

1. Symmetry of QM

2. Theory of Angular momentum.

3. Scattering theory.

Symmetry of transformation of space  $\mathbb{R}^3$

Wave function  $\psi(\vec{r}) \quad \vec{r} \in \mathbb{R}^3$

Consider linear transf.  $Q: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$Q\vec{r} = \vec{r}' \quad \vec{r}: 3d \text{ column vector}$

$Q: 3 \times 3 \text{ matrix}$

$Q$  is a symmetry of the space, if it preserves the inner product

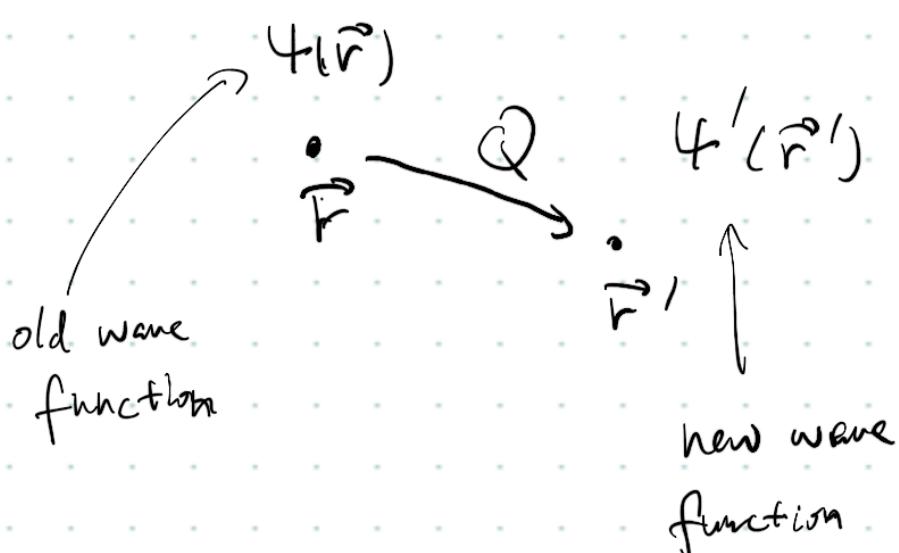
$$\vec{r}_1 \cdot \vec{r}_2 = (Q\vec{r}_1) \cdot (Q\vec{r}_2)$$

In particular,  $Q$  preserves the distance in  $\mathbb{R}^3$

$$\vec{r} \cdot \vec{r} = (Q\vec{r}) \cdot (Q\vec{r})$$

✓

$Q$  induces transformation of  $\psi(\vec{r})$



$$\text{s.t. } \dot{\gamma}(\vec{r}') = \dot{\gamma}(\vec{r})$$

$$4'(\vec{qr}) = 4(\vec{r})$$

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Q1 r1

$$4'(\vec{r}) = 4(Q \frac{\vec{r}}{r})$$

$$\equiv \widehat{D}(Q) + (\vec{r})$$

$$Q : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$\mathcal{D}(Q) : \mathcal{H} \rightarrow \mathcal{H} \quad \quad \quad \mathcal{H} = L^2(\mathbb{R}^3, d^3x)$$

$$|4\rangle \rightarrow |4'\rangle \quad \psi'(\vec{r}) = D(Q) \psi(\vec{r})$$

$$= -\vec{r} (Q^{-1} \vec{r})$$

properties of  $D(Q)$  : linear operator on  $H$ .

- Whiteness :  $\langle \psi_1 | \psi_2 \rangle = \int d^3 \vec{r} \overline{\psi_1(\vec{r})} \psi_2(\vec{r})$

change of

variable

$$= \int d^3(\vec{Q}) \overrightarrow{\psi_1(Q\vec{r})} \psi_2(Q\vec{r})$$

$$\vec{r} \rightarrow Q^{-1} \vec{r}$$

Lemma  $\det Q = \pm 1$

Pf :  $\vec{r} \cdot \vec{r}' = \underbrace{\delta_{ij} r^i r'^j}_{\text{||}} \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow \delta_{ij} Q^i_k Q^j_l = \delta_{kl}$

$Q \vec{r} \cdot Q \vec{r}' = \underbrace{\delta_{ij} Q^i_k r^k Q^j_l r'^l}_{\text{||}} \quad \left. \begin{array}{l} \\ \end{array} \right\} = \delta_{kl}$

$$Q^T Q = I_{3 \times 3}$$

$$\Rightarrow Q^T = Q^{-1}, \quad (\det Q)^2 = 1$$

$$\det Q = \pm 1$$

$$d^3(Q^{-1} \vec{r}) = d^3 \vec{r} \quad |\det Q^{-1}| = d^3 \vec{r}$$

$$\begin{aligned} \langle \psi_1 | \psi_2 \rangle &= \int d^3 \vec{r} \overrightarrow{\psi_1(Q^T \vec{r})} \psi_2(Q^T \vec{r}) \\ &= \langle D(Q) \psi_1 | D(Q) \psi_2 \rangle \end{aligned}$$

$\Rightarrow D(Q)$  is unitary operator.

