**3.20** Find the state-variable model for the system shown in Figure P2.22, when the energy stored in each spring is a separate output.

* **3.21** Find the state-variable model for the system shown in Figure P2.23. The outputs are the momentum of each mass and the tensile force in the cable.

**3.22** Find the state-variable model for the system shown in Figure P2.24 when the elongation of each spring is a separate output.

**3.23** Redo Example 3.11 by using the $p$ operator to obtain
  **a)** The input-output equation for $x_1$.
  **b)** The input-output equation for $x_2$.

* **3.24** Use the $p$ operator to obtain the input-output equation relating $x_1$ and $x_2(t)$ for the system shown in Figure 3.7(a), starting with
  **a)** The state-variable equations for $x_1$ and $v_1$ given in (18).
  **b)** The state-variable equations for $x_1$ and $q$ given in (22).

**3.25** **a)** Starting with (2.19) and using the $p$ operator, obtain an input-output differential equation for $x$ in the system model developed in Example 2.4. To simplify the calculations, let all parameters have values of unity in the appropriate units.
  **b)** Repeat part (a) by starting with (2.20) to find the input-output equation for $x_1$. Compare the answer to that which you obtained in part (a).

* **3.26** Set all of the parameters to unity in (2.25) and then use the $p$ operator to obtain the input-output equation relating $x_1$ to the inputs $f_a(t)$ and $g$ for the system shown in Figure 2.17(a).

**3.27** **a)** Repeat Problem 3.26 for (2.29), describing the system shown in Figure 2.19(a).
  **b)** Obtain the input-output equation for $x_2$.

**3.28** Write in matrix form the state-variable model developed in Example 3.8, with the input $x_2(t)$ and the outputs $x_1$ and $v_1$. Identify the matrices **A**, **B**, **C**, and **D**, the state vector **q**, the input vector **u**, and the output vector **y**. Also give the values of $n$, $m$, and $p$.

* **3.29** Repeat Problem 3.28 for the system modeled by (8) and (9) in Example 3.5, with the input $f_a(t)$ and the outputs $f_{K_2}$ and $m_T$.

**3.30** Repeat Problem 3.29 using (10) and (11) in place of (8) and (9).

**3.31** Repeat Problem 3.28 for the system modeled by (3) and (4) in Example 3.2, with the input $f_a(t)$ and the outputs $f_{K_2}$ and $m_T$.

**3.32** Repeat Problem 3.28 for the state-variable model developed in Example 3.6, with the input $f_a(t)$ and the outputs $f_{K_2}$, $m_T$, and $v_2$.

CHAPTER 4

# ROTATIONAL MECHANICAL SYSTEMS

In Chapter 2 we presented the laws governing translational systems and introduced the use of free-body diagrams as an aid in writing equations describing the motion. In Chapter 3 we showed how to rearrange the equations and develop state-variable and input-output models.

Extending these procedures to rotational systems requires little in the way of new concepts. We first introduce the three rotational elements that are analogs of mass, friction, and stiffness in translational systems. Two other elements, levers and gears, are characterized in a somewhat different way. The use of interconnection laws and free-body diagrams is very similar to their use for translational systems. In the examples, we shall seek models consisting of sets of state-variable and output equations or else input-output equations that contain only a single unknown variable. The chapter closes with examples of combined translational and rotational systems.

## 4.1 VARIABLES

For rotational mechanical systems, the symbols used for the variables are

$\theta$, angular displacement in radians (rad)

$\omega$, angular velocity in radians per second (rad/s)

$\alpha$, angular acceleration in radians per second per second (rad/s$^2$)

$\tau$, torque in newton-meters (N·m)

all of which are functions of time. Angular displacements are measured with respect to some specified reference angle, often the equilibrium orientation of the body or point in question. We shall always choose the reference arrows for the angular displacement, velocity, and acceleration of a body to be in the same direction so that the relationships

$$\omega = \dot{\theta}$$

$$\alpha = \dot{\omega} = \ddot{\theta}$$

hold. The conventions used are illustrated in Figure 4.1, where $\tau$ denotes an external torque applied to the rotating body by means of some unspecified mechanism, such as by a gear on the supporting shaft. Because of the convention that the assumed positive directions for $\theta$, $\omega$, and $\alpha$ are the same, it is not necessary to show all three reference arrows explicitly.

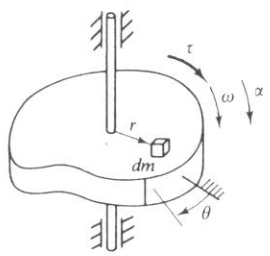



**FIGURE 4.1** Conventions for designating rotational variables.

The power supplied to the rotating body in Figure 4.1 is

$$p = \tau\omega \tag{1}$$

The power is the derivative of the energy $w$, and the energy supplied to the body up to time $t$ is

$$w(t) = w(t_0) + \int_{t_0}^{t} p(\lambda)d\lambda$$

## 4.2 ELEMENT LAWS

The elements used to represent physical devices in rotational systems are moment of inertia, friction, stiffness, levers, and gears. We shall restrict our consideration to elements that rotate about fixed axes in an inertial reference frame.



### Moment of Inertia

When Newton's second law is applied to the differential mass element $dm$ in Figure 4.1 and the result is integrated over the entire body, we obtain

$$\frac{d}{dt}(J\omega) = \tau \tag{2}$$

where $J\omega$ is the angular momentum of the body and where $\tau$ denotes the net torque applied about the fixed axis of rotation. The symbol $J$ denotes the **moment of inertia** in kilogram-meters$^2$ (kg·m$^2$). We can obtain it by carrying out the integration of $r^2\,dm$ over the entire body.

The moment of inertia for a body whose mass $M$ can be considered to be concentrated at a point is $ML^2$, where $L$ is the distance from the point to the axis of rotation. Figure 4.2 shows a slender bar and a solid disk, each of which has a total mass $M$ that is uniformly distributed throughout the body. In parts (a) and (b) of the figure, expressions for $J$ are given for the case where the axis of rotation passes through the center of mass. The results for other common shapes can be found in basic physics and mechanics books. If we have an axis that does not pass through the center of mass, we can use the **parallel-axis theorem**. Let $J_0$ denote the moment of inertia about the parallel axis that passes through the center of mass, and let $a$ be the distance between the two axes. Then the desired moment of inertia is

$$J = J_0 + Ma^2 \tag{3}$$

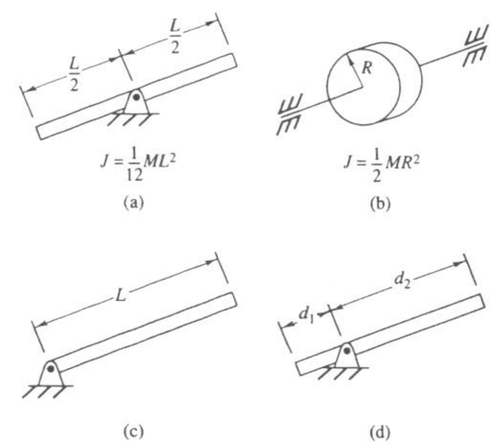



**FIGURE 4.2** Moments of inertia. (a) Slender bar. (b) Disk. (c), (d) Slender bar where axis of rotation does not pass through the center of mass.

For the slender bar shown in Figure 4.2(c) with the axis of rotation at one end, (3) gives

$$J = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2 \tag{4}$$

For two or more components rotating about the same axis, we can find the total moment of inertia by summing the individual contributions. If the uniform bar shown in Figure 4.2(d) has a total mass $M$, then the masses for the sections of length $d_1$ and $d_2$ will be $M_1 = d_1M/(d_1 + d_2)$ and $M_2 = d_2M/(d_1 + d_2)$. Using (4), we see that the total moment of inertia is

$$J = \frac{1}{3}M_1d_1^2 + \frac{1}{3}M_2d_2^2 = \frac{M(d_1^3 + d_2^3)}{3(d_1 + d_2)} \tag{5}$$

We consider only nonrelativistic systems and constant moments of inertia, so (2) reduces to

$$J\dot{\omega} = \tau \tag{6}$$

where $\dot{\omega}$ is the angular acceleration. As is the case for a mass having translational motion, a rotating body can store energy in both kinetic and potential forms. The kinetic energy is

$$w_k = \frac{1}{2}J\omega^2 \tag{7}$$

and for a uniform gravitational field, the potential energy is

$$w_p = Mgh \tag{8}$$

where $M$ is the mass, $g$ the gravitational constant, and $h$ the height of the center of mass above its reference position. If the fixed axis of rotation is vertical or passes through the center of mass, there is no change in the potential energy as the body rotates, and (8) is not needed. To find the complete response of a dynamic system containing a rotating body, we must know its initial angular velocity $\omega(t_0)$. If its potential energy can vary or if we want to find $\theta(t)$, then we must also know $\theta(t_0)$.

## Friction

A **rotational friction** element is one for which there is an algebraic relationship between the torque and the relative angular velocity between two surfaces. **Rotational viscous friction** arises when two rotating bodies are separated by a film of oil. Figure 4.3(a) shows two concentric rotating cylinders separated by a thin film of oil, where the angular velocities of the cylinders are $\omega_1$ and $\omega_2$ and the relative angular velocity is $\Delta\omega = \omega_2 - \omega_1$. The torque

$$\tau = B\Delta\omega \tag{9}$$



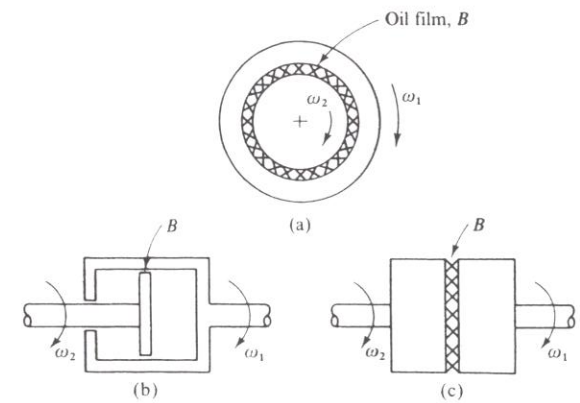



**FIGURE 4.3** Rotational devices characterized by viscous friction.

will be exerted on each cylinder, in directions that tend to reduce the relative angular velocity $\Delta\omega$. Hence the positive sense of the frictional torque must be counterclockwise on the inner cylinder and clockwise on the outer cylinder. The friction coefficient $B$ has units of newton-meter-seconds. Note that the same symbol is used for translational viscous friction, where it has units of newton-seconds per meter.

Equation (9) also applies to a rotational dashpot that might be used in modeling a fluid drive system, such as shown in Figure 4.3(b) or Figure 4.3(c). The inertia of the parts is assumed to be negligible or else is accounted for in the mathematical model by separate moments of inertia. If the rotational friction element is assumed to have no inertia, then when a torque $\tau$ is applied to one side, a torque of equal magnitude but opposite direction must be exerted on the other side (by a wall, the air, or some other component), as shown in Figure 4.4(a), where $\tau = B(\omega_2 - \omega_1)$. Thus in Figure 4.4(b), the torque $\tau$ passes through the first friction element and is exerted directly on the moment of inertia $J$.

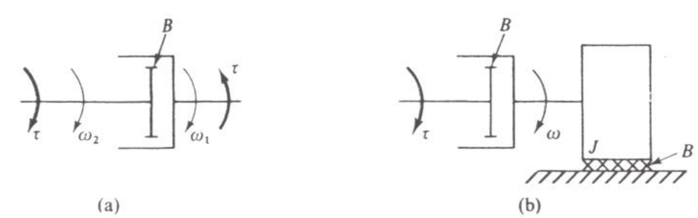



**FIGURE 4.4** (a) Dashpot with negligible inertia. (b) Torque transmitted through a friction element.

Other types of friction, such as the damping vanes shown in Figure 4.5(a), may exert a retarding torque that is not directly proportional to the angular velocity but that may be described by a curve of $\tau$ versus $\Delta\omega$, as shown in Figure 4.5(b). For a linear element, the curve must be a straight line passing through the origin. The power supplied to the friction element, $\tau\Delta\omega$, is immediately lost to the mechanical system in the form of heat.

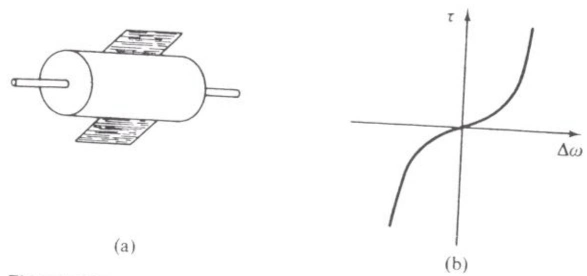



**FIGURE 4.5** (a) Rotor with damping vanes. (b) Nonlinear friction characteristic.

## Stiffness

**Rotational stiffness** is usually associated with a torsional spring, such as the mainspring of a clock, or with a relatively thin shaft. It is an element for which there is an algebraic relationship between $\tau$ and $\theta$, and it is generally represented as shown in Figure 4.6(a). The angle $\theta$ is the relative angular displacement of the ends of the element from the positions corresponding to no applied torque. If both ends of the device can move as shown in Figure 4.6(b), then the element law relates $\tau$ and $\Delta\theta$, where $\Delta\theta = \theta_2 - \theta_1$. For a linear torsional spring or flexible shaft,

$$\tau = K\Delta\theta \tag{10}$$

where $K$ is the spring constant with units of newton-meters (N·m), in contrast to newtons per meter for the parameter $K$ in translational systems. For a thin shaft, $K$ is directly proportional to the shear modulus of the material and to the square of the cross-sectional area and is inversely proportional to the length of the shaft.

