C H A P T E R 7

THE LAPLACE TRANSFORM

Frequently, we can transform a mathematical problem that is difficult to solve as it stands into an equivalent problem whose solution is much easier. If the solution of the transformed problem can be converted back to the framework of the original problem, then going through the process of transformation, solution, and conversion of the solution may be more attractive than attempting to solve the original problem directly.

A familiar example of such a process is the use of logarithms. Logarithms transform numbers into other numbers, but there are also procedures for transforming functions of one variable into functions of another variable. Among these procedures are the Fourier series, the Fourier transform, and the Laplace transform. Because the Laplace transform converts fixed linear differential equations into algebraic equations, it is of considerable value in analyzing the response of a dynamic system whose model is fixed and linear.

In this chapter, we first define the Laplace transform and derive the transforms of several common functions of time. We then develop properties that are useful in finding the transforms of specific time functions and in transforming the equations that describe dynamic models. We complete the initial mathematical development by showing how to convert a transform back to the corresponding time function. In the next section, we use Laplace transforms to solve for the response of a variety of models. We conclude the chapter by introducing and illustrating several additional transform properties.

7.1 TRANSFORMS OF FUNCTIONS

The **Laplace transform** converts a function of a real variable, which will always be time t in our applications, into a function of a complex variable that is denoted by s. The transform of $f(t)$ is represented symbolically by either $\mathscr{L}[f(t)]$ or $F(s)$, where the symbol $\mathscr{L}$ stands for "the Laplace transform of." One can think of the Laplace transform as providing a means of transforming a given problem from the **time domain**, where all variables are functions of t, to the **complex-frequency domain**, where all variables are functions of s.

The defining equation for the Laplace transform[1] is

$$F(s) = \int_0^{\infty} f(t)\epsilon^{-st}dt \tag{1}$$

Some authors take the lower limit of integration at $t = 0-$ rather than at $t = 0$. Either convention is acceptable provided that the transform properties are developed in a manner consistent with the defining integral.

The integration in (1) with respect to t is carried out between the limits of zero and infinity, so the resulting transform is not a function of t. In the factor ϵ^{-st} appearing in the integrand, we treat s as a constant in carrying out the integration.

The variable s is a complex quantity, which we can write as

$$s = \sigma + j\omega$$

where σ and ω are the real and imaginary parts of s, respectively. We can write the factor ϵ^{-st} in (1) as

$$\epsilon^{-st} = \epsilon^{-\sigma t}\epsilon^{-j\omega t}$$

Because the magnitude of $\epsilon^{-j\omega t}$ is always unity, $|\epsilon^{-st}| = \epsilon^{-\sigma t}$. By placing appropriate restrictions on σ, we can ensure that the the integral converges in most cases of practical interest, even if the function $f(t)$ becomes infinite as t approaches infinity. For all time functions considered in this book, convergence of the transform integral can be achieved.

To gain familiarity with the use of the transform definition and to begin the development of a table of transforms, we shall derive the Laplace transforms of several common functions. The results, along with others, are included in Appendix B.

[1] Equation (1) defines the one-sided Laplace transform. There is a more general, two-sided Laplace transform, which is useful for theoretical work but is seldom used for solving for system responses.

Step Function

The unit step function $U(t)$ is unity for all $t > 0$, so it follows from (1) that

$$\mathscr{L}[U(t)] = \int_0^\infty \epsilon^{-st}\,dt$$

$$= \left.\frac{\epsilon^{-st}}{-s}\right|_{t=0}^{t\to\infty} \tag{2}$$

In order for the integral to converge, $|\epsilon^{-st}|$ must approach zero as t approaches infinity. Because $|\epsilon^{-st}| = \epsilon^{-\sigma t}$, the integral will converge provided that $\sigma > 0$. Hence the expression for the transform of the unit step function converges for all values of s in the right half of the complex plane, and (2) becomes

$$\mathscr{L}[U(t)] = \frac{1}{s} \tag{3}$$

Note that the variable t has disappeared in the integration process and that the result is strictly a function of s.

For all functions of time with which we shall be concerned, the transform definition will converge for all values of s to the right of some vertical line in the complex s-plane. Although knowledge of the region of convergence is required in some advanced applications, we do not need such knowledge for the applications in this book. When evaluating Laplace transforms, we shall assume that σ, the real part of s, is sufficiently large to ensure convergence. If you are interested in regions of convergence, you should consult one of the more advanced references on system theory cited in Appendix D.

Because $U(t) = 1$ for all $t > 0$, the constant 1 is equivalent to $U(t)$ over the interval $t > 0$ and has the same transform as the unit step function. Thus

$$\mathscr{L}[1] = \frac{1}{s} \tag{4}$$

Exponential Function

Depending on the value of the parameter a, the function $f(t) = \epsilon^{-at}$ represents an exponentially decaying function, a constant, or an exponentially growing function for $t > 0$, as shown in Figure 7.1. In any case, the Laplace transform of the exponential function is

$$\mathscr{L}[\epsilon^{-at}] = \int_0^\infty \epsilon^{-at}\epsilon^{-st}\,dt$$

$$= \int_0^\infty \epsilon^{-(s+a)t}\,dt$$

$$= \left.\frac{\epsilon^{-(s+a)t}}{-(s+a)}\right|_{t=0}^{t\to\infty}$$

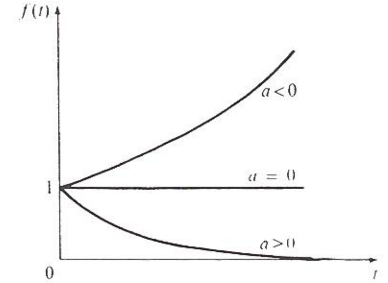


FIGURE 7.1 The exponential function ϵ^{-at} for various values of a.

The upper limit vanishes when σ, the real part of s, is greater than $-a$, so

$$\mathscr{L}[\epsilon^{-at}] = \frac{1}{s+a} \tag{5}$$

Note that if $a = 0$, the exponential function ϵ^{-at} reduces to the constant value of 1. Likewise, (5) reduces to (4), as it must for this value of a.

Ramp Function

The unit ramp function is defined to be $f(t) = t$ for $t > 0$. Substituting for $f(t)$ in (1), we have

$$\mathscr{L}[t] = \int_0^\infty t\epsilon^{-st}\,dt \tag{6}$$

To evaluate the integral in (6), we use the formula for integration by parts:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du \tag{7}$$

where the limits a and b apply to the variable t. Making the identifications $u = t$ and $v = \epsilon^{-st}/(-s)$, $a = 0$, and $b \to \infty$, we can rewrite (6) as

$$\mathscr{L}[t] = \left.\frac{t\epsilon^{-st}}{(-s)}\right|_0^\infty - \frac{1}{(-s)}\int_0^\infty \epsilon^{-st}\,dt \tag{8}$$

It can be shown that

$$\lim_{t\to\infty} t\epsilon^{-st} = 0 \tag{9}$$

provided that σ, the real part of s, is positive. Thus

$$\mathscr{L}[t] = 0 - 0 + \frac{1}{s}\int_0^\infty \epsilon^{-st}\,dt \tag{10}$$

Because the integral in (10) is $\mathscr{L}[U(t)]$, which from (3) has the value $1/s$, (10) simplifies to

$$\mathscr{L}[t] = \frac{1}{s^2} \tag{11}$$

Rectangular Pulse

The rectangular pulse shown in Figure 7.2 has unit height and a duration of L. Its Laplace transform is

$$\begin{aligned} F(s) &= \int_0^L \epsilon^{-st} dt + \int_L^\infty 0\,\epsilon^{-st} dt \\ &= \frac{1}{(-s)} \epsilon^{-st} \Big|_0^L + 0 \\ &= \frac{1}{s}(1 - \epsilon^{-sL}) \end{aligned} \tag{12}$$

As the pulse duration L becomes infinite, the pulse approaches the unit step function $U(t)$, and the term ϵ^{-sL} in (12) approaches zero for all values of s having a positive real part. Thus as L approaches infinity, $F(s)$ given by (12) approaches $1/s$, which is $\mathscr{L}[U(t)]$.

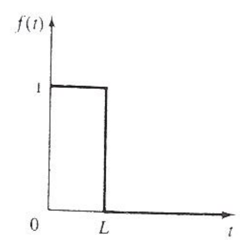


FIGURE 7.2 The rectangular pulse of unit height and duration L.

The Impulse

From (1), we can write the transform of the unit impulse as

$$\mathscr{L}[\delta(t)] = \int_0^\infty \delta(t)\epsilon^{-st} dt$$

This expression presents a dilemma because when the impulse was introduced in Section 6.3, we adopted the convention that $\delta(t)$ is an impulse at $t = 0$, which is the lower limit of integration in (1). To avoid this problem, we shall now adopt the convention that when we are working with Laplace transforms, the unit impulse $\delta(t)$ occurs just after $t = 0$, namely at $t = 0+$.

With this stipulation, (6.39) is applicable with $t_a = 0+$, $b = 0$, and $c \to \infty$, and we can say that

$$\int_0^\infty \delta(t)\epsilon^{-st} dt = \epsilon^{-st}\Big|_{t=0+} = 1$$

so

$$\mathscr{L}[\delta(t)] = 1 \tag{13}$$

In books wherein the lower limit of the transform integral in (1) is taken to be $t = 0-$, the impulse can be considered to occur at $t = 0$. This convention also yields (13).

Trigonometric Functions

When $f(t)$ is replaced by $\sin \omega t$ or $\cos \omega t$ in the definition of the Laplace transform, we can evaluate the resulting integral by using the identities given in Table 6.1, by performing integration by parts, or by consulting a table of integrals. Using an integral table, we find that

$$\begin{aligned} \mathscr{L}[\sin \omega t] &= \int_0^\infty \sin \omega t\ \epsilon^{-st} dt \\ &= \frac{\epsilon^{-st}}{(-s)^2 + \omega^2}(-s \sin \omega t - \omega \cos \omega t)\Big|_0^\infty \\ &= \frac{\omega}{s^2 + \omega^2} \end{aligned} \tag{14}$$

By a similar procedure, we can show that

$$\mathscr{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2} \tag{15}$$

7.2 TRANSFORM PROPERTIES

The Laplace transform has a number of properties that are useful in finding the transforms of functions in terms of known transforms and in solving for the responses of dynamic models. We shall state, illustrate, and in most cases derive those properties that will be useful in later work. They are tabulated in Appendix B.

Throughout this section, the symbols $F(s)$ and $G(s)$ denote the Laplace transforms of the arbitrary time functions $f(t)$ and $g(t)$, and a and b denote arbitrary constants. Our object is to express the transforms of various functions of $f(t)$ and $g(t)$ in terms of $F(s) = \int_0^\infty f(t)\epsilon^{-st} dt$ and $G(s) = \int_0^\infty g(t)\epsilon^{-st} dt$.

Multiplication by a Constant

To express $\mathcal{L}[af(t)]$ in terms of $F(s)$, where a is a constant and where $F(s) = \mathcal{L}[f(t)]$, we use (1) to write

$$\begin{aligned}\mathcal{L}[af(t)] &= \int_0^\infty af(t)\epsilon^{-st}dt \\ &= a\int_0^\infty f(t)\epsilon^{-st}dt \\ &= aF(s) \end{aligned} \tag{16}$$

Thus multiplying a function of time by a constant multiplies its transform by the same constant.

Superposition

The transform of the sum of the two functions $f(t)$ and $g(t)$ is

$$\begin{aligned}\mathcal{L}[f(t)+g(t)] &= \int_0^\infty [f(t)+g(t)]\epsilon^{-st}dt \\ &= \int_0^\infty f(t)\epsilon^{-st}dt + \int_0^\infty g(t)\epsilon^{-st}dt \\ &= F(s)+G(s) \end{aligned} \tag{17}$$

Using (16) and (17), we have the general superposition property

$$\mathcal{L}[af(t)+bg(t)] = aF(s)+bG(s) \tag{18}$$

for any constants a and b and any transformable functions $f(t)$ and $g(t)$. As an illustration of the superposition property, we can evaluate $\mathcal{L}[2+3\sin 4t]$ by using (18) with (4) and (14) to write

$$\begin{aligned}\mathcal{L}[2+3\sin 4t] &= \frac{2}{s} + 3\left(\frac{4}{s^2+4^2}\right) \\ &= \frac{2s^2+12s+32}{s^3+16s}\end{aligned}$$

Multiplication by an Exponential

If we replace $f(t)$ in (1) by the function $f(t)\epsilon^{-at}$, we have

$$\begin{aligned}\mathcal{L}[f(t)\epsilon^{-at}] &= \int_0^\infty f(t)\epsilon^{-at}\epsilon^{-st}dt \\ &= \int_0^\infty f(t)\epsilon^{-(s+a)t}dt \\ &= F(s+a) \end{aligned} \tag{19}$$

In words, (19) states that multiplying a function $f(t)$ by ϵ^{-at} is equivalent to replacing the variable s by the quantity $s+a$ wherever it occurs in $F(s)$.

With this property, we can derive several of the transforms in Appendix B rather easily from other entries in the table. Specifically, because $\mathcal{L}[\cos\omega t] = s/(s^2+\omega^2)$ and $\mathcal{L}[\sin\omega t] = \omega/(s^2+\omega^2)$, we can write

$$\mathcal{L}[\epsilon^{-at}\cos\omega t] = \frac{s+a}{(s+a)^2+\omega^2}$$

$$\mathcal{L}[\epsilon^{-at}\sin\omega t] = \frac{\omega}{(s+a)^2+\omega^2}$$

Also, because $\mathcal{L}[t] = 1/s^2$, it follows that

$$\mathcal{L}[t\epsilon^{-at}] = \frac{1}{(s+a)^2}$$

Multiplication by Time

We obtain the transform of the product of $f(t)$ and the variable t by differentiating the transform $F(s)$ with respect to the complex variable s and then multiplying by -1:

$$\mathcal{L}[tf(t)] = -\frac{d}{ds}F(s) \tag{20}$$

To prove (20), we note that

$$\begin{aligned}\frac{d}{ds}F(s) &= \frac{d}{ds}\left[\int_0^\infty f(t)\epsilon^{-st}dt\right] \\ &= -\int_0^\infty tf(t)\epsilon^{-st}dt \\ &= -\mathcal{L}[tf(t)] \end{aligned} \tag{21}$$

Multiplying both sides of (21) by -1 results in (20).

