

Theorem of orthonormal basis

$$D_i : G \rightarrow L(\mathcal{H})$$

$$D_j : G \rightarrow L(\mathcal{H})$$

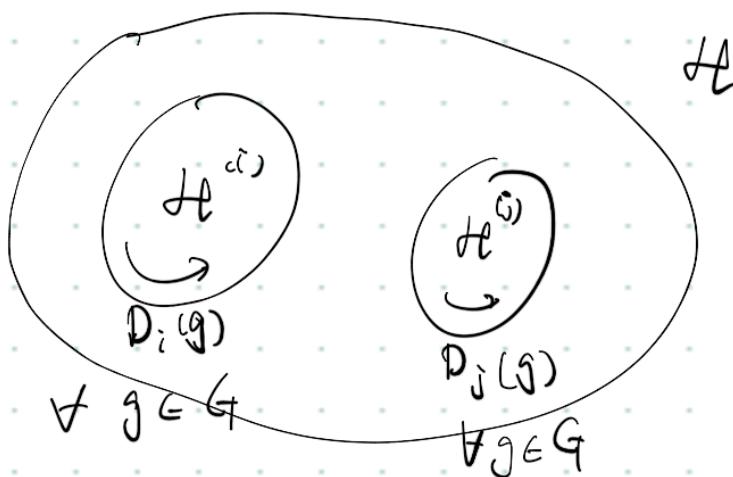
D_i, D_j are unitary irrep

if D_i, D_j are inequivalent

$$\Rightarrow \exists \quad \mathcal{H}^{(i)} \subset \mathcal{H} \quad D_i : G \rightarrow L(\mathcal{H}^{(i)})$$

$$\mathcal{H}^{(j)} \subset \mathcal{H} \quad D_j : G \rightarrow L(\mathcal{H}^{(j)})$$

and $\mathcal{H}^{(i)} \perp \mathcal{H}^{(j)}$



inequivalent unitary
irreps are carried
by mutually orthogonal
subspaces in \mathcal{H}

Theorem (orthogonality of matrix elements)

Let's assume G to be a finite group, i.e., G has only a finite number of elements

$$\begin{aligned} D^{(i)} : G \rightarrow L(H^{(i)}) \\ D^{(j)} : G \rightarrow L(H^{(j)}) \end{aligned} \quad \left. \begin{array}{l} \text{unitary irrep.} \\ \text{orthogonal basis in } H^{(i)} \end{array} \right\}$$

$f_\alpha^{(i)} = 1 \dots \dim(H^{(i)})$ orthogonal basis in $H^{(i)}$

$f_\beta^{(j)} = 1 \dots \dim(H^{(j)})$ orthogonal basis in $H^{(j)}$

$$\langle f_\alpha^{(i)} | f_\beta^{(j)} \rangle = \delta_{ij} \delta_{\alpha\beta} r$$

constant.

matrix elements of rep.

$$\left. \begin{array}{l} \text{functions} \\ \text{by } G \\ (\text{rep. functions}) \end{array} \right\} \begin{aligned} D_{\alpha\beta}^{(i)}(g) &= \langle f_\alpha^{(i)} | D^{(i)}(g) | f_\beta^{(i)} \rangle \\ D_{\alpha\beta}^{(j)}(g) &= \langle f_\alpha^{(j)} | D^{(j)}(g) | f_\beta^{(j)} \rangle \end{aligned}$$

orthogonality of rep. functions

$$\underbrace{\sum_{g \in G} D_{\alpha\beta}^{(i)}(g)^* D_{\gamma\delta}^{(j)}(g)}_{= \delta_{ij} \delta_{\alpha\beta} \delta_{\gamma\delta}} \xrightarrow{\text{rank of the group}} \frac{h}{\dim(\mathcal{H}^{(i)})}$$