- Consider 2 transformations  $Q_1, Q_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$Q = Q_1 \circ Q_2 \quad Q \vec{r} = Q_1 \circ Q_2 \vec{r}$$

$$\begin{aligned}
 D(Q_1) D(Q_2) \psi(\vec{r}) &= D(Q_1) \psi(Q_2^{-1} \vec{r}) \\
 &= \psi(Q_2^{-1}(Q_1^{-1} \vec{r})) \\
 &= \psi(Q_2^{-1} Q_1^{-1} \vec{r}) = \psi((Q_1 Q_2)^{-1} \vec{r}) \\
 &\stackrel{?}{=} D(Q_1 Q_2) \psi(\vec{r})
 \end{aligned}$$

$$D(Q_1) D(Q_2) = D(Q_1 Q_2)$$

if we view  $D$  to be a map from symmetry transf. on  $\mathbb{R}^3$   
to linear operators on  $H$ .

$D$  respects the product of transf.

- $D(Q) D(Q^{-1}) = D(Q Q^{-1}) = D(1_{3 \times 3})$

$$D(1) \psi(\vec{r}) = \psi(1 \vec{r}) = \psi(\vec{r})$$

$$D(1) = 1_{\text{op}}$$

$$D(Q)D(Q^{-1}) = I_H$$

$$\boxed{D(Q)^{-1} = D(Q^\top)}$$

$D$  respects  
the inverse.

Def A group is a set  $G$  together with a binary operation

$$a \cdot b \in G \quad \forall a, b \in G$$

called group multiplication

s.t. (1)  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$   $\forall a, b, c \in G$   
 (associativity)

(2)  $\exists ! e \in G$ , s.t.  $e \cdot a = a$   $\forall a \in G$   
 (identity)

(3)  $\forall a \in G$ ,  $\exists a^{-1} \in G$  s.t.  $a \cdot a^{-1} = a^{-1} \cdot a = e$   
 (inverse)

Example (1)  $\mathbb{R} \setminus \{0\}$  with multiplication

(2) set of symmetry transf.  $Q$ 's on  $\mathbb{R}^3$

multiplication : composition of  $Q_1, Q_2$

- $Q$ :  $3 \times 3$  matrix  $\rightarrow$  associativity

- identity  $1_{3 \times 3}$

- inverse  $Q^T Q = 1 \quad Q^{-1} = Q^T$

the set of all  $3 \times 3$  matrices with  $Q^{-1} = Q^T$

is called  $O(3)$  group.

(Orthogonal group on  $\mathbb{R}^3$ )

(3) We have found a set of  $D(Q)$  unitary operators on  $\mathcal{H}$

$$\{D(Q) \mid Q \in O(3)\} = D(O(3))$$

multiplication, composition of operators  
(product)  $D(Q_1) D(Q_2)$

- $D(Q)$  <sup>one</sup> operators  $\rightarrow$  associativity

- identity operators  $D(1_{3 \times 3}) = 1_{\mathcal{H}}$

$$\cdot \text{ inverse } D(Q^{-1}) = D(Q)^{-1}$$

$$D(Q_1)D(Q_2) = D(Q_1Q_2)$$

multiplication                      multiplication

$$\text{on } D(O(3)) \quad \text{on } O(3)$$

Def, given 2 groups  $G, H$ , a group homomorphism

is a map  $h: G \rightarrow H$

$$\text{s.t. } h(g_1) h(g_2) = h(g_1g_2) \quad \forall g_1, g_2 \in G$$

$$\begin{matrix} & \uparrow \\ \text{multiplication} & & \downarrow \\ \text{in } H & & \text{multiplication} \\ & & \text{in } G \end{matrix}$$

$D: D(O(3)) \rightarrow D(O(3))$  is a homomorphism.

in terms of Dirac bracket ,  $|4\rangle \in \mathcal{H}$

$$4(\vec{r}) := \langle \vec{r} | 4 \rangle \quad \hat{\vec{R}}(\vec{r}) = \vec{r} | \vec{r} \rangle$$

$$4'(\vec{r}) := \langle \vec{r} | 4' \rangle$$

$$|4'\rangle = D(Q)|4\rangle$$

$$\left\{ \begin{array}{l} 4'(\vec{r}) = \langle \vec{r} | D(Q) | 4 \rangle = \langle D(Q)^+ \vec{r} | 4 \rangle \\ \text{||} \end{array} \right.$$

$$4(Q^{-1}\vec{r}) = \langle Q^{-1}\vec{r} | 4 \rangle \quad \langle D(Q^{-1})\vec{r} | 4 \rangle$$

$$\langle + | \hat{O} | 4 \rangle = \langle \hat{O}^+ + | 4 \rangle$$

$$D(Q)^+ = D(Q)^{-1} = D(Q^{-1})$$

$$D(Q^{-1})|\vec{r}\rangle = |Q^{-1}\vec{r}\rangle$$

$$D(Q)|\vec{r}\rangle = |Q\vec{r}\rangle$$

$$D(Q) \vec{r} = \vec{r}'$$

$Q \rightarrow D(Q)$  transformations  
 transf transf of operators on  
 on  $\mathbb{R}^3$  on  $\mathcal{H}$   $\mathcal{H}$