We can illustrate the use of this property by deriving the entry in Appendix B for $\mathcal{L}[t^n]$, where n is any positive integer. Because $\mathcal{L}[1] = \mathcal{L}[U(t)] = 1/s$, it follows that

$$\mathcal{L}[t] = -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

$$\mathcal{L}[t^2] = -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2}{s^3}$$

$$\mathcal{L}[t^3] = -\frac{d}{ds}\left(\frac{2}{s^3}\right) = \frac{2\cdot 3}{s^4}$$

and, for the general case,

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \tag{22}$$

Differentiation

Because we shall need to take the Laplace transform of each term in a differential equation when solving system models for their responses, we must derive expressions for the transforms of derivatives of arbitrary order. We shall first develop and illustrate the formula for obtaining the transform of df/dt in terms of $F(s) = \mathcal{L}[f(t)]$. Then we shall use this result to derive expressions for the transforms of higher derivatives.

First Derivative. From the transform definition (1), we can write the transform of df/dt as

$$\mathcal{L}[\dot{f}] = \int_0^\infty \left(\frac{df}{dt}\right) \epsilon^{-st} dt \tag{23}$$

We can rewrite (23) by using the formula for integration by parts given by (7), with $a = 0$ and $b \to \infty$. The result is

$$\int_0^\infty u\, dv = uv\Big|_0^\infty - \int_0^\infty v\, du$$

If we let $u = \epsilon^{-st}$ and $dv = (df/dt)dt$, then $du = -s\epsilon^{-st}dt$, $v = f(t)$, and (23) becomes

$$\mathcal{L}[\dot{f}] = \epsilon^{-st} f(t)\Big|_0^\infty - \int_0^\infty f(t)(-s\epsilon^{-st})dt$$

For all the functions we shall encounter, there will be values of s for which $\epsilon^{-st} f(t)$ approaches zero as t approaches infinity, so

$$\begin{aligned} \mathcal{L}[\dot{f}] &= [0 - f(0)] + s\int_0^\infty f(t)\epsilon^{-st}dt \\ &= sF(s) - f(0) \end{aligned} \tag{24}$$

To illustrate the application of (24), let $f(t) = \sin \omega t$. Then $F(s) = \omega/(s^2 + \omega^2)$ from (14) and $f(0) = \sin 0 = 0$, so

$$\mathcal{L}[\dot{f}] = s\left(\frac{\omega}{s^2 + \omega^2}\right) - 0 = \frac{s\omega}{s^2 + \omega^2} \tag{25}$$

We can verify this result by noting that $\dot{f} = \omega \cos \omega t$ and that

$$\mathcal{L}[\omega \cos \omega t] = \omega \mathcal{L}[\cos \omega t] = \omega \left(\frac{s}{s^2 + \omega^2}\right)$$

which agrees with (25).

Deriving and using (24) are straightforward when $f(t)$ is continuous at $t = 0$. However, when $f(t)$ is discontinuous at $t = 0$, we must take care in applying (24) because of the potential ambiguity in evaluating $f(0)$. To avoid such problems, we shall adopt the convention that $f(0) = f(0-)$, which is equivalent to saying that any discontinuity at the time origin is considered to occur just after $t = 0$. In such cases $f(0) \neq f(0+)$, and

df/dt will contain an impulse of weight $f(0+) - f(0)$ occurring at $t = 0+$. This impulse is within the integration interval $0 \le t < \infty$ that we used for (1), and thus it will contribute to the expression for $\mathcal{L}[f]$. This approach is consistent with our definition of the unit step function in Chapter 6 as

$$U(t) = \begin{cases} 0 & \text{for } t \le 0 \\ 1 & \text{for } t > 0 \end{cases}$$

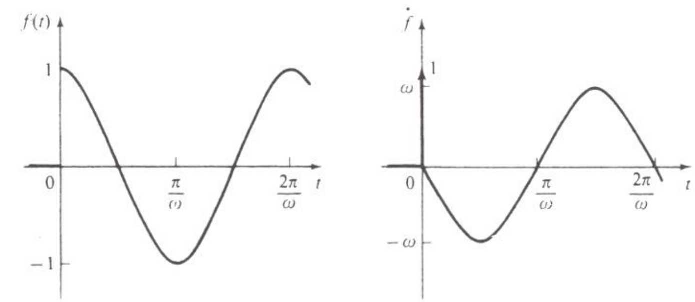
and with our derivation of the transform of $\delta(t)$.

To illustrate the application of (24) when $f(t)$ has a discontinuity at $t = 0+$, consider the product of $\cos \omega t$ and the unit step function $U(t)$:

$$\begin{aligned} f(t) &= [\cos \omega t]U(t) \\ &= \begin{cases} 0 & \text{for } t \le 0 \\ \cos \omega t & \text{for } t > 0 \end{cases} \end{aligned} \tag{26}$$

which is shown in Figure 7.3(a). Because $f(t)$ differs from $\cos \omega t$ only at the point $t = 0$ within the interval $0 \le t < \infty$, its transform is the same[2] as that of $\cos \omega t$:

$$F(s) = \frac{s}{s^2 + \omega^2} \tag{27}$$



(a) (b)

FIGURE 7.3 The function $f(t) = [\cos \omega t]U(t)$ and its derivative.

We see from (26), however, that $f(0) = 0$ because of the presence of the unit step function. Thus (24) gives

$$\begin{aligned} \mathcal{L}[\dot{f}] &= s\left(\frac{s}{s^2 + \omega^2}\right) - 0 \\ &= \frac{s^2}{s^2 + \omega^2} \end{aligned} \tag{28}$$

[2]The value of a definite integral, and hence the result of using (1), is not affected if the value of the integrand at a single point is changed to another finite value.

To check this result, consider $\dot{f}$, the derivative of (26), which is shown in Figure 7.3(b). Because of the discontinuity in $f(t)$ at $t = 0+$, the function $\dot{f}$ contains a unit impulse at $t = 0+$. Consistent with the facts that $\dot{f}(t) = 0$ for all $t < 0$ and that the impulse occurs at $t = 0+$, we define $\dot{f}(0) = 0$. Thus

$$\dot{f} = \begin{cases} 0 & \text{for } t \leq 0 \\ \delta(t) - \omega \sin \omega t & \text{for } t > 0 \end{cases} \tag{29}$$

Transforming (29) gives

$$\begin{aligned} \mathscr{L}[\dot{f}] &= 1 - \omega\left(\frac{\omega}{s^2 + \omega^2}\right) \\ &= \frac{s^2}{s^2 + \omega^2} \end{aligned}$$

which is in agreement with (28).

For comparison, consider the related function

$$g(t) = \cos \omega t \qquad \text{for all } t$$

which is continuous for all t and for which $g(0) = 1$. From (24), the transform of its derivative is

$$\begin{aligned} \mathscr{L}[\dot{g}] &= s\left(\frac{s}{s^2 + \omega^2}\right) - 1 \\ &= -\frac{\omega^2}{s^2 + \omega^2} \end{aligned}$$

The derivative of $g(t)$ is $\dot{g} = -\omega \sin \omega t$ for all t. Taking the transform of this expression gives $-\omega^2/(s^2+\omega^2)$, which agrees with the foregoing result.

In summary, whenever we use (24) we shall consider any discontinuities in $f(t)$ at the time origin to occur at $t = 0+$. We shall now develop formulas for finding the transforms of second and higher derivatives of $f(t)$ in terms of $F(s)$.

Second and Higher Derivatives. If we write (24) in terms of the function $g(t)$, it becomes

$$\mathscr{L}[\dot{g}] = sG(s) - g(0) \tag{30}$$

Now let $g(t) = \dot{f}(t)$. Then $\dot{g}(t) = \ddot{f}(t)$, and it follows from (30) and (24) that

$$\begin{aligned} \mathscr{L}[\ddot{f}] &= s\mathscr{L}[\dot{f}(t)] - \dot{f}(0) \\ &= s[sF(s) - f(0)] - \dot{f}(0) \\ &= s^2F(s) - sf(0) - \dot{f}(0) \end{aligned} \tag{31}$$

If $\dot{f}$ is discontinuous at the time origin, we must use for $\dot{f}(0)$ in (31) the limit of $\dot{f}$ as t approaches zero from the left. Consider, for example,

$$f(t) = [\sin \omega t]U(t)$$

which is shown in Figure 7.4(a). The first and second derivatives of this function are

$$\dot{f} = [\omega \cos \omega t]U(t) \tag{32a}$$

$$\ddot{f} = \omega\delta(t) - [\omega^2 \sin \omega t]U(t) \tag{32b}$$

which are shown in Figure 7.4(b) and Figure 7.4(c). To be consistent with our previous development, we consider the discontinuity in $\dot{f}$ to occur at $t = 0+$ and assume that $\dot{f}(0) = 0$. Then (31) gives

$$\begin{aligned} \mathscr{L}[\ddot{f}] &= s^2\left(\frac{\omega}{s^2 + \omega^2}\right) - s \cdot 0 - 0 \\ &= \frac{\omega s^2}{s^2 + \omega^2} \end{aligned}$$

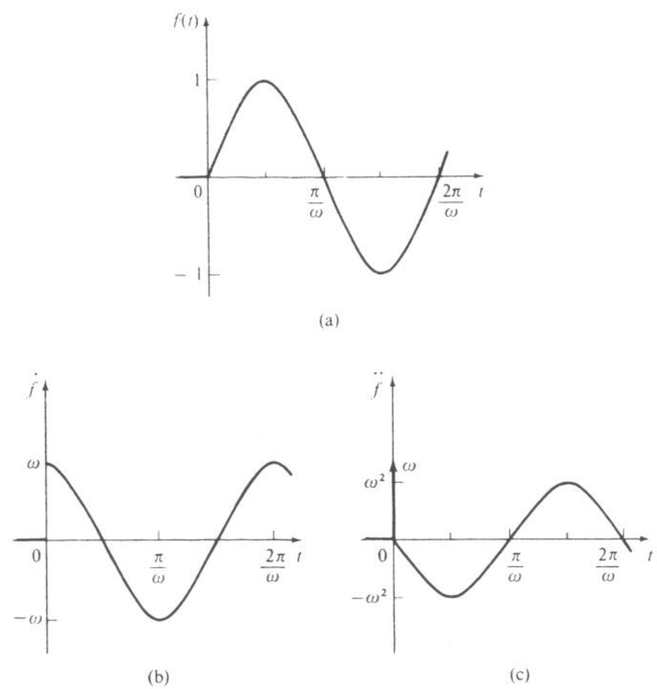


FIGURE 7.4 The function $f(t) = [\sin \omega t]U(t)$ and its first two derivatives.

By a direct transformation of the right side of (32b), we obtain

$$\mathcal{L}[\ddot{f}] = \omega - \omega^2\left(\frac{\omega}{s^2+\omega^2}\right)$$
$$= \frac{\omega s^2}{s^2+\omega^2}$$

which is the same result.

Equation (31) can be generalized to give the formula for the transform of the nth derivative of $f(t)$. The result is

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0) \tag{33}$$

where $f^{(n-1)}(0)$ denotes $d^{n-1}f/dt^{n-1}$ evaluated at $t = 0$. If any of the derivatives of $f(t)$ has a discontinuity at the time origin, we use its value at $t = 0-$ in the corresponding initial-condition term in (33).

Integration

The definite integral $\int_0^t f(\lambda)d\lambda$ will be a function of t because of the upper limit. From the transform definition given by (1), we can write

$$\mathcal{L}\left[\int_0^t f(\lambda)d\lambda\right] = \int_0^\infty \left[\int_0^t f(\lambda)d\lambda\right]\epsilon^{-st}dt$$

To evaluate the double integral on the right side of this expression, we use integration by parts with $u = \int_0^t f(\lambda)d\lambda$ and $dv = \epsilon^{-st}dt$. Then $du = f(t)dt$, $v = \epsilon^{-st}/(-s)$, and

$$\mathcal{L}\left[\int_0^t f(\lambda)d\lambda\right] = \left[\left(\frac{\epsilon^{-st}}{-s}\right)\int_0^t f(\lambda)d\lambda\right]\Bigg|_{t=0}^{t\to\infty} - \int_0^\infty \left(\frac{\epsilon^{-st}}{-s}\right) f(t)dt$$
$$= [0-0] + \frac{1}{s}\int_0^\infty f(t)\epsilon^{-st}dt$$
$$= \frac{1}{s}F(s) \tag{34}$$

For example, if $f(t) = \cos\omega t$, then $F(s) = s/(s^2+\omega^2)$, so

$$\mathcal{L}\left[\int_0^t \cos\omega\lambda \; d\lambda\right] = \frac{1}{s}\left(\frac{s}{s^2+\omega^2}\right)$$
$$= \frac{1}{s^2+\omega^2}$$

To check this result, we note that

$$\int_0^t \cos\omega\lambda\; d\lambda = \frac{1}{\omega}\sin\omega t$$

and

$$\mathcal{L}\left[\frac{1}{\omega}\sin\omega t\right] = \frac{1}{\omega}\left(\frac{\omega}{s^2+\omega^2}\right) = \frac{1}{s^2+\omega^2}$$

which agrees with the result we obtained by using (34).

