02: Random signal time-domain characteristics

EE21: Random Signal Processing

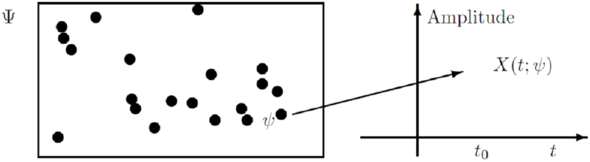
March 2017

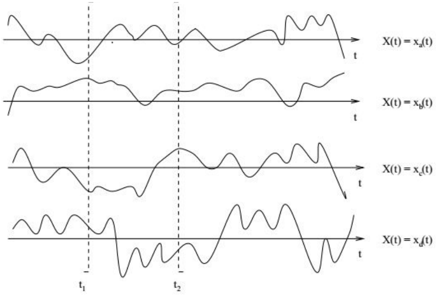
Zhigang SU
Professor
Sino-European Institute of Aviation Engineering
Civil Aviation University of China

Outline

Definition of random process

A stochastic process $X(t, \omega)$ is a family of time functions depending on the parameter ω or, equivalently, a function of t and ω .





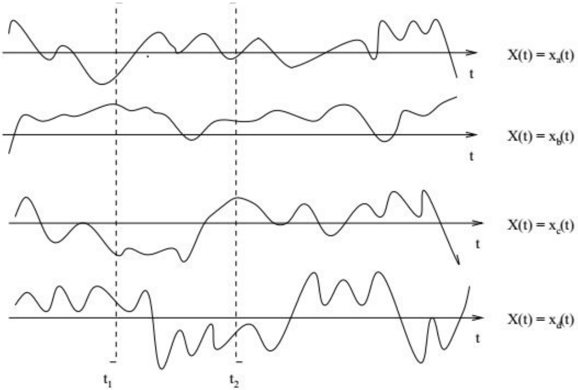
A stochastic process $X(t, \omega)$ is a family of time functions depending on the parameter ω or,equivalently,a function of t and ω .

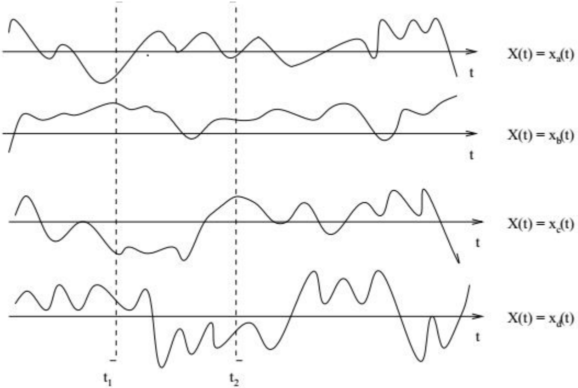
- The domain of ω is the set of all experimental outcomes
- and the domain of t is a set R of real numbers.
  - If R is the real axis,then $X(t)$ is a $continuous - time$ process.
  - If R is the set of integers,then $X(t)$ is a $discrete - time$ process.
- $X(t)$ is a $discrete - state$ process if its values are countable.
- Otherwise, it is a $continuous - state$ process.

We shall use the notation $X(t)$ to represent a stochastic process omitting, as in the case of random variables,its dependences on ω. Thus $X(t)$ has the following interpretations:

- It is **a family (or an** $ensemble$**) of function** $X(t, \omega)$**. In this interpretation,** t **and** ω **are variables.**
- **It is a single time function (or a sample of the given process). In this case,** t **is a variable and** ω **is fixed.**
- **If** t **is fixed and** ω **is variable,then** $X(t)$**is a random variable equal to the** $state$ **of the given process at time** t**.**
- **If** t **and** ω **are fixed, then** $X(t)$ **is a** $number$

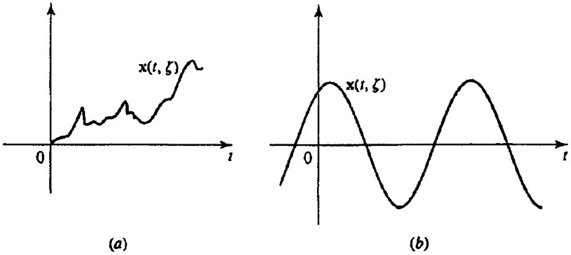
A random process is not just one signal but rather an ensemble of signals, as illustrated schematically in Figure below, for which the outcome of the probabilistic experiment could be any of the four waveforms indicated. Each waveform is deterministic, but the process is probabilistic or random because it is not known a $priori$ which waveform will be generated by the probabilistic experiment.





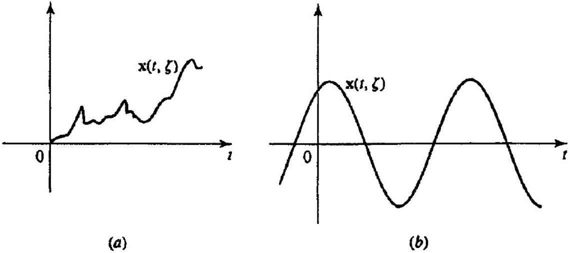
Consequently, prior to obtaining the outcome of the probabilistic experiment, many aspects of the signal are unpredictable, since there is uncertainty associated with which signal will be produced. After the experiment, or a posteriori, the outcome is totally determined.

