1. Symmetry of QM

2. Theory of Angular momentum.

3. Scattering theory.

Symmetry of transformation of space $\mathbb{R}^3$

wave function $\psi(\vec{r})$ $\quad \vec{r} \in \mathbb{R}^3$.

consider linear transf. $Q: \mathbb{R}^3 \to \mathbb{R}^3$

$$Q\vec{r} = \vec{r}'$$

$\vec{r}$: 3d column vector

Q : 3×3 matrix

Q is a symmetry of the space, if it preserves the inner product

$$\vec{r}_1 \cdot \vec{r}_2 = (Q\vec{r}_1) \cdot (Q\vec{r}_2)$$

in particular, Q preserves the distance in $\mathbb{R}^3$

$$\vec{r} \cdot \vec{r} = (Q\vec{r}) \cdot (Q\vec{r})$$

Q induces transformation of $\psi(\vec{r})$

$\psi(\vec{r})$

$\vec{r} \xrightarrow{Q} \vec{r}\,'$ $\quad \psi'(\vec{r}\,')$

old wave function

new wave function

s.t. $\psi'(\vec{r}\,') = \psi(\vec{r})$

$$\underbrace{\psi'(Q\vec{r})}_{Q\vec{r}\,=\,\vec{r}\,'} = \psi(\vec{r}), \qquad \vec{r} = Q^{-1}\vec{r}\,'$$

$$\boxed{\begin{aligned} \psi'(\vec{r}) &= \psi(Q^{-1}\vec{r}) \\ &\equiv \hat{D}(Q)\psi(\vec{r}) \end{aligned}}$$

$Q : \mathbb{R}^3 \to \mathbb{R}^3$

$\hat{D}(Q) : \mathcal{H} \to \mathcal{H} \qquad \mathcal{H} = L^2(\mathbb{R}^3, d^3x)$

$|\psi\rangle \to |\psi'\rangle \qquad \psi'(\vec{r}) = D(Q)\,\psi(\vec{r})$

$$= \psi(Q^{-1}\vec{r})$$

properties of $D(Q)$: linear operator on $\mathcal{H}$.

- unitarity : $\langle \psi_1 | \psi_2 \rangle = \int d^3\vec{r}\; \overline{\psi_1(\vec{r})}\; \psi_2(\vec{r})$

change of variable $\vec{r} \to Q^{-1}\vec{r}$

$$= \int d^3(Q^{-1}\vec{r})\; \overline{\psi_1(Q^{-1}\vec{r})}\; \psi_2(Q^{-1}\vec{r})$$

<u>Lemma</u> $\det Q = \pm 1$

Pf: $\vec{r} \cdot \vec{r}' = \underbrace{\delta_{ij} r^i r'^j}$

$\quad\quad \shortparallel$

$Q\vec{r} \cdot Q\vec{r}' = \underbrace{\delta_{ij} Q^i{}_k r^k Q^j{}_l r'^l}$ $\Bigg\} \Rightarrow \delta_{ij} Q^i{}_k Q^j{}_l = \delta_{kl}$

$\Updownarrow$

$Q^T Q = 1_{3\times3}$

$\Rightarrow Q^T = Q^{-1}$, $(\det Q)^2 = 1$

$$\det Q = \pm 1$$

$$d^3(Q^{-1}\vec{r}) = d^3\vec{r}\ |\det Q^{-1}| = d^3\vec{r}$$

$$\langle \psi_1 | \psi_2 \rangle = \int d^3\vec{r}\ \overline{\psi_1(Q^{-1}\vec{r})}\ \psi_2(Q^{-1}\vec{r})$$

$$= \langle D(Q)\psi_1 | D(Q)\psi_2 \rangle$$

$\Rightarrow$ $D(Q)$ is unitary operator.

- Consider 2 transformations $Q_1, Q_2 : \mathbb{R}^3 \to \mathbb{R}^3$

$$Q = Q_1 \circ Q_2 \qquad Q\vec{r} = Q_1 \circ Q_2 \vec{r}$$

$$D(Q_1)\,D(Q_2)\,\psi(\vec{r}) = D(Q_1)\,\psi(Q_2^{-1}\vec{r})$$

$$= \psi(Q_2^{-1}(Q_1^{-1}\vec{r}))$$

$$= \psi(Q_2^{-1}Q_1^{-1}\vec{r}) = \psi((Q_1Q_2)^{-1}\vec{r})$$

$$= D(Q_1Q_2)\,\psi(\vec{r})$$

$$\boxed{D(Q_1)\,D(Q_2) = D(Q_1Q_2)}$$

if we view D to be a map from symmetry transf. on $\mathbb{R}^3$ to linear operators on $\mathcal{H}$.

