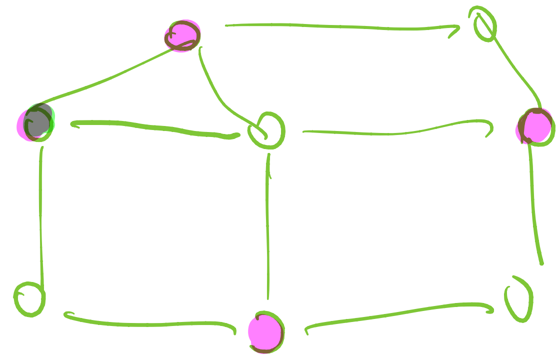
$G = (V, E)$

VC = vertex cover



$S \subseteq V \to$ vertex cover

$\forall uv \in E : u \in S$ or $v \in S$

$VC_k = \exists S \subseteq V : |S| \leq k$, $S = VC$?

$\hookrightarrow O(\underbrace{1.46^{k^2}}_{f(k)} + poly(n))$

$VC \in NPC$

BASELINE takes

$O(\binom{n}{k} poly(n))$

$\sim O(n^{k+const})$

$VC^* = k_{min}$ s.t. $\nexists S \subseteq V$, $|S| < k_{min}$, $S \neq VC$

Binary search on k is too expensive as the largest k is at the exponent

Better to go "trivially": test VC_k for $k = 1, 2, \ldots$

we surely stop at $k = k_{min} \Rightarrow O(1.46^{k_{min}^2} + \underbrace{\underset{\leq n}{k_{min}} \cdot poly(n)}_{poly(n)})$ time

Glimpse on approximation

- we want poly(n) anyway

2-approx

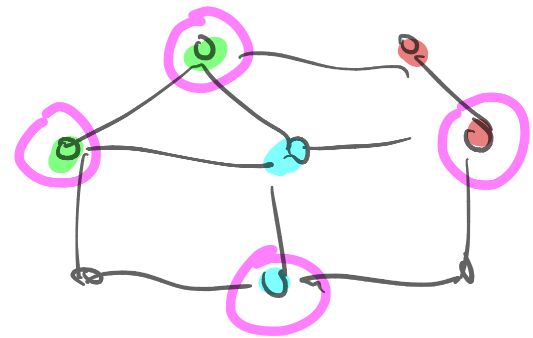
- we relax optimality:

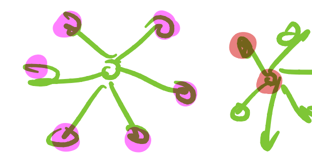
  instead of looking for VC $S^* \subseteq V$ s.t. $|S^*| = k_{min}$

  we are happy if we find a VC $\tilde{S} \subseteq V$ s.t. $|\tilde{S}| \leq 2 \cdot k_{min}$

For VC this is doable:

```
S̃ = {};
for each uv ∈ E do
    if u ∉ S̃ and v ∉ S̃ then S̃ = S̃ ∪ {u,v}
return S̃
```





1) $\tilde{S}$ is a VC: by design

2) $|\tilde{S}| \leq 2 \cdot k_{min} = 2 \cdot |S^*|$. Consider S^*: $\forall uv \in E$: $u \in S^*$ or $v \in S^*$

Consider $\tilde{S}$: the chosen edges are a "matching": for every two edges, no common endpoint; moreover each such edge has at least a node in S^*.

$\Big\}$ # edges in matching $= \frac{|\tilde{S}|}{2} \leq |S^*| \Rightarrow |\tilde{S}| \leq 2|S^*|$

RANDOMIZED FTP algorithms

↳ fixed-parameter algorithms
$O(f(k)\, n^c)$ time

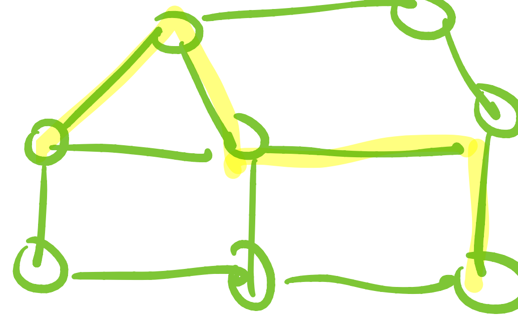
2 techniques
randomized
coloring

COLOR CODING

RANDOMIZED SEPARATION
(2-coloring)

COLOR CODING

k-path = path of k nodes, which are all distinct



$k=5$

≠ walk (edges, as nodes/edges can be traversed more than once)

k-path is NPC in general as for $k=|V|$ it is the Hamiltonian path (HAM)