Example: Brownian Motion



The resulting process $X(t)$ consists of the motions of all particles. A single realization $X(t, \omega_i)$ of this process is the motion of a specific particle(sample).

Example: AC Generator



The voltage

$$X(t) = r\cos(\omega t + \varphi)$$

with random amplitude r and phase φ. In this case, the process $X(t)$ consists of a family of pure sine waves and a single sample is the function

$$X(t, \omega_i) = r(\omega_i)\cos[\omega t + \varphi(\omega_i)]$$

Definition of Equality

Equality.

We shall say that two stochastic processes $X(t)$and $Y(t)$ are equal(everywhere) if their respective samples $X(t,\omega)$and $Y(t,\omega)$ are identical for every ω.

Similarly, the equality $Z(t) = X(t) + Y(t)$ means that

$$Z(t,\omega) = X(t,\omega) + Y(t,\omega) \qquad \text{for every } \omega.$$

Derivatives integrals, or any other operations involving stochastic processes are defined similarly in terms of the corresponding operations for each sample.

As in the case of limits, the above definitions can be relaxed. We give below the meaning of MS equality. Two processes $X(t)$ and $Y(t)$ are equal in the MS sense iff

$$E\{|X(t) - Y(t)|^2\} = 0 \tag{1}$$

for every t Equality in the MS sense.

Notation:

- A_t denotes the set of outcomes ω such that $X(t,\omega) = Y(t,\omega)$ for a specific t,
- and A_∞ denotes the set of outcomes ω such that $X(t,\omega) = Y(t,\omega)$ for every t.

From

$$E\{|X(t) - Y(t)|^2\} = 0$$

it follows that $X(t,\omega) - Y(t,\omega) = 0$ with probability 1; hence

$$P(A_t) = P(S) = 1.$$

It does not follow, however, that $P(A_\infty) = 1$. In fact,since A_∞ is the intersection of all sets A_t as t ranges over the entire axis, $P(A_\infty)$ might even equal 0.

Outline

Statistics of Stochastic Processes

A stochastic process is a noncountable infinity of random variables, one for each t. For a specific t, $X(t)$ is a random variable with distribution

$$F_X(x,t) = P\{X(t) \leq x\} \tag{2}$$

This function depends on t, and it equals the probability of the event $X(t) \leq x$ consisting of all outcomes ω such that, at the specific time t, the samples $X(t,\omega)$ of the given process do not exceed the number x. The function $F_X(x,t)$ will be called the *first-order distribution* of the process $X(t)$. Its derivative with respect to x

$$f_X(x,t) = \frac{\partial F_X(x,t)}{\partial x} \tag{3}$$

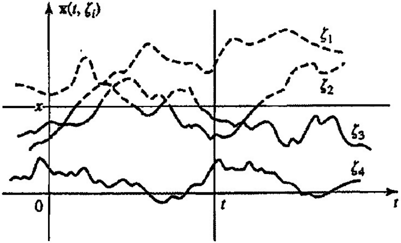
is the *first-order density* of $X(t)$.

Frequency interpretation

If the experiment is performed n times, then n functions $X(t, \omega_i)$ are observed,one for each trial.

Denoting by $n_t(x)$ the number of trials such that at time t the ordinates of the observed functions do not exceed x (solid lines),we conclude that

$$F_X(x,t) \simeq \frac{n_t(x)}{n} \quad (4)$$



The $second - order\ distribution$ of the process $X(t)$ is the joint distribution of the random variables $X(t_1)$ and $X(t_2)$.

$$F_X(x_1, x_2; t_1, t_2) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2\} \tag{5}$$

The corresponding density equals

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2 F_X(x_1, x_2; t_1, t_2)}{\partial x_1 \partial x_2} \tag{6}$$

We note that(consistency conditions)

$$F_X(x_1; t_1) = F_X(x_1, \infty; t_1, t2) \qquad f_X(x_1, t_1) = \int_{-\infty}^{\infty} f_X(x_1, x_2; t_1, t_2) dx_2$$

The $nth - order\ distribution$ of $X(t)$is the joint distribution $F_X(x_1, \cdots, x_n; t_1, \cdots t_n)$ of the random variables $X(t_1), \cdots, X(t_n)$.

Outline

Second-Order Properties

Mean

The mean $\mu(t)$ of $X(t)$ is the expected value of the random variable $X(t)$:

$$\mu(t) = E\{X(t)\} = \int_{-\infty}^{\infty} x f_X(x,t)dx \quad (7)$$

Autocorrelation (自相关)

The autocorrelation $R_X(t_1, t_2)$ of $X(t)$ is the expected value of the product $X(t_1)X(t_2)$:

$$R_X(t_1, t_2) = E\{X(t_1)X(t_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2; t_1, t_2)dx_1 dx_2 \quad (8)$$

The value of $R_X(t_1, t_2)$ on the diagonal $t_1 = t_2 = t$ is the average power of $X(t)$:

$$E\{X^2(t)\} = R_X(t, t)$$

Autocovariance (自协方差)

The *autocovariance* $C_X(t_1, t_2)$ of $X(t)$ is the covariance of the random variables $X(t_1)$ and $X(t_2)$:

$$C_X(t_1, t_2) = R_X(t_1, t_2) - \mu(t_1)\mu(t_2) \tag{9}$$

and its value $C_X(t, t)$ on the diagonal $t_1 = t_2 = t$ equals the variance of $X(t)$.

$$E\{[X(t_1) + X(t_2)]^2\} = R_X(t_1, t_1) + 2R(t_1, t_2) + R_X(t_2, t_2)$$

Example

A stochastic process is a deterministic signal $X(t) = f(t)$. In this case, to determine the mean, autocorrelation of $X(t)$.

