Capped Weak Plane Plasticity

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Contents

1 Introduction and the yield functions

Weak-plane plasticity is designed to simulate a layered material. Each layer can slip over adjacent layers, and be separated from those adjacent layers. An example of particular interest to CSIRO is of stratified rocks, which consist of large sheets of fairly strong rock separated by weak and very thin joints. Upon application of stress, the joints can fail, either by slipping or separating. The idea is to use one finite element that potentially contains many layers, and prescribe "weak plane plasticity" for that finite element, so that it can fail by joint separation and joint slipping.

Denote the normal to the layers by z, and the tangential directions by x and y. It is convenient to introduce two new stress variables in terms of the stress tensor σ:

$$p = \sigma_{zz} \quad \text{and} \quad q = \sqrt{\sigma_{xz}^2 + \sigma_{yz}^2} \ . \tag{1.1}$$

In standard elasticity, the stress tensor is symmetric, so an equivalent definition of q is $q = \sqrt{\frac{1}{2}(\sigma_{xz}+\sigma_{zx})^2 + \frac{1}{2}(\sigma_{yz}+\sigma_{zy})^2}$, however the symmeterisation is deliberately not written in Eqn (1.1) and below so that the equations also hold for the Cosserat case of Section 6.2.

The joint slipping is assumed to be governed by a Drucker-Prager type of plasticity with a cohesion C, and friction angle ϕ:

$$f_0 = q + p\tan\phi - C \ . \tag{1.2}$$

The parameter C and ϕ may be constants, or they may harden or soften (more on this later).

Joints may also open, and this type of failure is assumed to be governed by a tensile failure yield function:

$$f_1 = p - S_T \ , \tag{1.3}$$

where S_T is the tensile strength, which may be contant or harden or soften.

Joints may also close, and this type of failure is assumed to be governed by a compressive failure yield function:

$$f_2 = -p - S_C \ , \tag{1.4}$$

where S_C is the compressive strength (a positive quantity), which may be contant or harden or soften.

The yield functions f_1 and f_2 place "caps" on the shear yield function f_0, as shown in Figure 1.1. The combined yield function is simply

$$f = \max(f_0, f_1, f_2) \ , \tag{1.5}$$

which defines the admissible domain where all yield functions are non-positive, and the inadmissible domain where at least one yield function is positive.

Figure 1.1: Placeholder for a figure that shows an example of a non-smoothed capped weak-plane plasticity yield surface, including f_0, f_1, f_2 and f, and the admissible and inadmissible regions.

One of the features of this plasticity is the ability to model cyclic behaviour. For instance, the compressive strength may be initially very high. However, after tensile failure, the compressive strength can soften to zero in order to model the fact that the material now contains open joints which cannot support any compressive load. If the material then fails in compression (eg, because it gets squashed) and the joints close then the compressive strength can be made high again.

2 Flow rules and hardening

This plasticity is nonassociative. Define the dilation angle ψ, which may be constant, or harden or soften. The shear flow potential is

$$g_0 = q + p\tan\psi. \tag{2.1}$$

The tensile flow potential is

$$g_1 = p\ , \tag{2.2}$$

and the compressive flow potential is

$$g_2 = -p\ . \tag{2.3}$$

The overall flow potential is

$$g = \begin{cases} g_0 & \text{if } f = f_0 \\ g_1 & \text{if } f = f_1 \\ g_2 & \text{if } f = f_2\ . \end{cases} \tag{2.4}$$

Obviously there are problems here where g is not defined properly at the corners where $f_0 = f_1$ and $f_0 = f_2$ (or even $f_0 = f_1 = f_2$). This is resolved by using smoothing (more on this later).

This plasticity model contains two internal parameters, denote by i_0 and i_1. It is assumed that

$$\begin{aligned} C &= C(i_0)\ , & (2.5)\\ \phi &= \phi(i_0)\ , & (2.6)\\ \psi &= \psi(i_0)\ , & (2.7)\\ S_T &= S_T(i_1)\ , & (2.8)\\ S_C &= S_C(i_1)\ . & (2.9) \end{aligned}$$

That is, i_0 can be thought of as the "shear" internal parameter, while i_1 is the "tensile" internal parameter.