D respects the product of transf.

- $D(Q)\,D(Q^{-1}) = D(QQ^{-1}) = D(1_{3\times3})$

$$D(1)\,\psi(\vec{r}) = \psi(1\vec{r}) = \psi(\vec{r})$$

$$D(1) = 1_{\mathcal{H}}$$

$$D(Q)D(Q^{-1}) = 1_{\mathcal{H}}$$

$$\boxed{D(Q)^{-1} = D(Q^{-1})}$$

D respects the inverse.

Def A group is a set G together with a binary operation

$$a \cdot b \in G \quad \forall a, b \in G$$

called group multiplication

s.t. (1) $(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \forall a, b, c \in G$

(associativity)

(2) $\exists! \, e \in G$, s.t. $e \cdot a = a \quad \forall a \in G$

(identity)

(3) $\forall a \in G$, $\exists \, a^{-1} \in G$ s.t. $a \cdot a^{-1} = a^{-1} \cdot a = e$

(inverse)

Example (1) $\mathbb{R}^1 \setminus \{0\}$ with multiplication

(2) set of symmetry transf. Q's on $\mathbb{R}^3$

multiplication ; composition of Q_1, Q_2

- Q ; 3×3 matrix $\rightarrow$ associativity
- identity $\mathbb{1}_{3\times3}$
- inverse $Q^T Q = \mathbb{1}$ $\quad Q^{-1} = Q^T$

the set of all 3×3 matrices with $Q^{-1} = Q^T$ is called $O(3)$ group.

(orthogonal group on $\mathbb{R}^3$)

(3) We have found a set of $D(Q)$ unitary operators on $\mathcal{H}$

$$\{ D(Q) \mid Q \in O(3) \} \equiv D(O(3))$$

multiplication, composition of operators $\underline{D(Q_1)D(Q_2)}$ (product)

- $D(Q)$ are operators $\rightarrow$ associativity
- identity operators $D(\mathbb{1}_{3\times3}) = \mathbb{1}_{\mathcal{H}}$

- inverse $D(Q^{-1}) = D(Q)^{-1}$

$$D(Q_1)\,D(Q_2) = D(Q_1 Q_2)$$

multiplication on $D(O(3))$ multiplication on $O(3)$

Def, given 2 groups G, H, a group homomorphism is a map $h: G \to H$

s.t. $h(g_1)\,h(g_2) = h(g_1 g_2)$ $\forall g_1, g_2 \in G$

multiplication in H multiplication in G

$D: O(3) \to D(O(3))$ is a homomorphism.

in terms of Dirac bra-ket, $|\psi\rangle \in \mathcal{H}$

$$\psi(\vec{r}) := \langle \vec{r} | \psi \rangle \qquad \hat{\vec{R}} |\vec{r}\rangle = \vec{r} |\vec{r}\rangle$$

$$\psi'(\vec{r}) := \langle \vec{r} | \psi' \rangle$$

$$|\psi'\rangle = D(Q) |\psi\rangle$$

$$\psi'(\vec{r}) = \langle \vec{r} | D(Q) | \psi \rangle = \langle D(Q)^{\dagger} \vec{r} | \psi \rangle$$

$$\shortparallel \qquad\qquad\qquad \shortparallel$$

$$\psi(Q^{-1}\vec{r}) = \langle Q^{-1} \vec{r} | \psi \rangle \qquad \langle D(Q^{-1}) \vec{r} | \psi \rangle$$

$$\langle \psi | \hat{O} | \phi \rangle = \langle \hat{O}^{\dagger} \psi | \phi \rangle$$