Because we assume that the moment of inertia of a stiffness element either is negligible or is represented by a separate element, the torques exerted on the two ends of a stiffness element must be equal in magnitude and opposite in direction, as was indicated in Figure 4.6(b). Thus for the system shown in Figure 4.7, the applied torque $\tau$ passes through the first shaft and is exerted directly on the body that has moment of inertia $J$.



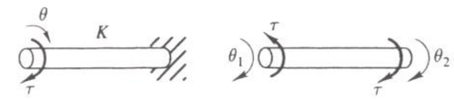



**FIGURE 4.6** (a) Rotational stiffness element with one end fixed. (b) Rotational stiffness element with $\Delta\theta = \theta_2 - \theta_1$.

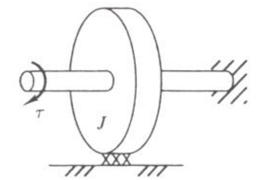



**FIGURE 4.7** Torque transmitted through a shaft.

Potential energy is stored in a twisted stiffness element and can affect the response of the system at later times. For a linear spring or shaft, the potential energy is

$$w_p = \frac{1}{2}K(\Delta\theta)^2 \tag{11}$$

The relative angular displacement at time $t_0$ is one of the initial conditions we need in order to find the response of a system for $t \geq t_0$.

## The Lever

An **ideal lever** is assumed to be a rigid bar pivoted at a point and having no mass, no friction, no momentum, and no stored energy. In all our examples the pivot point will be fixed. If the magnitude of the angle of rotation is small (say less than 0.25 rad), the motion of the ends can be considered strictly translational. In Figure 4.8, let $\theta$ denote the angular displacement of the lever from the horizontal position. For a rigid lever with a fixed pivot,

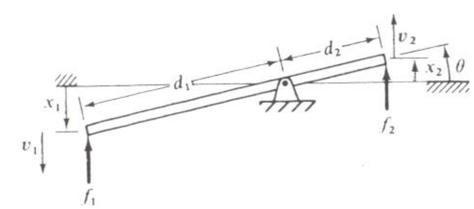



**FIGURE 4.8** The lever.

the displacements of the ends are given by $x_1 \simeq d_1\theta$ and $x_2 \simeq d_2\theta$, where $\theta$ is in radians. Thus for small displacements we have

$$x_2 = \left(\frac{d_2}{d_1}\right) x_1 \tag{12}$$

and, by differentiating the foregoing equation, we find

$$v_2 = \left(\frac{d_2}{d_1}\right) v_1 \tag{13}$$

Because the sum of the moments about the pivot point vanishes, as required by the assumed absence of mass, it follows that $f_2d_2 - f_1d_1 = 0$, or

$$f_2 = \left(\frac{d_1}{d_2}\right) f_1 \tag{14}$$

The pivot exerts a downward force of $f_1 + f_2$ on the lever, but this does not enter into the derivation of (14), because this force exerts no moment about the pivot. The lever differs from the mass, friction, and stiffness elements in that the algebraic relationships given by (12) through (14) involve pairs of the same types of variables: two displacements, two velocities, or two forces.

If the lever's mass cannot be neglected, then we must include its moment of inertia when summing moments about the pivot point, and (14) is no longer valid. If the bar in Figure 4.8 has mass $M$, then its moment of inertia is given by (5). The examples in Section 4.4 include a system with a massless lever and one in which the lever's mass is not negligible.

## Gears

Consider next the pair of gears shown in Figure 4.9. In order to develop the basic geometric and torque relationships, we shall assume **ideal gears**, which have no moment of inertia, no stored energy, no friction, and a perfect meshing of the teeth. Any inertia or bearing friction in an actual pair of gears can be represented by separate lumped elements in the free-body diagrams.

The relative sizes of the two gears result in a proportionality constant for the angular displacements, angular velocities, and transmitted torques of the respective shafts. For purposes of analysis, it is convenient to visualize the pair of ideal gears as two circles, shown in Figure 4.10(a), that are tangent at the contact point and rotate without slipping. The spacing between teeth must be equal for each gear in a pair, so the radii of the gears are proportional to the number of teeth. Thus if $r$ and $n$ denote the radius and number of teeth, respectively, then

$$\frac{r_2}{r_1} = \frac{n_2}{n_1} = N \tag{15}$$

where N is called the **gear ratio**.



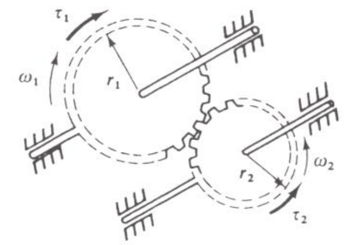



**FIGURE 4.9** A pair of gears.

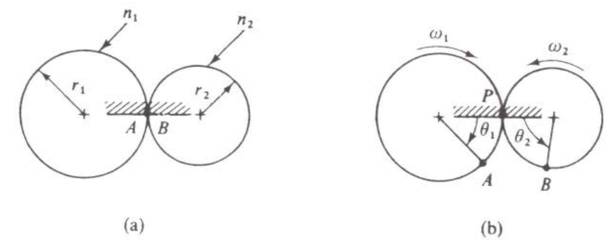



**FIGURE 4.10** Ideal gears. (a) Reference position. (b) After rotation.

Let points $A$ and $B$ in Figure 4.10(a) denote points on the circles that are in contact with each other at some reference time $t_0$. At some later time, points $A$ and $B$ will have moved to the positions shown in Figure 4.10(b), where $\theta_1$ and $\theta_2$ denote the respective angular displacements from their original positions. Because the arc lengths $PA$ and $PB$ must be equal,

$$r_1\theta_1 = r_2\theta_2 \tag{16}$$

which we can rewrite as

$$\frac{\theta_1}{\theta_2} = \frac{r_2}{r_1} = N \tag{17}$$

By differentiating (16) with respect to time, we see that the angular velocities are also related by the gear ratio:

$$\frac{\omega_1}{\omega_2} = \frac{r_2}{r_1} = N \tag{18}$$

Note that the positive directions of $\theta_1$ and $\theta_2$, and likewise of $\omega_1$ and $\omega_2$, are taken in opposite directions in the figure. Otherwise a negative sign would be introduced into (16), (17), and (18).

We can derive the torque relationship for a pair of gears by drawing a free-body diagram for each gear, as shown in Figure 4.11. The external torques applied to the gear shafts are denoted by $\tau_1$ and $\tau_2$. The force

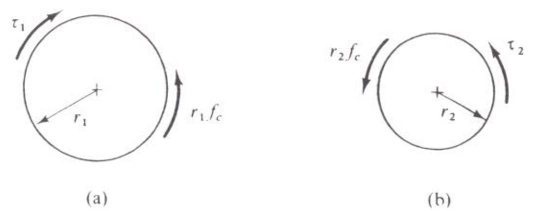



**FIGURE 4.11** Free-body diagrams for a pair of ideal gears.

exerted by each gear at the point of contact by its mate is $f_c$. By the law of reaction forces, the arrows must be in opposite directions for the two gears. The corresponding torques $r_1 f_c$ and $r_2 f_c$ are shown on the diagram. In addition to the contact force $f_c$, each gear must be supported by a bearing force of equal magnitude and opposite direction, because the gears have no translational motion. However, because the bearing support forces act through the center of the gear, they do not contribute to the torque and hence have been omitted from the figure. Because the gears have no inertia, the sum of the torques on each of the gears must be zero. Thus, from Figure 4.11,

$$\begin{aligned} f_c r_1 - \tau_1 &= 0 \\ f_c r_2 + \tau_2 &= 0 \end{aligned} \tag{19}$$

Eliminating the contact force $f_c$, the value of which is seldom of interest, we obtain

$$\frac{\tau_2}{\tau_1} = -\frac{r_2}{r_1} = -N \tag{20}$$

The minus sign in (20) should be expected, because both $\tau_1$ and $\tau_2$ in Figure 4.11 are shown as driving torques; that is, the reference arrows indicate that positive values of $\tau_1$ and $\tau_2$ both tend to make the gears move in the positive direction. The gears are assumed to have no inertia, so the two torques must actually be in opposite directions, and either $\tau_1$ or $\tau_2$ must be negative at any instant.

An alternative derivation of (20) makes use of the fact that the power supplied to the first gear is $p_1 = \tau_1\omega_1$ and the power supplied to the second gear is $p_2 = \tau_2\omega_2$. Since no energy can be stored in the ideal gears and since no power can be dissipated as heat due to the assumed absence of friction, conservation of energy requires that $p_1 + p_2 = 0$. Thus

$$\tau_1\omega_1 + \tau_2\omega_2 = 0$$

or

$$\frac{\tau_2}{\tau_1} = -\frac{\omega_1}{\omega_2} = -N$$

which agrees with (20).



## 4.3 INTERCONNECTION LAWS

The interconnection laws for rotational systems involve laws for torques and angular displacements that are analogous to (2.13) and (2.14) for translational mechanical systems. The law governing reaction torques has an important modification when compared to the one governing reaction forces.

### D'Alembert's Law

For a body with constant moment of inertia rotating about a fixed axis, we can write (6) as

$$\sum_i (\tau_{\text{ext}})_i - J\dot{\omega} = 0 \tag{21}$$

where the summation over $i$ includes all the torques acting on the body. Like the translational version of D'Alembert's law, the term $-J\dot{\omega}$ can be considered an **inertial torque**. When it is included along with all the other torques acting on the body, (21) reduces to the form appropriate for a body in equilibrium:

$$\sum_i \tau_i = 0 \tag{22}$$

In the application of (22), the torque $J\dot{\omega}$ is directed opposite to the positive sense of $\theta$, $\omega$, and $\alpha$. D'Alembert's law can also be applied to a junction point that has no moment of inertia, in which case the $J\dot{\omega}$ term vanishes.

### The Law of Reaction Torques

For bodies that are rotating about the same axis, any torque exerted by one element on another is accompanied by a **reaction torque** of equal magnitude and opposite direction on the first element. In Figure 4.12, for example, a counterclockwise torque $K_1\theta_1$ exerted by the shaft $K_1$ on the right disk is accompanied by a clockwise torque $K_1\theta_1$ exerted by the disk on the shaft. However, for bodies not rotating about the same axis, the magnitudes of the two torques are not necessarily equal. For the pair of gears shown in Figure 4.9, Figure 4.10, and Figure 4.11, the contact forces where the gears

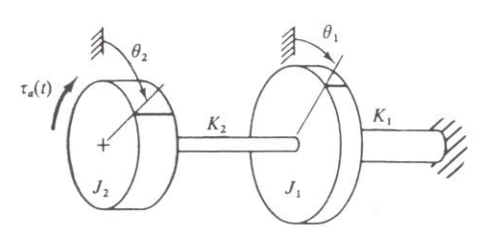



**FIGURE 4.12** Rotational system to illustrate the laws for reaction torques and angular displacements.

mesh are equal and opposite, but because the gears have different radii, the torque exerted by the first gear on the second has a different magnitude from the torque exerted by the second on the first.

### The Law for Angular Displacements

When examining the interconnection of elements in a system, we automatically use the fact that we can express the motions of some of the elements in terms of the motions of other elements. In Figure 4.12, the angular velocity of the left end of the shaft $K_1$ is identical to the angular velocity of the disk labeled $J_1$. Suppose that, in the same figure, the reference marks on the rims are at the top of the two disks when no torque is applied, but make the angles $\theta_1$ and $\theta_2$ with the vertical reference when the torque $\tau_a(t)$ is applied to the left disk. Then the net angular displacement for the shaft $K_2$ with respect to its unstressed condition is $\theta_2 - \theta_1$.

One way to summarize these facts in the form of a general interconnection law is to say that at any instant the algebraic sum of the differences in angular displacement around any closed path is zero. The equation

$$\sum_i (\Delta\theta)_i = 0 \quad \text{around any closed path} \tag{23}$$

is analogous to (2.14) for translational systems. It is understood that the signs in the summation take into account the direction in which the path is being traversed and that all measurements are with respect to reference positions that correspond to a single equilibrium condition for the system. If in Figure 4.12, for example, the relative differences in angular displacement are summed going from the vertical reference to disk 1, then to disk 2, and finally back to the vertical reference, (23) gives

$$(\theta_1 - 0) + (\theta_2 - \theta_1) + (0 - \theta_2) = 0$$

Although this is a rather trivial result, it does provide a formal basis for stating that the relative angular displacements for the shafts $K_1$ and $K_2$ are $\theta_1$ and $\theta_2 - \theta_1$, respectively. If for some reason the references for the angular displacements did not correspond to a single equilibrium condition, then the summation in (23) would be a constant other than zero.

In formulating the system equations, we normally use (23) automatically to reduce the number of displacement variables that must be shown on the diagram. Equation (23) could be differentiated to yield a comparable equation governing the relative velocities.

## 4.4 OBTAINING THE SYSTEM MODEL

The methods for using the element and interconnection laws to develop an appropriate mathematical model for a rotational system are the same as those discussed in Chapter 2 and Chapter 3 for translational systems. We indicate the assumed positive directions for the variables, and the assumed senses for the angular displacement, velocity, and acceleration of a body are chosen to be the same. We automatically use (23) to avoid introducing more symbols than are necessary to describe the motion. For each mass or junction point whose motion is unknown beforehand, we normally draw a free-body diagram showing all torques, including the inertial torque. We express all the torques except inputs in terms of angular displacements, velocities, or accelerations by means of the element laws. Then we apply D'Alembert's law, as given in (22), to each free-body diagram.



