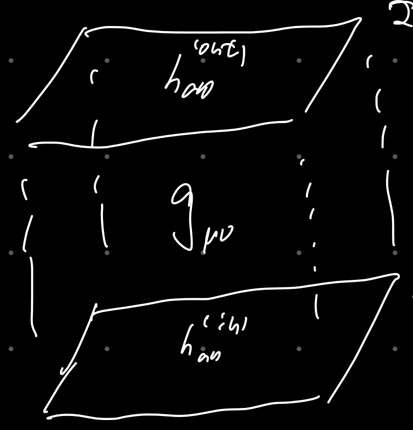
Covariant dynamics of LQG

Path integral of 4d gravity : $Z[h_{ab}^{(in)}, h_{ab}^{(out)}] = \int \mathcal{D}g_{\mu\nu}\, e^{\frac{i}{\ell_p^2}\int R + \cdots}$

$g_{\mu\nu}|_{\Sigma^{out}} = h_{ab}^{out}$

$g_{\mu\nu}|_{\Sigma^{in}} = h_{ab}^{(in)}$

Σ^{out} : 3d



M_4 = 4-manifold

Σ^{in} : 3d

$g_{\mu\nu}$: evolution of 3-geometry from $h_{ab}^{(in)}$ to $h_{ab}^{(out)}$

Z : sum over 4d metric w/ bdy condition $h_{ab}^{(in)}$ $h_{ab}^{(out)}$

$\parallel$

sum over history of 3d geometries.

LQG : 3d geometries are quantized

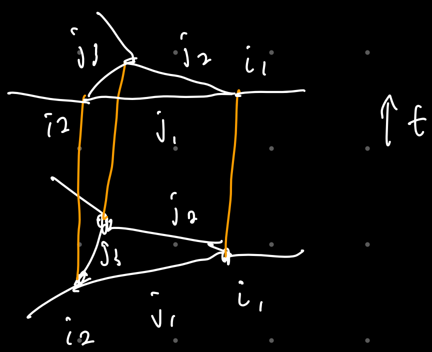
3d quantum geometry = spin-network states

(Γ, j, i)

$\uparrow$ graph $\quad \uparrow$ spin on links $\quad \uparrow$ intertwiner at nodes

quantum 4-geometry = history of spin-network.