Idea: random k-coloring

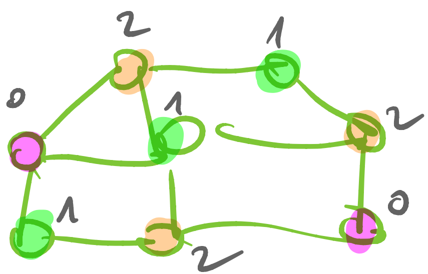
Graph $G=(V,E)$, k-coloring $X: V \to [k]$

random k-coloring: $Pr[X(v)=c] = \frac{1}{k}$ $\forall v \in V$ $\forall c \in [k]$

Suppose X random is chosen uniformly

$V' \subseteq V$ subset of nodes: COLORFUL iff

$\forall u,v \in V', u \neq v : X(u) \neq X(v)$



$k=3$

prop 1 : A colorful walk of k nodes is a k-path
(vice versa is not necessarily true)

prop 2 : A walk of k nodes is a k-path iff
$\exists$ coloring χ s.t. the walk is colorful.

one coloring χ is not enough
for all k-paths

$V' \subseteq V$ s.th. $|V'| = k$

$$\boxed{\Pr(V' \text{ is COLORFULL})} = \frac{\#\text{ colorings } \chi \text{ that make } V' \text{ colorful}}{\#\text{ all possible colorings of } G}$$

$$\underbrace{\overset{k}{\circ}\ \overset{\times(k-1)}{\circ}\ \circ\ \circ\ \overset{2}{\circ}\ \overset{1}{\circ}}_{V'} = k!$$

$$= \frac{\overbrace{k^{n-k}}^{V \setminus V'} \cdot \overbrace{k!}^{V'}}{k^n} = \frac{k!}{k^k} \boxed{\geq e^{-k}}$$

$k! \geq \left(\frac{k}{e}\right)^k$

e = Euler constant

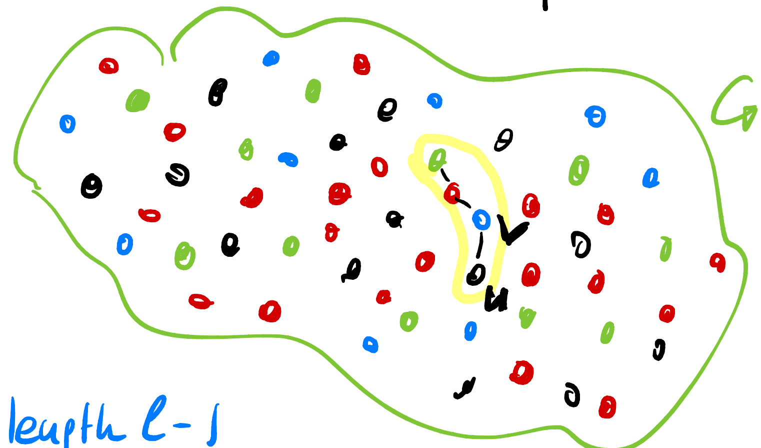
Q: Given random coloring $\mathcal{X}$, how do we find $\underbrace{\text{colourful paths}}_{\text{k-paths}}$?

Dynamic programming:

1) any subpath of a colourful path is clearly colourful

2) A colourful path of length ℓ ends in a node u iff $\exists v \in N(u)$ s.t. a colourful path of length $\ell-1$ ends in v and none of the nodes in the shorter path have color $\mathcal{X}(u)$

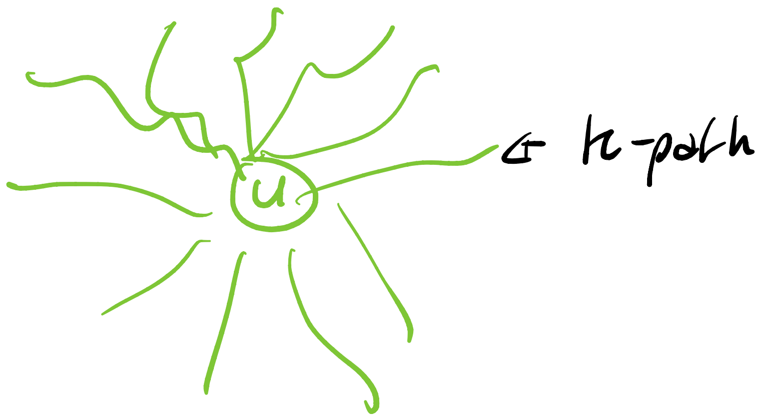
3) The set of $\ell-1$ colours matters, not their order or paths



$k=4$

WHY COLOR CODING IS POWERFUL?

take a node u and consider all k-paths ending in u



potentially, we may have up to $\Theta(n^k)$ such paths, and a significant fraction of them are colorful for χ

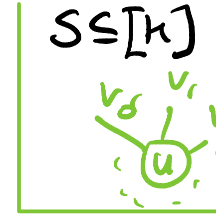
for each colorful k-path, the set of colors is the same, namely $[k]$

$\Rightarrow$ just <u>one</u> set of colors represent all of them!