If we seek a set of state-variable equations, we identify the state variables and write those equations (such as $\dot{\theta} = \omega$) that do not require a free-body diagram. We write the equations obtained from the free-body diagrams by D'Alembert's law in terms of the state variables and inputs and manipulate them into the standard form of (3.31) or (3.36). Each of the state-variable equations should express the derivative of a different state variable as an *algebraic* function of the state variables, inputs, and possibly time. For any output of the system that is not one of the state variables, we need a separate algebraic equation.

If, on the other hand, we seek an input-output differential equation, we normally write the equations from the free-body diagrams in terms of only angular displacements or only angular velocities. Then we combine the equations to eliminate all variables except the input and output, which can be done by the $p$ operator described in Section 3.2. For single-input, single-output systems, the equation has the form of (3.25).

We shall now consider a number of examples involving free-body diagrams, state-variable equations, and input-output equations for rotational systems. Although these examples involve linear friction and stiffness elements, it is a simple matter to replace the appropriate terms by the nonlinear element laws when we encounter a nonlinear system. For example, a linear frictional torque $B\omega$ would become $\tau_B(\omega)$, which is a single-valued algebraic function that describes the nonlinear relationship.

### ▶ *EXAMPLE 4.1*

Derive the state-variable model for the rotational system shown in Figure 4.13(a) when the outputs are the angular acceleration of the disk and the counterclockwise torque exerted on the disk by the flexible shaft. Also find the input-output equation relating $\theta$ and $\tau_a(t)$.

**Solution**

The first step is to draw a free-body diagram of the disk. This is done in Figure 4.13(b), where the left face of the disk is shown. The torques acting on the disk are the spring torque $K\theta$, the viscous-frictional torque $B\omega$, and the applied torque $\tau_a(t)$. In addition, the inertial torque $J\dot{\omega}$ is indicated by

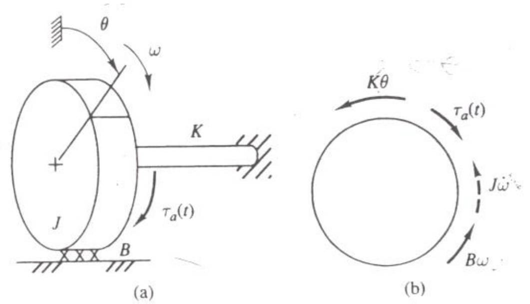



**FIGURE 4.13** (a) Rotational system for Example 4.1. (b) Free-body diagram.

the dashed arrow. By D'Alembert's law,

$$J\dot{\omega} + B\omega + K\theta = \tau_a(t) \tag{24}$$

The normal choices for the state variables are $\theta$ and $\omega$, which are related to the energy stored in the shaft and in the disk, respectively. One state-variable equation is $\dot{\theta} = \omega$, and the other can be found by solving (24) for $\dot{\omega}$. Thus

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= \frac{1}{J}[-K\theta - B\omega + \tau_a(t)] \end{aligned} \tag{25}$$

The output equations are

$$\begin{aligned} \alpha_M &= \frac{1}{J}[-K\theta - B\omega + \tau_a(t)] \\ \tau_K &= K\theta \end{aligned}$$

For the input-output equation with $\theta$ designated as the output, we merely rewrite (24) with all terms on the left side expressed in terms of $\theta$ and its derivatives. Then the desired result is

$$J\ddot{\theta} + B\dot{\theta} + K\theta = \tau_a(t)$$

### ▶ *EXAMPLE 4.2*

In the system shown in Figure 4.14(a), the two shafts are assumed to be flexible, with stiffness constants $K_1$ and $K_2$. The two disks, with moments of inertia $J_1$ and $J_2$, are supported by bearings whose friction is negligible compared with the viscous-friction elements denoted by the coefficients $B_1$ and $B_2$. The reference positions for $\theta_1$ and $\theta_2$ are the positions of the reference marks on the rims of the disks when the system contains no stored



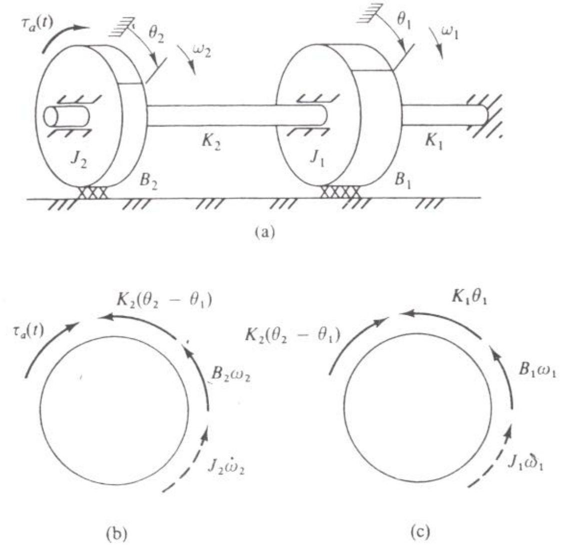



**FIGURE 4.14** (a) Rotational system for Example 4.2. (b), (c) Free-body diagrams.

energy. Find a state-variable model when the outputs are the counterclockwise torque exerted on $J_2$ by the shaft $K_2$ and the total angular momentum of the disks. Also find an input-output equation relating $\theta_2$ and $\tau_a(t)$.

**Solution**

The system has two inertia elements with independent angular velocities and two shafts with independent angular displacements, so four state variables are required. We choose $\theta_1$, $\theta_2$, $\omega_1$, and $\omega_2$, because they reflect the potential energy stored in each of the shafts and the kinetic energy stored in each disk.

The resulting free-body diagrams are shown in Figure 4.14(b) and Figure 4.14(c), where only the torque $K_2(\theta_2 - \theta_1)$, which is the reaction torque on the left disk by the shaft connecting the two disks, should require an explanation. The corresponding reference arrow has arbitrarily been drawn counterclockwise in Figure 4.14(b), implying that the torque is being treated as a retarding torque on disk 2. The actual torque will act in the counterclockwise direction only when $\theta_2 > \theta_1$, so the torque must be labeled $K_2(\theta_2 - \theta_1)$ and not $K_2(\theta_1 - \theta_2)$. By the law of reaction torques, the effect of the connecting shaft on disk 1 is a torque $K_2(\theta_2 - \theta_1)$ with its positive sense in the clockwise direction. We can reach the same conclusion by first selecting a clockwise sense for the arrow in Figure 4.14(c), thereby treating that torque as a driving torque on disk 1, and then noting that disk 2 will tend to drive disk 1 in the positive direction only if $\theta_2 > \theta_1$. Thus the correct expression is $K_2(\theta_2 - \theta_1)$. Of course, if we had selected a counterclockwise arrow in Figure 4.14(c), we would have labeled the arrow either $-K_2(\theta_2 - \theta_1)$ or $K_2(\theta_1 - \theta_2)$.

For each of the free-body diagrams, the algebraic sum of the torques may be set equal to zero by D'Alembert's law, giving the pair of equations

$$\begin{aligned} J_1\dot{\omega}_1 + B_1\omega_1 + K_1\theta_1 - K_2(\theta_2 - \theta_1) &= 0 \\ J_2\dot{\omega}_2 + B_2\omega_2 + K_2(\theta_2 - \theta_1) - \tau_a(t) &= 0 \end{aligned} \tag{26}$$

Two of the state-variable equations are $\dot{\theta}_1 = \omega_1$ and $\dot{\theta}_2 = \omega_2$, and we can find the other two by solving the two equations in (26) for $\dot{\omega}_1$ and $\dot{\omega}_2$, respectively. Thus

$$\begin{aligned} \dot{\theta}_1 &= \omega_1 \\ \dot{\omega}_1 &= \frac{1}{J_1}[-(K_1 + K_2)\theta_1 - B_1\omega_1 + K_2\theta_2] \\ \dot{\theta}_2 &= \omega_2 \\ \dot{\omega}_2 &= \frac{1}{J_2}[K_2\theta_1 - K_2\theta_2 - B_2\omega_2 + \tau_a(t)] \end{aligned} \tag{27}$$

The output equations are

$$\begin{aligned} \tau_{K_2} &= K_2(\theta_2 - \theta_1) \\ m_T &= J_1\omega_1 + J_2\omega_2 \end{aligned}$$

where $m_T$ is the total angular momentum.

To obtain an input-output equation, we rewrite (26) in terms of the angular displacements $\theta_1$ and $\theta_2$. A slight rearrangement of terms yields

$$J_1\ddot{\theta}_1 + B_1\dot{\theta}_1 + (K_1 + K_2)\theta_1 - K_2\theta_2 = 0 \tag{28a}$$

$$-K_2\theta_1 + J_2\ddot{\theta}_2 + B_2\dot{\theta}_2 + K_2\theta_2 = \tau_a(t) \tag{28b}$$

Neither of the equations in (28) can be solved separately, but we want to combine them into a single differential equation that does not contain $\theta_1$. Because $\theta_1$ appears in (28b) but none of its derivatives do, we rearrange that equation to solve for $\theta_1$ as

$$\theta_1 = \frac{1}{K_2}[J_2\ddot{\theta}_2 + B_2\dot{\theta}_2 + K_2\theta_2 - \tau_a(t)]$$

Substituting this result into (28a) gives

$$\begin{aligned} J_1J_2\theta_2^{(\text{iv})} + (J_1B_2 + J_2B_1)\theta_2^{(\text{iii})} + (J_1K_2 + J_2K_1 + J_2K_2 + B_1B_2)\ddot{\theta}_2 \\ + (B_1K_2 + B_2K_1 + B_2K_2)\dot{\theta}_2 + K_1K_2\theta_2 \\ = J_1\ddot{\tau}_a + B_1\dot{\tau}_a + (K_1 + K_2)\tau_a(t) \end{aligned} \tag{29}$$



which is the desired result. Equation (29) is a fourth-order differential equation relating $\theta_2$ and $\tau_a(t)$, in agreement with the fact that four state variables appear in (27).

---

In the last example, note the signs when like terms are gathered together in the torque equations corresponding to the free-body diagrams. In (28a) for $J_1$, all the terms involving $\theta_1$ and its derivatives have the same sign. Similarly in (28b) for $J_2$, the signs of all the terms with $\theta_2$ and its derivatives are the same. This is consistent with the comments made after Example 2.2 and can be used as a check on the work. Some insight into the reason for this can be obtained from the discussion of stability in Chapter 6.

### ▶ *EXAMPLE 4.3*

Find state-variable and input-output models for the system shown in Figure 4.14(a) and studied in Example 4.2, but with the shaft connecting disk 1 to the wall removed.

#### Solution

This example is analogous to Example 3.5. Because there are now only three energy-storing elements, we expect to need only three state variables. Two of these are chosen to be $\omega_1$ and $\omega_2$, which are related to the kinetic energy stored in the disks. The relative displacement of the ends of the connecting shaft is $\theta_2 - \theta_1$, which is related to the potential energy in that element. Hence we select as the third state variable

$$\theta_R = \theta_2 - \theta_1$$

although $\theta_1 - \theta_2$ would have been an equally good choice.

The free-body diagrams for each of the disks, with torques labeled in terms of the state variables and input, are shown in Figure 4.15. By D'Alembert's law,

$$\begin{aligned} J_1\dot{\omega}_1 + B_1\omega_1 - K_2\theta_R &= 0 \\ J_2\dot{\omega}_2 + B_2\omega_2 + K_2\theta_R &= \tau_a(t) \end{aligned} \tag{30}$$

We obtain one of the state-variable equations by noting that $\dot{\theta}_R = \dot{\theta}_2 - \dot{\theta}_1 = \omega_2 - \omega_1$, and we find the other two by rearranging the last two equations. Thus the third-order state-variable model is

$$\dot{\theta}_R = \omega_2 - \omega_1 \tag{31a}$$

$$\dot{\omega}_1 = \frac{1}{J_1}(-B_1\omega_1 + K_2\theta_R) \tag{31b}$$

$$\dot{\omega}_2 = \frac{1}{J_2}[-K_2\theta_R - B_2\omega_2 + \tau_a(t)] \tag{31c}$$

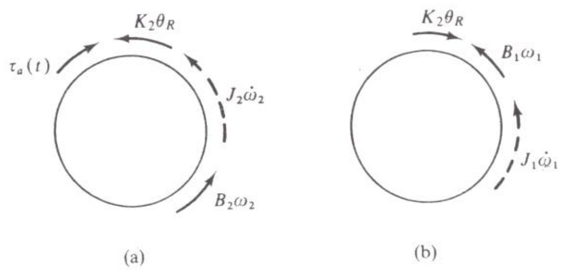



**FIGURE 4.15** Free-body diagrams for Example 4.3.

The output equations for $\tau_{K_2}$ and $m_T$ are

$$\tau_{K_2} = K_2\theta_R$$
$$m_T = J_1\omega_1 + J_2\omega_2$$

If one of the outputs were the angular displacement $\theta_2$, it would not be possible to write an output equation for $\theta_2$ as an algebraic function of $\theta_R$, $\omega_1$, $\omega_2$, and $\tau_a(t)$. In that case we would need four state variables. For the state-variable equations, we could either use (27) with $K_1 = 0$ or add to (31) the equation $\dot{\theta}_2 = \omega_2$.