= a spin foam

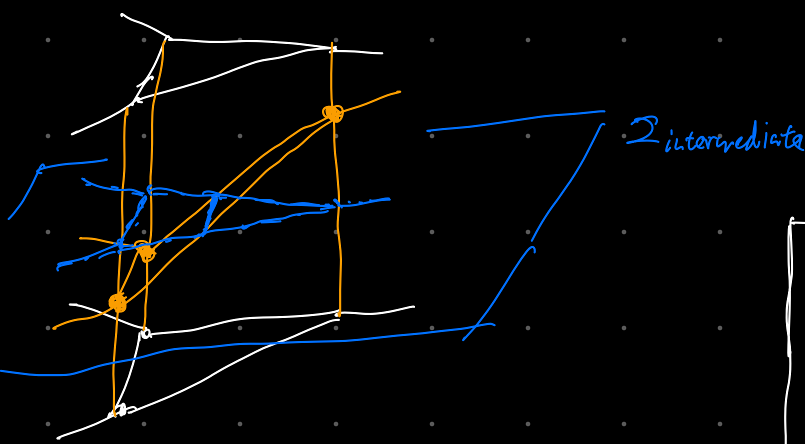


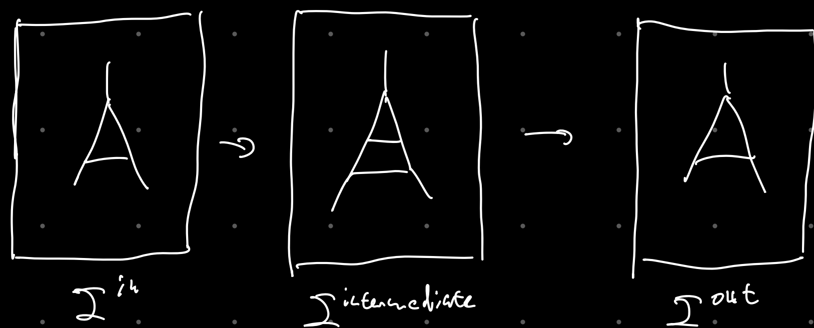
- evolution of link → spinfoam face

  spin j on link → spin j on face

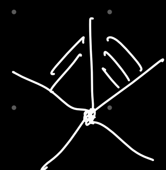
- evolution of node → spinfoam edge

  intertwiner i at node → intertwiner on edge





nontrivial process : spinfoam vertex (interaction among edges and faces)



→ creating 4d quantum geometry

Simple spinfoam model of BF theory

classical BF theory : G : matrix Lie group , M_D : D-manifold

$$S_{BF} = \int_{M_D} \mathrm{tr}\Big(\underbrace{B}_{\mathfrak{g}\text{-valued } (D-2)\text{-form}} \wedge \underbrace{F(A)}_{\text{curvature 2-form of } \mathfrak{g}\text{-connection } A} \Big)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

BF theory is a topological field theory

$$\delta S_{BF} = 0 \Rightarrow d_A B = 0, \quad F(A) = 0$$

gauge inv: $\begin{cases} B \to g B g^{-1} \\ A \to g^{-1} A g + g^{-1} dg \end{cases}$ $g(x) \in G$

$\begin{cases} B \to B + d_A f \\ A \to A \end{cases}$ D-3 form

e.g. $M_D = \mathbb{R}^D$, flat conn. $\Rightarrow A = 0$ up to gauge transf.

$$0 = d_A B = dB \quad B \text{ is close}$$

$$\Rightarrow B \text{ is exact}; \ B = df$$

$$\Rightarrow B = 0 \quad \text{up to gauge transf.}$$

<u>quantum theory</u>:

$$Z = \int DB\, DA\, e^{i \int_{M_D} \mathrm{Tr}(B \wedge F)} = \int DA\, \delta(F(A))$$

integral over flat connections on M.

Consider loop holonomy in M_D

ℓ : loop

S : 2-surface

$\partial S = \ell$

$$h_\ell(A) = \mathcal{P} e^{\oint_\ell A} = \mathcal{P}_S e^{\int_S g F g^{-1}}$$

non abelian Stokes Thm.

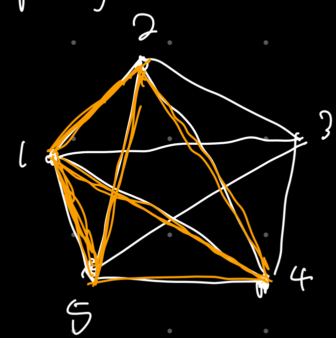
$$F(A) = 0 \iff h_\ell(A) = \mathbb{1} \quad \forall \ell$$

$$Z = \int \mathcal{D}A \prod_\ell \delta_G(h_\ell(A))$$

$D = 4$, $G = SU(2)$

simplicial complex in 4d (triangulation of 4-manifold)

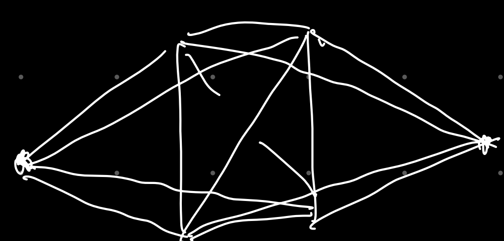
elementary cell in 4d triangulation : 4-simplex