When solving dynamic models, we may need to transform a term such as $g(t) = g(0) + \int_0^t f(\lambda)d\lambda$. To do this, we note that $g(0)$ is a constant and thus has the transform $g(0)/s$, whereas we can transform the integral part of the term by using (34). The result is

$$G(s) = \frac{g(0)}{s} + \frac{F(s)}{s} \tag{35}$$

7.3 TRANSFORM INVERSION

When we use Laplace transforms to solve for the response of a system, we find the transform $F(s)$ of a particular variable, such as the output, first. The final step in the process, known as **transform inversion**, is to determine the corresponding time function $f(t)$ where $f(t) = \mathcal{L}^{-1}[F(s)]$, which is read "$f(t)$ is the inverse transform of $F(s)$."

For the types of problems encountered in this book, we do not need a completely general method of transform inversion. In this section, we present and illustrate a method that uses a partial-fraction expansion of $F(s)$. An extension of this method that permits handling an additional category of transformed functions appears in Section 7.5.

Assume that we can write the transform $F(s)$ as the ratio of two polynomials $N(s)$ and $D(s)$, such that

$$F(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0} \tag{36}$$

Functions that can be written in the form of (36) are called **rational functions**. A **proper rational function** is one for which $m \le n$; while a **strictly proper rational function** is one for which $m < n$.

The equation $D(s) = 0$ will have n roots denoted by $s_1, s_2, \ldots, s_n$, and $D(s)$ can be written in factored form as

$$D(s) = (s-s_1)(s-s_2)\cdots(s-s_n) \tag{37}$$

Note that the coefficient of the highest power of s in the denominator polynomial $D(s)$ has been assumed to be unity. If this is not the case for a given $F(s)$, we can always make the coefficient unity by dividing both $N(s)$ and $D(s)$ by a constant. The quantities $s_1, s_2, \ldots, s_n$ are called the **poles** of $F(s)$ and are those values of s for which $F(s)$ becomes infinite.[3]

[3]It is assumed that $N(s) \neq 0$ at any of the poles.

The **method of partial-fraction expansion** is applicable to any strictly proper rational function. Briefly, it allows us to express a known transform $F(s)$ as the sum of less complicated transforms. Using the table in Appendix B, we can identify the time functions that correspond to the individual transforms in the expansion and then use the superposition theorem to write $f(t)$.

We consider first the case where all the poles of $F(s)$ are distinct. We next modify the procedure to include repeated poles, where two or more of the quantities $s_1, s_2, \ldots, s_n$ are equal. Then we examine the case where the poles are complex numbers and discuss what to do if the degree of $N(s)$ is not less than that of $D(s)$. Throughout the section, we assume that the polynomial $D(s)$ has been factored, so that the values of the n poles are known.

Distinct Poles

The partial-fraction expansion theorem states that if $F(s)$ is a strictly proper rational function with distinct poles, it can be written as

$$F(s) = \frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \cdots + \frac{A_n}{s-s_n} \tag{38}$$

where $A_1, A_2, \ldots, A_n$ are constants. We can write (38) with a summation sign as follows:

$$F(s) = \sum_{i=1}^{n} A_i \left(\frac{1}{s-s_i}\right) \tag{39}$$

From (5), the term $1/(s-s_i)$ is the transform of the time function $\epsilon^{s_i t}$. Then, from the superposition formula given in (18), it follows that for $t > 0$,

$$\begin{aligned} f(t) &= A_1\epsilon^{s_1 t} + A_2\epsilon^{s_2 t} + \cdots + A_n\epsilon^{s_n t} \\ &= \sum_{i=1}^{n} A_i\epsilon^{s_i t} \end{aligned} \tag{40}$$

We can find the n poles s_i by factoring the denominator of $F(s)$ or, more generally, by finding the roots of $D(s) = 0$. We shall now develop a procedure for evaluating the n coefficients A_i so that we can write $f(t)$ as a sum of exponential time functions by using (40).

Multiplying both sides of (38) by the term $(s - s_1)$ yields

$$(s-s_1)F(s) = A_1 + A_2\frac{(s-s_1)}{(s-s_2)} + \cdots + A_n\frac{(s-s_1)}{(s-s_n)}$$

Because this equation must be a mathematical identity for all values of the variable s, we can set s equal to s_1 throughout the equation. The poles are distinct, so $s_1 \neq s_j$ for $j = 2, 3, \ldots, n$. Thus each term on the right side

will vanish except the term A_1, and we can write

$$A_1 = (s-s_1)F(s)|_{s=s_1} \tag{41}$$

To make it clear why (41) does not give a value of zero, we write $F(s)$ as $N(s)/D(s)$ with $D(s)$ in factored form. Then

$$A_1 = \left.\frac{\cancel{(s-s_1)}N(s)}{\cancel{(s-s_1)}(s-s_2)\cdots(s-s_n)}\right|_{s=s_1}$$

The term $(s - s_1)$ in the numerator will be canceled by the corresponding term in $D(s)$ before s is replaced by s_1.

Repeating the above process with s_1 replaced by s_2, we have

$$A_2 = (s-s_2)F(s)|_{s=s_2}$$

The general expression for the coefficients is

$$A_i = (s-s_i)F(s)|_{s=s_i} \qquad i = 1, 2, \ldots, n \tag{42}$$

In a numerical problem we can check the calculation of $A_1, A_2, \ldots, A_n$ by combining the terms on the right side of (38) over a common denominator, which should give the original function $F(s)$.

▶ *EXAMPLE 7.1*

Find the inverse transform of

$$F(s) = \frac{-s+5}{(s+1)(s+4)}$$

Solution

Comparing $F(s)$ to (36), we see that $N(s) = -s + 5$ so $m = 1$, and $D(s) = (s+1)(s+4)$ so $n = 2$. Because $D(s)$ is already in factored form, we see by inspection that the poles are $s_1 = -1$ and $s_2 = -4$. Thus we can rewrite the transform $F(s)$ in the form of (38) as

$$F(s) = \frac{A_1}{s+1} + \frac{A_2}{s+4}$$

Using (42) with $i = 1$ and 2, we find the coefficients of the partial-fraction expansion to be

$$A_1 = \left.\frac{\cancel{(s+1)}(-s+5)}{\cancel{(s+1)}(s+4)}\right|_{s=-1} = \frac{6}{3} = 2$$

$$A_2 = \left.\frac{\cancel{(s+4)}(-s+5)}{(s+1)\cancel{(s+4)}}\right|_{s=-4} = \frac{9}{-3} = -3$$

Hence the partial-fraction expansion of the transform is

$$F(s) = \frac{2}{s+1} - \frac{3}{s+4}$$

and, from (40), the time function $f(t)$ is

$$f(t) = 2\epsilon^{-t} - 3\epsilon^{-4t} \qquad \text{for } t > 0$$

Repeated Poles

If two or more of the n roots of $D(s) = 0$ are identical, these roots, which are poles of $F(s)$, are said to be **repeated**. When $F(s)$ contains repeated poles, (38) and (40) no longer hold. If $s_1 = s_2$, for example, the first two terms in (40) become $A_1\epsilon^{s_1 t}$ and $A_2\epsilon^{s_1 t}$, which are identical except for the multiplying constant, and (40) is not valid. If $s_1 = s_2$ and if the remaining poles are distinct, then (38) must be modified to be

$$F(s) = \frac{A_{11}}{(s-s_1)^2} + \frac{A_{12}}{s-s_1} + \frac{A_3}{s-s_3} + \cdots + \frac{A_n}{s-s_n} \tag{43}$$

Referring to Appendix B, we see that the first term on the right-hand side of (43) is the transform of $A_{11}t\epsilon^{s_1 t}$, whereas the second term has the same form as the remaining ones. Thus the inverse transform is

$$f(t) = A_{11}t\epsilon^{s_1 t} + A_{12}\epsilon^{s_1 t} + A_3\epsilon^{s_3 t} + \cdots + A_n\epsilon^{s_n t} \tag{44}$$

Note that the repeated pole at $s = s_1$ introduces into the time function a term of the form $t\epsilon^{s_1 t}$.

In order to evaluate any of the $n-2$ coefficients $A_3, \ldots, A_n$ we can use the procedure given by (42). This formula cannot be used to find A_{11} and A_{12} because of the term $(s-s_1)^2$ in the denominator of $F(s)$. However, multiplying both sides of (43) by $(s-s_1)^2$ gives

$$(s-s_1)^2F(s) = A_{11} + A_{12}(s-s_1) + A_3\frac{(s-s_1)^2}{(s-s_3)} + \cdots + A_n\frac{(s-s_1)^2}{(s-s_n)} \tag{45}$$

Setting s equal to s_1 throughout (45) results in

$$A_{11} = (s-s_1)^2F(s)|_{s=s_1} \tag{46}$$

To find A_{12}, we note that the right-hand side of (45) has the terms $A_{11} + A_{12}(s-s_1)$ and that all the remaining terms contain $(s-s_1)^2$ in their numerators. Thus if we differentiate both sides of (45) with respect to s, we have

$$\frac{d}{ds}[(s-s_1)^2F(s)] = A_{12} + (s-s_1)G(s) \tag{47}$$

where $G(s)$ is a rational function without a pole at $s = s_1$. Note that in (47) the coefficient A_{11} is not present, A_{12} stands alone, and the function $G(s)$ that contains all the other coefficients is multiplied by the quantity $s - s_1$.

Hence setting s equal to s_1 in (47) gives

$$A_{12} = \left\{\frac{d}{ds}[(s-s_1)^2F(s)]\right\}\bigg|_{s=s_1} \tag{48}$$

where the differentiation must be performed before s is set equal to s_1.

If $F(s)$ has two or more pairs of identical poles, then each pair of poles contributes terms of the form $\epsilon^{s_i t}$ and $t\epsilon^{s_i t}$, and we can evaluate the coefficients of these terms by using (46) and (48) with the appropriate indices. If $F(s)$ has three or more identical poles, (43) must be modified further. For example, if $s_1 = s_2 = s_3$ and if the remaining poles are distinct, the partial-fraction expansion has the form

$$F(s) = \frac{A_{11}}{(s-s_1)^3} + \frac{A_{12}}{(s-s_1)^2} + \frac{A_{13}}{s-s_1} + \frac{A_4}{s-s_4} + \cdots + \frac{A_n}{s-s_n} \tag{49}$$

By a procedure similar to that used in the derivation of (46) and (48), we find that

$$\begin{aligned} A_{11} &= (s-s_1)^3F(s)|_{s=s_1} \\ A_{12} &= \left\{\frac{d}{ds}[(s-s_1)^3F(s)]\right\}\bigg|_{s=s_1} \\ A_{13} &= \frac{1}{2!}\left\{\frac{d^2}{ds^2}[(s-s_1)^3F(s)]\right\}\bigg|_{s=s_1} \end{aligned} \tag{50}$$

and

$$A_i = (s-s_i)F(s)|_{s=s_i} \qquad \text{for } i = 4, 5, \ldots, n.$$

The time function corresponding to (49) and (50) is

$$f(t) = \left(\frac{1}{2!}A_{11}t^2 + A_{12}t + A_{13}\right)\epsilon^{s_1 t} + A_4\epsilon^{s_4 t} + \cdots + A_n\epsilon^{s_n t} \tag{51}$$

▶ ***EXAMPLE 7.2***

Find the inverse Laplace transform of

$$F(s) = \frac{5s+16}{(s+2)^2(s+5)}$$

Solution

The denominator of $F(s)$ is given in factored form, so we note that the poles are $s_1 = s_2 = -2$ and $s_3 = -5$. Because of the repeated pole, the partial-fraction expansion of $F(s)$ has the form

$$F(s) = \frac{A_{11}}{(s+2)^2} + \frac{A_{12}}{s+2} + \frac{A_3}{s+5}$$

Using (46), (48), and (42) in order, we find that the coefficients are

$$A_{11} = (s+2)^2F(s)\Big|_{s=-2} = \left.\frac{5s+16}{s+5}\right|_{s=-2} = 2$$

$$A_{12} = \left.\left\{\frac{d}{ds}\left[\frac{5s+16}{s+5}\right]\right\}\right|_{s=-2} = \left.\frac{9}{(s+5)^2}\right|_{s=-2} = 1$$

$$A_3 = (s+5)F(s)\Big|_{s=-5} = \left.\frac{5s+16}{(s+2)^2}\right|_{s=-5} = -1$$

Using numerical values for the coefficients gives the partial-fraction expansion of the transform as

$$F(s) = \frac{2}{(s+2)^2} + \frac{1}{s+2} - \frac{1}{s+5}$$

The corresponding time function is

$$f(t) = 2t\epsilon^{-2t} + \epsilon^{-2t} - \epsilon^{-5t} \quad \text{for } t > 0$$

Complex Poles

The form of $f(t)$ given by (40) is valid for complex poles as well as for real poles. When (40) is used directly with complex poles, however, it has the disadvantage that the functions ϵ^{s_it} and the coefficients A_i are complex. Hence, further developments are necessary in order to write $f(t)$ directly in terms of real functions and real coefficients. We shall present two methods for finding $f(t)$ that lead to slightly different but equivalent forms. First we show how mathematical identities are used to combine any complex terms in the partial-fraction expansion of $F(s)$ into real functions of time. Then we present the alternative method known as completing the square.

For simplicity, we shall first assume that $F(s)$ has only two complex poles and that the order of its numerator is less than the order of its denominator. Any other poles, whether real or complex, will lead to additional terms in the partial-fraction expansion. With these restrictions, we can write the transform as

$$F(s) = \frac{Bs+C}{(s+a-j\omega)(s+a+j\omega)} \tag{52}$$

which has poles at $s_1 = -a + j\omega$ and $s_2 = -a - j\omega$.