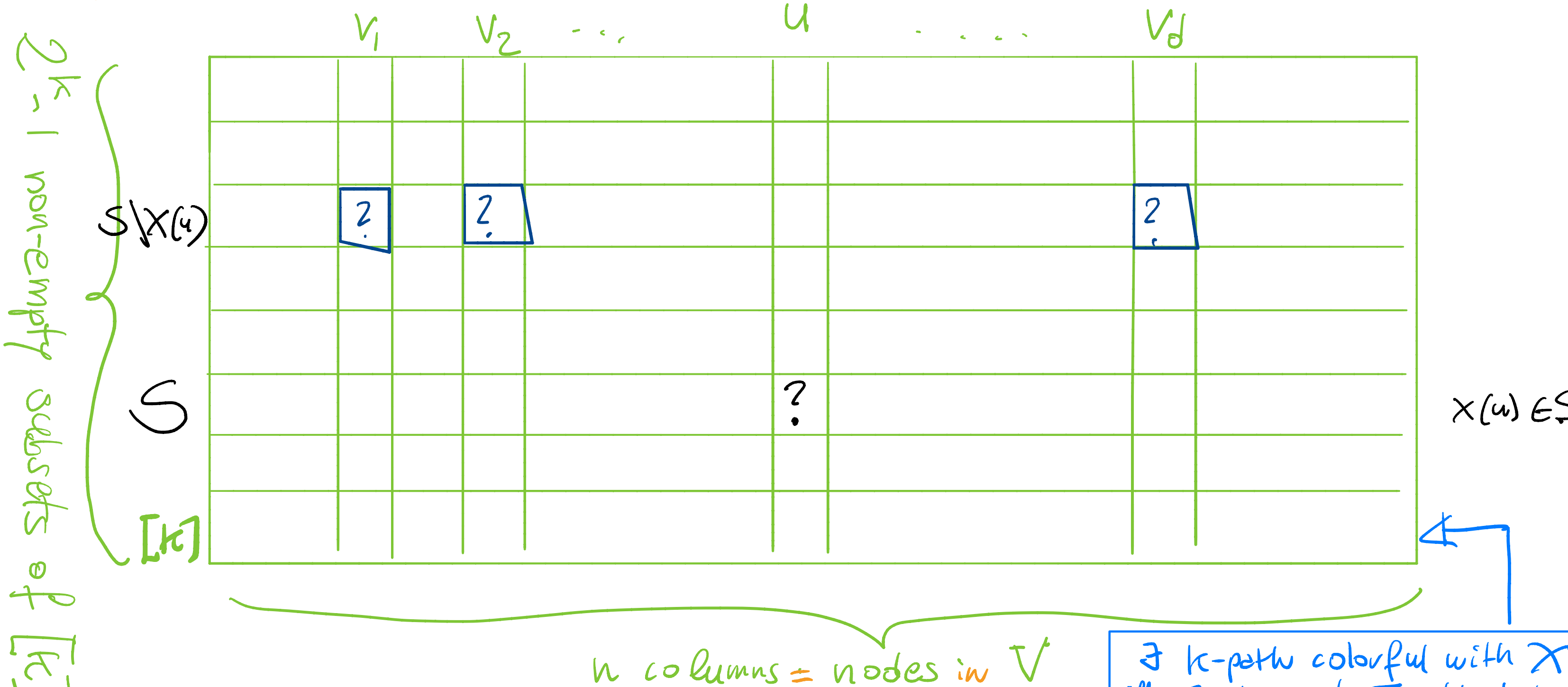
In general, for ℓ-paths, where $1 \leq \ell \leq k$, we have $\binom{k}{\ell}$ set of colors to remember, instead of $O(n^\ell)$ ℓ-paths

$$
\mathrm{PATH}[S][u] = \begin{cases} T & \text{if } S = \{X(u)\} \\ \operatorname{OR}_{v \in N(u)} \mathrm{PATH}[S \setminus X(u)][v] & \text{if } X(u) \in S,\ |S| \geq 2 \\ F & \text{otherwise} \end{cases}
$$