Solution

$$\mu(t) = E\{f(t)\} = f(t)$$

$$R_X(t_1, t_2) = E\{f(t_1)f(t_2)\} = f(t_1)f(t_2)$$

Example

Suppose that $X(t)$ is a process with

$$\mu(t) = 3 \qquad R_X(t_1, t_2) = 9 + 4e^{-0.2|t_1 - t_2|}$$

We shall determine the mean, the variance, and the covariance of the random variables $Z = X(5)$ and $W = X(8)$.

Solution

Clearly, $E\{Z\} = \mu(5) = 3$ *and* $E\{W\} = \mu(8) = 3$*. Furthermore,*

$$E\{Z^2\} = R(5,5) = 13 \qquad E\{W^2\} = R(8,8) = 13$$

$$E\{ZW\} = R(5,8) = 9 + 4e^{-0.6} = 11.195$$

Thus Z *and* W *have the same variance* $\sigma^2 = 4$ *and their covariance equals* $C(5,8) = 4e^{-0.6} = 2.195$

Riemann integral

The integral

$$S = \int_a^b X(t)dt$$

of a stochastic process $X(t)$ is a random variable S and its value $S(\omega)$ for a specific outcome ω is the area under the curve $X(t,\omega)$ in the interval(a,b).We conclude from the linearity of expected values that

$$\mu_S = E\{S\} = \int_a^b E\{X(t)\}dt = \int_a^b \mu(t)dt \tag{10}$$

积分的均值 $=$ 均值的积分
Similarly, since

$$S^2 = \int_a^b \int_a^b X(t_1)X(t_2)dt_1dt_2$$

we conclude using again the linearity of expected values,that

$$E\{S^2\} = \int_a^b \int_a^b E\{X(t_1)X(t_2)\}dt_1dt_2 = \int_a^b \int_a^b R_X(t_1,t_2)dt_1dt_2 \tag{11}$$

Example

To determine the autocorrelation $R_X(t_1, t_2)$ of the process

$$X(t) = r\cos(\omega t + \varphi)$$

where the random variables r and φ are independent and φ is uniform in the interval $(-\pi, \pi)$.

Solution

Using simple trigonometric identifies,We find

$$E\{X(t_1)X(t_2)\} = \frac{1}{2}E\{r^2\}E\{\cos\omega(t_1 - t_2) + \cos(\omega t_1 + \omega t_2 + 2\varphi)\}$$

and since

$$E\{\cos(\omega t_1 + \omega t_2 + 2\varphi)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(\omega t_1 + \omega t_2 + 2\varphi)d\varphi = 0$$

We conclude that

$$R_X(t_1, t_2) = \frac{1}{2}E\{r^2\}\cos\omega(t_1 - t_2) \tag{12}$$

Outline

General Properties

The statistical properties of a real stochastic process $X(t)$ are completely determined in terms of its n^{th}-order distribution

$$F_X(x_1 \cdots, x_n; t_1 \cdots, t_n) = P\{X(t_1) \leq x_1, \cdots, X(t_n) \leq x_n\} \quad (13)$$

The joint statistics of two real processes $X(t)$ and $Y(t)$ are determined in terms of the joint distribution of the random variables

$$X(t_1) \cdots, X(t_n), Y(t_1') \cdots, Y(t_m')$$

The complex process $Z(t) = X(t) + jY(t)$ is specified in terms of the joint statistics of the real processes $X(t)$ and $Y(t)$.

A vector process (n-dimensional process) is a family of n stochastic processes.

Outline

Correlation

The autocorrelation of a process $X(t)$, real or complex, is by definition the mean of the product $X(t_1)X^*(t_2)$. This function will be denoted by $R(t_1, t_2)$ or $R_X(t_1, t_2)$ or $R_{XX}(t_1, t_2)$. Thus

$$R_{XX}(t_1, t_2) = E\{X(t_1)X^*(t_2)\} \tag{14}$$

where the conjugate tennis associated with the second variable in $R_{XX}(t_1, t_2)$. From this it follows that

$$R_X(t_2, t_1) = E\{X(t_2)X^*(t_1)\} = R^*(t_1, t_2) \tag{15}$$

We note, further, that

$$R_X(t, t) = E\{|X(t)|^2\} \geq 0 \tag{16}$$

The autocorrelation $R(t_1, t_2)$ of a stochastic process $X(t)$ is a positive definite (p.d.) function, that is, for any a_i and a_j :

$$\sum_{i,j} a_i a_j^* R(t_i, t_j) \geq 0 \tag{17}$$

This is a consequence of the identity

$$0 \leq E\{|\sum_i a_i X(t_i)|^2\} = \sum_{i,j} a_i a_j^* E\{X(t_i)X^*(t_j)\}$$

Example

(a) If $X(t) = ae^{j\omega t}$, then

$$R_X(t_1, t_2) = E\{ae^{j\omega t_1} a^* e^{-j\omega t_2}\} = E\{|a|^2\} e^{j\omega(t_1 - t_2)}$$

(b)Suppose that the random variables a_i are uncorrelated with zero mean and variance σ_i^2. If

$$X(t) = \sum_i a_i e^{j\omega_i t}$$

then

$$R_X(t_1, t_2) = \sum_i \sigma_i^2 e^{j\omega_i(t_1 - t_2)}$$

Autocovariance

The autocovariance $C_X(t_1, t_2)$ of a process $X(t)$ is the covariance of the random variables $X(t_1)$ and $X(t_2)$:

$$C_X(t_1, t_2) = R_X(t_1, t_2) - \mu(t_1)\mu^*(t_2) \tag{18}$$

where, $\mu(t) = E\{X(t)\}$ is the mean of $X(t)$.

