Compute CG coefficients (sketch)

$$|j_1 j_2 j m\rangle = \sum_{m_1 m_2} |j_1 j_2 m_1 m_2\rangle \, C^{j_1 j_2}_{m_1 m_2\, j m}$$

at $m = j$

$$\underline{|j_1 j_2 j j\rangle = \sum_{m_1 m_2} |j_1 j_2 m_1 m_2\rangle} \, \underbrace{C^{j_1 j_2}_{m_1 m_2\, j j}}_{\equiv a_{m_1 m_2} \delta_{j, m_1+m_2}}$$

$$0 = \hat{J}_+ |j_1 j_2 j j\rangle = (\hat{J}_{1+} + \hat{J}_{2+}) |j_1 j_2 j j\rangle$$

$$= \sum_{m_1 m_2} |j_1 j_2 m_1+1, m_2\rangle \sqrt{(j_1 - m_1)(j_1 + m_1 + 1)} \; a_{m_1 m_2} \delta_{j\, m_1+m_2}$$

$$+ \sum_{m_1 m_2} |j_1 j_2 m_1 m_2+1\rangle \sqrt{(j_2 - m_2)(j_2 + m_2 + 1)} \; a_{m_1 m_2} \delta_{j, m_1+m_2}$$

1st term: $m_2 \to m_2' + 1$

2nd term: $m_1 \to m_1' + 1$

$$0 = \sum_{m_1 m_2} |j_1 j_2 m_1+1 \; m_2'+1\rangle \sqrt{(j_1 - m_1)(j_1 + m_1 + 1)} \; a_{m_1 m_2'+1} \delta_{j, m_1+m_2'+1}$$

$$+ \sum_{m_1 m_2} |j_1 j_2 m_1'+1 \; m_2+1\rangle \sqrt{(j_2 - m_2)(j_2 + m_2 + 1)} \; a_{m_1+1\, m_2} \delta_{j, m_1'+m_2+1}$$

$$\Rightarrow \sqrt{(j_1-m_1)(j_1+m_1+1)}\ a_{m_1, m_2+1}\ \cancel{\delta_{j\ m_1+m_2+1}}$$

$$= -\sqrt{(j_2-m_2)(j_2+m_2+1)}\ a_{m_1+1, m_2}\ \cancel{\delta_{j\ m_1+m_2+1}}$$

$m_2 \to m_2 - 1$

$$\Rightarrow \quad a_{m_1 m_2} = -\sqrt{\frac{(j_2-m_2+1)(j_2+m_2)}{(j_1-m_1)(j_1+m_1+1)}}\ a_{m_1+1\ m_2-1}$$

recursion relation for $a_{m_1 m_2}$

$$a_{m_1 m_2} = (-1)\sqrt{\frac{(j_2-m_2+1)(j_2+m_2)}{(j_1-m_1)(j_1+m_1+1)}}\ (-1)$$

$$\sqrt{\frac{(j_2-m_2+2)(j_2+m_2-1)}{(j_1-m_1-1)(j_1+m_1+2)}}\ a_{m_1+2\ m_2-2}$$

$$= \cdots$$

$$= (-1)^{j_1-m_1}\sqrt{\frac{(j_1+m_1)!\,(j_2+m_2)!}{(j_1-m_1)!\,(j_2-m_2)!}}\ \sqrt{\frac{(j_1+j_2-j)!}{(j_2-j_1+j)!\,(2j)!}}$$

$$\times\ a_{j_1, j-j_1}$$

by normalization of $|\hat{j}_1 \hat{j}_2 \hat{j} \hat{j}\rangle$ $\quad$ $\underbrace{\text{a only dep. on } j}$

$$\Rightarrow a = \sqrt{\frac{(2\hat{j}+1)!\,(\hat{j}_1+\hat{j}_2-\hat{j})!}{(\hat{j}+\hat{j}_1+\hat{j}_2+1)!\,(\hat{j}+\hat{j}_1-\hat{j}_2)!\,(\hat{j}-\hat{j}_1+\hat{j}_2)!}}$$

$$S^{\hat{j}_1\hat{j}_2}_{m_1 m_2 \hat{j}\,\underline{\underline{\hat{j}}}} = a_{m_1 m_2}\,\delta_{\hat{j},\,m_1+m_2}.$$

$$= \delta_{\hat{j},\,m_1+m_2}\,(-1)^{\hat{j}_1-m_1}\sqrt{\frac{(2\hat{j}+1)!\,(\hat{j}_1+\hat{j}_2-\hat{j})!}{(\hat{j}+\hat{j}_1+\hat{j}_2+1)!\,(\hat{j}+\hat{j}_1-\hat{j}_2)!\,(\hat{j}-\hat{j}_1+\hat{j}_2)!}}$$

