03: The Power Spectrum

EE21: Random Signal Processing

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Outline

Power Spectrum

Definition

The power spectrum (or *spectral density*) of a WSS process $X(t)$, real or complex, is the Fourier transform $S(\omega)$ of its autocorrelation $R(\tau) = E\{X(t+\tau)X^*(t)\}$:

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}d\tau \qquad (1)$$

Since $R(-\tau) = R^*(\tau)$ it follows that $S(\omega)$ is a real function of ω.

From the Fourier inversion formula, it follows that

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{j\omega\tau}d\omega \qquad (2)$$

If $X(t)$ is a real process, then $R(\tau)$ is real and even; hence $S(\omega)$ is also real and even. In this case,

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\cos\omega\tau d\tau = 2\int_{0}^{\infty} R(\tau)\cos\omega\tau d\tau \tag{3}$$

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\cos\omega\tau d\omega = \frac{1}{\pi}\int_{0}^{\infty} S(\omega)\cos\omega\tau d\omega \tag{4}$$

Cross-power Spectrum

Definition

The cross-power spectrum of two processes $X(t)$ and $Y(t)$ is the Fourier transform $S_{XY}(\omega)$ of their cross-correlation $R_{XY}(\tau) = E\{X(t+\tau)Y^*(t)\}$:

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}d\tau \quad (5)$$

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{j\omega\tau}d\omega \quad (6)$$

The function $S_{XY}(\omega)$ is, in general, complex even when both processes $X(t)$ and $Y(t)$ are real. In all cases,

$$S_{XY}(\omega) = S^*_{YX}(\omega) \quad (7)$$

because $R_{XY}(-\tau) = E\{X(t-\tau)Y^*(t)\} = R^*_{YX}(\tau)$.

Autocorrelations and the corresponding Spectra

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{j\omega\tau}d\omega \leftrightarrow S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}d\tau$$

- $\delta(\tau) \leftrightarrow 1$
- $1 \leftrightarrow 2\pi\delta(\omega)$
- $e^{j\beta\tau} \leftrightarrow \pi\delta(\omega-\beta)$
- $\cos\beta\tau \leftrightarrow \pi\delta(\omega-\beta) + \pi\delta(\omega+\beta)$
- $e^{j\alpha|\tau|} \leftrightarrow \frac{2\alpha}{\alpha^2+\omega^2}$
- $e^{-\alpha\tau^2} \leftrightarrow \sqrt{\frac{\pi}{\alpha}}e^{-\omega^2/4\alpha}$
- $e^{-\alpha|\tau|}\cos\beta \leftrightarrow \frac{\alpha}{\alpha^2+(\omega-\beta)^2} + \frac{\alpha}{\alpha^2+(\omega+\beta})$
- $2e^{-\alpha\tau^2}\cos\beta\tau \leftrightarrow \sqrt{\frac{\pi}{\alpha}}[e^{-(\omega-\beta)^2/4\alpha} + e^{-(\omega+\beta)^2/4\alpha}]$
- $1-\frac{|\tau|}{T} \quad |\tau|<T \leftrightarrow \frac{4\sin^2(\omega T/2)}{T\omega^2}$
- $\frac{\sin\sigma\tau}{\pi\tau} \leftrightarrow \begin{cases} 1 & |\omega|<\sigma \\ 0 & |\omega|>\sigma \end{cases}$

Example

A random telegraph signal is a process $X(t)$ taking the values $+1$ and -1:

$$X(t) = \begin{cases} 1 & t_{2i} < t < t_{2i+1} \\ -1 & t_{2i-1} < t < t_{2i} \end{cases}$$

where t_i is a set of Poisson points with average density λ. Its autocorrelation equals

$$R(\tau) = e^{-2\lambda|\tau|}.$$

Hence

$$S(\omega) = \frac{4\lambda}{4\lambda^2 + \omega^2}$$

For most processes $R(\tau) \to \mu^2$, where $\mu = E\{X(t)\}$.

If, therefore, $\mu \neq 0$, then $S(\omega)$ contains an impulse at $\omega = 0$.

To avoid this, it is often convenient to express the spectral properties of $X(t)$ in terms of the Fourier transform $S^c(\omega)$ of its autocovariance $C(\tau)$.

Since $R(\tau) = C(\tau) + \mu^2$, it follows that

$$S(\omega) = S^c(\omega) + 2\pi\mu^2\delta(\omega) \tag{8}$$

The function $S^c(\omega)$ is called the covariance spectrum of $X(t)$.

Example

We have shown that the autocorrelation of the Poisson impulses

$$Z(t) = \frac{d}{dt}\sum_i U(t - t_i) = \sum_i \delta(t - t_i)$$

equals $R_Z(\tau) = \lambda^2 + \lambda\delta(\tau)$.