To complete the definition of this plasticity model, the increments of i_0 and i_1 during the return-map process must be defined. The return-map process involves being provided with a trial stress σ^{trial} and an existing value of the internal parameters i^{old}, and finding a "returned" stress, σ, and internal parameters, i, that satisfy

$$\begin{aligned} 0 &= f(\sigma, i)\ . & (2.10)\\ \sigma &= \sigma^{\text{trial}} - E\gamma\frac{\partial g}{\partial \sigma}\ , & (2.11) \end{aligned}$$

where E is the elasticity tensor, and γ is a "plastic multiplier", that must be positive. The former expresses that the stress must be admissible, while the latter is called the "normality condition".

Loosely speaking, the returned stress lies at a position on the yield surface where the normal points to the trial stress (actually E and $\partial g/\partial\sigma$ must be used to define the "normal direction").

Let us express the normality condition in (p,q) space. The zz component is easy:

$$p = \sigma_{zz} = \sigma_{zz}^{\mathrm{trial}} - E_{zzij}\gamma\frac{\partial g}{\partial\sigma_{ij}} = \sigma_{zz}^{\mathrm{trial}} - E_{zzzz}\gamma\frac{\partial g}{\partial p} \ , \tag{2.12}$$

where the last equality holds by assumption (see full list of assumptions below). The xz and yz components are similar:

$$\sigma_{xz} = \sigma_{xz}^{\mathrm{trial}} - E_{xzxz}\gamma\frac{\partial g}{\partial q}\frac{\partial q}{\partial\sigma_{xz}} \ . \tag{2.13}$$

Another assumption has been made about E. The final term is

$$\frac{\partial q}{\partial\sigma_{xz}} = \frac{\sigma_{xz}}{q} \ . \tag{2.14}$$

This means that Eqn (2.13) can be re-written

$$\sigma_{xz}^2\left(1 + E_{xzxz}\gamma\frac{\partial g}{\partial q}\frac{1}{q}\right)^2 = \left(\sigma_{xz}^{\mathrm{trial}}\right)^2 \ . \tag{2.15}$$

A similar equation holds for the yz component, and these can be summed and rearranged to yield

$$q = q^{\mathrm{trial}} - E_{xzxz}\gamma\frac{\partial g}{\partial q} \ . \tag{2.16}$$

Equations (2.11), (2.12) and (2.16) are the three conditions that need to be satisfied, and the three variables to be found are p, q and γ.

Consider the case of returning to the shear yield surface, as shown in Figure 2.1. Since $\partial g/\partial p = \tan\psi$ and $\partial g/\partial q = 1$ for this flow, the return-map process must solve the following system of equations

$$\begin{aligned} 0 &= q + p\tan\phi - C \ , & (2.17)\\ p &= p^{\mathrm{trial}} - E_{zzzz}\gamma\tan\psi \ , & (2.18)\\ q &= q^{\mathrm{trial}} - E_{xzxz}\gamma \ . & (2.19)\end{aligned}$$

The solution satisfied $p^{\mathrm{trial}} - p = E_{zzzz}\gamma\tan\psi$ and $q^{\mathrm{trial}} - q = E_{xzxz}\gamma$.

Figure 2.1: Placeholder for a figure that shows flow back to the shear yield surface, via $\tan\psi$, showing $(p^{\mathrm{trial}}, q^{\mathrm{trial}})$, and (p,q), and γ for E having a zero Poisson's ratio.

