

Compute CG coefficients (sketch)

$$|\tilde{j}_1 \tilde{j}_2 \tilde{J}^m\rangle = \sum_{m_1 m_2} |\tilde{j}_1 \tilde{j}_2 m_1 m_2\rangle \underbrace{\sum_{m_1 m_2} \tilde{J}^m}_{\tilde{J}^m}$$

at  $m = j$

$$|\tilde{j}_1 \tilde{j}_2 jj\rangle = \sum_{m_1 m_2} |\tilde{j}_1 \tilde{j}_2 m_1 m_2\rangle \underbrace{\sum_{m_1 m_2} \tilde{J}^j}_{\tilde{J}^j}$$

$$\equiv a_{m_1 m_2} \delta_{j, m_1 + m_2}$$

$$0 = \hat{J}_+ |\tilde{j}_1 \tilde{j}_2 jj\rangle = (\hat{J}_{1+} + \hat{J}_{2+}) |\tilde{j}_1 \tilde{j}_2 jj\rangle$$

$$= \sum_{m_1 m_2} |\tilde{j}_1 \tilde{j}_2 m_1 + 1, m_2\rangle \sqrt{(\tilde{j}_1 - m_1)(\tilde{j}_1 + m_1 + 1)} a_{m_1 m_2}$$

$$+ \sum_{m_1 m_2} |\tilde{j}_1 \tilde{j}_2 m_1, m_2 + 1\rangle \sqrt{(\tilde{j}_2 - m_2)(\tilde{j}_2 + m_2 + 1)} a_{m_1 m_2}$$

1st term:  $m_2 \rightarrow m_2' + 1$

$$\delta_{\tilde{j}_2, m_2 + m_2'}$$

2nd term:  $m_1 \rightarrow m_1' + 1$

$$0 = \sum_{m_1 m_2} |\tilde{j}_1 \tilde{j}_2 m_1 + 1, m_2' + 1\rangle \sqrt{(\tilde{j}_1 - m_1)(\tilde{j}_1 + m_1 + 1)} a_{m_1 m_2' + 1}$$

$$\delta_{\tilde{j}_2, m_2 + m_2' + 1}$$

$$+ \sum_{m_1 m_2} |\tilde{j}_1 \tilde{j}_2 m_1' + 1, m_2 + 1\rangle \sqrt{(\tilde{j}_2 - m_2)(\tilde{j}_2 + m_2 + 1)} a_{m_1' + 1 m_2}$$

$$\delta_{\tilde{j}_1, m_1' + m_2 + 1}$$

$$\Rightarrow \sqrt{(\hat{j}_1 - m_1)(\hat{j}_1 + m_1 + 1)} a_{m_1, m_2 + 1} \delta_{\hat{j}_1, m_1 + m_2 + 1}$$

$$= - \sqrt{(\hat{j}_2 - m_2)(\hat{j}_2 + m_2 + 1)} a_{m_2 + 1, m_2} \delta_{\hat{j}_2, m_1 + m_2 + 1}$$

$$m_2 \rightarrow m_2 - 1$$

$$\Rightarrow a_{m_1, m_2} = - \sqrt{\frac{(\hat{j}_2 - m_2 + 1)(\hat{j}_2 + m_2)}{(\hat{j}_1 - m_1)(\hat{j}_1 + m_1 + 1)}} a_{m_1 + 1, m_2 - 1}$$

Recursion relation for  $a_{m_1, m_2}$

$$a_{m_1, m_2} = (-1) \sqrt{\frac{(\hat{j}_2 - m_2 + 1)(\hat{j}_2 + m_2)}{(\hat{j}_1 - m_1)(\hat{j}_1 + m_1 + 1)}} (-1)$$

$$\sqrt{\frac{(\hat{j}_2 - m_2 + 2)(\hat{j}_2 + m_2 - 1)}{(\hat{j}_1 - m_1 - 1)(\hat{j}_1 + m_1 + 2)}} a_{m_1 + 2, m_2 - 2}$$

$\vdots \dots$

$$= (-1)^{\hat{j}_1 - m_1} \sqrt{\frac{(\hat{j}_1 + m_1)! (\hat{j}_2 + m_2)!}{(\hat{j}_1 - m_1)! (\hat{j}_2 - m_2)!}} \sqrt{\frac{(\hat{j}_1 + \hat{j}_2 - \hat{j})!}{(\hat{j}_2 - \hat{j}_1 + \hat{j})! / 2!}} x a_{\hat{j}_1, \hat{j}_2 - \hat{j}}$$

by normalization of  $| \hat{j}_1 \hat{j}_2 \hat{j} \bar{j} \rangle$   $\underbrace{\quad}_{\text{a only dep. on } j}$

$$\Rightarrow a = \sqrt{\frac{(2\hat{j}+1)! (j_1 + j_2 - \hat{j})!}{(\hat{j} + j_1 + j_2 + 1)! (j + j_1 - j_2)! (j - j_1 + j_2)!}}$$

$$S_{m_1 m_2 \hat{j} \bar{j}}^{\hat{j}_1 \hat{j}_2} = a_{m_1 m_2} \delta_{\hat{j}, m_1 + m_2}$$

$$= \delta_{\hat{j}, m_1 + m_2} (-1)^{\hat{j}_1 - m_1} \sqrt{\frac{(2\hat{j}+1)! (j_1 + j_2 - \hat{j})!}{(\hat{j} + j_1 + j_2 + 1)! (j + j_1 - j_2)! (j - j_1 + j_2)!}}$$

