Randomized algorithm for MIN-CUT in graphs

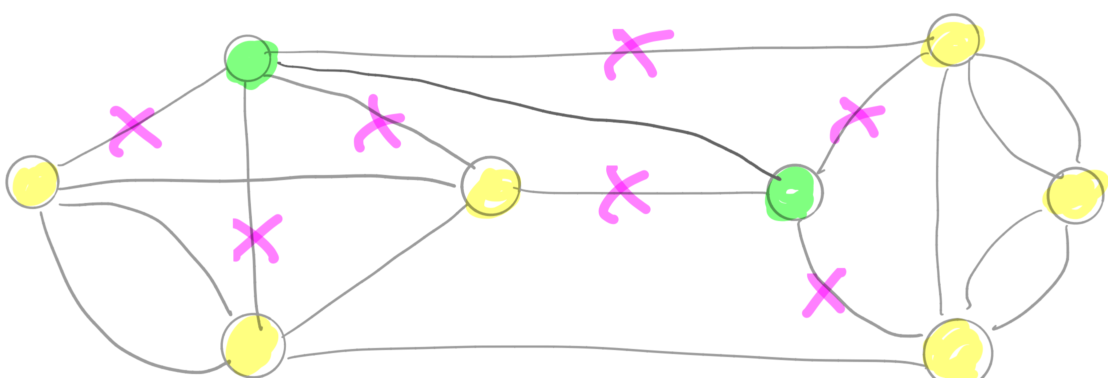
Graph $G = (V, E)$ (multiple edges)

$n = |V| = \#\text{nodes}$

$m = |E| = \#\text{edges}$ (multiplicities of the same edges)

- $C = (V_1, V_2)$ is a $\boxed{\text{CUT}}$ if $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$

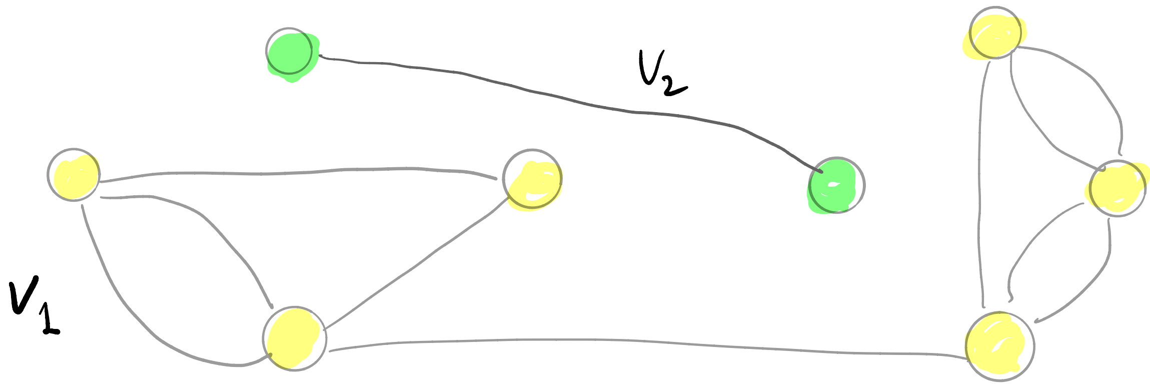
$\boxed{\text{CUTSET}}$ $E(V_1, V_2) = \{ uv \in E : u \in V_1, v \in V_2 \text{ or } u \in V_2, v \in V_1 \} \subseteq E$



