1. Symmetry of QM

2. Theory of Angular momentum.

3. Scattering theory.

Symmetry of transformation of space $\mathbb{R}^3$

wave function $\psi(\vec{r})$ $\quad \vec{r} \in \mathbb{R}^3$.

consider linear transf. $Q: \mathbb{R}^3 \to \mathbb{R}^3$

$$Q\vec{r} = \vec{r}' \qquad \vec{r}: \text{3d column vector}$$

$$Q: 3\times 3 \text{ matrix}$$

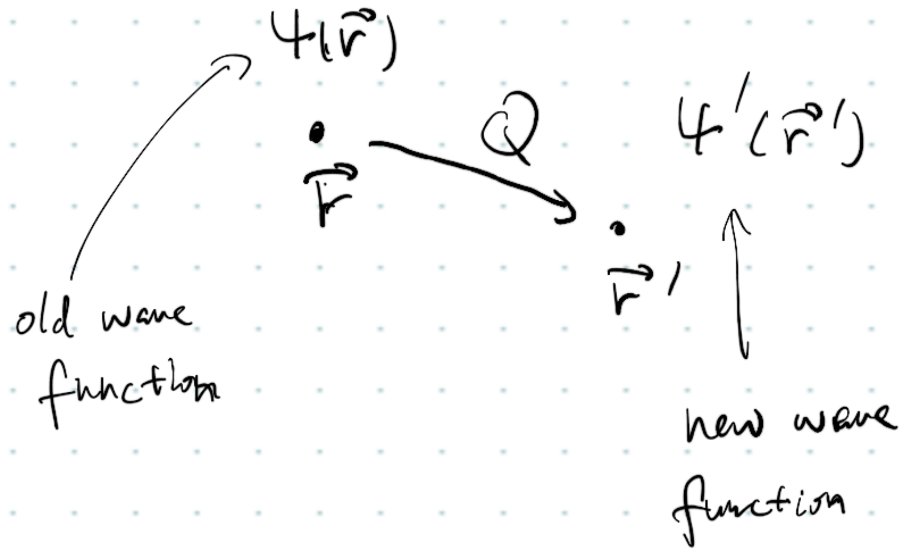
Q is a symmetry of the space, if it preserves the inner product

$$\vec{r}_1 \cdot \vec{r}_2 = (Q\vec{r}_1)\cdot(Q\vec{r}_2)$$

in particular, Q preserves the distance in $\mathbb{R}^3$ ✓

$$\vec{r}\cdot\vec{r} = (Q\vec{r})\cdot(Q\vec{r})$$

Q induces transformation of $\psi(\vec{r})$



s.t. $\psi'(\vec{r}\,') = \psi(\vec{r})$

$$\underset{\vec{r}\,'}{\underbrace{\psi'(Q\vec{r})}} = \underset{Q^{-1}\vec{r}\,'}{\underbrace{\psi(\vec{r})}}$$

$$\boxed{\begin{aligned}\psi'(\vec{r}) &= \psi(Q^{-1}\vec{r}) \\ &\equiv \hat{D}(Q)\psi(\vec{r})\end{aligned}}$$

$Q : \mathbb{R}^3 \to \mathbb{R}^3$

$\hat{D}(Q) : \mathcal{H} \to \mathcal{H} \qquad \mathcal{H} = L^2(\mathbb{R}^3, d^3x)$

$|\psi\rangle \to |\psi'\rangle \qquad \psi'(\vec{r}) = D(Q)\,\psi(\vec{r})$

$$= \psi(Q^{-1}\vec{r})$$

properties of $D(Q)$: linear operator on $\mathcal{H}$.

- unitarity : $\langle \psi_1 | \psi_2 \rangle = \int d^3\vec{r}\, \overline{\psi_1(\vec{r})}\, \psi_2(\vec{r})$

change of variable $\vec{r} \to Q^{-1}\vec{r}$

$$= \int d^3(Q^{-1}\vec{r})\, \overline{\psi_1(Q^{-1}\vec{r})}\, \psi_2(Q^{-1}\vec{r})$$

<u>Lemma</u> $\det Q = \pm 1$

Pf: $\vec{r}\cdot\vec{r}' = \underbrace{\delta_{ij} r^i r'^j}$

$\quad\quad \| $

$Q\vec{r}\cdot Q\vec{r}' = \underbrace{\delta_{ij} Q^i{}_k r^k Q^j{}_l r'^l}$

$\Rightarrow \delta_{ij} Q^i{}_k Q^j{}_l = \delta_{kl}$

$\Updownarrow$

$Q^T Q = 1_{3\times 3}$

$$\Rightarrow Q^T = Q^{-1}, \quad (\det Q)^2 = 1$$

$$\det Q = \pm 1$$

$$d^3(Q^{-1}\vec{r}) = d^3\vec{r}\,|\det Q^{-1}| = d^3\vec{r}$$

$$\langle \psi_1 | \psi_2 \rangle = \int d^3\vec{r}\, \overline{\psi_1(Q^{-1}\vec{r})}\, \psi_2(Q^{-1}\vec{r})$$

$$= \langle D(Q)\psi_1 | D(Q)\psi_2 \rangle$$

$\Rightarrow$ $D(Q)$ is unitary operator.

- Consider 2 transformations $Q_1, Q_2 : \mathbb{R}^3 \to \mathbb{R}^3$

$$Q = Q_1 \circ Q_2 \qquad Q\vec{r} = Q_1 \circ Q_2 \vec{r}$$

$$D(Q_1)\, D(Q_2)\, \psi(\vec{r}) = D(Q_1)\, \psi(Q_2^{-1}\vec{r})$$
$$= \psi(Q_2^{-1}(Q_1^{-1}\vec{r}))$$
$$= \psi(Q_2^{-1}Q_1^{-1}\vec{r}) = \psi((Q_1Q_2)^{-1}\vec{r})$$
$$= D(Q_1Q_2)\, \psi(\vec{r})$$

$$\boxed{D(Q_1)\, D(Q_2) = D(Q_1Q_2)}$$

if we view D to be a map from symmetry transf. on $\mathbb{R}^3$ to linear operators on $\mathcal{H}$.