The input-output differential equation relating $\theta_2$ and $\tau_a(t)$ can be obtained by letting $K_1 = 0$ in (29):

$$J_1J_2\theta_2^{(iv)} + (J_1B_2 + J_2B_1)\theta_2^{(iii)} + (J_1K_2 + J_2K_2 + B_1B_2)\ddot{\theta}_2 + (B_1 + B_2)K_2\dot{\theta}_2 = J_1\ddot{\tau}_a + B_1\dot{\tau}_a + K_2\tau_a(t)$$

Although this could be viewed as a third-order differential equation in the variable $\dot{\theta}_2$, which is the angular velocity $\omega_2$, we would need four initial conditions if we wanted to determine $\theta_2$ rather than $\omega_2$. This is consistent with the fact that four state variables are needed when the output is $\theta_2$.

We would expect the input-output equation relating $\theta_R$ and $\tau_a(t)$ to be strictly third-order, because only three state variables are needed for the output $\theta_R$. We see from (31a) that $\omega_1 = \omega_2 - \dot{\theta}_R$. Substituting this expression into (30), we obtain

$$J_1\dot{\omega}_2 + B_1\omega_2 - (J_1\ddot{\theta}_R + B_1\dot{\theta}_R + K_2\theta_R) = 0$$
$$J_2\dot{\omega}_2 + B_2\omega_2 + K_2\theta_R = \tau_a(t)$$

We can eliminate the variable $\omega_2$ from this pair of equations by using the $p$-operator method described in Section 3.2. When this is done, we find that

$$J_1J_2\dddot{\theta}_R + (J_1B_2 + J_2B_1)\ddot{\theta}_R + (J_1K_2 + J_2K_2 + B_1B_2)\dot{\theta}_R + (B_1 + B_2)K_2\theta_R = J_1\dot{\tau}_a + B_1\tau_a(t)$$



## ▶ EXAMPLE 4.4

The shaft supporting the disk in the system shown in Figure 4.16 is composed of two sections that have spring constants $K_1$ and $K_2$. Show how to replace the two sections by an equivalent stiffness element, and derive the state-variable model. The outputs of interest are the angular displacements $\theta$ and $\theta_A$.

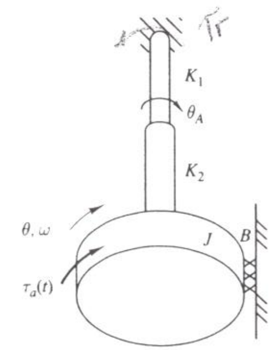



**FIGURE 4.16** Rotational system for Example 4.4.

### Solution

Free-body diagrams for each of the sections of the support shaft and for the inertia element $J$ are shown in Figure 4.17. No inertial torques are included on the shafts, because their moments of inertia are assumed to be negligible. The quantity $\tau_r$ is the reaction torque applied by the support on the top of

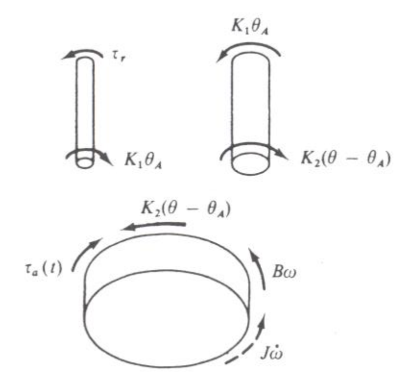



**FIGURE 4.17** Free-body diagrams for Example 4.4.

the shaft. Summing the torques on each of the free-body diagrams gives

$$K_1\theta_A - \tau_r = 0 \tag{32a}$$

$$K_2(\theta - \theta_A) - K_1\theta_A = 0 \tag{32b}$$

$$J\dot{\omega} + B\omega + K_2(\theta - \theta_A) - \tau_a(t) = 0 \tag{32c}$$

We need the first of these equations only if we wish to find $\tau_r$, which is not normally the case. Equations (32b) and (32c) can also be obtained by considering the free-body diagrams for just the disk and the massless junction of the two shafts. The corresponding free-body diagrams are shown in Figure 4.18. Applying D'Alembert's law to them yields (32b) and (32c).

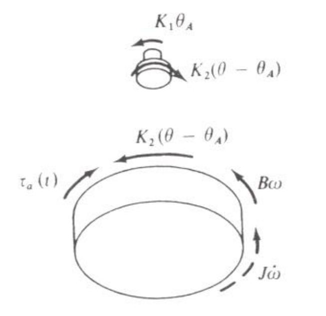



**FIGURE 4.18** Alternative free-body diagrams for Example 4.4.

We see from (32b) that $\theta_A$ and $\theta$ are proportional to each other. Specifically,

$$\theta_A = \left(\frac{K_2}{K_1 + K_2}\right)\theta \tag{33}$$

Substituting (33) into (32c) yields

$$J\dot{\omega} + B\omega + K_{eq}\theta = \tau_a(t) \tag{34}$$

where

$$K_{eq} = \frac{K_1K_2}{K_1 + K_2}$$

The parameter $K_{eq}$ can be regarded as an equivalent spring constant for the series combination of the two shafts. Selecting $\theta$ and $\omega$ as the state variables and using (34), we can write the state-variable model as

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= \frac{1}{J}[-K_{eq}\theta - B\omega + \tau_a(t)] \end{aligned} \tag{35}$$



The output $\theta$ is one of the state variables. The output equation for $\theta_A$ is given by (33).

---

## ▶ *EXAMPLE 4.5*

The system shown in Figure 4.19 consists of a moment of inertia $J_1$ corresponding to the rotor of a motor or a turbine, which is coupled to the moment of inertia $J_2$ representing a propeller. Power is transmitted through a fluid coupling with viscous-friction coefficient $B$ and a shaft with spring constant $K$. A driving torque $\tau_a(t)$ is exerted on $J_1$, and a load torque $\tau_L(t)$ is exerted on $J_2$. If the output is the angular velocity $\omega_2$, find the state-variable model and also the input-output differential equation.

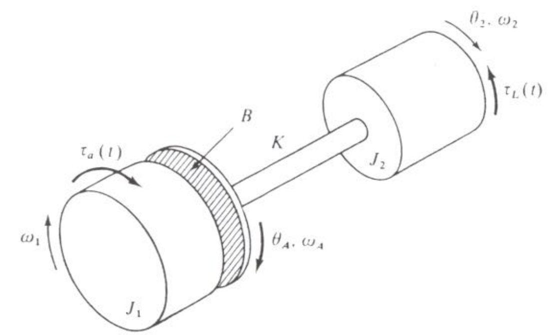



**FIGURE 4.19** Rotational system for Example 4.5.

### Solution

There are three independent energy-storing elements, so we select as state variables $\omega_1$, $\omega_2$, and the relative displacement $\theta_R$ of the two ends of the shaft, where

$$\theta_R = \theta_A - \theta_2 \tag{36}$$

Note that the equation

$$\dot{\theta}_R = \omega_A - \omega_2 \tag{37}$$

is not yet a state-variable equation because of the symbol $\omega_A$ on the right side.

Next we draw the free-body diagrams for the two inertia elements and for the shaft, as shown in Figure 4.20. Note that the moment of inertia of the right side of the fluid coupling element is assumed to be negligible. The directions of the arrows associated with the torque $B(\omega_1 - \omega_A)$ are consistent with the law of reaction torques and also indicate that the frictional torque tends to retard the relative motion

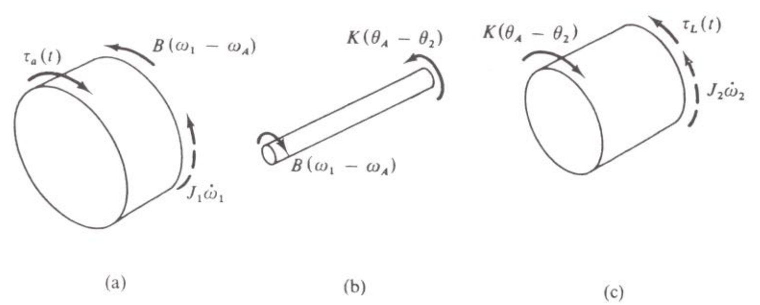



**FIGURE 4.20** Free-body diagrams for Example 4.5.

Setting the algebraic sum of the torques on each diagram equal to zero yields the three equations

$$J_1\dot{\omega}_1 + B(\omega_1 - \omega_A) - \tau_a(t) = 0 \tag{38a}$$

$$B(\omega_1 - \omega_A) - K(\theta_A - \theta_2) = 0 \tag{38b}$$

$$J_2\dot{\omega}_2 - K(\theta_A - \theta_2) + \tau_L(t) = 0 \tag{38c}$$

Using (36), we can rewrite (38) as

$$J_1\dot{\omega}_1 + B(\omega_1 - \omega_A) - \tau_a(t) = 0 \tag{39a}$$

$$B(\omega_1 - \omega_A) = K\theta_R \tag{39b}$$

$$J_2\dot{\omega}_2 - K\theta_R + \tau_L(t) = 0 \tag{39c}$$

Substituting (39b) into (39a) and repeating (39c) give

$$\begin{aligned} J_1\dot{\omega}_1 + K\theta_R - \tau_a(t) &= 0 \\ J_2\dot{\omega}_2 - K\theta_R + \tau_L(t) &= 0 \end{aligned} \tag{40}$$

Also from (39b),

$$\omega_A = \omega_1 - \frac{K}{B}\theta_R \tag{41}$$

Substituting (41) into (37) and rearranging (40) give the three state-variable equations

$$\dot{\theta}_R = -\frac{K}{B}\theta_R + \omega_1 - \omega_2 \tag{42a}$$

$$\dot{\omega}_1 = \frac{1}{J_1}[-K\theta_R + \tau_a(t)] \tag{42b}$$

$$\dot{\omega}_2 = \frac{1}{J_2}[K\theta_R - \tau_L(t)] \tag{42c}$$



Because the specified output is one of the state variables, a separate output equation is not needed as part of the state-variable model.

To obtain the input-output equation, we first rewrite (38) in terms of the angular velocities $\omega_1$, $\omega_2$, and $\omega_A$ and the torques $\tau_A(t)$ and $\tau_L(t)$. Differentiating (38b) and (38c) and noting that $\dot{\theta}_2 = \omega_2$ and $\dot{\theta}_A = \omega_A$, we have

$$J_1\dot{\omega}_1 + B(\omega_1 - \omega_A) = \tau_a(t)$$

$$B(\dot{\omega}_1 - \dot{\omega}_A) - K(\omega_A - \omega_2) = 0$$

$$J_2\ddot{\omega}_2 - K(\omega_A - \omega_2) + \dot{\tau}_L = 0$$

By using the $p$-operator technique to eliminate $\omega_A$ and $\omega_1$ from these equations, we can obtain the input-output equation

$$\dddot{\omega}_2 + \frac{K}{B}\ddot{\omega}_2 + K\left(\frac{1}{J_1} + \frac{1}{J_2}\right)\dot{\omega}_2 = \frac{K}{J_1J_2}\tau_a(t) - \frac{1}{J_2}\ddot{\tau}_L - \frac{K}{BJ_2}\dot{\tau}_L - \frac{K}{J_1J_2}\tau_L(t) \tag{43}$$

Although this result can be viewed as a second-order differential equation in $\dot{\omega}_2$, we will need three initial conditions if we are to determine $\omega_2$ rather than the acceleration $\dot{\omega}_2$. Note that if the load torque $\tau_L(t)$ were given as an algebraic function of $\omega_2$, as it would be in practice, $\omega_2$ would appear in (43). Then the input-output equation would be strictly third-order.

The next three examples contain either a lever or a pendulum that does not rotate about its midpoint. In Example 4.6 the mass of the lever is assumed to be negligible. The pendulum in the next example is approximated by a point mass at the end of a rigid bar. The final example includes a lever whose mass is uniformly distributed along the bar.

▶ ***EXAMPLE 4.6***

Find the state-variable equations for the system shown in Figure 4.21(a). Also find the output equation when the output is defined to be the force exerted on the pivot by the lever. The input is the displacement $x_4(t)$ of the right end of the spring $K_2$; it affects the mass $M$ through the lever. The lever has a fixed pivot and is assumed to be massless yet rigid. Its angular rotation $\theta$ is small so that only horizontal motion need be considered. In a practical situation, the springs $K_1$ and $K_2$ might represent the stiffness of the lever and of associated linkages that have a certain degree of flexibility.

**Solution**

The displacements $x_2$ and $x_3$ are directly proportional to the angle $\theta$ and hence to one another. Furthermore, the two springs appear to form a series

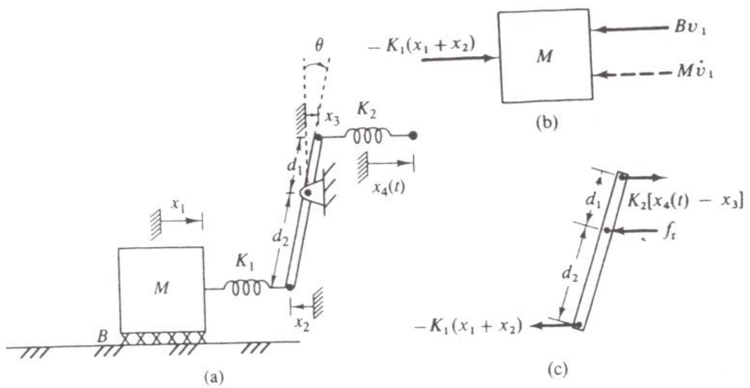



**FIGURE 4.21** (a) Translational system containing a lever. (b), (c) Free-body diagrams.

combination somewhat similar to the one shown in Figure 2.24(a), because the lever has no mass. Hence we can express $x_2, x_3$, and $\theta$ as algebraic functions of $x_1$ and $x_4(t)$. Thus we will select only $x_1$ and $v_1$ as state variables, with $x_4(t)$ being the input. By inspection, we determine that one of the required state-variable equations is $\dot{x}_1 = v_1$.

