The theory of rational choice

The theory of rational choice is that a decision-maker chooses the best action according to her preferences, among all the actions available to her. The action chosen by a decision-maker is at least as good, according to her preferences, as every other available action.
The theory is based on a model with two components: a set A consisting of all the **actions** that, under some circumstances, are available to the decision-maker, and a specification of the decision-maker's **preferences**. In any given situation the decision-maker is faced with a subset of A, from which she must choose a single element. The decision-maker knows this subset of available choices, and takes it as given; in particular, the subset of actions is not influenced by the decision-maker's preferences.
As to preferences, we assume that the decision-maker, when presented with any pair of actions, knows which of the pair she prefers, or knows that she regards both actions as equally desirable. Preferences can be seen as a metric.
Instead of specifying preferences for each possible pair of actions, we can "represent" the preferences by a **payoff function** u() (aka utility function), which associates a number with each action in such a way that actions with higher numbers are preferred. For any actions a in A and b in A, u(a) > u(b) if and only if the decision-maker prefers a to b.

EXAMPLE 5.2 (Payoff function representing preferences) A person is faced with the choice of three vacation packages, to Havana, Paris, and Venice. She prefers the package to Havana to the other two, which she regards as equivalent. Her preferences between the three packages are represented by any payoff function that assigns the same number to both Paris and Venice and a higher number to Havana. For example, we can set u(Havana) = 1 and u(Paris) = u(Venice) = 0.
A decision-maker's preferences, in the sense used here, **convey only ordinal information**. They may tell us that the decision-maker prefers the action a to the action b to the action c, for example, but they do not tell us "how much" she prefers a to b, or whether she prefers a to b "more" than she prefers b to c.
It may be tempting to think that the payoff numbers attached to actions by a payoff function convey intensity of preference—that if, for example, a decision-maker's preferences are represented by a payoff function u for which u(a) = 0, u(b) = 1, and u(c) = 100, then the decision-maker likes c a lot more than b but finds little difference between a and b. But a payoff function contains no such information! The only conclusion we can draw from the fact that u(a) = 0, u(b) = 1, and u(c) = 100 is that the decision-maker prefers c to b to a.

if we assume that a decision-maker who is indifferent between two actions sometimes chooses one action and sometimes the other, not every collection of choices for different sets of available actions is consistent with the theory. Suppose, for example, we observe that a decision-maker chooses a whenever she faces the set {a, b}, but sometimes chooses b when facing the set {a, b, c}.
The fact that she always chooses a when faced with {a, b} means that she prefers a to b (if she were indifferent then she would sometimes choose b). But then when she faces the set {a, b, c} she must choose either a or c, never b. Thus, her choices are inconsistent with the theory.
EXAMPLE: if you choose the same dish from the menu of your favorite lunch spot whenever there are no specials then, regardless of your preferences, it is inconsistent for you to choose some other item from the menu on a day when there is an off-menu special.

Strategic games

A strategic game is a model of interacting decision-makers. In recognition of the interaction, we refer to the decision-makers as **players**. Each player has a set of possible **actions**. The model captures interaction between the players by allowing each player to be affected by the actions of all players, not only her own action. Specifically, each player has preferences about an **action profile** (i.e. a combination of choices of player' choices).

DEFINITION 11.1 (Strategic game with ordinal preferences) A strategic game (with ordinal preferences) consists of
• a set of players
• for each player, a set of actions
• for each player, preferences over the set of action profiles.

EXAMPLE: two players play even/odd game.

- Players: player1 (even), player2 (odd)
- Actions: say an odd number, say an even number)
- Action profile (even-even, even-odd, odd-even, odd-odd).
  - Preferences of player1 (even-even, odd-odd, even-odd, odd-even)
  - Preferences of player2 (even-odd, odd-even even-even, odd-odd)

As in the model of rational choice by a single decision-maker, it is convenient to specify the players' preferences by giving payoff functions that represent them. **These payoffs have only ordinal significance**.

- Time is absent from the model.
- The players choose their actions "simultaneously" in the sense that no player is informed, when she chooses her action, of the action chosen by any other player.

Prisoner's Dilemma

Two suspects in a major crime are held in separate cells. There is enough evidence to convict each of them of a minor offense, but not enough evidence to convict either of them of the major crime unless one of them acts as an informer against the other (finks). If they both stay quiet, each will be convicted of the minor offense and spend one year in prison. If one and only one of them finks, she will be freed and used as a witness against the other, who will spend four years in prison. If they both fink, each will spend three years in prison.

