLE 8.5***

Find the form of the zero-state response of a system for which

$$H(s) = \frac{9s + 14}{3(s + 1)(s + 3)}$$

when the input is $u(t) = 6\cos 2t$ for $t > 0$. Identify the transient and steady-state terms.

Solution

The transform of the input is

$$U(s) = \frac{6s}{s^2 + 4}$$

When $H(s)$ and $U(s)$ are substituted into (19), it follows that

$$Y(s) = \frac{9s+14}{3(s+1)(s+3)} \cdot \frac{6s}{(s^2+4)} \tag{29}$$

Referring to the procedures in Section 7.3, we can write the partial-fraction expansion for $Y(s)$ in either of the following two equivalent forms:

$$Y(s) = \frac{A_1}{s+1} + \frac{A_2}{s+3} + \frac{A_3}{s-j2} + \frac{A_3^*}{s+j2} \tag{30a}$$

$$Y(s) = \frac{A_1}{s+1} + \frac{A_2}{s+3} + \frac{A_4 s + A_5}{s^2+4} \tag{30b}$$

where A_3 is a complex number and A_3^* denotes the complex conjugate. Using the methods of Section 7.3 to evaluate the constants, we find that $A_1 = -1$, $A_2 = -3$, $A_3 = 2\sqrt{2}\epsilon^{-j\pi/4}$, $A_4 = 4$, and $A_5 = 8$. Then, by the entries in Appendix B, we obtain the following expressions for the inverse transform for $t > 0$.

$$y(t) = -\epsilon^{-t} - 3\epsilon^{-3t} + 4\sqrt{2}\cos(2t - \pi/4) \tag{31a}$$

$$y(t) = -\epsilon^{-t} - 3\epsilon^{-3t} + 4\cos 2t + 4\sin 2t \tag{31b}$$

By using Table 6.1, it is easy to show that these two forms for $y(t)$ are equivalent. For either form, the first two terms constitute the transient response. These terms come from the poles of $H(s)$, which are entirely within the left half of the s-plane. The rest of the expressions represent the steady-state response, which is a constant-amplitude oscillation resulting from the poles of $U(s)$, which are on the imaginary axis of the s-plane.

Differentiating the Input

We now consider a property that we used without a complete proof in Section 6.3. For a linear system that contains no initial stored energy, suppose that we know the response to a certain input. If we substitute a new input that is the derivative of the old input, then the new response is the derivative of the old response.

To prove this statement, we denote the original input and output by $u_1(t)$ and $y_1(t)$, respectively. Because we are dealing with the zero-state response, we can assume that $u_1(t)$, $y_1(t)$, and all their derivatives are zero for $t \leq 0$. Any discontinuities are assumed to occur at $t = 0+$. By (19),

$$Y_1(s) = H(s)U_1(s)$$

Let the new input and output be $u_2(t)$ and $y_2(t)$, where $u_2(t) = \dot{u}_1$. Then with zero initial conditions, $U_2(s) = sU_1(s) - u_1(0) = sU_1(s)$, so

$$Y_2(s) = H(s)U_2(s) = sH(s)U_1(s) = sY_1(s)$$

Comparing this result with

$$\mathcal{L}[\dot{y}_1] = sY_1(s) - y_1(0) = sY_1(s)$$

we see that

$$y_2(t) = \dot{y}_1 \tag{32}$$

which confirms that the new output is the derivative of the old output.

8.3 THE COMPLETE RESPONSE

When we are to evaluate the response of a fixed linear system to the combination of nonzero initial conditions and a nonzero input for $t > 0$, the principle of superposition enables us to add the zero-input and zero-state responses to give the complete response. Likewise, we can evaluate the Laplace transforms of the two responses separately and add them to give the transform of the complete response.

For the nth-order system described by (17), (3) gives the transform of the zero-input response, that portion due to the initial conditions. The transform of the zero-state response, that portion due to the input, is given by (19). Thus the transform of the total response is

$$Y(s) = \frac{F(s)}{P(s)} + H(s)U(s) \tag{33}$$

where $F(s)$ is the polynomial given by (4) and involves the initial conditions, and where $P(s)$ is the characteristic polynomial given by (5). If we write $H(s)$ and $U(s)$ as ratios of polynomials, then (33) becomes

$$Y(s) = \frac{F(s)}{P(s)} + \frac{G(s)}{P(s)} \cdot \frac{N(s)}{D(s)} \tag{34}$$

where $G(s)$ is the numerator of $H(s)$ and where $U(s) = N(s)/D(s)$. The first term in (34) is the transform of the zero-input response; the second is the transform of the zero-state response.

Let $p_1, p_2, \ldots, p_n$ denote the poles of $H(s)$, which are the roots of $P(s) = 0$. Let $p_{n+1}, \ldots, p_q$ denote the poles of the transformed input—that is, the roots of $D(s) = 0$. If the poles are distinct, we can write the partial-fraction expansion of $Y(s)$ as

$$Y(s) = \left[\frac{K_1}{s-p_1} + \frac{K_2}{s-p_2} + \cdots + \frac{K_n}{s-p_n}\right] + \left[\frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \cdots + \frac{A_n}{s-p_n} + \frac{A_{n+1}}{s-p_{n+1}} + \cdots + \frac{A_q}{s-p_q}\right]$$

where the expression in the first set of brackets is the transformed zero-input response, and the expression in the second set is the transformed zero-state response. Only the constants $K_1, K_2, \ldots, K_n$ depend on the initial conditions. The complete time response is

$$y(t) = y_H(t) + y_P(t)$$

where

$$y_H(t) = (K_1 + A_1)\epsilon^{p_1 t} + \cdots + (K_n + A_n)\epsilon^{p_n t}$$
$$y_P(t) = A_{n+1}\epsilon^{p_{n+1} t} + \cdots + A_q\epsilon^{p_q t}$$

Comparing these expressions with (28), we see that the complete response is the same as the zero-state response, except that the terms in $y_H(t)$ have different coefficients. For a stable system with an input that does not decay to zero, $y_P(t)$ is the steady-state response, which is unaffected by the initial conditions.

We can rewrite (34) with a common denominator as

$$Y(s) = \frac{F(s)D(s) + G(s)N(s)}{P(s)D(s)} \tag{35}$$

The poles of $Y(s)$, the transform of the complete response, are the combined poles of the transfer function and the transformed input. The zeros of $Y(s)$ depend on the four polynomials $F(s)$, $D(s)$, $G(s)$, and $N(s)$.

▶ *EXAMPLE 8.6*

The system we discussed in Example 8.5 obeys the differential equation

$$3\ddot{y} + 12\dot{y} + 9y = 9\dot{u} + 14u(t) \tag{36}$$

Find the complete response when the initial conditions are $y(0) = 2$, $\dot{y}(0) = 0$ and when the input is $u(t) = 3\epsilon^{-2t}$ for $t > 0$ and zero otherwise.

Solution

Transforming (36) with the specified initial conditions and taking $u(0) = 0$, we obtain

$$3s^2Y(s) - 6s + 12sY(s) - 24 + 9Y(s) = (9s + 14)U(s)$$

where $U(s) = 3/(s+2)$. Solving for $Y(s)$, the transform of the complete response, and writing the result in the form of (33), we find that

$$Y(s) = \frac{6s + 24}{P(s)} + \frac{9s + 14}{P(s)} \cdot \frac{3}{s+2}$$

where the characteristic polynomial is

$$P(s) = 3s^2 + 12s + 9 = 3(s+1)(s+3)$$

Combining terms over a common denominator, we have

$$Y(s) = \frac{2s^2 + 21s + 30}{(s+1)(s+2)(s+3)}$$

Carrying out the partial-fraction expansion gives

$$Y(s) = \frac{11/2}{s+1} + \frac{4}{s+2} - \frac{15/2}{s+3}$$

so for $t > 0$,

$$y(t) = \frac{11}{2}\epsilon^{-t} + 4\epsilon^{-2t} - \frac{15}{2}\epsilon^{-3t}$$

Note that each term in $y(t)$ decays to zero as t approaches infinity. This is because all the poles of $H(s)$ are in the left half of the s-plane and because the input $u(t)$ decays to zero with increasing time. Hence the steady-state response is zero.

▶ *EXAMPLE 8.7*

Repeat Example 8.6 when $y(0) = 2$ and $\dot{y}(0) = 0$ and when $u(t) = 6\cos 2t$ for $t > 0$ and zero otherwise.

Solution

Transforming (36) with the specified conditions and then solving for the transform of the output, we obtain

$$Y(s) = \frac{6s + 24}{3(s+1)(s+3)} + \frac{9s + 14}{3(s+1)(s+3)} \cdot \frac{6s}{s^2 + 4}$$

We could combine these terms over a common denominator. However, we recognize the second of the two terms as the transformed zero-state response, whose inverse transform we found in Example 8.5. The partial-fraction expansion for the first of the two terms is found to be

$$\frac{2s + 8}{(s+1)(s+3)} = \frac{3}{s+1} - \frac{1}{s+3}$$

whose inverse transform is

$$3\epsilon^{-t} - \epsilon^{-3t}$$

By adding this expression to the answer for Example 8.5, we obtain for the complete response

$$y(t) = 2\epsilon^{-t} - 4\epsilon^{-3t} + 4\cos 2t + 4\sin 2t$$

This is, of course, the same as the response in Example 8.5, except that the transient terms have different coefficients.

8.4 STEP AND IMPULSE RESPONSES

In the analysis and design of dynamic systems, it is helpful to specify and evaluate the response of a system to a specific input in order to have a common point of reference. Three inputs that stand out because of their usefulness and acceptance are (1) the unit impulse $\delta(t)$, (2) the unit step function $U(t)$, and (3) the sinusoidal function $\sin \omega t$. We shall examine the responses of fixed linear systems to these three functions by using Laplace transforms and the system transfer function $H(s)$. In the case of the impulse and the step function, we shall be interested in the zero-state response, including both the transient and the steady-state portions, for both stable and unstable systems. For the sinusoidal input, which will be considered in the next section, we shall limit our attention to the steady-state response of stable systems.

The basis for our study of the responses to these special inputs will be (19), which is

$$Y(s) = H(s)U(s) \tag{37}$$

where $Y(s)$ is the transform of the zero-state response, $H(s)$ is the system's transfer function, and $U(s)$ is the transform of the input.

Impulse Response

The transform of the unit impulse is unity, so $U(s) = 1$ and (37) reduces to

$$Y(s) = H(s)$$

In words, the transform of the unit impulse response is merely the system's transfer function $H(s)$. In addition, $h(t)$ must be zero for all $t \le 0$, because it is the zero-state response and the impulse does not occur until $t = 0+$. Using the symbol $h(t)$ to denote the response to the unit impulse, we can write

$$h(t) = \begin{cases} 0 & \text{for } t \le 0 \\ \mathcal{L}^{-1}[H(s)] & \text{for } t > 0 \end{cases} \tag{38}$$

which is a significant result in spite of its simplicity. In particular, (38) serves to tie together the system's time-domain characterization in terms of $h(t)$ and its complex-frequency-domain characterization $H(s)$. For example, one can relate the locations in the s-plane of the poles and zeros of the transfer function to the character of the system's impulse response. Furthermore, although the subject is beyond the scope of this book, the impulse response can be related to the response to arbitrary inputs by an integral expression known as the convolution integral.