Partial-Fraction Expansion. As indicated in (52), the complex poles of $F(s)$ always occur in complex conjugate pairs. Using the partial-fraction expansion of (38) for distinct poles, we can write $F(s)$ as

$$F(s) = \frac{K_1}{s+a-j\omega} + \frac{K_2}{s+a+j\omega} \tag{53}$$

where

$$K_1 = (s+a-j\omega)F(s)|_{s=-a+j\omega} \tag{54a}$$

$$K_2 = (s+a+j\omega)F(s)|_{s=-a-j\omega} \tag{54b}$$

Because (54a) is identical to (54b) except for the sign of the imaginary term $j\omega$ wherever it appears, the coefficient K_2 is the complex conjugate of K_1; that is, $K_2 = K_1^*$. Hence we can write K_1 and K_2 in polar form as $K_1 = K\epsilon^{j\phi}$ and $K_2 = K\epsilon^{-j\phi}$, where K and ϕ are the magnitude and angle, respectively, of the complex number K_1. Thus both K and ϕ are real quantities, and $K \geq 0$.

Rewriting (53) with K_1 and K_2 in polar form, we have

$$F(s) = \frac{K\epsilon^{j\phi}}{s+a-j\omega} + \frac{K\epsilon^{-j\phi}}{s+a+j\omega} \tag{55}$$

which has the form of (39), although the coefficients and poles are complex. From (40) with $n = 2$ and with the appropriate values substituted for A_i and s_i, the complex form of the time function is

$$f(t) = K\epsilon^{j\phi}\epsilon^{(-a+j\omega)t} + K\epsilon^{-j\phi}\epsilon^{(-a-j\omega)t}$$

To obtain $f(t)$ as a real function of time, we factor the term $2K\epsilon^{-at}$ out of each term on the right-hand side of this equation and combine the remaining complex exponentials. This yields

$$f(t) = 2K\epsilon^{-at}\left[\frac{\epsilon^{j(\omega t+\phi)} + \epsilon^{-j(\omega t+\phi)}}{2}\right]$$

Recognizing from Table 6.1 that the term in the brackets is $\cos(\omega t + \phi)$, we can write

$$f(t) = 2K\epsilon^{-at}\cos(\omega t + \phi) \quad \text{for } t > 0 \tag{56}$$

where the parameters a, ω, K, and ϕ are all real. Note that (55) and (56) constitute one of the entries in Appendix B.

Completing the Square. To develop the second of the two forms of $f(t)$ when $F(s)$ has complex poles, we multiply the denominator factors in (52) to get

$$F(s) = \frac{Bs+C}{s^2+2as+a^2+\omega^2}$$

Next we write the denominator as the sum of the perfect square $(s+a)^2$ and the constant ω^2, so that

$$F(s) = \frac{Bs+C}{(s+a)^2+\omega^2} \tag{57}$$

Now we rearrange the numerator of $F(s)$ to have the term $(s+a)$ appear:

$$F(s) = \frac{B(s+a)+(C-aB)}{(s+a)^2+\omega^2}$$
$$= B\left[\frac{s+a}{(s+a)^2+\omega^2}\right] + \left(\frac{C-aB}{\omega}\right)\left[\frac{\omega}{(s+a)^2+\omega^2}\right]$$

Referring to Appendix B, we see that the quantities within the brackets are the transforms of $\epsilon^{-at}\cos\omega t$ and $\epsilon^{-at}\sin\omega t$, respectively. Thus

$$f(t) = B\epsilon^{-at}\cos\omega t + \left(\frac{C-aB}{\omega}\right)\epsilon^{-at}\sin\omega t \qquad (58)$$

Equations (57) and (58) form another one of the entries in Appendix B. Although (56) and (58) look somewhat different, we can always use Table 6.1 to show that they are equivalent functions of time.

EXAMPLE 7.3

Use each of the two methods that we have described to find $f(t)$ when

$$F(s) = \frac{4s+8}{s^2+2s+5}$$

Solution

The poles of $F(s)$ are the roots of $s^2+2s+5=0$, namely $s_1=-1+j2$ and $s_2=-1-j2$. Hence $a=1$, $\omega=2$, and the partial-fraction expansion of $F(s)$ is

$$F(s) = \frac{K_1}{s+1-j2} + \frac{K_2}{s+1+j2}$$

Solving for K_1 according to (54a), we find

$$K_1 = \left.\frac{\cancel{(s+1-j2)}(4s+8)}{\cancel{(s+1-j2)}(s+1+j2)}\right|_{s=-1+j2}$$
$$= 2-j1$$
$$= \sqrt{5}\,\epsilon^{-j0.4636}$$

Thus $K=\sqrt{5}$ and $\phi=-0.4636$ rad. Substituting the known values of a, ω, K and ϕ into (56) yields

$$f(t) = 2\sqrt{5}\,\epsilon^{-t}\cos(2t-0.4636) \qquad \text{for } t>0 \qquad (59)$$

Having found $f(t)$ from the partial-fraction expansion of $F(s)$, we repeat the problem by the method of completing the square. Because the denominator of $F(s)$ is s^2+2s+5, we can obtain a perfect-square term by adding and subtracting 1 to give

$$(s^2+2s+1)+(5-1) = (s+1)^2+4$$

so

$$F(s) = \frac{4s+8}{(s+1)^2+2^2}$$

By comparison with (57), we see that $B=4, C=8, a=1$, and $\omega=2$. With these values, (58) gives

$$f(t) = 4\epsilon^{-t}\cos 2t + 2\epsilon^{-t}\sin 2t \qquad \text{for } t>0$$

which we can convert to the form given by (59) by using the appropriate entry in Table 6.1.

Comparison of the two methods for finding $f(t)$ indicates that the partial-fraction expansion requires the manipulation of complex numbers. It is frequently employed, however, and we shall need it for a derivation in the next chapter. The method of completing the square has the advantage of avoiding the use of complex numbers. The following example illustrates the two methods when $F(s)$ has more than the two poles specified by (52) and (57).

EXAMPLE 7.4

Find the inverse transform of

$$F(s) = \frac{5s^2+8s-5}{s^2(s^2+2s+5)}$$
$$= \frac{5s^2+8s-5}{s^2(s+1-j2)(s+1+j2)}$$

Solution

In order to use (56), we write the partial-fraction expansion as

$$F(s) = \frac{A_{11}}{s^2} + \frac{A_{12}}{s} + \frac{K_1}{s+1-j2} + \frac{K_1^*}{s+1+j2} \qquad (60)$$

where

$$A_{11} = \left.\frac{5s^2+8s-5}{s^2+2s+5}\right|_{s=0} = -1$$

$$A_{12} = \left.\left\{\frac{d}{ds}\left[\frac{5s^2+8s-5}{s^2+2s+5}\right]\right\}\right|_{s=0} = \left.\frac{2s^2+60s+50}{(s^2+2s+5)^2}\right|_{s=0} = 2$$

$$K_1 = \left.\frac{5s^2+8s-5}{s^2(s+1+j2)}\right|_{s=-1+j2} = \frac{-28-j4}{16-j12} = \frac{-7-j1}{4-j3}$$

Although it is possible to evaluate the quotient of two complex numbers directly on many hand calculators, an alternative procedure is to rationalize the fraction by multiplying both halves by the complex conjugate of the denominator. Then

$$K_1 = \frac{-7-j1}{4-j3}\cdot\frac{4+j3}{4+j3} = \frac{1}{25}(-25-j25)$$
$$= -1-j1 = \sqrt{2}\epsilon^{-j2.356}$$

Inserting the values of A_{11}, A_{12}, and K_1 into (60) and then using (11), (4) and (56), we obtain

$$f(t) = -t + 2 + 2\sqrt{2}\epsilon^{-t}\cos(2t - 2.356) \quad \text{for } t > 0 \tag{61}$$

In order to avoid complex numbers, we may choose not to factor the quadratic $s^2 + 2s + 5$ that appears in the denominator of $F(s)$. For the corresponding term in a partial-fraction expansion, however, we must then assume a numerator of the form $Bs + C$. Thus, for an alternative solution to this example, we write

$$F(s) = \frac{5s^2 + 8s - 5}{s^2(s^2 + 2s + 5)}$$
$$= \frac{A_{11}}{s^2} + \frac{A_{12}}{s} + \frac{Bs + C}{s^2 + 2s + 5}$$

Recombining the terms on the right side of this equation over a common denominator gives

$$F(s) = \frac{(A_{12} + B)s^3 + (A_{11} + 2A_{12} + C)s^2 + (2A_{11} + 5A_{12})s + 5A_{11}}{s^2(s^2 + 2s + 5)}$$

By equating corresponding coefficients in the numerators, we have

$$A_{12} + B = 0$$
$$A_{11} + 2A_{12} + C = 5$$
$$2A_{11} + 5A_{12} = 8$$
$$5A_{11} = -5$$

from which $A_{11} = -1$, $A_{12} = 2$, $B = -2$, and $C = 2$. If we also complete the square in the factor $s^2 + 2s + 5$, we may write the partial-fraction expansion as

$$F(s) = \frac{-1}{s^2} + \frac{2}{s} + \frac{-2s + 2}{(s+1)^2 + (2)^2}$$

From (11), (4), and (58), the corresponding function of time is

$$f(t) = -t + 2 - 2\epsilon^{-t}\cos 2t + 2\epsilon^{-t}\sin 2t \quad \text{for } t > 0 \tag{62}$$

It is easy to show by Table 6.1 that (61) and (62) are equivalent.

Preliminary Step of Long Division

Remember that the techniques discussed so far are subject to the restriction that $F(s)$ is a strictly proper rational function—in other words, that m, the degree of the numerator polynomial $N(s)$, is less than n, the degree of the denominator polynomial $D(s)$. Otherwise, the partial-fraction expansion given for distinct poles by (39) or for a pair of repeated poles by (43) is not valid. In order to find the inverse transform of $F(s)$ when $m = n$, we must first write $F(s)$ as the sum of a constant[4] and a fraction whose numerator is of degree $n - 1$ or less. We can accomplish this by dividing the numerator by the denominator so that the remainder is of degree $n - 1$ or less. Then we may write

$$F(s) = A + F'(s) \tag{63}$$

where A is a constant and is the transform of $A\delta(t)$. The function $F'(s)$ is a ratio of polynomials having the same denominator as $F(s)$ but a numerator of degree less than n. We can find the inverse transform of $F'(s)$ by using the techniques described previously. Thus

$$f(t) = A\delta(t) + \mathscr{L}^{-1}[F'(s)] \tag{64}$$

▶ *EXAMPLE 7.5*

Find $f(t)$ when

$$F(s) = \frac{2s^2 + 7s + 8}{s^2 + 3s + 2}$$

Solution

Both the numerator and the denominator of $F(s)$ are quadratic in s, so we have $m = n = 2$. Before carrying out the partial-fraction expansion, we must rewrite $F(s)$ in the form of (63) by dividing its numerator by its denominator, as follows:

$$\begin{array}{r@{}l} & \;\; 2 \\ s^2 + 3s + 2 \,\big) & \overline{\,2s^2 + 7s + 8} \\ & \;\underline{2s^2 + 6s + 4} \\ & \qquad\quad s + 4 \end{array}$$

Because the poles of $F(s)$ are $s_1 = -1$ and $s_2 = -2$, we can write

$$F(s) = 2 + \frac{s + 4}{(s+1)(s+2)}$$
$$= 2 + \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

[4] When $m > n$, we can write $F(s)$ as the sum of a polynomial in s and a strictly proper rational function, but we shall not encounter such cases here.

where

$$A_1 = \left.\frac{\cancel{(s+1)}(s+4)}{\cancel{(s+1)}(s+2)}\right|_{s=-1} = 3$$

$$A_2 = \left.\frac{\cancel{(s+2)}(s+4)}{(s+1)\cancel{(s+2)}}\right|_{s=-2} = -2$$

Thus

$$F(s) = 2 + \frac{3}{s+1} - \frac{2}{s+2}$$

and

$$f(t) = 2\delta(t) + 3\epsilon^{-t} - 2\epsilon^{-2t} \quad \text{for } t > 0$$

which is the sum of an impulse at $t = 0+$ and two decaying exponentials.

7.4 SOLVING FOR THE RESPONSE

The preceding sections of this chapter have laid the groundwork for an efficient means of solving for the responses of fixed linear systems. The analysis of first-order systems is not difficult enough to warrant transform methods. However, for second- and higher-order systems, using transform methods is generally easier than using the methods discussed in Chapter 6. In addition, acquaintance with the transform approach will enable us to develop important concepts in Chapter 8. Before outlining the general technique, we shall consider a simple, first-order example, which we would not normally solve by transform methods.

▶ *EXAMPLE 7.6*

The capacitor in the circuit shown in Figure 7.5(a) is uncharged for $t < 0$, and the switch closes at $t = 0$. Using Laplace transforms, find $e_C(t)$ for $t > 0$.

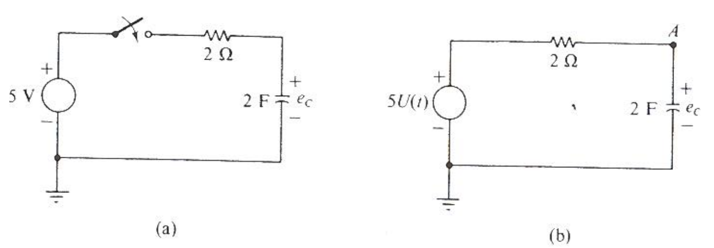


FIGURE 7.5 (a) Circuit for Example 7.6. (b) Equivalent circuit without the switch.