Players: The two suspects.
Actions: {Quiet, Fink}.
Preferences:

- Suspect 1's ordering of the action profiles, from best to worst, is (Fink, Quiet) (she finks and suspect 2 remains quiet, so she is freed), (Quiet, Quiet) (she gets one year in prison), (Fink, Fink) (she gets three years in prison), (Quiet, Fink) (she gets four years in prison).
- Suspect 2's ordering is (Quiet, Fink),(Quiet, Quiet), (Fink, Fink), (Fink, Quiet).

Using payoff function we can do this:

- Player1: u1(Fink, Quiet) = 3, u1(Quiet, Quiet) = 2, u1(Fink, Fink) = 1, u1(Quiet, Fink) = 0.
- Player2: u2(Quiet, Fink) = 3, u2(Quiet, Quiet) = 2, u2(Fink, Fink) = 1, u2(Fink, Quiet) = 0.

		Suspect 2	
		Quiet	Fink
Suspect 1	Quiet	2,2	0,3
	Fink	3,0	1,1

The Prisoner's Dilemma models a situation in which there are gains from cooperation (each player prefers that both players choose Quiet than they both choose Fink) but each player has an incentive to "free ride" (choose Fink) whatever the other player does.

Pollution game

This game is the extension of Prisoner's Dilemma to the case of many players. Assume that there are n countries and that each country faces the choice of either passing legislation to control pollution or not. Assume that pollution control has a cost of 3 for the country, but each country that pollutes adds 1 to the cost of all countries (in terms of added basic solution concepts and computational issues health costs, etc.). The cost of controlling pollution (which is 3) is considerably larger than the cost of 1 a country pays for being socially irresponsible.

Suppose that k countries choose not to control pollution. Clearly, the cost incurred by each of these countries is k. On the other hand, the cost incurred by the remaining n − k countries is k + 3 each, since they have to pay the added cost for their own pollution control. The only stable solution is the one in which no country controls pollution, having a cost of n for each country. In contrast, if they all had controlled pollution, the cost would have been only 3 for each country.

Bach or Stravinsky?

In the Prisoner's Dilemma the main issue is whether or not the players will cooperate (choose Quiet). In the following game the players agree that it is better to cooperate than not to cooperate, but disagree about the best outcome.

EXAMPLE 16.2 (Bach or Stravinsky?) Two people wish to go out together. Two concerts are available: one of music by Bach, and one of music by Stravinsky. One person prefers Bach and the other prefers Stravinsky. If they go to different concerts, each of them is equally unhappy independently of the composer.

Players: The two people.
Actions: {Back, Stravinsky}
Preferences: (Stravinsky, Stravinsky), (Back, Stravinsky), (Stravinsky, Back), (Back, Back)

- Player 1: u_1(Back, Back) = 2, u_1(Stravinsky, Stravinsky) = 1, u_1(Back, Stravinsky) = u_1(Stravinsky, Back) = 0
- Player 2: u_1(Stravinsky, Stravinsky) = 2, u_1(Back, Back) = 1, u_1(Back, Stravinsky) = u_1(Stravinsky, Back) = 0

		Player 2	
		Back	Stravinsky
Player 1	Back	2,1	0,0
	Stravinsky	0,0	1,2

Matching Pennies
Aspects of both conflict and cooperation are present in both the Prisoner's Dilemma and BoS. The next game is purely conflictual (such a game is called "strictly competitive").

Two people choose, simultaneously, whether to show the Head or the Tail of a coin. If they show the same side, person 2 pays person 1 a dollar; if they show different sides, person 1 pays person 2 a dollar. Each person cares only about the amount of money she receives.

		Player 2	
		Head	Tail
Player 1	Head	1, -1	-1, 1
	Tail	-1, 1	1, -1

Stag Hunt
Each of a group of hunters has two options: remain attentive to the pursuit of a stag, or catch a hare. If all hunters pursue the stag, they catch it and share it equally; if any hunter devotes her energy to catching a hare, the stag escapes, and the hare belongs to the defecting hunter alone. Each hunter prefers a share of the stag to a hare.

