CHAPTER 12

HYDRAULIC SYSTEMS

A hydraulic system is one in which liquids, generally considered incompressible, flow. Hydraulic systems commonly appear in chemical processes, automatic control systems, and actuators and drive motors for manufacturing equipment. Such systems are usually interconnected to mechanical systems through pumps, valves, and movable pistons. A turbine driven by water and used for driving an electric generator is an example of a system with interacting hydraulic, mechanical, and electrical elements. We will not discuss here the more general topic of fluid systems, which would include compressible fluids such as gases and air.

An exact analysis of hydraulic systems is usually not feasible because of their distributed nature and the nonlinear character of the resistance to flow. For our dynamic analysis, however, we can obtain satisfactory results by using lumped elements and linearizing the resulting nonlinear mathematical models. On the other hand, the design of chemical processes requires a more exact analysis wherein static, rather than dynamic, models are used.

In most cases, hydraulic systems operate with the variables remaining close to a specific operating point. Thus we are generally interested in models involving incremental variables. This fact is particularly helpful because such models are usually linear, although the model in terms of the total variables may be quite nonlinear.

In the next two sections, we shall define the variables to be used and introduce and illustrate the element laws. Then we will present a variety of examples to demonstrate the modeling process and the application of the analytical techniques discussed in previous chapters, including Laplace transforms and transfer functions.

12.1 VARIABLES

Because hydraulic systems involve the flow and accumulation of liquid, the variables used to describe their dynamic behavior are

w, flow rate in cubic meters per second (m^3/s)

v, volume in cubic meters (m^3)

h, liquid height in meters (m)

p, pressure in newtons per square meter (N/m^2)

Unless otherwise noted, a pressure will be the **absolute pressure**. In addition, we shall sometimes find it convenient to express pressures in terms of gauge pressures. A **gauge pressure**, denoted by p^*, is defined to be the difference between the absolute pressure and the atmospheric pressure p_a:

$$p^*(t) = p(t) - p_a \tag{1}$$

A pressure difference, denoted by Δp, is the difference between the pressures at two points.

12.2 ELEMENT LAWS

Hydraulic systems exhibit three types of characteristics that can be approximated by lumped elements: capacity, resistance to flow, and inertance. In this section we shall discuss the first two. The inertance, which accounts for the kinetic energy of a moving fluid stream, is usually negligible, and we will not consider it. A brief discussion of centrifugal pumps that act as hydraulic sources appears at the end of this section.

Capacitance

When liquid is stored in an open vessel, there is an algebraic relationship between the volume of the liquid and the pressure at the base of the vessel. If the cross-sectional area of the vessel is given by the function $A(h)$, where h is the height of the liquid level above the bottom of the vessel, then the liquid volume v is the integral of the area from the base of the vessel to the top of the liquid. Hence,

$$v = \int_0^h A(\lambda)d\lambda \tag{2}$$

where λ is a dummy variable of integration. For a liquid of density ρ expressed in kilograms per cubic meter, the absolute pressure p and the liquid height h are related by

$$p = \rho g h + p_a \tag{3}$$

where g is the gravitational constant (9.807 m/s^2) and where p_a is the atmospheric pressure, which is taken as 1.013×10^5 N/m^2.

Equations (2) and (3) imply that for any vessel geometry, liquid density, and atmospheric pressure, there is a unique algebraic relationship between the pressure p and the liquid volume v. A typical characteristic curve describing this relationship is shown in Figure 12.1(a).

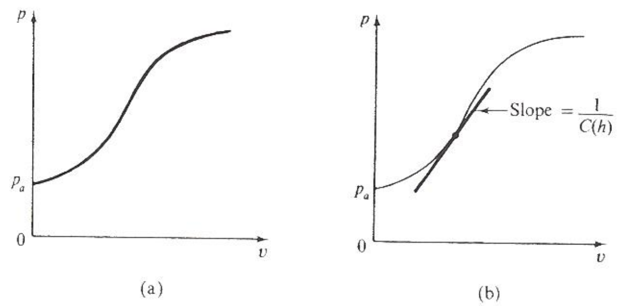


FIGURE 12.1 Pressure versus liquid volume for a vessel with variable cross-sectional area $A(h)$.

If the tangent to the pressure-versus-volume curve is drawn at some point, as shown in Figure 12.1(b), then the reciprocal of the slope is defined to be the **hydraulic capacitance**, denoted by $C(h)$. As indicated by the h in parentheses, the capacitance depends on the point on the curve being considered and hence on the liquid height h. Now

$$C(h) = \frac{1}{\dfrac{dp}{dv}} = \frac{dv}{dp}$$

and, from the chain rule of differentiation,

$$C(h) = \frac{dv}{dh}\frac{dh}{dp}$$

We see that $dv/dh = A(h)$ from (2) and that $dh/dp = 1/\rho g$ from (3). Thus for a vessel of arbitrary shape,

$$C(h) = \frac{A(h)}{\rho g} \tag{4}$$

which has units of m^4·s^2/kg or, equivalently, m^5/N.

For a vessel with constant cross-sectional area A, (2) reduces to $v = Ah$. We can substitute the height $h = v/A$ into (3) to obtain the pressure in terms of the volume:

$$p = \frac{\rho g}{A} v + p_a \tag{5}$$

Equation (5) yields a linear plot of pressure versus volume, as shown in Figure 12.2. The slope of the line is the reciprocal of the capacitance C, where

$$C = \frac{A}{\rho g} \tag{6}$$

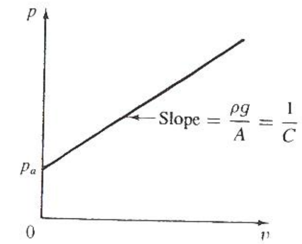


FIGURE 12.2 Pressure versus liquid volume for a vessel with constant A.