$$D(Q)^{\dagger} = D(Q)^{-1} = D(Q^{-1})$$

$$D(Q^{-1}) |\vec{r}\rangle = |Q^{-1} \vec{r}\rangle$$

$$\boxed{D(Q) |\vec{r}\rangle = |Q\vec{r}\rangle}$$

$$\underline{D(Q)\,\psi(\vec{r}) = \psi(Q^{-1}\vec{r})}$$

$$Q \longrightarrow D(Q) \longrightarrow \text{transformations}$$

transf on $\mathbb{R}^3$ | transf on $\mathcal{H}$ | of operators on $\mathcal{H}$

$$D(Q):\quad \begin{tikzcd} \hat{A}\,D(Q)^{-1}|\psi\rangle & |\psi\rangle \\ \mathcal{H}_{(1)} \arrow[r] \arrow[loop below, "\hat{A}"] & \mathcal{H}_{(2)} \arrow[loop below, "\hat{A}'"] \end{tikzcd}$$

$\uparrow$ old $\qquad\qquad$ $\uparrow$ new

$$\forall\, |\psi\rangle \in \mathcal{H}_{(2)}, \qquad D(Q)^{-1}|\psi\rangle \in \mathcal{H}_{(1)}$$

$$\hat{A}\,D(Q)^{-1}|\psi\rangle \in \mathcal{H}_{(1)}$$

$$\left(D(Q)\,\hat{A}\,D(Q)^{-1}\right)|\psi\rangle \in \mathcal{H}_{(2)}$$

$$\hat{A}' := D(Q)\,\hat{A}\,D(Q)^{-1}$$

$\hat{A} \mapsto \hat{A}'$ transf. of operators induced by Q.

$Q \longrightarrow D(Q) \longrightarrow$ transf. of operators

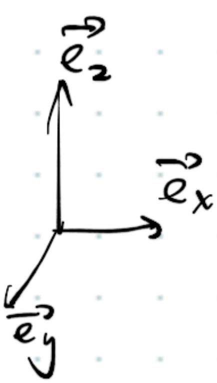
$Q\vec{r} = \vec{r}\,'$ $D(Q)|\psi\rangle = |\psi'\rangle$ $D(Q)\,\hat{A}\,D(Q)^{-1} = \hat{A}'$

conjugate.

position operator: $\hat{\vec{R}}$ $\hat{\vec{R}}\,|\vec{r}\rangle = \vec{r}\,|\vec{r}\rangle$

$$\hat{\vec{R}} = \hat{R}_1\,\vec{e}_x + \hat{R}_2\,\vec{e}_y + \hat{R}_3\,\vec{e}_z$$

$$= \sum_{i=1}^{3} \hat{R}_i\,\vec{e}_i$$



$$\hat{R}_i\,|\vec{r}\rangle = r_i\,|\vec{r}\rangle$$

$i = 1, 2, 3$

$$\hat{\vec{R}} \to D(Q)\,\hat{\vec{R}}\,D(Q)^{-1} \equiv \hat{\vec{R}}'$$

$$\hat{\vec{R}}\,|\vec{r}\rangle = \vec{r}\,|\vec{r}\rangle$$

$D(Q)^{-1}D(Q)$

$D(Q)$

$$\Rightarrow \underbrace{D(Q)\,\hat{\vec{R}}\,D(Q)^{-1}}\,D(Q)\,|\vec{r}\rangle = D(Q)\,\vec{r}\,|\vec{r}\rangle$$

$$\hat{\vec{R}}'\,\underbrace{D(Q)|\vec{r}\rangle}_{\text{new eigenstate}} = \vec{r}\,\underbrace{D(Q)|\vec{r}\rangle}$$

$$\underline{\hat{\vec{R}}'\,|Q\vec{r}\rangle = \vec{r}\,|Q\vec{r}\rangle}$$

On the other hand, $\hat{\vec{R}}\,|Q\vec{r}\rangle = Q\vec{r}\,|Q\vec{r}\rangle$

$$\Rightarrow \underline{\underbrace{Q^{-1}\hat{\vec{R}}}\,|Q\vec{r}\rangle = \vec{r}\,|Q\vec{r}\rangle}$$