Players: The n hunters.
Actions: {Stag, Hare}
Preferences:

- Player_i All Stag → 2, Player I hare (no matther the others) → 1, player I stag but at least another hare → 0

When n = 2 we can use the table

		Hunter 2	
		Stag	Hare
Hunter 1	Stag	2, 2	0, 1
	Hare	1, 0	1, 1

Nash equilibrium

In a game, the best action for any given player depends, in general, on the other players' actions. So, when choosing an action, a player must have in mind the actions the other players will choose. That is, she must form a belief about the other players' actions. The assumption underlying is that each player's belief is **derived from her past experience playing the game**, and that this experience is sufficiently extensive that she knows how her opponents will behave.
Although we assume that each player has experience playing the game, we assume that she views each play of the game in isolation. She does not become familiar with the behavior of specific opponents and consequently does not condition her action on the opponent she faces; nor does she expect her current action to affect the other players' future behavior.

Assumptions:

- belief derived from experience playing the game
- play of the game in isolation

- not familiar with the behavior of specific opponents
- her action according to the model of rational choice, given her belief about the other players' actions.
- every player's belief about the other players' actions is **correct.**

A **Nash equilibrium** is an action profile $a^* = \{a_1^*, \ldots a_n^*\}$ with the property that no player i can do better by choosing an action different from a_i^*, given that every other player j adheres to a_j^*

Nash equilibrium of strategic game with ordinal preferences: The action profile $a^* = \{a_1^*, \ldots a_n^*\}$ in a strategic game with ordinal preferences is a Nash equilibrium if, for every player i and every action a_i of player i, a^* is at least as good according to player i's preferences as the action profile (a_i , $a^*[-i]$) in which player i chooses $a_i \neq a_i^*$, while every other player j chooses a_j^* .
Equivalently, for every player I $u_i(a^*) \geq u_i(\mathrm{a_i}, \mathrm{a}^*[-\mathrm{i}])$ for every action a_i of player i, where u_i is a payoff function that represents player i's preferences.

- Nash equilibrium embodies a stable "social norm": if everyone else adheres to it, no individual wishes to deviate from it.
- The players know each others' preferences, and considers what each player can deduce about the other players' actions from their rationality and their knowledge of each other's rationality.

Let's see again our examples:

Prisoner's Dilemma

		Suspect 2	
		Quiet	Fink
Suspect 1	Quiet	2,2	0,3
	Fink	3,0	1,1

- (Quiet, Quiet) is not a Nash equilibrium because
  - if player 2 chooses Quiet, player 1's payoff to Fink exceeds her payoff to Quiet
  - if player 1 chooses Quiet, player 2's payoff to Fink exceeds her payoff to Quiet
- (Quiet, Fink) is not a Nash equilibrium because if player 2 finks it is better for player 1 to fink instead of being quite (finking she makes 3 years of prison while being quiet she stays for 4 years)
- (Fink, Quiet) is not a Nash equilibrium as the previous case
- (Fink, Fink) is a Nash equilibrium as if the other player finks the best is finking.

NOTICE: Nash equilibrium does not imply the best solution for all players.

Bach or Stravinsky?

		Player 2	
		Back	Stravinsky
Player 1	Back	2,1	0,0
	Stravinsky	0,0	1,2

- (Back, Stravinsky) and (Stravinsky, Back) are not Nash equilibria since it is better to agree with the other player and go together than split.
- (Back, Back) and (Stravinsky, Stravinsky) are Nash equilibria even if players have different opinion about which is better.

Coordination game: a variant of Bach or Stravinsky?
What if both the two players prefer going to Back instead of Stravinsky?

- As in the original game cooperation gives higher payoff
- Coordination would give best payoff

		Player 2	
		Back	Stravinsky
Player 1	Back	2,2	0,0
	Stravinsky	0,0	1,1

Again, however
- (Back, Stravinsky) and (Stravinsky, Back) are not Nash equilibria since it is better to agree with the other player and go together than split.
- (Back, Back) and (Stravinsky, Stravinsky) are **BOTH** Nash equilibria.

Is Stravinsky, Stravinsky possible? Yes, because in the case Stravinsky, Stravinsky still no player has interest to change its action.

Matching Pennies

		Player 2	
		Head	Tail
Player 1	Head	1, -1	-1, 1
	Tail	-1, 1	1, -1

- (Head, Head) and (Tail, Tail) are not Nash equilibria since in both cases it would be better for player 2 to change her choice
- (Head, Tail) and (Tail, Head) are not Nash equilibria since in both cases it would be better for player 1 to change her choice

This game has no Nash equilibria.