The volume of liquid in a vessel at any instant is the integral of the net flow rate into the vessel plus the initial volume. Hence we can write

$$v(t) = v(0) + \int_0^t [w_{\text{in}}(\lambda) - w_{\text{out}}(\lambda)]d\lambda$$

which can be differentiated to give the alternative form

$$\dot{v} = w_{\text{in}}(t) - w_{\text{out}}(t) \tag{7}$$

To obtain expressions for the time derivatives of the pressure p and the liquid height h that are valid for vessels with variable cross-sectional areas, we use the chain rule of differentiation to write

$$\frac{dv}{dt} = \frac{dv}{dh}\frac{dh}{dt}$$

where dv/dt is given by (7) and where $dv/dh = A(h)$. Thus the rate of change of the liquid height depends on the net flow rate according to

$$\dot{h} = \frac{1}{A(h)}[w_{\text{in}}(t) - w_{\text{out}}(t)] \tag{8}$$

Alternatively, we can write dv/dt as

$$\frac{dv}{dt} = \frac{dv}{dp}\frac{dp}{dt}$$

where $dv/dp = C(h)$. Hence the rate of change of the pressure at the base of the vessel is

$$\dot{p} = \frac{1}{C(h)}[w_{\text{in}}(t) - w_{\text{out}}(t)] \tag{9}$$

where $C(h)$ is given by (4).

Because any of the variables v, h, and p can be used as a measure of the amount of liquid in a vessel, we generally select one of them as a state variable. Then (7), (8), or (9) will yield the corresponding state-variable equation when w_{in} and w_{out} are expressed in terms of the state variables and inputs.

If the cross-sectional area of the vessel is variable, then the coefficient $A(h)$ in (8) will be a function of h, and the system model will be nonlinear. To develop a linearized model, we must find the operating point, define the incremental variables, and retain the first two terms in the Taylor-series expansion. Likewise, the term $C(h)$ in (9) will cause the differential equation to be nonlinear because the capacitance varies with h, which in turn is a function of the pressure.

▶ *EXAMPLE 12.1*

Consider a vessel formed by a circular cylinder of radius R and length L that contains a liquid of density ρ in units of kilograms per cubic meter. Find the hydraulic capacitance of the vessel when the cylinder is vertical, as shown in Figure 12.3(a). Then evaluate the capacitance when the cylinder is on its side, as shown in Figure 12.3(b).

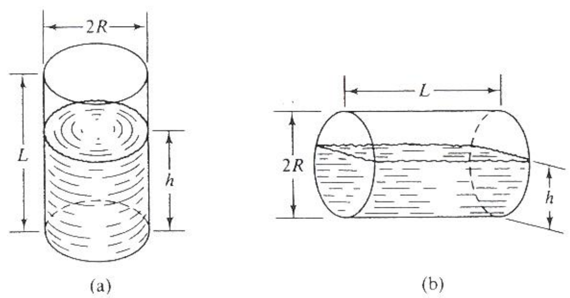


FIGURE 12.3 Cylindrical vessel for Example 12.1. (a) Cylinder vertical. (b) Cylinder horizontal.

Solution

For the configuration shown in Figure 12.3(a), the cross-sectional area is πR^2 and so is independent of the liquid height. Thus we can use (6), and the vessel's hydraulic capacitance is $C_a = \pi R^2/\rho g$.

When the vessel is on its side, as shown in Figure 12.3(b), the cross-sectional area is a function of the liquid height h. You can verify that the width of the liquid surface is $2\sqrt{R^2 - (R-h)^2}$, which is zero when $h = 0$ and $h = 2R$ and which has a maximum value of $2R$ when $h = R$. Using

(4), we find that the capacitance is

$$C_b = \frac{2L}{\rho g}\sqrt{R^2 - (R-h)^2}$$

which is shown in Figure 12.4.

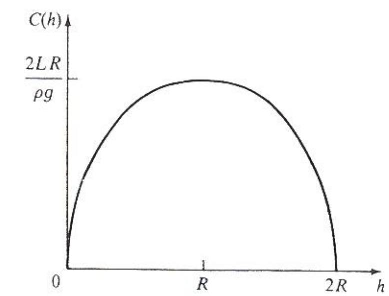


FIGURE 12.4 Capacitance of the vessel shown in Figure 12.3(b).

Resistance

As liquid flows through a pipe, there is a drop in the pressure of the liquid over the length of pipe. There is likewise a pressure drop if the liquid flows through a valve or through an orifice. The change in pressure associated with a flowing liquid results from the dissipation of energy and usually obeys a nonlinear algebraic relationship between the flow rate w and the pressure difference Δp. The symbol for a valve is shown in Figure 12.5; it can also be used for other energy-dissipating elements. A positive value of w indicates that liquid is flowing in the direction of the arrow; a positive value of Δp indicates that the pressure at the end marked + is higher than the pressure at the other end. The expression

$$w = k\sqrt{\Delta p} \tag{10}$$

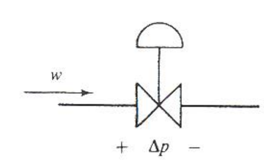


FIGURE 12.5 Symbol for a hydraulic valve.

describes an orifice and a valve and is a good approximation for turbulent flow through pipes. We can treat all situations of interest to us by using a nonlinear element law of the form of (10). In this equation, k is a constant that depends on the characteristics of the pipe, valve, or orifice. A typical curve of flow rate versus pressure difference is shown in Figure 12.6(a).

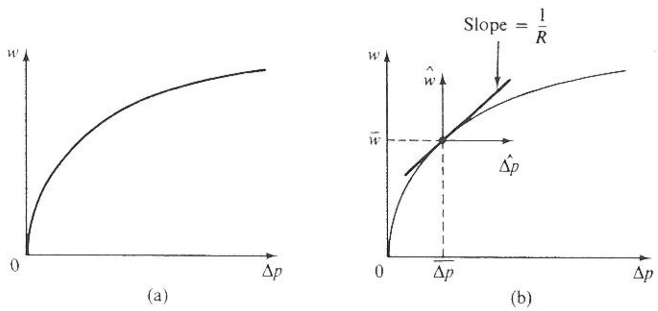


FIGURE 12.6 (a) Flow rate versus pressure difference given by (10). (b) Geometric interpretation of hydraulic resistance.

Because (10) is a nonlinear relationship, we must linearize it about an operating point in order to develop a linear model of a hydraulic system. If we draw the tangent to the curve of w versus Δp at the operating point, the reciprocal of its slope is defined to be the **hydraulic resistance** R. Figure 12.6(b) illustrates the geometric interpretation of the resistance, which has units of newton-seconds per meter^5.