In Section 6.3, we stated that the unit impulse response could be found by either (1) finding the appropriate initial conditions and solving the homogeneous differential equation or (2) finding the response to a unit step function and differentiating the result. Now an easier and more direct method is available. We can find the transfer function and then take its inverse transform. From (17) and (20), we know that we can write $H(s)$ by inspection of the system's input-output differential equation, although we may encounter some computational difficulties in factoring its denominator if the system is of third or higher order. In the event that the system model is available as two or more simultaneous equations (for example, as a set of state-variable equations and an output equation), we can find the transfer function by first transforming the simultaneous equations with zero initial conditions and then solving algebraically for the ratio $Y(s)/U(s)$, which is the transfer function.

▶ ***EXAMPLE 8.8***

Find the unit impulse response of the first-order system described by $\dot{y} + (1/\tau)y = u(t)$.

Solution

The transfer function $Y(s)/U(s)$ is

$$H(s) = \frac{1}{s + 1/\tau}$$

Recalling that $\mathcal{L}[\epsilon^{-at}] = 1/(s+a)$, we can write

$$h(t) = \begin{cases} 0 & \text{for } t \le 0 \\ \epsilon^{-t/\tau} & \text{for } t > 0 \end{cases} \tag{39}$$

which is shown in Figure 8.7.

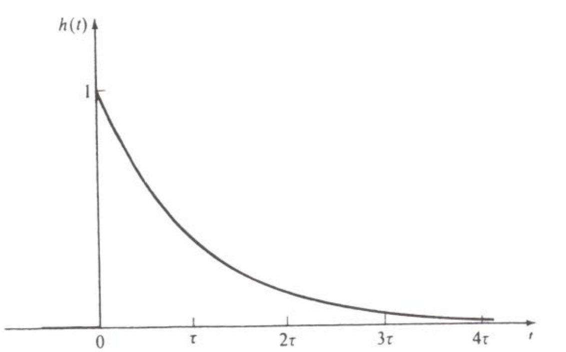

FIGURE 8.7 Impulse response for $\dot{y} + (1/\tau)y = u(t)$.

It is instructive to apply the initial-value theorem to $H(s)$ to evaluate $h(0+)$, getting

$$h(0+) = \lim_{s\to\infty} \frac{s}{s + 1/\tau} = 1$$

which agrees with (39) and Figure 8.7. It is easy to verify that the final-value theorem yields a value of zero.

▶ ***EXAMPLE 8.9***

Find and sketch the unit impulse response of the system whose input-output equation is

$$\dddot{y} + 4\ddot{y} + 3\dot{y} = 2\dot{u} + u(t)$$

Solution

By inspection of the input-output differential equation, we see that the transfer function is

$$H(s) = \frac{2s+1}{s^3 + 4s^2 + 3s} = \frac{2s+1}{s(s+1)(s+3)}$$

Carrying out the partial-fraction expansion, we find that

$$H(s) = \frac{\frac{1}{3}}{s} + \frac{\frac{1}{2}}{s+1} - \frac{\frac{5}{6}}{s+3}$$

Thus the unit impulse response is

$$h(t) = \begin{cases} 0 & \text{for } t \le 0 \\ \frac{1}{3} + \frac{1}{2}\epsilon^{-t} - \frac{5}{6}\epsilon^{-3t} & \text{for } t > 0 \end{cases}$$

which is shown in Figure 8.8. In this case,

$$h(0+) = \lim_{s\to\infty} \frac{s(2s+1)}{s(s^2+4s+3)} = 0$$

and

$$\lim_{t\to\infty} h(t) = \lim_{s\to 0} \frac{s(2s+1)}{s(s^2+4s+3)} = \frac{1}{3}$$

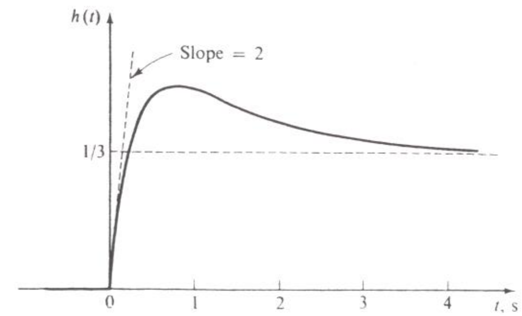

FIGURE 8.8 Impulse response for the system in Example 8.9.

both of which agree with the sketch of $h(t)$. Here $h(t)$ approaches a nonzero constant value because of the single pole at $s = 0$ in $H(s)$.

Step Response

Because the transform of the unit step function is $1/s$, it follows from (37) that the transform of $y_U(t)$, the zero-state response of a system to a unit step-function input, is

$$Y_U(s) = H(s) \cdot \frac{1}{s} \tag{40}$$

Thus the unit step response $y_U(t)$ is zero for $t \le 0$, and for $t > 0$ it is the inverse transform of $H(s)/s$. The poles of $Y_U(s)$ consist of the poles of $H(s)$ and a pole at $s = 0$. Provided that $H(s)$ has no pole at $s = 0$, the time functions that make up $y_U(t)$ will be the mode functions of the system's zero-input response and a constant term resulting from the pole of $Y_U(s)$ at $s = 0$. In fact, if the system is stable, all the mode functions will decay to zero as t approaches infinity, and the steady-state portion of the step response will be due entirely to the pole at $s = 0$. The coefficient in the partial-fraction expansion of this steady-state term, and thus $(y_U)_{ss}$ itself, will be

$$(y_U)_{ss} = sY_U(s)|_{s=0} = H(0)$$

We can also get this result by applying the final-value theorem to $Y_U(s)$ as given by (40):

$$y_U(\infty) = \lim_{s\to 0} sY_U(s) = H(0)$$

Because the initial conditions do not affect the steady-state response of a stable system, the result applies even when the initial stored energy is not zero. If $H(s)$ has a single pole at $s = 0$, then $Y_U(s)$ as given by (40) will have a double pole at $s = 0$, and the step response will contain a ramp function in addition to a constant term. Then $y_U(t)$ will grow without bound as t approaches infinity.

To show that a system's unit step response is the integral of its unit impulse response, we use the theorem for the transform of an integral to write

$$\frac{1}{s}H(s) = \mathcal{L}\left[\int_0^t h(\lambda)d\lambda\right]$$

where λ is a dummy variable of integration. Because $\mathcal{L}[y_U(t)] = H(s)/s$, it follows that

$$y_U(t) = \int_0^t h(\lambda)d\lambda \tag{41}$$

In graphical terms, (41) states that the unit step response $y_U(t)$ is the area under the curve of the unit impulse response $h(t)$. By the same token, the impulse response is the derivative of the step response—that is,

$$h(t) = \frac{d}{dt}y_U(t) \tag{42}$$

which agrees with (6.41) and (32).

▶ *EXAMPLE 8.10*

Evaluate the unit step response of the first-order system $\dot{y} + (1/\tau)y = u(t)$.

Solution

From Example 8.8, the system's transfer function is $H(s) = 1/(s + 1/\tau)$. Thus (40) indicates that the transform of the unit step response is

$$\begin{aligned} Y_U(s) &= \frac{1}{s(s+1/\tau)} \\ &= \frac{\tau}{s} - \frac{\tau}{s+1/\tau} \end{aligned}$$

Evaluating the inverse transform of $Y_U(s)$ and noting that the step response must be zero for $t \le 0$, we find that

$$y_U(t) = \begin{cases} 0 & \text{for } t \le 0 \\ \tau(1 - \epsilon^{-t/\tau}) & \text{for } t > 0 \end{cases} \tag{43}$$

which is shown in Figure 8.9. You should verify that (43) is indeed the integral of the unit impulse response, which was given by (39) and is repeated in Figure 8.9. Finally, we note that the steady-state value of the unit step response is

$$(y_U)_{ss} = H(0) = \tau$$

which agrees with (43) and Figure 8.9.

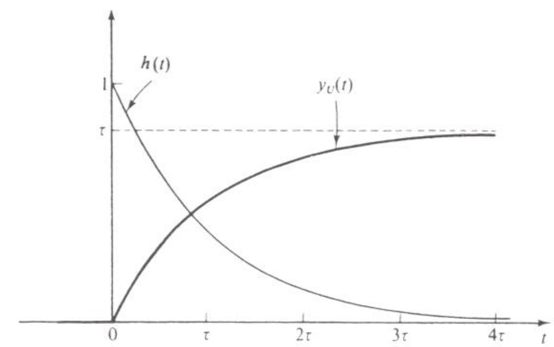

FIGURE 8.9 Step response for $\dot{y} + (1/\tau)y = u(t)$.

▶ *EXAMPLE 8.11*

Use Laplace transforms to find the unit step response of the system described by $\dddot{y} + 4\ddot{y} + 3\dot{y} = 2\dot{u} + u(t)$, for which we found the impulse response in Example 8.9.

Solution

Inspection of the system's differential equation reveals the transfer function to be

$$H(s) = \frac{2s+1}{s^3 + 4s^2 + 3s}$$

Hence the transform of the unit step response is

$$Y_U(s) = \frac{2s+1}{s^2(s^2+4s+3)}$$

whose partial-fraction expansion can be shown to be

$$Y_U(s) = \frac{\frac{1}{3}}{s^2} + \frac{\frac{2}{9}}{s} - \frac{\frac{1}{2}}{s+1} + \frac{\frac{5}{18}}{s+3}$$

Thus

$$y_U(t) = \frac{1}{3}t + \frac{2}{9} - \frac{1}{2}\epsilon^{-t} + \frac{5}{18}\epsilon^{-3t} \quad \text{for } t > 0$$

As shown in Figure 8.10, the steady-state portion of the unit step response contains a ramp component with a slope of $\frac{1}{3}$, in addition to the constant of $\frac{2}{9}$. This is because $H(s)$ has a pole at $s = 0$. This pole combines with the pole at $s = 0$ that is due to the step-function input to give a double pole of $Y(s)$ at $s = 0$.