$$Q^{-1}\hat{\vec{R}} = \sum_{i=1}^{3} \hat{R}_i \left(Q^{-1}\vec{e}_i\right)$$

$$\hat{\vec{R}}' = Q^{-1}\hat{\vec{R}}$$

$$\boxed{\underline{D(Q)\hat{\vec{R}}D(Q)^{-1}} = Q^{-1}\hat{\vec{R}}}$$

$$\hat{\vec{R}} = \sum_{i=1}^{3}\hat{R}_i\,\vec{e}_i$$

$$Q^{-1}\hat{\vec{R}} = \sum_{i=1}^{3}\hat{R}_i\,Q^{-1}\vec{e}_i$$

Parity transformation

$P \in O(3)$: $P\vec{r} = -\vec{r}$

$$D(P)\,|\vec{r}\rangle = |P\vec{r}\rangle = |-\vec{r}\rangle$$

$$D(P) \equiv \hat{P}$$

$\{P,\ \mathbb{1}_{3\times3}\}$ is a subgroup in $O(3)$

$P^2 = \mathbb{1}$, $P\cdot\mathbb{1} = P = \mathbb{1}\cdot P$.

$\hat{P}|\psi\rangle = |\psi'\rangle$

$$\boxed{\langle \vec{r} | \psi' \rangle = \langle \vec{r} | \hat{P} | \psi \rangle = \langle -\vec{r} | \psi \rangle}$$

$$\langle \vec{r} | \hat{P} | \psi \rangle = \langle \hat{P}^{+} | \vec{r} \rangle, \psi \rangle = \langle \hat{P} | \vec{r} \rangle, \psi \rangle$$

$$\hat{P}^{+} = \hat{P}^{-1} \qquad \boxed{\begin{aligned} \hat{P}^2 &= D(P) D(P) = D(P^2) \\ &= D(\mathbb{1}) = \mathbb{1}_{\mathcal{H}} \end{aligned}}$$

$$\psi'(\vec{r}) = \psi(-\vec{r})$$

$$\boxed{\hat{P} : \psi(\vec{r}) \to \psi'(\vec{r}) = \psi(-\vec{r})}$$

$\hat{P}^2 = \mathbb{1}_{\mathcal{H}} \Rightarrow \hat{P}$ has eigenvalue ± 1

eigenstates $\psi_{\pm}$

$$\hat{P} \psi_{\pm}(\vec{r}) = \underline{\psi_{\pm}(-\vec{r}) = \pm \psi_{\pm}(\vec{r})}$$

ψ_{+} : even parity

ψ_{-} : odd parity

$\hat{P}^\dagger = \hat{P}^{-1} = \hat{P}$ $\quad$ $\hat{P}$ is Hermitian & unitary

transf. of operators under parity

$$\hat{\vec{R}} \to \hat{P}\,\hat{\vec{R}}\,\hat{P}$$

$$\hat{P}\,\hat{\vec{R}}\,\hat{P}\,|\vec{r}\rangle = \hat{P}\,\hat{\vec{R}}\,|-\vec{r}\rangle = (-\vec{r})\,\hat{P}\,|-\vec{r}\rangle$$

$$= (-\vec{r})\,|\vec{r}\rangle$$

$$= -\hat{\vec{R}}\,|\vec{r}\rangle$$

$$\hat{P}\,\hat{\vec{R}}\,\hat{P} = -\hat{\vec{R}}$$

momentum operator $\quad \hat{\vec{P}}\,|\vec{p}\rangle = \vec{p}\,|\vec{p}\rangle$

$$\langle \vec{r}|\vec{p}\rangle = e^{\frac{i}{\hbar}\vec{r}\cdot\vec{p}} = e^{\frac{i}{\hbar}(-\vec{r})\cdot(-\vec{p})}$$

$$= \langle -\vec{r}|-\vec{p}\rangle$$

$\hat{\vec{P}} \to \hat{P}\,\hat{\vec{P}}\,\hat{P}$ $\qquad$ $\hat{P}\,|\vec{p}\rangle = \hat{P}\underbrace{\int d^3\vec{r}\,|\vec{r}\rangle\langle\vec{r}|}_{}\vec{p}\rangle$

resolution of identity

$$= \int d^3\vec{r}\, |-\vec{r}\rangle\langle-\vec{r}|-\vec{p}\rangle$$

$$d^3\vec{r} = d^3(-\vec{r}) \qquad = \underbrace{\int d^3(-\vec{r})\, |-\vec{r}\rangle\langle-\vec{r}|}\,-\vec{p}\rangle$$