Expanding (10) in a Taylor series about the operating point gives

$$w = \overline{w} + \left.\frac{dw}{d\,\Delta p}\right|_{\overline{\Delta p}} (\Delta p - \overline{\Delta p}) + \cdots$$

The incremental variables $\hat{w}$ and $\widehat{\Delta p}$ are defined by

$$\hat{w} = w - \overline{w} \tag{11a}$$

$$\widehat{\Delta p} = \Delta p - \overline{\Delta p} \tag{11b}$$

and the second- and higher-order terms in the expansion are dropped. Thus the incremental model becomes

$$\hat{w} = \frac{1}{R}\widehat{\Delta p} \tag{12}$$

where

$$\frac{1}{R} = \left.\frac{dw}{d\,\Delta p}\right|_{\overline{\Delta p}}$$

We can express the resistance R in terms of either $\overline{\Delta p}$ or $\overline{w}$ by carrying out the required differentiation using (10). Specifically,

$$\frac{1}{R} = \left.\frac{d}{d\,\Delta p}(k\,\Delta p^{1/2})\right|_{\overline{\Delta p}} = \frac{k}{2\sqrt{\overline{\Delta p}}}$$

so

$$R = \frac{2\sqrt{\overline{\Delta p}}}{k} \tag{13}$$

To express the resistance in terms of $\overline{w}$, we note from (10) that

$$\overline{w} = k\sqrt{\overline{\Delta p}} \tag{14}$$

Substituting (14) into (13) gives the alternative equation for the hydraulic resistance as

$$R = \frac{2\overline{w}}{k^2} \tag{15}$$

Because liquids typically flow through networks composed of pipes, valves, and orifices, we must often combine several relationships of the form of (10) into a single equivalent expression. We use linearized models in much of our analysis of hydraulic systems, so it is important to develop rules for combining the resistances of linearized elements that occur in series and parallel combinations. In the following example, we consider the relationship of flow versus pressure difference and the equivalent resistance for two valves in series. The parallel-flow situation is treated in one of the end-of-chapter problems.

▶ *EXAMPLE 12.2*

Figure 12.7(a) shows a series combination of two valves through which liquid flows at a rate of w and across which the pressure difference is Δp. An equivalent valve is shown in Figure 12.7(b). The two valves obey

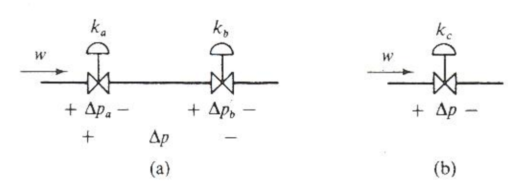


FIGURE 12.7 (a) Two valves in series. (b) Equivalent valve.

the relationships $w = k_a\sqrt{\Delta p_a}$ and $w = k_b\sqrt{\Delta p_b}$, respectively. Find the coefficient k_c of an equivalent valve that obeys the relationship $w = k_c\sqrt{\Delta p}$. Also evaluate the resistance R_c of the series combination in terms of the individual resistances R_a and R_b.

Solution

Because the two valves are connected in series, they have the same flow rate w, and the total pressure difference is $\Delta p = \Delta p_a + \Delta p_b$. To determine k_c, we write Δp in terms of w as

$$\Delta p = \Delta p_a + \Delta p_b = \left(\frac{1}{k_a^2} + \frac{1}{k_b^2}\right) w^2$$

and then solve for w in terms of Δp. After some manipulations, we find that

$$w = \left(\frac{k_a k_b}{\sqrt{k_a^2 + k_b^2}}\right)\sqrt{\Delta p} \tag{16}$$

Comparing (16) with (10) reveals that the equivalent valve constant is

$$k_c = \frac{k_a k_b}{\sqrt{k_a^2 + k_b^2}} \tag{17}$$

Using (15) for the resistance of the linearized model of the equivalent valve, we can write

$$R_c = \frac{2\overline{w}}{k_c^2} = 2\overline{w}\left(\frac{1}{k_a^2} + \frac{1}{k_b^2}\right) \tag{18}$$

However, by applying (15) to the individual valves, we see that their resistances are $R_a = 2\overline{w}/k_a^2$ and $R_b = 2\overline{w}/k_b^2$, respectively. Using these expressions for R_a and R_b, we can rewrite (18) as

$$R_c = R_a + R_b \tag{19}$$

which is identical to the result for a linear electrical circuit.

Sources

In most hydraulic systems, the source of energy is a pump that derives its power from an electric motor. Here we shall consider the centrifugal pump driven at a constant speed, which is widely used in chemical processes. The symbolic representation of a pump is shown in Figure 12.8. Typical input-output relationships for a centrifugal pump being driven at three different constant speeds are shown in Figure 12.9(a). Pump curves of Δp versus w

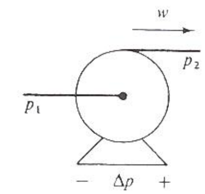


FIGURE 12.8 Symbolic representation of a pump.

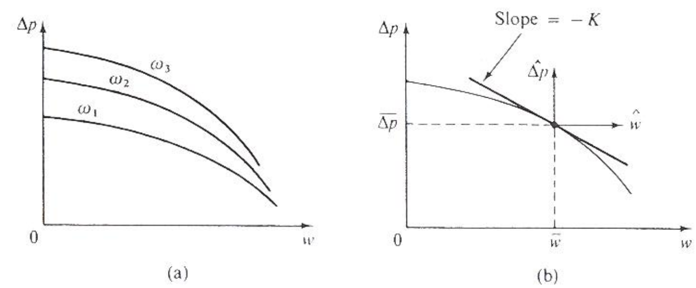


FIGURE 12.9 Typical centrifugal pump curves where $\Delta p = p_2 - p_1$. (a) For three different pump speeds ($\omega_1 < \omega_2 < \omega_3$). (b) Showing the linear approximation.

are determined experimentally under steady-state conditions and are quite nonlinear. To include a pump being driven at constant speed in a linear dynamic model, we first determine the operating point for the particular pump speed by calculating the values of $\overline{\Delta p}$ and $\overline{w}$. Then we find the slope of the tangent to the pump curve at the operating point and define it to be $-K$, which has units of newton-seconds per meter[5]. Having done this, we can express the incremental pressure difference $\widehat{\Delta p}$ in terms of the incremental flow rate $\hat{w}$ as

$$\widehat{\Delta p} = -K\hat{w} \tag{20}$$

where the constant K is always positive. Solving (20) for $\hat{w}$ yields

$$\hat{w} = -\frac{1}{K}\widehat{\Delta p} \tag{21}$$

Figure 12.9(b) illustrates the relationship of the linearized approximation to the nonlinear pump curve.