$$\hat{A} D(Q)^{-1} |4\rangle$$

$$D(Q) : \mathcal{H}_{(1)} \rightarrow \mathcal{H}_{(2)}$$



$$|4\rangle \in \mathcal{H}_{(2)}, \quad D(Q)^{-1} |4\rangle \in \mathcal{H}_{(1)}$$

$$\hat{A} D(Q)^{-1} |4\rangle \in \mathcal{H}_{(1)}$$

$$(D(Q) \hat{A} D(Q)^{-1}) |4\rangle \in \mathcal{H}_{(2)}$$

$$\hat{A}' := D(Q) \hat{A} D(Q)^{-1}$$

$\hat{A} \mapsto \hat{A}'$  transf. of operators induced  
by  $Q$ .

$Q \rightarrow D(Q) \rightarrow$  transf. of operators

$$Q\vec{r} = \vec{r}'$$

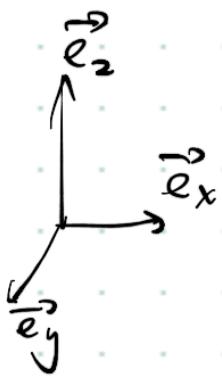
$$D(Q)|\psi\rangle = |\psi'\rangle$$

$$D(Q) \hat{A} D(Q)^{-1} = \hat{A}'$$

conjugate.

position operator:  $\hat{\vec{R}}$   $\hat{\vec{R}} |\vec{r}\rangle = \vec{r} |\vec{r}\rangle$

$$\begin{aligned} \hat{\vec{R}} &= \hat{R}_1 \vec{e}_x + \hat{R}_2 \vec{e}_y + \hat{R}_3 \vec{e}_z \\ &= \sum_{i=1}^3 \hat{R}_i \vec{e}_i \end{aligned}$$



$$\hat{R}_i |\vec{r}\rangle = r_i |\vec{r}\rangle$$

$$i = 1, 2, 3$$

$$\hat{R} \rightarrow D(Q) \hat{R} D(Q)^{-1} = \hat{R}'$$

$$\hat{R}' |\vec{r}\rangle = \vec{r} |\vec{r}\rangle$$

$$D(Q)$$

$\nearrow$        $\uparrow$   
 $D(Q)^{-1} D(Q)$

$$\Rightarrow \underbrace{D(Q) \hat{R} D(Q)^{-1}}_{\hat{R}'} D(Q) |\vec{r}\rangle = D(Q) \vec{r} |\vec{r}\rangle$$

$$\hat{R}' \underbrace{D(Q) |\vec{r}\rangle}_{\text{new eigenstate}} = \vec{r} \underbrace{D(Q) |\vec{r}\rangle}_{\text{new eigenstate}}$$

$$\underbrace{\hat{R}' |\vec{Q}\vec{r}\rangle}_{\hat{R}' |\vec{Q}\vec{r}\rangle} = \vec{r} |\vec{Q}\vec{r}\rangle$$

$$\text{On the other hand, } \hat{R} |\vec{Q}\vec{r}\rangle = Q \vec{r} |\vec{Q}\vec{r}\rangle$$

$$\Rightarrow \underbrace{Q^{-1} \hat{R} |\vec{Q}\vec{r}\rangle}_{Q^{-1} \hat{R} |\vec{Q}\vec{r}\rangle} = \vec{r} |\vec{Q}\vec{r}\rangle$$

$$Q^{-1} \hat{R} = \sum_{i=1}^3 \hat{R}_i \underbrace{(Q^{-1} \vec{e}_i)}_{\vec{e}_i}$$

$$\hat{\vec{R}}' = Q^{-1} \hat{\vec{R}}$$

$$D(Q) \hat{\vec{R}} D(Q)^{-1} = Q^{-1} \hat{\vec{R}}$$

$$\hat{\vec{R}} = \sum_{i=1}^3 \hat{R}_i \hat{e}_i$$

$$Q^{-1} \hat{\vec{R}} = \sum_{i=1}^3 \hat{R}_i Q^{-1} \hat{e}_i$$

Parity transformation

$$P \in O(3) : P\vec{r} = -\vec{r}$$

$$D(P) |\vec{r}\rangle = |P\vec{r}\rangle = |-\vec{r}\rangle$$

$$D(P) \equiv \hat{P}$$

$\{P, 1_{3 \times 3}\}$  is a subgroup in  $O(3)$

$$P^2 = 1, P \cdot 1 = P = 1 \cdot P.$$

$$\hat{P} |4\rangle = |4'\rangle$$

$$\boxed{\langle \vec{r} | \psi' \rangle = \langle \vec{r} | \hat{P} |\psi\rangle = \langle \vec{r} | \psi \rangle}$$

$$\langle \vec{r} | \hat{P} |\psi\rangle = \langle \hat{P}^+ | \vec{r} \rangle, \psi \rangle = \langle \hat{P}^+ | \vec{r} \rangle, \psi \rangle$$