Lie group (infinite group)

$$\sum_{g \in G} \rightarrow \int_G dg$$

Character : group G , rep $D : G \rightarrow L(\mathcal{H})$
 $g \mapsto D(g) \in L(\mathcal{H})$

$$\text{character} : \chi(g) = \text{tr}(D(g))$$

$$= \sum_{\alpha} D_{\alpha\alpha}(g)$$

Properties : (1) Conjugacy class : group conjugate
 \uparrow

$$g \mapsto hgh^{-1} \quad \forall g, h \in G$$

orbit of group conjugate

$$g \rightarrow \text{orbit of } hgh^{-1}$$

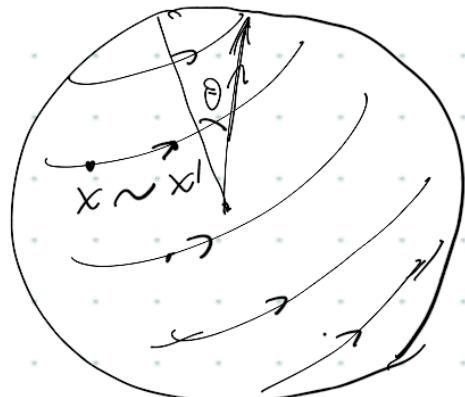
= ||

$$\{hgh^{-1} \mid \forall h \in G\}$$

$$\begin{aligned}\chi(g) &\rightarrow \chi(hgh^{-1}) = \text{tr}(D(hgh^{-1})) \\&= \text{tr}(D(h) D(g) D(h^{-1})) \\&= \text{tr}(\underbrace{D(h)}_{\text{ }} D(g) \underbrace{D(h)^{-1}}_{\text{ }}) \\&= \text{tr}(D(g)) = \chi(g)\end{aligned}$$

i.e. $\chi(g)$ inv. under conjugate

χ is function of conjugacy classes. in G



an orbit = ^a circle with
const. θ

$$S^2$$

= one class.

$$\{ \text{classes} \} = \{ \text{orbits} \}$$

$$f(\theta, \varphi)$$

$\cos \theta$ is a function of $\Omega_{C, \overline{\alpha}}$

is a function of classes
(orbits)

2) unitary reps D' is equivalent to D

$$\forall g \in G, D'(g) = U D(g) U^{-1} \quad U: \mathcal{H} \rightarrow \mathcal{H} \text{ unitary}$$

$$\chi'(g) = \text{tr } D'(g) = \text{tr}(U D(g) U^{-1})$$

$$= \text{tr } D(g) = \chi(g)$$

equivalent reps have the same character.

3) reducible rep. $D = D_1 \oplus D_2 \oplus \dots$

$$\forall g \in G \quad \chi(g) = \text{tr } D(g) = \sum_i \text{tr } D_i(g) = \sum_i \chi_i(g)$$

$$D(g) = \begin{pmatrix} [D_1(g)] & & \\ & \ddots & \\ & & [D_n(g)] \end{pmatrix}$$

Character of reducible rep = sum of characters of irreps.

Theorem (Orthogonality of χ)

(i) G finite group, $D^{(i)}, D^{(j)}$ irrep, then

$$\sum_{g \in G} \underbrace{\chi^{(i)}(g)^* \chi^{(j)}(g)}_{\parallel} = h \delta^{ij}$$

↑ rank of the group.

$$\sum_{C \in \text{(conjugacy classes)}} h_C \chi^{(i)}(C)^* \chi^{(j)}(C)$$

$h_C = \# \text{ of elements in conjugacy class } C.$

(2) $D^{(i)}$ irrep of G (finite group)

$i \in I \leftarrow$ set of all irreps of G .

$$\sum_{i \in I} \chi^{(i)}(c_l)^* \chi^{(i)}(c_m) = \sum_{l=1}^h \delta_{lm}$$

l th conjugacy class

h_l : # of elements in C_l

Symmetries of Schrödinger eqn

Let G be the symmetry group of the system

i.e. $\forall g \in G : D(g)$: unitary transf. on \mathcal{H}

(D : unitary rep G , carried by \mathcal{H})

$$\text{s.t. } \underline{D(g) \hat{H} D(g)^{-1} = \hat{H}} \quad \forall g \in G$$

eigen-eqn of \hat{H} $\hat{H} | \psi_n \rangle = E_n | \psi_n \rangle$

$$i = 1 \dots d(n)$$

$d(n)$: degeneracy of energy
level E_n

$\{|4_{ni}\rangle\}_{i=1 \dots d(n)}$ span the eigen space of \hat{H} at
the eigenvalue E_n

$$|4_{ni}\rangle \rightarrow D(g)|4_{ni}\rangle \equiv |4'_{ni}\rangle \quad \forall g \in G$$

$$\underbrace{\hat{H} D(g) |4_{ni}\rangle} = D(g) \underbrace{\hat{H} |4_{ni}\rangle} = E_n D(g) |4_{ni}\rangle$$

$D(g)|4_{ni}\rangle$ belongs to the eigen space $\mathcal{H}^{(n)}$

($D(g)$ leaves $\mathcal{H}^{(n)}$ inv) $\forall g \in G$

$\{|4_{ni}\rangle\}_{i=1 \dots d(n)}$ basis in $\mathcal{H}^{(n)}$

$$D(g)|4_{ni}\rangle = \sum_j |4_{nj}\rangle D_{ji}^{(n)}(g)$$

↑
rep matrix.

