Lecture 1 Formal correspondence in Coq
Lecture notes 1 - 2 - 3

(In class/next notes)

Lecture 4 Simply typed lambda calculus (Monday, Sept 1)

§ Abstract definition

Logical operators:

$\wedge, \vee, \top, \rightarrow$

Rules: Introduction & elimination

$\wedge$-intro $\dfrac{A \quad B}{A \wedge B}$ $\wedge$-elim $\dfrac{A \wedge B}{A}$ $\wedge$-elim $\dfrac{A \wedge B}{B}$

$\top$-intro $\dfrac{}{\top}$ $\vee$-intro $\dfrac{A}{A \vee B}$ $\vee$-intro $\dfrac{B}{A \vee B}$

$\rightarrow$-intro $\dfrac{A \vdash B}{A \rightarrow B}$ $\rightarrow$-elim $\dfrac{A \rightarrow B \quad A}{B}$

$A \rightarrow B = \neg A \vee B$

Law of excluded middle

LEM $\dfrac{}{A \vee \neg A}$

Rule of Peirce

$\dfrac{\neg A \rightarrow B}{A \vee B}$

$\dfrac{A \vee B \quad A \vee \neg B}{A \vee B}$ $\dfrac{A \vee B \quad A \vee \neg B}{A \vee B}$

$\dfrac{A \vee B \quad A \vee \neg B}{A \vee B}$

$\vdash A \vee \neg A \Rightarrow \vdash A \vee B$

§ Simply typed λ-calculus

Syntax

$\Gamma \quad \Delta$

$\Gamma \vdash M : A$ type level

- Judgmental equality

$\Gamma \vdash M \equiv N : A$

"equal by computation"

Eg. $\{\}, \{x : \text{bool}\} \vdash x \equiv x$

$x \rightarrow \text{and} \rightarrow \text{or}$

$\text{true} \rightarrow \text{true} \rightarrow x$

$\text{true} \rightarrow \text{false}$

Judgments are decidable (by algorithm)

- Variables

$\dfrac{x : A \in \Gamma}{\Gamma \vdash x : A}$ (var) $\quad \dfrac{x : A \in \Gamma \quad x : A \in \Gamma}{x : A \in \Gamma}$ (var)

$\dfrac{x : A \in \Gamma}{x \equiv x : A}$ (var)

$\dfrac{x : A \in \Gamma \quad x : A \in \Gamma}{x \equiv x : A}$ (var)

Compare with

$\rightarrow$ most deduction

(equalities)

$\Gamma \vdash x \equiv x : A$

$\Gamma \vdash x : A$

- Functions

$\dfrac{x : A \in \Gamma \quad \Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x : A . M : A \rightarrow B}$ (lam)

$\dfrac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B}$ (app)

$\dfrac{\Gamma \vdash \lambda x : A . M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash (\lambda x : A . M)\ N : B}$ (app)

$\Gamma \vdash \lambda x : A . M : A \rightarrow B$

$\Gamma, x : A \vdash M : B$

$\Gamma \vdash N : A$

$\Gamma \vdash (\lambda x : A . M)\ N : B$

$\Gamma \vdash M : A \rightarrow B$

$\Gamma \vdash N : A$

$\Gamma \vdash M\ N : B$

- Problem

$\{x : \text{bool}\} \vdash x : \text{bool}$

$\{x : \text{bool}\} \vdash \text{and}\ x : \text{bool} \rightarrow \text{bool}$

$\{x : \text{bool}\} \vdash \text{and}\ x\ x : \text{bool}$

$\{x : \text{bool}\} \vdash \text{and}\ x\ \text{true} : \text{bool}$

$\{x : \text{bool}\} \vdash \text{and}\ x\ \text{false} : \text{bool}$

$\{x : \text{bool}\} \vdash \text{and}\ \text{true}\ x : \text{bool}$

$\{x : \text{bool}\} \vdash \text{and}\ \text{false}\ x : \text{bool}$

$\{x : \text{bool}\} \vdash \text{and}\ \text{true}\ \text{true} : \text{bool}$

$\{x : \text{bool}\} \vdash \text{and}\ \text{true}\ \text{false} : \text{bool}$

$\{x : \text{bool}\} \vdash \text{and}\ \text{false}\ \text{true} : \text{bool}$

$\{x : \text{bool}\} \vdash \text{and}\ \text{false}\ \text{false} : \text{bool}$