$$\hat{P}^+ = \hat{P}^- \quad \boxed{\begin{aligned} \hat{P}^2 &= D(P)D(P) = D(P^2) \\ &= D(1) = \mathbb{1}_{\mathcal{H}} \end{aligned}}$$

$$\psi'(\vec{r}) = \psi(-\vec{r})$$

$$\boxed{\hat{P}: \psi(\vec{r}) \rightarrow \psi'(\vec{r}) = \psi(-\vec{r})}$$

$$\hat{P}^2 = \mathbb{1}_{\mathcal{H}} \Rightarrow \hat{P} \text{ has eigenvalue } \pm 1$$

eigenstates  $\psi_{\pm}$

$$\hat{P} \psi_{\pm}(\vec{r}) = \underbrace{\psi_{\pm}(-\vec{r})}_{\pm \psi_{\pm}(\vec{r})}$$

$\psi_+$ : even parity

$\psi_-$ : odd parity

$$\hat{P}^+ = \hat{P}^\dagger = \hat{P} \quad \hat{P} \text{ is Hermitian \& unitary}$$

transf. of operators under parity

$$\hat{\vec{R}} \rightarrow \hat{P} \hat{\vec{R}} \hat{P}$$

$$\begin{aligned} \hat{P} \hat{\vec{R}} \hat{P} |\vec{r}\rangle &= \hat{P} \hat{\vec{R}} |-\vec{r}\rangle = (-\vec{r}) \hat{P} |-\vec{r}\rangle \\ &= (-\vec{r}) |\vec{r}\rangle \\ &\approx -\hat{\vec{R}} |\vec{r}\rangle \end{aligned}$$

$$\hat{P} \hat{\vec{R}} \hat{P} = -\hat{\vec{R}}$$

momentum operator  $\hat{\vec{P}} |\vec{p}\rangle = \vec{p} |\vec{p}\rangle$

$$\begin{aligned} \langle \vec{r} | \vec{p} \rangle &= e^{\frac{i}{\hbar} \vec{r} \cdot \vec{p}} = e^{\frac{i}{\hbar} (-\vec{r}) \cdot (-\vec{p})} \\ &= \langle -\vec{r} | -\vec{p} \rangle \end{aligned}$$

$$\hat{\vec{P}} \rightarrow \hat{P} \hat{\vec{P}} \hat{P} \hat{\vec{P}}$$

$$\hat{P} |\vec{p}\rangle = \hat{P} \underbrace{\int d^3r |r\rangle \langle r|}_{\hat{P}} |\vec{p}\rangle$$

resolution  
 of identity

$$\begin{aligned}
 d^3\vec{r} &= d^3(-\vec{r}) & = \int d^3\vec{r} |-\vec{r}\rangle \langle -\vec{r}| - \vec{p} \rangle \\
 && = \underbrace{\int d^3(-\vec{r}) |-\vec{r}\rangle \langle -\vec{r}| - \vec{p} \rangle}_{= 1 - \vec{p} \rangle}
 \end{aligned}$$

$$\begin{aligned}
 \hat{p} \hat{p} \hat{p} | \vec{p} \rangle &= \hat{p} \hat{p} \hat{p} | -\vec{p} \rangle = (-\vec{p}) \hat{p} | -\vec{p} \rangle \\
 &= (-\vec{p}) | \vec{p} \rangle = -\hat{p} | \vec{p} \rangle \\
 \Rightarrow \hat{p} \hat{p} \hat{p} &= -\hat{p}
 \end{aligned}$$

angular momentum operator:

$$\hat{L} = \hat{R} \times \hat{p}$$

$$\begin{aligned}
 \hat{p} \hat{L} \hat{p} &= \hat{p} (\hat{R} \times \hat{p}) \hat{p} = \hat{p} \hat{R} \hat{p} \times \hat{p} \hat{p} \hat{p} \\
 &= (-\hat{R}) \times (-\hat{p}) \\
 &= \hat{L}
 \end{aligned}$$

$$\hat{\vec{V}} = \sum_{i=1}^3 \hat{V}_i \hat{e}_i$$

$\hat{\vec{V}}$  is vector operator if  $\hat{p} \hat{\vec{V}} \hat{p} = -\hat{\vec{V}}$ , like,  $\hat{\vec{R}}$

" pseudo-vector operator if  $\hat{p} \hat{\vec{V}} \hat{p} = \hat{\vec{V}}$

like  $\hat{\vec{L}}$ ,  $\hat{\vec{s}}$  —————  
 ↑  
 spin

$\hat{\vec{O}}$  scalar operator if  $\hat{p} \hat{\vec{O}} \hat{p} = \hat{\vec{O}}$  e.g.

Hamiltonian of central potential

pseudo-scalar operator if  $\hat{p} \hat{\vec{O}} \hat{p} = -\hat{\vec{O}}$  e.g.