Any eigenspace $\mathcal{H}^{(n)}$ of \hat{H} is a rep space of the

Symmetry group

$$\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}^{(n)}$$

$$D = \bigoplus_{n=1}^{\infty} D^{(n)}$$

(\mathcal{H}, D) is a reducible rep of G .

Example 1) central potential $\hat{H} = \frac{\hat{p}^2}{2m} + V(r)$

$G = SO(3)$

group of 3d rotations.

on the hand, $\hat{H} \psi_{nlm}(r, \theta, \varphi) = E_{nl} \psi_{nlm}(r, \theta, \varphi)$

$$\psi_{nlm}(r, \theta, \varphi) = R_{nl}^l(r) Y_{lm}(\theta, \varphi)$$

$= \underbrace{\phantom{R_{nl}^l(r) Y_{lm}(\theta, \varphi)}}_{\substack{\uparrow \\ \text{spherical harmonics}}}$

$$l = 0, 1, 2 \dots$$

$$m = -l, -l+1 \dots l$$

eigenspace $\underline{\mathcal{H}}^{(n, l)}$ is spanned by $\{\Psi_{lm}\}_{m=-l \dots l}$

$$\dim \underline{\mathcal{H}}^{(n, l)} = 2l + 1$$

$\underline{\mathcal{H}}^{(n, l)}$ are irrep. of $G = SO(3)$

all irreps of $SO(3)$ are $(\underline{\mathcal{H}}_L, D_L)$

$$\text{s.t. } \dim \underline{\mathcal{H}}_L = 2L + 1$$

$\{|l, m\rangle\}_{m=-L \dots L}$ spans $\underline{\mathcal{H}}_L$ carrier space of irrep of $SO(3)$

(2) hydrogen atom : $\hat{H} = \frac{\hat{p}^2}{2m} - \frac{e^2}{r}$

$$\hat{H} \Psi_{nlm}(r, \theta, \varphi) = E_n \Psi_{nlm}(r, \theta, \varphi)$$

$$\Psi_{nlm}(r, \theta, \varphi) = R_{nl}(r) \underline{\Psi_{lm}(\theta, \varphi)}$$

eigen space $\underline{\mathcal{H}}^{(n)}$ is spanned by $\{R_{nl}(r) \underline{\Psi_{lm}(\theta, \varphi)}\}$

$$l=0, 1 \dots n-1$$

$$m = -l \dots l$$

$$\dim \mathcal{H}^{(n)} = \sum_{l=0}^{n-1} (2l+1) = n^2$$

$$\rightarrow \mathcal{H}^{(n)} = \bigoplus_{l=0}^{n-1} \mathcal{H}^{(n,l)}$$

↑
reducible
rep of $SO(3)$

↑
irrep of $SO(3)$

$$\dim(\mathcal{H}^{(n,l)}) = 2l+1$$

for any system, for any eigenspace $\mathcal{H}^{(n)}$ of the hamiltonian,
there exists a symmetry group G , s.t. $\mathcal{H}^{(n)}$ is an
irrep of G

for hydrogen atom, $G \neq SO(3)$

we expect there is a larger group $G \supset SO(3)$

$H \subset G$, H is a subgroup

Any irrep D of G is also a rep. D' of H

D' may be a reducible rep of H .