D respects the product of transf.

- $D(Q)\, D(Q^{-1}) = D(Q Q^{-1}) = D(1_{3\times3})$

$$D(1)\, \psi(\vec{r}) = \psi(1\vec{r}) = \psi(\vec{r})$$
$$D(1) = 1_{\mathcal{H}}$$

$$D(Q)D(Q^{-1}) = 1_{H}$$

$$\boxed{D(Q)^{-1} = D(Q^{-1})}$$ D respects the inverse.

<u>Def</u> A group is a set G together with a binary operation

$a \cdot b \in G \quad \forall a, b \in G$

called group multiplication

s.t. (1) $(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \forall a, b, c \in G$

(associativity)

(2) $\exists!\, e \in G$, s.t. $e \cdot a = a \quad \forall a \in G$

(identity)

(3) $\forall a \in G$, $\exists\, a^{-1} \in G$ s.t. $a \cdot a^{-1} = a^{-1} \cdot a = e$

(inverse)

<u>Example</u> (1) $\mathbb{R}^{1} \setminus \{0\}$ with multiplication

(2) set of symmetry transf. Q's on $\mathbb{R}^3$

multiplication ; composition of Q_1, Q_2

- Q ; 3×3 matrix $\rightarrow$ associativity
- identity $\mathbb{1}_{3\times3}$
- inverse $Q^T Q = \mathbb{1}$ $\quad Q^{-1} = Q^T$

the set of all 3×3 matrices with $Q^{-1} = Q^T$

is called $O(3)$ group.

(orthogonal group on $\mathbb{R}^3$)

(3) We have found a set of $D(Q)$ unitary operators on $\mathcal{H}$

$$\{ D(Q) \mid Q \in O(3) \} \equiv D(O(3))$$

multiplication, composition of operators $D(Q_1)D(Q_2)$
(product)

- $D(Q)$ are operators $\rightarrow$ associativity
- identity operators $D(\mathbb{1}_{3\times3}) = \mathbb{1}_{\mathcal{H}}$

- inverse $D(Q^{-1}) = D(Q)^{-1}$

$$\underbrace{D(Q_1)D(Q_2)}_{\text{multiplication on } D(O(3))} = D(\underbrace{Q_1 Q_2}_{\text{multiplication on } O(3)})$$

<u>Def</u>, given 2 groups G, H, a group homomorphism is a map $h: G \to H$

s.t. $\underbrace{h(g_1)\,h(g_2)}_{\text{multiplication in } H} = h(\underbrace{g_1 g_2}_{\text{multiplication in } G}) \quad \forall g_1, g_2 \in G$

$D: O(3) \to D(O(3))$ is a homomorphism.

in terms of Dirac bra-ket , $|\psi\rangle \in \mathcal{H}$

$$\psi(\vec{r}) := \langle \vec{r} | \psi \rangle \qquad \hat{\vec{R}} |\vec{r}\rangle = \vec{r} |\vec{r}\rangle$$

$$\psi'(\vec{r}) := \langle \vec{r} | \psi' \rangle$$

$$|\psi'\rangle = D(Q) |\psi\rangle$$

$$\psi'(\vec{r}) = \langle \vec{r} | D(Q) | \psi \rangle = \langle D(Q)^{+} \vec{r} | \psi \rangle$$

$$\psi(Q^{-1}\vec{r}) = \langle Q^{-1} \vec{r} | \psi \rangle \qquad \langle D(Q^{-1}) \vec{r} | \psi \rangle ?$$

$$\langle \psi | \hat{O} | \phi \rangle = \langle \hat{O}^{+} \psi | \phi \rangle$$

$$D(Q)^{+} = D(Q)^{-1} = D(Q^{-1})$$

$$D(Q^{-1}) |\vec{r}\rangle = |Q^{-1} \vec{r}\rangle$$

$$\boxed{D(Q) |\vec{r}\rangle = |Q \vec{r}\rangle}$$

$$D(Q)\,\psi(\vec r) = \psi(Q^{-1}\vec r)$$

$$Q \longrightarrow D(Q) \longrightarrow \text{transformations of operators on } \mathcal{H}$$

Q: transf on $\mathbb{R}^3$

$D(Q)$: transf on $\mathcal{H}$

$$\begin{array}{ccccc} & \hat A\, D(Q)^{-1}|\psi\rangle & & |\psi\rangle \\ D(Q): & \mathcal{H}_{(1)} & \longrightarrow & \mathcal{H}_{(2)} \\ & \circlearrowright & & \circlearrowright \\ & \hat A & & \hat A' \\ & \uparrow & & \uparrow \\ & \text{old} & & \text{new} \end{array}$$

$$\forall\, |\psi\rangle \in \mathcal{H}_{(2)}, \quad D(Q)^{-1}|\psi\rangle \in \mathcal{H}_{(1)}$$

$$\hat A\, D(Q)^{-1}|\psi\rangle \in \mathcal{H}_{(1)}$$

$$\left(D(Q)\,\hat A\, D(Q)^{-1}\right)|\psi\rangle \in \mathcal{H}_{(2)}$$

$$\hat{A}' := D(Q)\,\hat{A}\,D(Q)^{-1}$$

$\hat{A} \mapsto \hat{A}'$ transf. of operators induced by Q.