Boundary of 4-simplex : $C_5^4 = 5$ tetrahedra

$C_5^3 = 10$ triangles

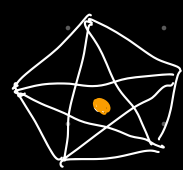
4d simplical complex : glue 4-simplices by identifying a pair of tetrahedra



simplicial complex associates a unique dual 2-complex

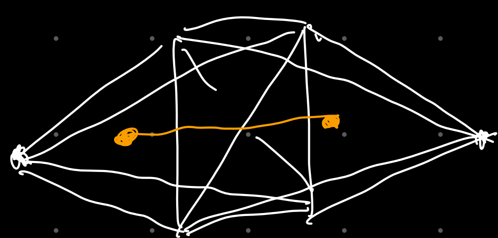
simplicial complex			dual 2-complex	
4-simplex	4d object	$\longleftrightarrow$	0d object	vertex
tetrahedron	3d object	$\longleftrightarrow$	1d object	edge
triangle	2d object	$\longleftrightarrow$	2d object	face

4d - 0d : 4 - simplex $\longleftrightarrow$ a vertex at the center of the 4-simplex



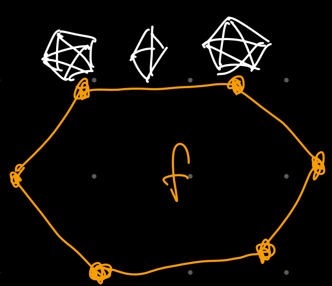
3d - 1d : tetrahedron shared by 2 4-simplices

$\longleftrightarrow$ edge connecting 2 vertices



2d - 2d : triangle shared by N 4-simplices

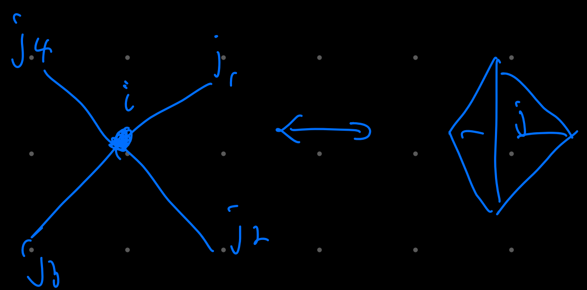
$\longleftrightarrow$ face with N vertices along ∂f



Example :

an edge connecting 2 vertices $\Longleftrightarrow$ a tetrahedron shared by 2 4-simplices

4 faces connecting to the edge $\Longleftrightarrow$ 4 triangles on the bdy of tetrahedron

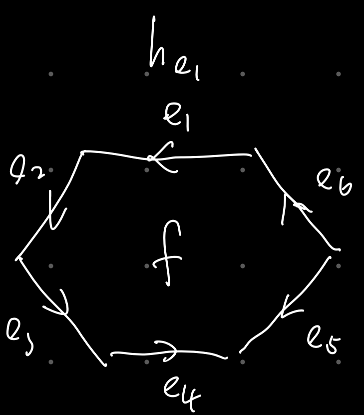


Regularized path integral: $Z = \int DA \prod_{l} \delta_G(h_l(A))$

$$\to \int DA \prod_{f} \delta_G(h_{\partial f}(A))$$

Haar measure

faces in 2-complex



$$DA = \prod_{e} d\mu_H(h_e(A))$$

edges in 2-complex

$L^2(G)$

$\pi^j_{mn}(h)$

$G = SU(2)$

$$\delta_G(h) = \sum_{j=0}^{\infty} \dim(j)\, \chi_j(h) \qquad \chi_j = tr_j(h)$$

$$\chi_j(h_{\partial f}) = tr_j(h_{\partial f}) = \pi^j_{mk}(h_{e_n})\, \pi^j_{kl}(h_{e_{n-1}}) \cdots \pi^j_{pm}(h_{e_1})$$

$$Z = \int \prod_e d\mu_{H}(h_e) \prod_f \sum_{j_f=0}^{\infty} \dim(j_f)\, \pi^{j_f}_{n_f k_f}(h_{e_n})\, \pi^{j_f}_{k_f l_f}(h_{e_{n-1}}) \cdots \pi^{j_f}_{p_f n_f}(h_{e_1})$$

$$\int dh_e\, \pi^{j_1}_{m_1 n_1}(h_e)\, \pi^{j_2}_{m_2 n_2}(h_e)\, \pi^{j_3}_{m_3 n_3}(h_e)\, \pi^{j_4}_{m_4 n_4}(h_e)$$

$$= P^{j_1 \cdots j_4}_{m_1 \cdots m_4\, n_1 \cdots n_4} = \sum_i \underline{i^*_{m_1 \cdots m_4}}\ \underline{i_{n_1 \cdots n_4}}$$