From this it follows that

$$S_Z(\omega) = \lambda + 2\pi\lambda^2\delta(\omega)$$

$$S_Z^c(\omega) = \lambda$$

Outline

We shall show that given an arbitrary positive function $S(\omega)$, we can find a process $X(t)$ with power spectrum $S(\omega)$.
(a) Consider the process

$$X(t) = ae^{j(\omega t - \varphi)} \tag{9}$$

where a is a real constant, ω is a random variable with density $f_\omega(\omega)$, and φ is a random variable independent of ω and uniform in the interval $(0, 2\pi)$.

As we know, this process is WSS with zero mean and autocorrelation

$$R_X(\tau) = a^2 E\{e^{j\omega\tau}\} = a^2 \int_{-\infty}^{\infty} f_\omega(\omega) e^{j\omega\tau} d\omega$$

From this and the uniqueness property of Fourier transforms, it follows that the power spectrum of $X(t)$ equals

$$S_X(\omega) = 2\pi a^2 f_\omega(\omega) \tag{10}$$

If, therefore,

$$f_\omega(\omega) = \frac{S(\omega)}{2\pi a^2} \qquad a^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)d\omega = R(0)$$

then $f_\omega(\omega)$ is a density and $S_X(\omega) = S(\omega)$.

To complete the specification of $X(t)$, it suffices to construct a random variable ω with density $S(\omega)/2\pi a^2$ and insert it into(9)

$$X(t) = ae^{j(\omega t - \varphi)}$$

(b)We show next that if $S(-\omega) = S(\omega)$, we can find a real process with power spectrum $S(\omega)$.

To do so, we form the process

$$Y(t) = a\cos(\omega t + \varphi) \tag{11}$$

In this case

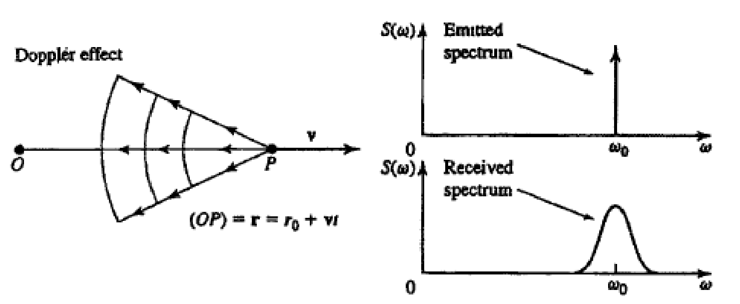
$$R_Y(\tau) = \frac{a^2}{2}E\{\cos\omega\tau\} = \frac{a^2}{2}\int_{-\infty}^{\infty} f(\omega)\cos\omega\tau d\omega$$

From this it follows that if $f_\omega(\omega) = S(\omega)/\pi a^2$, then

$$S_Y(\omega) = S(\omega)$$

Example

A harmonic oscillator located at point P of the x axis moves in the x direction with velocity **v**.



The emitted signal equals $e^{j\omega_0 t}$ and the signal received by an observer located at point O equals

$$\mathbf{s}(t) = ae^{j\omega_0(t-\mathbf{r}/c)}$$

where c is the velocity of propagation and $\mathbf{r} = r_0 + \mathbf{v}t$.

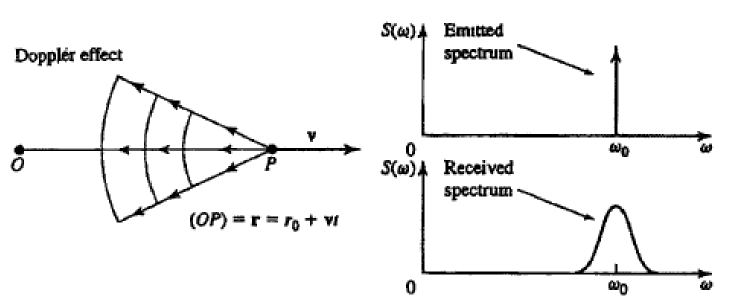
Solution

We assume that $\mathbf{v}$ *is a random variable with density* $f_v(v)$*. Clearly.*

$$\mathbf{s}(t) = ae^{j(\omega t - \varphi)} \qquad \omega = \omega_0\left(1 - \frac{\mathbf{v}}{c}\right) \qquad \varphi = \frac{r_0\omega_0}{c}$$

hence the spectrum of the received signal is given

$$S(\omega) = 2\pi a^2 f_\omega(\omega) = \frac{2\pi a^2 c}{\omega_0} f_v\left[\left(1 - \frac{\omega}{\omega_0}\right)c\right] \tag{12}$$



Solution

Note that if $\mathbf{v} = 0$, *then*

$$\mathbf{s}(t) = ae^{j(\omega_0 t - \varphi)} \qquad R(\tau) = a^2 e^{j\omega_0\tau} \qquad S(\omega) = 2\pi a^2 \delta(\omega - \omega_0)$$

This is the spectrum of the emitted signal.