$SO(4)$

IS (local)

Hydrogen atom has a larger symmetry group $G \simeq \text{Spin}(4)$

\cup

(dynamical symmetry)

$SO(3)$

there exists hidden symmetry of \hat{H}

reverse procedure, reduce the symmetry $\hat{H} \rightarrow G_{\text{small}}$

by adding perturbations to \hat{H}

$H^{(n)}$ irrep of G

\cup

G_{small}

$H^{(n)} = H_1^{(n)} \oplus H_2^{(n)} \oplus \dots$

\uparrow

reducible
rep of
 G_{small}

irrep
of G_{small}

$$\hat{H} \rightarrow \hat{H}_{\text{new}} = \hat{H}_0 + \hat{H}'$$

\uparrow

perturbation

$$E_n \rightarrow E_n + \Delta E_n;$$

\uparrow

different values

at $H_i^{(n)}$

$$\hat{H}_{\text{new}} | \psi_{n_i} \rangle = (E_n + \Delta E_{n_i}) | \psi_{n_i} \rangle$$

$$| \psi_{n_i} \rangle \in \mathcal{H}_i^{(n)}$$

the total Hilbert space \mathcal{H} , \hat{H}_0 , E_n ,

degeneracy at E_n : $d(n)$,

Symmetry of \hat{H}_0 : G , i.e. $\forall g \in G$

$$D(g) \hat{H}_0 D(g)^{-1} = \hat{H}_0$$

let's assume eigenspace $\mathcal{H}^{(n)}$ of E_n is an irrep of G

add perturbation: $\hat{H} = \hat{H}_0 + \hat{H}'$

- if \hat{H}' doesn't break the symmetry, i.e. $\forall g \in G$,

$$D(g) \hat{H}' D(g)^{-1} = \hat{H}' \Rightarrow D(g) \hat{H} D(g)^{-1} = \hat{H}$$

energy level $E_n \rightarrow E_n + \Delta E_n$

but degeneracy $d(n)$ must unchanged,

because the eigenspace $\mathcal{H}^{(h)}$ must still be the irrep of G $\rightarrow \dim(\mathcal{H}^{(h)})$ is fixed

more generally, \hat{H}' may break some symmetry

i.e. $\exists g \in G$, s.t. $D(g)\hat{H}'D(g)^{-1} \neq \hat{H}'$

G is broken into subgroup $G_{\text{new}} \subset G$

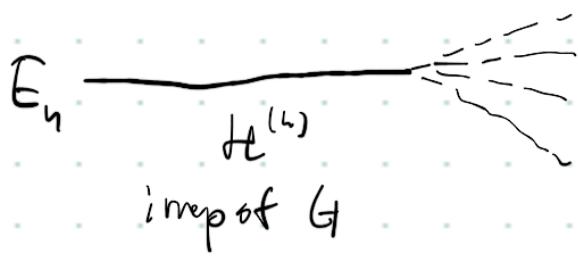
$$\text{irrep of } G : \mathcal{H}^{(h)} = \bigoplus_{i=1}^{k_m} \overline{\mathcal{H}_i}^{(h)}$$

eigen space
of \hat{H}_0

irreps of G_{new}

"
eigenspace of $\hat{H} = \hat{H}_0 + \hat{H}'$

(correspond to different
energy levels)



$$E_n + \Delta E_{n,1}$$

$$\vdots$$

$$E_n + \Delta E_{n,k}$$

$$\hat{H}_0$$

$$\hat{H}_0 + \hat{H}_1$$

splitting of energy levels predicted by group theory.

splitting of the energy level,
each new energy level has smaller
degeneracy

Example : an atom in a crystal.

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + V(r)$$

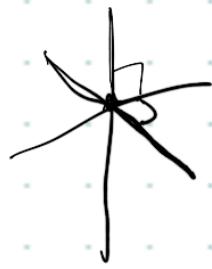
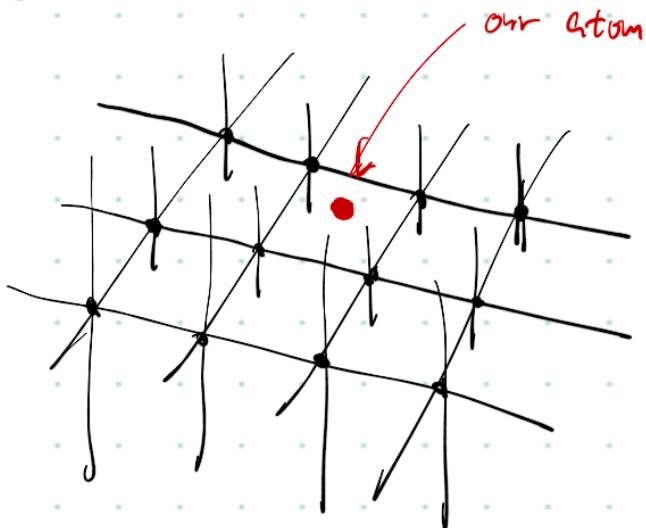
$$\hat{H}_0 + \hat{V}_{\text{lattice}} = E_{nl} + \hat{V}_{\text{lattice}}$$