(equations)

$\{x : \text{bool}\} \vdash x \equiv x : \text{bool}$

$\{x : \text{bool}\} \vdash \text{and}\ x\ x \equiv x : \text{bool}$

$\{x : \text{bool}\} \vdash \text{and}\ x\ \text{true} \equiv x : \text{bool}$

$\{x : \text{bool}\} \vdash \text{and}\ x\ \text{false} \equiv \text{false} : \text{bool}$

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$\{x : \text{bool}\} \vdash \text{and}\ \text{true}\ \text{true} \equiv \text{true} : \text{bool}$

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$\{x : \text{bool}\} \vdash \text{and}\ \text{false}\ \text{true} \equiv \text{false} : \text{bool}$

$\{x : \text{bool}\} \vdash \text{and}\ \text{false}\ \text{false} \equiv \text{false} : \text{bool}$

Assume types: bool

$\text{and} : \text{bool} \rightarrow \text{bool} \rightarrow \text{bool}$

$\text{true} : \text{bool}$

$\text{false} : \text{bool}$

Take propositions as types

propositions: and, or

- Equations (Algebra axioms)

$\dfrac{x : A}{x \equiv x : A}$ $\dfrac{x \equiv x : A}{x \equiv x : A}$

$\dfrac{x : A \rightarrow B \quad y : A}{x\ y : B}$ $\dfrac{x \equiv x : A \rightarrow B \quad y \equiv y : A}{x\ y \equiv x\ y : B}$ (eq)

$\{x : \text{bool}\} \vdash x \equiv \text{true}$

$\{x : \text{bool}\} \vdash \text{and}\ x\ \text{true} \equiv x$

$\{x : \text{bool}\} \vdash \text{and}\ x\ \text{false} \equiv \text{false}$

$\{x : \text{bool}\} \vdash \text{and}\ \text{true}\ x \equiv x$

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Assume types: bool

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$\{x : \text{bool}\} \vdash x \equiv \text{true}$

$\{x : \text{bool}\} \vdash \text{and}\ x\ \text{true} \equiv x$

$\{x : \text{bool}\} \vdash \text{and}\ x\ \text{false} \equiv \text{false}$

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$\{x : \text{bool}\} \vdash \text{and}\ \text{true}\ \text{false} \equiv \text{false}$

$\{x : \text{bool}\} \vdash \text{and}\ \text{false}\ \text{true} \equiv \text{false}$

$\{x : \text{bool}\} \vdash \text{and}\ \text{false}\ \text{false} \equiv \text{false}$

- Problem of simply typed λ-calculus

Can we express propositions. We can. $\text{bool} \equiv \text{prop}$

- Type for "0"/"1"

- Type for "$\rightarrow$"

§ Dependently typed λ-calculus

- Dependent function

$\dfrac{x : A \vdash B : \text{Type}}{\Pi x : A . B : \text{Type}}$ (Π-intro)

$\dfrac{x : \text{bool} \vdash \text{Nat}}{\Pi x : \text{bool} . \text{Nat} : \text{Type}}$

$\text{Nat} : \text{Type}$

$\text{Vec}\ n\ \text{Nat}$

Eg. $\text{eq}_{\text{nat}} : \Pi (x : \text{Nat}) \Pi (y : \text{Nat}) . \text{Type}$

$\text{Vec}\ n\ \text{Nat} : \text{Type}$

$\text{Vec}\ n\ \text{Nat} : \text{Type}$

Where Π consists $\rightarrow$ $\rightarrow$. $\{\}, x : \text{bool} \vdash x : A \rightarrow B$

$\dfrac{x : A \quad \Gamma \vdash M : B[x/x]}{\lambda x : A . M : \Pi x : A . B}$ (Π-intro)

$\Gamma \vdash \text{Type}$

$\Gamma \vdash \text{Type}$

$\Gamma \vdash \text{Type}$