$|E(V_1, V_2)| = 7$

$n = 8, \; m = 18$

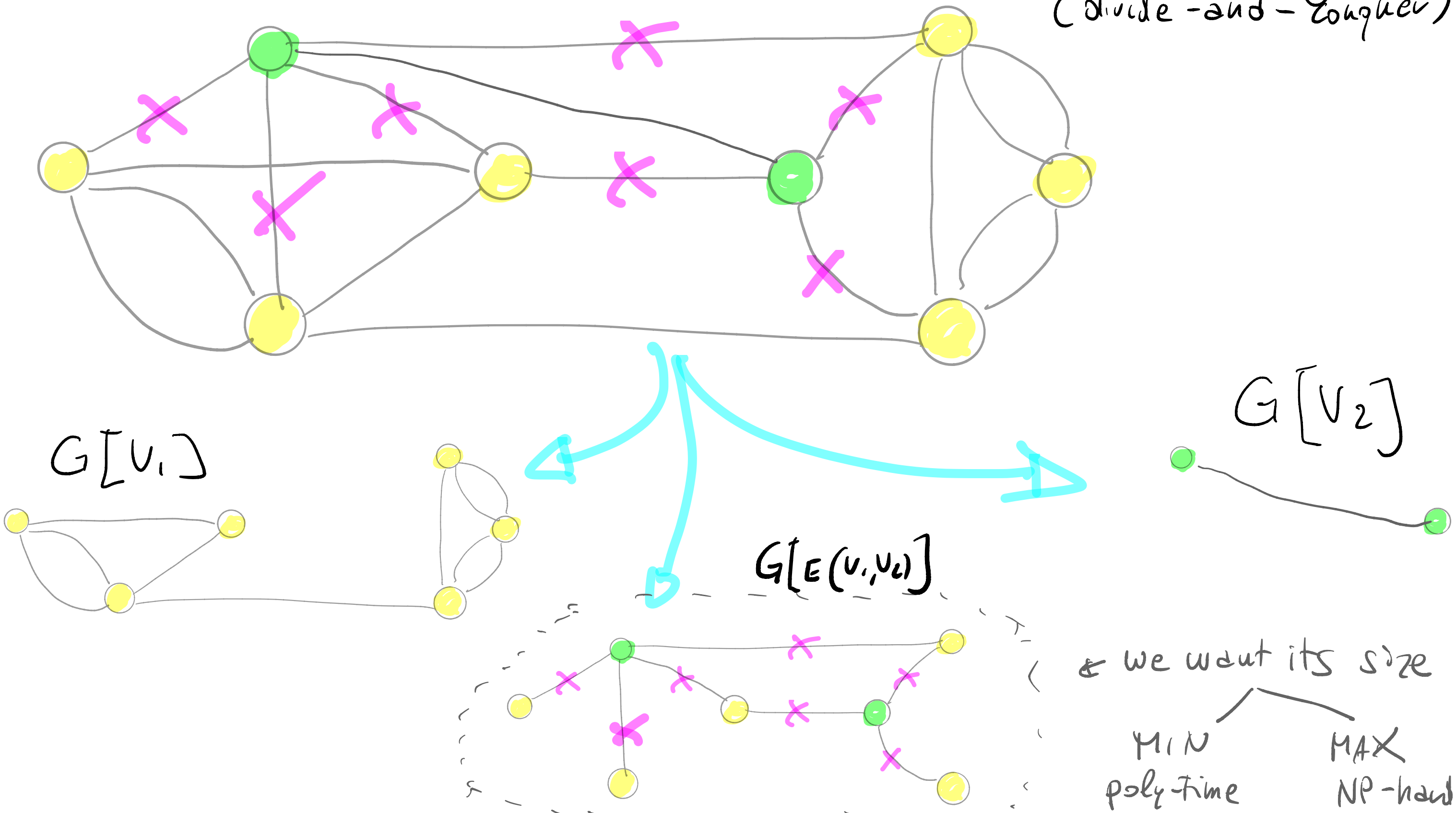
2-coloring of the nodes $\equiv$ CUT



$G[V_i] = \{uv \in E : u,v \in V_i\}$ node induced graph