The ratio

$$\rho(t_1, t_2) = \frac{C_X(t_1, t_2)}{\sqrt{C_X(t_1, t_1)C_X(t_2, t_2)}} \tag{19}$$

is the *correlation coefficient* of the process $X(t)$.

Notes

The autocovariance $C_X(t_1, t_2)$ of a process $X(t)$is the autocorrelation of the $centered\ process$

$$\overline{X}(t) = X(t) - \mu(t)$$

Hence it is the p.d.

The correlation coefficient $\rho(t_1, t_2)$ of $X(t)$ is the autocovariance of the $normalized\ process$ $X(t)/\sqrt{C_X(t,t)}$; hence it is also p.d.

$$|\rho(t_1, t_2)| \leqslant 1 \qquad \rho(t,t) = 1 \tag{20}$$

Cross-correlation

The cross-correlation of two process $X(t)$ and $Y(t)$ is the function

$$R_{XY}(t_1, t_2) = E\{X(t_1)Y^*(t_2)\} = R^*_{YX}(t_2, t_1) \quad (21)$$

Similarly,

$$C_{XY}(t_1, t_2) = R_{XY}(t_1, t_2) - \mu_X(t_1)\mu^*_Y(t_2) \quad (22)$$

is their $cross - covariance$.

Two processes $X(t)$ and $Y(t)$ are called (mutually) orthogonal if

$$R_{XY}(t_1, t_2) = 0 \qquad \text{for every} \quad t_1 \quad \text{and} \quad t_2 \quad (23)$$

They are called $uncorrelated$ if

$$C_{XY}(t_1, t_2) = 0 \qquad \text{for every} \quad t_1 \quad \text{and} \quad t_2 \quad (24)$$

a-dependent processes a依赖过程

In general, the values $X(t_1)$ and $X(t_2)$ of a stochastic process $X(t)$ are statistically dependent for any t_1 and t_2.

However, in most cases this dependence decreases as $|t_1 - t_2| \to \infty$.

This leads to the following concept: A stochastic process $X(t)$ is called $a - dependent$ if all its values $X(t)$ for $t < t_0$ and for $t > t_0 + a$ are mutually $independent$.

From this it follows that

$$C_X(t_1, t_2) = 0 \qquad for \quad |t_1 - t_2| > a \tag{25}$$

A process $X(t)$ is called $correlation\ a - dependented$ if its autocorrelation satisfies (25).

Clearly, if $X(t)$ is correlation a-dependent, then any linear combination of its values for $t < t_0$ is uncorrelated with any linear combination of its values for $t > t_0 + a$.

White noise

We shall say that a process $V(t)$ is white noise if its values $V(t_i)$ and $V(t_j)$ are uncorrelated for every t_i and $t_j \neq t_i$;

$$C_V(t_i, t_j) = 0 \qquad t_i \neq t_j$$

The autocovariance of a nontrivial white-noise process must be of the form

$$C_V(t_1, t_2) = q(t_1)\delta(t_1 - t_2) \qquad q(t) \geqslant 0 \tag{26}$$

If the random variables $V(t_i)$ and $V(t_j)$ are not uncorrelated but also independent, then $V(t)$ will be called $strictly$ white noise. Unless otherwise stated, it will be assumed that the mean of a white-noise process is identically 0.

Example

Suppose that $V(t)$ is white noise and

$$X(t) = \int_0^t v(\alpha) d\alpha \tag{27}$$

Inserting (26) into (27), we obtain

$$E\{X^2(t)\} = \int_0^t \int_0^t q(t_1)\delta(t_1 - t_2) dt_2 dt_1 = \int_0^t q(t_1) dt_1 \tag{28}$$

because

$$\int_0^t \delta(t_1 - t_2) dt_2 = 1 \qquad for \quad 0 < t_1 < t$$

Uncorrelated and indpendent increments (不相关独立增量)

If the increments $X(t_2) - X(t_1)$ and $X(t_3) - X(t_4)$ of a process X_t are uncorrelated (independent) for any $t_1 < t_2 < t_3 < t_4$, then we say that $X(t)$ is a process with **uncorrelated (independent) increments**.

The Poisson process is a process with independent increments.

The integral

$$X(t) = \int_0^t v(\alpha)d\alpha$$

of white noise is a process with uncorrelated increments.

Independent process

If two processes $X(t)$ and $Y(t)$ are such that the random variables $X(t_1), \cdots, X(t_n)$ and $Y(t_1^{'}, \cdots, Y(t_n^{'}))$ are mutually independent, then these processes are called independent.

Outline

Normal Processes

A process $X(t)$ is called normal, if the random variables $X(t_1), \cdots, X(t_n)$ are jointly normal for any n and $t_1, \cdots, t_n$.