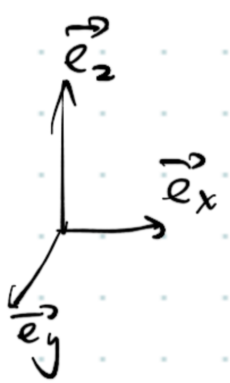
$Q \longrightarrow D(Q) \longrightarrow$ transf. of operators

$Q\vec{r} = \vec{r}'$ $\quad D(Q)|\psi\rangle = |\psi'\rangle$ $\quad D(Q)\,\hat{A}\,D(Q)^{-1} = \hat{A}'$

conjugate.

position operator: $\hat{\vec{R}}$ $\quad \hat{\vec{R}}\,|\vec{r}\rangle = \vec{r}\,|\vec{r}\rangle$

$$\hat{\vec{R}} = \hat{R}_1\vec{e}_x + \hat{R}_2\vec{e}_y + \hat{R}_3\vec{e}_z = \sum_{i=1}^{3}\hat{R}_i\,\vec{e}_i$$



$$\hat{R}_i\,|\vec{r}\rangle = r_i\,|\vec{r}\rangle$$

$i = 1, 2, 3$

$$\hat{\vec{R}} \to D(Q)\,\hat{\vec{R}}\,D(Q)^{-1} \equiv \hat{\vec{R}}'$$

$$\hat{\vec{R}}\,|\vec{r}\rangle = \vec{r}\,|\vec{r}\rangle$$

$D(Q)$ $D(Q)^{-1}D(Q)$

$$\Rightarrow \underbrace{D(Q)\,\hat{\vec{R}}\,D(Q)^{-1}}\,D(Q)\,|\vec{r}\rangle = D(Q)\,\vec{r}\,|\vec{r}\rangle$$

$$\hat{\vec{R}}'\,\underbrace{D(Q)|\vec{r}\rangle}_{\text{new eigenstate}} = \vec{r}\,\underbrace{D(Q)|\vec{r}\rangle}$$

$$\underline{\hat{\vec{R}}'\,|Q\vec{r}\rangle = \vec{r}\,|Q\vec{r}\rangle}$$

On the other hand, $\hat{\vec{R}}\,|Q\vec{r}\rangle = Q\vec{r}\,|Q\vec{r}\rangle$

$$\Rightarrow \underline{Q^{-1}\,\hat{\vec{R}}\,|Q\vec{r}\rangle = \vec{r}\,|Q\vec{r}\rangle}$$

$$Q^{-1}\hat{\vec{R}} = \sum_{i=1}^{3} \hat{R}_i \left(Q^{-1}\vec{e}_i\right)$$

$$\hat{\vec{R}}' = Q^{-1}\hat{\vec{R}}$$

$$\boxed{\underline{D(Q)\hat{\vec{R}}D(Q)^{-1}} = Q^{-1}\hat{\vec{R}}}$$

$$\hat{\vec{R}} = \sum_{i=1}^{3}\hat{R}_i\,\vec{e}_i$$

$$Q^{-1}\hat{\vec{R}} = \sum_{i=1}^{3}\hat{R}_i\,Q^{-1}\vec{e}_i$$

<u>Parity transformation</u>

$$P \in O(3) \quad : \quad P\vec{r} = -\vec{r}$$

$$D(P)\,|\vec{r}\rangle = |P\vec{r}\rangle = |-\vec{r}\rangle$$

$$D(P) \equiv \hat{P}$$

$\{P,\ \mathbb{1}_{3\times3}\}$ is a subgroup in $O(3)$

$$P^2 = \mathbb{1}\ ,\quad P\cdot\mathbb{1} = P = \mathbb{1}\cdot P$$

$$\hat{P}\,|\psi\rangle = |\psi'\rangle$$

$$\boxed{\langle \vec{r} | \psi' \rangle = \langle \vec{r} | \hat{P} | \psi \rangle = \langle -\vec{r} | \psi \rangle}$$

$$\langle \vec{r} | \hat{P} | \psi \rangle = \langle \hat{P}^{+} | \vec{r} \rangle , \psi \rangle = \langle \hat{P} | \vec{r} \rangle , \psi \rangle$$

$$\hat{P}^{+} = \hat{P}^{-1}$$

$$\boxed{\begin{aligned} \hat{P}^2 &= D(P) D(P) = D(P^2) \\ &= D(\mathbb{1}) = \mathbb{1}_{\mathcal{H}} \end{aligned}}$$

$$\psi'(\vec{r}) = \psi(-\vec{r})$$

$$\boxed{\hat{P} : \psi(\vec{r}) \to \psi'(\vec{r}) = \psi(-\vec{r})}$$

$\hat{P}^2 = \mathbb{1}_{\mathcal{H}} \Rightarrow \hat{P}$ has eigenvalue ± 1

eigenstates $\psi_{\pm}$

$$\hat{P} \psi_{\pm}(\vec{r}) = \underline{\psi_{\pm}(-\vec{r}) = \pm \psi_{\pm}(\vec{r})}$$

ψ_{+} : even parity

ψ_{-} : odd parity

$\hat{P}^{+} = \hat{P}^{-1} = \hat{P}$ $\quad$ $\hat{P}$ is Hermitian & unitary

transf. of operators under parity

$$\hat{\vec{R}} \to \hat{P}\,\hat{\vec{R}}\,\hat{P}$$

$$\hat{P}\,\hat{\vec{R}}\,\hat{P}\,|\vec{r}\rangle = \hat{P}\hat{\vec{R}}\,|-\vec{r}\rangle = (-\vec{r})\,\hat{P}\,|-\vec{r}\rangle$$

$$= (-\vec{r})\,|\vec{r}\rangle$$

$$= -\hat{\vec{R}}\,|\vec{r}\rangle$$

$$\hat{P}\,\hat{\vec{R}}\,\hat{P} = -\hat{\vec{R}}$$

momentum operator $\quad \hat{\vec{P}}\,|\vec{p}\rangle = \vec{p}\,|\vec{p}\rangle$

$$\langle\vec{r}|\vec{p}\rangle = e^{\frac{i}{\hbar}\vec{r}\cdot\vec{p}} = e^{\frac{i}{\hbar}(-\vec{r})\cdot(-\vec{p})}$$