$$= |-\vec{p}\rangle$$

$$\hat{P}\hat{\vec{p}}\hat{P}|\vec{p}\rangle = \hat{P}\hat{\vec{p}}|-\vec{p}\rangle = (-\vec{p})\,P|-\vec{p}\rangle$$

$$= (-\vec{p})|\vec{p}\rangle = -\hat{\vec{p}}|\vec{p}\rangle$$

$$\Rightarrow \hat{P}\hat{\vec{p}}\hat{P} = -\hat{\vec{p}}$$

angular momentum operator: $\hat{\vec{L}} = \hat{\vec{R}} \times \hat{\vec{p}}$

$$\hat{P}\hat{\vec{L}}\hat{P} = \hat{P}(\hat{\vec{R}} \times \hat{\vec{p}})\hat{P} = \hat{P}\hat{\vec{R}}\hat{P} \times \hat{P}\hat{\vec{p}}\hat{P}$$

$$= (-\hat{\vec{R}}) \times (-\hat{\vec{p}})$$

$$= \hat{\vec{L}}$$

$$\hat{\vec{V}} = \sum_{i=1}^{3} \hat{V}_i \hat{e}_i$$

$\hat{\vec{V}}$ is vector operator if $\hat{P} \hat{\vec{V}} \hat{P} = -\hat{\vec{V}}$, like $\hat{\vec{R}}$, $\hat{\vec{P}}$

" " pseudo-vector operator if $\underline{\hat{P} \hat{\vec{V}} \hat{P} = \hat{\vec{V}}}$

like $\hat{\vec{L}}$, $\hat{\vec{S}}$

spin

$\hat{O}$ scalar operator if $\hat{P} \hat{O} \hat{P} = \hat{O}$ e.g.

Hamiltonian of central potential.

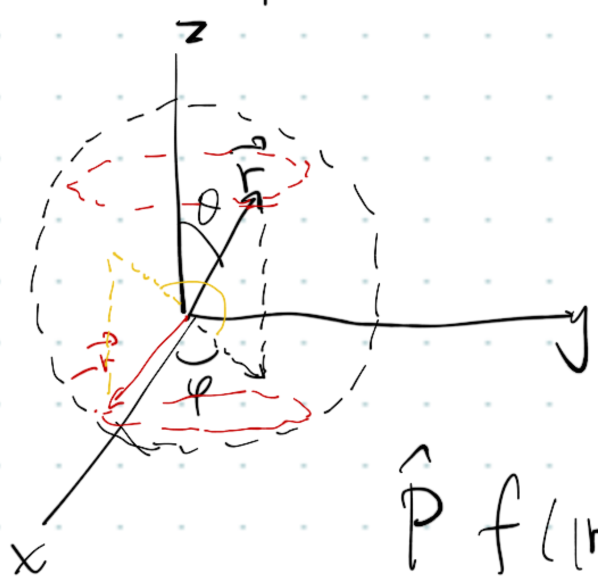
pseudo-scalar operator if $\hat{P} \hat{O} \hat{P} = -\hat{O}$ e.g.

helicity operator $\hat{\vec{S}} \cdot \hat{\vec{P}} / |\hat{\vec{P}}| \equiv \hat{h}$

because $\hat{P}\hat{\vec{L}}\hat{P} = \hat{\vec{L}}$ i.e. $[\hat{P}, \hat{\vec{L}}] = 0$

they share the same eigenbasis.

indeed in spherical coordinate.



$P: \vec{r} \to -\vec{r}$

$\theta \to \pi - \theta$

$\varphi \to \varphi + \pi$

$$\hat{P} f(|r|, \theta, \varphi) = f(|r|, \pi-\theta, \varphi+\pi)$$

$$\begin{cases} \hat{P} Y_{lm}(\theta, \varphi) = (-1)^l Y_{lm}(\theta, \varphi) \\ \hat{L}^2 Y_{lm}(\theta, \varphi) = \hbar^2 l(l+1) Y_{lm}(\theta, \varphi) \\ \hat{L}_z Y_{lm}(\theta, \varphi) = \hbar m Y_{lm}(\theta, \varphi) \end{cases}$$