HARDNESS P, NP, NP-complete

P = problems that can be solved in poly(n) time (e.g. sorting, min-cut)
$\cap$
NP = problems that admit a poly(n)-time certificate (e.g. SAT, VC)
↳ non-deterministic

$NP \subseteq P$? BIG QUESTION

NP-Complete (NPC)

$\pi \in NP$ s.t. $\forall \pi' \in NP$ "polynomially reduces to" π (VC, SAT, etc.)

Th Cook-Levin

$SAT \in NPC$

EXACT SOLUTIONS → parameterized algorithms (FPT)
↳ fixed-parameter tractable

APPROXIMATED SOLUTIONS → optimization problems
↳ min cost
↳ max benefit

We saw VC = vertex cover (minimize # nodes)

▶ r-approximation = algorithm which provides a solution $\tilde{S}$ s.t.

min : $\dfrac{cost(\tilde{S})}{OPT} \leq r$

max : $\dfrac{OPT}{cost(\tilde{S})} \leq r$

$r > 1$

$cost(\tilde{S}) \leq 2 \cdot OPT$

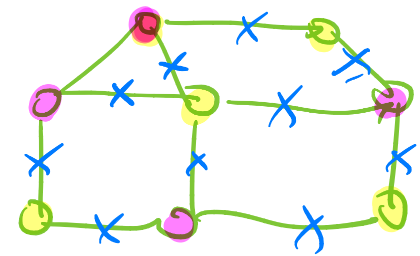
VC is a 2-approx

MAX-CUT (NP) is a 2-appr

$Cost(\tilde{S}) \geq \frac{1}{2} \cdot OPT$

MAX-CUT

We saw MIN-CUT



cut : $V = V_1 \cup V_2$ (2-coloring)

$V_1 \cap V_2 = \phi$

cutset $E(V_1, V_2) = \{uv \in E : u \in V_1 \text{ and } v \in V_2\}$

MIN-CUT	vs	MAX-CUT
min $E(V_1,V_2)$		max $E(V_1,V_2)$
P		NPC

3 ways to get a 2-approx for MAX-CUT

$$\text{cost}(\tilde{S}) = |E(V_1,V_2)| \geq \underbrace{\frac{1}{2}}_{} \text{OPT}$$

0.5 (the higher, the better)

the best know is 0.878

Goemans, Williamson

1. local search
2. greedy
3. randomization

① LOCAL SEARCH

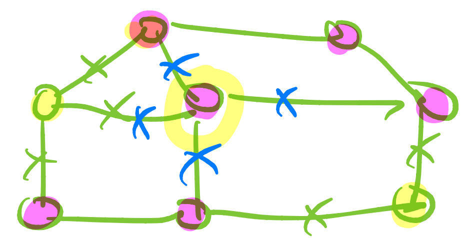
$S = V_1 \quad V_2 = V - S$

$S = \{\}$

<u>while</u> $\exists u \in V$ changing its set $\begin{pmatrix} u \in S \to u \in V-S \\ u \in V-S \to u \in S \end{pmatrix}$ s.t. $\boxed{E(S, V-S) \text{ STRICTLY increases}}$

<u>do</u> <u>if</u> $u \in S$ then $S \leftarrow S \setminus \{u\}$
else $S \leftarrow S \cup \{u\}$

return $V_1 = S$, $V_2 = V - S$



a) the algorithm terminates as $|E(S,V-S)|$ strictly increases
and $|E(S,V-S)| \leq |E|$
no more than $|E|$ steps

LOCAL MAXIMUM

b) It provides a 2-approximation
we do not know OPT ... so we establish an upper bound $UB \geq OPT$
and prove that $cost(S) \geq \frac{1}{2} UB$ (hence, $\geq \frac{1}{2} OPT$) $\quad UB = |E|$