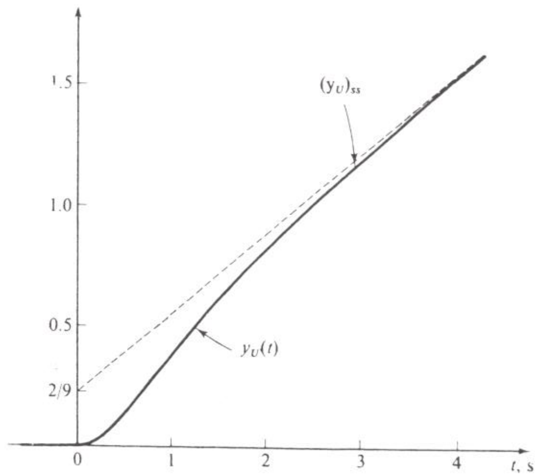

FIGURE 8.10 Step response for Example 8.11.

As an aid in sketching the unit step response, we note that in addition to $y_U(0+) = 0$, the initial slope is zero, because $h(t)$ is the derivative of the step response and (from Figure 8.8) $h(0+) = 0$.

8.5 FREQUENCY RESPONSE

In discussing the impulse and step responses, we were interested in both the transient and steady-state components. However, when considering the sinusoidal input

$$u(t) = \sin \omega t \tag{44}$$

we are normally interested only in the forced response. From our discussion of the particular solution in Section 6.1, we know that the forced response will have the same frequency as the input but that, in general, its amplitude and phase will differ from those of the input. For stable systems with a sinusoidal input, the forced response is also the steady-state response. Denoting the amplitude and phase angle of the steady-state response by A

and ϕ, respectively, we can write

$$y_{ss}(t) = A \sin(\omega t + \phi) \tag{45}$$

which is referred to as the **sinusoidal-steady-state response**. It plays a key role in many aspects of system analysis, including electronic circuits and feedback control systems. We shall now show how we can express A and ϕ in terms of the transfer function $H(s)$.

The initial conditions do not affect the steady-state response of a stable system, so we start with the transform of the zero-state response as given by (37). Because $\mathcal{L}[\sin \omega t] = \omega/(s^2 + \omega^2)$, the transformed response to (44) is

$$Y(s) = H(s)\left[\frac{\omega}{s^2 + \omega^2}\right] \tag{46}$$

Hence the poles of $Y(s)$ will be the poles of $H(s)$ plus the two imaginary poles of $U(s)$ at $s = j\omega$ and $s = -j\omega$. Provided that the system is stable, all the poles of $H(s)$ will be in the left half of the complex plane, and all the terms in the response corresponding to these poles will decay to zero as time increases. Thus the steady-state response will result from the imaginary poles of $U(s)$ and will be a sinusoid at the frequency ω. To determine the steady-state component of the response, we can write (46) as

$$\begin{aligned} Y(s) &= H(s)\left[\frac{\omega}{(s - j\omega)(s + j\omega)}\right] \\ &= \frac{C_1}{s - j\omega} + \frac{C_2}{s + j\omega} + [\text{terms corresponding to poles of } H(s)] \end{aligned} \tag{47}$$

where

$$\begin{aligned} C_1 &= (s - j\omega)Y(s)\big|_{s=j\omega} \\ &= H(s)\frac{\omega}{(s + j\omega)}\bigg|_{s=j\omega} \\ &= \frac{H(j\omega)}{2j} \end{aligned}$$

The constant C_2 is the complex conjugate of C_1; that is,

$$C_2 = -\frac{H(-j\omega)}{2j}$$

Because all the terms in (47) corresponding to the poles of $H(s)$ will decay to zero if the system is stable, the transform of the steady-state response is

$$Y_{ss}(s) = \frac{H(j\omega)/2j}{s - j\omega} - \frac{H(-j\omega)/2j}{s + j\omega} \tag{48}$$

In general, $H(j\omega)$ is a complex quantity and may be written in polar form as

$$H(j\omega) = M(\omega)\epsilon^{j\theta(\omega)} \tag{49}$$

where $M(\omega)$ is the magnitude of $H(j\omega)$ and $\theta(\omega)$ is its angle. Both M and θ depend on the value of ω, as is emphasized by the ω within the parentheses. The quantity $H(-j\omega)$ is the complex conjugate of $H(j\omega)$, and we can write it as

$$H(-j\omega) = M(\omega)\epsilon^{-j\theta(\omega)} \tag{50}$$

Substituting (49) and (50) into (48) gives

$$Y_{ss}(s) = \frac{M}{2j}\left(\frac{\epsilon^{j\theta}}{s - j\omega} - \frac{\epsilon^{-j\theta}}{s + j\omega}\right)$$

Taking the inverse transforms of the two terms in $Y_{ss}(s)$, we find that

$$\begin{aligned} y_{ss}(t) &= \frac{M}{2j}(\epsilon^{j\theta}\epsilon^{j\omega t} - \epsilon^{-j\theta}\epsilon^{-j\omega t}) \\ &= \frac{M}{2j}[\epsilon^{j(\omega t+\theta)} - \epsilon^{-j(\omega t+\theta)}] \end{aligned}$$

Using the exponential form of the sine function, as in Table 6.1, we obtain

$$y_{ss}(t) = M\sin(\omega t + \theta) \tag{51}$$

which has the form predicted in (45), where $A = M(\omega)$ and $\phi = \theta(\omega)$. One can use a similar derivation to show that for a stable system, the steady-state response to $u(t) = B\sin(\omega t + \phi_1)$ is

$$y_{ss}(t) = BM\sin(\omega t + \phi_1 + \theta) \tag{52}$$

Likewise, the steady-state response to $u(t) = B\cos(\omega t + \phi_2)$ is

$$y_{ss}(t) = BM\cos(\omega t + \phi_2 + \theta) \tag{53}$$

In summary, the sinusoidal-steady-state response of a stable linear system is a sinusoid having the same frequency as the input, an amplitude that is $M(\omega)$ times that of the input, and a phase angle that is $\theta(\omega)$ plus the input angle, where $M(\omega)$ and $\theta(\omega)$ are the magnitude and angle of $H(j\omega)$, respectively. As a consequence, the function $H(j\omega)$, which is the transfer function evaluated for $s = j\omega$, is known as the **frequency-response function**.

Calculating and interpreting the frequency-response function are illustrated in the following examples. In the first example, we find the steady-state response to a sinusoidal input that has a specified frequency. In the next two examples, we sketch curves of $M(\omega)$ and $\theta(\omega)$ as functions of ω. Such curves indicate how the magnitude and angle of the sinusoidal steady-state response change as the frequency of the input is changed.

▶ *EXAMPLE 8.12*

Use (49) and (53) to find the steady-state response of the system described by the transfer function

$$H(s) = \frac{9s + 14}{3(s^2 + 4s + 3)}$$

to the input $u(t) = 6\cos 2t$, and compare the result to that from Example 8.5.

Solution

We replace s by $j\omega$ with $\omega = 2$ rad/s, the frequency of the input, to form

$$H(j2) = \frac{j18 + 14}{3(-4 + j8 + 3)} = \frac{14 + j18}{3(-1 + j8)}$$

The magnitude of $H(j2)$ is the magnitude of its numerator divided by the magnitude of its denominator:

$$M = \frac{|14 + j18|}{3|-1 + j8|} = \frac{22.80}{3(8.062)} = 0.9427$$

The angle of $H(j2)$ is the angle of its numerator minus the angle of its denominator:

$$\begin{aligned} \theta &= \tan^{-1}(18/14) - \tan^{-1}(-8) \\ &= 0.9098 - 1.695 = -0.7853 \text{ rad} \end{aligned}$$

Thus, by (53),

$$\begin{aligned} y_{ss}(t) &= 6(0.9427)\cos(2t - 0.7853) \\ &= 5.656\cos(2t - 0.7853) \end{aligned}$$

which agrees with the steady-state portion of the response found in Example 8.5.

▶ *EXAMPLE 8.13*

Evaluate and sketch the magnitude and phase angle of the frequency-response function for the system described by the first-order equation $\dot{y} + (1/\tau)y = u(t)$.

Solution

The system transfer function is $H(s) = 1/(s + 1/\tau)$. When we replace s by $j\omega$, we find the system's frequency-response function to be

$$H(j\omega) = \frac{1}{j\omega + 1/\tau} \tag{54}$$

In order to identify the magnitude $M(\omega)$ and angle $\theta(\omega)$ for (54), we convert it to polar form. To find $M(\omega)$, we divide the magnitude of the numerator of (54) by the magnitude of the denominator, obtaining

$$M(\omega) = \frac{1}{[\omega^2 + (1/\tau)^2]^{1/2}}$$

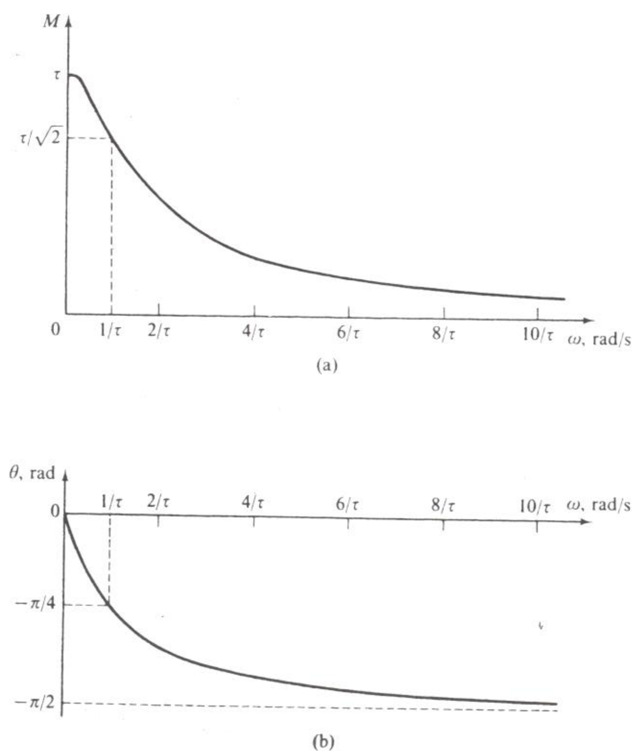

FIGURE 8.11 Frequency-response function for $\dot{y} + (1/\tau)y = u(t)$. (a) Magnitude $M(\omega)$. (b) Phase angle $\theta(\omega)$.

Subtracting the angle of the denominator from the numerator angle, we find that the phase angle[3] of $H(j\omega)$ is

$$\begin{aligned}\theta(\omega) &= \arg[1] - \arg[j\omega + 1/\tau] \\ &= 0 - \tan^{-1}\omega\tau \\ &= -\tan^{-1}\omega\tau\end{aligned}$$

To assist us in sketching M versus ω and θ versus ω, we note that at $\omega = 0$ we have $M = \tau$ and $\theta = 0$. As ω approaches infinity, M diminishes monotonically to zero, and θ decreases to $-\frac{1}{2}\pi$ rad. At $\omega = 1/\tau$, $M = \tau/\sqrt{2}$ and $\theta = -\frac{1}{4}\pi$ rad. The actual functions are shown in Figure 8.11.