Solution

For convenience of analysis, we can redraw the original circuit as shown in Figure 7.5(b), where the switch and constant voltage source have been replaced by the step-function input $5\,U(t)$. Summing the currents leaving node A gives the first-order differential equation

$$\frac{1}{2}[e_C - 5U(t)] + 2\dot{e}_C = 0$$

Transforming this equation term by term, we have

$$\frac{1}{2}\{\mathcal{L}[e_C(t)] - \mathcal{L}[5U(t)]\} + 2\mathcal{L}[\dot{e}_C(t)] = 0$$

From Appendix B, we see that $\mathcal{L}[5U(t)] = 5/s$. Because $e_C(t)$ is unknown, we cannot write down its transform explicitly, so we use $E_C(s)$ to denote $\mathcal{L}[e_C(t)]$.[5] Using the expression for the transform of a derivative, we rewrite the transformed equation as

$$\frac{1}{2}\left[E_C(s) - \frac{5}{s}\right] + 2[sE_C(s) - e_C(0)] = 0$$

Note that the original differential equation has been converted into an algebraic equation. Because the capacitor is uncharged for $t < 0$, $e_C(0) = (0)$. Solving this equation algebraically for $E_C(s)$ gives

$$E_C(s) = \frac{2.5/s}{2s + 0.5} = \frac{1.25}{s(s+0.25)}$$

To find the output voltage $e_C(t)$, we take the inverse transform by expanding $E_C(s)$ in its partial-fraction expansion and using Appendix B to identify the individual time functions. The expansion is

$$E_C(s) = \frac{A_1}{s} + \frac{A_2}{s+0.25}$$

where

$$A_1 = sE_C(s)|_{s=0} = 5$$

$$A_2 = (s+0.25)E_C(s)|_{s=-0.25} = -5$$

Thus

$$E_C(s) = \frac{5}{s} - \frac{5}{s+0.25}$$

[5]As a rule, functions of time are denoted by lowercase letters and their Laplace transforms by the corresponding capital letters followed by the transform variable s in parentheses.

and

$$e_C(t) = 5 - 5\epsilon^{-0.25t} \quad \text{for } t > 0$$

Because $e_C(t) = 0$ for all $t \leq 0$, we can write the solution in the alternative form

$$e_C(t) = 5[1 - \epsilon^{-0.25t}]U(t)$$

which is valid for all t. We can check this answer by solving the differential equation by the classical method of Chapter 6.

General Procedure

Application of the Laplace transform to solve for the response of dynamic systems of any order consists of the following three steps:

1. Write and immediately transform the differential or integral-differential equations describing the system for $t > 0$, evaluating all the initial-condition terms that appear in the transformed equations.
2. Solve these algebraic equations for the transform of the output.
3. Evaluate the inverse transform to obtain the output as a function of time.

The first step transforms the original equations into a set of algebraic equations in the variable s. Then we may solve the algebraic equations by any convenient method to obtain the transform of the response. The inverse transform includes both the steady-state and the transient components and, unlike the classical approach, contains no unknown constants that must still be evaluated. Any necessary initial-condition terms will automatically appear in step 1.

If we transform immediately the equations obtained by summing forces or torques shown on the free-body diagrams or by summing currents at the nodes of a circuit, the only initial-condition terms normally encountered are for variables that cannot change instantaneously. However, if any variable or one of its derivatives does have a discontinuity at the origin, we define its value at time zero to be the value approached through negative values of time. Thus $f(0) = f(0-)$ and $\dot{f}(0) = \dot{f}(0-)$, where the value at $t = 0-$ is the value at the left side of any discontinuity. This is consistent with the treatment in previous sections; examples include our definition of $U(0) = 0$ and our interpretation of $\delta(t)$ as a unit impulse occurring at $t = 0+$. Any initial-condition terms, such as $f(0)$ and $\dot{f}(0)$, that appear in the transformed equations are due entirely to conditions for $t < 0$ and are not affected by a discontinuity in the input at time zero.

The remainder of this section contains a variety of examples. The next two examples illustrate the application of the foregoing general procedure for a mechanical and an electrical system.

▶ *EXAMPLE 7.7*

The mechanical system shown in Figure 7.6 has the parameter values $M = 1$ kg, $B = 4$ N·s/m, and $K = 3$ N/m, and the applied force is $f_a(t) = 9$ N for all $t > 0$. The mass has no initial velocity, but it is released from a position 1 m to the right of its equilibrium position at the instant the force is applied. Find the displacement $x(t)$ for all $t > 0$.

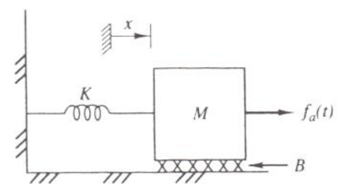


FIGURE 7.6 Mechanical system for Example 7.7.

Solution

The differential equation describing the system for $t > 0$ is $M\ddot{x} + B\dot{x} + Kx = f_a(t)$, which becomes

$$\ddot{x} + 4\dot{x} + 3x = 9$$

when the parameter and input values are substituted. Transforming the equation term by term gives

$$[s^2X(s) - sx(0) - \dot{x}(0)] + 4[sX(s) - x(0)] + 3X(s) = \frac{9}{s}$$

Substituting the specified initial conditions $x(0) = 1$ and $\dot{x}(0) = 0$, we solve algebraically to obtain

$$X(s) = \frac{s + 4 + 9/s}{s^2 + 4s + 3} = \frac{s^2 + 4s + 9}{s(s + 1)(s + 3)}$$

Noting that $X(s)$ has three distinct real poles and that $m < n$, we write $X(s)$ in the form

$$X(s) = \frac{A_1}{s} + \frac{A_2}{s + 1} + \frac{A_2}{s + 3}$$

where

$$A_1 = sX(s)|_{s=0} = 3$$
$$A_2 = (s + 1)X(s)|_{s=-1} = -3$$
$$A_3 = (s + 3)X(s)|_{s=-3} = 1$$

Thus

$$X(s) = \frac{3}{s} - \frac{3}{s+1} + \frac{1}{s+3}$$

and the displacement is

$$x(t) = 3 - 3\epsilon^{-t} + \epsilon^{-3t} \quad \text{for } t > 0$$

The response $x(t)$ reduces to the specified initial condition $x(0) = 1$ m and approaches a steady-state value of $x_{ss} = 3$ m. The two transient terms decay exponentially with time constants of 1 s and $\frac{1}{3}$ s.

EXAMPLE 7.8

After steady-state conditions have been reached, the switch in Figure 7.7(a) opens at $t = 0$. Find the voltage e_o across the capacitor for all $t > 0$.

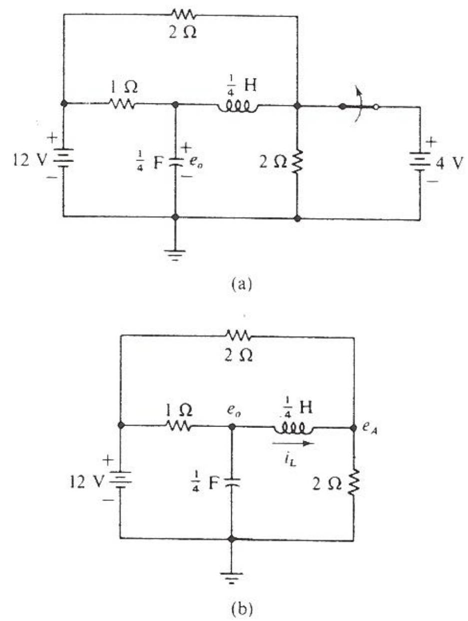


FIGURE 7.7 Circuit for Example 7.8 (a) Original circuit. (b) Circuit valid for $t > 0$.

Solution

The circuit for $t > 0$ is shown in Figure 7.7(b), with the switch open and with the node voltages e_o and e_A shown. The current-law equations at these

two nodes are

$$e_o - 12 + \frac{1}{4}\dot{e}_o + i_L(0) + 4\int_0^t (e_o - e_A)\,d\lambda = 0$$

$$\frac{1}{2}(e_A - 12) - i_L(0) + 4\int_0^t (e_A - e_o)\,d\lambda + \frac{1}{2}e_A = 0$$

Transforming these equations, we have

$$E_o(s) - \frac{12}{s} + \frac{1}{4}[sE_o(s) - e_o(0)] + \frac{i_L(0)}{s} + \frac{4}{s}[E_o(s) - E_A(s)] = 0$$

$$\frac{1}{2}\left[E_A(s) - \frac{12}{s}\right] - \frac{i_L(0)}{s} + \frac{4}{s}[E_A(s) - E_o(s)] + \frac{1}{2}E_A(s) = 0$$

We must find the numerical values of the initial conditions $e_o(0)$ and $i_L(0)$ that appear in these transformed equations. Because e_o and i_L are measures of the energy stored in the capacitor and inductor, respectively, they cannot change instantaneously and do not have discontinuities at the time origin.

When the circuit is in the steady state with the switch closed, the capacitor and inductor may be replaced by open and short circuits, respectively. This is done in Figure 7.8(a), from which we find that $e_o(0) = 4$ V and $i_L(0) = 8$ A. Substituting these initial conditions into the transformed equations and collecting like terms, we get

$$\left[\frac{1}{4}s + 1 + \frac{4}{s}\right]E_o(s) - \frac{4}{s}E_A(s) = 1 + \frac{4}{s} \tag{65a}$$

$$-\frac{4}{s}E_o(s) + \left[1 + \frac{4}{s}\right]E_A(s) = \frac{14}{s} \tag{65b}$$

We want to find the capacitor voltage $e_o(t)$, so the next step is to solve (65) for $E_o(s)$ by eliminating $E_A(s)$. Noting from (65b) that

$$E_A(s) = \frac{4E_o(s) + 14}{s + 4}$$

and substituting this expression into (65a), we find that

$$E_o(s) = \frac{4(s^2 + 8s + 72)}{s^3 + 8s^2 + 32s}$$

One pole of $E_o(s)$ is $s = 0$, and the remaining two are the roots of $s^2 + 8s + 32 = 0$. Hence the poles are $s_1 = 0$, $s_2 = -4 + j4$, and $s_3 = -4 - j4$, and we can expand the transformed output into the form

$$E_o(s) = \frac{A_1}{s} + \frac{A_2}{s + 4 - j4} + \frac{A_3}{s + 4 + j4} \tag{66}$$

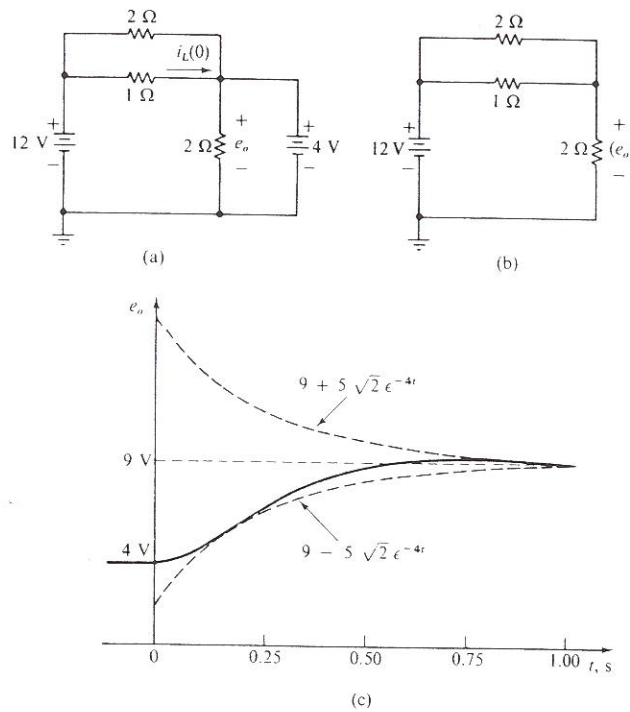


FIGURE 7.8 (a) Equivalent circuit for Example 7.8 just before the switch opens. (b) Equivalent circuit valid as t approaches infinity. (c) Complete response.

where the coefficients are

$$A_1 = sE_o(s)|_{s=0} = \frac{(4)(72)}{32} = 9$$

$$\begin{aligned}A_2 &= (s+4-j4)E_o(s)|_{s=-4+j4}\\ &= \frac{4[(-4+j4)^2+8(-4+j4)+72]}{(-4+j4)(-4+j4+4+j4)}\\ &= \frac{4(40)}{(-4+j4)(j8)} = \frac{5}{-(1+j1)} = \frac{5}{\sqrt{2}}\epsilon^{j3\pi/4}\end{aligned}$$

and

$$A_3 = A_2^* = \frac{5}{\sqrt{2}}\epsilon^{-j3\pi/4}$$

Comparing the second and third terms on the right side of (66) to (55), we see that $a = 4$, $\omega = 4$, $K = 5/\sqrt{2}$, and $\phi = \frac{3}{4}\pi$ rad. Using (56) to write

the part of the response that corresponds to the pair of complex poles, we have

$$e_o(t) = 9 + 5\sqrt{2}\epsilon^{-4t}\cos\left(4t + \frac{3}{4}\pi\right) \qquad \text{for } t > 0$$

As a check on the work, note that as t approaches infinity, this expression gives a constant value of 9. We can also find the steady-state behavior with the switch open from Figure 7.8(b), where the capacitor and inductor have again been replaced by open and short circuits, respectively. The parallel combination of the 2-Ω and 1-Ω resistors is equivalent to $(2)(1)/(2+1) = \frac{2}{3}$ Ω. By the voltage-divider rule,

$$e_o(\infty) = \frac{2}{2+\frac{2}{3}}(12) = 9 \text{ V}$$

which agrees with the general equation for $t > 0$.

The complete response is shown in Figure 7.8(c). The transient component has an envelope that decays with a time constant of 0.25 s and has a period of $\frac{1}{2}\pi$ s.

When the input is an impulse, it can instantaneously change the energy stored within the system. Because of this, when finding the unit impulse response in Chapter 6, we first found and then differentiated the unit step response. With the Laplace transform, however, there is no difficulty in finding the impulse response directly.

