04: Discrete-time random signal characterization and transmission

EE21: Random Signal Processing

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Outline

1. **Sampling**
2. Autocorrelation and Autocovariance
3. Power Spectrum

Sampling

Given an analog process $X(t)$, we form the digital process

$$\mathbf{X}[n] = \mathbf{X}(nT)$$

where T is a given constant. From this it follows that

$$\mu[n] = \mu_a(nT) \qquad R[n_1, n_2] = R_a(n_1T, n_2T) \tag{1}$$

where $\mu_a(t)$ is the mean and $R_a(t_1, t_2)$ the autocorrelation of $\mathbf{X}(t)$.

If $\mathbf{X}(t)$ is a stationary process, then $\mathbf{X}[n]$ is also stationary with mean $\mu = \mu_a$ and autocorrelation

$$R[m] = R_a(mT)$$

From this it follows that the power spectrum of $X[n]$ equals

$$\mathbf{S}_X(e^{j\omega}) = \sum_{m=-\infty}^{\infty} R_a(mT)e^{-jm\omega} = \frac{1}{T}\sum_{n=-\infty}^{\infty} S_a(\frac{\omega+2\pi n}{T}) \tag{2}$$

where $S_a(\omega)$ is the power spectrum of $X(t)$. The above is a consequence of Poisson's sum formula.

Example

Suppose that $X(t)$ is a WSS process consisting of M exponentials as:

$$\mathbf{X}(t) = \sum_{i=1}^{M} \sigma_i e^{j\omega_i t} \qquad S_a(\omega) = 2\pi \sum_{i=1}^{M} \sigma_i^2 \delta(\omega - \omega_i)$$

where $\sigma_i^2 = E\{\mathbf{c}_i^2\}$. We shall determine the power spectrum $\mathbf{S}(e^{j\omega})$ of the process $\mathbf{X}[n] = \mathbf{X}(nT)$. Since $\delta(\omega/T) = T\delta(\omega)$, it follows from (2) that

$$\mathbf{S}(e^{j\omega}) = 2\pi \sum_{n=-\infty}^{\infty} \sum_{i=1}^{M} \sigma_i^2 \delta(\omega - T\omega_i + 2\pi n)$$

In the interval $(-\pi, \pi)$, this consists of M lines:

$$\mathbf{S}(e^{j\omega}) = 2\pi \sum_{i=1}^{M} \sigma_i^2 \delta(\omega - \beta_i) \qquad |\omega| < \pi$$

where $\beta_i = T\omega_i - 2\pi n_i$ and such that $|\beta_i| < \pi$.

Outline

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2. **Autocorrelation and Autocovariance**
3. Power Spectrum

Autocorrelation and Autocovariance

The autocorrelation and autocovariance of $X[n]$ are given by

$$R[n_1, n_2] = E\{X[n_1]X^*[n_2]\} \quad (3)$$
$$C[n_1, n_2] = R[n_1, n_2] - \mu[n_1]\mu^*[n_2] \quad (4)$$

respectively where $\mu[n] = E\{X[n]\}$ is the mean of $X[n]$.

A process $X[n]$ is SSS if its statistical properties are invariant to a shift of the origin.

A process $X[n]$ is WSS if $\mu[n] = \mu =$ constant and

$$R[n+m, n] = E(X[n+m]X^*[n]) = R[m] \quad (5)$$

1. A process $X[n]$ is strictly white noise if the random variables $X[n_i]$ are independent.
2. It is white noise if the random variables $X[n_i]$ are uncorrelated.
3. The autocorrelation of a white-noise process with zero mean is thus given by

$$R[n_1, n_2] = q[n_1]\delta[n_1 - n_2] \qquad \text{where} \quad \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \tag{6}$$

   and $q[n] = E\{X^2[n]\}$.
4. If $X[n]$ is also stationary, then $R[m] = q\delta[m]$. Thus a WSS white noise is a sequence of i.i.d. random variables with variance q.