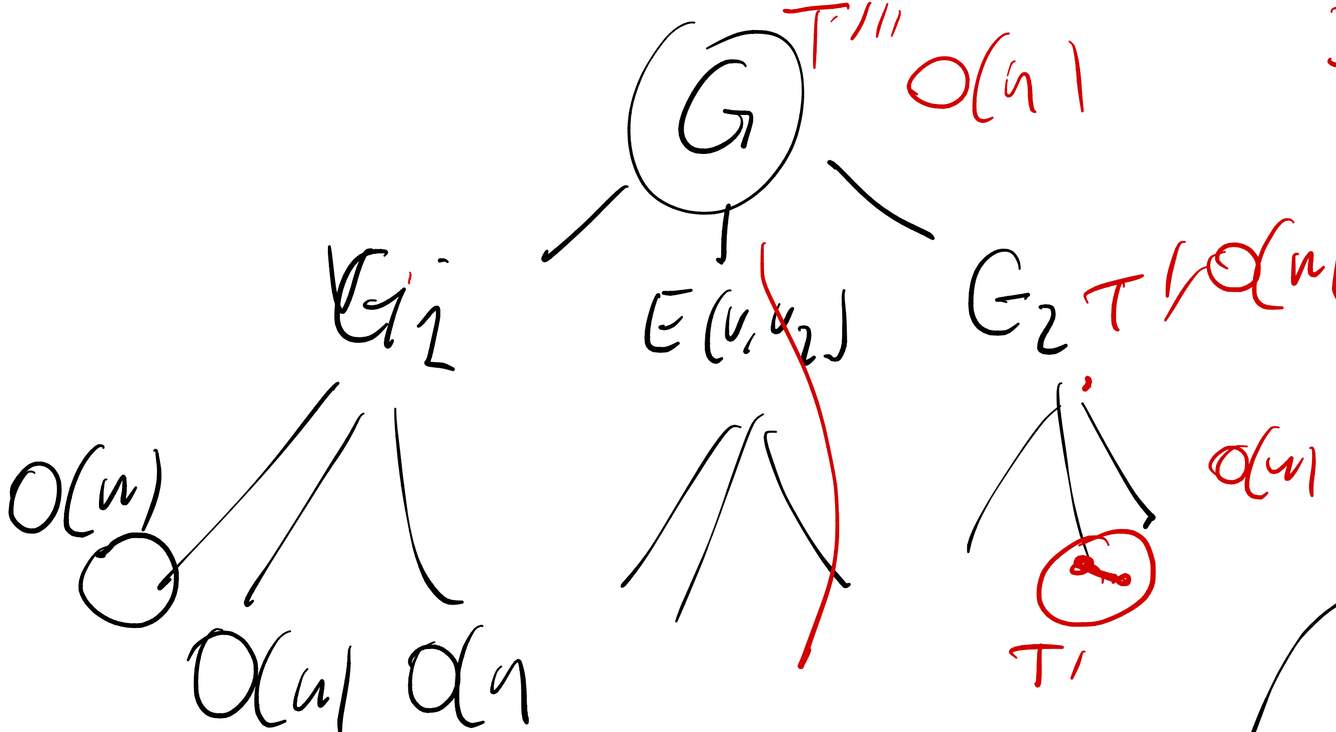
$G[E(V_1,V_2)] = (V_{12}, E(V_1,V_2))$ $V_{12} = \{u \in V : uv \in E(V_1,V_2)\}$

A cut divides G into 3 parts: applications e.g. recursion on graphs (divide-and-conquer)