▶ ***EXAMPLE 8.14***

The output for the circuit shown in Figure 8.12 is i_o. Find the frequency-response function $H(j\omega)$ and sketch its magnitude versus frequency.

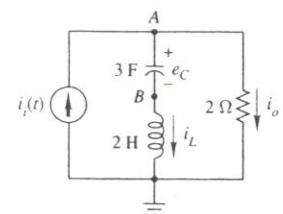

FIGURE 8.12 Circuit for Example 8.14.

Solution

We start by evaluating the circuit's transfer function $H(s)$, which is the ratio $I_o(s)/I_i(s)$ when the initial stored energy is zero. Applying Kirchhoff's current law at nodes A and B gives

$$3(\dot{e}_A - \dot{e}_B) + \frac{1}{2}e_A - i_i(t) = 0$$

$$3(\dot{e}_B - \dot{e}_A) + i_L(0) + \frac{1}{2}\int_0^t e_B(\lambda)d\lambda = 0$$

The output current is given by

$$i_o = \frac{1}{2}e_A$$

[3]The notation $\arg[z]$ denotes the angle of the complex quantity z. It is expressed in radians.

Transforming these equations with zero initial voltage across the capacitor and zero initial current through the inductor gives

$$\left(3s + \frac{1}{2}\right)E_A(s) - 3sE_B(s) = I_i(s)$$

$$-3sE_A(s) + \left(3s + \frac{1}{2s}\right)E_B(s) = 0$$

$$I_o(s) = \frac{1}{2}E_A(s)$$

We can solve this set of three transformed equations for $H(s) = I_o(s)/I_i(s)$. The result is

$$H(s) = \frac{6s^2 + 1}{6s^2 + 6s + 1}$$

We find the system's frequency-response function by setting s equal to $j\omega$ in $H(s)$. Noting that $(j\omega)^2 = -\omega^2$, we get

$$H(j\omega) = \frac{-6\omega^2 + 1}{-6\omega^2 + j6\omega + 1}$$

The magnitude function is

$$\begin{aligned} M(\omega) &= \frac{|1 - 6\omega^2|}{|1 - 6\omega^2 + j6\omega|} \\ &= \frac{|1 - 6\omega^2|}{[(1 - 6\omega^2)^2 + (6\omega)^2]^{1/2}} \end{aligned}$$

which is plotted versus ω in Figure 8.13. The phase angle $\theta(\omega)$ is given by

$$\theta(\omega) = \arg[1 - 6\omega^2] - \arg[1 - 6\omega^2 + j6\omega]$$

which could also be plotted versus frequency.

Note that when $\omega = 0$, $M(\omega)$ is unity. A sinusoidal function for which $\omega = 0$ reduces to a constant, so $H(0)$ is the steady-state response to the unit step function, which is consistent with our earlier discussion. We can check the fact that $H(0) = 1$ for Figure 8.12 by noting from Section 6.2 that when finding the steady-state response to a constant input, we can replace the inductor and capacitor by short and open circuits, respectively.

We see from Figure 8.13 that the sinusoidal-steady-state response is zero at a frequency of $\omega_1 = 1/\sqrt{6}$ rad/s. Because of the shape of the plot of $M(\omega)$ versus ω, such a circuit is often called a *notch filter*. For sinusoidal inputs with frequencies $\omega \ll \omega_1$ or $\omega \gg \omega_1$, the magnitude of the frequency-response function is approximately unity, which means that the amplitude of $(i_o)_{ss}$ will be close to that of $i_i(t)$. For inputs with $\omega \simeq 1/\sqrt{6}$ rad/s, however, the amplitude of $(i_o)_{ss}$ will be much less than that of $i_i(t)$. Hence sinusoidal inputs in this frequency range are substantially attenuated

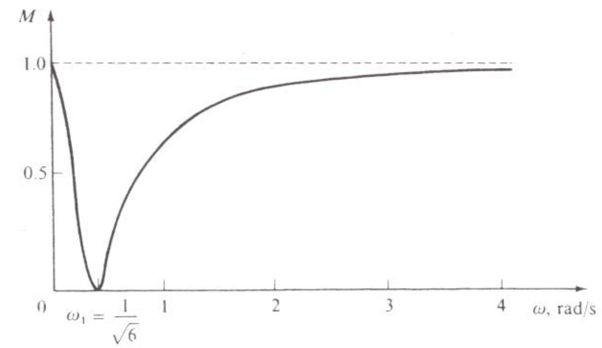

FIGURE 8.13 Frequency-response magnitude for the circuit in Example 8.14.

by the circuit. Such a circuit can be used to filter out unwanted signals that have frequencies close to ω_1 without significantly affecting signals at other frequencies.

Curves showing how the magnitude and angle of $H(j\omega)$ change with frequency constitute one of the important tools for the analysis and design of feedback control systems and will be used extensively in Chapter 14. It will become convenient to plot the curves somewhat differently, in the form of Bode diagrams, but the basic procedure is illustrated by the previous two examples.

8.6 IMPEDANCES

To find the transfer function for any linear system, we can transform the system's equations while assuming that there is no initial energy in the energy-storing elements. We can then solve for the transform of the output and after that, write the transfer function down by (21). However, when this needs to be done for the electrical part of a system, an easier method can be developed.

Consider the element laws for the resistor, inductor, and capacitor:

$$\begin{aligned} e(t) &= Ri(t) \\ e(t) &= L\frac{di}{dt} \\ i(t) &= C\frac{de}{dt} \end{aligned} \tag{55}$$

Transforming these equations, using the symbols $E(s) = \mathscr{L}[e(t)]$ and $I(s) = \mathscr{L}[i(t)]$, gives

$$\begin{aligned} E(s) &= RI(s) \\ E(s) &= L[sI(s) - i(0)] \qquad (56) \\ I(s) &= C[sE(s) - e(0)] \end{aligned}$$

When determining the transfer function, we always assume that there is no initial stored energy. In the case of an inductor, this means that $i(0) = 0$; for a capacitor, it means that $e(0) = 0$. Under these circumstances, the previous transformed equations for the resistor, inductor, and capacitor reduce to

$$E(s) = RI(s) \qquad (57a)$$

$$E(s) = sLI(s) \qquad (57b)$$

$$E(s) = \frac{1}{sC} I(s) \qquad (57c)$$

Equation (57a) looks exactly like Ohm's law for a resistor, except that transformed quantities are used for the voltage and current. Note that (57b) and (57c) have this same form, except that R is replaced by sL and $1/sC$, respectively. Thus we are led to define the **impedance** $Z(s)$ as the ratio of the transformed voltage to the transformed current when there is no initial stored energy. For the three types of passive circuit elements,

$$Z_R(s) = R \qquad (58a)$$

$$Z_L(s) = sL \qquad (58b)$$

$$Z_C(s) = \frac{1}{sC} \qquad (58c)$$

Note that the impedance depends on the element value and on s. However, (57) and (58) are *algebraic* equations, with no derivative or integral signs and with no initial-condition terms. Furthermore, transforming the algebraic equations given by the interconnection laws (Kirchhoff's voltage and current laws) yields the same equations, but with the variables replaced by their Laplace transforms. When finding the transfer function, therefore, we can use all the rules for resistive circuits, even if the elements are not of the same type. This includes the procedures in Section 5.5 for series and parallel combinations, such as equivalent resistance and the voltage-divider and current-divider rules. For this reason, the impedance $Z(s)$ is sometimes viewed as a generalized resistance. It can be treated as a resistance, even though it is a function of s.

In order to avoid writing any time-domain equations, we redraw the circuit with the passive elements characterized by their impedances and the voltages and currents characterized by their Laplace transforms. The result

is called the s-domain circuit. We can obtain the transfer function directly from it, using only the techniques for resistive circuits.

▶ *EXAMPLE 8.15*

Find the transfer function $H(s) = E_o(s)/E_i(s)$ for the circuit shown in Figure 8.14(a).

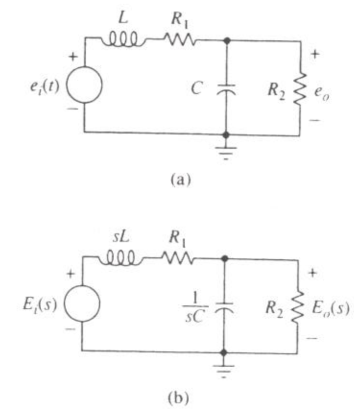

FIGURE 8.14 Circuit for Example 8.15. (a) Time-domain circuit. (b) s-domain circuit.

Solution

The s-domain circuit is drawn in Figure 8.14(b). The passive elements have been characterized by their impedances, as given in (58), and the input and output voltages by their Laplace transforms. The two right-hand elements are in parallel and can be replaced by the equivalent impedance

$$Z_2(s) = \frac{R_2/sC}{R_2 + 1/sC} = \frac{R_2}{1 + sCR_2}$$

Then, by the voltage-divider rule,

$$\begin{aligned} E_o(s) &= \frac{Z_2(s)}{sL + R_1 + Z_2(s)} E_i(s) \\ &= \frac{R_2}{(1 + sCR_2)(sL + R_1) + R_2} E_i(s) \\ &= \left[\frac{R_2}{LCR_2 s^2 + (L + R_1R_2C)s + (R_1 + R_2)} \right] E_i(s) \qquad (59) \end{aligned}$$

The quantity inside the brackets in (59) is the transfer function $H(s)$.

We saw from (17) and (20) how the transfer function and the input-output differential equation are directly related to one another. In the last example, the input-output equation for the transfer function in (59) would be

$$LCR_2\ddot{e}_o + (L + R_1R_2C)\dot{e}_o + (R_1 + R_2)e_o = R_2e_i(t)$$

Note that for a constant input voltage $e_i(t) = A$, we can find the steady-state response by setting the derivatives of the output equal to zero, which yields

$$(e_0)_{ss} = \frac{R_2}{R_1 + R_2}A \tag{60}$$

This same result is achieved by replacing s by zero for the transfer function in (59). Alternatively, if we look at the original circuit in Figure 8.14(a) with the inductor and capacitor replaced by short and open circuits, respectively, we see that (60) follows immediately from the voltage-divider rule.

In the design of electrical networks, it is sometimes convenient to represent individual subnetworks by their impedances. Once a general expression for the transfer function has been found, we can then decide what to put inside the subnetworks in order to obtain a specific result. We conclude this section with two such examples involving an operational amplifier.

▶ *EXAMPLE 8.16*

Find the transfer function $H(s)$ for the s-domain circuit shown in Figure 8.15.

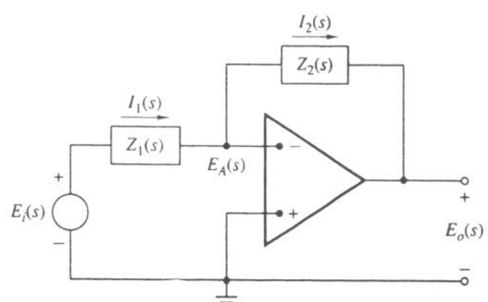

FIGURE 8.15 Circuit for Example 8.16.