Symmetry group $SO(3)$

eigenspace of E_{nl} : $\mathcal{H}^{(n,l)}$
(irrep of $SO(3)$)

crystal, cubic lattice

$$\dim \mathcal{H}^{(n,l)} = 2l+1$$



$$\hat{H} = \hat{H}_0 + V_{\text{lattice}}(\vec{r})$$

break $SO(3)$ symmetry
down to $O \subset SO(3)$

O group has 24 elements

(5 conjugacy classes)

• identity E

• $8 C_3$

• $6 C_2$

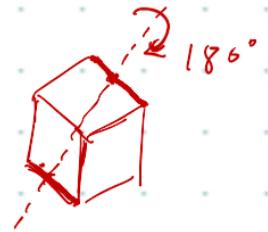
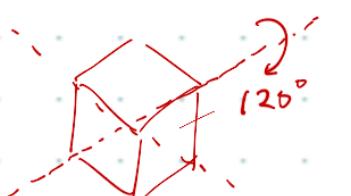
• $3 C_2$

• $6 C_4$

irrep of $SO(3)$

irreps of O

O group:
Symmetry group
of cubic
lattice
(finite group)



$$E_{nl} \xrightarrow{2l+1}$$

$$E_{nl}^{(i)} \quad d_{nl}^{(i)}$$

$$\sum_i d_{nl}^{(i)} = d(n, l)$$

$$= 2l+1$$

H_0

$$\mathcal{H}^{(n,l)} = \bigoplus_i \mathcal{H}_i^{(n,l)}$$

↑ ↑

irrep of $SO(3)$ irrep of O

$$D^{(n,l)} = \bigoplus_i D_i^{(n,l)}$$

Character of O group

$$\begin{aligned}\chi(E) &= \text{tr } D(E) \\ &= \text{tr } \mathbb{1} = \dim(D)\end{aligned}$$

<u>Value of $\chi(g)$</u>	E	$8C_3$	$6C_2$	$3C_2$	$6C_4$
T_1	1	1	1	1	1
T_2	1	1	-1	1	-1
T_3	2	-1	0	2	0
T_4	3	0	-1	-1	1
T_5	3	0	1	-1	-1

$$\underline{\chi_L(g) = \sum_i \chi_i^{(l)}(g)} \quad \forall g \in O$$

Value of characters of $SO(3)$ for elements in $O \subset SO(3)$

$\chi_l(g)$	E	$8C_3$	$6C_2$	$3C_2$	$6C_4$
$L=0, S\text{-state}$	1	1	1	1	1
$L=1, P$	3	0	-1	-1	1
$L=2, d$	5	-1	1	1	-1
$L=3, f$	7	1	-1	-1	1
$L=4, g$	9	0	1	1	1
⋮	⋮				

• $L=0 \quad D^{(L=0)} = T_1 \quad \chi_{L=0}(g) = \chi_{T_1}(g)$

$$E^{(n,0)} \xrightarrow{\quad} \dots \xrightarrow{\quad} E_{\text{new}}^{(n,0)}$$

no split for
 S - state,

• $L=1 \quad D^{(L=1)} = T_4 \quad \chi_{L=1}(g) = \chi_{T_4}(g) \quad \forall g \in O$

$$E^{(n,1)} \xrightarrow{l=1} \dots \xrightarrow{T_4} E_{\text{new}}^{(n,1)}$$

• $L=2$: $D^{(L=2)} = T_3 \oplus T_5$

$(d=5)$ $(d=2)$ $(d=3)$

• $L=3$: $D^{(L=3)} = T_2 \oplus T_4 \oplus T_5$

$(d=7)$ $d=1$ $d=3$ $d=3$

• $L=4$ $D^{(L=3)} = T_1 \oplus T_3 \oplus T_4 \oplus T_5$

$(d=9)$ $d=1$ $d=2$ $d=3$ $d=3$

HW check the splitting at $L=2, 3, 4,$

We obtain energy level splitting & degeneracy just by symmetry & group theory, no need to solve Schrödinger eqn

Dynamical symmetry

$2L+1$

hydrogen atom : $\hat{H} = \frac{\hat{p}^2}{2m} - \frac{e^2}{r}$ $SO(3)$
sym.