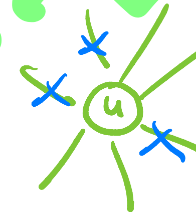
To see this: after the algorithm ends, each node u has $\geq \frac{d_u}{2}$ incident edges in $E(S, V-S)$, where d_u = degree of u

we prove that
$$|E(S,V-S)| \geq \frac{1}{2}|E|$$

$$|E(S,V-S)| = \frac{\sum_{u \in V} \#\text{edges incident to } u \text{ in } E(S,V-S)}{2} \geq$$

$$\geq \frac{\sum_{u \in V} d_u/2}{2}$$

$$= \frac{2|E|/2}{2} = \frac{|E|}{2}$$



→ at least $d_u/2$ are in the cutset, as otherwise changing u set will strictly increase $|E(S,V-S)|$ but this is not possible as the algorithm terminates

② Greedy

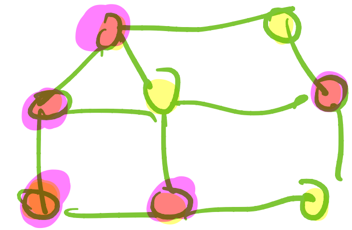
number the nodes from 1 to n (any order is fine)

$S = \{\}$

for $u = 1, 2, \ldots, n$:

$\left.\begin{array}{l} u \in S \to |E(S, V-S)| \\ u \in V-S \to |E(S, V-S)| \end{array}\right\}$ choose the largest

return $V_1 = S$, $V_2 = V - S$

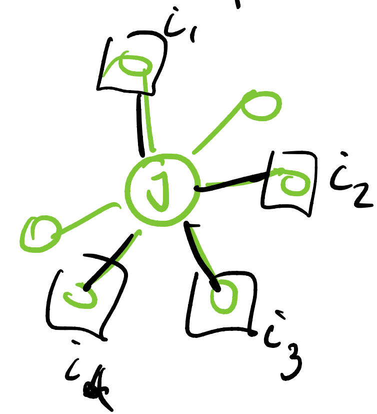


a) it takes $n = |V|$ steps

b) $|E(S, V-S)| \geq \frac{|E|}{2}$

Let's consider edge i — j and w.l.o.g. $i < j$

The "fate" of edge ij is decided by the color of j rather than the color of i



- node j is "responible" for edge ij iff $i < j$

$$r_u = \left| \{ v \in V : uv \in E \text{ and } u > v \} \right|$$

<u>Claim</u> #edges for which u is responsible and belong to $E(S, V-S)$ final $\geq \frac{r_u}{2}$

$i_1, i_2, i_3, i_4 < j$

$r_j = 4$

$$|E(S,V-S)| = \sum_{u \in V} \text{\#edges for which } u \text{ is responsible and belong to the cutsize} \geq \sum_{u \in V} \frac{r_u}{2} = \frac{|E|}{2}$$

we do not divide by 2 as each edge has exactly <u>one</u> responsible node

$$\sum r_u = |E|$$

③ Randomization

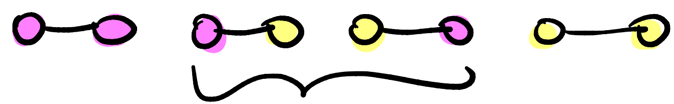
Assign a random color $\chi : V \to \{0,1\}$ uniformly and independently

$uv \in E$ is in $E(S, V-S)$ iff $\chi(u) \neq \chi(v)$

$S = \{u \in V : \chi(u) = 0\}$, $V_1 = S$, $V_2 = V - S$

$\in E(S, V-S)$

$\Pr(u - v \in E(S, V-S)) = \frac{1}{2}$



$\in E(S, V-S)$

Indicator variable

$$X_{uv} = \begin{cases} 1 & \text{if } uv \in E(S, V-S) \\ 0 & \text{o.w.} \end{cases}$$

$$|E(S, V-S)| = \sum_{uv \in E} X_{uv} \Rightarrow E\big[|E(S, V-S)|\big] = \sum_{uv \in E} E[X_{uv}] = \sum_{uv \in E} \frac{1}{2} = \frac{|E|}{2}$$