Solution

From the virtual short concept discussed near the end of Section 5.7 for the ideal op-amp, we know that $E_A(s) = 0$ and that $I_1(s) = I_2(s)$. Thus

$$\frac{1}{Z_1(s)}E_i(s) = -\frac{1}{Z_2(s)}E_o(s)$$

from which the transfer function is seen to be

$$H(s) = \frac{E_o(s)}{E_i(s)} = -\frac{Z_2(s)}{Z_1(s)}$$

If, in the last example, we wanted $H(s) = K/s$, corresponding to an integrator, we could put a resistor R in the first subnetwork and a capacitor C in the second one. Then $Z_1(s) = R$, $Z_2(s) = 1/sC$, and $H(s) = -1/RCs$. The basic configuration in Figure 8.15 can be used to obtain a variety of useful transfer functions, as illustrated by some of the problems at the end of the chapter.

▶ *EXAMPLE 8.17*

Find the transfer function $H(s) = E_o(s)/E_i(s)$ for the s-domain circuit shown in Figure 8.16. Let $K = (R_1 + R_2)/R_1$.

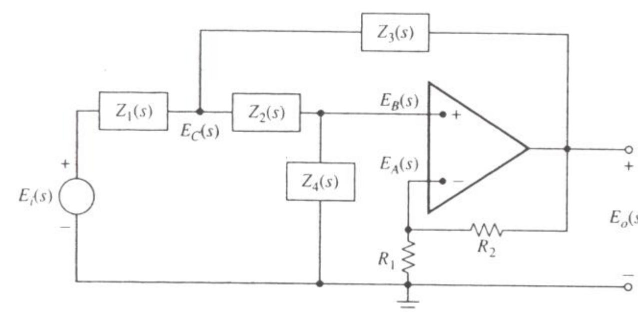

FIGURE 8.16 Circuit for Example 8.17.

Solution

This circuit can be relatively complex, with perhaps several elements inside each of the four boxes, but the general analysis is simplified by the use of impedances. Because no current flows into the input terminals of the op-amp, we can use the voltage-divider rule to write

$$E_A(s) = \frac{R_1}{R_1 + R_2}E_o(s) = \frac{1}{K}E_o(s)$$

$$E_B(s) = \frac{Z_4(s)}{Z_2(s) + Z_4(s)}E_C(s)$$

Using the virtual-short concept for the ideal op-amp, we know that $E_A(s) = E_B(s)$, so

$$E_C(s) = \frac{Z_2(s) + Z_4(s)}{KZ_4(s)}E_o(s) \tag{61}$$

Summing the currents at node C yields

$$\frac{1}{Z_1(s)}[E_C(s) - E_i(s)] + \frac{1}{Z_3(s)}[E_C(s) - E_o(s)] + \left[\frac{1}{Z_2(s) + Z_4(s)}\right] E_C(s) = 0 \quad (62)$$

Inserting (61) into (62), solving for $E_0(s)$, and simplifying, we obtain (after a little algebra) the following result:

$$H(s) = \frac{E_o(s)}{E_i(s)} = \frac{K Z_3(s) Z_4(s)}{Z_1(s)Z_2(s) + (1-K)Z_1(s)Z_4(s) + Z_3(s)[Z_1(s) + Z_2(s) + Z_4(s)]} \quad (63)$$

Even by considering very simple subnetworks for the blocks labeled $Z_1(s)$ through $Z_4(s)$ in Figure 8.16, we can obtain a number of useful results. Suppose, for example, that we want the denominator of the transfer function to be a quadratic in s and the numerator to be just a constant. We can put a single resistor inside blocks 1 and 2 and a single capacitor inside blocks 3 and 4. For simplicity, let $Z_1(s) = Z_2(s) = R$ and $Z_3(s) = Z_4(s) = 1/sC$. Then the general expression for the transfer function in (63) becomes

$$H(s) = \frac{\dfrac{K}{(RC)^2}}{s^2 + \dfrac{1}{RC}(3-K)s + \left(\dfrac{1}{RC}\right)^2}$$

We can control the position of the poles of $H(s)$ by varying the parameter K. If, for example, $K = 3$ (corresponding to $R_2 = 2R_1$), the poles are on the imaginary axis of the complex plane. If $K = 1$ (corresponding to $R_2 = 0$), both the poles are on the negative real axis at $s = -1/RC$. Examples of the importance of controlling the pole and zero positions will be given in Chapter 14.

8.7 TRANSFORM SOLUTION OF MATRIX STATE-VARIABLE EQUATIONS

For fixed linear systems, the matrix form of the state-variable and output equations is given by (6.62):

$$\dot{\mathbf{q}} = \mathbf{Aq} + \mathbf{Bu} \quad (64a)$$

$$\mathbf{y} = \mathbf{Cq} + \mathbf{Du} \quad (64b)$$

where the elements in the coefficient matrices **A**, **B**, **C**, and **D** are constants. Methods for solving these equations in the time domain were developed in Section 6.6. In this section, we use the Laplace transform to obtain a solution.

The Zero-Input Response

We start with the **zero-input response** for which $\mathbf{u} = \mathbf{0}$ for all $t > 0$ and for which the initial state $\mathbf{q}(0)$ has at least one nonzero element. Then the matrix form of the system model given by (64) reduces to

$$\dot{\mathbf{q}} = \mathbf{Aq} \quad (65a)$$

$$\mathbf{y} = \mathbf{Cq} \quad (65b)$$

where $\mathbf{q}(0)$ is specified and **A** and **C** are constant. Because (65a) represents a set of simultaneous linear differential equations, we can use the Laplace transform to solve them for **q**. Once we have found the state vector, we need only premultiply it by the matrix **C** to find the output vector **y**.

The transform of a vector is defined to be that vector whose elements are the Laplace transforms of the corresponding time functions in the original vector.[4] For example, we denote the transform vector $\mathscr{L}[\mathbf{q}(t)]$ by $\mathbf{Q}(s)$ and define it to be

$$\mathbf{Q}(s) = \begin{bmatrix} Q_1(s) \\ Q_2(s) \\ \vdots \\ Q_n(s) \end{bmatrix}$$

where $Q_i(s) = \mathscr{L}[q_i(t)]$ for $i = 1, 2, \ldots, n$.

To transform the state-variable equation (65a), we first note that because the matrix **A** is constant, the transform of the right side is just $\mathbf{AQ}(s)$. As for the left side of the equation, $\mathscr{L}[\dot{\mathbf{q}}]$ is a vector whose elements are the transforms of the corresponding state-variable derivatives. Recalling from Chapter 7 that $\mathscr{L}[\dot{q}_i] = sQ_i(s) - q_i(0)$, we can write

$$\mathscr{L}[\dot{\mathbf{q}}] = s\mathbf{Q}(s) - \mathbf{q}(0)$$

where $\mathbf{q}(0)$ is the vector of initial conditions. Thus, when transformed, (65a) becomes

$$s\mathbf{Q}(s) - \mathbf{q}(0) = \mathbf{AQ}(s)$$

which can be rearranged to give

$$s\mathbf{Q}(s) - \mathbf{AQ}(s) = \mathbf{q}(0) \quad (66)$$

[4]Capital letters followed by (s) are used for Laplace transforms, even when the time function is a vector.

To combine the two terms on the left side of (66), we note that $s\mathbf{Q}(s)$ can be written as $s\mathbf{I}\mathbf{Q}(s)$, where $\mathbf{I}$ is the identity matrix of order n. Hence we can rewrite (66) as

$$(s\mathbf{I} - \mathbf{A})\mathbf{Q}(s) = \mathbf{q}(0) \tag{67}$$

where

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & s - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & s - a_{nn} \end{bmatrix}$$

We can find the transform of the state vector, $\mathbf{Q}(s)$, from (67) by premultiplying both of its sides by the inverse of the matrix $s\mathbf{I} - \mathbf{A}$, provided that this inverse exists—that is, provided that the matrix $s\mathbf{I} - \mathbf{A}$ is not singular. A square matrix is nonsingular if its determinant is nonzero, so we can solve for $\mathbf{Q}(s)$ provided that $|s\mathbf{I} - \mathbf{A}| \neq 0$. Hence the inverse of the matrix $s\mathbf{I} - \mathbf{A}$ will exist for all points in the complex s-plane except those points that coincide with the characteristic values (eigenvalues) of the matrix $\mathbf{A}$. This restriction is of no consequence in our solution for $\mathbf{q}$ because, as we shall see, these points (there are at most n of them) are the poles of the transform vector $\mathbf{Q}(s)$. A Laplace transform is infinite when s equals one of its poles, so we should not expect to obtain finite values for $\mathbf{Q}(s)$ at its poles by matrix inversion or any other method.

Except for values of s that coincide with a characteristic value of $\mathbf{A}$, we can write

$$\mathbf{Q}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{q}(0)$$

where $(s\mathbf{I} - \mathbf{A})^{-1}$ is an $n \times n$ matrix known as the **resolvent matrix**, each element of which is a function of s. Denoting the resolvent matrix by the symbol $\mathbf{\Phi}(s)$, we have

$$\mathbf{Q}(s) = \mathbf{\Phi}(s)\mathbf{q}(0) \tag{68}$$

where

$$\mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1} \tag{69}$$

All that remains is to take the inverse Laplace transform of $\mathbf{Q}(s)$, which gives

$$\begin{aligned} \mathbf{q}(t) &= \mathcal{L}^{-1}\{\mathbf{Q}(s)\} \\ &= \mathcal{L}^{-1}\{\mathbf{\Phi}(s)\mathbf{q}(0)\} \end{aligned}$$

Because the vector $\mathbf{q}(0)$ does not involve the variable s, we may calculate the inverse transform of $\mathbf{\Phi}(s)$ first and then multiply it by $\mathbf{q}(0)$, obtaining

$$\mathbf{q}(t) = \mathcal{L}^{-1}\{\mathbf{\Phi}(s)\}\mathbf{q}(0)$$

When we denote the inverse transform of $\mathbf{\Phi}(s)$ by $\phi(t)$, which will be an $n \times n$ matrix, the desired expression for the zero-input response is

$$\mathbf{q}(t) = \phi(t)\mathbf{q}(0) \tag{70}$$

where

$$\phi(t) = \mathcal{L}^{-1}\{\mathbf{\Phi}(s)\}$$

The State-Transition Matrix

The matrix $\phi(t)$ is known as the **state-transition matrix**. The inverse of a matrix can be written as its adjoint matrix divided by its determinant (see Appendix C), so we can write the resolvent matrix from (69) as

$$\mathbf{\Phi}(s) = \frac{1}{|s\mathbf{I} - \mathbf{A}|}\operatorname{adj}[s\mathbf{I} - \mathbf{A}] \tag{71}$$

where $\operatorname{adj}[s\mathbf{I} - \mathbf{A}]$ denotes the adjoint matrix of $s\mathbf{I} - \mathbf{A}$. Thus the state-transition matrix $\phi(t)$ can be written as

$$\phi(t) = \mathcal{L}^{-1}\left[\frac{1}{|s\mathbf{I} - \mathbf{A}|}\operatorname{adj}[s\mathbf{I} - \mathbf{A}]\right] \tag{72}$$

Evaluating $(s\mathbf{I} - \mathbf{A})^{-1}$ is difficult computationally because the matrix to be inverted is a function of s. Methods exist that are better suited for this task than using the adjoint matrix. However, our principal objective is to develop certain theoretical properties and to show the form of $\phi(t)$. For these tasks and for computational problems with matrices of order three or less, the adjoint method is acceptable.