$$\times \sqrt{\frac{(j_1 + m_1)! (j_2 + m_2)!}{(j_1 - m_1)! (j_2 - m_2)!}}$$

general CG coefficients  $S_{m_1 m_2 \hat{j} \bar{m}}^{\hat{j}_1 \hat{j}_2}$ :

$$S_{m_1 m_2 \hat{j} \bar{m}}^{\hat{j}_1 \hat{j}_2} = \langle \hat{j}_1 \hat{j}_2 m_1 m_2 | \hat{j}_1 \hat{j}_2 \hat{j} \bar{m} \rangle$$

$$= \sqrt{\frac{(\hat{j} + m)!}{(2\hat{j})! (\hat{j} - m)!}} \underbrace{\langle \hat{j}_1 \hat{j}_2 m_1 m_2 | (\hat{j}_-)^{\hat{j} - m} | \hat{j}_1 \hat{j}_2 \hat{j} \bar{j} \rangle}_{\hat{j}_- = \hat{j}_{1-} + \hat{j}_{2-}}$$

$$\sum_{m_1, m_2, \tilde{J}, \tilde{m}}^{\tilde{j}_1, \tilde{j}_2} = \sqrt{\frac{(\tilde{j} + m)!}{(2\tilde{j})! (\tilde{j} - m)!}} \underbrace{\langle \tilde{j}_1, \tilde{j}_2 |}_{\tilde{j}_1, \tilde{j}_2} \underbrace{\tilde{j} j |}_{\tilde{j}_1, \tilde{j}_2} \left( \hat{J}_{1+} + \hat{J}_{2+} \right)^{\tilde{j}-m} |$$

$$(\hat{J}_{1+} + \hat{J}_{2+})^{\tilde{j}-m} | \tilde{j}_1, \tilde{j}_2, m_1, m_2 \rangle$$

$$= \sum_s \frac{(\tilde{j} - m)!}{s! (\tilde{j} - m - s)!} \hat{J}_{1+}^s \hat{J}_{2+}^{\tilde{j} - m - s} | \tilde{j}_1, \tilde{j}_2, m_1, m_2 \rangle$$

$$= \sum_s \frac{(\tilde{j} - m)!}{s! (\tilde{j} - m - s)!} | \tilde{j}_1, \tilde{j}_2, m_1 + s, m_2 + \tilde{j} - m - s \rangle$$

$$X \sqrt{\frac{(\tilde{j}_1 - m_1)! (\tilde{j}_1 + m_1 + s)!}{(\tilde{j}_1 - m_1 - s)! (\tilde{j}_1 + m_1)!}} \sqrt{\frac{(\tilde{j}_2 - m_2)! (\tilde{j}_2 + m_2 + \tilde{j} - m - s)!}{(\tilde{j}_2 - m_2 - \tilde{j} + m + s)! (\tilde{j}_2 + m_2)!}}$$

$$\langle \tilde{j}_1, \tilde{j}_2 | \tilde{j} j | \tilde{j}_1, \tilde{j}_2, m_1, m_2 \rangle = \sum_{m_1, m_2, \tilde{J}, \tilde{j}}^{\tilde{j}_1, \tilde{j}_2} \tilde{j} j$$

$$S_{m_1, m_2, \vec{j}, \vec{m}}^{j_1, j_2} = \delta_{m_1, m_2 + m_2} \underbrace{\frac{(j_1 + j_2 - j)!, (j_1 - m_1)!, (j_2 - m_2)!, (j + m_1)!, (j - m_1)!}{(2j + 1)!}}_{\frac{(j + j_1 + j_2 + 1)!, (j + j_1 - j_2)!, (j - j_1 + j_2)!, (j_1 + m_1)!, (j_2 + m_2)!, (j_1 - m_1)!, (j_2 - m_2)!, (j - m_1 - m_2)!, (j + m_1 + m_2)!, (j_1 + j_2 + m_1 + m_2)!}}$$

$$\times \sum_s (-1)^{j_1 + m_1 + s} \frac{(j_1 + m_1 + s)!, (j + j_2 - m_1 - s)!}{s! (j - m_1 - s)!, (j_1 - m_1 - s)!, (j_2 - j + m_1 + s)!}$$

Edmonds formula of CG coefficients.

$$(x)! = \begin{cases} x! & x \in \mathbb{N}_+ \\ 0 & \text{otherwise} \end{cases}$$

$\sum_s$  is a finite sum so that all  $(\dots)! \neq 0$

Relation between CG coefficients and D-matrix

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

rotation in  $\mathcal{H}_1$ ,  $D_1(\vec{n}, \varphi) |j_1, m_1\rangle = e^{-\frac{i}{\hbar} \vec{\varphi} \cdot \hat{\vec{j}}_1} |j_1, m_1\rangle$

$$= \sum_{m'_1} |j_1, m'_1\rangle D_{m'_1, m_1}^{j_1}(\vec{n}, \varphi)$$

$$\text{in } \mathcal{H}_2 \quad D_2(\vec{n}, \varphi) |j_2 m_2\rangle = e^{-\frac{i}{\hbar} \varphi \vec{n} \cdot \hat{\vec{j}}_2} |j_2 m_2\rangle$$

$$= \sum_{m'_2} |j_2 m'_2\rangle D_{m'_2 m_2}^{j_2}(\vec{n}, \varphi)$$

$$\text{in } \mathcal{H}_1 \otimes \mathcal{H}_2, \quad D(\vec{n}, \varphi) = D_1(\vec{n}, \varphi) \otimes D_2(\vec{n}, \varphi)$$