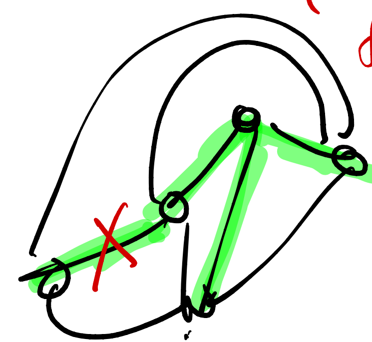


subgraph induced by the cutset $E(V_1,V_2) = E(C)$

Dynamic Spanning tree



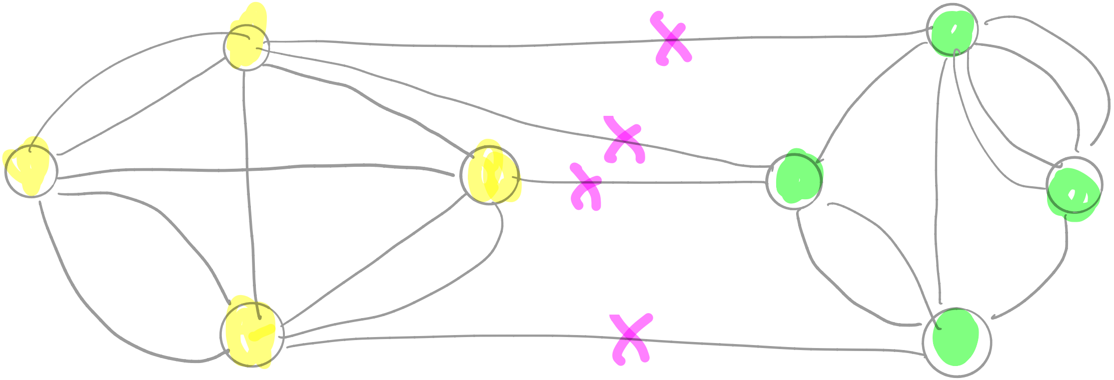
O(m) time

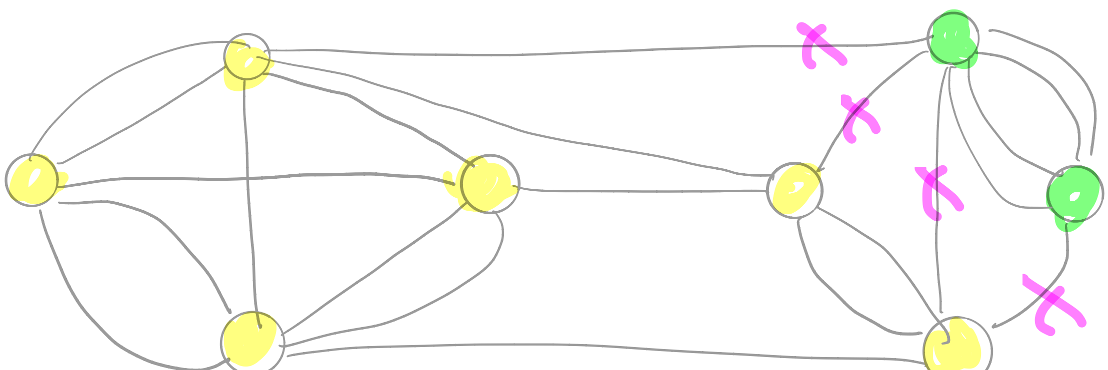


$\underline{h}$ = height

$O(n \cdot h)$ time

$C = (V_1, V_2)$ is a $\boxed{\text{MIN-CUT}}$ if $|E(C)| \leq |E(C')|$ for any cut C'





$\boxed{1}$ MIN-CUT is not unique

$\boxed{2}$ $|E(C)| \leq deg(v)$ $\forall v \in V$

(as otherwise we could choose a smaller cutset)

EDGE CONTRACTION

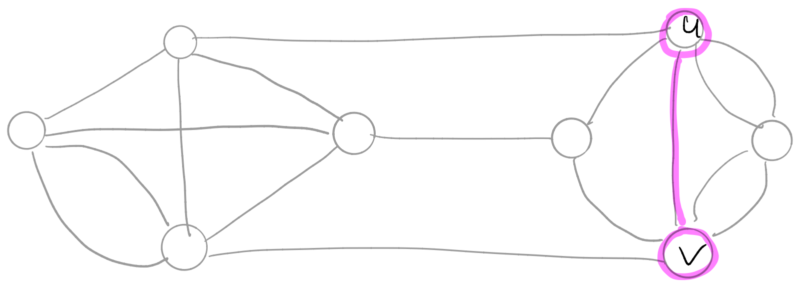
$uv \in E$ edge

$G/uv = (V', E')$ where $V' = V - \{u,v\} \cup \{uv\}$

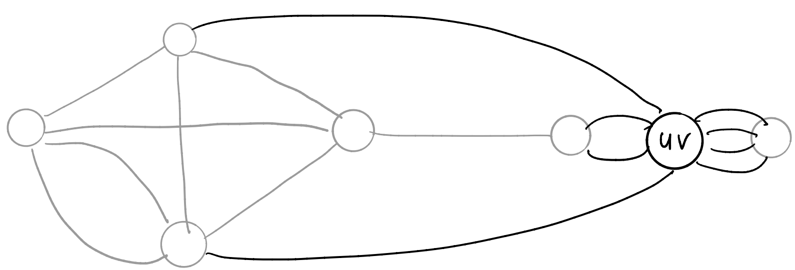
$E' = E - \{uv\} \cup \{uv\,z : uz \in E \text{ or } vz \in E\}$

they can be multiple edges between u and v

G



$G/\{u,v\}$



INTUITION: we are conceptually assigning the same color to u and v in the final cut, so we can drop all the multiedges uv