We can write the Taylor-series expansion for a pump driven at a constant speed as

$$w = \overline{w} + \left.\frac{dw}{d\,\Delta p}\right|_{\overline{\Delta p}} (\Delta p - \overline{\Delta p}) + \cdots$$

where the coefficient $(dw/d\,\Delta p)|_{\overline{\Delta p}}$ is the slope of the tangent to the curve of w versus Δp, measured at the operating point, and has the value $-1/K$. By dropping the second- and higher-order terms in the expansion and using the incremental variables $\hat{w}$ versus $\widehat{\Delta p}$, we obtain the linear relationship (21).

The manner in which a constant-speed pump can be incorporated into the dynamic model of a hydraulic system is illustrated in Example 12.4 in the following section. References in Appendix D contain more comprehensive discussions of pumps and their models, including ones where variations in the pump speed are significant.

12.3 DYNAMIC MODELS OF HYDRAULIC SYSTEMS

In this section, we apply the element laws presented in Section 12.2 and use many of the analytical techniques from previous chapters. We will develop and analyze dynamic models for a single vessel with a valve; a combination of a pump, vessel, and valve; and finally two vessels with two valves and a pump. In each case, we will derive the nonlinear model and then develop and analyze a linearized model.

▶ *EXAMPLE 12.3*

Figure 12.10 shows a vessel that receives liquid at a flow rate $w_i(t)$ and loses liquid through a valve that obeys the nonlinear flow-pressure relationship $w_o = k\sqrt{p_1 - p_a}$. The cross-sectional area is A and the liquid density is ρ. Derive the nonlinear model obeyed by the absolute pressure p_1 at the bottom of the vessel. Then develop the linearized version that is valid in the vicinity of the operating point, and find the transfer function relating the transforms of the incremental input $\hat{w}_i(t)$ and the incremental pressure $\hat{p}_1$.

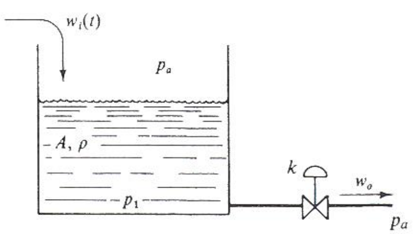


FIGURE 12.10 Hydraulic system for Example 12.3.

Having developed the system models in literal form, determine in numerical form the operating point, the transfer function, and the response to a 10% step-function increase in the input flow rate for the following parameter values:

$$A = 2 \text{ m}^2$$
$$\rho = 1000 \text{ kg/m}^3$$
$$k = 5.0 \times 10^{-5} \text{ m}^4/\text{s} \cdot N^{1/2}$$
$$\overline{w}_i = 6.0 \times 10^{-3} \text{ m}^3/\text{s}$$
$$p_a = 1.013 \times 10^5 \text{ N/m}^2$$

Solution

Taking the pressure p_1 as the single state variable, we use (9), with $C(h)$ replaced by the constant $C = A/\rho g$, to write

$$\dot{p} = \frac{1}{C}[w_{\text{in}}(t) - w_{\text{out}}(t)] \tag{22}$$

where

$$w_{\text{in}}(t) = w_i(t) \tag{23a}$$
$$w_{\text{out}}(t) = k\sqrt{p_1 - p_a} \tag{23b}$$

Substituting (23) into (22) gives the nonlinear system model as

$$\dot{p} = \frac{1}{C}[-k\sqrt{p_1 - p_a} + w_i(t)] \tag{24}$$

To develop a linearized model, we rewrite (24) in terms of the incremental variables $\hat{p}_1 = p_1 - \overline{p}_1$ and $\hat{w}_i(t) = w_i(t) - \overline{w}_i$. The nominal values $\overline{p}_1$ and $\overline{w}_i$ must satisfy the algebraic equation

$$k\sqrt{\overline{p}_1 - p_a} = \overline{w}_i$$

or

$$\overline{p}_1 = p_a + \frac{1}{k^2}\overline{w}_i^2 \tag{25}$$

which corresponds to an outflow rate equal to the inflow rate, resulting in a constant liquid level and pressure. The nominal height of the liquid is

$$\overline{h} = \frac{(\overline{p}_1 - p_a)}{\rho g} \tag{26}$$

Making the appropriate substitutions into (24) and using (12) with (15) for the linearized valve equation, we obtain

$$\dot{\hat{p}}_1 + \frac{1}{RC}\hat{p}_1 = \frac{1}{C}\hat{w}_i(t) \tag{27}$$

where $R = 2\overline{w}_i/k^2$. Transforming (27) with $\hat{p}_1(0) = 0$, which corresponds to $p_1(0) = \overline{p}_1$, we find that the system's transfer function $H(s) = \hat{P}_1(s)/\hat{W}_i(s)$ is

$$H(s) = \frac{\frac{1}{C}}{s + \frac{1}{RC}} \tag{28}$$

which has a single pole at $s = -1/RC$.

For the parameter values specified, the operating point given by (25) reduces to

$$\overline{p}_1 = 1.013 \times 10^5 + \left(\frac{6.0 \times 10^{-3}}{5.0 \times 10^{-5}}\right)^2 = 1.157 \times 10^5 \text{ N/m}^2$$

and from (26), the nominal liquid height is

$$\overline{h} = \frac{1.440 \times 10^4}{1000 \times 9.807} = 1.468 \text{ m}$$

The numerical values of the hydraulic resistance and capacitance are, respectively,

$$R = \frac{2 \times 6.0 \times 10^{-3}}{(5.0 \times 10^{-5})^2} = 4.80 \times 10^6 \text{ N}\cdot\text{s/m}^5$$

$$C = \frac{2.0}{1000 \times 9.807} = 2.039 \times 10^{-4} \text{ m}^5\text{/N}$$

Substituting these values of R and C into (28), we obtain the numerical form of the transfer function as

$$H(s) = \frac{4904.}{s + 1.0216 \times 10^{-3}} \tag{29}$$

If $w_i(t)$ is originally equal to its nominal value of $\overline{w}_i = 6.0 \times 10^{-3}$ m^3/s and undergoes a 10% step-function increase, then $\hat{w}_i(t) = [0.60 \times 10^{-3}]U(t)$ m^3/s and $\hat{W}_i(s) = 0.60 \times 10^{-3}\ (1/s)$. Thus

$$\hat{P}_1(s) = \frac{4904. \times 0.60 \times 10^{-3}}{s(s + 1.0216 \times 10^{-3})}$$

$$= \frac{2.942}{s(s + 1.0216 \times 10^{-3})}$$

From the final-value theorem, the steady-state value of $\hat{p}_1$ is $s\hat{P}_1(s)$ evaluated at $s = 0$, namely

$$\lim_{t\to\infty} \hat{p}_1(t) = \frac{2.942}{1.0216 \times 10^{-3}} = 2880. \text{ N/m}^2$$