$$= e^{-\frac{i}{\hbar} \varphi \vec{n} \cdot (\hat{\vec{j}}_1 + \hat{\vec{j}}_2)}$$

$$\underbrace{D(\vec{n}, \varphi) |j_1 j_2 m_1 m_2\rangle}_{= e^{-\frac{i}{\hbar} \varphi \vec{n} \cdot \hat{\vec{j}}_1} |j_1 m_1\rangle \otimes e^{-\frac{i}{\hbar} \varphi \vec{n} \cdot \hat{\vec{j}}_2} |j_2 m_2\rangle}$$

$$= \sum_{m'_1 m'_2} |j_1 j_2 m'_1 m'_2\rangle \underbrace{D_{m'_1 m_1}^{j_1}(\varphi, \vec{n}) D_{m'_2 m_2}^{j_2}(\varphi, \vec{n})}_{=}$$

$$\underbrace{D(\vec{n}, \varphi) |j_1 j_2 j_m\rangle}_{\text{fix } (j_1 j_2), \text{ fix } j} = e^{-\frac{i}{\hbar} \varphi \vec{n} \cdot (\underbrace{\hat{\vec{j}}_1 + \hat{\vec{j}}_2}_{\hat{\vec{j}}})} |j_1 j_2 j_m\rangle$$

$$\left\{ |j_1 j_2 j_m\rangle \right\}_{m=-j}^j = \sum_{m'} |j_1 j_2 j_m\rangle \underbrace{D_{m' m}^{j_1 j_2}(\varphi, \vec{n})}_{=}$$

spans irrep of  $SU(2)$

$$\text{apply } |j_1, j_2 m_1 m_2\rangle = \sum_{j'm} |j_1 j_2 j' m\rangle \left(S^{j_1 j_2}\right)^{-1}_{j'm, m_1 m_2}$$

$$|j_1 j_2 j' m\rangle = \sum_{m_1 m_2} |j_1 j_2 m_1 m_2\rangle S^{j_1 j_2}_{m_1 m_2 j' m}$$

$$D(Q) |j_1 j_2 j' m\rangle \underset{\uparrow}{=} D(Q) \sum_{m_1 m_2} |j_1 j_2 m_1 m_2\rangle S^{j_1 j_2}_{m_1 m_2 j' m}$$

$$Q \in SU(2) \quad = \sum_{\substack{m_1 m_2 \\ m'_1 m'_2}} |j_1 j_2 m'_1 m'_2\rangle \underbrace{D^{j_1}_{m'_1 m'_1}(Q)}_{m'_1 m'_2} \underbrace{D^{j_2}_{m'_2 m'_2}(Q)}_{m'_1 m'_2} S^{j_1 j_2}_{m'_1 m'_2 j' m}$$

$$\sum_{j'm'} |j_1 j_2 j' m'\rangle \left(S^{j_1 j_2}\right)^{-1}_{j'm' m'_1 m'_2}$$

$$= \sum_{j'm'} |j_1 j_2 j' m'\rangle \left[ \sum_{\substack{m_1 m_2 \\ m'_1 m'_2}} \left( S^{j_1 j_2} \right)^{-1}_{j'm' m'_1 m'_2} \underbrace{D^{j_1}_{m'_1 m'_1}(Q)}_{m'_1 m'_2} \underbrace{D^{j_2}_{m'_2 m'_2}(Q)}_{m'_1 m'_2} S^{j_1 j_2}_{m'_1 m'_2 j' m} \right]$$

$$\left[ S^{-1} \left( D^{j_1}(Q) \otimes D^{j_2}(Q) \right) S \right]_{j'm', j'm}$$

/

$$\left[ \underset{=}{\underbrace{S^{\hat{j}}(D^{j_1}(Q) \otimes D^{j_2}(Q))S}} \right]_{\hat{j}'m', \hat{j}m} = \delta_{\hat{j}\hat{j}'} \underset{=}{{D}_{m'm}^{\hat{j}}(Q)}$$

$\mathcal{H}$  is spanned by  $\{(j_1, j_2, j_m)\}_{\substack{j_1+j_2 \\ j=-|j_1-j_2| \\ m=-j}}$   
 ↑  
 Reducible rep.

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \quad D^{j_1}, D^{j_2} \text{ 2 irreps of } SU(2)$$

$$(j_1, m_1) \quad (j_2, m_2) \\ \text{---} \quad \text{---} \\ D^{j_1} \quad D^{j_2}$$

$D^{j_1} \otimes D^{j_2}$  is also rep of  $SU(2)$

(tensor product rep of  $SU(2)$ )

$$\text{rep. matrix } D_{m_1 m_2}^{\hat{j}}(Q) \quad D_{m'_1 m'_2}^{\hat{j}}(Q)$$

$$D^{j_1} \otimes D^{j_2} = \bigoplus_{\hat{j}=|j_1-j_2|}^{|j_1+j_2|} D^{\hat{j}}$$

$\forall Q \in SU(2)$

decomposition of tensor product  
rep into irreps

How  $D$ -matrix in the irrep is expressed in terms of

$D$ -matrix in tensor product rep.