$$= \langle -\vec{r}|-\vec{p}\rangle$$

$\hat{\vec{p}} \to \hat{P}\,\hat{\vec{p}}\,\hat{P}$ $\qquad$ $\hat{P}\,|\vec{p}\rangle = \hat{P}\underbrace{\int d^3\vec{r}\,|\vec{r}\rangle\langle\vec{r}|}\vec{p}\rangle$

resolution of identity

$$= \int d^3\vec r \, |-\vec r\rangle\langle -\vec r|-\vec p\rangle$$

$$d^3\vec r = d^3(-\vec r) \qquad = \underbrace{\int d^3(-\vec r)\, |-\vec r\rangle\langle -\vec r|}\,-\vec p\rangle$$

$$= |-\vec p\rangle$$

$$\hat P \hat{\vec P} \hat P |\vec p\rangle = \hat P \hat{\vec P} |-\vec p\rangle = (-\vec p)\, P |-\vec p\rangle$$

$$= (-\vec p)|\vec p\rangle = -\hat{\vec P}|\vec p\rangle$$

$$\Rightarrow \hat P \hat{\vec P} \hat P = -\hat{\vec P}$$

angular momentum operator: $\hat{\vec L} = \hat{\vec R} \times \hat{\vec P}$

$$\hat P \hat{\vec L} \hat P = \hat P (\hat{\vec R} \times \hat{\vec P}) \hat P = \hat P \hat{\vec R} \hat P \times \hat P \hat{\vec P} \hat P$$

$$= (-\hat{\vec R}) \times (-\hat{\vec P})$$

$$= \hat{\vec L}$$

$$\hat{\vec{V}} = \sum_{i=1}^{3} \hat{V}_i \hat{e}_i$$

$\hat{\vec{V}}$ is vector operator if $\hat{P} \hat{\vec{V}} \hat{P} = -\hat{\vec{V}}$, like, $\hat{\vec{R}}$, $\hat{\vec{P}}$

" " pseudo-vector operator if $\hat{P} \hat{\vec{V}} \hat{P} = \hat{\vec{V}}$

like $\hat{\vec{L}}$, $\hat{\vec{S}}$ ← spin

$\hat{O}$ scalar operator if $\hat{P} \hat{O} \hat{P} = \hat{O}$ e.g. Hamiltonian of central potential.

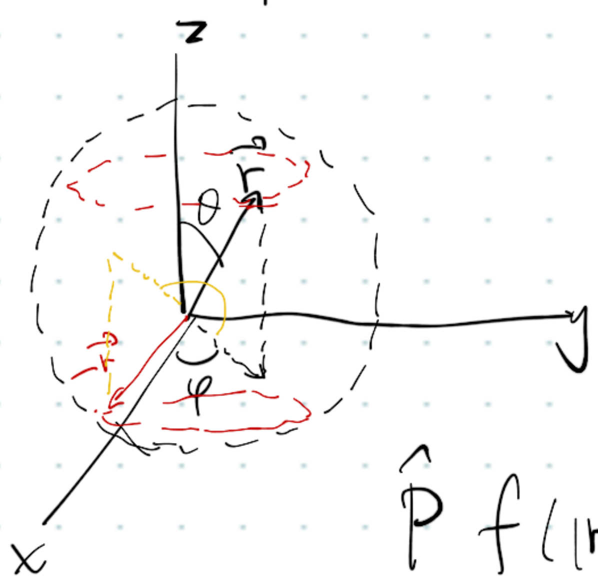
pseudo-scalar operator if $\hat{P} \hat{O} \hat{P} = -\hat{O}$ e.g.

helicity operator $\hat{\vec{S}} \cdot \hat{\vec{P}} / |\hat{\vec{P}}| \equiv \hat{h}$

because $\hat{P}\hat{\vec{L}}\hat{P} = \hat{\vec{L}}$ i.e. $[\hat{P}, \hat{\vec{L}}] = 0$

they share the same eigenbasis.

indeed in spherical coordinate.



$P: \vec{r} \to -\vec{r}$

$\theta \to \pi - \theta$

$\varphi \to \varphi + \pi$

$$\hat{P} f(|r|, \theta, \varphi) = f(|r|, \pi-\theta, \varphi+\pi)$$

$$\begin{cases} \hat{P} Y_{lm}(\theta, \varphi) = (-1)^l Y_{lm}(\theta, \varphi) \\ \hat{L}^2 Y_{lm}(\theta, \varphi) = \hbar^2 l(l+1) Y_{lm}(\theta, \varphi) \\ \hat{L}_z Y_{lm}(\theta, \varphi) = \hbar m Y_{lm}(\theta, \varphi) \end{cases}$$

space translation

$O(3)$: $Q\vec{r} = \vec{r}\,'$

space translation : $\underbrace{\vec{r} \to \vec{r}\,'}_{Q(\vec{\lambda})} = \underline{\vec{r} + \vec{\lambda}}$

$O(3) \cup \{\text{space translations}\} = E(3)$ Euclidean group.

a general transf. in $E(3)$

$$\vec{r} \to \vec{r}\,' = \underset{\substack{\uparrow \\ \Lambda \in O(3)}}{\Lambda}\,\vec{r} + \vec{\lambda}$$

$O(3) \subset E(3)$ subgroup.