Now consider the case of returning to the tensile yield surface. The equations are

$$\begin{aligned} 0 &= p - S_T \ , & (2.20)\\ p &= p^{\mathrm{trial}} - E_{zzzz}\gamma \ , & (2.21)\\ q &= q^{\mathrm{trial}} \ . & (2.22)\end{aligned}$$

Comparing these two types of return, it is obvious that $q^{\mathrm{trial}} - q$ quantifies the amount of shear failure. Therefore, the following definitions are used in this plasticity model

$$
\begin{aligned}
i_0 &= i_0^{\mathrm{old}} + \frac{q^{\mathrm{trial}} - q}{E_{xzxz}} , && (2.23)\\
i_1 &= i_1^{\mathrm{old}} + \frac{p^{\mathrm{trial}} - p}{E_{zzzz}} - \frac{(q^{\mathrm{trial}} - q)\tan\psi}{E_{xzxz}} . && (2.24)
\end{aligned}
$$

The final term ensures that i_1 does not increase during pure shear failure. The scaling by E ensures that these internal parameters are dimensionless.

In summary, this plasticity model is defined by the yield function of Equation (1.5), the flow potential of Equation (2.4), and the following return-map problem.

Return-map problem At any given MOOSE timestep, given the old stress $\sigma_{ij}^{\mathrm{old}}$, and a total strain increment $\delta\varepsilon$ (that comes from the nonlinear solver proposing displacements) the trial stress is

$$
\sigma_{ij}^{\mathrm{trial}} = \sigma_{ij}^{\mathrm{old}} + E_{ijkl}\delta\varepsilon_{kl} , \qquad (2.25)
$$

This gives p^{trial} and q^{trial}. If p^{trial}, q^{trial}, i_0^{old} and i_1^{old} are such that $f(p^{\mathrm{trial}}, q^{\mathrm{trial}}, i^{\mathrm{old}}) > 0$, the return-map problem is: find p, q, γ, i_0 and i_1 such that

$$
\begin{aligned}
0 &= f(p,q,i) , \\
p &= p^{\mathrm{trial}} - E_{zzzz}\gamma\frac{\partial g}{\partial p} , \\
q &= q^{\mathrm{trial}} - E_{xzxz}\gamma\frac{\partial g}{\partial q} , \\
i_0 &= i_0^{\mathrm{old}} + \frac{q^{\mathrm{trial}} - q}{E_{xzxz}} , \\
i_1 &= i_1^{\mathrm{old}} + \frac{p^{\mathrm{trial}} - p}{E_{zzzz}} - \frac{(q^{\mathrm{trial}} - q)\tan\psi}{E_{xzxz}} . && (2.26)
\end{aligned}
$$

The latter two equations are assumed to hold in the smoothed situation (discussed below) too, and note that $\psi = \psi(i_0)$, so the these two equations are not completely trivial.

After the return-map problem has been solved, the stress components are $\sigma_{ij} = \sigma_{ij}^{\mathrm{trial}}$, except for the following

$$
\begin{aligned}
\sigma_{xx} &= \sigma_{xx}^{\mathrm{trial}} - E_{zzxx}\gamma\frac{\partial g}{\partial p} , && (2.27)\\
\sigma_{yy} &= \sigma_{yy}^{\mathrm{trial}} - E_{zzyy}\gamma\frac{\partial g}{\partial p} , && (2.28)\\
\sigma_{zz} &= p , && (2.29)\\
\sigma_{zx} &= \sigma_{zx}^{\mathrm{trial}} q/q^{\mathrm{trial}} , && (2.30)\\
\sigma_{xz} &= \sigma_{xz}^{\mathrm{trial}} q/q^{\mathrm{trial}} , && (2.31)\\
\sigma_{zy} &= \sigma_{zy}^{\mathrm{trial}} q/q^{\mathrm{trial}} , && (2.32)\\
\sigma_{yz} &= \sigma_{yz}^{\mathrm{trial}} q/q^{\mathrm{trial}} . && (2.33)
\end{aligned}
$$

The plastic strain is

$$\varepsilon_{ij}^{\text{plastic}} = \varepsilon_{ij}^{\text{plastic,old}} + \delta\varepsilon_{ij} + E_{ijkl}^{-1}(\sigma_{kl}^{\text{old}} - \sigma_{kl}) = \varepsilon_{ij}^{\text{plastic,old}} + E_{ijkl}\gamma\frac{\partial g}{\partial\sigma_{kl}} \ . \tag{2.34}$$