The time constant of the linearized model is $\tau = RC$, which becomes

$$\tau = (4.80 \times 10^6)(2.039 \times 10^{-4})$$

$$= 978.7 \text{ s}$$

which is slightly over 16 minutes. Thus the response of the incremental pressure is

$$\hat{p}_1 = 2880.(1 - \epsilon^{-t/978.7})$$

The change in the incremental level is $\hat{p}_1/\rho g$, which becomes

$$\hat{h} = 0.2937(1 - \epsilon^{-t/978.7})$$

To obtain the responses of the actual pressure and liquid level, we merely add the nominal values $\overline{p}_1 = 1.157 \times 10^5$ N/m^2 and $\overline{h} = 1.468$ m to these incremental variables. It is interesting to note that because of the nonlinear valve, the 10% increase in the flow rate results in a 20% increase in both the gauge pressure $p_1 - p_a$ and the height h.

▶ *EXAMPLE 12.4*

Find the linearized model of the hydraulic system shown in Figure 12.11(a), which consists of a constant-speed centrifugal pump feeding a vessel from which liquid flows through a pipe and valve obeying the relationship $w_o = k\sqrt{p_1 - p_a}$. The pump characteristic for the specified pump speed $\overline{\omega}$ is shown in Figure 12.11(b).

Solution

The equilibrium condition for the system corresponds to

$$\overline{w}_i = \overline{w}_o \tag{30}$$

where $\overline{w}_i$ and $\overline{\Delta p} = \overline{p}_1 - p_a$ must be one of the points on the pump curve in Figure 12.11(b), and where $\overline{w}_o$ obeys the nonlinear flow relationship

$$\overline{w}_o = k\sqrt{\overline{\Delta p}} \tag{31}$$

To determine the operating point, we find the solution to (30) graphically by plotting the valve characteristic (31) on the pump curve. Doing this gives Figure 12.12(a), where the operating point is the intersection of the valve curve and the pump curve, designated as point A in the figure. Once we have located the operating point, we can draw the tangent to the pump curve as shown in Figure 12.12(b) and determine its slope $-K$ graphically.

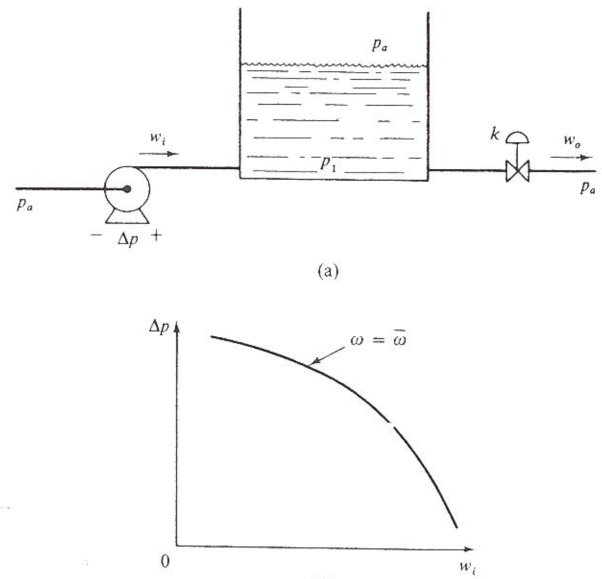


FIGURE 12.11 (a) System for Example 12.4. (b) Pump curve.

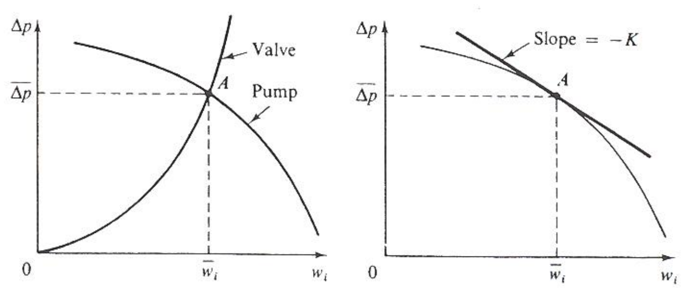


FIGURE 12.12 (a) Combined pump and valve curves for Example 12.4. (b) Pump curve with linear approximation.

Following this preliminary step, we can use (9) to write the model of the system as

$$\dot{p}_1 = \frac{1}{C}(w_i - w_o) \tag{32}$$

where, from (12), the approximate flow rate through the valve is

$$w_o = \overline{w}_o + \frac{1}{R}\widehat{\Delta p} \tag{33}$$

and where, from (21), the approximate flow rate through the pump is

$$w_i = \overline{w}_i - \frac{1}{K}\widehat{\Delta p} \tag{34}$$

Substituting (33) and (34) into (32), using $\dot{p}_1 = \dot{\hat{p}}_1$ and (30), and noting that $\widehat{\Delta p} = \hat{p}_1$ because p_a is constant, we find the incremental model to be

$$\dot{\hat{p}}_1 = \frac{1}{C}\left(-\frac{1}{K} - \frac{1}{R}\right)\hat{p}_1$$

which we can write as the homogeneous first-order differential equation

$$\dot{\hat{p}}_1 + \frac{1}{C}\left(\frac{1}{K} + \frac{1}{R}\right)\hat{p}_1 = 0 \tag{35}$$

Inspection of (35) indicates that the magnitude of the slope of the pump curve at the operating point enters the equation in exactly the same manner as the resistance associated with the valve. Hence, if we evaluate the equivalent resistance R_{eq} according to

$$R_{eq} = \frac{RK}{R + K}$$

(35) is the same as (27), which was derived for a vessel and a single valve, except for the absence of an input flow rate.

▶ *EXAMPLE 12.5*

The valves in the hydraulic system shown in Figure 12.13 obey the flow-pressure relationships $w_1 = k_1\sqrt{p_1 - p_2}$ and $w_2 = k_2\sqrt{p_2 - p_a}$. The atmospheric pressure is p_a, and the capacitances of the vessels are C_1 and C_2. Find the equations that determine the operating point, and show how the pump curve is used to solve them. Derive a linearized model that is valid about the operating point.