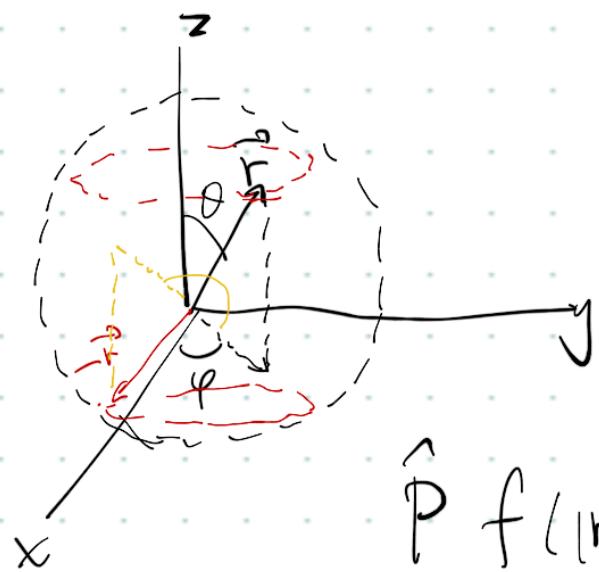
helicity operator

$$\hat{\vec{s}} \cdot \frac{\hat{\vec{p}}}{|\hat{\vec{p}}|} = \hat{h}$$

because  $\hat{P} \hat{L} \hat{P} = \hat{L}$  i.e.  $[\hat{P}, \hat{L}] = 0$

they share the same eigenbasis.

indeed in spherical coordinate.



$$P: \vec{r} \rightarrow -\vec{r}$$

$$\theta \rightarrow \pi - \theta$$

$$\varphi \rightarrow \varphi + \pi$$

$$\hat{P} f(r, \theta, \varphi) = f(r, \pi - \theta, \varphi + \pi)$$

$$\left\{ \begin{array}{l} \hat{P} Y_{lm}(\theta, \varphi) = (-1)^l Y_{lm}(\theta, \varphi) \\ \hat{L}^2 Y_{lm}(\theta, \varphi) = \hbar^2 l(l+1) Y_{lm}(\theta, \varphi) \end{array} \right.$$

$$\hat{L}_z Y_{lm}(\theta, \varphi) = \hbar m Y_{lm}(\theta, \varphi)$$

$$\hat{L}_z Y_{lm}(\theta, \varphi) = \hbar m Y_{lm}(\theta, \varphi)$$

## space translation

$$O(3) \quad , \quad Q \vec{r} = \vec{r}'$$

space translation:  $\vec{r} \rightarrow \vec{r}' = \vec{r} + \vec{\lambda}$

$$O(3) \cup \{ \text{space translations} \} = E(3) \quad \text{Euclidean group.}$$

a general transf. in  $E(3)$

$$\vec{r} \rightarrow \vec{r}' = \Lambda \vec{r} + \vec{\lambda}$$

$$O(3) \subset E(3) \quad \text{subgroup.}$$

$$\{ \text{space translations} \} \subset E(3) \quad \text{sub group.}$$

!!

$$\{ Q(\vec{\lambda}) \mid \vec{\lambda} \in \mathbb{R}^3 \}$$

$$D: Q(\vec{\lambda}) \rightarrow D(Q(\vec{\lambda}))$$

$$\mathbb{R}^3$$

$$\mathcal{H}$$

$$D(Q(\vec{\lambda})) : \psi(\vec{r}) \rightarrow \psi'(\vec{r})$$

$$\underline{\psi'(\vec{r}) = \psi(Q(\vec{\lambda})^{-1} \vec{r}) = \psi(\vec{r} - \vec{\lambda})}$$

assume

$$\begin{aligned} \vec{\lambda} \text{ is infinitesimal} &= \psi(\vec{r}) - \vec{\lambda} \cdot \vec{\nabla} \psi(\vec{r}) + O(\lambda^2) \\ &= \left(1 - \frac{i}{\hbar} \vec{\lambda} \cdot \hat{\vec{p}}\right) \psi(\vec{r}) + O(\lambda^2) \end{aligned}$$

finite translation

= composition of many infinitesimal translations

$$\begin{aligned} D(Q(\lambda)) &= \lim_{n \rightarrow \infty} \left(1 - \frac{i}{\hbar} \frac{\vec{\lambda}}{n} \cdot \hat{\vec{p}}\right)^n \\ &= e^{-\frac{i}{\hbar} \vec{\lambda} \cdot \hat{\vec{p}}} \end{aligned}$$

$\hat{p}$  momentum operator is the generator of space translation.

in terms of Dirac bracket

$$D(Q(\lambda)) |4\rangle = |4'\rangle$$

$$D(Q(\lambda)) |\vec{r}\rangle = |Q(\vec{\lambda}) \vec{r}\rangle = |\vec{r} + \vec{\lambda}\rangle$$

transformations of operator.