$$\sum_j (S D^j(Q) S^\dagger) = D^{j_1}(Q) \otimes D^{j_2}(Q)$$

multiply  $D^j(Q)^*$  and integrate  $Q$  over  $Q \in SU(2)$

$$S_{m_1 m_2 j' m'}^{j_1 j_2} S_{j m_1 m_2}^{j_1 j_2} \sim \frac{16\pi^2}{2j'+1}$$

$$= \int D_{m_1 m_1}^{j_1}(Q) D_{m_2 m_2}^{j_2}(Q) D_{m' m'}^{j'}(Q)^* dQ$$

$$dQ = \sin\beta \, dx \, d\beta \, dy$$

from this we can solve

$$S_{m_1 m_2 j' m'}^{j_1 j_2} = S_{m_1 + m_2 m}$$

$$\frac{(2j'+1)(j+j_1-j_2)! (j-j_1+j_2)! (j_1+j_2-j)! (j+m_1)! (j-m)!}{(j+j_1+j_2+1)! (j_1+m_1)! (j_1-m_1)! (j_2+m_2)! (j_2-m_2)!}$$

$$\sum_n (-1)^{n+j_1+m_1} \frac{(j+j_2+m_1-n)! (j_1-m_1+n)!}{(j-j_1+j_2-n)! (j+m_1-n)! n! (n+j_1-j_2-m_1)!}$$

Wigner's formulae of CG Coefficients.

CG Coefficients &  $3j$  symbol

properties of  $S_{m_1 m_2 j_m}^{j_1 j_2} = \langle j_1 j_2 m_1 m_2 | j_m \rangle$

1)  $j_1 + j_2 + j \in \text{integer}$ ,  $j = j_1 + j_2, j_1 + j_2 - 1$

in order that

$$S_{m_1 m_2 j_m}^{j_1 j_2} \neq 0$$

triangle inequality

$$j_1 + j_2 - j \geq 0$$

$$\begin{aligned} j_1 - j_2 + j &\geq 0 \\ -j_1 + j_2 + j &\geq 0 \end{aligned} \quad \left. \begin{array}{l} j \geq |j_1 - j_2| \\ j \leq j_1 + j_2 \end{array} \right\}$$

$$-j \leq j_1 - j_2 \leq j$$

$$m_1 + m_2 = m$$

2)  $S_{m_1 m_2 j_m}^{j_1 j_2} \in \mathbb{R}$

3), Unitarity

$$S_{m_1 m_2 \bar{j} m}^{j_1 j_2} \equiv \left( S_{\bar{j} m}^{j_1 j_2} \right)^*_{j_1 m_1 m_2}$$

$$S^+ S = S S^+ = 1$$

4)  $S_{m_1 m_2 \bar{j} m}^{j_1 j_2} = (-1)^{\bar{j}_1 + \bar{j}_2 - \bar{j}} S_{m_2 m_1 \bar{j} m}^{j_2 j_1}$

recurrence relation

$$\sqrt{j(j+1) - m(m+1)} S_{m_1 m_2 \bar{j} m+1}^{j_1 j_2} = \sqrt{j_1(j_1+1) - m_1(m_1+1)} S_{m_1 m_2 \bar{j} m}^{j_1 j_2}$$

$$+ \sqrt{j_2(j_2+1) - m_2(m_2+1)} S_{m_1 m_2 - 1 \bar{j} m}^{j_1 j_2}$$

$$\sqrt{j(j+1) - m(m-1)} S_{m_1 m_2 \bar{j} m-1}^{j_1 j_2} = \sqrt{j_1(j_1+1) - m_1(m_1+1)} S_{m_1 m_2 \bar{j} m}^{j_1 j_2}$$

$$+ \sqrt{j_2(j_2+1) - m_2(m_2+1)} S_{m_1 m_2 + 1 \bar{j} m}^{j_1 j_2}$$

# Wigner's 3j-Symbol

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \frac{(-1)^{j_1+j_2+m}}{\sqrt{2j+1}} \langle j_1 j_2 m_1 m_2 | j_3 j_1 j_2 j_3 m_3 \rangle$$

Symmetry properties:

- $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix}$

- $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_3 & m_1 & m_2 \end{pmatrix}$

- $\begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = \underbrace{\begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix}}_{\text{swap } j_1 \text{ and } j_2} = \underbrace{\begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix}}_{\text{swap } j_1 \text{ and } j_2}$   
 $= (-1)^{j_1+j_2+j_3} \underbrace{\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}}_{\text{original}}$

$$\begin{pmatrix} j_1, \hat{j}_2, \hat{j}_3 \\ -m_1, -m_2, -m_3 \end{pmatrix} = (-1)^{\hat{j}_1 + \hat{j}_2 + \hat{j}_3} \begin{pmatrix} \hat{j}_1, \hat{j}_2, \hat{j}_3 \\ m_1, m_2, m_3 \end{pmatrix}$$

change of basis

$$|j_1, j_2, m\rangle = (-1)^{\hat{j}_2 - j - m} \sum_{m_1, m_2} \sqrt{2\hat{j}+1} \begin{pmatrix} \hat{j}_1, \hat{j}_2, \hat{j} \\ m_1, m_2, m \end{pmatrix}$$

$$|\hat{j}_1, \hat{j}_2, m_1, m_2\rangle = \sum_{\hat{j}m} (-1)^{\hat{j}_2 - \hat{j}_1 - m} \sqrt{2\hat{j}+1} \begin{pmatrix} \hat{j}_1, \hat{j}_2, \hat{j} \\ m_1, m_2, -m \end{pmatrix}$$