The statistics of a normal process are completely determined in terms of its mean $\mu_X(t)$ and autocovariance $C_X(t_1, t_2)$. Indeed, since

$$E\{X(t)\} = \mu_X(t) \qquad \sigma_X^2(t) = C_X(t, t)$$

we conclude that the first-order density $f_X(x, t)$ of $X(t)$ is the normal density $N[\mu_X(t); \sqrt{C_X(t, t)}]$.

Similarly, since the function $\rho(t_1, t_2)$ is the correlation coefficient of the random variables $X(t_1)$ and $X(t_2)$, the second-order density $f_X(x_1, x_2; t_1, t_2)$ of $X(t)$ is the jointly normal density

$$N[\mu_X(t_1), \mu_X(t_2); \sqrt{C_X(t_1, t_2)}, \sqrt{C_X(t_1, t_2)}; \rho(t_1, t_2)]$$

The n^{th} order characteristic function of the process $X(t)$ is given

$$\exp\{j \sum_i \mu_X(t_i)\omega_i - \frac{1}{2} \sum_{i,k} C_X(t_i, t_k)\omega_i\omega_k\} \tag{29}$$

Its inverse $f_X(x_1, ..., x_n, t_1, ..., t_n)$ is the n^{th} order density of $X(t)$.

Existence Theorem

Given an arbitrary function $\mu_X(t)$ and a p.d. function $C_X(t_1, t_2)$, we can construct a normal process with mean $\mu_X(t)$ and autocovariance $C_X(t_1, t_2)$.

This follows if we use in

$$\exp\{j \sum_i \mu(t_i)\omega_i - \frac{1}{2} \sum_{i,k} C_X(t_i, t_k)\omega_i\omega_k\}$$

the given functions $\mu_X(t)$ and $C_X(t_1, t_2)$.

The inverse of the resulting characteristic function is a density because the function $C_X(t_1, t_2)$ is p.d. by assumption.

Suppose that $X(t)$ is a normal process with

$$\mu_X(t) = 3 \qquad C_X(t_1, t_2) = 4e^{-0.2|t_1 - t_2|}$$

(a) Find the probability that $X(5) \leq 2$.
Clearly, $X(5)$ is a normal random variable with mean $\mu_X(5) = 3$ and variance $C_X(5,5) = 4$. Hence

$$P\{X(5) \leq 2\} = G(-1/2) = 0.309$$

(b) Find the probability that $|X(8) - X(5)| \leq 1$.
The difference $S = X(8) - X(5)$ is a normal random variable with mean $\mu_X(8) - \mu_X(5) = 0$ and variance

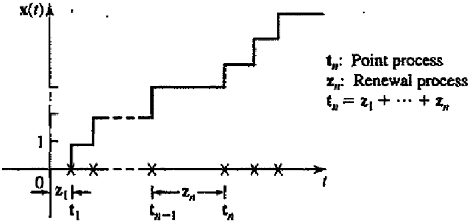
$$C_S(8,8) = C_X(5,5) - 2C_X(8,5) = 8(1 - e^{-0.6}) = 3.608$$

Hence

$$P\{|X(8) - X(5)| \leq 1\} = 2G(1/1.9) - 1 = 0.4$$

Outline

Point and Renewal Process



A *point process* is a set of random points t_i on the time axis. To every point process we can associate a stochastic process $X(t)$ equal to the number of points t_i in the interval $(0, t)$.

An example is the Poisson process.

To every point process t_i we can associate a sequence of random variables Z_n such that

$$Z_1 = t_1 \qquad Z_2 = t_2 - t_1 \cdots Z_n = t_n - t_{n-1}$$

where t_1 is the first random point to the right of the origin. This sequence is called a $renewal\ process$.

An example is the life history of lightbulbs that are replaced as soon as they fail. In this case, Z_i is the total time the ith bulb is in operation and t_i is the time of its failure.

We have thus established a correspondence between the following three concepts:

(a) a point process t_i,
(b) a discrete-state stochastic process X_t increasing in unit steps at the points t_i,
(c) a renewal process consisting of the random variables Z_i and such that $t_n = Z_1 + \cdots + Z_n$.

Outline

Outline

Strict Sense Stationary

A stochastic process $X(t)$ is called strict-sense stationary (SSS) if its statistical properties are invariant to a shift of the origin. This means that the processes $X(t)$ and $X(t+c)$ have the same statistics for any c.

Two process $X(t)$ and $Y(t)$ are called jointly stationary if the joint statistics of $X(t)$ and $Y(t)$ are the same as the joint statistics of $X(t+c)$ and $Y(t+c)$ for any c.

A complex process $Z(t) = X(t) + jY(t)$ is stationary if the process $X(t)$ and $Y(t)$ are jointly stationary.

From the definition it follows that the n^{th}-order density of an SSS process must be such that

$$f_X(x_1, ..., x_n; t_1, ..., t_n) = f_X(x_1, ..., x_n; t_1 + c, ..., t_n + c) \quad (30)$$

for any c.

From this it follows that $f_X(x;t) = f_X(x;t+c)$ for any c. Hence the first-order density of $X(t)$ is independent of t:

$$f_X(x;t) = f_X(x) \tag{31}$$

Similarly, $f_X(x_1, x_2; t_1 + c, t_2 + c)$ is independent of c for any c, in particular for $c = -t_2$. This leads to the conclusion that

$$f_X(x_1, x_2, t_1, t_2) = f_X(x_1, x_2; \tau) \qquad \tau = t_1 - t_2 \tag{32}$$

Thus the joint density of the random variables $X(t+\tau)$ and $X(t)$ is independent of t and it equals $f_X(x_1, x_2; \tau)$.