Each edge contraction decreases # nodes by 1

RELATION BETWEEN EDGE CONTRACTIONS AND CUTS

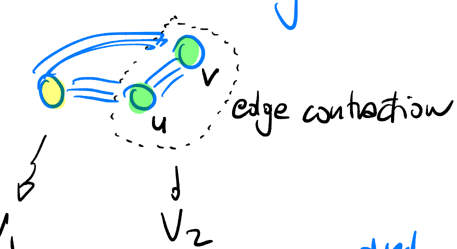
G connected

a) Any sequence of edge contractions leaving $\boxed{\text{just 2 nodes}}$ gives a cut V_1, V_2

By induction on the number n of nodes

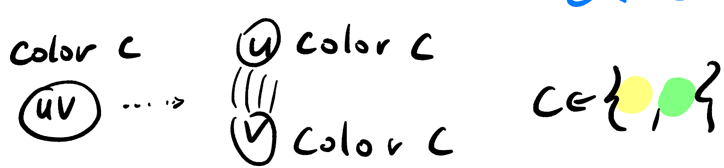
- $n = 2$: V_1 = one node V_2 = other node

- $n > 2$: take the last edge contraction, done on 3 nodes

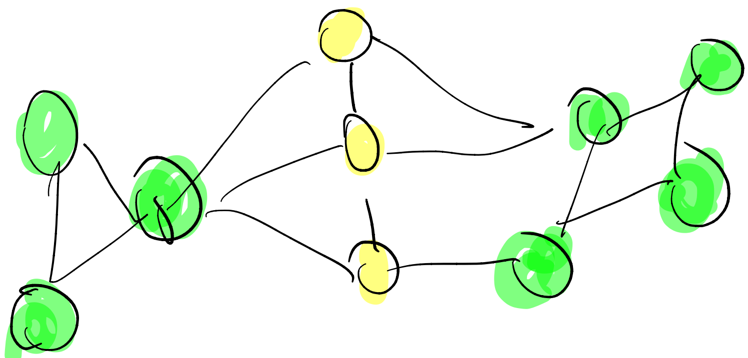


and so on for 4, 5, 6, .. n

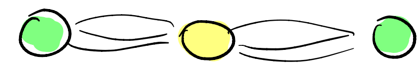
General rule :



(b) Given a cut $C=(V_1, V_2)$, it does <u>not</u> necessarily exist a sequence of edge contractions that lead to just 2 nodes



sequence of edge contractions



here we cannot apply any edge contraction that keeps this 2-coloring

(clearly, by (a), we can obtain another 2-coloring)

ALGORITHM GUESSMINCUT(G):

number of current nodes

while $|V(G)| > 2$ do:

1) choose uniformly and randomly an edge $uv \in E$

2) $G \leftarrow G/uv$

return $|E(G)|$

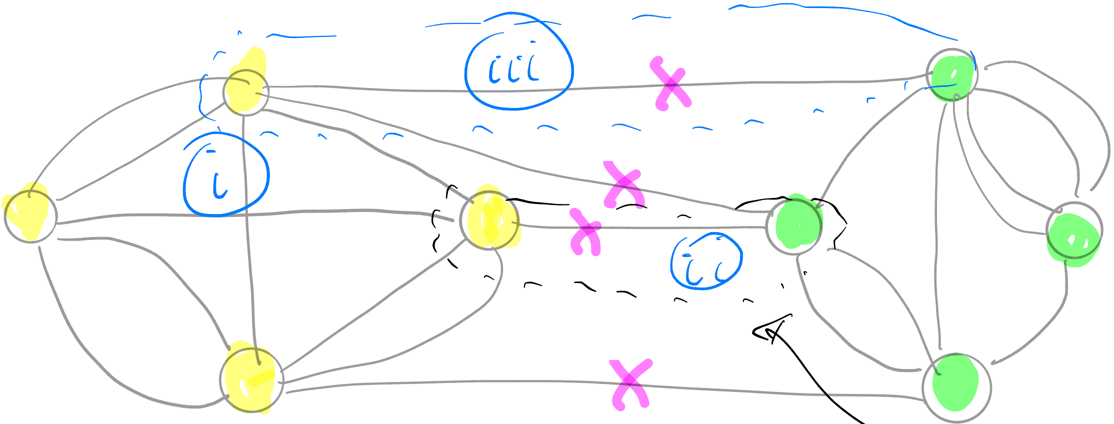
number of multiple edges when only 2 nodes remain

(how?)