$$|\hat{j}, \hat{j}_2, \hat{j}m\rangle$$

Unitarity

$$\sum_{m_1, m_2} \begin{pmatrix} \hat{j}_1, \hat{j}_2, \hat{j}_3 \\ m_1, m_2, m_3 \end{pmatrix} \begin{pmatrix} \hat{j}_1, \hat{j}_2, \hat{j}'_3 \\ m_1, m_2, m'_3 \end{pmatrix} = \frac{1}{2\hat{j}_3 + 1} \delta_{\hat{j}_3, \hat{j}'_3} \delta_{m_3, m'_3}$$

$$\sum_{\hat{j}_3, m_3} (2\hat{j}_3 + 1) \begin{pmatrix} \hat{j}_1, \hat{j}_2, \hat{j}_3 \\ m_1, m_2, m_3 \end{pmatrix} \begin{pmatrix} \hat{j}_1, \hat{j}_2, \hat{j}_3 \\ m'_1, m'_2, m'_3 \end{pmatrix} = \delta_{m_1, m'_1} \delta_{m_2, m'_2}$$

## Coupling 3 angular momenta

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \quad \vec{\mathbb{J}}_i \in \mathcal{H}_i$$

$$\vec{\mathbb{J}} = \vec{\mathbb{J}}_1 + \vec{\mathbb{J}}_2 + \vec{\mathbb{J}}_3 \quad \begin{matrix} \\ \text{imp of} \\ \text{SU}(2) \end{matrix}$$

$$\vec{\mathbb{J}} \in \mathcal{H} \quad \text{w/ } \vec{j}_i$$

decoupled basis in  $\mathcal{H}$

$$|\vec{j}_1, \vec{j}_2, \vec{j}_3, m_1, m_2, m_3\rangle = |\vec{j}_1, m_1\rangle \otimes |\vec{j}_2, m_2\rangle \otimes |\vec{j}_3, m_3\rangle$$

$$\begin{matrix} \vec{\mathbb{J}}_i^2 & (\vec{\mathbb{J}}_i)_z \end{matrix} \quad \begin{matrix} \vec{\mathbb{J}}_i^2 \\ \vec{\mathbb{J}}_i \end{matrix} |\vec{j}_i, m_i\rangle = \hbar^2 (j_i + 1) |\vec{j}_i, m_i\rangle$$

$$(\vec{\mathbb{J}}_i)_z |\vec{j}_i, m_i\rangle = \hbar m_i |\vec{j}_i, m_i\rangle$$

Recoupling schemes : Scheme 1 : firstly couple  $\vec{j}_1 \& \vec{j}_2$

$$\vec{\mathbb{J}}_{12} = \vec{\mathbb{J}}_1 + \vec{\mathbb{J}}_2$$

$$\text{Common eigenbasis } \begin{matrix} \vec{\mathbb{J}}_1^2 & \vec{\mathbb{J}}_2^2 & \vec{\mathbb{J}}_{12}^2 \\ (\vec{\mathbb{J}}_{12})_z \end{matrix}$$

$$|\vec{j}_1, \vec{j}_2, \vec{j}_{12}, m_{12}\rangle = \sum_{m_1, m_2} |\vec{j}_1, \vec{j}_2, m_1, m_2\rangle$$

$$\langle \vec{j}_1, \vec{j}_2, m_1, m_2 | \vec{j}_1, \vec{j}_2, \vec{j}_{12}, m_{12} \rangle$$

then couple  $J_{12}$  and  $J_3$  :  $\vec{J} = \vec{J}_{12} + \vec{J}_3$

$$\Rightarrow J_1 + J_2 + J_3$$

$$|(j_1 j_2) \hat{j}_{12} \hat{j}_3 \hat{j}_m\rangle = \sum_{m_{12} m_3} \underbrace{|j_1 j_2 \hat{j}_{12} \hat{j}_3 m_{12} m_3\rangle}_{\text{common eigenbasis}} \underbrace{\underbrace{|j_1 j_2 \hat{j}_{12} m_{12}\rangle \otimes |j_3 m_3\rangle}_{\text{coupling}}}_{\text{of } J_1^2 J_2^2 J_3^2 \hat{J}_{12}^2 \hat{J}^2 \hat{J}_2}$$

$$\langle (j_1 j_2) \hat{j}_{12} \hat{j}_3 m_{12} m_3 | (j_1 j_2) \hat{j}_{12} \hat{j}_3 \hat{j}_m \rangle = \sum_{m_{12} m_3} \langle j_1 j_2 \hat{j}_{12} m_{12} m_3 | j_1 j_2 \hat{j}_{12} m_{12} \rangle \langle j_1 j_2 \hat{j}_{12} m_{12} | j_3 m_3 \rangle$$

$$= \sum_{\substack{m_{12} m_3 \\ m_1 m_2}} \langle j_1 j_2 \hat{j}_3 m_1 m_2 m_3 \rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 \hat{j}_{12} m_{12} \rangle \langle j_{12} \hat{j}_3 m_3 | j_{12} \hat{j}_3 \hat{j}_m \rangle$$