$$\times\sqrt{\frac{(\hat{j}_1+m_1)!\,(\hat{j}_2+m_2)!}{(\hat{j}_1-m_1)!\,(\hat{j}_2-m_2)!}}$$

general CG coefficients $S^{\hat{j}_1\hat{j}_2}_{m_1 m_2 \hat{j} m}$:

$$S^{\hat{j}_1\hat{j}_2}_{m_1 m_2 \hat{j} m} = \langle \hat{j}_1 \hat{j}_2\, m_1 m_2 \,|\, \underline{\hat{j}_1 \hat{j}_2\, \hat{j}\, m}\rangle$$

$$= \sqrt{\frac{(\hat{j}+m)!}{(2\hat{j})!\,(\hat{j}-m)!}}\;\langle \hat{j}_1 \hat{j}_2\, m_1 m_2 \,|\, (\hat{J}_-)^{\hat{j}-m}\,|\,\hat{j}_1 \hat{j}_2\, \hat{j}\, \hat{j}\rangle$$

$$\hat{J}_- = \hat{J}_{1-} + \hat{J}_{2-}$$

$$S^{j_1 j_2 \, *}_{m_1 m_2 j m} = \sqrt{\frac{(j+m)!}{(2j)!(j-m)!}} \, \underline{\langle j_1 j_2 j j|} \underbrace{\left(\hat{J}_{1+} + \hat{J}_{2+}\right)^{j-m} |j_1 j_2 m_1 m_2\rangle}$$

$$\left(\hat{J}_{1+} + \hat{J}_{2+}\right)^{j-m} |j_1 j_2 m_1 m_2\rangle$$

$$= \sum_s \frac{(j-m)!}{s!(j-m-s)!} \hat{J}_{1+}^{s} \hat{J}_{2+}^{j-m-s} |j_1 j_2 m_1 m_2\rangle$$

$$= \sum_s \frac{(j-m)!}{s!(j-m-s)!} \underline{|j_1 j_2 m_1+s, m_2+j-m-s\rangle}$$

$$\times \sqrt{\frac{(j_1-m_1)!(j_1+m_1+s)!}{(j_1-m_1-s)!(j_1+m_1)!}} \sqrt{\frac{(j_2-m_2)!(j_2+m_2+j-m-s)!}{(j_2-m_2-j+m+s)!(j_2+m_2)!}}$$

$$\langle j_1 j_2 j j | j_1 j_2 m_1 m_2\rangle = \underline{S^{j_1 j_2}_{m_1 m_2 j j}}$$

$$S^{j_1 j_2}_{m_1 m_2 j m} = \delta_{m, m_1+m_2} \sqrt{(2j+1) \frac{(j_1+j_2-j)!\,(j_1-m_1)!\,(j_2-m_2)!\,(j+m)!\,(j-m)!}{(j+j_1+j_2+1)!\,(j+j_1-j_2)!\,(j-j_1+j_2)!\,(j_1+m_1)!\,(j_2+m_2)!}}$$

$$\times \sum_s (-1)^{j_1+m_1+s} \frac{(j_1+m_1+s)!\,(j+j_2-m_1-s)!}{s!\,(j-m-s)!\,(j_1-m_1-s)!\,(j_2-j+m_1+s)!}$$

Edmonds formula of CG coefficients.

$$(x)! = \begin{cases} x! & x \in \mathbb{N}_+ \\ 0 & \text{otherwise} \end{cases}$$

$\sum_s$ is a finite sum so that all $(\cdots)! \neq 0$

Relation between CG coefficients and D-matrix

$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

rotation in $\mathcal{H}_1$ $\quad D_1(\vec{n}, \varphi)\, |j_1, m_1\rangle = e^{-\frac{i}{\hbar}\varphi \vec{n}\cdot\hat{\vec{J}}_1} |j_1, m_1\rangle$

$$= \sum_{m_1'} |j_1, m_1'\rangle\, D^{j_1}_{m_1' m_1}(\vec{n}, \varphi)$$

$\sim\sim$, in $\mathcal{H}_2$ $\quad D_2(\vec{n},\varphi)|j_2 m_2\rangle = e^{-\frac{i}{\hbar}\varphi\vec{n}\cdot\hat{\vec{J}}_2}|j_2 m_2\rangle$