As summarized in Appendix C, the adjoint of an $n \times n$ matrix is itself an $n \times n$ matrix that is the transposed matrix of cofactors, where the cofactor of an element is a determinant of order $n - 1$. It follows that $\operatorname{adj}[s\mathbf{I} - \mathbf{A}]$ will be an $n \times n$ matrix whose elements are polynomials in s of degree $n - 1$ or less. Furthermore, the determinant of $s\mathbf{I} - \mathbf{A}$ is

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & s - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & s - a_{nn} \end{vmatrix}$$

which will always reduce to a polynomial in s of degree n. This polynomial is the **characteristic polynomial** of the matrix $\mathbf{A}$. Hence the matrix $\mathbf{\Phi}(s)$ defined by (71) is an $n \times n$ matrix whose elements are rational functions of s having numerators of lower degree than that of the denominator. Specifically, a general element of $\mathbf{\Phi}(s)$ will be of the form

$$\frac{\beta_{n-1}s^{n-1} + \cdots + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0}$$

where the numerator coefficients $\beta_{n-1}, \ldots, \beta_0$ depend on the specific row and column of the element. The denominator, however, which comes from the evaluation of $|s\mathbf{I}-\mathbf{A}|$, will be the characteristic polynomial and thus will be the same for each element of $\Phi(s)$.

In principle, at least, the characteristic polynomial can be factored into the form

$$\begin{aligned}|s\mathbf{I}-\mathbf{A}| &= s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0 \\ &= (s-s_1)(s-s_2)\cdots(s-s_n) \qquad (73)\end{aligned}$$

where the quantities $s_1, s_2, \ldots, s_n$ are known as the **characteristic values (eigenvalues)** of the matrix $\mathbf{A}$ (see Appendix C). Having factored the characteristic polynomial, we can evaluate the inverse transform of $\Phi(s)$ by performing a partial-fraction expansion of each of its elements. If the characteristic values of $\mathbf{A}$ are distinct, a typical element of the resolvent matrix will have as its expansion

$$\Phi_{ij}(s) = \frac{\gamma_1}{s-s_1} + \frac{\gamma_2}{s-s_2} + \cdots + \frac{\gamma_n}{s-s_n}$$

and the corresponding term in the state-transition matrix will be

$$\phi_{ij}(t) = \gamma_1\epsilon^{s_1t} + \gamma_2\epsilon^{s_2t} + \cdots + \gamma_n\epsilon^{s_nt} \qquad (74)$$

where, of course, the values of $\gamma_1, \gamma_2, \ldots, \gamma_n$ depend on the element being considered. Thus the state-transition matrix $\phi(t)$ is an $n \times n$ matrix each of whose elements is a linear combination of the n mode functions $\epsilon^{s_1t}, \ldots, \epsilon^{s_nt}$, where the s_i are the characteristic values of the matrix $\mathbf{A}$.

▶ *EXAMPLE 8.18*

Evaluate the state-transition matrix for a system that obeys the state-variable equation $\dot{\mathbf{q}} = \mathbf{A}\mathbf{q}$, where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

and identify the corresponding mode functions.

Solution

First we write the matrix $s\mathbf{I}-\mathbf{A}$ in numerical form and evaluate its determinant and inverse. Thus

$$s\mathbf{I}-\mathbf{A} = \begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix}$$

and

$$|s\mathbf{I}-\mathbf{A}| = s^2 + 5s + 6 = (s+2)(s+3)$$

Before taking the inverse of $s\mathbf{I}-\mathbf{A}$, we note that the characteristic polynomial associated with $\mathbf{A}$ is $P(s) = (s+2)(s+3)$. Hence the characteristic values, which must satisfy $P(s) = 0$, are $s_1 = -2$ and $s_2 = -3$. This means that the mode functions of which the zero-input response is composed are ϵ^{-2t} and ϵ^{-3t}.

The next step is to find the resolvent matrix $\Phi(s) = (s\mathbf{I}-\mathbf{A})^{-1}$. Following the steps outlined here and illustrated in Appendix C for taking the inverse of a matrix, we find the adjoint of $s\mathbf{I}-\mathbf{A}$ to be

$$\text{adj}\,[s\mathbf{I}-\mathbf{A}] = \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}$$

Dividing each element of the adjoint matrix by $|s\mathbf{I}-\mathbf{A}|$, we find that

$$\Phi(s) = \begin{bmatrix} \dfrac{s+5}{(s+2)(s+3)} & \dfrac{1}{(s+2)(s+3)} \\ \dfrac{-6}{(s+2)(s+3)} & \dfrac{s}{(s+2)(s+3)} \end{bmatrix} \qquad (75)$$

Carrying out a partial-fraction expansion of each of the four elements on the right side of (75), we get

$$\Phi(s) = \begin{bmatrix} \dfrac{3}{s+2} - \dfrac{2}{s+3} & \dfrac{1}{s+2} - \dfrac{1}{s+3} \\ -\dfrac{6}{s+2} + \dfrac{6}{s+3} & -\dfrac{2}{s+2} + \dfrac{3}{s+3} \end{bmatrix} \qquad (76)$$

By taking the inverse transform of each element of $\Phi(s)$, we find the state-transition matrix to be

$$\phi(t) = \begin{bmatrix} 3\epsilon^{-2t} - 2\epsilon^{-3t} & \epsilon^{-2t} - \epsilon^{-3t} \\ -6\epsilon^{-2t} + 6\epsilon^{-3t} & -2\epsilon^{-2t} + 3\epsilon^{-3t} \end{bmatrix} \qquad (77)$$

The state-transition matrix corresponding to the matrix $\mathbf{A}$ used in the last example was also obtained in Example 6.10, where the solution was carried out entirely in the time domain. The expression for $\phi(t)$ in (77) is identical to that for $\epsilon^{\mathbf{A}t}$ in (6.73). A number of important properties of the state-transition matrix were discussed in Section 6.6, including the fact that $\phi(t) = \epsilon^{\mathbf{A}t}$.

Once $\phi(t)$ is known, the state vector $\mathbf{q}(t)$ can be found from (70) in terms of the initial state vector $\mathbf{q}(0)$. Each of the state variables $q_i(t)$ will be a linear combination of the system's mode functions, with the relative weightings dependent on the initial state vector $\mathbf{q}(0)$ and on the partial fraction coefficients $\gamma_1, \gamma_2, \ldots, \gamma_n$ in (74). The output vector $\mathbf{y}(t)$ then follows directly from (65b). This procedure was illustrated in some detail in Example 6.12.

Solution for the Zero-State Response

Having analyzed the zero-input response of a fixed linear system described in state-variable form, we now consider the **zero-state response** for which $\mathbf{q}(0) = \mathbf{0}$ but the input $\mathbf{u}$ is not zero. The system model is given by (64), which is repeated here:

$$\dot{\mathbf{q}} = \mathbf{Aq} + \mathbf{Bu} \tag{78a}$$

$$\mathbf{y} = \mathbf{Cq} + \mathbf{Du} \tag{78b}$$

If (78a) is transformed with $\mathbf{q}(0) = \mathbf{0}$, we get

$$s\mathbf{Q}(s) = \mathbf{AQ}(s) + \mathbf{BU}(s)$$

where $\mathbf{U}(s) = \mathscr{L}[\mathbf{u}(t)]$. As we did for the zero-input response, we solve the transformed equation for $\mathbf{Q}(s)$, obtaining

$$\begin{aligned}\mathbf{Q}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{BU}(s) \\ &= \mathbf{\Phi}(s)\mathbf{BU}(s)\end{aligned} \tag{79}$$

where $(s\mathbf{I} - \mathbf{A})^{-1}$ is the resolvent matrix $\mathbf{\Phi}(s)$. Hence we can write the zero-state response of $\mathbf{q}$ as

$$\mathbf{q}(t) = \mathscr{L}^{-1}[\mathbf{\Phi}(s)\mathbf{BU}(s)]$$

As we shall see, the zero-state response consists of a combination of the free response and the forced response. Rather than proceed further with the response of the state vector $\mathbf{q}$, we shall obtain $\mathbf{Y}(s)$, the transform of the output vector. Transforming the output equation (78b) and substituting (79) give

$$\begin{aligned}\mathbf{Y}(s) &= \mathbf{C\Phi}(s)\mathbf{BU}(s) + \mathbf{DU}(s) \\ &= [\mathbf{C\Phi}(s)\mathbf{B} + \mathbf{D}]\mathbf{U}(s)\end{aligned} \tag{80}$$

Recall that the system transfer function $H(s)$ was defined as the ratio $Y(s)/U(s)$, where $Y(s)$ is the transform of the zero-state response. As a consequence of this definition, we could write $Y(s) = H(s)U(s)$. When dealing with matrices as we are here, we cannot divide $Y(s)$ by $U(s)$. However, we can define the **transfer-function matrix** $\mathbf{H}(s)$ to be the quantity in brackets on the right side of (80), which when multiplied by $\mathbf{U}(s)$ gives $\mathbf{Y}(s)$.

Thus the transfer-function matrix is

$$\mathbf{H}(s) = \mathbf{C\Phi}(s)\mathbf{B} + \mathbf{D} \tag{81}$$

and the Laplace transform of the zero-state output vector is

$$\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{U}(s) \tag{82}$$

Often the output $\mathbf{y}$ is a function of only the state vector $\mathbf{q}$, in which case $\mathbf{D} = \mathbf{0}$ and (81) simplifies to

$$\mathbf{H}(s) = \mathbf{C\Phi}(s)\mathbf{B} \tag{83}$$

We now define the **impulse-response matrix** $\mathbf{h}(t)$ as the inverse Laplace transform of the transfer-function matrix $\mathbf{H}(s)$. Thus

$$\mathbf{h}(t) = \mathscr{L}^{-1}[\mathbf{H}(s)] \tag{84}$$

is a $p \times m$ matrix of time functions such that $h_{ij}(t)$ is the response of y_i when u_j is a unit impulse, when all other inputs are zero, and when the initial value of the state vector is $\mathbf{q}(0) = \mathbf{0}$.