Scheme 2 first  $\vec{J}_{23} = \vec{J}_2 + \vec{J}_3$  then

$$\vec{J} = \vec{J}_1 + \vec{J}_{23}$$

common eigenbasis of  $J_1^2 J_2^2 J_3^2 \hat{J}_{23}^2 \hat{J}^2 \hat{J}_2$

$$|\hat{j}_1(\hat{j}_2\hat{j}_3)\hat{j}_{23}\hat{j}^m\rangle$$

$$= \sum_{\substack{m_1 m_2 m_3 \\ m_{23}}} |\hat{j}_1 \hat{j}_2 \hat{j}_3 m_1 m_2 m_3\rangle \langle \hat{j}_2 \hat{j}_3 m_1 m_2 | \hat{j}_2 \hat{j}_3 \hat{j}_{23} m_{23}\rangle$$

$$\langle \hat{j}_1 \hat{j}_{23} m_1 m_{23} | \hat{j}_1 \hat{j}_{23} \hat{j}^m \rangle$$

two orthonormal basis

$$|(\hat{j}_1 \hat{j}_2) \hat{j}_3 \hat{j}_{12} \hat{j}^m\rangle \xleftarrow{\text{unitary transf.}} |\hat{j}_1(\hat{j}_2 \hat{j}_3) \hat{j}_{23} \hat{j}^m\rangle$$

$$|\hat{j}_1(\hat{j}_2 \hat{j}_3) \hat{j}_{23} \hat{j}^m\rangle = \sum_{\hat{j}_{12}} |\epsilon(\hat{j}_1 \hat{j}_2) \hat{j}_{12} \hat{j}_3 \hat{j}^m\rangle$$

$$\langle (\hat{j}_1 \hat{j}_2) \hat{j}_{12} \hat{j}_3 \hat{j}^m | \hat{j}_1(\hat{j}_2 \hat{j}_3) \hat{j}_{23} \hat{j}^m \rangle$$

$$\equiv \sqrt{(2\hat{j}_{12}+1)(2\hat{j}_{23}+1)}$$

$$W(\hat{j}_1 \hat{j}_2 \hat{j}_3; \hat{j}_{12} \hat{j}_{23})$$

Racah coefficients

explicit formula of Racah coefficients

$$W(a, b, c, d; ef) = (-1)^{a+b+c+d} \Delta(abc) \Delta(acf)$$

$$\Delta(bdf) \Delta(cde) \times$$

$$\times \sum_z \frac{(-1)^z (z+1)!}{(z-a-b-c)! (z-c-d-e)! (z-a-c-f)! (z-b-d-f)!}$$

$$\times \frac{1}{(a+b+c+d-z)! (a+d+e+f-z)! (b+c+e+f-z)!}$$

$$\Delta(abc) = \sqrt{\frac{(a+b+c)! (a-b+c)! (b+c-a)!}{(a+b+c+1)!}}$$

$6j$  symbol.

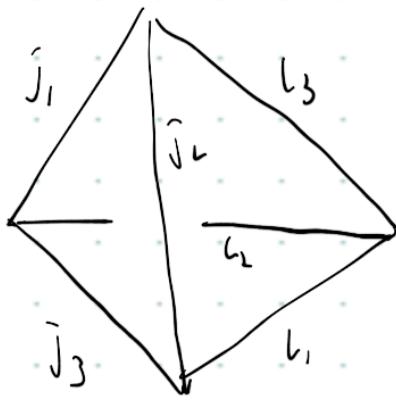
$$\left\{ \begin{matrix} a & b & c \\ d & e & f \end{matrix} \right\} = (-1)^{a+b+c+d} W(abcde; ef)$$

$\nearrow$

$6 \text{ spins}$        $\left\{ \begin{matrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{matrix} \right\}$        $j_i, l_i = 0, \frac{1}{2}, 1, \dots$

(1)  $\left\{ \begin{matrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{matrix} \right\} \neq 0$     if  $j_i, l_i$  form a tetrahedron  
 s.t.  $j_i, l_i$  are edge lengths of

# the tetrahedron



i.e.,  $j_1, j_2, j_3$  satisfy triangle inequality

$$j_1 \ l_2 \ l_3 = \dots$$

$$j_3 \ l_1 \ l_2 = \dots$$

$$j_2 \ l_1 \ l_3 = \dots$$

(2) i.e., under permuting columns

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = \begin{Bmatrix} j_2 & j_1 & j_3 \\ l_2 & l_1 & l_3 \end{Bmatrix} = \dots$$

(3) i.e., under top-down flip in any pair of columns:

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = \begin{Bmatrix} l_1 & l_2 & j_3 \\ j_1 & j_2 & l_3 \end{Bmatrix} = \dots$$

(4) orthogonality / unitarity

$$\sum_{j_3} (2j_3 + 1) \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l'_3 \end{Bmatrix} = \frac{1}{2l_3 + 1} \delta_{l_3 l'_3}$$

$$\sum_{l_3} (-1)^{j_1+j_2+j_3} (2l_3+1) \left\{ \begin{matrix} j_1 j_2 j_3 \\ l_1 l_2 l_3 \end{matrix} \right\} \left\{ \begin{matrix} j_1 l_1 \bar{j} \\ j_2 l_2 l_3 \end{matrix} \right\} = \left\{ \begin{matrix} j_1 j_2 j_3 \\ l_2 l_1 \bar{j} \end{matrix} \right\}$$

(5) a formula to relate 3j symbols

$$\sum_{\mu_1 \mu_2 \mu_3} (-1)^{l_1+l_2+l_3+\mu_1+\mu_2+\mu_3} \left( \begin{matrix} j_1 l_1 l_2 \\ \mu_1 \mu_2 - \mu_3 \end{matrix} \right) \left( \begin{matrix} l_1 j_2 l_3 \\ -\mu_1 \mu_2 \mu_3 \end{matrix} \right) \left( \begin{matrix} l_1 l_2 j_3 \\ \mu_1 - \mu_2 \mu_3 \end{matrix} \right)$$

$$= \left( \begin{matrix} j_1 j_2 j_3 \\ m_1 m_2 m_3 \end{matrix} \right) \left\{ \begin{matrix} j_1 j_2 j_3 \\ l_1 l_2 l_3 \end{matrix} \right\}$$