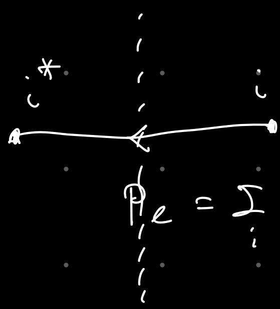
is a projection operator onto the intertwiner space $\mathrm{Inv}_{SU(2)}(V_{j_1} \otimes V_{j_2} \otimes V_{j_3} \otimes V_{j_4})$

i is basis of intertwiner

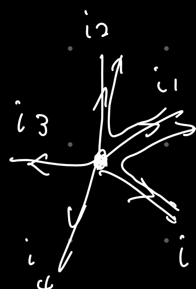
$\Rightarrow$ each edge is associated w/ a projection $\left(P_e^{j_1 \cdots j_4}\right)_{\vec{m}, \vec{n}}$

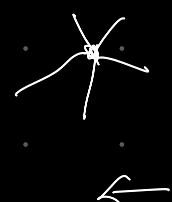
$$Z = \sum_{\{j_f\}} \prod_f \dim(j_f)\ \mathrm{tr}\left(\prod_e P_e\right)$$

contracting indices acording to faces



$$P_e = \sum_i i^* i$$

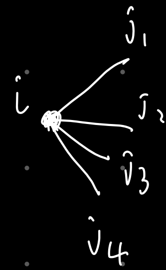




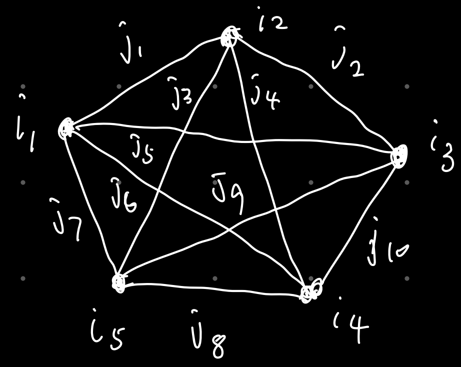
2 edge $\to$ a face

5 edge $\to$ 10 faces

$\to$ 10 contractions



$\equiv i^{j_1 \cdots j_4}_{m_1 \cdots m_4}$



$\equiv \{15j\}$

15j symbol

$$Z = \sum_{j_f, i_e}^{\infty} \prod_f \dim(j_f) \prod_v \{15j\}$$

Ooguri model

face amplitude

vertex amplitude "topological invariant"

EPRL model

classical : Plebanski formulation of gravity

$I, J = 0, \cdots 3$

$$S_{pl} = \int_{M_4} \left(B^{IJ} + \frac{1}{r} \varepsilon^{IJ}{}_{KL} B^{KL} \right) \wedge F_{IJ} + \varphi^{IJKL} B_{IJ} \wedge B_{KL}$$

Barbero - Immirzi

Lagrangian multiplier

B^{IJ}, F^{IJ} are $sl(2, \mathbb{C})$ -valued

φ^{IJKL} anti symm

$\varphi^{IJKL} = \varphi^{KLIJ}$

$F^{IJ} = F^{IJ}(A)$

$\varphi^{IJKL} \epsilon_{IJKL} = 0$

$\delta S_{pl} = 0 \Rightarrow$ Einstein eqn.

$$Z_{pl} = \int DA DB D\varphi \, e^{\frac{i}{l_p^2} S_{pl}}$$

$V = \frac{1}{4!} \varepsilon_{IJKL} B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma} \varepsilon^{\mu\nu\rho\sigma}$

integrate out φ $\rightarrow$ Simplicity constraint

$$\varepsilon^{\mu\nu\rho\sigma} B^{IJ}_{\mu\nu} B^{KL}_{\rho\sigma} = V \varepsilon^{IJKL}$$