The elastic strain is

$$\varepsilon_{ij}^{\text{elastic}} = \varepsilon_{ij}^{\text{elastic,old}} + \delta\varepsilon_{ij} - \varepsilon_{ij}^{\text{plastic}} + \varepsilon_{ij}^{\text{plastic,old}} \ . \tag{2.35}$$

3 Smoothing

The shear yield function, f_0, describes a cone in $(\sigma_{yz}, \sigma_{xz}, \sigma_{zz})$ space, as depicted in Figure 3.1. The cone's tip is problematic for the return-map process (the derivative is not defined there) and there are two main ways of getting around this. Firstly, a multi-surface technique can be used to define the return-map process. Secondly, the cone's tip can be smoothed. This plasticity model uses the second technique. The yield function is defined to be

$$f_0 = \sqrt{q^2 + s_t^2} + p \tan\phi - C \ , \tag{3.1}$$

and the flow potential is

$$g_0 = \sqrt{q^2 + s_t^2} + p \tan\psi \ . \tag{3.2}$$

Figure 3.1: The shear cone in stress space and its 2D representation including the smoothed version.

The vertices where the shear yield surface meets the tensile and compressive yield surfaces also need to be handled. Smoothing is also used here. This uses a new type of smoothing that I invented for this type of situation and hopefully I'll publish at some stage soon. For the case at hand only two yield surfaces and flow potentials need to be smoothed (there are no points where three or more yield surfaces get close to each other) and only in 2D space, and a single parameter s can be used. The parameter s has the units of stress. At any point (p, q, i) order the 3 yield function values, and denote the largest by A, the second largest by B and the smallest by C:

$$A \geq B \geq C \tag{3.3}$$

Then the single, smoothed yield function is defined to be

$$f = \begin{cases} A & \text{if } A \geq B + s \\ \frac{A+B+s}{2} - \frac{s}{\pi}\cos\left(\frac{(B-A)\pi}{2s}\right) & . \end{cases} \tag{3.4}$$

The derivative of the flow potential is smoothed similarly. Figure 3.2 shows an example of this smoothing, with arrows indicating the flow-potential gradients.

Figure 3.2: Level sets of the smoothed yield function, and associated gradients of the flow potential (as arrows)

4 Constraints and assumptions concerning parameters

The friction angle and cohesion should be positive, and the dilation angle should be non-negative. Furthermore, the MOOSE user must ensure that

$$\psi \leq \phi \ . \tag{4.1}$$

These conditions should be satisfied for all values of the internal parameter i_0. MOOSE checks that these conditions hold for $i_0 = 0$ only.

The tensile and compressive strength must satisfy

$$S_T \geq -S_C \ , \tag{4.2}$$

otherwise the "caps" are swapped and the assumption of a convex yield surface is violated. MOOSE checks this condition holds for $i_1 = 0$ only: the MOOSE user must ensure that it actually holds for all values of the internal parameter

The smoothing parameter s must be chosen carefully. At no time should the tensile cap mix with the compressive cap via smoothing, otherwise this typically means that no stress is admissible and MOOSE will never converge. For instance, if $S_T = 1 = S_C$, then a smoothing parameter of 0.1 is fine, but a smoothing parameter ≥ 2 will cause mixing of tension with compression. The MOOSE user must ensure that this holds for all values of the internal parameters.

The tip-smoothing parameter s_t is important, even if the tensile cap completely chops off the shear-cone's tip. This is because MOOSE can explore regions of parameter space where the cone's tip is exposed.

It is vital that the smoothing parameters s and s_t are chosen so that the yield surface is not wildly varying around $q = 0$, otherwise poor convergence of the return-map process will occur.

It is assumed that the elasticity tensor has the following symmetries:

$$E_{ijkl} = E_{jikl} = E_{ijlk} = E_{klij} \ , \tag{4.3}$$

and that

$$0 = E_{zzij} \quad \text{if} \quad i \neq j \ , \tag{4.4}$$

and that

$$E_{xzxz} = E_{yzyz} \ , \tag{4.5}$$

and that

$$0 = E_{xzij} \quad \text{unless } (i,j) = (z,x) \quad \text{or } (i,j) = (x,z) \ . \tag{4.6}$$

These are quite standard conditions that hold for all non-Cosserat materials I am aware of.