Solution

Because the pump and the two vessels are in series at equilibrium conditions, we define the operating point by equating the three flow rates $\overline{w}_p$, $\overline{w}_1$, and

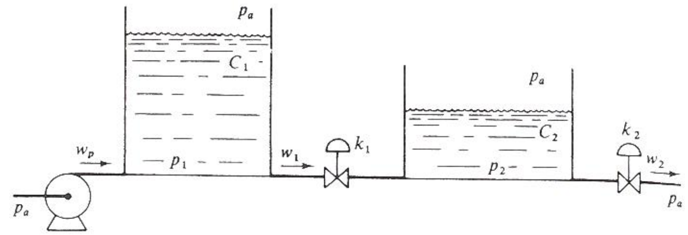


FIGURE 12.13 Hydraulic system with two vessels considered in Example 12.5.

$\overline{w}_2$. The flow rates through the two valves are given by

$$\overline{w}_1 = k_1\sqrt{\overline{p}_1 - \overline{p}_2} \tag{36a}$$

$$\overline{w}_2 = k_2\sqrt{\overline{p}_2 - p_a} \tag{36b}$$

The flow rate $\overline{w}_p$ through the pump and the pressure difference $\overline{\Delta p}_1 = \overline{p}_1 - p_a$ must correspond to a point on the pump curve. Equations (36a) and (36b) are those of two valves in series, and, as shown in Example 12.2, we can replace such a combination of valves by an equivalent valve specified by (16). Hence

$$\overline{w}_p = k_{\text{eq}}\sqrt{\overline{\Delta p}_1} \tag{37}$$

where, from (17),

$$k_{\text{eq}} = \frac{k_1 k_2}{\sqrt{k_1^2 + k_2^2}}$$

Plotting (37) on the pump curve as shown in Figure 12.12(a) yields the values of $\overline{\Delta p}_1$ and $\overline{w}_p$, from which we can find the other nominal values.

With this information, we can develop the incremental model. Using (9), (12), and (21), we can write the pair of linear differential equations

$$\dot{\hat{p}}_1 = \frac{1}{C_1}\left[-\frac{1}{K}\hat{p}_1 - \frac{1}{R_1}(\hat{p}_1 - \hat{p}_2)\right]$$

$$\dot{\hat{p}}_2 = \frac{1}{C_2}\left[\frac{1}{R_1}(\hat{p}_1 - \hat{p}_2) - \frac{1}{R_2}\hat{p}_2\right] \tag{38}$$

where the valve resistances are given by $R_1 = 2\overline{w}_1/k_1^2$ and $R_2 = 2\overline{w}_2/k_2^2$, and where $-K$ is the slope of the pump curve at the operating point. As indicated by (38), the incremental model has no inputs and hence can respond only to nonzero initial conditions—that is, $\hat{p}_1(0) \neq 0$ and/or $\hat{p}_2(0) \neq 0$.

In practice, there might be additional liquid streams entering either vessel, or the pump speed might be changed. It is also possible for either of the two valves to be opened or closed slightly. Such a change would modify the respective hydraulic resistance.

In the absence of an input, we can transform (38) to find the Laplace transform of the zero-input response. Doing this, we find that after rearranging,

$$\left[C_1 s + \left(\frac{1}{K} + \frac{1}{R_1}\right)\right]\hat{P}_1(s) = \frac{1}{R_1}\hat{P}_2(s) + C_1\hat{p}_1(0) \tag{39a}$$

$$\left[C_2 s + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right]\hat{P}_2(s) = \frac{1}{R_1}\hat{P}_1(s) + C_2\hat{p}_2(0) \tag{39b}$$

We can find either $\hat{P}_1(s)$ or $\hat{P}_2(s)$ by combining these two equations into a single transform equation. The corresponding inverse transform will yield the zero-input response in terms of $\hat{p}_1(0)$ and $\hat{p}_2(0)$. The denominator of either $\hat{P}_1(s)$ or $\hat{P}_2(s)$ will be the characteristic polynomial of the system, which, as you may verify, is

$$s^2 + \left[\frac{1}{C_1}\left(\frac{1}{K} + \frac{1}{R_1}\right) + \frac{1}{C_2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right]s + \frac{1}{C_1 C_2}\left(\frac{1}{KR_1} + \frac{1}{KR_2} + \frac{1}{R_1 R_2}\right)$$

SUMMARY

The basic variables in a hydraulic system are flow rate and pressure. Other variables that are equivalent to the pressure at the bottom of a container are the volume of the liquid and the liquid height.

Because hydraulic systems are generally nonlinear, especially in the resistance to fluid flow, we developed linearized models valid in the vicinity of an operating point. We introduced the passive elements of hydraulic capacitance and hydraulic resistance in constructing such models. The former is associated with the potential energy of a fluid in a vessel, the latter with the energy dissipated when fluid flows through valves, orifices, and pipes. Another possible passive element is inertance, which is associated with the kinetic energy of fluids in motion. Because the inertance is normally negligible, we did not include it in our models.

A source may consist of a specified flow rate into a vessel. However, most practical hydraulic energy sources are mechanically driven pumps. We generally describe such pumps by a nonlinear relationship between pressure and flow rate, rather than specifying one of these two variables independently. This contrasts with the ideal force, velocity, voltage, and current sources used in earlier chapters. Because of the algebraic relationship between pressure and flow rate, a pump enters into the linearized system equations in a way somewhat similar to hydraulic resistance.

The pressure in each vessel (or else the volume or the liquid height) is normally chosen to be a state variable. The procedures for finding and solving hydraulic models, including state-variable equations and transfer functions, are generally the same as those used in the earlier chapters.

PROBLEMS

* **12.1** Figure P12.1 shows a conical vessel that has a circular cross section and contains a liquid. Evaluate and sketch the hydraulic capacitance as a function of the liquid height h. Also evaluate and sketch the gauge pressure $p^* = p - p_a$ at the base of the vessel as a function of the liquid volume v.

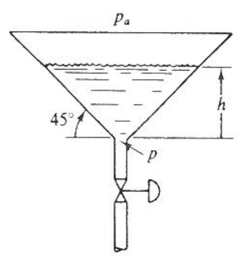
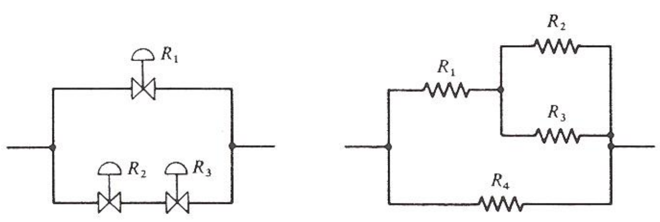


FIGURE P12.1

12.2 Find the equivalent hydraulic resistances for the linear models of hydraulic networks shown in Figure P12.2. Express your answers as single fractions.