Solution:
$$\begin{cases} \mathrm{I}: & \exists\, e^{I}_{\mu} \text{ s.t. } B^{IJ} = \pm\, e^{I}\wedge e^{J} \\ \mathrm{II}: & \exists\, e^{I}_{\mu} \text{ s.t. } B^{IJ} = \pm\, \varepsilon^{IJ}{}_{KL}\, e^{K}\wedge e^{L} \\ \text{deg}: & V = 0 \end{cases}$$

$$\underline{Z_{pl} = \int DADB\; \underline{\delta(\text{simplicity})}\; e^{\frac{i}{l_p^2}\int_{M_4}\left(B^{IJ}+\frac{1}{\gamma}*B^{IJ}\right)\wedge F_{IJ}}}$$

gravity is a constrained BF theory.

<u>spinfoam quantization on simplicial complex in 4d</u>

1. put Z_{pl} in a single 4-simplex. the result is the spinfoam vertex amplitude

2. firstly quantize BF theory,

A

$$\int DADB\; e^{\frac{i}{l_p^2}\int \overbrace{\underline{\left(B^{IJ}+\frac{1}{\gamma}*B^{IJ}\right)}}^{J}\wedge F_{IJ}}$$

$$= Z_{BF}(\text{bdy data})$$

$\uparrow$

bdy Hilbert space:

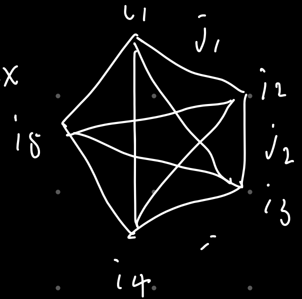
Hilbert space of bdy data

3. quantize simplicity constraint & impose to BF bdy Hilbert space

It gives restriction to bdy data.

4. $A_v = Z_{BF}\left(\text{bdy data}\Big|_{\text{simplicity}}\right)$

SU(2) BF : vertex



$= \{15j\}$ 5 tetrahedra : $i_{a=1\cdots5}$

10 triangles : j_{ab} $a<b$

bdy Hilbert space : $L^2(SU(2)^{10})/\text{gauge}$

SL(2, ℂ) BF : bdy Hilbert space $L^2(SL(2,\mathbb{C})^{10})/\text{gauge}$

bdy data 5 tetrahedra : SL(2, ℂ) intertwiners

10 triangles : SL(2, ℂ) irrep.

Quantization of B-field : discretization $B_f^{IJ} = \int_f B^{IJ}$

$\uparrow$ triangle of 4-simplex

$J_f^{IJ} = B_f + \frac{1}{\gamma} * B_f$ is quantized as the right inv. vector field on SL(2, ℂ)

$J_f^{0i} \equiv K_f^i$ boost generator

$\frac{1}{2} J_f^{kl} \varepsilon^j{}_{kl} = L_f^j$ rotation generator

linear simplicity constraint : For each tetrahedron, ∃ internal timelike vector N^I

(discretization & modification of $B = *(e\wedge e)$) s.t. ∀ f triangle of the tetrahedron

$N_I * B_f^{IJ} = 0$

EPRL 0711.0146

time gauge $\quad N_I = (1, 0, 0, 0)$

$$* \hat{B}^{0i}_f = 0 \quad \forall f \subset \text{tetrahedron}$$

$$\rightarrow \text{"}\underbrace{\left(\hat{L}^i_f + \frac{1}{\gamma}\hat{K}^i_f\right)}_{\hat{C}^i_f} \Psi = 0\text{"}$$

However $\hat{C}^i_f$ is 2nd class $\quad [\hat{C}^i_f, \hat{C}^j_f] = \varepsilon^{ijk}\left(L^k - \frac{1}{\gamma^2}L^k - \frac{1}{\gamma}K^k\right)$

nonvanishing on constraint surface.