it say if there is a colorful path ending in u and using all the colors in S



$N(u) = \{v_1, v_2, \dots, v_d\}$



∃ k-path colorful with X iff. ∃ a true value T in the last ro

PATH has 2^k-1 rows and $n=|V|$ columns

each entry PATH[S][u] takes $O(|N(u)|)$ time to fill

Total cost is $\sum_u |N(u)| = O(m)$ per row, $m=|E|$

$\hookrightarrow O(2^k m) = O(2^k n^2)$ time

Final touch: we repeat the above for e^k times, each time randomly choosing a coloring χ

We stop when we find "T" in the last row of PATH

Union bound $\Rightarrow$ constant probability of success (use CB to boost as usual)

Running time is $O(e^k 2^k n^2) = O((2e)^k n^2)$ FPT? $\Rightarrow$ MONTECARLO

RANDOMIZED SEPARATION

IDEA: random 2-coloring χ $\Rightarrow$ you split the graph into groups with the same color

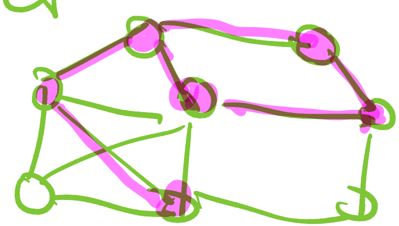
SUBGRAPH ISOMORPHISM (k-paths are a special case)

IN: $G=(V,E)$, $H=(V_H,E_H)$

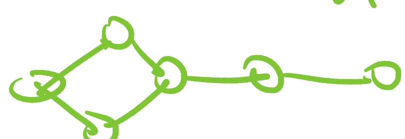
OUT: $\hat{H}=(\hat{V},\hat{E})$ subgraph of G, $\hat{V}\subseteq V$, $\hat{E}\subseteq E$, isomorphic to H

↳ $\hat{H}$ isomorphic to H iff $\exists$ 1-1 mapping $\varphi: V_H \to \hat{V}$ s.t. $(u,v)\in E_H$ iff $(\varphi(u),\varphi(v))\in\hat{E}$

G



H



↓ $O(k!k^2)$

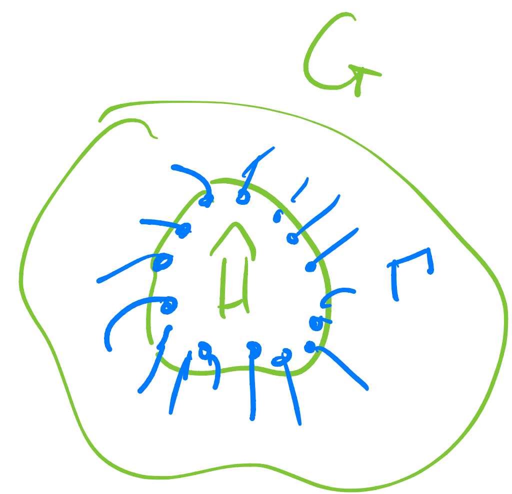
checking is exponential in $|\hat{V}|=|V_H|=k$

FPT algorithm: which parameter?

- $k = |V_H| \Rightarrow$ no FPT unless $P = NP$
- $k = |V_H|$, $\Delta =$ max degree$(G) \Rightarrow$ FPT alg. $O(2^{\Delta k} k!\, n^{O(1)})$ time

BOUNDED DEGREE

Suppose $\hat{H}$ isomorphic to H exists in G



$\Gamma = \{(u,v) \in E \setminus \hat{E} : u \in \hat{V} \text{ or } v \in \hat{V}\}$

If we cut Γ, we get $\hat{H}$ as a connected component (cc)

Prop $|\Gamma| + |\hat{E}| \le |\hat{V}| \cdot \Delta = k\Delta$

disjoint by def.

FPT

1) check if Δ = max degree

2) Repeat $\boxed{2^{\Delta k}}$ times.

2.a χ = random 2-coloring $\quad \chi : V \to [2]$

2.b G_0 = part of G colored 0 (induced subgraphs)

2.c. G_1 = " " " 1

2.d find the CCs in G_0 and G_1, and check if any CC is isomorphic to $\hat{H}$

$$\Pr(\chi \text{ gives } \hat{H}) = \frac{2}{2^{|\hat{V}|+|\hat{E}|}} \geq \boxed{\frac{1}{2^{\Delta k}}}$$