E_n degeneracy $d(n) = n^2$

$H^{(n)}$ is reducible rep of $SO(3)$

there should be a larger symmetry group

$$G \supset SO(3)$$

classical mechanics $V(r) \propto \frac{1}{r}$ Kepler problem

Conserved quantities : energy, angular momentum,

$$[\hat{x}, \hat{L}] = i\hbar \hat{L}$$

Runge-Lenz vector.

$$[\hat{L}_i, \hat{L}_j] = i\hbar \epsilon_{ijk} \hat{L}_k$$

$$\vec{M} = \frac{1}{me^2} \vec{p} \times \hat{L} - \frac{\vec{r}}{r}$$

Quantization : $\hat{M} = \frac{1}{2me^2} (\hat{p} \times \hat{L} - \hat{L} \times \hat{p}) - \frac{\hat{N}}{N}$

$$\frac{\hat{N}}{N} = \frac{\hat{R}}{\vec{R}}$$

orientation
operator

$$(1) [\hat{P}, \hat{H}] = 0$$

$$(2) \hat{\vec{P}} \times \hat{\vec{P}} = -i\hbar \frac{2}{me^4} \hat{\vec{L}} \hat{H}$$

$$(3) \hat{P}^2 = \frac{2}{me^4} \hat{H} (\hat{L}^2 + \hbar^2) + 1$$

HW show (1), (2), (3)

(submit HW before 17th Feb)

Given a energy level, E_n ($E_n < 0$), eigenspace $\mathcal{H}^{(n)}$

$$\hat{H} \psi_n = E_n \psi_n \quad \forall \psi_n \in \mathcal{H}^{(n)}$$

define $\hat{K} = \sqrt{\frac{me^4}{2(-\hat{H})}} \hat{\vec{P}}$

on $\mathcal{H}^{(n)}$, $\hat{K} = \sqrt{\frac{me^4}{2(-E_n)}} \hat{\vec{P}}$

$$(2) \rightarrow \hat{\vec{P}} \times \hat{\vec{P}} = -i\hbar \frac{2}{me^4} \hat{\vec{L}} E_n$$

$$\hat{K} \times \hat{L} = i\hbar \hat{L}$$

$$\text{i.e., } [\hat{K}, \hat{L}_j] = i\hbar \epsilon_{ijk} \hat{L}_k$$

$$\underline{\text{Hw check}}: [\hat{L}_i, \hat{K}_j] = i\hbar \epsilon_{ijk} \hat{K}_k$$

$$\text{we know that 1) } [H, \hat{P}] = 0$$

$$2) [H, \hat{T}] = 0$$

both \hat{K}, \hat{L} leaves $H^{(4)}$ inv.

we obtain operators $\hat{K}_{i=1,2,3}, \hat{L}_{i=1,2,3}$ with

closed commutator algebra

$$[L_i, L_j] = i\hbar \epsilon_{ijk} L_k \quad \leftarrow$$

$$[L_i, K_j] = i\hbar \epsilon_{ijk} K_k$$

$$[K_i, K_j] = i\hbar \epsilon_{ijk} L_k$$

$$\hat{L}_i \rightarrow D(\varphi, \vec{\eta}) = e^{-\frac{i}{\hbar} \varphi \vec{\eta} \cdot \hat{L}}, \varphi, \vec{\eta}$$

3 parameters

unitary rep of $SO(3) \xrightarrow{\text{local}} SU(2)$

$$\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \quad \alpha, \beta \in \mathbb{C}$$

$$\det = 1$$

$$\hat{L}_i, \hat{K}_i \rightarrow D(\varphi, \vec{n}, \vec{\theta}) = e^{-\frac{i}{\hbar} \varphi \vec{n} \cdot \hat{L}} e^{-\frac{i}{\hbar} \vec{\theta} \cdot \hat{K}}$$

$\uparrow \quad \uparrow$
 $3 + 3 = 6$

parameters

We obtain a larger group rep. on $\mathcal{H}^{(l)}$

Let's determine this larger group.

