Likelihood ratio test for samples from exponentially-distributed populations

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Definitions

Let Y_{ij} denote the jth observation from the i treatment group, where $i = 1, 2, 3, ..., m$ and $j = 1, 2, 3, ..., n_i$.

Let:

$$
\begin{aligned}
n &= \sum_{i=1}^{m} n_i \\
\overline{Y} &= \frac{1}{n} \sum_{i=1}^{m} \sum_{j=1}^{n_i} Y_{ij} \\
\overline{Y}_i &= \frac{1}{n_i} \sum_{j=1}^{n_i} Y_{ij}.
\end{aligned}
$$

Hypotheses

H_0: $Y_{ij} \sim \text{Exp}(\theta)$

H_A: $Y_{ij} \sim \text{Exp}(\theta_i)$, where $\theta_i \neq \theta_k$ for some combination of i and k values.

Let us denote the parameter space under the null hypothesis as $\Omega_0 = \{(\theta) : \ 0 < \theta < \infty\}$ and the parameter space under the alternative hypothesis as $\Omega_a = \{(\theta_i) : \ 0 < \theta_i < \infty, \ \theta_i \neq \theta_k \text{ for at least one pair of } i \text{ and } k \text{ values}\}$. The unrestricted parameter space is thus $\Omega = \Omega_0 \cup \Omega_a = \{(\theta_i) : \ 0 < \theta_i < \infty\}$.

Derivation of the maximum likelihood under the null

$$
\begin{aligned}
L(\Omega_0) &= \prod_{i=1}^{m} \prod_{j=1}^{n_i} \frac{1}{\theta} \exp\left(-\frac{Y_{ij}}{\theta}\right) \\
&= \theta^{-n} \exp\left(-\sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{Y_{ij}}{\theta}\right) \\
&= \theta^{-n} \exp\left(-\frac{n\overline{Y}}{\theta}\right). \qquad (1)
\end{aligned}
$$

Therefore the log-likelihood under the null is:

$$
\ln L(\Omega_0) = -n \ln \theta - \frac{n\overline{Y}}{\theta}.
$$

Differentiating with respect to θ and setting to zero:

$$\frac{\partial \ln \Omega_0}{\partial \theta}\Big|_{\theta=\widehat{\theta}} = -\frac{n}{\widehat{\theta}} + \frac{n\overline{Y}}{\widehat{\theta}^2} = 0.$$

Multiplying by $\frac{\widehat{\theta}^2}{n}$ yields:

$$\begin{aligned} 0 &= -\widehat{\theta} + \overline{Y} \\ \implies \widehat{\theta} &= \overline{Y}. \end{aligned} \tag{2}$$

Substituting Equation 2 into Equation 1 therefore yields the maximum likelihood under the null:

$$\begin{aligned} L(\widehat{\Omega_0}) &= \overline{Y}^{-n} \exp\left(-\frac{n\overline{Y}}{\overline{Y}}\right) \\ &= \overline{Y}^{-n} \exp(-n) \\ &= (\overline{Y}e)^{-n}. \end{aligned} \tag{3}$$

Derivation of the unrestricted maximum likelihood

$$\begin{aligned} L(\Omega) &= \prod_{i=1}^{m}\prod_{j=1}^{n_i} \frac{1}{\theta_i} \exp\left(-\frac{Y_{ij}}{\theta_i}\right) \\ &= \prod_{i=1}^{m} \theta_i^{-n_i} \exp\left(-\sum_{j=1}^{n_i} \frac{Y_{ij}}{\theta_i}\right) \\ &= \prod_{i=1}^{m} \theta_i^{-n_i} \exp\left(-\frac{n_i\overline{Y}_i}{\theta_i}\right) \\ &= \left(\prod_{i=1}^{m} \theta_i^{-n_i}\right) \exp\left(-\sum_{i=1}^{m} \frac{n_i\overline{Y}_i}{\theta_i}\right). \end{aligned} \tag{4}$$

Thus the log-likelihood is:

$$\ln L(\Omega) = -\sum_{i=1}^{m} n_i \ln\theta_i - \sum_{i=1}^{m} \frac{n_i\overline{Y}_i}{\theta_i} \tag{5}$$

$$= -\sum_{i=1}^{m} n_i \left(\ln\theta_i + \frac{\overline{Y}_i}{\theta_i}\right). \tag{6}$$

Taking the partial derivative of Equation 6 with respect θ_k and setting to zero:

$$\frac{\partial \ln L(\Omega)}{\partial\theta_k}\Big|_{\theta_l=\widehat{\theta}_l} = -\sum_{i=1}^{m} n_i \left(\frac{1}{\widehat{\theta}_i} - \frac{\overline{Y}_i}{\widehat{\theta}_i^2}\right)\delta_{ik} = 0$$

$$-n_k \left(\frac{1}{\widehat{\theta}_k} - \frac{\overline{Y}_k}{\widehat{\theta}_k^2}\right) = 0. \tag{7}$$

Multiplying Equation 7 by $-\frac{\widehat{\theta}_k^2}{n_k}$ yields:

$$\widehat{\theta}_k - \overline{Y}_k = 0$$

$$\implies \widehat{\theta}_k = \overline{Y}_k. \tag{8}$$

Substituting Equation 8 into Equation 4 should yield the unrestricted maximum likelihood:

$$L(\widehat{\Omega}) = \left(\prod_{i=1}^{m} \overline{Y}_i^{-n_i}\right) \exp\left(-\sum_{i=1}^{m} \frac{n_i\overline{Y}_i}{\overline{Y}_i}\right)$$

$$= \left(\prod_{i=1}^{m} \overline{Y}_i^{-n_i}\right) \exp(-n). \tag{9}$$

Likelihood ratio

$$\begin{aligned}
\lambda &= \frac{L(\widehat{\Omega_0})}{L(\widehat{\Omega})} \\
&= \frac{(\overline{Y}e)^{-n}}{\left(\prod_{i=1}^{m} \overline{Y}_i^{-n_i}\right)\exp(-n)} \\
&= \overline{Y}^{-n} \prod_{i=1}^{m} \overline{Y}_i^{n_i} \\
\therefore \quad -2\ln\lambda &= -2\left(-n\ln\overline{Y} + \sum_{i=1}^{m} n_i \ln\overline{Y}_i\right) \\
&= 2n\ln\overline{Y} - 2\sum_{i=1}^{m} n_i \ln\overline{Y}_i. \qquad (10)
\end{aligned}$$

And we know under the null hypothesis that $-2\ln\lambda \sim \chi^2_{m-1}$.