Thus the motion causes broadening of the spectrum.

Suppose that the emitter is a particle in a gas of temperature T. *In this case, the* x *component of its velocity is a normal random variable with zero mean and variance* kT/m. *Inserting into (12), we conclude that*

$$\begin{aligned} S(\omega) &= \frac{2\pi a^2 c}{\omega_0\sqrt{2\pi kT/m}} \exp\left\{ -\frac{mc^2}{2kT}\left(1 - \frac{\omega}{\omega_0}\right)^2 \right\} \\ R(\tau) &= a^2 \exp\left\{ -\frac{kT\omega_0^2\tau^2}{2mc^2} \right\} e^{j\omega_0\tau} \end{aligned}$$

Outline

Wiener-Khinchin Theorem

It follows that

$$E\{X^2(t)\} = R(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)d\omega \geq 0 \tag{13}$$

This shows that the area of the power spectrum of any process is positive. We shall show that

$$S(\omega) \geq 0 \tag{14}$$

for every ω.

Proof.We form an ideal bandpass system with system function

$$H(\omega) = \begin{cases} 1 & \omega_1 < \omega < \omega_2 \\ 0 & otherwise \end{cases}$$

and apply $X(t)$ to its input. The power spectrum $S_{YY}(\omega)$, of the resulting output $Y(t)$ equals

$$S_{YY}(\omega) = \begin{cases} S(\omega) & \omega_1 < \omega < \omega_2 \\ 0 & otherwise \end{cases}$$

Hence

$$0 \leq E\{Y^2(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)d\omega = \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} S(\omega)d\omega \quad (15)$$

Thus the area of $S(\omega)$ in any interval is positive. This is possible only if $S(\omega) \geq 0$ everywhere.

Outline

1. Definitions
2. Existence Theorem
3. Wiener-Khinchin Theorem
4. **Integrated spectrum**
5. Properties of Correlations
6. Periodic Theorem
7. Continuity
   - MS continuity
   - MS periodicity

Integrated spectrum

In mathematics, the spectral properties of a process $X(t)$ are expressed in terms of the integrated spectrum $F(\omega)$ defined as the integral of $S(\omega)$:

$$F(\omega) = \int_{-\infty}^{\infty} S(\alpha)d\alpha \tag{16}$$

From the positivity of $S(\omega)$, it follows that $F(\omega)$ is a nondecreasing function of ω.

Integrating the inversion formula

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{j\omega\tau} d\omega$$

by parts, we can express the autocorrelation $R(\tau)$ of $X(t)$ as a Riemann-Stieltjes integral:

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega\tau} dF(\omega) \tag{17}$$

This approach avoids the use of singularity functions in the spectral representation of $R(\tau)$ even when $S(\omega)$ contains impulses.

If $S(\omega)$ contains the terms $\beta_i \delta(\omega - \omega_i)$, then $F(\omega)$ is discontinuous at ω_i and the discontinuity jump equals β_i.

The integrated covariance spectrum $F^c(\omega)$ is the integral of the covariance spectrum. From

$$S(\omega) = S^c(\omega) + 2\pi\mu^2\delta(\omega)$$

it follows that

$$F(\omega) = F^c(\omega) + 2\pi\mu^2 U(\omega).$$

Outline

Properties of Correlations

If a function $R(\tau)$ is the autocorrelation of a WSS process $X(t)$, then its Fourier transform $S(\omega)$ is positive.

Furthermore. if $R(\tau)$ is a function with positive Fourier transform, we can find a process $X(t)$ as in

$$X(t) = ae^{j(\omega t - \varphi)}$$

with autocorrelation $R(\tau)$.

Thus a necessary and sufficient condition for a function $R(\tau)$ to be an autocorrelation is the positivity of its Fourier transform.

The conditions for a function $R(\tau)$ to be an autocorrelation can be expressed directly in terms of $R(\tau)$. The autocorrelation $R(\tau)$ of a process $X(t)$ is p.d., that is,

$$\sum_{i,j} a_i a_j^* R(\tau_i - \tau_j) \geq 0 \tag{18}$$

for every a_i, a_j, τ_i, and τ_j.

It can be shown that the converse is also true: If $R(\tau)$ is a p.d. function, then its Fourier transform is positive. Thus a function $R(\tau)$ has a positive Fourier transform iff it is positive definite.

Sufficient condition

To establish whether $R(\tau)$ is p.d., we must show either that it satisfies (18) or that its transform is positive.
Polya's criterion. It can be shown that a function $R(\tau)$ is p.d. if it is concave for $\tau > 0$ and it tends to a finite limit as $\tau \to \infty$.