(a) (b)

FIGURE P12.2

12.3 Two valves that obey the relationships $w_a = k_a\sqrt{\Delta p}$ and $w_b = k_b\sqrt{\Delta p}$ are connected in parallel, as indicated in Figure P12.3.

a) Determine the equivalent valve coefficient k_c in terms of k_a and k_b such that the total flow rate is given by $w = k_c\sqrt{\Delta p}$.

b) Show that the hydraulic resistance of the equivalent linearized model is $R_c = 2\sqrt{\overline{\Delta p}}/k_c$, where $\overline{\Delta p}$ denotes the nominal pressure drop across the valves.

c) Show that $R_c = R_aR_b/(R_a + R_b)$, where R_a and R_b are the hydraulic resistances of the individual valves evaluated at the nominal pressure difference $\overline{\Delta p}$.

d) Assume curves for w_a versus Δp and w_b versus Δp, and sketch the corresponding curve for w_c. Indicate the linearized approximations at a typical value of $\overline{\Delta p}$.

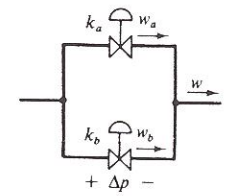


FIGURE P12.3

12.4 Consider the hydraulic system that was modeled in Example 12.3, consisting of a single vessel and a valve.

a) Obtain the linearized model in terms of the incremental pressure that is valid for the nominal flow rate $\overline{w}_i = 3.0 \times 10^{-3}$ m^3/s, and evaluate the transfer function $\hat{P}_1(s)/\hat{W}_i(s)$. Also find $\overline{p}_1$.

b) Rewrite the model and transfer function you found in part (a) in terms of the incremental volume $\hat{v}$. Also find $\overline{v}$.

c) Solve for and sketch the incremental volume versus time when $\hat{w}_i(t) = 0.5 \times 10^{-3}U(t)$ m^3/s and when the system starts at the nominal conditions found in part (a).

* **12.5** Write the state-variable equations for the system shown in Figure P12.5, using the incremental pressures $\hat{p}_1$ and $\hat{p}_2$ as state variables, where the pump obeys the relationship

$$w_p = \overline{w}_p - \frac{1}{K}(\hat{p}_2 - \hat{p}_1)$$

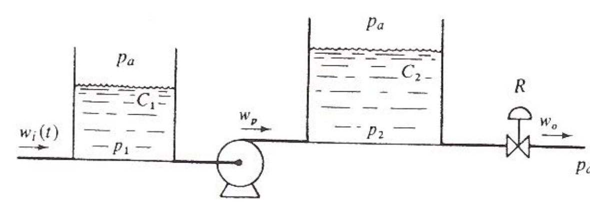


FIGURE P12.5

12.6 For the hydraulic system shown in Figure P12.6, the incremental input is $\hat{w}_i(t)$ and the incremental output is $\hat{w}_o$.

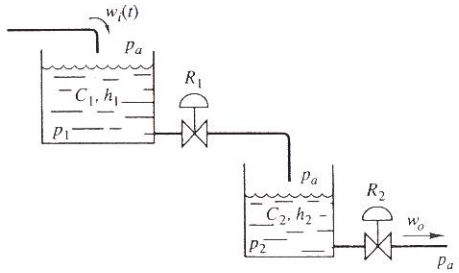


FIGURE P12.6

a) Verify that the following equations represent an appropriate state-variable model in terms of the incremental state variables $\hat{p}_1$ and $\hat{p}_2$.

$$\dot{\hat{p}}_1 = \frac{1}{C_1}\left[-\frac{1}{R_1}\hat{p}_1 + \hat{w}_i(t)\right]$$

$$\dot{\hat{p}}_2 = \frac{1}{C_2}\left[\frac{1}{R_1}\hat{p}_1 - \frac{1}{R_2}\hat{p}_2\right]$$

$$\hat{w}_o = \frac{1}{R_2}\hat{p}_2$$

b) Derive the system transfer function $\hat{W}_o(s)/\hat{W}_i(s)$.
c) Define the time constants $\tau_1 = R_1C_1$ and $\tau_2 = R_2C_2$, and evaluate the unit step response for the case $\tau_1 \neq \tau_2$. Sketch the response for the case where $\tau_1 = 2\tau_2$.
d) Evaluate the transfer function $\hat{H}_2(s)/\hat{W}_i(s)$ that relates the transform of $\hat{w}_i(t)$ to the transform of the incremental liquid height $\hat{h}t_2$.

* **12.7** The hydraulic system shown in Figure P12.7 has the incremental pressure $\hat{p}_i(t)$ as its input and the incremental flow rate $\hat{w}_o$ as its output.
a) Verify that the following equations represent an appropriate state-variable model in terms of the incremental state variables $\hat{p}_1$ and $\hat{p}_2$.

$$\dot{\hat{p}}_1 = -\frac{1}{R_3C_1}\hat{p}_1 + \frac{1}{R_1C_1}\hat{p}_i(t)$$

$$\dot{\hat{p}}_2 = \frac{1}{R_3C_2}\hat{p}_1 - \frac{1}{R_4C_2}\hat{p}_2 + \frac{1}{R_2C_2}\hat{p}_i(t)$$

$$\hat{w}_o = \frac{1}{R_4}\hat{p}_2$$

b) Find the transfer function $\hat{W}_o(s)/\hat{P}_i(s)$.
c) Solve for the steady-state value of the unit step response.

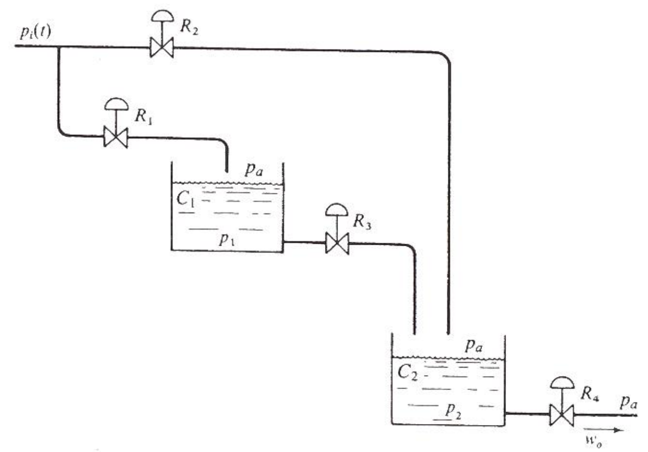


FIGURE P12.7

12.8 Figure P12.8 shows three identical tanks having capacitance C connected by identical lines having resistance R. An input stream flows into tank 1 at the incremental flow rate $\hat{w}_i(t)$, and an output stream flows from tank 3 at the incremental flow rate $\hat{w}_o(t)$.