The next step is to draw free-body diagrams for the mass $M$ and the lever, as shown in Figure 4.21(b) and Figure 4.21(c). We must pay particular attention to the signs of the force arrows and to the expressions for the elongations of the springs. For example, the elongation of spring $K_1$ is $-(x_1 + x_2)$ because of the manner in which the displacements have been defined. Summing the forces on the mass $M$ yields

$$M\dot{v}_1 + Bv_1 + K_1(x_1 + x_2) = 0 \tag{44}$$

The forces on the lever are those exerted by the springs and the reaction force $f_r$ of the pivot. Because the lever's angle of rotation $\theta$ is small, the motion of the lever ends can be considered to be translational, obeying the relationships $\theta = x_2/d_2 = x_3/d_1$ and

$$x_3 = \left(\frac{d_1}{d_2}\right)x_2 \tag{45}$$

To obtain a second lever equation that involves $x_1$ and $x_2$ but not $f_r$, we sum moments about the pivot point, getting

$$K_2[x_4(t) - x_3]d_1 - K_1(x_1 + x_2)d_2 = 0 \tag{46}$$

Equations (45) and (46) can also be obtained directly from (12) and (14). In order to solve (44) for $\dot{v}_1$ as a function of only the state variables and the input, we must first express $x_2$ in terms of $x_1$ and $x_4(t)$. Substituting (45) into (46) and solving for $x_2$, we obtain the algebraic expression



$$x_2 = \frac{(d_1/d_2)K_2x_4(t) - K_1x_1}{K_1 + (d_1/d_2)^2K_2} \tag{47}$$

We find the second of the two state-variable equations by substituting (47) into (44) and rearranging terms so as to solve for $\dot{v}_1$. Doing this, we find that the state-variable equations are

$$\begin{aligned}\dot{x}_1 &= v_1\\ \dot{v}_1 &= -\frac{1}{M}\left[Bv_1 + \alpha K_1x_1 + \alpha\left(\frac{d_2}{d_1}\right)K_1x_4(t)\right]\end{aligned} \tag{48}$$

where

$$\alpha = \frac{1}{1 + \dfrac{K_1}{K_2}\left(\dfrac{d_2}{d_1}\right)^2} \tag{49}$$

To develop the output equation expressing $f_r$ as an algebraic function of the state variables and the input, we first sum the forces on the lever in Figure 4.21(c) to obtain

$$f_r = K_2[x_4(t) - x_3] + K_1(x_1 + x_2) \tag{50}$$

Substituting (45) and (47) into (50) gives

$$f_r = \left[K_1 + \left(K_2 - \frac{d_2}{d_1}K_1\right)\left(\frac{\alpha d_2 K_1}{d_1 K_2}\right)\right]x_1 + \left[K_2 + \alpha\left(\frac{d_2}{d_1}K_1 - K_2\right)\right]x_4(t)$$

which has the desired form.

In the last example it is instructive to consider the special case where $d_1 = d_2$, which corresponds to having the lever pivoted at its midpoint. From (49), $\alpha K_1 = K_1K_2/(K_1 + K_2)$, which is the equivalent spring constant for a series connection of the two springs, as in (2.37). Furthermore, the reader can readily verify that (48) reduces to the equations describing the system shown in Figure 4.22(a), in which the massless junction $A$ replaces the lever. In turn, this system is equivalent to that shown in Figure 4.22(b), where a single spring with the coefficient $K_{eq} = K_1K_2/(K_1+K_2)$ replaces the series spring connection. Note, in determining the motion of $M$, that the lever changes the effective direction of the displacement input $x_4(t)$.

In each of the next two examples, an object with mass $M$ rotates about an axis that does not pass through the center of mass. In order to obtain a complete description of all the forces acting on such an object, we would have to consider the motion of its center of mass. The acceleration of this point has one component tangential to the direction of motion and another component perpendicular to it, directed toward the pivot point. The force corresponding to the tangential acceleration is accounted for by the

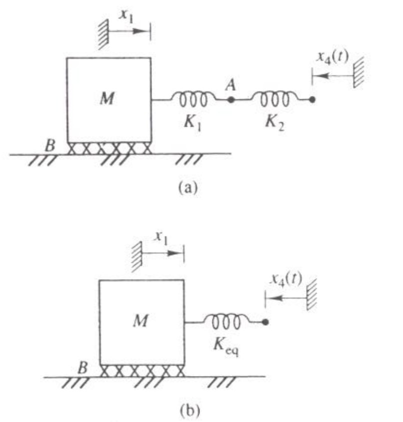



**FIGURE 4.22** Systems equivalent to Figure 4.21(a) when $d_1 = d_2$.

D'Alembert torque $J\dot{\omega}$. Corresponding to the perpendicular component of acceleration, there is a centrifugal force directed away from the pivot point. However, this centrifugal force does not result in a torque about the pivot. Similarly, the forces exerted by the fixed pivot do not contribute to such a torque.

Obtaining expressions for the forces corresponding to the acceleration of the center of mass can become cumbersome, and the interested reader should consult a book on mechanics. When drawing free-body diagrams for rotating bodies, we shall generally consider only those forces that yield a torque about the axis of rotation and shall not include the centrifugal and pivot forces. We shall not seek expressions for the pivot forces except when the mass of the rotating body is negligible or when the acceleration of the center of mass is zero. The first of these special cases was illustrated by Example 4.6 and Figure 4.21(c). The second occurs when the axis of rotation passes through the center of mass.

### ▶ *EXAMPLE 4.7*

The pendulum sketched in Figure 4.23(a) can be considered a point mass $M$ attached to a rigid massless bar of length $L$, which rotates about a pivot at the other end. An external torque $\tau_a(t)$ is applied to the bar by a mechanism that is not shown. Let $B$ denote the rotational viscous friction at the pivot point. Derive the input-output differential equation relating the angular displacement $\theta$ to the applied torque $\tau_a(t)$. Also write a set of state-variable equations. Simplify the expressions for the case where $|\theta|$ is restricted to small values.



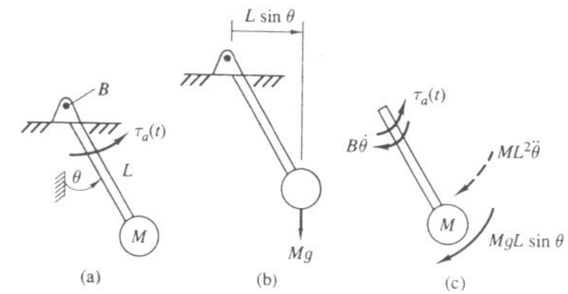



**FIGURE 4.23** (a) Pendulum for Example 4.7. (b) Partial diagram to determine the torque produced by the weight. (c) Free-body diagram showing all torques about the pivot.

**Solution**

As can be seen from Figure 4.23(b), the weight of the mass results in a clockwise torque $MgL\sin\theta$ on the bar. Because of the assumption of a point mass, its moment of inertia about the pivot is $J = ML^2$. Applying D'Alembert's law to the free-body diagram in Figure 4.23(c) yields the input-output equation

$$ML^2\ddot{\theta} + B\dot{\theta} + MgL\sin\theta = \tau_a(t) \tag{51}$$

In order to write a set of state-variable equations, we note that $\dot{\theta} = \omega$ and $\ddot{\theta} = \dot{\omega}$, where $\omega$ is the angular velocity. Then from (51) we have

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= \frac{1}{ML^2}[-MgL\sin\theta - B\omega + \tau_a(t)] \end{aligned} \tag{52}$$

Equations (51) and (52) are nonlinear because of the factor $\sin\theta$. In Chapter 9 we shall discuss methods for approximating nonlinear systems by linear models. For the present, we note that $\sin\theta \simeq \theta$ for small values of $\theta$. The quality of this approximation is reasonably good for $|\theta| \leq 0.5$ rad. This is indicated by the fact that when $\theta = 0.5$ rad, the deviation of $\theta$ from $\sin\theta$ is only 4.2%. Thus for small values of $\theta$, we can approximate (51) and (52) by the linear models

$$ML^2\ddot{\theta} + B\dot{\theta} + MgL\theta = \tau_a(t)$$

and

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= \frac{1}{ML^2}[-MgL\theta - B\omega + \tau_a(t)] \end{aligned}$$

▶ *EXAMPLE 4.8*

Find the state-variable equations for the system shown in Figure 4.21(a) when the lever's moment of inertia cannot be neglected. Continue to assume that the friction at the pivot point can be neglected and also that the angular rotation $\theta$ is small, so that the ends of the lever move essentially horizontally.

**Solution**

Let $J$ denote the lever's moment of inertia about the pivot point. There are now four independent energy-storing elements: $M$, $K_1$, $J$, and $K_2$. We choose the state variables to be $x_1$, $v_1$, $\theta$, and $\omega$, where $\omega$ denotes the clockwise angular velocity of the lever. Two of the four state-variable equations are $\dot{x}_1 = v_1$ and $\dot{\theta} = \omega$. The other two equations can be obtained from the free-body diagrams in Figure 4.24. The diagram for the mass is the same as in the previous example, but the lever is labeled with torques rather than forces.

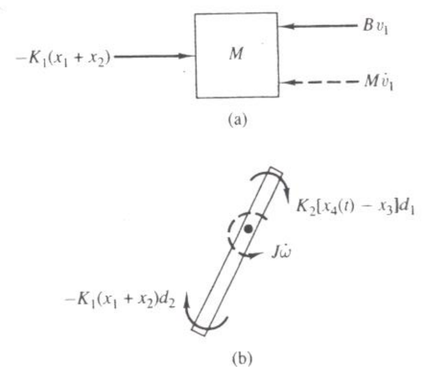



**FIGURE 4.24** Free-body diagrams for Example 4.8. (a) Forces on $M$. (b) Torques on the lever.

Summing forces on the mass $M$, we again have

$$M\dot{v}_1 + Bv_1 + K_1(x_1 + x_2) = 0$$

Summing torques about the lever's pivot point gives

$$J\dot{\omega} = K_2 d_1[x_4(t) - x_3] - K_1 d_2(x_1 + x_2)$$

With the substitutions $x_2 = d_2\theta$ and $x_3 = d_1\theta$, these two equations become

$$M_1\dot{v}_1 + Bv_1 + K_1x_1 + K_1d_2\theta = 0$$
$$J\dot{\omega} + K_2d_1^2\theta + K_1d_2^2\theta + K_1d_2x_1 = K_2d_1x_4(t)$$



Thus the state-variable equations are

$$\dot{x}_1 = v_1$$
$$\dot{v}_1 = \frac{1}{M_1}[-K_1x_1 - Bv_1 - K_1d_2\theta]$$
$$\dot{\theta} = \omega$$
$$\dot{\omega} = \frac{1}{J}[-K_1d_2x_1 - (K_1d_2^2 + K_2d_1^2)\theta + K_2d_1x_4(t)]$$

Each of the following two examples involves a pair of gears. The first example shows how gears can change the magnitude of the torque applied to a rotating body. When gears are used to couple two rotational subsystems, as in the second example, the parameters $J$, $B$, and $K$ are sometimes reflected in an appropriate manner from one side of the pair of gears to the other.

▶ *EXAMPLE 4.9*

Derive the state-variable equations for the gear-driven disk shown in Figure 4.25(a). The torque $\tau_a(t)$ is applied to a gear with radius $r_1$. The mating

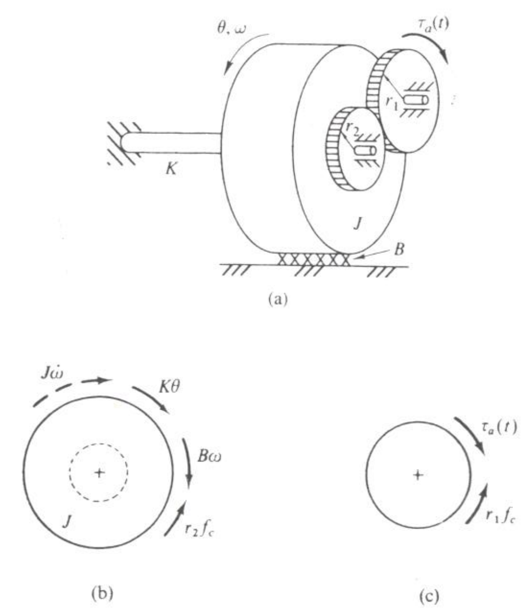



**FIGURE 4.25** (a) System for Example 4.9. (b), (c) Free-body diagrams.

gear with radius $r_2$ is rigidly connected to the moment of inertia $J$, which in turn is restrained by the flexible shaft $K$ and viscous damping $B$.

**Solution**

The free-body diagram for the disk and the gear attached to it is shown in Figure 4.25(b), and the diagram for the other gear is shown in Figure 4.25(c). The contact force where the gears mesh is denoted by $f_c$, and the corresponding torques are included on the diagrams. Rather than drawing a separate free-body diagram for the shaft, we show the torque $K\theta$ that it exerts on the disk in Figure 4.25(b). By D'Alembert's law,

$$J\dot{\omega} + B\omega + K\theta - r_2 f_c = 0 \qquad (53a)$$

$$r_1 f_c = \tau_a(t) \qquad (53b)$$

Solving (53b) for $f_c$ and substituting the result into (53a), we have

$$J\dot{\omega} + B\omega + K\theta = N\tau_a(t)$$

where $N = r_2/r_1$. If $\theta$ and $\omega$ are chosen as the state variables, we write

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= \frac{1}{J}[-K\theta - B\omega + N\tau_a(t)] \end{aligned} \qquad (54)$$

Note that (25), which describes a system identical to this one except for the gears, is identical to (54) with $N = 1$. Hence, as expected, the only effect of the gears is to multiply the applied torque $\tau_a(t)$ by the gear ratio $N$ and to reverse its direction on the disk.