We have to impose $\hat{C}^i_f$ weakly

EPRL reformulation : $\quad \hat{L}^i_f + \frac{1}{\gamma}\hat{K}^i_f = 0$

EPRL paper & Ding, Han, Rovelli 1011.2149 $\Rightarrow$

(1) $\quad M_f = \left(\vec{L}_f + \frac{1}{\gamma}\vec{K}_f\right)^2 = 0$

(2) $\quad C_{ff} = *B_f \cdot B_f = *J_f \cdot J_f\left(1 - \frac{1}{\gamma^2}\right) + \frac{2}{\gamma} J_f \cdot J_f = 0$

$X \cdot Y = X^{IJ} Y_{IJ}$

in terms of $\underbrace{SL(2,\mathbb{C})}$ Casmir
quadratic

$$\hat{C}_1 = J \cdot J = 2(L^2 - K^2)$$

$$\hat{C}_2 = *J \cdot J = -4 L \cdot K$$

$$\hat{C}_{ff} = \hat{C}_2\left(1 - \frac{1}{\gamma^2}\right) + \frac{2}{\gamma}\hat{C}_1 = 0$$

$$\hat{M}_f = 0 \Rightarrow \hat{L}^2\left(1 + \frac{1}{\gamma}\right) - \frac{1}{2\gamma^2}\hat{C}_1 - \frac{1}{2\gamma}\hat{C}_2 = 0$$

$$\Rightarrow \quad \hat{C}_2 = 4\gamma \hat{L}^2 \qquad (*)$$

unitary irrep of $SL(2,\mathbb{C})$
(principal series)

$\mathcal{H}_{(\rho,n)}$ infinite-dim Hilbert space

$\rho \in \mathbb{R}$, $n \in \mathbb{N}_0$

$SU(2) \subset SL(2,\mathbb{C})$

$\mathcal{H}_{(\rho,n)}$ is reducible by $SU(2)$

$$\mathcal{H}_{(\rho,n)} = \bigoplus_{j=\frac{n}{2}}^{\infty} V_j \qquad \text{canonical basis } |\rho, n, j, m\rangle$$

quadratic casimir

$$C_1 = \frac{1}{2}(n^2 - \rho^2 - 4)$$

$$C_2 = n\rho$$

$$\hat{C}_{ff} = 0 \quad \Rightarrow \quad n\rho\left(1 - \frac{1}{\gamma^2}\right) + \frac{1}{\gamma}(n^2 - \rho^2) = \frac{4}{\gamma}$$

$$(*) \quad \Rightarrow \quad n\rho = 4\gamma j(j+1)$$

Solution: $\rho = 2\gamma j\left[1 + O\left(\frac{1}{j}\right)\right]$, $n = 2j\left[1 + O\left(\frac{1}{j}\right)\right]$

we ignore $O\left(\frac{1}{j}\right)$ since it does not affect semiclassical limit

As the result, EPRL projection

(1) restrict principle series irrep $(\rho_f, n_f) = (2\gamma j_f, 2j_f)$

(2) Y-map, $Y: V_{j_f} \to \mathcal{H}_{(2\gamma j_f, 2j_f)} = \bigoplus_{k=j_f}^{\infty} V_k$

by identifying to the lowest level.

$$Y : |j_f, m_f\rangle \mapsto |(2\gamma j_f, 2j_f), j_f, m_f\rangle$$

EPRL intertwiner : restriction of $SL(2,\mathbb{C})$ intertwiner

Given $SU(2)$ intertwiner $i \in \mathrm{Inv}_{SU(2)}(V_{j_1} \otimes \cdots V_{j_4})$

$Y^{\otimes 4} \circ i$ is an $SL(2,\mathbb{C})$ tensor

$I(i) = P \circ Y^{\otimes 4} \circ i$ is EPRL intertwiner

$\uparrow$ projection onto $SL(2,\mathbb{C})$ intertwiners

Explicitly, $D^{(\rho,n)}_{j'm', jm}(g) = \langle \rho, n, j', m' | g | \rho, n, j, m\rangle$