Outline

Wide Sense Stationary

A stochastic process $X(t)$ is called wide-sense stationary (WSS) if its mean is constant

$$E\{X(t)\} = \mu \tag{33}$$

and its autocorrelation depends only on $\tau = t_1 - t_2$:

$$E\{X(t+\tau)X^*(t)\} = R(\tau) \tag{34}$$

Since τ is the distance from t to $t+\tau$, the function $R(\tau)$ can be written in the symmetrical form

$$R(\tau) = E\{X(t+\frac{\tau}{2})X^*(t-\frac{\tau}{2})\} \tag{35}$$

Note in particular that

$$E\{|X(t)|^2\} = R(0)$$

Thus the average power of a stationary process is independent of t and it equals $R(0)$.

Example

Suppose that $X(t)$ is a WSS process with autocorrelation

$$R(\tau) = Ae^{-\alpha|\tau|}$$

We shall determine the second moment of the random variable $X(8) - X(5)$.

Solution

Clearly,

$$\begin{aligned} E\{[X(8)-X(5)]^2\} &= E\{X^2(8)\}+E\{X^2(5)\}-2E\{X(8)X(5)\} \\ &= R(0)+R(0)-2R(3) = 2A-2Ae^{-3\alpha} \end{aligned}$$

The autocorrelation of a WSS process depends only on $\tau = t_1 - t_2$:

$$C(\tau) = R(\tau) - |\mu|^2 \tag{36}$$

and its correlation coefficient equals

$$\rho(\tau) = C(\tau)/C(0) \tag{37}$$

Thus $C(\tau)$ is the covariance, and $\rho(\tau)$ the correlation coefficient of the random variables $X(t+\tau)$ and $X(t)$.

Two processes $X(t)$ and $Y(t)$ are called jointly WSS if each is WSS and their cross-correlation depends only on $\tau = t_1 - t_2$:

$$R_{XY}(\tau) = E\{X(t+\tau)Y^*(t)\} \qquad C_{XY}(\tau) = R_{XY}(\tau) - \mu_X\mu_Y^* \tag{38}$$

If $X(t)$ is WSS white noise, then

$$C_X(\tau) = q\delta(\tau) \tag{39}$$

If $X(t)$ is an a-dependent process, then $C_X(\tau) = 0$ for $|\tau| > a$. In this case, the constant a is called the correlation time of $X(t)$. This term is also used for arbitrary processes and it is defined as the ratio

$$\tau_c = \frac{1}{C_X(0)} \int_0^{\infty} C_X(\tau)d\tau \tag{40}$$

In general $C_X(\tau) \neq 0$ for every τ. However, for most regular processes

$$C_X(\tau) \xrightarrow[|\tau|\to\infty]{} 0 \qquad R_X(\tau) \xrightarrow[|\tau|\to\infty]{} |\mu|^2$$

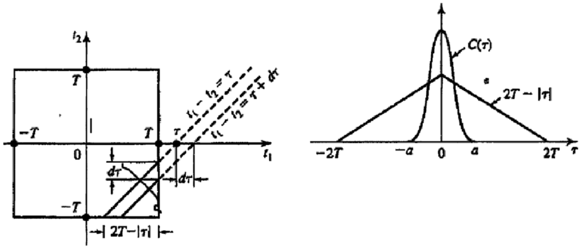
If $X(t)$ is WSS and

$$S = \int_{-T}^{T} X(t)dt$$

then

$$\sigma_S^2 = \int_{-T}^{T}\int_{-T}^{T} C_X(t_1 - t_2)dt_1 dt_2 = \int_{-2T}^{2T} (2T - |\tau|)C_X(\tau)d\tau \quad (41)$$

The last equality follows with $\tau = t_1 - t_2$ (see Fig); the details, however, are omitted.



Special cases

(a) If $C_X(\tau) = q\delta(\tau)$, then

$$\sigma_S^2 = \int_{-2T}^{2T} (2T - |\tau|) C_X(\tau) d\tau = 2Tq$$

(b) If the process $X(t)$ is a-dependent and $a \ll T$, then (41) yields

$$\sigma_S^2 = \int_{-2T}^{2T} (2T - |\tau|) C_X(\tau) d\tau \simeq 2T \int_{-a}^{a} C_X(\tau) d\tau$$

$$\int_{-T}^{T} \int_{-T}^{T} C_X(t_1 - t_2) dt_1 dt_2 = \int_{-2T}^{2T} (2T - |\tau|) C_X(\tau) d\tau$$

This shows that, in the evaluation of the variance of S, an a-dependent process with $a \ll T$ can be replaced by white noise with $q = \int_{-a}^{a} C_X(\tau) d\tau$

If a process is SSS, then it is also WSS, This follows readily from (31) and (32).
Indeed, suppose that $X(t)$ is a normal WSS process with mean μ_X and autocovariance $C_X(\tau)$. As we see from (29), its n^{th}-order characteristic function equals

$$\exp\{j\mu_X \sum_i \omega_i - \frac{1}{2}\sum_{i,k} C_X(t_i - t_k)\omega_i\omega_k\} \tag{42}$$

This function is invariant to a shift of the origin. And since it determines completely the statistics of $X(t)$, we conclude that $X(t)$ is SSS.