Stag Hunt
The table shows the case n = 2 but we need to consider every possible n

		Hunter 2	
		Stag	Hare
Hunter 1	Stag	2, 2	0, 1
	Hare	1, 0	1, 1

- (Stag, ..., Stag) is a Nash equilibrium because each player prefers this profile to that in which she alone chooses Hare.
- (Hare, …, Hare) is a Nash equilibrium because each player prefers this profile to that in which she alone pursues the stag.
- No other profile is a Nash equilibrium

Strict vs nonstrict Nash equilibria

In all the Nash equilibria of the games a deviation by a player leads to an outcome worse for that player than the equilibrium outcome. The definition of Nash equilibrium, however, requires only that the outcome of a deviation be no better for the deviant than the equilibrium outcome. And, indeed, some games have equilibria in which a player is indifferent between her equilibrium action and some other action, given the other players' actions.

For a general game, an equilibrium is **strict** if each player's equilibrium action is better than all her other actions, given the other players' actions. Precisely, action profile a^* is a strict Nash equilibrium if for every player i: $u_i(a^*) > u_i(\mathrm{a_i}\,, \mathrm{a}^*[-\mathrm{i}])$ for every action a_i of player i, where u_i is a payoff function that represents player i's preferences.

The following game shows the implications of nonstrict equilibria

		player 2		
		L	M	R
player 1	T	1, 1	1, 0	0, 1
	B	1, 0	0, 1	1, 0

This game has a unique Nash equilibrium, namely (T, L). When player 2 chooses L, as she does in this equilibrium, player 1 is equally happy choosing T or B; if she deviates to B then she is no worse off than she is in the equilibrium. We say that the Nash equilibrium (T, L) is not a strict equilibrium.

Best response functions

When the number of players and/or the number of actions becomes large inspecting all the possible action profiles to find the Nash equilibria can be not feasible.
Consider a player, say player i. For any given actions of the players other than i, player i's actions yield her various payoffs. We are interested in the **best actions**: namely the subset B of actions that yield her the highest payoff.
Formally B_i can be defined as the following function

$$B_i(a_{-i}) = \{a_i \in A_i : u_i(a_i, a[-i]) \geq u_i(a_i', a[-i]) \quad \forall a_i' \in A_i\}$$

Examples:

In the prisoner dilemma the best action for suspect1 when suspect2 finks is fink and also when suspect2 choses quiet is fink.

PROPOSITION: The **action profile** $a^* = \{a_1^*, \dots a_n^*\}$ is a Nash equilibrium of a strategic game with ordinal preferences if and only if every player's action is a best response to the other players' actions. Formally: $a_i^* \in B_i(a_{-i})$ for every player i.

The above proposition provides a simple algorithm to find a Nash equilibrium that reduces the number of player preferences to examinate.

Algorithm for finding Nash equilibria:

1. find the best response function of each player
2. find the action profiles $a_i^* \in B_i\big((a^*[-i])\big)$ for every player i.

Variant of Stag Hunt

Consider the following variant of n-hunter Stag Hunt:

- only m hunters, with 2 ⩽ m < n, need to pursue the stag in order to catch it.
- there is still a single stag
- stag is shared only by the hunters that catch it
- that each hunter prefers the fraction 1/n of the stag to a hare.

Let's see as for example the case with n=4 and m=2 but the same considerations apply for every value of n.

Hunter 1	S	S	S	S	S	S	S	S	H	H	H	H	H	H	H	H
Hunter 2	S	S	S	S	H	H	H	H	S	S	S	S	H	H	H	H
Hunter 3	S	S	H	H	S	S	H	H	S	S	H	H	S	S	H	H
Hunter 4	S	H	S	H	S	H	S	H	S	H	S	H	S	H	S	H

We highlight in red the best response for each player. In yellow the two Nash equilibria. Observe that using the algorithm for finding Nash equilibrium using best response we don't need to evaluate all the players' preferences but only the subset in common among the already observed players (Underlined elements)