4 - angular-momentum coupling & 9j symbol.

$$\vec{J} = \vec{J}_1 + \vec{J}_2 + \vec{J}_3 + \vec{J}_4 \quad \text{acting} \quad H = \frac{\vec{J}_1^2}{J_{12}} J_{12} \otimes H_1 \otimes H_2 \otimes H_3 \otimes H_4$$

Scheme 1

$$\begin{matrix} J_1 & J_2 & J_3 & J_4 \\ \checkmark & & \checkmark & \\ J_{12} & & J_{34} & \\ & \swarrow & & \end{matrix}$$

$$\begin{matrix} J_1^2 & J_2^2 & J_{12}^2 & J_3^2 & J_4^2 & J_{34}^2 & J_2^2 & J_2 \\ | & (j_1 j_2) j_{12} & (j_3 j_4) \bar{j}_{34} & \bar{j}_m \end{matrix}$$

Scheme 2

$$\begin{matrix} J_1 & J_2 & J_3 & J_4 \\ & \cancel{\checkmark} & \cancel{\checkmark} & \\ J_{13} & & J_{24} & \\ & \swarrow & & \end{matrix}$$

$$\begin{matrix} J_1^2 & J_3^2 & J_{13}^2 & J_2^2 & J_4^2 & J_{24}^2 & J_2^2 & J_2 \\ | & (j_1 j_3) \bar{j}_{13} & (j_2 j_4) \bar{j}_{24} & \bar{j}_m \end{matrix}$$

J

Unitary transf. between 2 basis : 9J symbol.

$$\left\{ \begin{array}{l} \hat{j}_1 \hat{j}_2 \hat{j}_{12} \\ \hat{j}_3 \hat{j}_4 \hat{j}_{34} \\ \hat{j}_{13} \hat{j}_{24} \hat{j} \end{array} \right\} = \frac{1}{\sqrt{(2\hat{j}_{12}+1)(2\hat{j}_{34}+1)(2\hat{j}_{13}+1)(2\hat{j}_{24}+1)}}$$

$$\langle (\hat{j}_1 \hat{j}_2) \hat{j}_{12} (\hat{j}_3 \hat{j}_4) \hat{j}_{34} \hat{j}^m | (\hat{j}_1 \hat{j}_3) \hat{j}_{13} (\hat{j}_2 \hat{j}_4) \hat{j}_{24} \hat{j}^n \rangle$$

$$= \sum_{j'} (-1)^{2j'} (2j'+1) \left\{ \begin{array}{l} \hat{j}_1 \hat{j}_2 \hat{j}_{12} \\ \hat{j}_{34} \hat{j} \hat{j}' \end{array} \right\} \left\{ \begin{array}{l} \hat{j}_3 \hat{j}_4 \hat{j}_{34} \\ \hat{j}_1 \hat{j}' \hat{j}_{24} \end{array} \right\} \left\{ \begin{array}{l} \hat{j}_{12} \hat{j}_{34} \hat{j} \\ \hat{j}' \hat{j}_1 \hat{j}_3 \end{array} \right\}$$

Ls coupling & jj coupling

two electrons with orbital angular momenta  $\vec{l}_1, \vec{l}_2$

spin angular momenta  $\vec{s}_1, \vec{s}_2$

total angular momentum :  $\vec{J} = \vec{l}_1 + \vec{s}_1 + \vec{l}_2 + \vec{s}_2$

$$\hat{H} = \hat{H}_{L_1} \otimes \hat{H}_{S_1} \otimes \hat{H}_{L_2} \otimes \hat{H}_{S_2}$$

$L_S$  coupling

$$\begin{matrix} \vec{L}_1 & \vec{S}_1 \\ \checkmark & \checkmark \\ \vec{L}_2 & \vec{S}_2 \\ \checkmark & \checkmark \\ \vec{J} & \checkmark \end{matrix}$$

$$| (L_1 L_2) L, (S_1 S_2) S \rangle_{jm} \rangle$$

$$\equiv | L S \rangle_{jm} \rangle$$

$J_J$  coupling

$$\begin{matrix} \vec{L}_1 & \vec{S}_1 \\ \checkmark & \checkmark \\ \vec{L}_2 & \vec{S}_2 \\ \checkmark & \checkmark \\ \vec{J} & \checkmark \end{matrix}$$

$$| (L_1 S_1) J_1, (L_2 S_2) J_2 \rangle_{jm} \rangle$$

$$| J, J_L \rangle_{jm} \rangle$$

Hamiltonian

$$\hat{H} = \hat{H}_{01} + \hat{H}_{02} + \frac{e^2}{r_{12}} + \underbrace{f(r_1) \vec{S}_1 \cdot \vec{L}_1 + f(r_2) \vec{S}_2 \cdot \vec{L}_2}_{R}$$

$$r_{12} = |\vec{r}_1 - \vec{r}_2|$$

spin-orbital  
interaction

$$\hat{H}_{0i} = \frac{1}{2m} \hat{p}_i^2 + V_i(r_i)$$

$$\hat{H} |4\rangle = E |4\rangle$$

if  $\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} \equiv \hat{H}_0$  no interaction

$$\left. \begin{array}{l} \text{commuting} \\ \text{operators} \end{array} \right\} \left. \begin{array}{l} \hat{H}_0, \vec{L}^2, S^2, J^2, J_2 \\ \hat{H}_0, J_1^2, \vec{J}_1 \cdot \vec{J}_2, J_2 \end{array} \right. \rightarrow \left. \begin{array}{l} |\chi LS j m\rangle \\ |\chi j_1 j_2 \bar{j} m\rangle \end{array} \right.$$