Example

We shall establish necessary and sufficient conditions for the stationary of the process

$$X(t) = a\cos\omega t + b\sin\omega t \tag{43}$$

The mean of this process equals

$$E\{X(t)\} = E\{a\}\cos\omega t + E\{b\}\sin\omega t$$

This function must be independent of t. Hence the condition

$$E\{a\} = E\{b\} = 0 \tag{44}$$

is necessary for both forms of stationary. We shall assume that it holds.

Example

Wild sense. The process $X(t)$ is WSS iff the random variables a and b are uncorrelated with equal variance:

$$E\{ab\} = 0 \qquad E\{a^2\} = E\{b^2\} = \sigma^2 \tag{45}$$

If this holds, then

$$R(\tau) = \sigma^2 \cos \omega\tau \tag{46}$$

Proof.

If $X(t)$ is WSS, then

$$E\{X^2(0)\} = E\{X^2(\pi/2\omega)\} = R(0)$$

But $X(0) = a$ and $X(\pi/2\omega) = b$; hence $E\{a^2\} = E\{b^2\}$. Using the above, we obtain

$$\begin{aligned} E\{X(t+\tau)X(t)\} &= E\{[a\cos\omega(t+\tau) + b\sin\omega(t+\tau)] \\ &\qquad [a\cos\omega t + b\sin\omega t]\} \\ &= \sigma^2\cos\omega\tau + E\{ab\}\sin\omega(2t+\tau) \end{aligned}$$

This is independent of t only if $E\{ab\} = 0$ and

$$E\{ab\} = 0 \qquad E\{a^2\} = E\{b^2\} = \sigma^2 \tag{47}$$

Conversely, the autocorrelation of $X(t)$ equals $\sigma^2\cos\omega t$; hence $X(t)$ is WSS. $\square$

Example

Strict sense. The process $X(t)$ is SSS iff the joint density $f(a,b)$ of the random variables a and b has circular symmetry (圆对称), that is, if

$$f(a,b) = f(\sqrt{a^2+b^2}) \tag{48}$$

Proof.

If $X(t)$ is SSS, then the random variables

$$X(0) = a \qquad X(\pi/2\omega) = b$$

and

$$X(t) = a\cos\omega t + b\sin\omega t \qquad X(t+\pi/2\omega) = b\cos\omega t - a\sin\omega t$$

have the same joint density for every t, Hence $f(a,b)$ must have circular symmetry.

We shall now show that, if $f(a,b)$ has circular symmetry, then $X(t)$ is SSS, With τ a given number and

$$a_1 = a\cos\omega\tau + b\sin\omega\tau \qquad b_1 = b\cos\omega\tau - a\sin\omega\tau$$

$\square$

Proof.

we form the process

$$X_1(t) = a_1 \cos \omega t + b_1 \sin \omega t = X(t + \tau)$$

Clearly, the statistics of $X(t)$ and $X_1(t)$ are determined in terms of the joint densities $f(a, b)$ and $f(a_1, b_1)$ of the random variables a, b and a_1, b_1 have the same joint density. Hence the processes $X(t)$ and $X(t + \tau)$ have the same statistics for every τ. $\square$

Corrolary

If the process $X(t)$ is SSS and the random variables a and b are independent, then they are normal.

Example

Given a random variable ω with density $f(\omega)$ and a random variable φ uniform in the interval $(-\pi, \pi)$ and independent of ω, we form the process

$$X(t) = a\cos(\omega t + \varphi) \tag{49}$$

We shall show that $X(t)$ is WSS with zero mean and autocorrelation

$$R(\tau) = \frac{a^2}{2} E\{\cos\omega\tau\} = \frac{a^2}{2} \Re e\{\phi_\omega(\tau)\} \tag{50}$$

where

$$\phi_\omega(\tau) = E\{e^{j\omega t}\} = E\{\cos\omega\tau\} + jE\{\sin\omega\tau\} \tag{51}$$

is the characteristic function of ω.

Proof.

Clearly

$$E\{\cos(\omega t+\varphi)\} = E\{E\{\cos(\omega t+\varphi)\}|\omega\}$$

From the independent of ω and φ, it follows that

$$E\{\cos(\omega t+\varphi)|\omega\} = \cos\omega t E\{\cos\varphi\} - \sin\omega t E\{\sin\varphi\}$$

Hence $E\{X(t)\} = 0$ because

$$\begin{aligned} E\{\cos\varphi\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos\varphi d\varphi = 0 \\ E\{\sin\varphi\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sin\varphi d\varphi = 0 \end{aligned}$$

Reasoning similarly, we obtain $E\{\cos(2\omega t+\omega\tau+2\varphi)\} = 0$. And since

$$2\cos[\omega(t+\tau)+\varphi]\cos(\omega t+\varphi) = \cos\omega\tau + \cos(2\omega t+\omega\tau+2\varphi)$$

□

Proof.