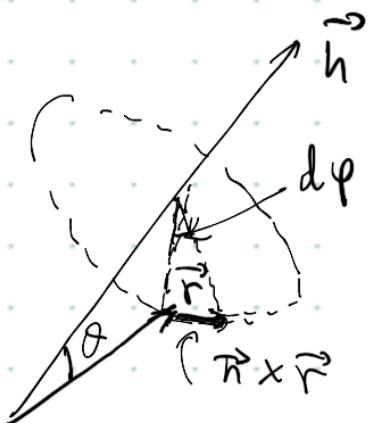
$$\begin{aligned}\hat{\vec{R}}' &= D(Q(\lambda)) \hat{\vec{R}} D(Q(\lambda))^{-1} = Q^{-1}(\vec{\lambda}) \hat{\vec{R}} \\ &= \hat{\vec{R}} - \vec{\lambda}\end{aligned}$$

$$\begin{aligned}\hat{\vec{P}}' &= D(Q(\lambda)) \vec{P} D(Q(\lambda))^{-1} = \vec{P} \\ \text{since } [\vec{P}, e^{-\frac{i}{\hbar} \vec{\lambda} \cdot \vec{P}}] &= 0\end{aligned}$$

Spatial Rotations  $O(3) \supset SO(3)$

infinitesimal rotations

$$\begin{aligned}\vec{r} \rightarrow \vec{r}' &= \vec{r} + \underbrace{d\varphi \vec{n} \times \vec{r}}_{Q(\vec{n}, d\varphi) \vec{r}} \\ &= Q(\vec{n}, d\varphi) \vec{r}\end{aligned}$$



$$d\vec{r} = r \sin \theta \, d\varphi \, \vec{n}$$

$$= \underline{\underline{|d\varphi \vec{n} \times \vec{r}|}}$$

on H :  $D(\vec{Q}(\vec{n}, d\varphi)) = D(\vec{n}, d\varphi)$

$$\begin{aligned}\psi'(\vec{r}) &= D(\vec{n}, d\varphi) \psi(\vec{r}) \\ &= \psi(Q^{-1}(\vec{n}, d\varphi) \vec{r}) \\ &= \psi(\vec{r} - d\varphi \vec{n} \times \vec{r}) \\ &= \psi(\vec{r}) - d\varphi (\vec{n} \times \vec{r}) \cdot \vec{\nabla} \psi(\vec{r}) \\ &= \left(1 - \frac{1}{h} d\varphi \vec{n} \cdot \hat{\vec{L}}\right) \psi(\vec{r})\end{aligned}$$

$$(\vec{n} \times \vec{r}) \cdot \vec{\nabla} = \vec{n} \cdot (\vec{r} \times \vec{\nabla})$$

$$\underline{\underline{=}}$$

$$D(\vec{n}, d\varphi) = 1 - \frac{i}{h} d\varphi \vec{n} \cdot \hat{\vec{L}}$$

$m \rightarrow \infty$

$$\text{finite rotation : } D(\vec{n}, \varphi) = \left(1 - \frac{i}{\hbar} \frac{\varphi}{m} \vec{n} \cdot \hat{L}\right)^m$$

$$= e^{-\frac{i}{\hbar} \varphi \vec{n} \cdot \hat{L}}$$

angular momentum operator is  
the generator of rotations.

transf. of  $\hat{R}$

$$\begin{aligned}\hat{R}' &= \underline{D(\vec{n}, d\varphi)} \hat{R} \underline{D(\vec{n}, d\varphi)^{-1}} \\ &= \left(1 - \frac{i}{\hbar} d\varphi \vec{n} \cdot \hat{L}\right) \hat{R} \left(1 + \frac{i}{\hbar} d\varphi \vec{n} \cdot \hat{L}\right) \\ &= \hat{R} - \frac{i}{\hbar} d\varphi [\vec{n} \cdot \hat{L}, \hat{R}] + O(d\varphi^2)\end{aligned}$$

$$\left([\hat{L}_i, \hat{R}_j] = i\hbar \epsilon_{ijk} \hat{R}_k\right)$$

$$= \hat{R} - d\varphi \vec{n} \times \underline{\hat{R}}$$

$$\hat{R}' = \underline{Q^{-1}(\vec{n}, d\varphi)} \hat{R}$$

l+tw prove  $\hat{P}' = D(\vec{n}, d\varphi) \hat{P} D(\vec{n}, d\varphi)^{-1} = Q'(\vec{n}, d\varphi) \hat{P}$

$$\hat{L}' = D(\vec{n}, d\varphi) \hat{L} D(\vec{n}, d\varphi)^{-1} = Q^{-1}(\vec{n}, d\varphi) \hat{L}$$

Scalar :  $\mathcal{S}$  const. inv. under rotations.