Consider, for example, the function $\omega(\tau) = e^{-\alpha|\tau|^c}$. If $0 < c < 1$, then $\omega(\tau) \to 0$ as $\tau \to \infty$ and $\omega''(\tau) > 0$ for $\tau > 0$; hence $\omega(\tau)$ is p.d. because it satisfies Polya's criterion.

Note, however, that it is p.d. also for $1 \leq c \leq 2$ even though it does not satisfy this criterion.

Necessary conditions

The autocorrelation $R(\tau)$ of any process $X(t)$ is maximum at the origin because

$$|R(\tau)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega = R(0) \tag{19}$$

We show next that if $R(\tau)$ is not periodic, it reaches its maximum only at the origin.

Outline

1. Definitions
2. Existence Theorem
3. Wiener-Khinchin Theorem
4. Integrated spectrum
5. Properties of Correlations
6. **Periodic Theorem**
7. Continuity
   - MS continuity
   - MS periodicity

If $R(\tau_1) = R(0)$ for some $\tau_1 \neq 0$, then $R(\tau))$ is periodic with period τ_1

$$R(\tau + \tau_1) = R(\tau) \qquad \forall \tau \tag{20}$$

Proof.

From Schwarz's inequality

$$E^2\{ZW\} \leq E\{Z^2\}E\{W^2\} \tag{21}$$

it follows that

$$\begin{aligned} & E^2\{[X(t+\tau+\tau_1) - X(t+\tau)]X(t)\} \\ \leq \quad & E\{[X(t+\tau+\tau_1) - X(t+\tau)]^2\}E\{X^2(t)\} \end{aligned} \tag{22}$$

Hence

$$[R(\tau+\tau_1) - R(\tau)]^2 \leq 2[R(0) - R(\tau_1)]R(0) \tag{23}$$

If $R(\tau_1) = R(0)$, then the right side is 0; hence the left side is also 0 for every τ. This yields

$$R(\tau+\tau_1) = R(\tau) \qquad \forall\tau.$$

$\square$

Corollary

If $R(\tau_1) = R(\tau_2) = R(0)$ and the numbers τ_1 and $\tau 2$ are noncommensurate, that is, their ratio is irrational, then $R(\tau)$ is constant.

Proof.

From the theorem it follows that $R(\tau)$ is periodic with periods τ_1 and τ_2. This is possible only if $R(\tau)$ is constant. $\square$

Outline

Continuity

If $R(\tau)$ is continuous at the origin, it is continuous for every τ.

Proof.

From the continuity of $R(\tau)$ at $\tau = 0$ it follows that $R(\tau_1) \to R(0)$; hence the left side of

$$\begin{aligned} & E^2\{[X(t+\tau+\tau_1) - X(t+\tau)]X(t)\} \\ \leq \quad & E\{[X(t+\tau+\tau_1) - X(t+\tau)]^2\}E\{X^2(t)\} \end{aligned}$$

also tends to 0 for every τ as $\tau_1 \to 0$. $\square$

Example

Using the theorem, we shall show that the truncated parabola

$$\omega(\tau) = \begin{cases} a^2 - \tau^2 & |\tau| < a \\ 0 & |\tau| > a \end{cases}$$

is not an autocorrelation.

If $\omega(\tau)$ is the autocorrelation of some process $X(t)$. then [see (22)] the function

$$-\omega''(\tau) = \begin{cases} 2 & |\tau| < a \\ 0 & |\tau| > a \end{cases}$$

is the autocorrelation of $X'(t)$. This is impossible because $-\omega''(\tau)$ is continuous for $\tau = 0$ but not for $\tau = a$.

Outline

MS continuity

We shall say that the process $X(t)$ is MS continuous if

$$E\{[X(t+\varepsilon) - X(t)]^2\} \to 0 \qquad as \;\; \varepsilon \to 0 \tag{24}$$

Since

$$E\{[X(t+\varepsilon) - X(t)]^2\} = 2[R(0) - R(\varepsilon)],$$

we conclude that if $X(t)$ is MS continuous, $R(0) - R(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Thus a WSS process $X(t)$ is MS continuous iff its autocorrelation $R(\tau)$ is continuous for all τ.

Outline

MS periodicity

We shall say that the process $X(t)$ is MS periodic with period τ_1 if

$$E\{[X(t+\tau_1) - X(t)]^2\} = 0 \tag{25}$$

Since the left side equals $2[R(0) - R(\tau_1)]$, we conclude that $R(\tau_1) = R(0)$; hence $R(\tau)$ is periodic.

This leads to the conclusion that a WSS process $X(t)$ is MS periodic iff its autocorrelation is periodic.