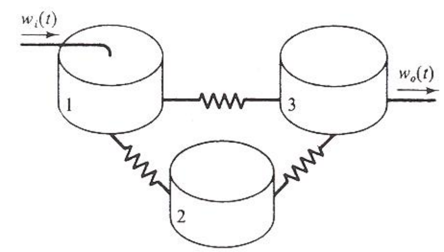


FIGURE P12.8

a) Verify that the following equations represent an appropriate state-variable model in terms of the incremental state variables $\hat{v}_1$, $\hat{v}_2$, and $\hat{v}_3$.

$$\dot{\hat{v}}_1 = -\frac{2}{RC}\hat{v}_1 + \frac{1}{RC}\hat{v}_2 + \frac{1}{RC}\hat{v}_3 + \hat{w}_i(t)$$

$$\dot{\hat{v}}_2 = \frac{1}{RC}\hat{v}_1 - \frac{2}{RC}\hat{v}_2 + \frac{1}{RC}\hat{v}_3$$

$$\dot{\hat{v}}_3 = \frac{1}{RC}\hat{v}_1 + \frac{1}{RC}\hat{v}_2 - \frac{2}{RC}\hat{v}_3 - \hat{w}_o(t)$$

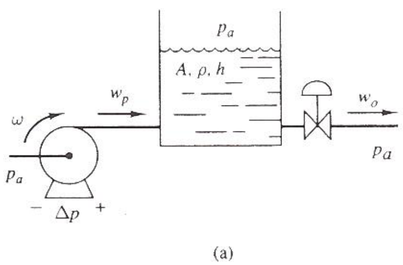
b) Let $RC = 1$ and find $\hat{V}_2(s)$ in terms of $\hat{W}_1(s)$ and $\hat{W}_o(s)$ by transforming the three state-variable equations and eliminating $\hat{V}_1(s)$ and $\hat{V}_3(s)$.

c) Solve for $\hat{v}_2$ as a function of time when $\hat{w}_i(t) = U(t)$ and $\hat{w}_o(t) = 0$. Repeat the solution when $\hat{w}_i(t) = 0$ and $\hat{w}_o(t) = U(t)$. Sketch the responses.

12.9 The valve and pump characteristics for the hydraulic system shown in Figure P12.9(a) are plotted in Figure P12.9(b). Curves are given for the two pump speeds 100 rad/s and 150 rad/s. The cross-sectional area of the vessel is 2.0 m^2, and the liquid density is 1000 kg/m^3.

a) Determine the steady-state flow rate, gauge pressure, and liquid height for each of the pump speeds for which curves are shown.

b) Derive the linearized models for each of the pump speeds in numerical form.



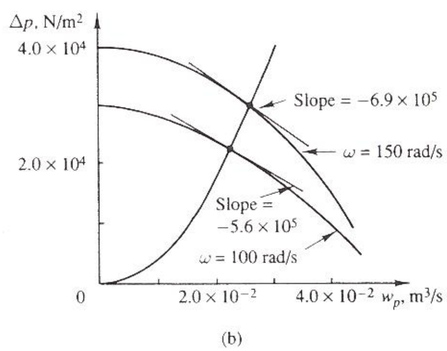


FIGURE P12.9

*** 12.10** As shown in Figure P12.10(a), liquid can flow into a vessel through a pump and a valve. The assumed pump characteristic is shown in Figure P12.10(b), where α and β are the maximum flow rate and the maximum pressure difference, respectively. The valve is shut for all $t < 0$, is opened at $t = 0$, and presents no resistance to the flow of the liquid for $t > 0$.

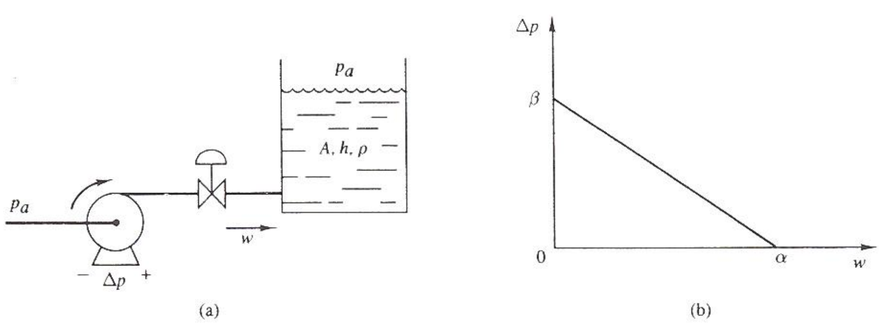


FIGURE P12.10

a) Verify that the differential equation obeyed by the liquid height h after the valve is opened is

$$\dot{h} + \left(\frac{\alpha\rho g}{\beta A}\right) h = \frac{\alpha}{A}$$

b) Write expressions for the time constant and the steady-state height.

c) Solve for $h(t)$ and sketch it versus time.

12.11 Consider the hydraulic system described in Problem 12.10 with a pump having the nonlinear pressure-flow relationship shown in Figure P12.11.

a) Verify that the differential equation obeyed by the liquid height h after the valve is opened is

$$\dot{h} + \frac{\alpha}{A}\left(\frac{\rho g}{\beta}\right)^2 h^2 = \frac{\alpha}{A}$$

b) Using a digital computer and a nonlinear simulation language such as ACSL, plot $h(t)$ and $w(t)$ versus time. Use the parameter values $A = 1.50$ m^2, $\rho = 1000$ kg/m^3, $\alpha = 0.120$ m^3/s, and $\beta = 3.0 \times 10^4$ N/m^2.

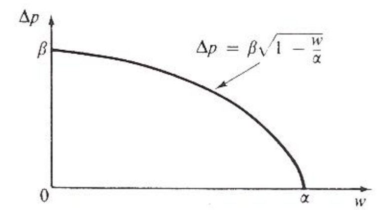


FIGURE P12.11