Vector :  $\hat{V} \rightarrow Q(\vec{n}, \varphi) \hat{V}$

Def : Scalar operator :  $\int \text{ s.t. } D(\vec{n}, \varphi) \int D(\vec{n}, \varphi)^{-1}$   
 $= \int [ \hat{S}, D(\vec{n}, \varphi) ] = 0 \quad \forall \vec{n}, \varphi$

e.g. Hamiltonian operator.  $\hat{H} = \frac{\hat{p}^2}{2m} + V(r)$

Vector operator :  $\hat{V} = (\hat{V}_1, \hat{V}_2, \hat{V}_3)$

s.t.  $D(\vec{n}, \varphi) \hat{V} D(\vec{n}, \varphi)^{-1} = Q^{-1}(\vec{n}, \varphi) \hat{V}$

e.g.  $\underbrace{\hat{P}}_{\text{polar}}, \underbrace{\hat{R}}_{\text{pseudo-vector}}$ ,  $\hat{L}$

polar vector operators pseudo-vector operator.

in infinitesimal version  $\varphi \rightarrow d\varphi$

$$\text{scalar: } [\hat{\mathcal{S}}, 1 - \frac{i}{\hbar} d\varphi \vec{n} \cdot \hat{\vec{L}}] = 0$$

$$\Leftrightarrow [\hat{\mathcal{S}}, \hat{\vec{L}}] = 0$$

$$P \hat{\vec{P}} P = - \hat{\vec{P}} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{polar vector}$$

$$P \hat{\vec{R}} P = - \hat{\vec{R}} \quad \left. \begin{array}{l} \\ \end{array} \right\}$$

$$P \hat{\vec{L}} P = \hat{\vec{L}} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{pseudo vector.}$$

$$\text{vector: } D(\vec{n}, d\varphi) \overset{\hat{\vec{V}}}{\vec{V}} D(\vec{n}, d\varphi) \sim$$

$$= V - \frac{i}{\hbar} d\varphi [\vec{n}, \hat{\vec{L}}, \hat{\vec{V}}]$$

$$Q^{-1}(\vec{n}, d\varphi) \overset{\hat{\vec{V}}}{\vec{V}} = \hat{\vec{V}} - d\varphi \vec{n} \times \hat{\vec{V}}$$

$$\Rightarrow [\vec{n} \cdot \hat{\vec{L}}, \hat{\vec{V}}] = -i\hbar \vec{n} \times \hat{\vec{V}}$$

$$n_m [\hat{\vec{L}}_m, \hat{\vec{V}}_k] = -i\hbar \epsilon_{kmn} n_m \hat{\vec{V}}_n$$

$$\Rightarrow [\hat{\vec{L}}_m, \hat{\vec{V}}_k] = -i\hbar \epsilon_{kmn} \hat{\vec{V}}_n = i\hbar \epsilon_{mkn} \hat{\vec{V}}_n$$

~~Rotation in spin space~~  
~~operators~~

$$\mathcal{H}(\mathbb{R}) \otimes \mathcal{Z} \leftarrow \mathcal{H} \otimes \mathbb{C}^2$$

$\downarrow$   $\uparrow$   
 $L^2(\mathbb{R}^3)$  spin hilbert  
space

$$\mathcal{Z} \subset \mathbb{C}^2$$

$$\hat{\vec{s}} = \frac{\hbar}{2} \hat{\vec{\tau}} \quad \text{spin operator.}$$

$\uparrow$   $(\tau_1, \tau_2, \tau_3)$  pauli matrices

the spin angular momentum

$$[s_i, s_j] = i\hbar \epsilon_{ijk} s_k$$

We define spin rotation by

$$D'(\vec{n}, \varphi) = e^{-\frac{i}{\hbar} \varphi \vec{n} \cdot \hat{\vec{s}}} : \mathbb{C}^2 \rightarrow \mathbb{C}^2$$

$\uparrow$   
 $2 \times 2$  matrix.

rotation in  $\vec{r}$ -space

$$D(\vec{n}, \varphi) = e^{-\frac{i}{\hbar} \varphi \vec{n} \cdot \hat{\vec{l}}} : \mathcal{H} \rightarrow \mathcal{H}$$

$$D(\vec{h}, \varphi) \otimes D'(\vec{h}, \varphi) = D(\vec{h}, \varphi)$$

$\mathcal{H} \otimes \mathbb{C}^2$       ↗       $\mathcal{H} \otimes \mathbb{C}^2$

same  
 $\vec{h}, \varphi.$

$$D(\vec{h}, \varphi) = \frac{e^{-\frac{i}{\hbar} \varphi \vec{h} \cdot \hat{\vec{L}}}}{e^{-\frac{i}{\hbar} \varphi \vec{h} \cdot \hat{\vec{J}}}}$$

(generator of total rotation)  
total angular momentum

to find  $\hat{\vec{J}}$ , infinitesimal rotation  $\varphi \rightarrow d\varphi$

$$\rightarrow \left( 1 - \frac{i}{\hbar} d\varphi \vec{h} \cdot \hat{\vec{L}} \right) \otimes \left( 1 - \frac{i}{\hbar} d\varphi \vec{h} \cdot \hat{\vec{J}} \right)$$

$$= 1_{\mathcal{H}} \otimes 1_{\mathbb{C}^2} - \frac{i}{\hbar} d\varphi \left( \vec{h} \cdot \hat{\vec{L}} \otimes 1_{\mathbb{C}} \right) - \frac{i}{\hbar} d\varphi \left( 1_{\mathcal{H}} \otimes \vec{h} \cdot \hat{\vec{J}} \right)$$