▶ ***EXAMPLE 4.10***

Find the state-variable equations for the system shown in Figure 4.26(a), in which the pair of gears couples two similar subsystems.

**Solution**

Because of the two moments of inertia and the two shafts, it might appear that we could choose $\omega_1$, $\omega_2$, $\theta_1$, and $\theta_2$ as state variables. However, $\theta_1$ and $\theta_2$ are related by the gear ratio, as are $\omega_1$ and $\omega_2$. Because the state variables must be independent, either $\theta_1$ and $\omega_1$ or $\theta_2$ and $\omega_2$ constitute a suitable set.

The free-body diagrams for each of the moments of inertia are shown in Figure 4.26(b) and Figure 4.26(c). As in Example 4.9, $f_c$ represents the contact force between the two gears. Summing the torques on each of the



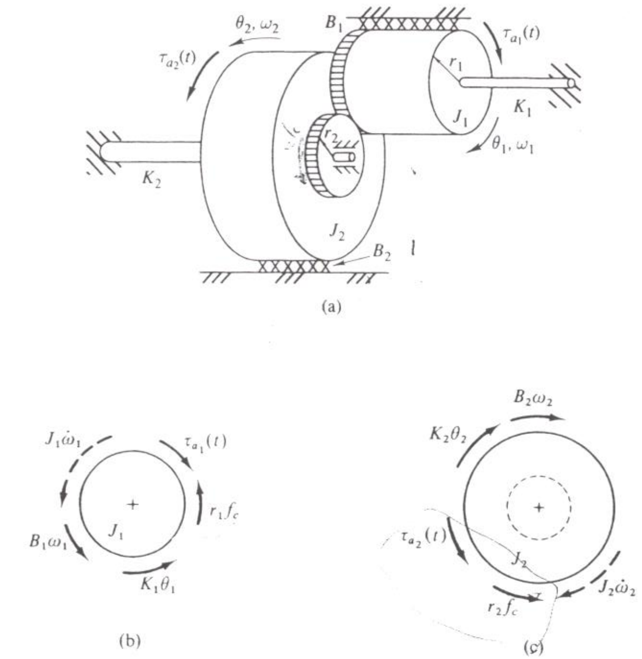



**FIGURE 4.26** (a) System for Example 4.10. (b), (c) Free-body diagrams.

free-body diagrams gives

$$J_1\dot{\omega}_1 + B_1\omega_1 + K_1\theta_1 + r_1 f_c = \tau_{a_1}(t) \qquad (55a)$$

$$J_2\dot{\omega}_2 + B_2\omega_2 + K_2\theta_2 - r_2 f_c = \tau_{a_2}(t) \qquad (55b)$$

By the geometry of the gears,

$$\begin{aligned} \theta_1 &= N\theta_2 \\ \omega_1 &= N\omega_2 \end{aligned} \qquad (56)$$

where $N = r_2/r_1$.

Selecting $\theta_2$ and $\omega_2$ as the state variables, we can write $\dot{\theta}_2 = \omega_2$ as the first state-variable equation and combine (55) and (56) to obtain the required equation for $\dot{\omega}_2$ in terms of $\theta_2$, $\omega_2$, $\tau_{a_1}(t)$, and $\tau_{a_2}(t)$. We first solve (55b) for $f_c$ and substitute that expression into (55a). Then, substituting (56) into the result gives

$$(J_2+N^2J_1)\dot{\omega}_2+(B_2+N^2B_1)\omega_2+(K_2+N^2K_1)\theta_2-N\tau_{a_1}(t)-\tau_{a_2}(t) = 0 \quad (57)$$

At this point, it is convenient to define the parameters

$$\begin{aligned} J_{2_{eq}} &= J_2 + N^2 J_1 \\ B_{2_{eq}} &= B_2 + N^2 B_1 \\ K_{2_{eq}} &= K_2 + N^2 K_1 \end{aligned} \tag{58}$$

which can be viewed as the combined moment of inertia, damping coefficient, and spring constant, respectively, when the combined system is described in terms of the variables $\theta_2$ and $\omega_2$. For example, it is common to say that $N^2 J_1$ is the equivalent inertia of disk 1 when that inertia is reflected to shaft 2. Similarly, $N^2 B_1$ and $N^2 K_1$ are the reflected viscous-friction coefficient and spring constant, respectively. Hence the parameters $J_{2_{eq}}$, $B_{2_{eq}}$, and $K_{2_{eq}}$ defined in (58) are the sums of the parameters associated with shaft 2 and the corresponding parameters reflected from shaft 1.

With the new notation, we can rewrite (57) as

$$J_{2_{eq}}\dot{\omega}_2 + B_{2_{eq}}\omega_2 + K_{2_{eq}}\theta_2 - N\tau_{a_1}(t) - \tau_{a_2}(t) = 0 \tag{59}$$

and the state-variable equations are

$$\begin{aligned} \dot{\theta}_2 &= \omega_2 \\ \dot{\omega}_2 &= \frac{1}{J_{2_{eq}}}\left[-K_{2_{eq}}\theta_2 - B_{2_{eq}}\omega_2 + N\tau_{a_1}(t) + \tau_{a_2}(t)\right] \end{aligned} \tag{60}$$

Note that the driving torque $\tau_{a_1}(t)$ applied to shaft 1 has the value $N\tau_{a_1}(t)$ when reflected to shaft 2.

If we wanted the system model in terms of $\theta_1$ and $\omega_1$, straightforward substitutions would lead to the equations

$$\begin{aligned} \dot{\theta}_1 &= \omega_1 \\ \dot{\omega}_1 &= \frac{1}{J_{1_{eq}}}\left[-K_{1_{eq}}\theta_1 - B_{1_{eq}}\omega_1 + \tau_{a_1}(t) + \frac{1}{N}\tau_{a_2}(t)\right] \end{aligned}$$

where the combined parameters with the elements associated with shaft 2 reflected to shaft 1 are

$$\begin{aligned} J_{1_{eq}} &= J_1 + \frac{1}{N^2}J_2 \\ B_{1_{eq}} &= B_1 + \frac{1}{N^2}B_2 \\ K_{1_{eq}} &= K_1 + \frac{1}{N^2}K_2 \end{aligned}$$



With the experience we have gained in deriving the mathematical models for separate translational and rotational systems, it is a straightforward matter to treat systems that combine both types of elements. The next example uses a rack and a pinion gear to convert rotational motion to translational motion. The final example combines translational and rotational systems via a cable attached to a disk.

### ▶ *EXAMPLE 4.11*

Derive the state-variable model for the system shown in Figure 4.27. The moment of inertia $J$ represents the rotor of a motor on which an applied torque $\tau_a(t)$ is exerted. The rotor is connected by a flexible shaft to a pinion gear of radius $R$ that meshes with the linear rack. The rack is rigidly attached to the mass $M$, which might represent the bed of a milling machine. The outputs of interest are the displacement and velocity of the rack and the contact force between the rack and the pinion.

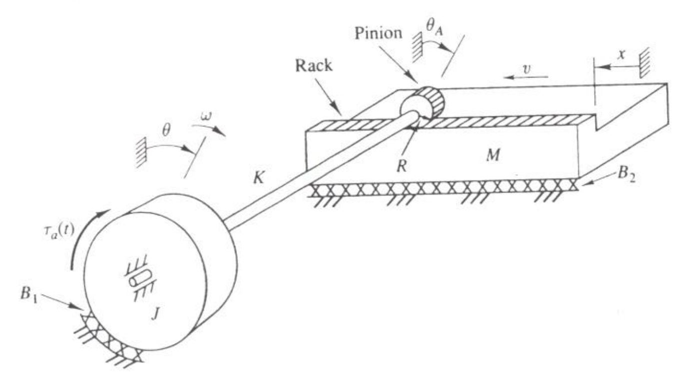



**FIGURE 4.27** System for Example 4.11 with rack and pinion gear.

#### Solution

The free-body diagrams for the moment of inertia $J$, the pinion gear, and the mass $M$ are shown in Figure 4.28. The contact force between the rack and the pinion is denoted by $f_c$. Forces and torques that will not appear in the equations of interest (such as the vertical force on the mass and the bearing forces on the rotor and pinion gear) have been omitted. Summing the torques in Figure 4.28(a) and Figure 4.28(b) and the forces in Figure 4.28(c) yields the three equations

$$J\dot{\omega} + B_1\omega + K(\theta - \theta_A) - \tau_a(t) = 0 \tag{61a}$$

$$Rf_c - K(\theta - \theta_A) = 0 \tag{61b}$$

$$M\dot{v} + B_2 v - f_c = 0 \tag{61c}$$

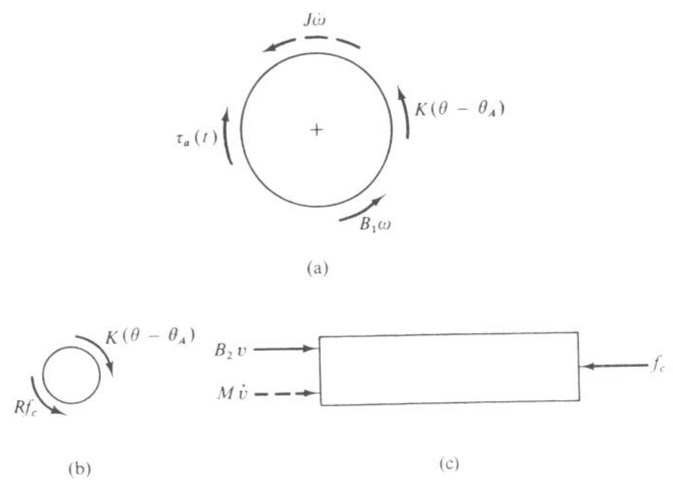



**FIGURE 4.28** Free-body diagrams for Example 4.11. (a) Rotor. (b) Pinion gear. (c) Mass.

In addition, the geometric relationship

$$R\theta_A = x \tag{62}$$

must hold because of the contact between the rack and the pinion gear.

The fact that there are three energy-storing elements corresponding to the parameters $J$, $M$, and $K$ suggests that the three variables $\omega$, $v$, and $\theta_R = \theta - \theta_A$ might constitute a satisfactory set of state variables. However, the displacement $x$ of the mass, which is usually of interest and which is one of the specified outputs, cannot be expressed as an algebraic function of $\omega$, $v$, $\theta_R$, and the input. Thus we need four state variables, which we choose to be $\theta$, $\omega$, $x$, and $v$. Using (62) to eliminate $\theta_A$ in (61a) gives

$$J\dot{\omega} + B_1\omega + K\theta - \frac{K}{R}x - \tau_a(t) = 0$$

and using (62) and (61b) to eliminate $f_c$ in (61c) results in

$$M\dot{v} + B_2 v + \frac{K}{R^2}x - \frac{K}{R}\theta = 0$$



Thus the desired state-variable equations are

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= \frac{1}{J}\left[-K\theta - B_1\omega + \frac{K}{R}x + \tau_a(t)\right] \\ \dot{x} &= v \\ \dot{v} &= \frac{1}{M}\left(\frac{K}{R}\theta - \frac{K}{R^2}x - B_2 v\right) \end{aligned} \tag{63}$$

The outputs $x$ and $v$ are also state variables. The output equation for $f_c$ can be found by substituting (62) into (61b). It is

$$f_c = \frac{K}{R}\left(\theta - \frac{x}{R}\right)$$

▶ ***EXAMPLE 4.12***

In the system shown in Figure 4.29, the mass and spring are connected to the disk by a flexible cable. Actually, the spring might be used to represent the stretching of the cable. The mass $M$ is subjected to the external force

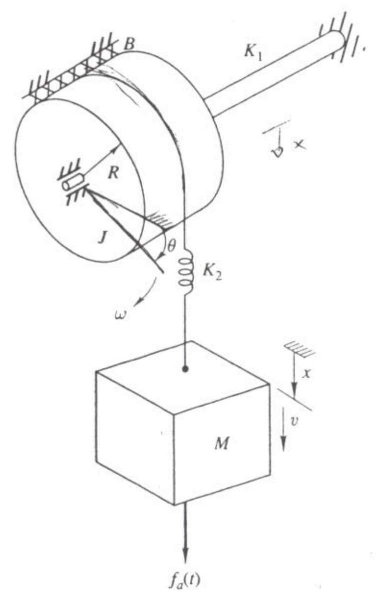



**FIGURE 4.29** System for Example 4.12 with translational and rotational elements.

$f_a(t)$ in addition to the gravitational force. Let $\theta$ and $x$ be measured from references corresponding to the position where the shaft $K_1$ is not twisted and the spring $K_2$ is not stretched. Find the state-variable model, treating $f_a(t)$ and the weight of the mass as inputs and the angular displacement $\theta$ and the tensile force in the cable as outputs.