5 Technical discussions

5.1 Unknowns and the convergence criterion

The return-map problem Eqn (2.26) is solved as a 3×3 system consisting of the first 3 equations, and substituting the fourth and fifth equations wherever needed. The three unknowns are p, q and $\gamma_E = \gamma E_{zzzz}$, which all have the same units. Convergence is deemed to be achieved when the sum of squares of the residuals of these 3 equations is less than a user-defined tolerance.

5.2 Iterative procedure and initial guesses

A Newton-Raphson process is used, along with a cubic line-search. The process may be initialised with the solution that is correct for perfect plasticity (no hardening) and no smoothing, if the user desires. Smoothing adds nonlinearities, so this initial guess will not always be the exact answer. For hardening, it is not always advantageous to initialise the Newton-Raphson process in this way, as the yield surfaces can move dramatically during the return process.

5.3 Substepping the strain increments

Because of the difficulties encountered during the Newton-Raphson process during rapidly hardening/softening moduli, it is possible to subdivide the applied strain increment, $\delta\epsilon$, into smaller substeps, and do multiple return-map processes, as shown in Figure 5.1. The final returned configuration will then be dependent on the number of substeps. While this is simply illustrating the non-uniqueness of plasticity problems, in my experience it does adversely affect MOOSE's nonlinear convergence as some Residual calculations will take more substeps than other Residual calculations: in effect this is reducing the accuracy of the Jacobian.

5.4 The consistent tangent operator

MOOSE's Jacobian depends on the derivative

$$H_{ijkl} = \frac{\delta\sigma_{ij}}{\delta\epsilon_{kl}} \ . \tag{5.1}$$

The quantity H is called the consistent tangent operator. For pure elasticity it is simply the elastic tensor, E, but it is more complicated for plasticity. Note that a small $\delta\epsilon_{kl}$ simply changes $\delta\sigma^{\mathrm{trial}}$, so H is capturing the change of the returned stress ($\delta\sigma$) with respect to a change in the trial stress

($\delta\sigma^{\mathrm{trial}}$). In (p,q) language, we need to the sx derivatives

$$\frac{\delta(p,q,\gamma)}{\delta(p^{\mathrm{trial}},q^{\mathrm{trial}})} \ . \tag{5.2}$$

The algebra is extremely tedious, but it is fairly easy for the computer. The MOOSE code contains two implementations of the consistent tangent operator. One is valie for any general (p,q) model, while the other is specialised to the weak-plane case.

5.5 General consistent tangent operator

The return-map algorithm provides

$$\sigma_{ij} = \sigma_{ij}^{\mathrm{trial}} - E_{ijmn}\gamma\frac{\partial g}{\partial\sigma_{mn}} \ . \tag{5.3}$$

Since $\sigma^{\mathrm{trial}} = E\varepsilon$, the consistent tangent operator is

$$\begin{aligned} H_{ijkl} &= E_{ijkl} - E_{ijmn}E_{pqkl}\frac{\partial}{\partial\sigma_{pq}^{\mathrm{trial}}}\gamma\frac{\partial g}{\partial\sigma_{mn}} \ . \\ &= E_{ijkl} - E_{ijmn}E_{pqkl}\left(\frac{\partial p^{\mathrm{trial}}}{\partial\sigma_{pq}^{\mathrm{trial}}}\frac{\partial}{\partial p^{\mathrm{trial}}} + \frac{\partial q^{\mathrm{trial}}}{\partial\sigma_{pq}^{\mathrm{trial}}}\frac{\partial}{\partial q^{\mathrm{trial}}}\right)\gamma\left(\frac{\partial g}{\partial p}\frac{\partial p}{\partial\sigma_{mn}} + \frac{\partial g}{\partial q}\frac{\partial q}{\partial\sigma_{mn}}\right) \end{aligned} \tag{5.4}$$