$\{\text{space translations}\} \subset E(3)$ sub group.

$$\{\text{space translations}\} = \{Q(\vec{\lambda}) \mid \vec{\lambda} \in \mathbb{R}^3\}$$

D : $Q(\vec{\lambda}) \longrightarrow D(Q(\vec{\lambda}))$

$\mathbb{R}^3$ $\qquad\qquad \mathcal{H}$

$D(Q(\vec{\lambda}))$: $\psi(\vec{r}) \to \psi'(\vec{r})$

$$\underline{\psi'(\vec{r}) = \psi(Q(\vec{\lambda})^{-1}\vec{r}) = \psi(\vec{r}-\vec{\lambda})}.$$

assume $\vec{\lambda}$ is infinitesimal

$$= \psi(\vec{r}) - \vec{\lambda}\cdot\vec{\nabla}\psi(\vec{r}) + O(\lambda^2)$$

$$= \left(1 - \frac{i}{\hbar}\vec{\lambda}\cdot\hat{\vec{p}}\right)\psi(\vec{r}) + O(\lambda^2)$$

finite translation

= composition of many infinitesimal translations

$$D(Q(\lambda)) = \lim_{n\to\infty}\left(1 - \frac{i}{\hbar}\frac{\vec{\lambda}}{n}\cdot\hat{\vec{p}}\right)^n$$

$$= e^{-\frac{i}{\hbar}\vec{\lambda}\cdot\hat{\vec{p}}}$$

momentum operator is the generator of space translation.

in terms of Dirac braket

$$D(Q(\lambda))|\psi\rangle = |\psi'\rangle$$

$$\underline{D(Q(\lambda))|\vec{r}\rangle = |Q(\vec{\lambda})\vec{r}\rangle = |\vec{r}+\vec{\lambda}\rangle}$$

transformations of operator.

$$\hat{\vec{R}}' = D(Q(\lambda))\,\hat{\vec{R}}\,D(Q(\lambda))^{-1} = Q^{-1}(\vec{\lambda})\hat{\vec{R}}$$
$$= \hat{\vec{R}} - \vec{\lambda}$$

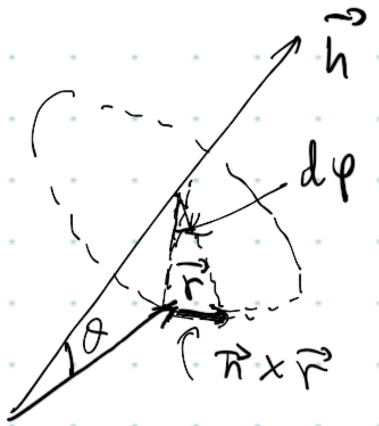
$$\hat{\vec{P}}' = D(Q(\lambda))\,\vec{P}\,D(Q(\lambda))^{-1} = \vec{P}$$

since $[\vec{P}, e^{-\frac{i}{\hbar}\vec{\lambda}\cdot\hat{\vec{P}}}] = 0$

<u>Spatial Rotations</u> $O(3) \supset SO(3)$

infinitesimal rotations

$$\vec{r} \to \vec{r}' = \vec{r} + \underline{\underline{d\varphi\,\vec{n}\times\vec{r}}}$$
$$= Q(\vec{n}, d\varphi)\vec{r}$$



$$dr = r\sin\theta\, d\varphi$$
$$= \left| \underline{\underline{d\varphi\, \vec{n}\times\vec{r}}} \right|$$

on $\mathcal{H}$: $D(Q(\vec{n}, d\varphi)) \equiv D(\vec{n}, d\varphi)$

$$\psi'(\vec{r}) = D(\vec{n}, d\varphi)\,\psi(\vec{r})$$
$$= \psi\left(Q^{-1}(\vec{n}, d\varphi)\,\vec{r}\right)$$
$$= \psi\left(\vec{r} - d\varphi\, \vec{n}\times\vec{r}\right)$$
$$= \psi(\vec{r}) - d\varphi\, (\vec{n}\times\vec{r})\cdot\vec{\nabla}\,\psi(\vec{r})$$
$$= \left(1 - \frac{i}{\hbar}\, d\varphi\, \vec{n}\cdot\hat{\vec{L}}\right)\psi(\vec{r})$$

$$(\vec{n}\times\vec{r})\cdot\vec{\nabla} = \vec{n}\cdot\underline{\underline{(\vec{r}\times\vec{\nabla})}}$$

$$D(\vec{n}, d\varphi) = 1 - \frac{i}{\hbar}\, d\varphi\, \vec{n}\cdot\hat{\vec{L}}$$

$m \to \infty$

finite rotation : $D(\vec{n}, \varphi) = \left(1 - \frac{i}{\hbar}\frac{\varphi}{m}\vec{n}\cdot\hat{\vec{L}}\right)^{m}$

$= e^{-\frac{i}{\hbar}\varphi\vec{n}\cdot\hat{\vec{L}}}$

angular momentum operator is the generator of rotations.

transf. of $\hat{\vec{R}}$

$$\hat{\vec{R}}' = \underline{D(\vec{n}, d\varphi)\,\hat{\vec{R}}\,D(\vec{n}, d\varphi)^{-1}}$$

$$= \left(1 - \frac{i}{\hbar}d\varphi\,\vec{n}\cdot\hat{\vec{L}}\right)\hat{\vec{R}}\left(1 + \frac{i}{\hbar}d\varphi\,\vec{n}\cdot\hat{\vec{L}}\right)$$

$$= \hat{\vec{R}} - \frac{i}{\hbar}d\varphi\,[\vec{n}\cdot\hat{\vec{L}}, \hat{\vec{R}}] + O(d\varphi^2)$$

$$\left([\hat{L}_i, \hat{R}_j] = i\hbar\,\varepsilon_{ijk}\hat{R}_k\right)$$

$$= \underline{\hat{\vec{R}} - d\varphi\,\vec{n}\times\hat{\vec{R}}}$$

$$\hat{\vec{R}}' = \underline{Q^{-1}(\vec{n}, d\varphi)\,\hat{\vec{R}}}$$

<u>HW</u> prove $\underline{\hat{\vec{P}}' = D(\vec{n}, d\varphi)\,\hat{\vec{P}}\,D(\vec{n}, d\varphi)^{-1} = Q^{-1}(\vec{n}, d\varphi)\,\hat{\vec{P}}}$

$$\hat{\vec{L}}' = D(\vec{n}, d\varphi)\,\hat{\vec{L}}\,D(\vec{n}, d\varphi)^{-1} = Q^{-1}(\vec{n}, d\varphi)\,\hat{\vec{L}}$$