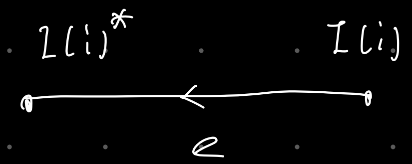
$\forall g \in SL(2,\mathbb{C})$

$$I(i) = \underline{\int_{SL(2,\mathbb{C})} dg} \prod_{i=1}^{4} D^{(2\gamma j_i,\, 2j_i)}_{j_i' m_i',\, \underline{j_i}\, \underline{m_i}}(g)\; i^{j_1 \cdots j_4}_{\underline{m_1 \cdots m_4}}$$

Contraction with Y-map

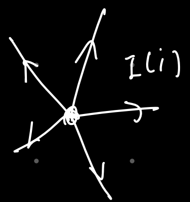
"Edge projector" : $P_e = \sum_i I(i)^* I(i)$ KKL 0909.0939 0912.0540

$\uparrow$ SU(2) intertwiner



EPRL spinfoam amplitude

$$Z = \sum_{\{j_f\}} \prod_f \dim(j_f)\, \mathrm{tr}\left(\prod_e P_e\right)$$



vertex amplitude $A_v = \mathrm{tr}\left(I(i_1) \otimes \cdots \otimes I(i_5)\right)$

$$Z_{EPRL} = \sum_{\{j_f, i_e\}} \prod_f \dim(j_f) \prod_v A_v(\vec{j}, \vec{i})$$

More on A_v

$$A_v = \int_{SL(2,\mathbb{C})} \prod_{a=1}^{5} dg_a \prod_{a<b} \langle j_{ab}\, \underline{m_{ab}} | Y^\dagger g_a^{-1} g_b Y | j_{ab}\, \underline{m_{ba}} \rangle$$

$$= \int_{SL(2,\mathbb{C})} \prod_{a=1}^{5} dg_a \prod_{a<b} \langle (2\gamma j_{ab}, 2j_{ab})\, j_{ab}\, m_{ab} | \underline{g_a^{-1} g_b} | (2\gamma j_{ab}, 2j_{ab})\, j_{ab}\, m_{ba} \rangle$$

$SL(2,\mathbb{C})$ gauge transf. $\quad g_a \to x g_a \quad x \in SL(2,\mathbb{C})$

cause the divergence of A_v

gauge fixing : $\displaystyle\int \prod_{a=1}^{5} dg_a \cdots \to \int \prod_{a=1}^{5} dg_a\, \delta(g_5) \cdots$

the resulting A_v is finite $\qquad$ 0805.4696 , 1010.5384

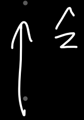
<u>Coherent state representation</u>

$V_j \circlearrowleft SU(2)$ $\qquad$ highest weight state $|j, j\rangle$

coherent state $|j,\xi\rangle = \begin{pmatrix}\xi^0 & -\bar\xi^1 \\ \xi^1 & \bar\xi^0\end{pmatrix} \triangleright |j,j\rangle$

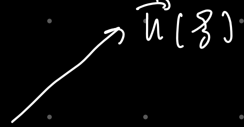
$\xi = \begin{pmatrix}\xi^0 \\ \xi^1\end{pmatrix}$ satisfying

$|j\,j\rangle$:



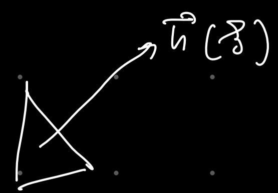
$\langle \xi, \xi\rangle = \bar\xi^0\xi^0 + \bar\xi^1\xi^1 = 1$

$|j\,\xi\rangle$:



$\vec n(\xi) = \langle \xi | \vec\sigma | \xi\rangle$

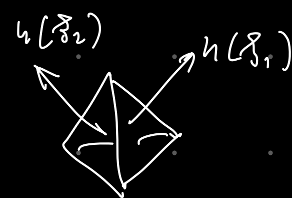
$\vec n(\xi)$ is the 3-normal of tetrahedron face



coherent intertwiner

0705.0674

$$\int dh \prod_{i=1}^{4} h\,|j_i\,\xi_i\rangle$$



peaked at $\sum_{i=1}^{4} j_i\,\vec n(\xi_i) = 0$

$$A_v(j\,\xi) = \int_{SL(2,\mathbb{C})} \prod_{a=1}^{5} dg_a\,\delta(g_5) \prod_{a<b} \langle j_{ab}\,\xi_{ab} | Y^\dagger g_a^{-1} g_b Y | j_{ab}\,\xi_{ba}\rangle$$