However, note that

$$\frac{\partial}{\partial p^{\mathrm{trial}}}\left(p - p^{\mathrm{trial}} + \gamma E_{pp}\frac{\partial g}{\partial p}\right) = 0 \ , \tag{5.5}$$

because the return-map algorithm gaurantees that the expression inside parenthases is zero. Therefore

$$\frac{\partial}{\partial p^{\mathrm{trial}}}\gamma\frac{\partial g}{\partial p} = \frac{1}{E_{pp}}\left(1 - \frac{\partial p}{\partial p^{\mathrm{trial}}}\right) \ . \tag{5.6}$$

A similar expression holds for three other cases. There are still terms that involve derivatives of $\partial p/\partial\sigma_{mn}$ and $\partial q/\partial\sigma_{mn}$, but these may be separated off as seen below.

The consistent tangent operator may therefore be written as

$$\begin{aligned} H_{ijkl} &= E_{ijkl} - E_{ijmn}E_{pqkl}\Bigg\{\frac{\partial p^{\mathrm{trial}}}{\partial\sigma_{pq}^{\mathrm{trial}}}\frac{1}{E_{pp}}\left(1 - \frac{\partial p}{\partial p^{\mathrm{trial}}}\right)\frac{\partial p}{\partial\sigma_{mn}} \\ &\quad \frac{\partial q^{\mathrm{trial}}}{\partial\sigma_{pq}^{\mathrm{trial}}}\frac{1}{E_{pp}}\left(-\frac{\partial p}{\partial q^{\mathrm{trial}}}\right)\frac{\partial p}{\partial\sigma_{mn}} + \frac{\partial p^{\mathrm{trial}}}{\partial\sigma_{pq}^{\mathrm{trial}}}\frac{1}{E_{qq}}\left(-\frac{\partial q}{\partial p^{\mathrm{trial}}}\right)\frac{\partial q}{\partial\sigma_{mn}} + \frac{\partial q^{\mathrm{trial}}}{\partial\sigma_{pq}^{\mathrm{trial}}}\frac{1}{E_{qq}}\left(1 - \frac{\partial q}{\partial q^{\mathrm{trial}}}\right)\frac{\partial q}{\partial\sigma_{mn}}\Bigg\} \\ &\quad \frac{\partial\sigma_{ab}}{\partial\varepsilon_{kl}}E_{ijmn}\gamma\left(\frac{\partial g}{\partial p}\frac{\partial^2 p}{\partial\sigma_{mn}\partial\sigma_{ab}} + \frac{\partial g}{\partial q}\frac{\partial^2 q}{\partial\sigma_{mn}\partial\sigma_{ab}}\right) \ . \end{aligned} \tag{5.7}$$

All terms but the final line have already been computed during the return-map process. The final line may be brought to the right-hand side (since $H_{ijkl} = \partial\sigma_{ij}/\partial\varepsilon_{kl}$) and the resulting expression multiplied inverse H's coefficient to finally yield H. This inversion, and all the multiplication of rank-four tensors may be computationally expensive, so a cheaper (but more lengthy looking) version is derived below for the capped weak-plane case.

5.6 Specialisation to the weak-plane case

The return-map equations Eqn (2.26) are obtaining (p,q) given the trial variables. Finding H is really just re-solving these equations for a slightly changed trial variable. Denote

$$\begin{pmatrix} R_0 \\ R_1 \\ R_2 \end{pmatrix} = \begin{pmatrix} -f \\ -p+p^{\mathrm{trial}} - E_{zzzz}\gamma\frac{\partial g}{\partial p} \\ -q+q^{\mathrm{trial}} - E_{xzxz}\gamma\frac{\partial g}{\partial q} \end{pmatrix} . \tag{5.8}$$