linear combination of \hat{L} and \hat{K}

$$\hat{J}^{(+)} = \frac{1}{2} (\hat{L} + \hat{K})$$

$$\hat{J}^{(-)} = \frac{1}{2} (\hat{L} - \hat{K})$$

$$\Rightarrow \left\{ \begin{array}{l} \left[\hat{\vec{J}}^{(+)}, \hat{\vec{J}}^{(-)} \right] = 0 \\ \left[J_i^{(+)} \quad J_j^{(+)} \right] = i\hbar \epsilon_{ijk} J_k^{(+)} \\ \left[J_i^{(-)} \quad J_j^{(-)} \right] = i\hbar \epsilon_{ijk} J_k^{(-)} \end{array} \right. \quad \begin{array}{l} \text{(two check} \\ \text{this,} \end{array}$$

$$D(\vec{\theta}^{(+)}, \vec{\theta}^{(-)}) = e^{-\frac{i}{\hbar} \vec{\theta}^{(+)} \cdot \hat{\vec{J}}^{(+)}} e^{-\frac{i}{\hbar} \vec{\theta}^{(-)} \cdot \hat{\vec{J}}^{(-)}}$$

↑ ↑

6 parameters

uni. rep. of $SU(2)$

uni. rep. of $SU(2)$

is Whitney rep. of $\underline{SO(2) \times SO(2)} \cong \text{Spin}(4)$

is local

S D(4)

۱

4×4 matrices 1

$$\Lambda^T \Lambda = I_{4 \times 4}$$

$$\det A = 1$$

explain degeneracy of $\mathcal{H}^{(b)}$

$\mathcal{H}^{(b)}$ is irrep of $SU(2) \times SU(2) \ni (\mathbf{j}_1, \mathbf{j}_2)$

$$\text{irrep. } D_{\mathbf{j}_1, \mathbf{j}_2}(\mathbf{j}) = D_{\mathbf{j}_1}(\mathbf{j}_1) \otimes D_{\mathbf{j}_2}(\mathbf{j}_2)$$

$$\begin{array}{ccc} \curvearrowleft & & \curvearrowright \\ & \mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} & \\ \curvearrowleft & & \curvearrowright \end{array}$$

$$\dim(\mathcal{H}_j) = 2j+1$$

$$\dim(\mathbf{j}_1, \mathbf{j}_2) = \underbrace{(2j_1+1)}_n \underbrace{(2j_2+1)}_m$$

$$\dim(\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2})$$

for hydrogen atom, $\hat{\mathbf{L}} \cdot \hat{\mathbf{L}} = \sqrt{\frac{me^4}{2(-E)}} \hat{\mathbf{L}} \cdot \hat{\mathbf{M}} = 0$

$$\hat{\mathbf{L}} \cdot \hat{\mathbf{L}} = 0$$

$$\hat{\mathbf{M}} \propto (\hat{\mathbf{P}} \times \hat{\mathbf{L}} - \hat{\mathbf{L}} \times \hat{\mathbf{P}}) - \frac{\hat{\mathbf{R}}}{R}$$

$$(\vec{V}_1 \times \vec{V}_2) \cdot \vec{V}_2 \approx 0 \quad \vec{L} = \vec{R} \times \vec{P}$$

$$\hat{J}^{(+)}^2 = \hat{J}^{(-)}^2 = (\hat{L}^2 + \hat{k}^2)/4$$

|| ||

$$J_1(J_1+1) \quad J_2(J_2+1) \quad \Rightarrow \quad J_1 = J_2 = j$$

$\mathcal{H}^{(n)}$ being irrep of $SO(4) \cong SU(2) \times SU(2)$

$$\dim \mathcal{H}^{(n)} = (2j+1)^2 = n^2$$

$n = 2j+1 = 1, 2, \dots$

algebraically determine the energy level.

$$\hat{J}^{(+)}^2 + \hat{J}^{(-)}^2 = \frac{1}{2} (\hat{L}^2 + \hat{k}^2) = \frac{1}{2} (\hat{L}^2 - \frac{me^4}{2E} \hat{M}^2)$$

recall $\hat{M}^2 = \frac{2}{me^2} (L^2 + k^2) \hat{H} + 1$

$$(\hat{J}^{(+)}^2 + \hat{J}^{(-)}^2) |4_{j,j}\rangle = \underline{2j(j+1)k} |+_{j,j}\rangle$$