Scalar : S const. inv. under rotations.

vector : $\vec{V} \to Q(\vec{n}, \varphi)\,\vec{V}$

Def : Scalar operator : $\hat{S}$ s.t. $D(\vec{n}, \varphi)\,\hat{S}\,D(\vec{n}, \varphi)^{-1} = \hat{S}$

$$[\hat{S}, D(\vec{n}, \varphi)] = 0 \quad \forall \vec{n}, \varphi$$

e.g. Hamiltonian operator. $\hat{H} = \frac{\hat{p}^2}{2m} + V(r)$

Vector operator : $\hat{\vec{V}} = (\hat{V}_1, \hat{V}_2, \hat{V}_3)$

s.t. $D(\vec{n}, \varphi)\,\hat{\vec{V}}\,D(\vec{n}, \varphi)^{-1} = Q^{-1}(\vec{n}, \varphi)\,\hat{\vec{V}}$

e.g. $\underbrace{\hat{\vec{P}}, \hat{\vec{R}}}_{\text{polar vector operators}}$, $\hat{\vec{L}}$ ← pseudo-vector operator.

infinitesimal version $\quad \varphi \to d\varphi$

scalar : $[\hat{S}, 1 - \frac{i}{\hbar} d\varphi\, \vec{n}\cdot\hat{\vec{L}}] = 0$

$$\Leftrightarrow \quad [\hat{S}, \hat{\vec{L}}] = 0$$

$$\left.\begin{aligned} P\hat{\vec{P}}P &= -\hat{\vec{P}} \\ P\hat{\vec{R}}P &= -\hat{\vec{R}} \end{aligned}\right\} \text{ polar vector}$$

$P\hat{\vec{L}}P = \hat{\vec{L}}$ $\quad$ pseudo vector.

vector : $D(\vec{n}, d\varphi)\,\hat{\vec{V}}\,D(\vec{n}, d\varphi)^{-1}$

$$= V - \frac{i}{\hbar} d\varphi\, [\vec{n}\cdot\hat{\vec{L}}, \hat{\vec{V}}]$$

$$Q^{-1}(\vec{n}, d\varphi)\,\hat{\vec{V}} = \hat{\vec{V}} - d\varphi\, \vec{n}\times\hat{\vec{V}}$$

$$\Rightarrow \quad [\vec{n}\cdot\hat{\vec{L}}, \hat{\vec{V}}] = -i\hbar\, \vec{n}\times\hat{\vec{V}}$$

$$n_m [\hat{L}_m, \hat{V}_k] = -i\hbar\, \varepsilon_{kmn} n_m \hat{V}_n$$

$$\Rightarrow \quad [\hat{L}_m, \hat{V}_k] = -i\hbar\, \varepsilon_{kmn} \hat{V}_n = i\hbar\, \varepsilon_{mkn} \hat{V}_n$$

Rotation in spin space

operators

$$\psi(\vec{r}) \otimes \xi \in \mathcal{H} \otimes \mathbb{C}^2$$

$L^2(\mathbb{R}^3)$ spin Hilbert space

$$\hat{\vec{S}} = \frac{\hbar}{2} \hat{\vec{\sigma}}$$ spin operator

$\xi \in \mathbb{C}^2$

$(\sigma_1, \sigma_2, \sigma_3)$ pauli matrices

the spin angular momentum

$$[S_i, S_j] = i\hbar \varepsilon_{ijk} S_k$$

We define spin rotation by

$$D^{1}(\vec{n}, \varphi) = e^{-\frac{i}{\hbar}\varphi \vec{n} \cdot \hat{\vec{S}}} \quad : \mathbb{C}^2 \to \mathbb{C}^2$$

2×2 matrix.

rotation in $\vec{r}$-space

$$D(\vec{n}, \varphi) = e^{-\frac{i}{\hbar}\varphi \vec{n} \cdot \hat{\vec{L}}} \quad : \mathcal{H} \to \mathcal{H}$$

$$D(\vec{n},\varphi) \otimes D'(\vec{n},\varphi) = \mathbb{D}(\vec{n},\varphi)$$ same $\vec{n}, \varphi$.

D acts on $\mathcal{H}$, D' acts on $\mathbb{C}^2$, $\mathbb{D}$ acts on $\mathcal{H} \otimes \mathbb{C}^2$

$$\mathbb{D}(\vec{n},\varphi) = \underline{e^{-\frac{i}{\hbar}\varphi \vec{n}\cdot\hat{\vec{L}}} \otimes e^{-\frac{i}{\hbar}\varphi \vec{n}\cdot\hat{\vec{S}}}}$$

$$= \underline{e^{-\frac{i}{\hbar}\varphi \vec{n}\cdot\hat{\vec{J}}}}$$ (generator of total rotation)

total angular momentum

to find $\hat{\vec{J}}$, infinitesimal rotation $\varphi \to d\varphi$

$$\underline{\left(1-\frac{i}{\hbar}d\varphi\,\vec{n}\cdot\hat{\vec{L}}\right)} \otimes \underline{\left(1-\frac{i}{\hbar}d\varphi\,\vec{n}\cdot\hat{\vec{S}}\right)}$$

$$= 1_{\mathcal{H}} \otimes 1_{\mathbb{C}^2} - \frac{i}{\hbar}d\varphi\left(\vec{n}\cdot\hat{\vec{L}} \otimes 1_{\mathbb{C}}\right) - \frac{i}{\hbar}d\varphi\left(1_{\mathcal{H}} \otimes \vec{n}\cdot\hat{\vec{S}}\right) + O(d\varphi^2)$$