Then

$$\frac{\partial R_0}{\partial p^{\mathrm{trial}}} = -\frac{\partial f}{\partial i_1}\frac{\partial i_i}{\partial p^{\mathrm{trial}}} , \tag{5.9}$$

$$\frac{\partial R_1}{\partial p^{\mathrm{trial}}} = 1 - E_{zzzz}\gamma\frac{\partial^2 g}{\partial p \partial i_i}\frac{\partial i_i}{\partial p^{\mathrm{trial}}} , \tag{5.10}$$

$$\frac{\partial R_2}{\partial p^{\mathrm{trial}}} = -E_{xzxz}\gamma\frac{\partial^2 g}{\partial q \partial i_i}\frac{\partial i_i}{\partial p^{\mathrm{trial}}} \tag{5.11}$$

In these equations

$$\frac{\partial i_1}{\partial p^{\mathrm{trial}}} = \frac{1}{E_{zzzz}} , \tag{5.12}$$

which comes from Eqn (2.26). The derivatives with respect to q^{trial} are similar but more lengthy due to both i_0 and i_1 being dependent on q^{trial}. The system to solve is

$$\begin{pmatrix} \frac{\partial R_0}{\partial \gamma} & \frac{\partial R_0}{\partial p} & \frac{\partial R_0}{\partial q} \\ \frac{\partial R_1}{\partial \gamma} & \frac{\partial R_1}{\partial p} & \frac{\partial R_1}{\partial q} \\ \frac{\partial R_2}{\partial \gamma} & \frac{\partial R_2}{\partial p} & \frac{\partial R_2}{\partial q} \end{pmatrix} \begin{pmatrix} \delta\gamma/\delta p^{\mathrm{trial}} \\ \delta p/\delta p^{\mathrm{trial}} \\ \delta q/\delta p^{\mathrm{trial}} \end{pmatrix} = \begin{pmatrix} \partial R_0/\delta p^{\mathrm{trial}} \\ \partial R_1/\delta p^{\mathrm{trial}} \\ \partial R_2/\delta p^{\mathrm{trial}} \end{pmatrix} \tag{5.13}$$

The 3×3 Jacobian matrix is identical to the one used in the Newton-Raphson process, but of course that process has completed before calculation of the consistent tangent operator. A similar system of equations gives the derivatives with respect to q^{trial}.

Once the six derivatives have been computed they need to be assembled into H. For instance,

$$\frac{\delta p^{\mathrm{trial}}}{\delta \varepsilon_{ii}} = E_{zzii} , \tag{5.14}$$

so that

$$H_{zzii} = \frac{\delta p}{\delta p^{\mathrm{trial}}} E_{zzii} , \tag{5.15}$$

and other more complicated expressions appear for other components, such as

$$\begin{aligned} H_{xxii} &= E_{xxii} - E_{zzxx}E_{zzii}\left(\frac{\delta\gamma}{\delta p^{\mathrm{trial}}}\frac{\partial g}{\partial p} + \gamma\frac{\partial^2 g}{\partial p^2}\frac{\delta p}{\delta p^{\mathrm{trial}}} + \gamma\frac{\partial^2 g}{\partial p \partial i_1}\frac{\delta q}{\delta p^{\mathrm{trial}}} + \gamma\frac{\partial^2 g}{\partial p \partial i_0}\frac{\partial i_0}{\partial q}\frac{\delta q}{\delta p^{\mathrm{trial}}} \right. \\ &\quad \left. + \gamma\frac{\partial^2 g}{\partial p \partial i_1}\left(\frac{\delta i_1}{\delta p^{\mathrm{trial}}} + \frac{\partial i_1}{\partial p}\frac{\delta p}{\delta p^{\mathrm{trial}}}\frac{\partial i_1}{\partial q}\frac{\delta q}{\delta p^{\mathrm{trial}}}\right)\right) . \end{aligned} \tag{5.16}$$

If you understand all this, give yourself a pat on the back and then go and check the code for bugs: there are plenty more nasty equations!