▶ *EXAMPLE 7.9*

Find the unit impulse response $h(t)$ for the system shown in Figure 7.6 when $M = 1$ kg, $B = 4$ N·s/m, and $K = 4$ N/m. The differential equation relating the displacement x to the applied force $f_a(t)$ is $\ddot{x} + 4\dot{x} + 4x = f_a(t)$.

Solution

Recall that the impulse response $h(t)$ is the zero-state response when $u(t) = \delta(t)$. If we rewrite the system equation as $\ddot{h} + 4\dot{h} + 4h = \delta(t)$ and assume that the impulse occurs at $t = 0+$, it follows that we should take $h(0)$ and $\dot{h}(0)$ as zero in evaluating the transforms of $\ddot{h}$ and $\dot{h}$.

Transforming both sides of the differential equation, using $\mathscr{L}[\delta(t)] = 1$ and denoting $\mathscr{L}[h(t)]$ by $H(s)$, we find that

$$[s^2H(s) - sh(0) - \dot{h}(0)] + 4[sH(s) - h(0)] + 4H(s) = 1$$

which reduces to

$$(s^2 + 4s + 4)H(s) = 1$$

when the initial conditions are set equal to zero. Hence

$$H(s) = \frac{1}{s^2+4s+4} = \frac{1}{(s+2)^2}$$

Referring to Appendix B and noting that $h(t) = 0$ for all $t \le 0$, we can write

$$h(t) = \begin{cases} 0 & \text{for } t \le 0 \\ t\epsilon^{-2t} & \text{for } t > 0 \end{cases} \tag{67}$$

which is shown in Figure 7.9.

Inspection of Figure 7.9 reveals that $\dot{h}(t)$ undergoes a discontinuity at $t = 0$. This means that the velocity of the mass changes instantaneously. You can verify from (67) that $\dot{h}(0+) = 1$, whereas we used $\dot{h}(0) = 0$ in taking the transform of the original differential equation. This distinction between the initial slopes is consistent with our assertion that the impulse occurs at $t = 0+$.

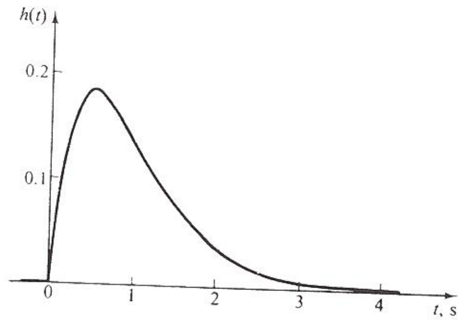


FIGURE 7.9 Impulse response found in Example 7.9.

The final three examples illustrate three different methods of finding a system's response to a given input. First, using the procedure recommended in most cases, we transform the original equations immediately. In the other two methods, we find either the state-variable model or the input-output model before applying the Laplace transform.

▶ *EXAMPLE 7.10*

The translational system shown in Figure 7.10(a) was modeled in Example 3.7 and in Example 3.11. Let $M_1 = 1$ kg, $B_1 = B_2 = 1$ N·s/m, and $K_1 = K_2 = 1$ N/m. The springs are undeflected when $x_1 = x_2 = 0$. Take the output to be the displacement x_1. Find the unit step response by transforming the equations obtained from the free-body diagrams.

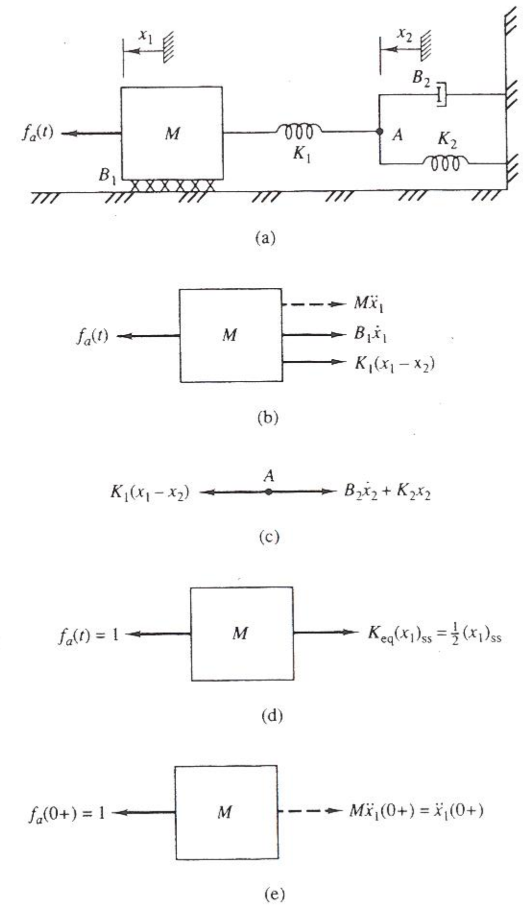


FIGURE 7.10 (a) Translational system for Example 7.10. (b), (c) General free-body diagrams. (d) Free-body diagram for the steady state. (e) Free-body diagram for $t = 0+$.

Solution

The free-body diagrams for the mass M and for the massless junction A are shown in parts (b) and (c) of the figure, respectively. Summing the forces on these diagrams gives

$$M\ddot{x}_1 + B_1\dot{x}_1 + K_1(x_1 - x_2) = f_a(t)$$
$$B_2\dot{x}_2 + K_2x_2 + K_1(x_2 - x_1) = 0$$

Transforming these equations, with all the element values equal to unity and with $\mathscr{L}[f_a(t)] = F_a(s)$, we have

$$[s^2X_1(s) - sx_1(0) - \dot{x}_1(0)] + [sX_1(s) - x_1(0)] + X_1(s) - X_2(s) = F_a(s)$$
$$[sX_2(s) - x_2(0)] + 2X_2(s) - X_1(s) = 0 \tag{68}$$

Because we assume that the initial stored energy is zero when finding the unit step response, the initial elongations of the springs and the initial velocity of the mass must be zero. Thus $x_1(0) = x_2(0) = 0$ and $\dot{x}_1(0) = 0$, so (68) becomes

$$(s^2 + s + 1)X_1(s) - X_2(s) = F_a(s)$$
$$-X_1(s) + (s+2)X_2(s) = 0$$

Eliminating $X_2(s)$ from these equations, we obtain

$$X_1(s) = \left[\frac{s+2}{s^3 + 3s^2 + 3s + 1}\right] F_a(s) \tag{69}$$

Factoring the denominator, noting that $F_a(s) = \mathscr{L}[U(t)] = 1/s$, and th writing a partial-fraction expansion, we have

$$X_1(s) = \frac{s+2}{s(s+1)^3} = \frac{A_0}{s} + \frac{A_{11}}{(s+1)^3} + \frac{A_{12}}{(s+1)^2} + \frac{A_{13}}{s+1} \tag{70}$$

where

$$A_0 = \left.\frac{s+2}{(s+1)^3}\right|_{s=0} = 2$$
$$A_{11} = \left.\frac{s+2}{s}\right|_{s=-1} = -1$$
$$A_{12} = \left[\frac{d}{ds}\frac{s+2}{s}\right]_{s=-1} = -2$$
$$A_{13} = \frac{1}{2}\left[\frac{d^2}{ds^2}\frac{s+2}{s}\right]_{s=-1} = -2$$

To find the output as a function of time, we insert these numbers into (70) and take the inverse transform of each term. This gives us

$$x_1(t) = 2 - (\tfrac{1}{2}t^2 + 2t + 2)\epsilon^{-t} \quad \text{for } t > 0 \tag{71}$$

This equation reveals that the steady-state response is $(x_1)_{ss} = 2$ m, which we can check by referring to Figure 7.10(a). When $f_a(t)$ has a constant value of 1, the mass will eventually become motionless and there will be no inertial or friction forces. Because the friction elements can be disregarded, the two springs are then in series and can be replaced by single equivalent spring for which $K_{eq} = \frac{1}{2}$ N/m. The only forces acting on the mass are those shown in Figure 7.10(d), so we can again conclude that $(x_1)_{ss} = 2$ m.

By differentiating (71), we can show that the velocity and acceleration of the mass are

$$\dot{x}_1 = (\tfrac{1}{2}t^2 + t)\epsilon^{-t} \quad \text{for } t > 0$$
$$\ddot{x}_1 = (-\tfrac{1}{2}t^2 + 1)\epsilon^{-t} \quad \text{for } t > 0 \tag{72}$$

Replacing t by zero in (71) and (72), we see that

$$x_1(0+) = 0 \tag{73a}$$
$$\dot{x}_1(0+) = 0 \tag{73b}$$
$$\ddot{x}_1(0+) = 1 \tag{73c}$$

The fact that x_1 and $\dot{x}_1$ remain zero at $t = 0+$ is expected; the elongations of the springs and the velocity of the mass cannot change instantaneously. Because of this, we also know that at $t = 0+$ there is no force on the mass from K_1 or B_1. Thus the only forces on the mass at $t = 0+$ are the applied and inertial forces, as shown in Figure 7.10(e). From that figure, we see that the initial acceleration must be $\ddot{x}_1(0+) = 1$ m/s², which serves as a check on (73c).

▶ *EXAMPLE 7.11*

Repeat Example 7.10 by transforming the state-variable equations.

Solution

The state-variable model for Figure 7.10(a) was found in Example 3.7. From (3.14), with all the element values set equal to unity, we have

$$\dot{x}_1 = v_1$$
$$\dot{v}_1 = -x_1 - v_1 + x_2 + f_a(t)$$
$$\dot{x}_2 = x_1 - 2x_2$$

Transforming these equations gives

$$sX_1(s) - x_1(0) = V_1(s) \tag{74a}$$
$$sV_1(s) - v_1(0) = -X_1(s) - V_1(s) + X_2(s) + F_a(s) \tag{74b}$$
$$sX_2(s) - x_2(0) = X_1(s) - 2X_2(s) \tag{74c}$$

where we can again set the initial-condition terms equal to zero. Then, by substituting (74a) into (74b) and rearranging the last two equations, we obtain

$$(s^2 + s + 1)X_1(s) - X_2(s) = F_a(s)$$
$$-X_1(s) + (s+2)X_2(s) = 0$$

from which

$$X_1(s) = \left[\frac{s+2}{s^3 + 3s^2 + 3s + 1}\right] F_a(s)$$

which agrees with (69).

▶ *EXAMPLE 7.12*

Repeat Example 7.10 by transforming the input-output differential equation.

Solution

The input-output equation for Figure 7.10(a) was derived in Example 3.11. From (3.27) with the element values set equal to unity, we have

$$\dddot{x}_1 + 3\ddot{x}_1 + 3\dot{x}_1 + x_1 = \dot{f}_a + 2f_a(t)$$

We transform this equation, getting

$$[s^3X_1(s) - s^2x_1(0) - s\dot{x}_1(0) - \ddot{x}_1(0)] + 3[s^2X_1(s) - sx_1(0) - \dot{x}_1(0)]$$
$$+ 3[sX_1(s) - x_1(0)] + X_1(s) = sF_a(s) - f_a(0) + 2F_a(s) \quad (75)$$

We note that (75) involves four different initial conditions: $x_1(0)$, $\dot{x}_1(0)$, $\ddot{x}_1(0)$, and $f_a(0)$. Because there is zero initial stored energy and because the step-function input is assumed to occur just after $t = 0$ (that is, at $t = 0+$), each of these initial conditions is zero. Then (75) reduces to

$$(s^3 + 3s^2 + 3s + 1)X_1(s) = (s+2)F_a(s)$$

which is equivalent to (69).

Although somewhat different, the methods used in the last three examples gave the same equations for the transformed output $X_1(s)$ and for $x_1(t)$ for all $t > 0$. When the equations from the free-body diagrams were transformed immediately or when the state-variable equations were transformed, the only initial conditions needed were for functions that did not change instantaneously.

This was not the case, however, when we obtained the input-output equation before applying the Laplace transform. In this method, it is not unusual for one or more of the initial conditions to involve functions that have discontinuities. For the acceleration $\ddot{x}_1$ in the last example, $\ddot{x}_1(0-) = 0$ but it was found that $\ddot{x}_1(0+) = 1$ m/s^2. Whenever there is a discontinuity at the time origin, we use the value at $t = 0-$ for the initial-condition term, for the reasons explained earlier in this section. This is in contrast to the classical solution of an input-output differential equation outlined in Section 6.1, which requires the values of the initial conditions at $t = 0+$. One advantage of the transform method is that evaluating the initial-condition terms is often easier.

In the last three examples, we chose to denote the transform of the unit step input $f_a(t)$ by $F_a(s)$ rather than to immediately replace it by $\mathcal{L}[U(t)] = 1/s$. Thus the expression for $X_1(s)$ in (69) was a somewhat more general result than necessary. Equation (69) holds for any input, provided only that the input has a Laplace transform and that the initial stored energy is zero. The quantity inside the brackets is called the transfer function and will become important in the next chapter. When finding the response to a particular input, however, we more often replace its transform immediately by the appropriate expression from Appendix B .

■ 7.5 ADDITIONAL TRANSFORM PROPERTIES

We now introduce several useful transform properties that were not needed for the examples in Section 7.4. The first two of these concern functions that are shifted in time. The others provide certain information about the time function directly from its transform, without our having to carry out the transform inversion.

Time Delay

If the function $f(t)$ is delayed by a units of time, we denote the delayed function as $f(t-a)$, where $a > 0$. In order to develop a general expression for the transform of the delayed function $f(t-a)$ in terms of the transform of $f(t)$, we must ensure that any part of $f(t)$ that is nonzero for $t < 0$ does not fall within the range $0 < t < \infty$ for the delayed function. Otherwise, a portion of the original time function will contribute to the transform of $f(t-a)$ but not to that of $f(t)$. To illustrate this point, consider the functions $f(t)$ and $f(t-a)$ shown in Figure 7.11. The shaded portion of $f(t)$ that is nonzero for $t < 0$ does not affect $\mathcal{L}[f(t)]$, because it is outside the limits of integration in (1). It does affect $\mathcal{L}[f(t-a)]$, however, because for $a > 0$, at least part of it falls within the interval $0 < t < \infty$, as shown in Figure 7.11(b).