↑  
label of radial-part  
wave function

- $\left( H_0 + \frac{e^2}{r_n} \right) |4\rangle = E |4\rangle$  if  $f(r_1), f(r_2)$  smaller than  $e^2$

↑ rotational invariance → commuting with  $\vec{L} = \vec{L}_1 + \vec{L}_2$

commuting operators:  $H_0 + \frac{e^2}{r_n}, \vec{L}^2, S^2, J^2, J_1, J_2$

$|4\rangle$  is LS coupling state

$$|4\rangle = |\chi LS j m\rangle$$

- $\left( H_0 + f(r_1) \vec{s}_1 \cdot \vec{L}_1 + f(r_2) \vec{s}_2 \cdot \vec{L}_2 \right) |4\rangle = E |4\rangle$

if  $e^2$  is smaller than  $f(r_1), f(r_2)$

$\vec{s}_i \cdot \vec{L}_i$  commutes with  $J_i^2$

$$\text{Commuting operators : } H = f(r_1) \vec{\sigma}_1 \cdot \vec{L}_1 + f(r_2) \vec{\sigma}_2 \cdot \vec{L}_2 \\ = J_1^2, J_2^2, J^2, J_z$$

|4> is  $\vec{j}\vec{j}$  coupling state.

$$|4> = |\alpha, J, J_L, J_M>$$

Irreducible tensor operator :

tensor and tensor operator : Vector and transformation under rotation

$$v'_k = \sum_i Q(\alpha, \beta, \gamma)_{ki} v_i \quad Q(\alpha, \beta, \gamma) \in SO(3) \\ k, i = 1, 2, 3$$

Tensor product :  $\vec{a}, \vec{b}$  Vectors

rank-2 tensor :  $\vec{a} \otimes \vec{b} \equiv ab$

$$(ab)_{k_1 k_2} = a_{k_1} b_{k_2}$$

$$\begin{pmatrix} a_1 b_1 & a_2 b_2 & a_3 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{pmatrix}$$

tensor transf,

$$\text{as : } (a' b')_{k_1 k_2} = a'_{k_1} b'_{k_2}$$

$$= \sum_{i_1, i_2} Q_{k_1 i_1} Q_{k_2 i_2} a_{i_1} b_{i_2}$$

$$= \sum_{i_1 i_2} Q_{k_1 i_1} Q_{k_2 i_2} \underline{(ab)_{i_1 i_2}}$$

in QM: vector operator

$$\hat{\vec{V}}' = D(Q) \hat{\vec{V}} D(Q)^{-1} = Q^{-1} \hat{\vec{V}}$$

( $Q \in SO(3)$ )

$$D(Q) \hat{\vec{V}}: D(Q)^{-1} = \sum_j Q_{ij}^{-1} \hat{V}_j \\ = \sum_j \hat{V}_j Q_{ji}$$

rank-2 tensor operator

$$D(Q) \hat{T}_{ij} D(Q)^{-1} = \sum_{kl} \hat{T}_{kl} Q_{ki} Q_{lj}$$

$$\hat{T}_{ij} = \hat{A}_i \hat{B}_j \text{ has } 3 \times 3 = 9 \text{ components.}$$

but some linear combinations have simpler transformation rules.

- $\hat{A}_1 \hat{B}_1 + \hat{A}_2 \hat{B}_2 + \hat{A}_3 \hat{B}_3$  is inv. under rotation.

$$D(Q) \sum_i \hat{A}_i \hat{B}_i D(Q)^{-1} = \sum_i \sum_j \hat{A}_j Q_{ji} \sum_k \hat{B}_k Q_{kj} \\ D(Q)^{-1} D(Q) = \sum_j \hat{A}_j \hat{B}_j$$

$$\begin{aligned} & (\hat{A}_2 \hat{B}_3 - \hat{A}_3 \hat{B}_2, \hat{A}_3 \hat{B}_1 - \hat{A}_1 \hat{B}_3, \hat{A}_1 \hat{B}_2 - \hat{A}_2 \hat{B}_1) \\ & = \hat{\vec{A}} \times \hat{\vec{B}} \quad \text{transf. as a vector.} \end{aligned}$$

there are parts in this tensor operator, transforming as scalar and vector.

We want to separate parts in  $\hat{T}_{ij}$  that transf. differently.

$Q(\alpha, \beta, \gamma)$  is 3-dim irrep of  $SO(3)$  carried by

$$\begin{array}{ccc} \mathbb{R}^3 & \ni & \vec{v}, \vec{a}, \vec{b} \\ \downarrow & & \\ \mathcal{H}_{j=1} & & \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \end{array}$$

$$\mathcal{H}_{j=1}$$

on  $\mathcal{H}_{j=1}$  we have  $D_{m_1 m_2}^{j=1}(\alpha, \beta, \gamma)$

$$|j, m\rangle \quad m = -1, 0, 1 \quad \uparrow$$

unitary equivalent

$$D^{j=1}(\alpha, \beta, \gamma) = U Q(\alpha, \beta, \gamma) U^{-1}$$

$$U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \quad \begin{array}{l} \text{unitary transf.} \\ \text{from } x, y, z\text{-basis} \\ \text{to } |\alpha\rangle, |\beta\rangle, |\gamma\rangle \end{array}$$