5.7 The consistent tangent operator and substepping strain increments

One extra complication arises from the potential substepping of the applied strain increment $\delta\varepsilon$. At each substep, the six derivatives must be computed. While this may seem expensive, in my experience it increases the accuracy of the Jacobian, and the main computational expense is building and solving the 3×3 system which is pretty quick for the computer to compared with the entire Newton-Raphson process. The substepping process is shown in Figure 5.1.

Figure 5.1: The substepping process

Let the n^{th} substep be

$$\delta\varepsilon^n = \lambda_n \delta\varepsilon \, , \tag{5.17}$$

with

$$1 = \sum_{n=1}^{N} \lambda_n \, , \tag{5.18}$$

where N is the total number of substeps. Denoting the initial stress by $(p^{\text{old}}, q^{\text{old}})$, and the returned stress at step $n-1$ by (p_{n-1}, q_{n-1}), of course the trial stress at step n is

$$(p_n^{\text{trial}}, q_n^{\text{trial}}) = (p_{n-1}, q_{n-1}) + \lambda_n (p^{\text{trial}} - p^{\text{old}}, q^{\text{trial}} - q^{\text{old}}) \, . \tag{5.19}$$

This means that

$$\frac{\partial p_n}{\partial p^{\text{trial}}} = \frac{\partial p_n}{\partial p_n^{\text{trial}}} \frac{\partial p_n^{\text{trial}}}{\partial p^{\text{trial}}} + \frac{\partial p_n}{\partial q_n^{\text{trial}}} \frac{\partial q_n^{\text{trial}}}{\partial p^{\text{trial}}} \tag{5.20}$$

$$= \frac{\partial p_n}{\partial p_n^{\text{trial}}} \left(\lambda_n + \frac{\partial p_{n-1}}{\partial p^{\text{trial}}} \right) + \frac{\partial p_n}{\partial q_n^{\text{trial}}} \frac{\partial q_{n-1}}{\partial p^{\text{trial}}} \, . \tag{5.21}$$

Similar inductive equations hold for the other derivatives, and note that $\partial p_0 / \partial p^{\text{trial}} = \partial p^{\text{old}} / \partial p^{\text{trial}} = 0$. The derivative of γ is slightly different: it is

$$\frac{\partial \gamma}{\partial p^{\text{trial}}} = \sum_{n=1}^{N} \frac{\partial \gamma_n}{\partial p_n^{\text{trial}}} \left(\lambda_n + \frac{\partial p_{n-1}}{\partial p^{\text{trial}}} \right) + \frac{\partial \gamma_n}{\partial q_n^{\text{trial}}} \frac{\partial q_{n-1}}{\partial p^{\text{trial}}} \, , \tag{5.22}$$

and similarly for the derivative with respect to q^{trial}.

6 Extensions

6.1 Arbitrary normal

The above presentation assumed the weak plane's normal is $(0,0,1)$. A version of capped weak-plane plasticity is available that accepts an arbitrary normal vector. It is called `ComputeCappedWeakInclinedPlaneS`
In the reference frame where the inclined plane's normal aligns with the z axis, the assumptions listed in Chapter 4 must hold.

6.2 Cosserat

The difference between the Cosserat and non-Cosserat case is that the stress tensor is potentially non-symmetric, and that only σ_{xz} and σ_{yz} enter into the definition of q. That is, Eqn (1.1) holds as it is written. This means that the return-map process often results in a non-symmetric stress tensor, even if the trial stress was symmetric. The equations of moment equilibrium then typically generate a spatially-varying moment stress, the precise nature of which depends on the problem at hand, such as the boundary conditions.

The differences between the above presentation for the non-Cosserat case, and the Cosserat case are only:

1. Eqn (2.30) and (2.32) do not hold. These components of stress take their trial values after the return-map problem has been solved: $\sigma_{zx} = \sigma_{zx}^{\mathrm{trial}}$ and $\sigma_{zy} = \sigma_{zy}^{\mathrm{trial}}$.

2. The elasticity tensor need not have the symmetries given in Eqn (4.3): it only needs to satisfy $E_{ijkl} = E_{klij}$.

The MOOSE Material is called `ComputeCappedWeakPlaneCosseratStress`.

7 Tests