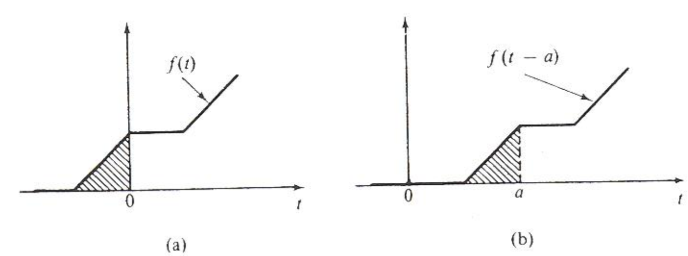


FIGURE 7.11 A function for which the time-delay theorem is not applicable.

We shall consider functions of the form

$$f_1(t) = f(t)U(t)$$

which are the product of any transformable function $f(t)$ and the unit step function $U(t)$. Because $U(t) = 0$ for all $t \leq 0$, the function $f_1(t)$ will be zero for all $t \leq 0$, and the delayed function

$$f_1(t-a) = f(t-a)U(t-a)$$

will be zero for all $t \leq a$. The functions $f_1(t)$ and $f_1(t-a)$ corresponding to the $f(t)$ defined in Figure 7.11(a) are shown in Figure 7.12.

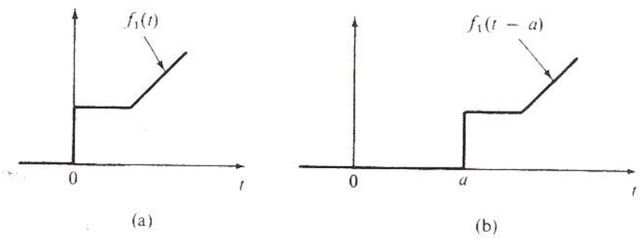


FIGURE 7.12 A function for which the time-delay theorem is applicable.

To express the transform of $f(t-a)U(t-a)$, where $a > 0$, in terms of $F(s) = \mathscr{L}[f(t)]$, we start with the transform definition in (1) and write

$$\mathscr{L}[f(t-a)U(t-a)] = \int_0^\infty f(t-a)U(t-a)\epsilon^{-st}\,dt$$

Because

$$U(t-a) = \begin{cases} 0 & \text{for } t \leq a \\ 1 & \text{for } t > a \end{cases}$$

we can rewrite the transform as

$$\begin{aligned} \mathscr{L}[f(t-a)U(t-a)] &= \int_a^\infty f(t-a)\epsilon^{-st}\,dt \\ &= \epsilon^{-sa}\int_a^\infty f(t-a)\epsilon^{-s(t-a)}\,dt \\ &= \epsilon^{-sa}\int_0^\infty f(\lambda)\epsilon^{-s\lambda}\,d\lambda \\ &= \epsilon^{-sa}F(s) \end{aligned} \tag{76}$$

where $F(s)$ is the transform of $f(t)$ and where $a > 0$. This theorem is one of the entries in Appendix B.

▶ ***EXAMPLE 7.13***

Use the time-delay theorem to derive the transform of the triangular pulse shown in Figure 7.13(a) and defined by the equation

$$f(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ At & \text{for } 0 < t \leq L \\ 0 & \text{for } t > L \end{cases}$$

Solution

Any pulse that consists of straight lines can be decomposed into a sum of step functions and ramp functions. The triangular pulse shown in Figure 7.13(a) can be regarded as the superposition of the three functions shown in Figure 7.13(b), Figure 7.13(c), and Figure 7.13(d). These are a ramp starting at $t = 0$, a delayed ramp starting at $t = L$, and a delayed step function starting

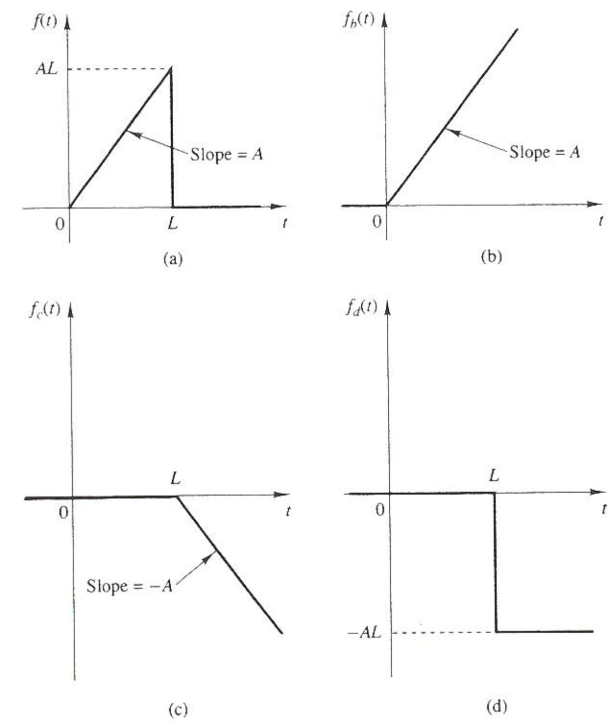


FIGURE 7.13 (a) Triangular pulse. (b), (c), (d) Its ramp and step components.

at $t = L$. Thus

$$f(t) = AtU(t) - A(t - L)U(t - L) - ALU(t - L) \tag{77}$$

From Appendix B, we note that

$$\mathcal{L}[AtU(t)] = \frac{A}{s^2} \tag{78}$$

Using (76) with (78), we have

$$\mathcal{L}[-A(t - L)U(t - L)] = -\frac{A\epsilon^{-sL}}{s^2}$$

From (76) and the fact that $\mathcal{L}[U(t)] = 1/s$,

$$\mathcal{L}[-ALU(t - L)] = -\frac{AL\epsilon^{-sL}}{s}$$

Using the superposition theorem, we obtain the transform of the triangular pulse as

$$F(s) = \frac{A}{s^2}(1 - \epsilon^{-sL}) - \frac{AL}{s}\epsilon^{-sL} \tag{79}$$

Inversion of Some Irrational Transforms

The use of a partial-fraction expansion to find an inverse transform is restricted to transforms that are rational functions of s. However, the transform given by (79) is not a rational function because of the factor ϵ^{-sL}. For transforms that would be rational functions except for multiplicative factors in the numerator such as ϵ^{-sL}, we may use the time-delay theorem in (76).

Assume that an irrational transform can be written as

$$F(s) = F_1(s) + F_2(s)\epsilon^{-sa} \tag{80}$$

where $F_1(s)$ and $F_2(s)$ are rational functions and where a is a positive constant. Then we can find the inverse transforms of $F_1(s)$ and $F_2(s)$ by using partial-fraction expansions and, if the order of the numerator is not less than that of the denominator, a preliminary step of long division. Denote the inverse transforms of $F_1(s)$ and $F_2(s)$ by $f_1(t)$ and $f_2(t)$, respectively. Then, by (76),

$$f(t) = f_1(t) + f_2(t - a)U(t - a) \tag{81}$$

▶ *EXAMPLE 7.14*

Find the inverse transform of

$$F(s) = \frac{A}{s^2}(1 - \epsilon^{-sL}) - \frac{AL}{s}\epsilon^{-sL}$$

Solution

We can rewrite the transform $F(s)$ in the form of (80) as

$$F(s) = \frac{A}{s^2} - \left(\frac{A}{s^2} + \frac{AL}{s}\right)\epsilon^{-sL}$$

Hence $F_1(s) = A/s^2$ and $f_1(t) = At$ for $t > 0$. The rational portion of the remaining term is

$$F_2(s) = -\left(\frac{A}{s^2} + \frac{AL}{s}\right)$$

which has as its inverse transform

$$f_2(t) = -At - AL \qquad \text{for } t > 0$$

Using (81) with $a = L$, we can write the complete inverse transform as

$$f(t) = At - A(t - L)U(t - L) - ALU(t - L) \qquad \text{for } t > 0$$

which agrees with (77) and Figure 7.13(a).

Initial-Value and Final-Value Theorems

It is possible to determine the limits of $f(t)$ as time approaches zero and infinity directly from its transform $F(s)$ without having to find $f(t)$ for all $t > 0$. First we consider the limit of $f(t)$ as time approaches zero through positive values (that is, from the right). This limit of $f(t)$ is denoted by $f(0+)$. To evaluate this limit directly from $F(s)$, we use the **initial-value theorem**, which states that

$$f(0+) = \lim_{s \to \infty} sF(s) \tag{82}$$

where the limit exists.

If $F(s)$ is a rational function, (82) will yield a finite value provided that the degree of the numerator polynomial is less than that of the denominator—in other words, provided that $m < n$. If we attempt to use (82) when $m = n$, the result will be infinite. Recall from Section 7.3 that to find $f(t)$ when $m = n$, we must use a preliminary step of long division to write $F(s)$ as the sum of a constant and a transform for which $m < n$. Because the inverse transform of the constant is an impulse at $t = 0+$, the value of $f(0+)$ is undefined when $m = n$.

The **final-value theorem** states that

$$f(\infty) = \lim_{s \to 0} sF(s) \tag{83}$$

provided that $F(s)$ has no poles in the right half of the complex plane and, with the possible exception of a single pole at the origin, has no poles on the imaginary axis. The symbol $f(\infty)$ denotes the limit of $f(t)$ as t approaches infinity.

To gain some insight into the effect of this restriction on the use of the final-value theorem, we recall that the forms of the terms in a partial-fraction expansion are dictated by the locations of the poles of $F(s)$. Suppose, for example, that

$$F(s) = \frac{A_1}{s} + \frac{A_2(s+\alpha)}{(s+\alpha)^2+\beta^2} + \frac{A_3}{s-b} + \frac{A_4\omega}{s^2+\omega^2}$$

where α, β, b, and ω are positive real constants. The expansion implies that the poles of $F(s)$ are $s_1 = 0$, $s_2 = -\alpha + j\beta$, $s_3 = -\alpha - j\beta$, $s_4 = b$, $s_5 = j\omega$, and $s_6 = -j\omega$. The corresponding time function for $t > 0$ is

$$f(t) = A_1 + A_2\epsilon^{-\alpha t}\cos\beta t + A_3\epsilon^{bt} + A_4 \sin\omega t$$

The limits of the first two terms as t approaches infinity are A_1 and zero, respectively. However, $A_3\epsilon^{bt}$ increases without limit, whereas $A_4 \sin\omega t$ oscillates continually without approaching a constant value. Thus, because of the poles of $F(s)$ at $s_4 = b$, $s_5 = j\omega$, and $s_6 = -j\omega$, the function $f(t)$ does not approach a limit as t approaches infinity. As another example, double poles of $F(s)$ at the origin will cause the partial-fraction expansion to have terms of the form

$$F(s) = \frac{A_{11}}{s^2} + \frac{A_{12}}{s} + \cdots$$

The corresponding time function for $t > 0$ is

$$f(t) = A_{11}t + A_{12} + \cdots$$

which again does not approach a limit.

The use of the initial-value and final-value theorems is illustrated in the following example and in the next chapter. In Example 7.16 we consider a transform for which neither theorem is applicable.

▶ *EXAMPLE 7.15*

Use the initial-value and final-value theorems to find $f(0+)$ and $f(\infty)$ when

$$F(s) = \frac{s^2+2s+4}{s^3+3s^2+2s} \tag{84}$$

Solution

From (82), the initial value of $f(t)$ is

$$f(0+) = \lim_{s\to\infty} \frac{s(s^2+2s+4)}{s^3+3s^2+2s}$$
$$= \lim_{s\to\infty} \frac{s^3+2s^2+4s}{s^3+3s^2+2s} \tag{85}$$

Because $f(0+)$ is the limit of a ratio of polynomials in s as s approaches infinity, we need consider only the highest powers in s in both the numerator and denominator. Hence (85) reduces to

$$f(0+) = \lim_{s\to\infty} \frac{s^3}{s^3} = 1$$

Before applying the final-value theorem, we must verify that the conditions necessary for it to be valid are satisfied. In this case, we can rewrite (84) with its denominator in factored form as

$$F(s) = \frac{s^2+2s+4}{s(s+1)(s+2)} \tag{86}$$

which has distinct poles at $s = 0, -1$, and -2. The function $F(s)$ has a single pole at the origin with the remaining poles inside the left half of the s-plane, so we can apply the final-value theorem. Using (83), we find that

$$f(\infty) = \lim_{s\to 0} \frac{s(s^2+2s+4)}{s(s^2+3s+2)}$$
$$= \lim_{s\to 0} \frac{s^2+2s+4}{s^2+3s+2} = 2$$

In this example, it is a simple task to evaluate $f(t)$ for all $t > 0$ by writing $F(s)$ as given by (86) in its partial-fraction expansion. The result is

$$f(t) = 2 - 3\epsilon^{-t} + 2\epsilon^{-2t} \qquad \text{for } t > 0$$

and it is apparent that the values we found for $f(0+)$ and $f(\infty)$ are correct.

▶ *EXAMPLE 7.16*

Explain why the initial-value and final-value theorems are not applicable to the transform

$$F(s) = \frac{s^3+2s^2+6s+8}{s^3+4s} \tag{87}$$

Solution

Attempting to apply the initial-value theorem, we would write

$$f(0+) = \lim_{s\to\infty} \frac{s(s^3+2s^2+6s+8)}{s^3+4s}$$
$$= \lim_{s\to\infty} \frac{s^4+2s^3+6s^2+8s}{s^3+4s}$$

which is infinite. This is expected, because $m = n = 3$ in (87).