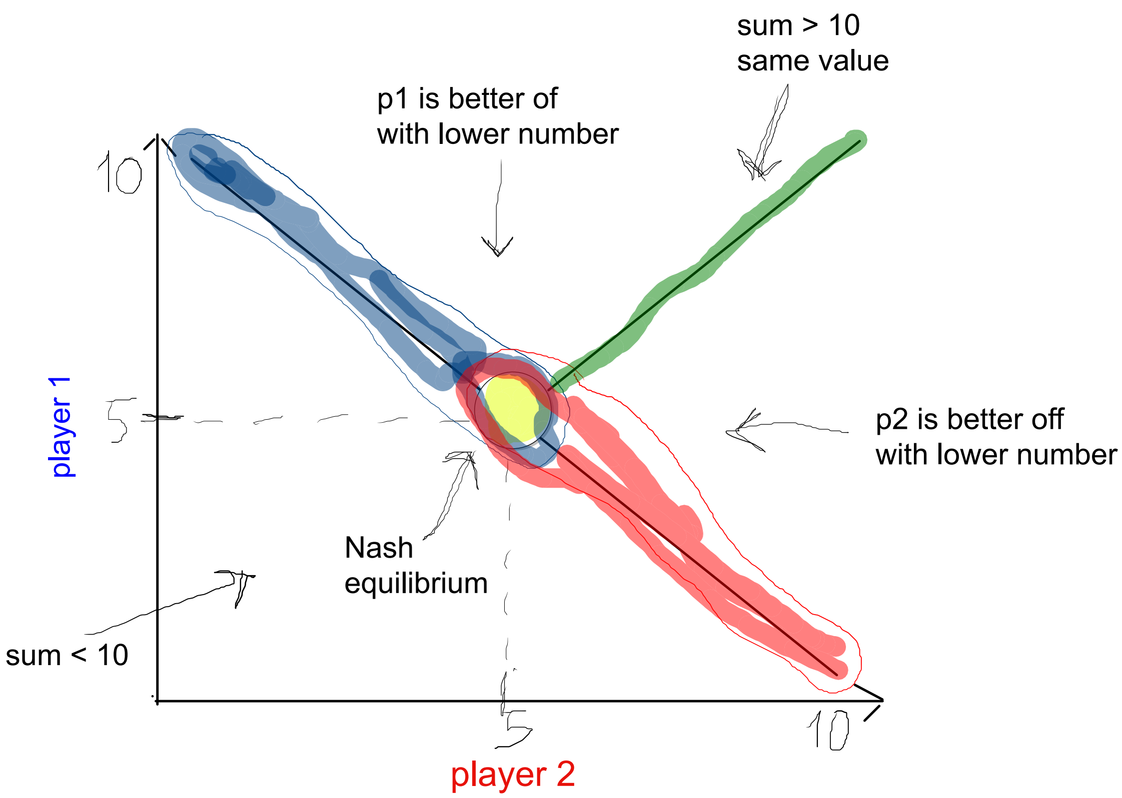
Homework: consider a further variation of the stag hare problem where the hunter prefers 1/k of the stag to the hare.

Dividing money

Two people use the following algorithm to divide between themselves \$10. Each person names an integer number in the range [0, 10].

- If the sum of the numbers is at most 10 then each person receives the amount of money she names and the remainder is destroyed.
- Otherwise (the sum of the numbers exceeds 10)
  - if the amounts named are different then the person who names the smaller amount receives that amount and the other person receives the remaining money.
  - If the amounts named are the same then each person receives \$5.

In this case the there is a too large number of action profiles. We can use a graphical schema to remove profiles not corresponding to best actions.



Best Response and Learning in Games

It would be desirable that natural game playing strategies quickly lead players to either find the equilibrium or at least converge to an equilibrium in the limit. Maybe the most natural "game playing" strategy is that of following "best response".

Consider an action profile a, and a player i. Using the action profile, a player i gets the value or utility $u_i(a_i, a_{-i})$. Changing the strategy a_i to some other strategy a_i', the player can change his utility to $u_i(a_i', a_{-i})$, assuming that all other players stick to their strategies in a_{-i} . We say that a change from strategy a_i, to a_i' is an **improving response** for player i if $u_i(a_i', a_{-i}) > u_i(a_i, a_{-i})$ and a best response if a_i' maximizes the player's utility. Playing a game by repeatedly allowing some player to make an improving or a best response move is perhaps the most natural game play.

In some games, such as the Prisoner's Dilemma or the Coordination Game, this dynamic leads the players to a Nash equilibrium in a few steps. In other games the players will not reach the equilibrium in a finite number of steps, but the action profile will converge to the equilibrium.

In other games, the play may cycle, and not converge. A simple example is matching pennies, where the players will cycle through the 4 possible action profiles if they alternate making best response moves. While this game play does not find a pure equilibrium (as none exists) in some sense we can still say that best response converges to the equilibrium: the average payoff for the two players converges to 0, which is the average payoff; and even the frequencies at which the 4 possible strategy vectors are played converge to the probabilities in equilibrium (1/4 each).