$$\forall |Y_{j,j}\rangle \in \mathcal{H}^{(n)} = \mathcal{H}_{j,j}$$

$$\begin{aligned} \hat{j}^{(+)^2} + \hat{j}^{(-)^2} &= \frac{1}{2} \left(\hat{l}^2 - \frac{1}{E} E (\hat{l}^2 + \hbar^2) \right. \\ &\quad \left. - \frac{me^4}{2E} \right) \\ &= \frac{1}{2} \left(-\hbar^2 - \frac{me^4}{2E} \right) \end{aligned}$$

$$2\hat{j}(j+1)\hbar = \frac{1}{2} \left(-\hbar^2 - \frac{me^4}{2E} \right)$$

$$\begin{aligned} E &= -\frac{me^4}{2\hbar^2} \frac{1}{(2j+1)^2} = -\frac{me^4}{2\hbar^2} \frac{1}{n^2} \\ n &= 1, 2, \dots \end{aligned}$$

time translation and reflection

in QM, position $\hat{\vec{R}}$ is a hermitian operator.

but time t is not a operator, but is a parameter.

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}(\hat{\vec{R}}, \hat{\vec{p}}, t) |\psi(t)\rangle$$

time translation $\psi(t) \xrightarrow{\quad} \psi'(t) \equiv \psi(t-\tau)$

$$\text{i.e. } \psi(t) = \psi'(t+\tau)$$

$$Q(\tau) : t \rightarrow t + \tau$$

$$D(Q(\tau)) \equiv D(\tau)$$

$$\begin{aligned} \psi'(\vec{r}, t) &= D(\tau) \psi(\vec{r}, t) = \psi(\vec{r}, Q^{-1}(\tau)t) \\ &= \psi(\vec{r}, t-\tau) \end{aligned}$$

$D(\tau)$ is

unitary : $\int \psi(t, \vec{r})^* \psi'(t, \vec{r}) d^3r$ is dep. of t .

transf. of operators $\hat{A} \rightarrow \hat{A}' = D(\tau) \hat{A} D(\tau)^{-1}$

$$\begin{aligned}
 i\hbar \frac{\partial}{\partial t} \psi'(\vec{r}, t) &= i\hbar \frac{\partial}{\partial t} D(\tau) \psi(\vec{r}, t) \\
 &= i\hbar D(\tau) \frac{\partial}{\partial t} \psi(\vec{r}, t) \\
 &= D(\tau) \hat{H}(\hat{\vec{R}}, \hat{\vec{P}}, t) \psi(\vec{r}, t) \\
 &= D(\tau) \hat{H}(\hat{\vec{R}}, \hat{\vec{P}}, t) D(\tau)^{-1} D(\tau) \psi(\vec{r}, t) \\
 &\quad \underbrace{\hat{H}'(\hat{\vec{R}}, \hat{\vec{P}}, t)}_{\text{if } \hat{H}'(\hat{\vec{R}}, \hat{\vec{P}}, t) = \hat{H}(\hat{\vec{R}}, \hat{\vec{P}}, t), \text{ Schrödinger eqn.}} \quad \underbrace{\psi'(\vec{r}, t)}
 \end{aligned}$$

if $\hat{H}'(\hat{\vec{R}}, \hat{\vec{P}}, t) = \hat{H}(\hat{\vec{R}}, \hat{\vec{P}}, t)$, Schrödinger eqn.
is time transl. inv.

when $\hat{H} = \hat{H}(\hat{\vec{R}}, \hat{\vec{P}})$ not explicitly dep. on t .

solution
of Schrödinger
eqn.

$$\psi(\vec{r}, t) = \sum_{\epsilon} \psi_{\epsilon}(\vec{r}) e^{-\frac{i}{\hbar} E_{\epsilon} t}$$

$$\hat{H} \psi_{\epsilon}(\vec{r}) = E_{\epsilon} \psi_{\epsilon}(\vec{r})$$

$$D(t) \psi(\vec{r}, t) = \sum_{\vec{E}} \psi_{\vec{E}}(\vec{r}) e^{-\frac{i}{\hbar} E(t-\tau)}$$

$$= e^{\frac{i}{\hbar} \hat{H}\tau} \psi(\vec{r}, t)$$

$$D(t) = e^{\frac{i}{\hbar} H\tau} \quad D(t) \hat{H} D(t)^{-1} = \hat{H}$$

\Rightarrow Schrödinger eqn. is time transl. inv.

$$[\hat{H}, \hat{H}] = 0 \rightarrow \text{energy conservation.}$$