**Solution**

The free-body diagrams for the disk and the mass are shown in Figure 4.30, where $f_2$ denotes the force exerted by the spring. The downward displacement of the top end of the spring is $R\theta$, so

$$f_2 = K_2(x - R\theta) \tag{64}$$

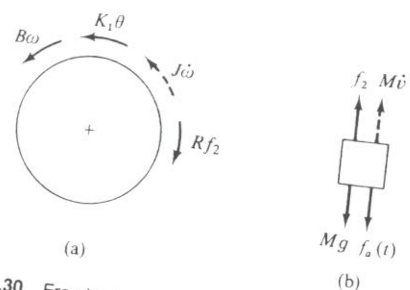



**FIGURE 4.30** Free-body diagrams for Example 4.12. (a) Disk. (b) Mass.

Because of the four energy-storing elements corresponding to the parameters $K_1$, $J$, $K_2$, and $M$, we select $\theta$, $\omega$, $x$, and $v$ as the state variables. From the free-body diagrams and with (64), we can write

$$J\dot{\omega} + B\omega + K_1\theta - RK_2(x - R\theta) = 0 \tag{65a}$$

$$M\dot{v} + K_2(x - R\theta) = f_a(t) + Mg \tag{65b}$$

Note that the reaction force $f_2 = K_2(x - R\theta)$ of the cable on the mass is not the same as the total external force $f_a(t) + Mg$ on the mass. As indicated by (65b), the difference is the inertial force $M\dot{v}$. Only if the mass were negligible would the external force be transmitted directly through the spring. From (65) and the identities $\dot{\theta} = \omega$ and $\dot{x} = v$, we can write the state-variable equations

$$\begin{aligned}
\dot{\theta} &= \omega \\
\dot{\omega} &= \frac{1}{J}[-(K_1 + K_2R^2)\theta - B\omega + K_2Rx] \\
\dot{x} &= v \\
\dot{v} &= \frac{1}{M}[K_2R\theta - K_2x + f_a(t) + Mg]
\end{aligned} \tag{66}$$



The only output that is not a state variable is the tensile force in the cable, for which the output equation is given by (64).

In order to emphasize the effect of the weight $Mg$, suppose that $f_a(t) = 0$ and that the mass and disk are not moving. Let $\theta_0$ denote the constant angular displacement of the disk and $x_0$ the constant displacement of the mass under these conditions. Then (65) becomes

$$\begin{aligned}
K_1\theta_0 &= RK_2(x_0 - R\theta_0) \\
K_2(x_0 - R\theta_0) &= Mg
\end{aligned} \tag{67}$$

from which

$$\begin{aligned}
\theta_0 &= \frac{RMg}{K_1} \\
x_0 &= \frac{Mg}{K_2} + \frac{R^2Mg}{K_1}
\end{aligned} \tag{68}$$

These expressions represent the constant displacements caused by the gravitational force $Mg$.

Now reconsider the case where $f_a(t)$ is nonzero and where the system is in motion. Let

$$\begin{aligned}
\theta &= \theta_0 + \phi \\
x &= x_0 + z
\end{aligned} \tag{69}$$

so that $\phi$ and $z$ represent the additional angular and vertical displacements caused by the input $f_a(t)$. Note that $\omega = \dot{\theta} = \dot{\phi}$ and $v = \dot{x} = \dot{z}$. Substituting (69) into (65) gives

$$J\dot{\omega} + B\omega + K_1(\theta_0 + \phi) - RK_2(x_0 + z - R\theta_0 - R\phi) = 0$$

$$M\dot{v} + K_2(x_0 + z - R\theta_0 - R\phi) = f_a(t) + Mg$$

Using (67) to cancel those terms involving $\theta_0$, $x_0$, and $Mg$, we are left with

$$J\dot{\omega} + B\omega + K_1\phi - RK_2(z - R\phi) = 0$$

$$M\dot{v} + K_2(z - R\phi) = f_a(t)$$

so the corresponding state-variable equations are

$$\begin{aligned}
\dot{\phi} &= \omega \\
\dot{\omega} &= \frac{1}{J}[-(K_1 + K_2R^2)\phi - B\omega + K_2Rz] \\
\dot{z} &= v \\
\dot{v} &= \frac{1}{M}[K_2R\phi - K_2z + f_a(t)]
\end{aligned} \tag{70}$$

We see that (66) and (70) have the same form, except that in the latter case the term $Mg$ is missing and $\theta$ and $x$ have been replaced by $\phi$ and $z$. As long as the stiffness elements are linear, we can ignore the gravitational force $Mg$ if we measure all displacements from the static-equilibrium positions corresponding to no inputs except gravity. This agrees with the conclusion reached in Examples 2.5 and 2.6. Note that if one of the desired outputs is the total tensile force in the cable, we must substitute (69) into (64) to get

$$f_2 = K_2(x_0 - R\theta_0) + K_2(z - R\phi) \tag{71}$$

where $x_0$ and $\theta_0$ are given by (68). The first of the two terms in (71) is the constant tensile force resulting only from the weight of the mass. The second term is the additional tensile force caused by the input $f_a(t)$.

## SUMMARY

In this chapter we extended the techniques of Chapters 2 and 3 to include bodies that rotate about fixed axes. Moment of inertia, friction, and stiffness elements are characterized by an algebraic relationship between the torque and the angular acceleration, velocity, or displacement, respectively. In contrast, levers and gears are described by algebraic equations that relate two variables of the same type (two displacements, for example, or two velocities). We showed how systems with levers, gears, or pulleys can have both rotational and translational motion.

As in Chapter 2, we drew free-body diagrams to help us obtain the basic equations governing the motion of the system. We then combined these equations into sets of state-variable equations or input-output equations. The same general modeling procedures are used for other types of systems in later chapters.

## PROBLEMS

**4.1** Write a differential equation for the system shown in Figure P4.1 and determine the equivalent spring constant.

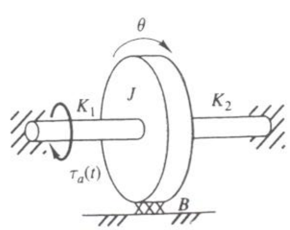



FIGURE P4.1



* **4.2** The left side of the fluid drive element denoted by $B$ in Figure P4.2 moves with the angular velocity $\omega_a(t)$. Find the input-output differential equation relating $\omega_2$ and $\omega_a(t)$.

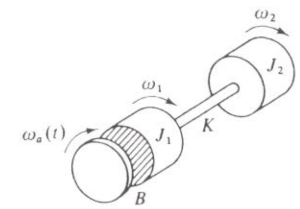



FIGURE P4.2

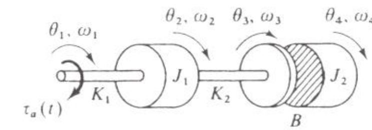



FIGURE P4.3

**4.3** **a)** Choose a set of state variables for the system shown in Figure P4.3, assuming that the values of the individual angular displacements $\theta_1, \theta_2, \theta_3$, and $\theta_4$ with respect to a fixed reference are not of interest. Then write the state-variable equations describing the system.
**b)** Find the input-output differential equation relating $\omega_4$ and $\tau_a(t)$.

**4.4** Repeat part (a) of Problem 4.3 if the torque input $\tau_a(t)$ is replaced by the angular velocity input $\omega_1(t)$.

**4.5** **a)** Find a state-variable model for the system shown in Figure P4.5. The input is the applied torque $\tau_a(t)$, and the output is the viscous torque on $J_2$, with the positive sense counterclockwise.
**b)** Write the input-output differential equation relating $\omega_2$ and $\tau_a(t)$ when $K_1 = K_2 = B = J_1 = J_2 = 1$ in a consistent set of units.

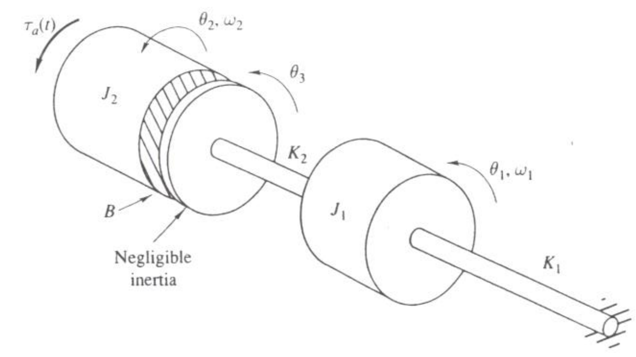



FIGURE P4.5

* **4.6** **a)** For the system shown in Figure P4.6, select state variables and write the model in state-variable form. The input is the applied torque $\tau_a(t)$, and the

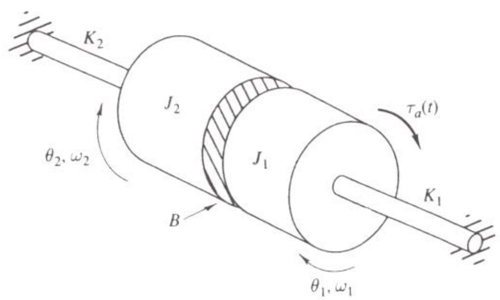
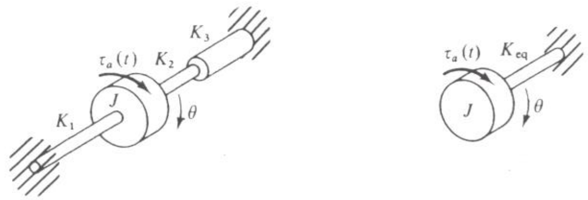



FIGURE P4.6

output is the viscous torque acting on $J_2$ through the fluid drive element, with the positive sense counterclockwise.

**b)** Write the input-output differential equation relating $\theta_2$ and $\tau_a(t)$ when $K_1 = K_2 = B = J_1 = J_2 = 1$ in a consistent set of units.

* **4.7** Find the value of $K_{eq}$ in Figure P4.7(b) such that the relationship between $\tau_a(t)$ and $\theta$ is the same as for Figure P4.7(a).




(a) (b)

FIGURE P4.7

**4.8** Use the $p$-operator technique to derive (43).

**4.9** For the system shown in Figure P4.9, the angular motion of the ideal lever from the vertical position is small, so the motion of the top and midpoint can be regarded as horizontal. The input is the force $f_a(t)$ applied at the top of the lever, and the output is the support force on the lever, taking the positive sense to the right.

**a)** Select a suitable set of state variables and write the corresponding state-variable model.

**b)** Find the input-output differential equation relating $x$ and $f_a(t)$.

* **4.10** In the mechanical system shown in Figure P4.10, the input is the applied force $f_a(t)$, and the output is the tensile force in the spring $K_2$. The lever is ideal and is horizontal when the system is in static equilibrium with $f_a(t) = 0$ and $M$ supported by the spring $K_1$. The displacements $x_1, x_2, x_3$, and $\theta$ are measured with respect to this equilibrium position. The lever angle $\theta$ remains small.



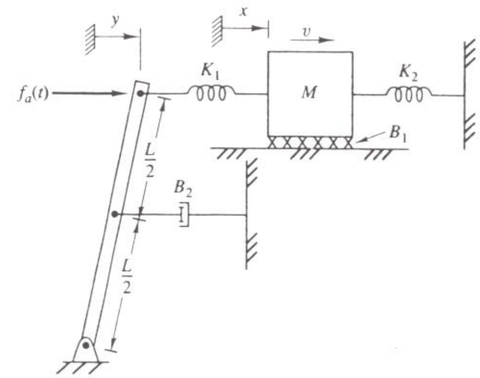



FIGURE P4.9

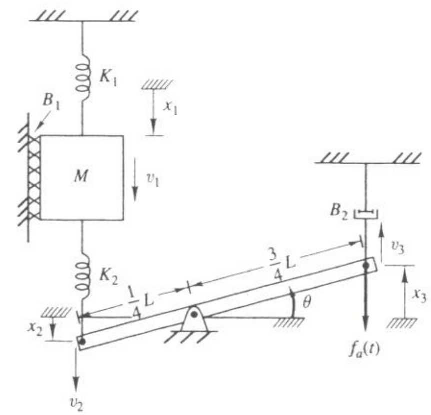



FIGURE P4.10

**a)** Taking $x_1$, $v_1$, and $\theta$ as state variables, write the state-variable model.

**b)** Also write the algebraic output equation for the reaction force of the pivot on the lever, taking the positive sense upward.

**4.11** For the system shown in Figure P4.11, the input is the force $f_a(t)$, and the output is $x_1$. Assume that the lever is ideal and that $|\theta|$ is small.

**a)** Select a suitable set of state variables and write the corresponding state-variable model.

**b)** Find the input-output differential equation when $a = 2$ and $b = K_1 = K_2 = B = M_1 = M_2 = 1$ in a consistent set of units.

**4.12** Repeat part (a) of Problem 4.9 when the moment of inertia of the lever cannot be neglected and the output is the displacement $x$.

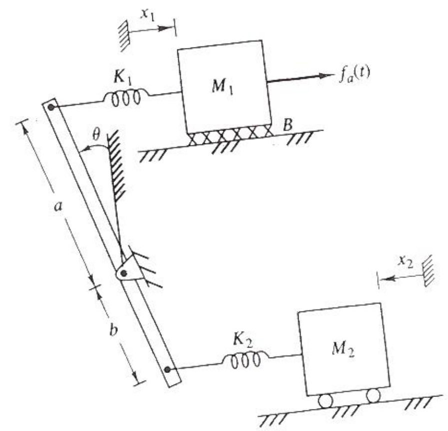



FIGURE P4.11

* **4.13** Repeat part (a) of Problem 4.10 when the moment of inertia of the lever canno[t] be neglected.

**4.14** Repeat part (a) of Problem 4.11 when the moment of inertia of the lever canno[t] be neglected.

**4.15** Figure P4.15 shows two pendulums suspended from frictionless pivots an[d] connected at their midpoints by a spring. Each pendulum may be considered a poi[nt] mass $M$ at the end of a rigid, massless bar of length $L$. Assume that $|\theta_1|$ and $|\theta$[…] are sufficiently small to allow use of the small-angle approximations $\sin\theta \simeq \theta$ an[d] $\cos\theta \simeq 1$. The spring is unstretched when $\theta_1 = \theta_2$.