$$= \sum_{m_2'} |j_2 m_2'\rangle D^{j_2}_{m_2' m_2}(\vec{n},\varphi)$$

in $\mathcal{H}_1 \otimes \mathcal{H}_2$, $\quad D(\vec{n},\varphi) = D_1(\vec{n},\varphi) \otimes D_2(\vec{n},\varphi)$

$$= e^{-\frac{i}{\hbar}\varphi\vec{n}\cdot(\hat{\vec{J}}_1+\hat{\vec{J}}_2)}$$

$$D(\vec{n},\varphi)\,\underline{|j_1 j_2 m_1 m_2\rangle} = e^{-\frac{i}{\hbar}\varphi\vec{n}\cdot(\vec{J}_1+\vec{J}_2)}|j_1 j_2 m_1 m_2\rangle$$

$$= e^{-\frac{i}{\hbar}\varphi\vec{n}\cdot\vec{J}_1}|j_1 m_1\rangle \otimes e^{-\frac{i}{\hbar}\varphi\vec{n}\cdot\vec{J}_2}|j_2 m_2\rangle$$

$$= \sum_{m_1' m_2'} |j_1 j_2 m_1' m_2'\rangle \underline{D^{j_1}_{m_1' m_1}(\varphi,\vec{n})\, D^{j_2}_{m_2' m_2}(\varphi,\vec{n})}$$

$$\underline{D(\vec{n},\varphi)|j_1 j_2 j m\rangle} = e^{-\frac{i}{\hbar}\varphi\vec{n}\cdot\underbrace{(\vec{J}_1+\vec{J}_2)}_{\vec{J}}}|j_1 j_2 j m\rangle$$

fix $(j_1 j_2)$, fix j

$$= \underline{\sum_{m'} |j_1 j_2 j m\rangle\, \underline{D^{j}_{m' m}}(\varphi,\vec{n})}$$

$\{|j_1 j_2 j m\rangle\}_{m=-j}^{j}$

spans irrep of $SU(2)$

apply $$|j_1 j_2 m_1 m_2\rangle = \sum_{jm} |j_1 j_2 j m\rangle \left(S^{j_1 j_2}\right)^{-1}_{\underline{jm},\, \underline{m_1 m_2}}$$

$$|j_1 j_2 j m\rangle = \sum_{m_1 m_2} |j_1 j_2 m_1 m_2\rangle S^{j_1 j_2}_{m_1 m_2 j m}$$

$$D(Q)\, |j_1 j_2 \underline{\underline{jm}}\rangle = D(Q) \sum_{m_1 m_2} |j_1 j_2 m_1 m_2\rangle S^{j_1 j_2}_{m_1 m_2 jm}$$

$\uparrow$

$Q \in SU(2)$

$$= \sum_{\substack{m_1 m_2 \\ m_1' m_2'}} \underbrace{|j_1 j_2 m_1' m_2'\rangle}_{\sum_{j'm'} |j_1 j_2 j' m'\rangle \left(S^{j_1 j_2}\right)^{-1}_{j'm'\, m_1' m_2'}} \underline{D^{j_1}_{m_1' m_1}(Q)\, D^{j_2}_{m_2' m_2}(Q)}\, S^{j_1 j_2}_{m_1 m_2\, \underline{jm}}$$

$$= \sum_{j'm'} |j_1 j_2 j' m'\rangle \underbrace{\left[\sum_{\substack{m_1 m_2 \\ m_1' m_2'}} \left(S^{j_1 j_2}\right)^{-1}_{j'm'\, m_1' m_2'} D^{j_1}_{m_1' m_1}(Q)\, D^{j_2}_{m_2' m_2}(Q)\, S^{j_1 j_2}_{m_1 m_2\, \underline{jm}} \right]}_{\underline{\underline{\left[S^{-1}\left(D^{j_1}(Q) \otimes D^{j_2}(Q)\right) S\right]_{j'm',\, jm}}}}$$

/

$$\left[\underset{=}{S}^{-1}\left(\underline{D^{j_1}(Q)\otimes D^{j_2}(Q)}\right)\underset{=}{S}\right]_{j'm',\,jm} = \delta_{jj'}\,\underline{D^{j}_{m'm}(Q)}$$

$\mathcal{H}$ is spanned by $\{|j_1 j_2 j m\rangle\}_{j=|j_1-j_2|}^{j_1+j_2}\,{}_{m=-j}^{j}$

reducible rep.