We conclude that

$$R(\tau) = a^2 E\{\cos[\omega(t+\tau)+\varphi]\cos(\omega t+\varphi)\} = \frac{a^2}{2}E\{\cos\omega\tau\}$$

Further, with ω and φ as above. The process

$$Z(t) = ae^{j(\omega t+\varphi)}$$

is WSS with zero mean and autocorrelation

$$E\{Z(t+\tau)Z^*(t)\} = a^2 E\{e^{j\omega t}\} = a^2\phi_\omega(\tau)$$

□

Centering

Given a process $X(t)$ with mean $\mu_X(t)$ and autocovariance $C_X(t_1, t_2)$, we form difference

$$\overline{X}(t) = X(t) - \mu_X(t) \tag{52}$$

This difference is called the centered process associated with the process $X(t)$.
Note that

$$E\{\overline{X}(t)\} = 0 \qquad R_{\overline{X}}(t_1, t_2) = C_X(t_1, t_2)$$

From this it follows that if the process $X(t)$ is covariance stationary, that is, if $C_X(t_1, t_2) = C_X(t_1 - t_2)$, then its centered process $\overline{X}(t)$ is WSS.

Other Forms of Stationary

A process $X(t)$ is asymptotically stationary (渐近平稳) if the statistics of the random variables $X(t_1+c), ..., X(t_n+c)$ do not depend on c if c is large. More precisely, the function

$$f_X(x_1, ..., x_n, t_1+c, ..., t_n+c)$$

tends to a limit (that dose not depend on c) as $c \to \infty$.

A process $X(t)$ is N^{th}-order starionary if

$$f_X(x_1, ..., x_n; t_1, ..., t_n) = f_X(x_1, ..., x_n; t_1+c, ..., t_n+c)$$

holds not for every n, but only for $n \leqslant N$.

A process $X(t)$ is stationary in a interval (区间平稳) if

$$f_X(x_1, ..., x_n; t_1, ..., t_n) = f_X(x_1, ..., x_n; t_1+c, ..., t_n+c)$$

holds for every t_i and $t_i + c$ in this interval.

We say that $X(t)$ is a process with stationary increments (增量平稳) if its increments $Y(t) = X(t+h) - X(t)$ form a stationary process for every h. The Poisson process is an example.

Mean Square Periodicity

A process $X(t)$ is called MS periodic if

$$E\{|X(t+T) - X(t)|^2\} = 0 \quad (53)$$

for every t, From this it follows that, for a specific t,

$$X(t+T) = X(t) \quad (54)$$

with probability 1.

It does not, however, follow that the set of outcomes ζ such that $X(t+T,\zeta) = X(t,\zeta)$ for all t has probability 1.

Theorem A process $X(t)$ is MS periodic iff its autocorrelation is $doubly\ periodic$, that is, if

$$R_X(t_1 + mT, t_2 + nT) = R_X(t_1, t_2) \quad (55)$$

for every integer m and n.

Proof: As we know

$$E^2\{ZW\} \le E\{Z^2\}E\{W^2\}$$

With $Z = X(t_1)$ and $W = X(t_2+T) - X(t_2)$ this yields

$$E^2\{X(t_1)[X(t_2+T) - X(t_2)]\} \le E\{X^2(t_1)\}E\{[X(t_2+T) - X(t_2)]^2\}$$

If $X(t)$ is MS periodic, then the last term in the last equation is 0. Equating the left side to 0, we obtain

$$R_X(t_1, t_2+T) - R_X(t_1, t_2) = 0$$

Repeated application of this yields (55).

Conversely, if (55) is true, then

$$R_X(t+T, t+T) = R_X(t+T, t) = R_X(t, t)$$

Hence

$$E\{[X(t+T) - X(t)]^2\} = R_X(t+T, t+T) + R_X(t, t) - 2R(t+T, t) = 0$$

therefore $X(t)$ is MS periodic.

Outline

We typically observe the outcome of a random process (e.g., we record a noise waveform) and want to **characterize the statistics of the random process by measurements on one ensemble member**.

For instance, we could consider the time-average of the waveform to represent the mean value of the process (assuming this mean is constant for all time). We could also construct histograms that represent the fraction of time (rather than the probability-weighted fraction of the ensemble)that the waveform lies in different amplitude bins, and this could be taken to reflect the probability density across the ensemble of the value obtained at a particular sampling time.

If the random process is such that the behavior of almost every particular realization over time is representative of the behavior down the ensemble, then the process is called **ergodic**.

Narrower notions of ergodicity may be defined. For example, if the time average

$$\langle X(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X(t)dt \tag{56}$$

almost always (i.e. for almost every realization or outcome) equals the ensemble average μ_X, then the process is termed **ergodic in the mean**.

It can be shown, for instance, that a WSS process with finite variance at each instant and with a covariance function that approaches 0 for large lags is ergodic in the mean.

Note that a (nonstationary) process with time-varying mean cannot be ergodic in the mean.

In our discussion of random processes, we will primarily be concerned with first-order and second-order moments of random processes.

While it is extremely difficult to determine in general whether a random process is ergodic, there are criteria(specified in terms of the moments of the process) that will establish ergodicity in the mean and in the autocorrelation.

Frequently, however, such ergodicity is simply assumed for convenience, in the absence of evidence that the assumption is not reasonable. Under this assumption, the mean and autocorrelation can be obtained from time-averaging on a single ensemble member, through the following equalities:

$$E\{X(t)\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)dt \quad (57)$$

and

$$E\{X(t)X(t+\tau)\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)X(t+\tau)dt \quad (58)$$

A random process for which (57) and (58) are true is referred as second-order ergodic.