▶ *EXAMPLE 8.19*

Evaluate the transfer-function matrix $\mathbf{H}(s)$ for the system described by the matrix equations

$$\begin{aligned}\dot{\mathbf{q}} &= \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}\mathbf{q} + \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{u} \\ \mathbf{y} &= \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\mathbf{q}\end{aligned} \tag{85}$$

Then use $\mathbf{H}(s)$ to find the zero-state response to the input vector

$$\mathbf{u} = \begin{bmatrix} \epsilon^{-t}U(t) \\ U(t) \end{bmatrix} \tag{86}$$

Solution

The system described by (85) has the same matrix $\mathbf{A}$ for which we found $\mathbf{\Phi}(s)$ in Example 8.18. Using (75) for $\mathbf{\Phi}(s)$, identifying $\mathbf{B}$ and $\mathbf{C}$ from (85), and noting that $\mathbf{D} = \mathbf{0}$, we can write (83) as

$$\begin{aligned}\mathbf{H}(s) &= \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \dfrac{s+5}{(s+2)(s+3)} & \dfrac{1}{(s+2)(s+3)} \\ \dfrac{-6}{(s+2)(s+3)} & \dfrac{s}{(s+2)(s+3)} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \dfrac{2s+9}{(s+2)(s+3)} & \dfrac{s+5}{(s+2)(s+3)} \\ \dfrac{-s-12}{(s+2)(s+3)} & \dfrac{-6}{(s+2)(s+3)} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{5s+30}{(s+2)(s+3)} & \dfrac{2s+16}{(s+2)(s+3)} \\ \dfrac{-s-12}{(s+2)(s+3)} & \dfrac{-6}{(s+2)(s+3)} \end{bmatrix}\end{aligned} \tag{87}$$

As indicated by (87), $\mathbf{H}(s)$ is a 2×2 matrix, each element of which is a transfer function having the same two poles (at $s_1 = -2$ and $s_2 = -3$) and zeros that depend on its position in the matrix.

To find $\mathbf{Y}(s)$, we transform the input vector defined by (86), getting

$$\mathbf{U}(s) = \begin{bmatrix} \dfrac{1}{s+1} \\ \dfrac{1}{s} \end{bmatrix} \tag{88}$$

and then use (82) and (87) to write

$$\mathbf{Y}(s) = \begin{bmatrix} \dfrac{5s+30}{(s+2)(s+3)} & \dfrac{2s+16}{(s+2)(s+3)} \\ \dfrac{-s-12}{(s+2)(s+3)} & \dfrac{-6}{(s+2)(s+3)} \end{bmatrix} \begin{bmatrix} \dfrac{1}{s+1} \\ \dfrac{1}{s} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{7s^2+48s+16}{s(s+1)(s+2)(s+3)} \\ \dfrac{-(s^2+18s+6)}{s(s+1)(s+2)(s+3)} \end{bmatrix}$$

We can now evaluate the zero-state responses of the two outputs y_1 and y_2 by performing a partial-fraction expansion of each element of $\mathbf{Y}(s)$. You can verify that the expansions of the two elements of $\mathbf{Y}(s)$ are

$$Y_1(s) = \frac{8/3}{s} + \frac{25/2}{s+1} - \frac{26}{s+2} + \frac{65/6}{s+3}$$

$$Y_2(s) = -\frac{1}{s} - \frac{11/2}{s+1} + \frac{13}{s+2} - \frac{13/2}{s+3}$$

Thus the zero-state response of the output vector is

$$\mathbf{y}(t) = \begin{bmatrix} \dfrac{8}{3} + \dfrac{25}{2}\epsilon^{-t} - 26\epsilon^{-2t} + \dfrac{65}{6}\epsilon^{-3t} \\ -1 - \dfrac{11}{2}\epsilon^{-t} + 13\epsilon^{-2t} - \dfrac{13}{2}\epsilon^{-3t} \end{bmatrix} \tag{89}$$

which holds for $t > 0$. Actually, (89) is valid for $t \geq 0$, because $\mathbf{y}(0) = \mathbf{y}(0+) = \mathbf{0}$.

Solution for the Complete Response

By the superposition property, the response of a linear system to a nonzero initial state vector $\mathbf{q}(0)$ and a nonzero input vector $\mathbf{u}$ is the sum of the zero-input and zero-state responses. It is known as the **complete response.** As an alternative to finding the zero-input and zero-state responses as functions of time and adding them, we can add their respective Laplace transforms to give the transform of the complete response. We shall develop this approach for the case when $\mathbf{D} = \mathbf{0}$ and then apply the result to the system considered in Example 8.19.

The transform of the zero-input state vector is given by (68) as

$$\mathbf{Q}(s) = \mathbf{\Phi}(s)\mathbf{q}(0)$$

With $\mathbf{D} = \mathbf{0}$, the output vector is

$$\mathbf{y} = \mathbf{Cq}$$

so the transform of the zero-input response, denoted here by $\mathbf{Y}_{zi}(s)$, is

$$\mathbf{Y}_{zi}(s) = \mathbf{C\Phi}(s)\mathbf{q}(0) \tag{90}$$

For the transform of the zero-state response, denoted here by $\mathbf{Y}_{zs}(s)$, we can use (82) and (83) to write

$$\mathbf{Y}_{zs}(s) = \mathbf{C\Phi}(s)\mathbf{BU}(s) \tag{91}$$

Because of the superposition property for linear systems and the superposition theorem for Laplace transforms, the transform of the complete response is

$$\begin{aligned} \mathbf{Y}(s) &= \mathbf{Y}_{zi}(s) + \mathbf{Y}_{zs}(s) \\ &= \mathbf{C\Phi}(s)\mathbf{q}(0) + \mathbf{C\Phi}(s)\mathbf{BU}(s) \\ &= \mathbf{C\Phi}(s)[\mathbf{q}(0) + \mathbf{BU}(s)] \end{aligned} \tag{92}$$

Having found $\mathbf{Y}(s)$ in numerical form by using (92), we can take its inverse Laplace transform element by element.

▶ *EXAMPLE 8.20*

Evaluate $\mathbf{y}$ for the system defined in Example 8.19 for the initial state vector

$$\mathbf{q}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and the input vector

$$\mathbf{u} = \begin{bmatrix} \epsilon^{-t}U(t) \\ U(t) \end{bmatrix}$$

Solution

Although the zero-input response of this system was found in general terms in Example 6.12, and the zero-state response for the specified input was found in Example 8.19, we shall use (92) to find $\mathbf{Y}(s)$ and take its inverse transform. You can use the functions of time found in Example 6.12 and Example 8.19 to verify that the result obtained here is correct.

Using (75) for $\mathbf{\Phi}(s)$ and transforming $\mathbf{u}$, we can substitute the appropriate matrices into (92) to write

$$\mathbf{Y}(s) = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{s+5}{(s+2)(s+3)} & \dfrac{1}{(s+2)(s+3)} \\ \dfrac{-6}{(s+2)(s+3)} & \dfrac{s}{(s+2)(s+3)} \end{bmatrix}$$

$$\times \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \dfrac{1}{s+1} \\ \dfrac{1}{s} \end{bmatrix} \right\}$$

$$= \begin{bmatrix} \dfrac{2s+16}{(s+2)(s+3)} & \dfrac{-s+2}{(s+2)(s+3)} \\ \dfrac{-6}{(s+2)(s+3)} & \dfrac{s}{(s+2)(s+3)} \end{bmatrix} \begin{bmatrix} \dfrac{-s^2+2s+1}{s(s+1)} \\ \dfrac{s^2}{s(s+1)} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{-3s^3 - 10s^2 + 34s + 16}{s(s+1)(s+2)(s+3)} \\ \dfrac{s^3 + 6s^2 - 12s - 6}{s(s+1)(s+2)(s+3)} \end{bmatrix} \tag{93}$$

Using partial-fraction expansions of the two elements of $\mathbf{Y}(s)$ leads to the complete response $\mathbf{y}$:

$$\mathbf{y}(t) = \begin{bmatrix} \frac{8}{3} + \frac{25}{2}\epsilon^{-t} - 34\epsilon^{-2t} + \frac{95}{6}\epsilon^{-3t} \\ -1 - \frac{11}{2}\epsilon^{-t} + 17\epsilon^{-2t} - \frac{19}{2}\epsilon^{-3t} \end{bmatrix} \tag{94}$$

This result is valid for $t > 0$. If we set $t = 0$ in (94), we get -3 and 1 for the two elements of $\mathbf{y}$, which is also the result of the matrix product $\mathbf{Cq}(0)$. The fact that these results are the same indicates that, in this example, $\mathbf{y}$ is continuous at $t = 0$.

In closing, it is worthwhile to comment on the computational task of evaluating the response of a multi-input, multi-output system. The second-order model used in the examples in this section is simple enough to permit us to evaluate the response in analytical form by hand. To do the same operations by hand for a third- or higher-order system would probably not be feasible. In such cases, we must usually resort to the digital computer. A direct solution can apply the methods described here or other algorithms that are better suited for computation. Alternatively, we can obtain a numerical simulation of the system's differential equation by using the methods presented in Chapter 15.

SUMMARY

The transform of the zero-input response has the form $Y(s) = F(s)/P(s)$, where $F(s)$ is a polynomial that depends on the initial conditions. The mode functions, which characterize the form of the zero-input response, are determined from the roots of the characteristic equation $P(s) = 0$. The

weighting of the mode functions depends on the initial energy in each of the energy-storing elements.

When the initial stored energy is zero, the transform of the output is given by $Y(s) = H(s)U(s)$, where $H(s)$ is the transfer function and $U(s)$ is the transform of the input. The transfer function can be written down by inspection of the input-output differential equation, can be found by transforming the system equations with the initial condition terms set equal to zero, or (in the case of electrical systems) can be found by using impedances. The poles and zeros are the values of s for which $H(s)$ becomes infinite or vanishes, respectively. The poles of $H(s)$ are identical to the roots of the characteristic equation $P(s) = 0$.

Two important special cases of the zero-state response are the unit impulse response $h(t)$ and the unit step response $y_U(t)$. Because they are found when the system is initially at rest, they are both zero for $t \le 0$. For $t > 0$, $h(t) = \mathscr{L}^{-1}[H(s)]$ and $y_U(t) = \mathscr{L}^{-1}[H(s)/s]$.

For a stable system with an input that does not decay to zero, the steady-state response does not depend on the initial conditions. For the steady-state response to sinusoidal inputs, we can use the frequency-response function $H(j\omega)$. If $M(\omega)$ and $\theta(\omega)$ denote the magnitude and angle of $H(j\omega)$, then the steady-state response to $u(t) = B\sin(\omega t + \phi_1)$ is $y_{ss}(t) = BM\sin(\omega t + \phi_1 + \theta)$.