$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

$|j_1 m_1\rangle$ $|j_2 m_2\rangle$

D^{j_1} D^{j_2}

D^{j_1}, D^{j_2} 2 irreps of $SU(2)$

$D^{j_1}\otimes D^{j_2}$ is also rep of $SU(2)$

(tensor product rep of $SU(2)$)

rep. matrix $D^{j}_{m_1'm_1}(Q)\, D^{j}_{m_2'm_2}(Q)$

$\forall\, Q \in SU(2)$

$$D^{j_1}\otimes D^{j_2} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} D^{j}$$

decomposition of tensor product rep into irreps

How D-matrix in the irrep is expressed in terms of D-matrix in tensor product rep.

$$\sum_{j} \left(S\, D^{j}(Q)\, S^{-1} \right) = D^{j_1}(Q) \otimes D^{j_2}(Q)$$

multiply $D^{j}(Q)^{*}$ and integrate Q over $Q \in SU(2)$

$$S^{j_1 j_2}_{m_1' m_2'\, j' m'}\, S^{j_1 j_2\;-1}_{j m\, m_1 m_2}\, \frac{16\pi^2}{2j'+1}$$

$$= \int D^{j_1}_{m_1' m_1}(Q)\, D^{j}_{m_2' m_2}(Q)\, D^{j}_{m' m}(Q)^{*}\, dQ$$

$$dQ = \sin\beta\, d\alpha\, d\beta\, d\gamma$$

from this we can solve

$$S^{j_1 j_2}_{m_1 m_2\, j m} = \delta_{m_1 + m_2\, m}$$

$$\sqrt{\frac{(2j+1)\,(j+j_1-j_2)!\,(j-j_1+j_2)!\,(j_1+j_2-j)!\,(j+m)!\,(j-m)!}{(j+j_1+j_2+1)!\,(j_1+m_1)!\,(j_1-m_1)!\,(j_2+m_2)!\,(j_2-m_2)!}}$$

$$\sum_{n} (-1)^{n+j_2+m_2}\, \frac{(j+j_2+m_1-n)!\,(j_1-m_1+n)!}{(j-j_1+j_2-n)!\,(j+m-n)!\,n!\,(n+j_1-j_2-m)!}$$

Wigner's formula of CG coefficients.

CG coefficients & 3j symbol

properties of $S^{j_1 j_2}_{m_1 m_2 j m} = \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle$

1) $j_1 + j_2 + j \in$ integer , $j = j_1 + j_2 \,,\, j_1 + j_2 - 1$
$\cdots |j_1 - j_2|$

in order that
$S^{j_1 j_2}_{m_1 m_2 j m} \neq 0$

triangle inequality $\quad j_1 + j_2 - j \geq 0$

$$\left.\begin{aligned} j_1 - j_2 + j &\geq 0 \\ -j_1 + j_2 + j &\geq 0 \end{aligned}\right\} \quad j \geq |j_1 - j_2|$$

$-j \leq j_1 - j_2 \leq j$

$m_1 + m_2 = m$

2) $S^{j_1 j_2}_{m_1 m_2 j m} \in \mathbb{R}$

3). unitarity $S^{j_1 j_2}_{m_1 m_2 j m} \equiv \left(S^{j_1 j_2 \, -1} \right)^{*}_{j m \, m_1 m_2}$

$$S^{\dagger} S = S S^{\dagger} = 1$$

4) $S^{j_1 j_2}_{m_1 m_2 j m} = (-1)^{j_1 + j_2 - j} S^{j_2 j_1}_{m_2 m_1 j m}$

recurrence relation

$$\sqrt{j(j+1) - m(m+1)}\, S^{j_1 j_2}_{m_1 m_2 j m+1} = \sqrt{j_1(j_1+1) - m_1(m_1-1)}\, S^{j_1 j_2}_{m_1 -1 m_2 j m}$$

$$+ \sqrt{j_2(j_2+1) - m_2(m_2-1)}\, S^{j_1 j_2}_{m_1 m_2 -1 j m}$$

$$\sqrt{j(j+1) - m(m-1)}\, S^{j_1 j_2}_{m_1 m_2 j m-1} = \sqrt{j_1(j_1+1) - m_1(m_1+1)}\, S^{j_1 j_2}_{m_1+1 m_2 j m}$$

$$+ \sqrt{j_2(j_2+1) - m_2(m_2+1)}\, S^{j_1 j_2}_{m_1 m_2+1 j m}$$

Wigner's 3j-symbol

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = \frac{(-1)^{j_1-j_2-m}}{\sqrt{2j+1}} \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m \rangle$$

symmetry properties ;

- $$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix}$$

- $$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_3 & m_1 & m_2 \end{pmatrix}$$