**a)** Draw a free-body diagram for each pendulum.

**b)** Define a set of state variables and write the state-variable equations.

**c)** Write an algebraic output equation for the spring force. Consider the for[ce] to be positive when the spring is in tension.

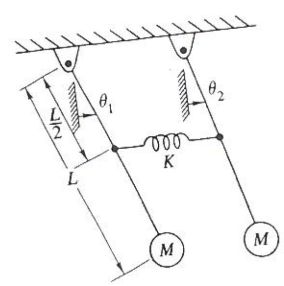



FIGURE P4.15

**4.16** Find the equivalent stiffness constant $K_{eq}$ such that the algebraic model o[f] gears and shafts shown in Figure P4.16 can be written as $\theta_1 = (1/K_{eq})\tau_a(t)$.



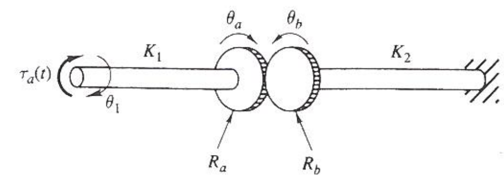



FIGURE P4.16

**4.17** In the system shown in Figure P4.17, a torque $\tau_a(t)$ is applied to the cylinder $J_1$. The gears are ideal with gear ratio $N = R_2/R_1$.

**a)** Write the input-output differential equation when the inertia $J_1$ is reflected to $J_2$ and $\omega_2$ is the output.

**b)** Write the equation when $J_2$ is reflected to $J_1$ and $\omega_1$ is the output.

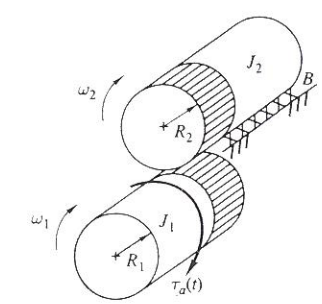



FIGURE P4.17

* **4.18** A torque input $\tau_a(t)$ is applied to the lower gear shown in Figure P4.18. There are two outputs: the total angular momentum and the viscous torque acting on $J_1$, with the positive sense clockwise.

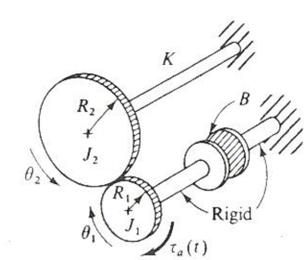



FIGURE P4.18

**a)** Write a pair of simultaneous differential equations describing the system, where the contact force between the gears is included.
**b)** Select state variables and write the model in state-variable form.
**c)** Derive an input-output differential equation relating $\theta_1$ and $\tau_a(t)$.

* **4.19** The input to the rotational system shown in Figure P4.19 is the applied torque $\tau_a(t)$ and the output is $\theta_3$. The gear ratio is $N = R_2/R_3$. Using $\theta_1, \omega_1, \theta_2$, and $\omega_2$ as the state variables, write the model in state-variable form.

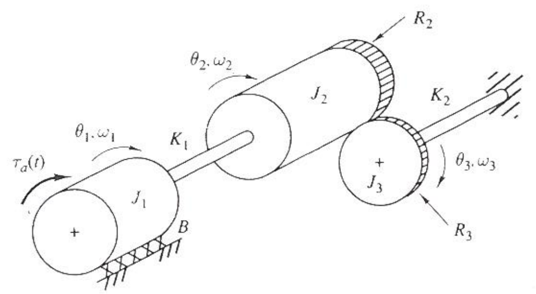



FIGURE P4.19

**4.20** Repeat Problem 4.19 when the state variables are $\theta_R = \theta_1 - \theta_2$, $\omega_1$, $\theta_3$, and $\omega_3$.

**4.21** The input for the drive system shown in Figure P4.21 is the applied torque $\tau_a(t)$, and a load attached to the moment of inertia $J_2$ produces the load torque $\tau_L = A|\omega_2|\omega_2$.
**a)** Taking as state variables $\omega_1$, $\omega_2$, $\omega_a$, $\phi_a = \theta_1 - \theta_a$ and $\phi_b = \theta_2 - \theta_b$, write the state-variable equations.
**b)** Write an algebraic output equation for the contact force on gear $a$, with the positive sense upward.

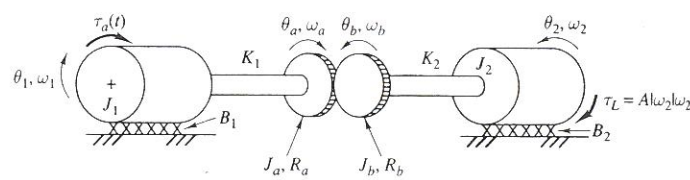



FIGURE P4.21

**4.22** The input to the combined translational and rotational system shown in Figure P4.22 is the force $f_a(t)$ applied to the mass $M$. The elements $K_1$ and $K_2$ are undeflected when $x = 0$ and $\theta = 0$.
**a)** Write a single input-output differential equation for $x$.
**b)** Write a single input-output differential equation for $\theta$.



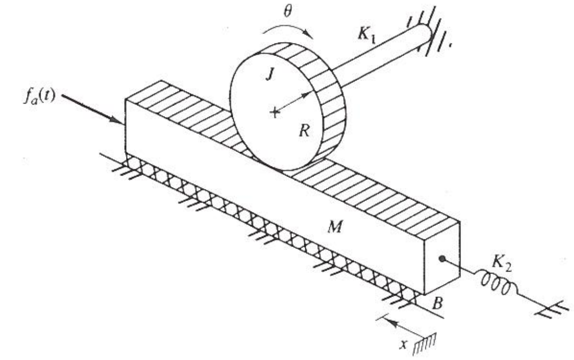



FIGURE P4.22

**4.23** Starting with (63), find the input-output differential equation for the system shown in Figure 4.27, taking $\tau_a(t)$ as the input and $x$ as the output.

**4.24** In the mechanical system shown in Figure P4.24, the cable is wrapped around the disk and does not slip or stretch. The input is the force $f_a(t)$, and the output is the displacement $x$. The springs are undeflected when $\theta = x = 0$.
**a)** Write the system model as a pair of differential equations involving only the variables $x$, $\theta$, $f_a(t)$, and their derivatives.
**b)** Select state variables and write the model in state-variable form.
**c)** Find an input-output differential equation relating $x$ and $f_a(t)$ when $K_1 = K_2 = K_3 = B_1 = B_2 = M = J = R = 1$ in a consistent set of units.

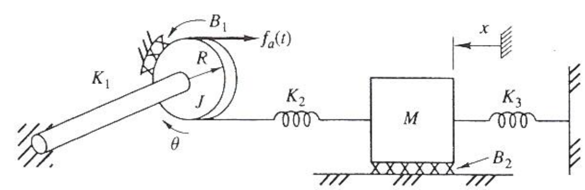



FIGURE P4.24

**4.25** A mass and a translational spring are suspended by cables wrapped around two sections of a drum as shown in Figure P4.25. The cables are assumed not to stretch, and the moment of inertia of the drum is $J$. The viscous-friction coefficient between the drum and a fixed surface is denoted by $B$. The spring is neither stretched nor compressed when $\theta = 0$.
**a)** Write a differential equation describing the system in terms of the variable $\theta$.
**b)** For what value of $\theta$ will the rotational element remain motionless?

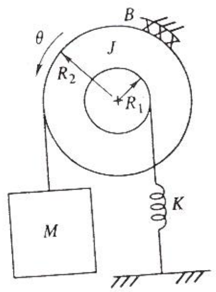



FIGURE P4.25

c) Rewrite the system's differential equation in terms of $\phi$, the relative angular displacement with respect to the static-equilibrium position you found in part (b).

* **4.26** In the system shown in Figure P4.26, the cable around the cylinder $J$ does not stretch and does not slip. The input is the applied force $f_a(t)$ and the outputs are $\theta$ and $z$, the additional elongation of $K$ from its static-equilibrium position due to $f_a(t)$. The spring is undeflected when $x_1 = x_2 = \theta = 0$. Select a set of state variables and write the model in state-variable form.

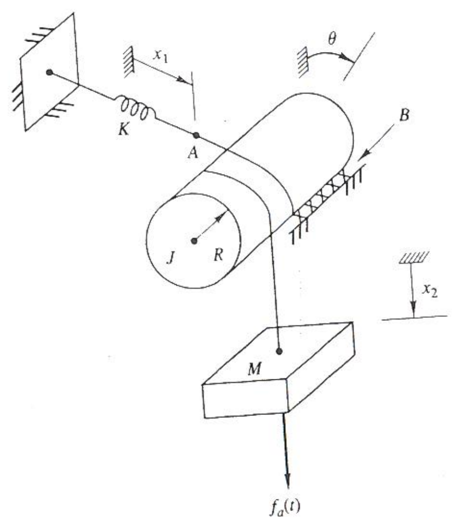



FIGURE P4.26



**4.27** In the system shown in Figure P4.27, the two masses are equal and the cable neither stretches nor slips. The springs are undeflected when $\theta = x_1 = x_2 = 0$. The input is the force $f_a(t)$ applied to the right mass, and the output is $x_1$. Select a set of state variables and write the model in state-variable form.

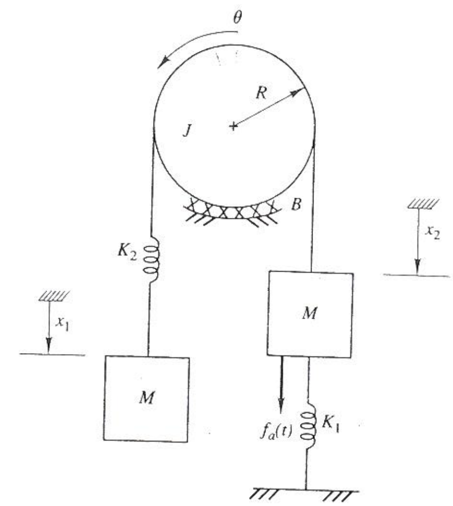



FIGURE P4.27

**4.28** The input to the system in Figure P4.28 is the applied torque $\tau_a(t)$. The shaft $K_1$ and the spring $K_2$ are undeflected when $\theta_1 = \theta_2 = x = 0$.

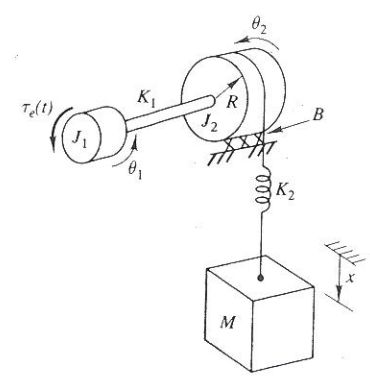



FIGURE P4.28

a) Write a set of differential equations describing the system in terms of $\theta_1, \theta_2, x$, and $\tau_a(t)$.
b) Select a set of state variables and write the state-variable equations.
c) Find the constant value of $\tau_a(t)$ for which the system will reach an equilibrium position with the mass remaining motionless. Determine the corresponding deflections of $K_1$ and $K_2$.

* **4.29** The input to the combined translational and rotational system shown in Figure P4.29 is the displacement $x_1(t)$. The output is the torque applied to the gear by the shaft, with the positive sense clockwise. The springs are undeflected when $\theta = x_1 = x_2 = 0$. Select an appropriate set of state variables and write the model in state-variable form.

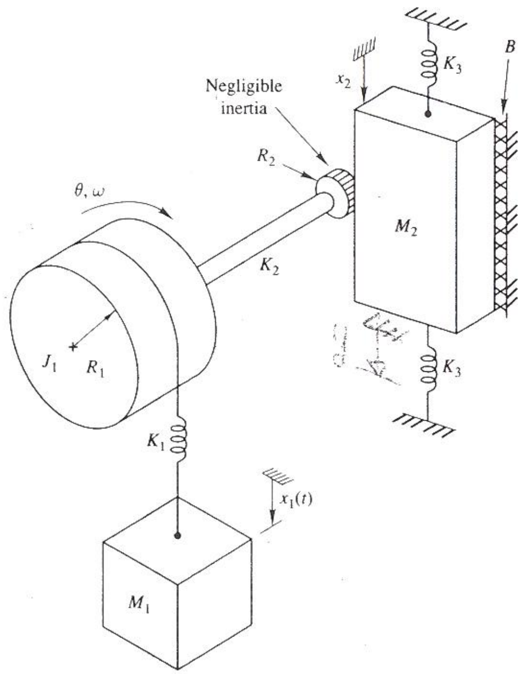



FIGURE P4.29

**4.30** Starting with (66), find the input-output differential equation for the system shown in Figure 4.29, taking $f_a(t)$ and the gravitational force as inputs and $\theta$ as the output.

**4.31** a) For the rotational mechanical system discussed in Example 4.3, write the state-variable equations (31) in matrix form. Show why we cannot obtain either $\theta_1$ or $\theta_2$ from this model.



b) Add $\theta_1$ as the fourth state variable and write the state-variable equations in matrix form.

* **4.32** For the combined translational and rotational system modeled in Example 4.12, write (66) in matrix form. Note that the system has two inputs, the gravitational constant $g$ and the applied force $f_a(t)$.

**4.33** For the combined translational and rotational system modeled in Example 4.12, write (70) in matrix form. Also write a matrix output equation for the displacements $\theta$ and $x$ defined in (69).