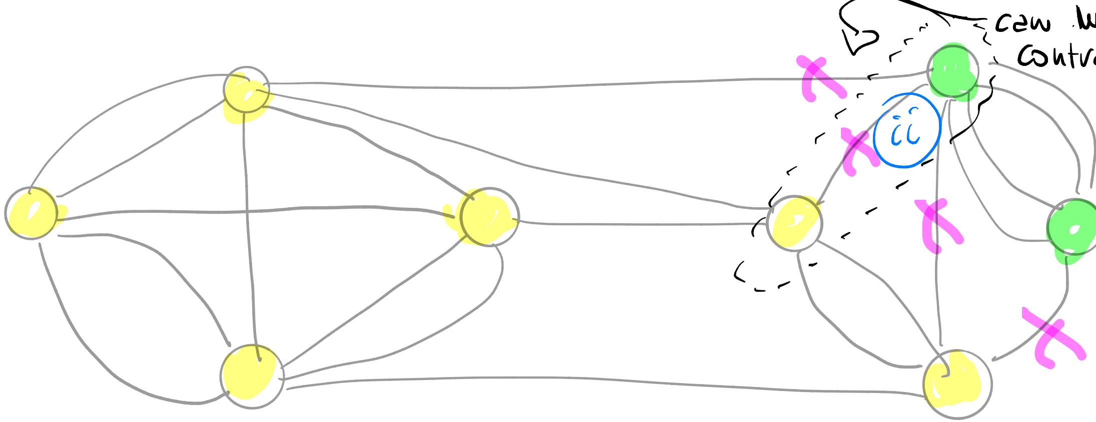
① choose u with $pr = \frac{deg(u)}{2m}$

② choose $v \in N(u)$ with $pr = \frac{1}{deg(u)}$

Q: when does the above algorithm fail?



can we safely contract this edge?



i

· if uv is not in a MIN-CUT SET, it is ok

ii

if uv is in a MIN-CUT SET but not in another MIN-CUT SET, it is ok

iii

if uv is in every MIN-CUT SET it is BAD

- Edge uv is $\boxed{\text{BAD}}$ if $uv \in E(C)$ for __every__ MIN-CUT C

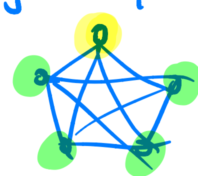
In other words, uv belongs to $\boxed{\text{all}}$ MIN CUT SETS!

Let $k = |E(C)|$ be the size of the MIN CUT SETS

__Prop__ There are at most k BAD edges in G

CRUCIAL PROPERTY: there could graphs in which each edge belongs to a MIN CUT SET but still, only $\leq k$ edges are BAD

↳ e.g. clique



as $\leq k$ BAD edges

$$\Pr[uv \text{ is BAD}] \leq \frac{k}{m} \leq \frac{2}{n\cancel{k}} \cdot \cancel{k} = \frac{2}{n}$$

$$\sum_{u \in V} \deg(u) = 2m \quad + \quad \forall u \in V: \deg(u) \geq k$$

handshaking lemma

$\boxed{2}$

$$\Downarrow$$

$$2m \geq nk \iff m \geq \frac{nk}{2}$$

<u>Note</u> This argument holds for multi edges as $\deg(u)$ takes into account their multiplicities.

$P(n)$ = probability that GUESSMINCUT correctly finds a MIN CUT in a graph with n nodes

$$P(n) \overset{2}{\geq} \begin{cases} 1 & n=2 \\ \underbrace{\left(1-\frac{2}{n}\right)}_{\text{the chosen edge } uv \text{ is NOT BAD}} \cdot P(n-1) & n>2 \end{cases}$$

conditional probability given the choice uv

$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B \mid A]$

Expanding it recursively...

$$P(n) \geq \left(\frac{n-2}{n}\right)\cdot\left(\frac{n-3}{n-1}\right)\cdot\left(\frac{n-4}{n-2}\right)\cdots\left(\frac{1}{3}\right)\cdot \underset{n=2}{1} = \frac{(n-2)!}{n!/2} = \frac{2}{n(n-1)}$$

Do we like this? Not yet, we can boost it?