- $$\underline{\begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix}} = \underline{\begin{pmatrix} j_1 & j_3 & j_2 \\ m_1 & m_3 & m_2 \end{pmatrix}} = \underline{\begin{pmatrix} j_3 & j_2 & j_1 \\ m_3 & m_2 & m_1 \end{pmatrix}}$$

$$= (-1)^{j_1+j_2+j_3} \underline{\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}}$$

- $\begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$

change of basis

$$|j_1 j_2 j m\rangle = (-1)^{j_2 - j - m} \sum_{m_1 m_2} \sqrt{2j+1} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}$$

$$|j_1 j_2 m_1 m_2\rangle = \sum_{jm} (-1)^{j_2 - j_1 - m} \sqrt{2j+1} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}$$

$|j_1 j_2 m_1 m_2\rangle$

$|j_1 j_2 j m\rangle$

unitarity

$$\sum_{m_1 m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \frac{1}{2j_3+1} \delta_{j_3 j_3'} \delta_{m_3 m_3'}$$

$$\sum_{j_3 m_3} (2j_3+1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3 \end{pmatrix} = \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

coupling 3 angular momenta

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \qquad \vec{J}_i \circlearrowleft \mathcal{H}_i$$

irrep of $SU(2)$ w/ j_i

$$\vec{J} = \underline{\vec{J}_1 + \vec{J}_2} + \underbrace{\vec{J}_2 + \vec{J}_3}$$

$$\vec{J} \circlearrowleft \mathcal{H}$$

decoupled basis in $\mathcal{H}$

$$|j_1 j_2 j_3 m_1 m_2 m_3\rangle = |j_1 m_1\rangle \otimes |j_2 m_2\rangle \otimes |j_3 m_3\rangle$$

$J_i^2 \quad (J_i)_z$

$$\hat{J}_i^2 |j_i m_i\rangle = \hbar^2 (j_i+1) j_i |j_i m_i\rangle$$

$$(\hat{J}_i)_z |j_i m_i\rangle = \hbar m_i |j_i m_i\rangle$$

recoupling schemes : Scheme 1 : firstly couple J_1 & J_2

$$\hat{\vec{J}}_{12} = \hat{\vec{J}}_1 + \hat{\vec{J}}_2$$

common eigenbasis $\hat{J}_1^2 \quad \hat{J}_2^2 \quad \hat{J}_{12}^2 \quad (\hat{J}_{12})_z$

$$\rightarrow \quad \underline{|j_1 j_2 j_{12} m_{12}\rangle} = \sum_{m_1 m_2} |j_1 j_2 m_1 m_2\rangle \underline{\langle j_1 j_2 m_1 m_2 | j_1 j_2 j_{12} m_{12}\rangle}$$

then couple J_{12} and J_3 : $\vec{J} = \vec{J}_{12} + \vec{J}_3$

$= J_1 + J_2 + J_3$

$| j_1 j_2 j_{12} j_3 m_{12} m_3 \rangle$

$$| (j_1 j_2) j_{12} j_3 j m \rangle = \sum_{m_{12} m_3} | j_1 j_2 j_{12} m_{12} \rangle \otimes | j_3 m_3 \rangle$$

common eigenbasis of $J_1^2 J_2^2 J_3^2 J_{12}^2 J^2 J_z$

$$\langle (j_1 j_2) j_{12} j_3 m_{12} m_3 | (j_1 j_2) j_{12} j_3 j m \rangle$$

$$= \sum_{\substack{m_{12} m_3 \\ m_1 m_2}} | j_1 j_2 j_3 m_1 m_2 m_3 \rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j_{12} m_{12} \rangle \langle j_{12} j_3 m_{12} m_3 | j_{12} j_3 j m \rangle$$

Scheme 2 first $\vec{J}_{23} = \vec{J}_2 + \vec{J}_3$ then

$$\vec{J} = \vec{J}_1 + \vec{J}_{23}$$

common eigenbasis of $J_1^2 J_2^2 J_3^2 J_{23}^2 J^2 J_z$

$$|j_1(j_2 j_3) j_{23}\, j\, m\rangle$$

$$= \sum_{\substack{m_1 m_2 m_3 \\ m_{23}}} |j_1 j_2 j_3\, m_1 m_2 m_3\rangle \langle j_2 j_3\, m_1 m_2 | j_2 j_3 j_{23}\, m_{23}\rangle$$

$$\langle j_1 j_{23}\, m_1 m_{23} | j_1 j_{23}\, j\, m\rangle$$