

Likelihood ratio test for samples from exponentially-distributed populations

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Contents

Definitions	2
Hypotheses	2
Derivation of the maximum likelihood under the null	2
Derivation of the unrestricted maximum likelihood	3
Likelihood ratio	5

Definitions

Let Y_{ij} denote the j th observation from the i th treatment group, where $i = 1, 2, 3, \dots, m$ and $j = 1, 2, 3, \dots, n_i$.

Let:

$$n = \sum_{i=1}^m n_i$$

$$\bar{Y} = \frac{1}{n} \sum_{i=1}^m \sum_{j=1}^{n_i} Y_{ij}$$

$$\bar{Y}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij}.$$

Hypotheses

H_0 : $Y_{ij} \sim \text{Exp}(\theta)$

H_A : $Y_{ij} \sim \text{Exp}(\theta_i)$, where $\theta_i \neq \theta_k$ for at least one combination of i and k values.

Let us denote the parameter space under the null hypothesis as $\Omega_0 = \{(\theta) : 0 < \theta < \infty\}$ and the parameter space under the alternative hypothesis as $\Omega_a = \{(\theta_i) : 0 < \theta_i < \infty, \theta_i \neq \theta_k \text{ for at least one pair of } i \text{ and } k \text{ values}\}$. The unrestricted parameter space is thus $\Omega = \Omega_0 \cup \Omega_a = \{(\theta_i) : 0 < \theta_i < \infty\}$.

Derivation of the maximum likelihood under the null

$$\begin{aligned} L(\Omega_0) &= \prod_{i=1}^m \prod_{j=1}^{n_i} \frac{1}{\theta} \exp\left(-\frac{Y_{ij}}{\theta}\right) \\ &= \theta^{-n} \exp\left(-\sum_{i=1}^m \sum_{j=1}^{n_i} \frac{Y_{ij}}{\theta}\right) \\ &= \theta^{-n} \exp\left(-\frac{n\bar{Y}}{\theta}\right). \end{aligned} \tag{1}$$

Therefore the log-likelihood under the null is:

$$\ln L(\Omega_0) = -n \ln \theta - \frac{n\bar{Y}}{\theta}.$$

Differentiating with respect to θ and setting to zero:

$$\frac{\partial \ln \Omega_0}{\partial \theta} \Big|_{\theta=\hat{\theta}} = -\frac{n}{\hat{\theta}} + \frac{n\bar{Y}}{\hat{\theta}^2} = 0.$$

Multiplying by $\frac{\hat{\theta}^2}{n}$ yields:

$$\begin{aligned} 0 &= -\hat{\theta} + \bar{Y} \\ \implies \hat{\theta} &= \bar{Y}. \end{aligned} \tag{2}$$

Substituting Equation 2 into Equation 1 therefore yields the maximum likelihood under the null:

$$\begin{aligned} L(\widehat{\Omega}_0) &= \bar{Y}^{-n} \exp\left(-\frac{n\bar{Y}}{\bar{Y}}\right) \\ &= \bar{Y}^{-n} \exp(-n) \\ &= (\bar{Y}e)^{-n}. \end{aligned} \tag{3}$$

Derivation of the unrestricted maximum likelihood

$$\begin{aligned} L(\Omega) &= \prod_{i=1}^m \prod_{j=1}^{n_i} \frac{1}{\theta_i} \exp\left(-\frac{Y_{ij}}{\theta_i}\right) \\ &= \prod_{i=1}^m \theta_i^{-n_i} \exp\left(-\sum_{j=1}^{n_i} \frac{Y_{ij}}{\theta_i}\right) \\ &= \prod_{i=1}^m \theta_i^{-n_i} \exp\left(-\frac{n_i \bar{Y}_i}{\theta_i}\right) \\ &= \left(\prod_{i=1}^m \theta_i^{-n_i}\right) \exp\left(-\sum_{i=1}^m \frac{n_i \bar{Y}_i}{\theta_i}\right). \end{aligned} \tag{4}$$

Thus the log-likelihood is:

$$\begin{aligned}
\ln L(\Omega) &= - \sum_{i=1}^m n_i \ln \theta_i - \sum_{i=1}^m \frac{n_i \bar{Y}_i}{\theta_i} \\
&= - \sum_{i=1}^m n_i \left(\ln \theta_i + \frac{\bar{Y}_i}{\theta_i} \right).
\end{aligned} \tag{5}$$

Taking the partial derivative of Equation 5 with respect θ_k and setting to zero:

$$\begin{aligned}
\frac{\partial \ln L(\Omega)}{\partial \theta_k} \Big|_{\theta_i = \hat{\theta}_i} &= - \sum_{i=1}^m n_i \left(\frac{1}{\hat{\theta}_i} - \frac{\bar{Y}_i}{\hat{\theta}_i^2} \right) \delta_{ik} = 0 \\
-n_k \left(\frac{1}{\hat{\theta}_k} - \frac{\bar{Y}_k}{\hat{\theta}_k^2} \right) &= 0.
\end{aligned} \tag{6}$$

Where δ_{ik} is the Kronecker delta. Multiplying Equation 6 by $-\frac{\hat{\theta}_k^2}{n_k}$ yields:

$$\begin{aligned}
\hat{\theta}_k - \bar{Y}_k &= 0 \\
\implies \hat{\theta}_k &= \bar{Y}_k.
\end{aligned} \tag{7}$$

Substituting Equation 7 into Equation 4 should yield the unrestricted maximum likelihood:

$$\begin{aligned}
L(\hat{\Omega}) &= \left(\prod_{i=1}^m \bar{Y}_i^{-n_i} \right) \exp \left(- \sum_{i=1}^m \frac{n_i \bar{Y}_i}{\bar{Y}_i} \right) \\
&= \left(\prod_{i=1}^m \bar{Y}_i^{-n_i} \right) \exp(-n).
\end{aligned} \tag{8}$$

Likelihood ratio

$$\begin{aligned}\lambda &= \frac{L(\widehat{\Omega}_0)}{L(\widehat{\Omega})} \\&= \frac{(\bar{Y}e)^{-n}}{\left(\prod_{i=1}^m \bar{Y}_i^{-n_i}\right) \exp(-n)} \\&= \bar{Y}^{-n} \prod_{i=1}^m \bar{Y}_i^{n_i} \\ \therefore -2 \ln \lambda &= -2 \left(-n \ln \bar{Y} + \sum_{i=1}^m n_i \ln \bar{Y}_i \right) \\&= 2n \ln \bar{Y} - 2 \sum_{i=1}^m n_i \ln \bar{Y}_i.\end{aligned}\tag{9}$$

And we know under the null hypothesis that $-2 \ln \lambda \sim \chi_{m-1}^2$, so we will use this to find a p-value.