($\vec{n}\cdot\hat{\vec{L}}$ acts on $\mathcal{H}$, $\vec{n}\cdot\hat{\vec{S}}$ acts on $\mathbb{C}^2$)

$$e^{-\frac{i}{\hbar}\vec{n}\cdot\hat{\vec{J}}} = 1_{\mathcal{H}} \otimes 1_{\mathbb{C}^2} - \frac{i}{\hbar}d\varphi\,\vec{n}\cdot\hat{\vec{J}} + O(d\varphi^2)$$

($\vec{n}\cdot\hat{\vec{J}}$ acts on $\mathcal{H} \otimes \mathbb{C}^2$)

$$\hat{\vec{J}} = \underbrace{\underset{\substack{\circlearrowleft \\ \mathcal{H}}}{\hat{\vec{L}}} \otimes \mathbb{1}_{\mathbb{C}^2} + \mathbb{1}_{\mathcal{H}} \otimes \underset{\substack{\circlearrowleft \\ \mathbb{C}^2}}{\hat{\vec{S}}}} \equiv \hat{\vec{L}} + \hat{\vec{S}}$$

spatial symmetry & conservation law

Schrödinger eqn. $i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$

Spatial symmetries $Q(\lambda) \in E(3)$

translation or rotation or parity.

$\lambda = \vec{\lambda}$ $\qquad$ $\lambda = (\vec{n}, \varphi)$

$Q \rightarrow D(Q) : \mathcal{H} \rightarrow \mathcal{H}$.

$$i\hbar \frac{\partial}{\partial t} \underbrace{D(Q) |\psi(t)\rangle}_{|\psi'(t)\rangle} = D(Q) \hat{H} |\psi(t)\rangle$$

$$= \underbrace{D(Q) \hat{H} D(Q)^{-1}}_{H'} \underbrace{D(Q) |\psi(t)\rangle}_{|\psi'(t)\rangle}$$

$$i\hbar \frac{\partial}{\partial t} |\psi'(t)\rangle = \hat{H}' |\psi'(t)\rangle$$

$$\hat{H}' = \hat{H} \quad \text{or} \quad D(Q)\,\hat{H}\,D(Q)^{-1} = \hat{H}$$

i.e., $[D(Q), \hat{H}] = 0$

if $[D(Q), \hat{H}] = 0$, we say $D(Q)$ is a symmetry of the physical system.

<u>Example</u> $\underline{\hat{H}_{free} = \frac{\hat{p}^2}{2m}}$ $\quad \underline{D(Q)\,\hat{\vec{p}}\,D(Q)^{-1} = Q^{-1}\hat{\vec{p}}}$

$Q \in E(3)$

$$\begin{aligned} D(Q)\hat{P}^2 D(Q)^{-1} &= \left(D(Q)\,\hat{\vec{P}}\,D(Q)^{-1}\right)\cdot\left(D(Q)\,\hat{\vec{P}}\,D(Q)^{-1}\right) \\ &= Q^{-1}\hat{\vec{P}} \cdot Q^{-1}\hat{\vec{P}} \\ &= \hat{\vec{P}}\cdot\hat{\vec{P}} = \hat{P}^2 \end{aligned}$$

$$\Rightarrow D(Q)\,\hat{H}_{free}\,D(Q)^{-1} = \hat{H}_{free} \quad \forall Q \in E(3)$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\vec{r})$$

$$[\hat{H}, D(Q)] = [V(\vec{r}), D(Q)]$$

symmetry is determined by $V(\vec{r})$

time evolution operator $U(t) = e^{\frac{i}{\hbar}\hat{H}t}$

$\forall$ operator $\hat{A} : \mathcal{H} \to \mathcal{H}$

$$\hat{A}(t) = U(t)\,\hat{A}\,U(t)^{-1}$$

$$i\hbar \frac{d}{dt}\hat{A}(t) = [\hat{A}(t), \hat{H}]$$

Heisenberg picture EOM

$[D(Q), \hat{H}] = 0 \quad \Rightarrow \quad \frac{d}{dt} D(Q) = 0$

$D(Q)$ is conserved

if $D(Q)$ is continuous symmetry $\Rightarrow$ its generator is conserved.

Example, (1) $D(\vec{\lambda}) = e^{-\frac{i}{\hbar}\vec{\lambda}\cdot\hat{\vec{p}}}$ spatial translation

if $[\hat{H}, D(\vec{\lambda})] = 0$, $\frac{d}{d\vec{\lambda}}\Big|_{\vec{\lambda}=0} [\hat{H}, e^{-\frac{i}{\hbar}\vec{\lambda}\cdot\hat{\vec{p}}}]$

$$= -\frac{i}{\hbar}[\hat{H}, \hat{\vec{p}}] = 0$$

$\Rightarrow$ momentum is conserved.

(2) $D(\vec{n}, \varphi) = e^{-\frac{i}{\hbar}\varphi\vec{n}\cdot\hat{\vec{J}}}$

if $[H, D(\vec{n}, \varphi)] = 0$,

$$\frac{d}{d\varphi}\Big|_{\varphi=0} [\hat{H}, e^{-\frac{i}{\hbar}\varphi\vec{n}\cdot\hat{\vec{J}}}] = 0$$

$$= +\frac{i}{\hbar}[\vec{n}\cdot\hat{\vec{J}}, \hat{H}]$$

$\Rightarrow$ anglar momentum $\vec{n}\cdot\hat{\vec{J}}$ is conserved.

(3) $[\hat{H}, \hat{P}] = 0$ parity inv., $\hat{P}$ is conserved

($\hat{P}$ is unitary and Hermitian, $\hat{P}$ can be a physical quantity)

symmetry group of $\hat{H}$

Given Hamiltonian $\hat{H}$, all unitary transf. U s.t.

$U\hat{H}U^{+} = \hat{H}$, form the symmetry group of $\hat{H}$ $\underbrace{\hphantom{of}}_{G}$.