As for the final-value theorem, we see from (87) that we can write the denominator of $F(s)$ as $s(s^2+4) = s(s-j2)(s+j2)$. Hence $F(s)$ has a pair of imaginary poles at $s_2 = j2$ and $s_3 = -j2$ and violates the requirements for the final-value theorem. If we carry out the partial-fraction expansion of $F(s)$, we find that

$$F(s) = 1 + \frac{2}{s} + \frac{2}{s^2+4}$$

which is the transform of the time function

$$f(t) = \delta(t) + 2 + \sin 2t \quad \text{for } t > 0$$

The initial-value theorem is invalid because of the impulse at $t = 0+$, and the final-value theorem is invalid because of the constant-amplitude sinusoidal term.

SUMMARY

The Laplace transform can be used to convert the integral-differential equations that describe fixed, linear dynamic systems into algebraic equations. The general procedure consists of three steps: transforming the system equations, solving the resulting algebraic equations for the transform of the output, and taking the inverse transform. Any initial conditions that need to be evaluated automatically appear in the first step. These initial conditions are often easier to find than those that are needed for a time-domain solution of the input-output equation.

Although the basic definition can be used to find the Laplace transform of a given function of time, transforms of the most common functions are tabulated in Appendix B. The inverse transform, which should be expressed as a real function of time, can usually be found by writing a partial-fraction expansion and by again using the table in Appendix B. The appendix also summarizes the properties and theorems developed throughout this chapter.

In addition to being an important tool for finding the response of a system to specified inputs, the Laplace transform can be used to develop a number of important general concepts. This further development will be carried out in the next chapter.

PROBLEMS

7.1 Using the definition of the Laplace transform in (1), evaluate the transforms of the following functions.

a) $f_1(t) = t^2$
b) $f_2(t) = \epsilon^{-at}\cos\omega t$
c) $f_3(t) = t\epsilon^{-at}$
d) $f_4(t) = \sin 2t$ for $0 < t < \pi$ and zero elsewhere

7.2 a) Derive the expressions given in (14) and (15) for the transforms of $\sin\omega t$ and $\cos\omega t$ by using the identities in Table 6.1 and then applying (5).
b) Using (14) and (15) with Table 6.1, derive expressions for the transforms of $\sin(\omega t + \phi)$ and $\cos(\omega t + \phi)$.

*** 7.3** Use the properties tabulated in Appendix B to find the Laplace transform of each of the following functions of time.

a) $f_1(t) = t\epsilon^{-2t}\cos 3t$
b) $f_2(t) = t^2\sin 2t$
c) $f_3(t) = \dfrac{d}{dt}(t^2\epsilon^{-t})$
d) $f_4(t) = \displaystyle\int_0^t \lambda^2\epsilon^{-\lambda}d\lambda$

7.4 a) Prove that $\mathcal{L}[f(t/a)] = aF(as)$.
b) Apply this property with $f(t) = \cos\omega t$ to find $\mathcal{L}[\cos 2\omega t]$.

For Problems 7.5 through 7.7, find $f(t)$ for the given $F(s)$.

7.5 a) $F(s) = \dfrac{2s^3 + 3s^2 + s + 4}{s^3}$
b) $F(s) = \dfrac{3s^2 + 9s + 24}{(s-1)(s+2)(s+5)}$
c) $F(s) = \dfrac{4}{s^2(s+1)}$
d) $F(s) = \dfrac{3s}{s^2 + 2s + 26}$

7.6 a) $F(s) = \dfrac{s}{s^2 + 8s + 16}$
b) $F(s) = \dfrac{1}{s(s^2 + \omega^2)}$
c) $F(s) = \dfrac{8s^2 + 20s + 74}{s(s^2 + s + 9.25)}$
d) $F(s) = \dfrac{2s^2 + 11s + 16}{(s+2)^2}$

*** 7.7** a) $F(s) = \dfrac{s^3 + 2s + 4}{s(s+1)^2(s+2)}$
b) $F(s) = \dfrac{4s^2 + 10s + 10}{s^3 + 2s^2 + 5s}$
c) $F(s) = \dfrac{3(s^3 + 2s^2 + 4s + 1)}{s(s+3)^2}$
d) $F(s) = \dfrac{s^3 - 4s}{(s+1)(s^2 + 4s + 4)}$

*** 7.8** a) Find the inverse transform of

$$F(s) = \frac{3s^2 + 2s + 2}{(s+2)(s^2 + 2s + 5)}$$

by writing a partial-fraction expansion and using (56).
b) Repeat part (a) by completing the square and using (58).

7.9 Find the inverse transform of $E_o(s)$ in Example 7.8 by first finding the constants A, B, and C in the expansion

$$E_o(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + 8s + 32}$$

and then completing the square to evaluate the inverse transform of the term having complex poles.

* **7.10** Use (34) and the results of Example 7.1 to find the inverse transform of

$$F(s) = \frac{-s + 5}{s(s+1)(s+4)}$$

Check your answer by writing a partial-fraction expansion for $F(s)$.

7.11 Use (24) and the results of Example 7.1 to find the inverse transform of

$$F(s) = \frac{s(-s+5)}{(s+1)(s+4)}$$

Check your answer by writing a partial-fraction expansion for $F(s)$.

7.12 Repeat all three parts of Example 6.1 using the Laplace transform.

7.13 Use the Laplace transform to solve for the response for $t > 0$ in Example 6.2.

* **7.14** For a given system described by the equation $\ddot{y} + 3\dot{y} + 2y = u(t)$, use the Laplace transform to find $y(t)$ for $t > 0$ when the input is $u(t) = 5t$ for $t > 0$ and the initial conditions are $y(0) = 1$ and $\dot{y}(0) = -1$. Sketch $y(t)$.

7.15 Repeat both parts of Example 6.3 using the Laplace transform.

7.16 Repeat Example 6.4 using the Laplace transform.

* **7.17** a) Find and sketch the unit step response for the circuit shown in Figure P5.1 by writing and transforming two node equations.
b) Repeat part (a) by transforming the input-output differential equation given in Problem 6.12.

7.18 Do part (d) of Problem 6.13 by using the Laplace transform.

7.19 Do part (c) of Problem 6.14 by using the Laplace transform.

7.20 Repeat Example 6.6 using the Laplace transform.

* **7.21** The mechanical system described in Problem 6.20 has two inputs, the gravitational force $M_2 g$ and the force $f_a(t)$ applied to M_1.
a) Solve for x_0, the constant displacement caused by gravity when the applied force $f_a(t)$ is zero.
b) Find the transformed output $X(s)$ when $x(0) = x_0$ and $f_a(t)$ is the unit step function.
c) Find $x(t)$ for the conditions in part (b) when $M_1 = M_2 = B_1 = B_2 = K_2 = 1$ in a consistent set of units. Check your answer by evaluating x_{ss} and $x(0+)$.

7.22 Use the Laplace transform to solve for the unit step response of the mechanical system described in Problem 6.21 when the parameter values are $M = J = R = K_1 = K_2 = 1$ and $B = 5$ in a consistent set of units.

* **7.23** Use the Laplace transform to find the zero-input responses of the pendulum modeled in Example 6.9 when there is no damping—that is, when $B = 0$. Give an expression for $\theta(t)$ in terms of $\theta(0)$ and $\dot{\theta}(0)$ that is valid about the equilibrium condition $\theta = 0$. Also give an expression for $\phi(t)$ in terms of $\phi(0)$ and $\dot{\phi}(0)$ that is valid about the equilibrium condition $\phi = 0$, which corresponds to $\theta = \pi$.

7.24 For the rotational system shown in Figure 4.19 and discussed in Example 4.5, let $J_1 = J_2 = 0.5$, $B = 1$, and $K = 2$ in a consistent set of units.
a) Find a general expression for $\Omega_2(s) = \mathcal{L}[\omega_2(t)]$ by transforming (4.42).
b) If the system is initially at rest and the inputs are $\tau_L(t) = 0$ and $\tau_a(t) = \epsilon^{-2t}$ for $t > 0$, find and sketch ω_2 for $t > 0$.

* **7.25** Find and sketch e_o versus t for the circuit shown in Figure 5.17 and discussed in Example 5.4. The parameter values are $C = 0.5$ F, $R_1 = 2\ \Omega$, $R_2 = R_3 = 1\ \Omega$, and $L = 0.5$ H. The inputs are $e_1(t) = 2$ V for all t and $e_2(t) = U(t)$ V. Steady-state conditions exist at $t = 0-$.

7.26 Use the Laplace transform to solve for the unit step response of the circuit described in Problem 6.22 when the parameter values are $C = L = R_1 = R_2 = 1$ in a consistent set of units.

7.27 a) For the circuit shown in Figure P7.27, verify that the application of Kirchhoff's current law to nodes A and O leads to the following pair of coupled equations.

$$\frac{1}{2}[e_A - e_i(t)] + \frac{1}{2}e_A + 2(\dot{e}_A - \dot{e}_o) = 0$$

$$2(\dot{e}_o - \dot{e}_A) + i_L(0) + 4\int_0^t e_o(\lambda)d\lambda + 2e_o = 0$$

b) Use the Laplace transform to find the unit impulse response when e_o is the output.
c) Use the Laplace transform to solve for the unit step response.

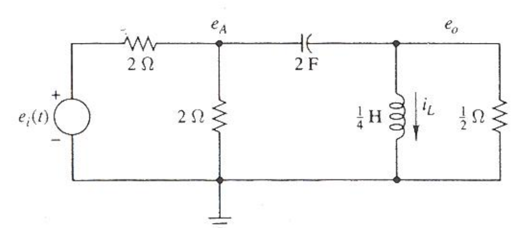


FIGURE P7.27

* **7.28** Consider the electrical circuit shown in Figure 5.15 and discussed in Example 5.3.
a) Verify that, for the parameter values given in the example, (5.19) can be written as

$$2\dot{e}_A + 2e_A - e_o = e_i(t)$$

$$-2e_A + 16\dot{e}_o + 3e_o + 8\int_0^t e_o(\lambda)d\lambda = -4i_L(0)$$

b) Find the Laplace transform of the zero-input response of the output e_o to an initial voltage $e_A(0)$ on the capacitor C_1. Give your answer as a ratio of polynomials.

7.29 Consider the circuit shown in Figure P7.29.
a) Verify that $E_o(s)$, the transform of the output voltage, can be written as

$$E_o(s) = \frac{-se_C(0) - 2i_L(0) + (s^2 + 2s/R)E_i(s)}{s^2 + (1 + 2/R)s + 16}$$

b) Determine the poles of $E_o(s)$ for $R = 2/9\ \Omega$, $2/7\ \Omega$, and $2/3\ \Omega$.

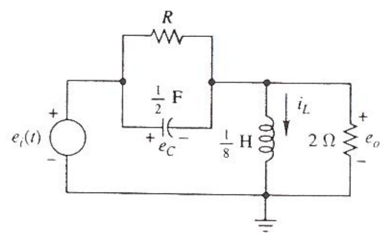


FIGURE P7.29

7.30 Consider the circuit shown in Figure 5.26(a) and modeled in Example 5.9.
a) Show that the Laplace transform of the zero-input response of $e_C(t)$ is

$$E_C(s) = \frac{15i_L(0) + (50s + 40)e_C(0)}{50s^2 + 70s + 42}$$

b) Find the Laplace transform of the zero-input response of $i_L(t)$.
c) Repeat part (b) for the output voltage e_o.

7.31 **a)** Use the time-delay theorem in (76) to derive the Laplace transform of the rectangular pulse shown in Figure 7.2.
b) Find the Laplace transform of the triangular pulse shown in Figure 7.13(a) by using (1) directly, rather than by decomposing the pulse into ramp and step components. Compare your results to (79).

* **7.32** Using the definition of the Laplace transform in (1), evaluate the transforms of the following functions.
a) The function $f_1(t)$ shown in Figure P7.32(a).
b) The function $f_2(t)$ shown in Figure P7.32(b).

7.33 Repeat Problem 7.32 by decomposing the functions into step functions and ramp functions, as appropriate.

7.34 Sketch the time functions corresponding to each of the following Laplace transforms.
a) $F(s) = \dfrac{1}{s^2 + 1}\left(1 + \epsilon^{-\pi s}\right)$
b) $F(s) = \dfrac{1}{s^2}\left(1 - 2\epsilon^{-s} + \epsilon^{-2s}\right)$

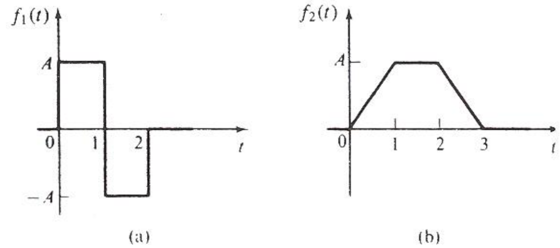


FIGURE P7.32

* **7.35** Apply the initial-value and final-value theorems to find $f(0+)$ and $f(\infty)$ for each of the four transforms in Problem 7.7. If either theorem is not applicable to a particular transform, explain why this is so.

7.36 Repeat Problem 7.35 for the following transforms.
a) $E_o(s)$ in Example 7.8.
b) The functions in part (c) and part (d) of Problem 7.6.
c) $F(s) = \dfrac{4s^3}{(5s^2 + 3)^2}$

7.37 **a)** Using (24) and the initial-value theorem, show that

$$\dot{f}(0+) = \lim_{s \to \infty} [s^2 F(s) - sf(0)]$$

provided that the limit exists.
b) Use the property derived in part (a) to find $\dot{f}(0+)$ for the transform $F(s)$ given in (84). Assume that $f(t) = 1$ for $t \le 0$. If this property is not applicable, explain why this is so.
c) Repeat part (b), assuming that $f(t) = 0$ for $t \le 0$.
d) Check your answers to part (b) and part (c) by differentiating the expression given for $f(t)$ in Example 7.15.