The time-domain solution of the matrix form of the state-variable model was included in Chapter 6. The corresponding transform solution is discussed in the final section of this chapter.

PROBLEMS

8.1 **a)** Repeat Example 8.1 for the parameter values $J = 1$ kg·m^2, $B = 4$ N·m·s/rad, and $K = 4$ N·m/rad and for no applied torque.
b) Identify the mode functions. For each mode function, give the restrictions on the initial conditions needed to eliminate it from the zero-input response.

* **8.2** Repeat Example 8.3 for the initial conditions $\theta_1(0) = \theta_2(0) = 0$ and $\dot{\theta}_1(0) = \dot{\theta}_2(0) = 0.5$ rad/s.

8.3 Use the expression for $E_o(s)$ in the statement of Problem 7.29 to obtain the zero-input response of the circuit shown in Figure P7.29 when $R = 2/9\ \Omega$. Identify the mode functions. For each mode function, give the restrictions on the initial conditions needed to eliminate it from the zero-input response.

8.4 Repeat Problem 8.3 when $R = 2/7\ \Omega$.

* **8.5** Repeat Problem 8.3 when $R = 2/3\ \Omega$.

8.6 **a)** Find the transfer function $H(s)$ for the circuit shown in Figure P7.29.
b) Find expressions for the damping ratio ζ and the undamped natural frequency ω_n in terms of the resistance R.
c) Find the zero-state response to the input $e_i(t) = 1 + \epsilon^{-2t}$ for $t > 0$ when $R = 2/7\ \Omega$. Explain why there is no steady-state response even though there

is a constant term in the input.

*** 8.7** Find the transfer function $X_R(s)/F_a(s)$ for the third-order mechanical system modeled in Example 3.5.

8.8 **a)** For the translational mechanical system described in Problem 2.4, let $M_1 = M_2 = B_1 = B_2 = B_3 = K_1 = K_2 = K_3 = 1$ in a consistent set of units. Verify that the system can be modeled by the pair of differential equations

$$\ddot{x}_1 + 2\dot{x}_1 + 2x_1 - \dot{x}_2 - x_2 = f_a(t)$$

$$-\dot{x}_1 - x_1 + \ddot{x}_2 + 2\dot{x}_2 + 2x_2 = 0$$

b) Find the transfer function when the output is x_1.
c) Repeat part (b) when the output is x_2. Comment on the similarities and dissimilarities between the two transfer functions.

8.9 **a)** Find the transfer function for the translational system modeled in Example 2.6 when the output is z_1, the displacement of the frame M_1 relative to its equilibrium position when the applied force is zero. Take the parameter values to be $M_1 = M_2 = B = K_1 = K_2 = K_3 = 1$ in a consistent set of units.
b) Repeat part (a) when the output is z_2.

*** 8.10** Find the transfer function of the rotational mechanical system modeled in Example 4.11 when the output is x.

8.11 Find the transfer function of the mechanical system modeled in Example 4.12, which has both translational and rotational elements, when the output is z.

*** 8.12** **a)** Find the Laplace transform of the zero-input response of the circuit modeled in Example 5.9.
b) Find the transfer function $E_o(s)/E_i(s)$.

8.13 **a)** Plot the pole-zero pattern for the transfer function

$$H(s) = \frac{s+1}{s^2+5s+6}$$

b) Write the general form of the zero-input response and find the system's input-output differential equation.
c) Use the initial-value and final-value theorems, if they are applicable, to determine the values of the unit step response at $t = 0+$ and when t approaches infinity.
d) Find the unit step response $y_U(t)$ and the unit impulse response $h(t)$ using $H(s)$.

*** 8.14** Repeat Problem 8.13 for

$$H(s) = \frac{s^2+2s+2}{s^2+4s+4}$$

8.15 Repeat Problem 8.13 for

$$H(s) = \frac{s+3}{s^3+7s^2+10s}$$

8.16 Repeat Problem 8.13 for

$$H(s) = \frac{12}{s(s^2+2s+4)}$$

*** 8.17** Repeat Problem 8.13 for

$$H(s) = \frac{s^2}{(2s+1)(s^2+4)}$$

8.18 Solve the differential equation in Problem 6.23 for the unit impulse response and the unit step response by using (38) and (41). Sketch both responses on the same axes.

8.19 Repeat Problem 8.18 for the differential equation in Problem 6.25.

*** 8.20** Repeat Problem 8.18 for the differential equation in Problem 6.24.

8.21 **a)** Find the transfer function for a system that obeys the equation

$$\ddot{y} + 4\dot{y} + 4y = u(t)$$

and use it to evaluate the unit step response.
b) Differentiate the answer to part (a) and compare the result to the unit impulse response found in Example 7.9.

8.22 Find the zero-state response of a system described by the transfer function in Problem 8.13 to the input te^{-t} for $t > 0$. Identify the free response $y_H(t)$, the forced response $y_P(t)$, and the transient terms.

*** 8.23** **a)** Verify that the transfer function for the circuit shown in Figure P8.23 is

$$H(s) = \frac{s^2+2s+1}{s^2+4s+4}$$

b) Find the unit impulse response.
c) Find the unit step response.

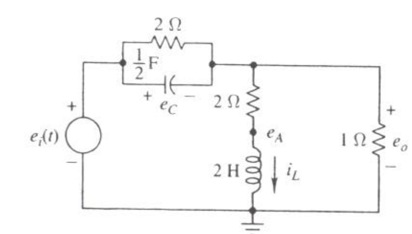

FIGURE P8.23

8.24 By a derivation similar to the one used to obtain (51), show that the steady-state response of a stable system to the input $u(t) = B\cos\omega t$ is $y_{ss}(t) = BM\cos(\omega t + \theta)$, where M and θ are given by (49).

8.25 Derive the expression given in (52) for the steady-state response to $u(t) = B\sin(\omega t + \phi_1)$.

8.26 For each of the following transfer functions, plot the pole-zero pattern, draw curves of $M(\omega)$ versus ω and $\theta(\omega)$ versus ω, and comment briefly on your results.

For the function in part (c), include the numerical values for $\omega = 9.9$, 10.0, and 10.1 rad/s.

a) $H(s) = \dfrac{2}{s^2 + 2s + 1}$

b) $H(s) = \dfrac{2s^2}{s^2 + 2s + 1}$

c) $H(s) = \dfrac{s}{s^2 + 0.2s + 100}$

*** 8.27** Repeat Problem 8.26 for the following transfer functions.

a) $H(s) = \dfrac{s-1}{s+1}$

b) $H(s) = \dfrac{(s+1)^2}{s(s+5)}$

c) $H(s) = \dfrac{s}{(s^2 + 0.2s + 100)^2}$

8.28 For the differential equation given in Example 6.1, find $H(j\omega)$ and $H(0)$. Also use (53) to verify the steady-state response to each of the inputs given in the example.

*** 8.29** The steady-state response of a stable system to the input

$$u(t) = \sin 5t + \sin 10t + \sin 15t$$

has the form

$$y_{ss}(t) = A\sin(5t + \theta_1) + B\sin(10t + \theta_2) + C\sin(15t + \theta_3)$$

For a system described by the transfer function in part (c) of Problem 8.26, use (51) to find the values of A, B, and C, and calculate the ratios A/B and C/B.

8.30 The steady-state response of a system described by the transfer function in part (a) of Problem 8.26 to the input

$$u(t) = 1 + \sin t + \sin 10t$$

has the form

$$y_{ss}(t) = A + B\sin(t + \theta_2) + C\sin(10t + \theta_3)$$

Calculate A, B, and C, and comment on the ratios B/A and C/A.

*** 8.31** **a)** For the series RLC circuit shown in Figure 5.13, find the transfer function $H(s) = I(s)/E_i(s)$.
b) By examining $H(j\omega)$, determine the value of ω such that the steady-state response to $e_i(t) = \sin\omega t$ is $i_{ss}(t) = (1/R)\sin\omega t$.

8.32 **a)** Find the transfer function $H(s) = E_o(s)/I_i(s)$ for the parallel RLC circuit shown in Figure 5.14.
b) By examining $H(j\omega)$, determine the value of ω for which the steady-state response to $i_i(t) = \sin\omega t$ is $e_o(t) = R\sin\omega t$.

8.33 **a)** Verify that when $\alpha \le 1$, the transfer function

$$H(s) = \frac{K}{s^2 + 2\alpha s + 100}$$

can be approximated by

$$H^*(s) = \frac{K}{(s + \alpha - j10)(s + \alpha + j10)}$$

b) Plot the pole-zero pattern for $H^*(s)$.
c) Using $H^*(j\omega)$, sketch to scale a curve of $M(\omega)/K$ versus ω, where M is defined by (49), when $\alpha = 0.2$ and $\alpha = 1.0$.
d) The transfer function $H^*(s)$ represents a band-pass filter having a center frequency of 10 rad/s and a bandwidth of 2α. Explain the appropriateness of these names in relation to your answers to part (c).
e) Discuss the behavior of $M(\omega)$ as α approaches zero.
f) Show that the circuit in Figure 5.13 has $H(s)$ as its transfer function with $K = 100$ if the output is e_C. Find the value of α in terms of the circuit elements.

8.34 Find the transfer function for the circuit shown in Figure 8.12 by characterizing the passive elements by their impedances.

*** 8.35** **a)** Draw the s-domain version of the circuit shown in Figure P8.23 using impedances.
b) Using the circuit drawn in part (a), determine the transfer function $E_o(s)/E_i(s)$.

8.36 **a)** Draw the s-domain circuit for Figure P5.1, with the passive elements characterized by their impedances.
b) Use the result of part (a) to determine the circuit's transfer function.
c) From the answer to part (b), write the input-output differential equation.

8.37 Repeat Problem 8.36 for the circuit shown in Figure P5.6.

*** 8.38** Repeat Problem 8.36 for the circuit shown in Figure P5.12.

8.39 **a)** Use impedances and the results of Example 8.16 to find the transfer function for the op-amp circuit shown in Figure P8.39.
b) Plot the pole-zero pattern.

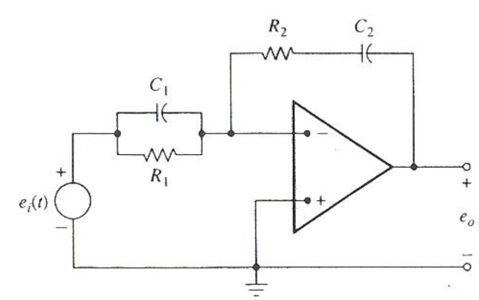

FIGURE P8.39

8.40 Repeat Problem 8.39 for the circuit shown in Figure P5.37.

In Problems 8.41 through 8.44, a system obeying the state-variable equation $\dot{\mathbf{q}} = \mathbf{A}\mathbf{q}$ has the given matrix $\mathbf{A}$.