- $U_1, U_2 \in G$, $\quad U_1\hat{H}U_1^{-1} = \hat{H}$

  $U_2\hat{H}U_2^{-1} = \hat{H}$ ←

  $\Rightarrow (U_1U_2)\hat{H}(U_1U_2)^{-1} = \hat{H}$

- operator product → associativity
- $1H1^{-1} = H \rightarrow 1 \in G$ is identity
- $\forall U \in G \quad \exists U^{-1} \in G$

Some results in the representation theory of group

Def: Give a group G, and a vector space V, and the set of invertible linear transformations $L(V)$ on

V, if $\exists$ group homomorphism i.e. $D(g_1)D(g_2) = D(g_1 g_2)$

$$D : G \to L(V)$$

$$\underset{\substack{\cap \\ G}}{g} \mapsto D(g) \quad \dim(V) \times \dim(V) \underbrace{\text{matrix}}_{\text{invertible}}$$

we say D is a representation of G.

V is the carrier space of the rep.

<u>Example</u>:

- trivial rep. $D(g) = \mathbb{1}_{n\times n}$ on n-dim vector space V $\forall g \in G$

- 3×3 matrices Q_{ij} s.t. $\vec{r} \to Q\vec{r}$ $\vec{r} \in \mathbb{R}^3$

$$Q\vec{r} \cdot Q\vec{r}' = \vec{r} \cdot \vec{r}'$$

$\{Q_{ij}\}$ rep. of $O(3)$ group

- $D(Q)$ $\overbrace{\text{operators}}^{\text{unitary}}$ operators on $\mathcal{H}$ (possibly ∞-dim) give rep. of $O(3)$ group (unitary rep.)

matrix form of $D(Q)$: orthonormal basis in $\mathcal{H}$

$f_i(\vec{r})$ s.t $\langle f_i | f_j \rangle = \delta_{ij}$

$$f_i'(\vec{r}) = D(Q) f_i(\vec{r})$$

$$= f_i(Q^{-1}\vec{r})$$

$Q \in$ rep of $O(3)$ on $\mathbb{R}^3$

$$\downarrow$$

$$\sum_j f_j(\vec{r}) D(Q)_{ji}$$

$$\Rightarrow D(Q)_{ji} = \langle f_j | D(Q) | f_i \rangle$$

$D(Q)$ is unitary $\rightarrow$ $D(Q)_{ji}$ is unitary matrix

$$D(Q)^\dagger = D(Q)^{-1}$$

we call it a unitary rep. of $O(3)$

Def Give $D : G \rightarrow L(\mathcal{H})$

$D' : G \rightarrow L(\mathcal{H})$ both are unitary rep.

$D : g \mapsto D(g)$ $\quad D(g) \neq D'(g)$

$D' : g \mapsto D'(g)$

if $\exists$ $U : \mathcal{H} \to \mathcal{H}$ unitary operator, s.t.

$$D'(g) = U D(g) U^{-1} \quad \forall g \in G$$

we say D and D' are unitary equivalent.

<u>Def</u> (direct sum)

Given reps. $D_1 : G \to L(V_1)$, $D_2 : G \to L(V_2)$ $\quad \dim(V_1) = n_1 \quad \dim(V_2) = n_2$

$D_3 : G \to L(V_3)$ ...

$\dim(V_3) = n_3$ ...

$\forall g \in G$, $\quad D_1(g) \quad n_1 \times n_1$ matrix

$D_2(g) \quad n_2 \times n_2$ matrix

$D_3(g) \quad n_3 \times n_3$ matrix

$\vdots$

define $D(g) : \sum_i n_i \times \sum_i n_i$ matrix

$$D(g)\,v = \begin{pmatrix} \boxed{D_1(g)} & & & \\ & \boxed{D_2(g)} & & \\ & & \boxed{D_3(g)} & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \end{pmatrix}$$

$$D(g) \in L(V_1 \oplus V_2 \oplus V_3 \oplus \cdots)$$

$$v \in V_1 \oplus V_2 \oplus V_3 \oplus \cdots$$

$$v = (\underset{\substack{\cap \\ V_1}}{v_1},\ \underset{\substack{\cap \\ V_2}}{v_2},\ \underset{\substack{\cap \\ V_3}}{v_3},\ \cdots)^T$$

$D: G \to L(V_1 \oplus V_2 \oplus V_3 \oplus \cdots)$ is direct sum rep.

$$D = D_1 \oplus D_2 \oplus D_3 \oplus \cdots$$

$$\forall\ v = (v_1, v_2, v_3 \cdots)^T \in V_1 \oplus V_2 \oplus V_3 \oplus \cdots$$

$$D(g)\,v := (D_1(g)\,v_1,\ D_2(g)\,v_2,\ D_3(g)\,v_3\ \cdots)^T$$

<u>Def</u> if a rep D is a direct sum, D is a reducible rep.

if a rep D cannot be written as direct sum of nontrivial reps, then D is irreducible rep.
(irrep)

Schur's Lemma

(1) Given $D: G \to L(V)$ irrep, for any matrix A satisfying

$$[A, D(g)] = 0 \quad \forall g \in G$$

then $A = \lambda \mathbb{1}_{\dim V \times \dim V} \quad \lambda \in \mathbb{C}$

(2) Given $D: G \to L(V)$ reducible

$\exists A$ satisfying $[A, D(g)] = 0 \quad \forall g \in G$

$$A \neq \lambda \mathbb{1}_{\dim V \times \dim V}.$$

$D = D_1 \oplus D_2 \cdots$

$$A = \lambda_1 \mathbb{1}_{V_1} \oplus \lambda_2 \mathbb{1}_{V_2} \oplus \cdots = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_1 & & & \\ & & & \lambda_2 & & \\ & & & & \ddots & \\ & & & & & \lambda_2 \\ & & & & & & \ddots \end{pmatrix}$$