Homework 1

If you are using technology (Matlab, Mathematica, etc.), include a printout of the relevant code and output or plot.

What I don't want to see is page after page of columns of numbers. That is just a waste of paper. If you are computing a numerical solution at 500 points, either print only the first few and last few numbers, or (even better) plot the curve.

1. Use Matlab (or some other program) to calculate the polynomial which interpolates the function

$$f(x) = \frac{1}{1 + 25x^2}$$

at equally spaced points between -1 and 1, in steps of 0.25. You can use a Matlab built-in to find the polynomial.

Plot the original function (in the default color blue) and the interpolating polynomial (in red), and put circles around the interpolation points. Make sure you use enough points so that you get smooth-looking curves.

2. **(a)** Given a point x_0 and stepsize h, use Lagrange polynomials to find the quadratic polynomial which interpolates a function $f(x)$ at the points $x_0 - 2h$, x_0, and $x_0 + h$.

(b) Integrate the polynomial on $[x_0 - 2h, x_0 + h]$ to produce an integration formula. Use the formula to estimate $\int_0^3 e^{-x}\,dx$.

3. This is a modified form of problem 2.3, page 31, in the Iserles book.

Recall that the k-step Adams-Bashforth and Adams-Moulton methods are based on

$$y(t_{n+k}) = y(t_{n+k-1}) + \int_{t_{n+k-1}}^{t_{n+k}} f(\tau, y(\tau))\,d\tau.$$

You interpolate f at $t_n, t_{n+1}, \ldots t_{n+k-1}$ (AB) or $t_n, t_{n+1}, \ldots, t_{n+k}$ (AM) by a polynomial, and integrate that.

You can also start integrating at t_{n+k-2} instead of t_{n+k-1}. For $k = 2$, Iserles calls these the *Nystrom* and *Milne* methods. That is, start with

$$y(t_{n+2}) = y(t_n) + \int_{t_n}^{t_{n+2}} f(\tau, y(\tau))\,d\tau,$$

and interpolate at t_n, t_{n+1} (explicit) or t_n, t_{n+1}, t_{n+2} (implicit).

Find the leading error term, and the order of the method, for both cases.

4. The IVP

$$\begin{aligned} y' &= -y + 2e^{-t}\cos(2t) \\ y(0) &= 0 \end{aligned}$$

has the true solution $y(t) = e^{-t}\sin(2t)$.

Solve it numerically, using Euler's method with 20, 40, 80, 160 steps from 0 to 1, and compute the global error at the endpoint for each stepsize. Verify that the error goes down approximately linearly.

5. Solve

$$\mathbf{y}''(t) = \begin{pmatrix} 0 \\ -10 \end{pmatrix} - 0.1\,\mathbf{y}'(t),$$
$$\mathbf{y}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$\mathbf{y}'(0) = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$$

by using Euler's method with stepsize $h = 0.1$, until y_2 becomes 0 or negative.

Hints: $\mathbf{y}(t)$ is a vector with 2 components, representing the position of an object at time t. The DE comes from Newton's law (force = mass · acceleration). The object is launched at time $t = 0$ from the origin, with an initial velocity $(10, 10)^T$ (speed $10\sqrt{2}$ at a 45° angle).

The forces acting on the object are gravity (first term on the right), and friction. I am assuming that friction is proportional to the speed $\|\mathbf{y}'\|$, and opposite to the direction of motion $\mathbf{y}'/\|\mathbf{y}'\|$ (second term).

What you need to do first is to convert this to a first order system with 4 components, and then use Euler's method in vector form.

Plot y_1 and y_2 as functions of t (horizontal and vertical distance as functions of time), and also plot y_2 versus y_1. That will show the path of the object. It should look a bit like a parabola, but compressed on the right because of friction.