& for $\mathcal{H}_{(2\gamma j,\,2j)}$

$\mathcal{H}_{(\rho,n)}$ c space of homogeneous func. of $(z^0, z^1) \equiv z$

$f(z,\bar z)$: $f(\lambda z, \bar\lambda \bar z) = \lambda^{-1+\frac{i\rho}{2}+\frac{n}{2}}\,\bar\lambda^{-1+\frac{i\rho}{2}-\frac{n}{2}}\,f(z,\bar z)$

L^2 inner product

$$\langle f, f' \rangle = \int_{\mathbb{CP}^1} \Omega_{z\bar{z}} \, \overline{f(z,\bar{z})} \, f'(z,\bar{z})$$

$$\Omega_{z\bar{z}} = \frac{i}{2} (z^0 dz^1 - z^1 dz^0) \wedge (\bar{z}^0 d\bar{z}^1 - \bar{z}^1 d\bar{z}^0)$$

$$Y|j\, j\rangle = |2\gamma j\; 2j, j, j\rangle : \sqrt{\frac{\dim(j)}{\pi}} \langle \bar{z}, z\rangle^{i\gamma j - 1 - j} (z^0)^{2j}$$

$$Y|j\, \xi\rangle = |2\gamma j\; 2j, j, j\rangle : \sqrt{\frac{\dim(j)}{\pi}} \langle \bar{z}, z\rangle^{i\gamma j - 1 - j} \langle \bar{z}, \xi\rangle^{2j}$$

$$\equiv f_\xi$$

$$\langle j_{ab}\, \xi_{ab} | Y^\dagger g_a^{-1} g_b Y | j_{ab}\, \xi_{ba} \rangle = \int_{\mathbb{CP}} \Omega_{zz'} \, \overline{f_{\xi_{ab}}(g_a^t z, \overline{g^t z})}$$

$$f_{\xi_{ba}}(g_b^t z, \overline{g_b^t z})$$

$$= \frac{\dim(j_{ab})}{\pi} \int_{\mathbb{CP}^1} \Omega_{z\bar{z}} \langle g_a^t \bar{z}, g_a^t z\rangle^{-i\gamma j_{ab} - 1 - j_{ab}} \langle \xi_{ab}, g_a^t \bar{z}\rangle^{2j_{ab}}$$

$$\langle g_a^t \bar{z}, g_a^t \bar{z}\rangle^{i\gamma j_{ab} - 1 - j_{ab}} \langle g_a^t \bar{z}, \xi_{ba}\rangle^{2j_{ab}}$$

$$= \frac{\dim(j_{ab})}{\pi} \int_{\mathbb{CP}^1} dz \; e^{S_{ab}}$$

$$dz = \frac{\Omega_{z\bar{z}}}{\langle Z_{ab}\, Z_{ab}\rangle \langle Z_{ba}, Z_{ba}\rangle} \qquad \begin{aligned} Z_{ab} &= g_a^\dagger \bar{z} \\ Z_{ba} &= g_b^\dagger \bar{z} \end{aligned}$$

$$S_{ab} = 2j_{ab} \ln \frac{\langle \xi_{ab} Z_{ab} \rangle \langle Z_{ba} \xi_{ba} \rangle}{\| Z_{ab} \| \, \| Z_{ba} \|} + i 2\gamma j_{ab} \ln \frac{\| Z_{ba} \|}{\| Z_{ab} \|}$$

$$A_v = \frac{\prod_{a<b} \dim(j_{ab})}{\pi^{10}} \int \prod_a dg_a \prod_{a<b} dz_{ab} \, \delta(g_5) \, e^{S}$$

$$S = \sum_{a<b} S_{ab}$$