$\mathcal{H}$       ↗       $\mathbb{C}^2$

$$e^{-\frac{i}{\hbar} \vec{h} \cdot \hat{\vec{J}}} + O(d\varphi^2)$$

$$= 1_{\mathcal{H}} \otimes 1_{\mathbb{C}^2} - \frac{i}{\hbar} d\varphi \vec{h} \cdot \hat{\vec{J}} + O(d\varphi^2)$$

↗       $\mathcal{H} \otimes \mathbb{C}^2$

$$\hat{\vec{J}} = \hat{\vec{L}} \otimes \mathbb{1}_{\mathbb{C}^2} + \mathbb{1}_{\mathcal{H}} \otimes \hat{\vec{S}} = \hat{\vec{L}} + \hat{\vec{S}}$$

$\mathbb{1}_{\mathcal{H}}$        $\mathbb{1}_{\mathbb{C}^2}$

Spatial symmetry & conservation law

Schrödinger eqn.  $i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$

Spatial symmetries  $Q(\lambda) \in E(3)$

Translation or rotation or parity.

$$x = \vec{x} \quad \lambda = (\vec{n}, \varphi)$$

$$Q \rightarrow D(Q) : \mathcal{H} \rightarrow \mathcal{H}$$

$$i\hbar \frac{\partial}{\partial t} D(Q) |\psi(t)\rangle = D(Q) \hat{H} |\psi(t)\rangle$$

$\underbrace{|\psi'(t)\rangle}_{H'}$        $\underbrace{D(Q) \hat{H} D(Q)^{-1}}_{-} D(Q) |\psi(t)\rangle$

$$i\hbar \frac{\partial}{\partial t} |\psi'(t)\rangle = \hat{H}' |\psi'(t)\rangle$$

$$\hat{H}' = \hat{H} \quad \text{or} \quad D(Q) \hat{H} D(Q)^{-1} = \hat{H}$$

$$\text{i.e., } [D(Q), \hat{H}] = 0$$

if  $[D(Q), \hat{H}] = 0$ , we say  $D(Q)$  is a symmetry of the physical system.

Example  $\hat{H}_{\text{free}} = \frac{\hat{p}^2}{2m}$

$$D(Q) \xrightarrow{\hat{p}} D(Q)^{-1} = Q^{-1} \xrightarrow{\hat{p}}$$

$Q \in E(3)$

$$\begin{aligned} D(Q) \hat{p}^2 D(Q)^{-1} &= (D(Q) \xrightarrow{\hat{p}} D(Q)^{-1}) \cdot (D(Q) \xrightarrow{\hat{p}} D(Q)^{-1}) \\ &= Q^{-1} \xrightarrow{\hat{p}} \cdot Q^{-1} \xrightarrow{\hat{p}} \\ &= \hat{p} \cdot \hat{p} = \hat{p}^2 \end{aligned}$$

$$\Rightarrow D(Q) \hat{H}_{\text{free}} D(Q)^{-1} = \hat{H}_{\text{free}} \quad \forall Q \in E(3)$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\vec{r})$$

$$[\hat{H}, D(Q)] = [V(\vec{r}), D(Q)]$$

Symmetry is determined by  $V(\vec{r})$

time evolution operator  $U(t) = e^{\frac{i}{\hbar} \hat{H} t}$

$A$  operator  $\hat{A} : \mathcal{H} \rightarrow \mathcal{H}$

$$\hat{A}(t) = U(t) \hat{A} U(t)^{-1}$$

$$i\hbar \frac{d}{dt} \hat{A}(t) = [\hat{A}(t), \hat{H}]$$

Heisenberg picture EOM.

$$[D(Q), \hat{H}] = 0 \Rightarrow \frac{d}{dt} D(Q) = 0$$

$D(Q)$  is conserved

if  $D(Q)$  is continuous symmetry  $\Rightarrow$  its generator is conserved.

Example, (1)  $D(\vec{x}) = e^{-\frac{i}{\hbar} \vec{x} \cdot \hat{\vec{p}}}$  spatial translation

$$\text{if } [\hat{H}, D(\vec{x})] = 0, \frac{d}{d\vec{x}} \Big|_{\vec{x}=0} [\hat{H}, e^{-\frac{i}{\hbar} \vec{x} \cdot \hat{\vec{p}}}] \\ = -\frac{i}{\hbar} [\hat{H}, \hat{\vec{p}}] = 0$$

$\Rightarrow$  momentum is conserved.

$$(2) D(\vec{\tau}, \varphi) = e^{-\frac{i}{\hbar} \varphi \vec{\tau} \cdot \frac{1}{J}}$$

$$\text{if } [\hat{H}, D(\vec{\tau}, \varphi)] = 0$$

$$\frac{d}{d\varphi} \Big|_{\varphi=0} [\hat{H}, e^{-\frac{i}{\hbar} \varphi \vec{\tau} \cdot \frac{1}{J}}] = 0$$

$$= +\frac{i}{\hbar} [\vec{\tau} \cdot \frac{1}{J}, \hat{H}]$$

$\Rightarrow$  angular momentum  $\vec{\tau} \cdot \frac{1}{J}$  is conserved.