Outline

Power Spectrum

Given a WSS process $X[n]$, we form the z transform $S(z)$ of its autocorrelation $R[m]$:

$$S(z) = \sum_{m=-\infty}^{\infty} R[m] z^{-m} \tag{7}$$

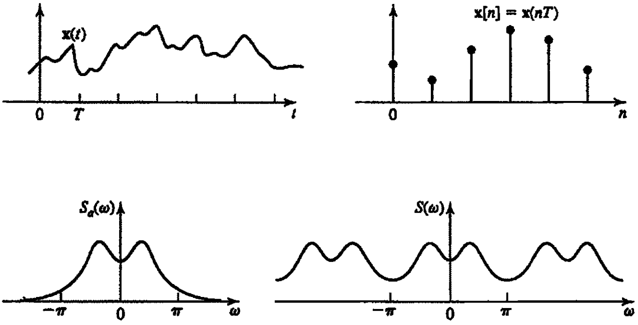
The power spectrum of $X[n]$ is the function

$$S(\omega) = S(e^{j\omega}) = \sum_{m=-\infty}^{\infty} R[m] e^{-jm\omega} \geq 0 \tag{8}$$

Thus $S(e^{j\omega})$ is the discrete Fourier transform (DFT) of $R[m]$. The function $S(e^{j\omega})$ is periodic with period 2π and Fourier series coefficients $R[m]$. Hence

$$R[m] = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(e^{j\omega}) e^{jm\omega} d\omega \tag{9}$$

It suffices, therefore, to specify $S(e^{j\omega})$ for $|w| < \pi$ only (see Fig.).



If $X[n]$ is a real process, then $R[-m] = R[m]$ and (8) yields

$$S(e^{j\omega}) = R[0] + 2\sum_{m=1}^{\infty} R[m]\cos m\omega \tag{10}$$

This shows that the power spectrum of a real process is a function of $\cos\omega$ because $\cos m\omega$ is a function of $\cos\omega$.

The nonnegativity condition can be expressed in terms of certain Hermitian Toeplitz matrices. Let

$$r_k \triangleq R[k] \tag{11}$$

and define

$$T_n = \begin{pmatrix} r_0 & r_1 & r_2 & . & \dots & r_n \\ r_1^* & r_0 & r_1 & r_2 & \dots & r_{n-1} \\ r_2^* & & r_0 & & \dots & r_{n-2} \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ r_n^* & r_{n-1}^* & . & \dots & r_1^* & r_0 \end{pmatrix} \tag{12}$$

In that case

$$S(\omega) \geq 0 \Leftrightarrow T_n \geq 0 \qquad n = 0 \to \infty \tag{13}$$

i.e., the nonnegative nature of the spectrum is equivalent to the nonnegativity of every Hemitian Toeplitz matrix T_n, $n = 0 \to \infty$

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Proof.

To prove this result due to Schur, first assume that $S(\omega) \geq 0$ in (8). Then letting

$$\boldsymbol{a} = [a_0, a_1, a_2, ..., a_n]^T \tag{14}$$

we have

$$\begin{aligned} \boldsymbol{a}^H T_n \boldsymbol{a} &= \sum_{i=0}^{n} \sum_{m=0}^{n} a_i^* a_m r_{i-m} \\ &= \sum_{i=0}^{n} \sum_{m=0}^{n} a_i^* a_m \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega) e^{j(i-m)\omega} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega) \left| \sum_{m=0}^{n} a_m e^{-jm\omega} \right|^2 d\omega \geq 0 \end{aligned} \tag{15}$$

Since $\boldsymbol{a}$ is arbitrary, this gives

$$S(\omega) \geq 0 \Rightarrow T_n \geq 0 \qquad n = 0 \to \infty \tag{16}$$

Conversely, assume that every T_n, $n = 0 \to \infty$ are nonnegative definite matrices.

Proof.

Further, for any ρ, $0 < \rho < 1$, and ω_0, $0 < \omega_0 < 2\pi$, define the vector $\boldsymbol{a}$ in (14) with

$$a_m = \sqrt{1-\rho^2}\rho^m e^{jm\omega_0}$$

Then $\boldsymbol{T}_n$ nonnegative implies that

$$0 \le \boldsymbol{a}^H \boldsymbol{T}_n \boldsymbol{a} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1-\rho^2)\left|\sum_{m=0}^{n}\rho^m e^{jm(\omega-\omega_0)}\right|^2 S(\omega)d\omega$$

and letting $n \to \infty$, the above intergrad tends to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(1-\rho^2)}{1-2\rho\cos(\omega-\omega_0)+\rho^2}S(\omega)d\omega \ge 0 \qquad (17)$$

The left-hand side of (17) represents the Poisson integral, and its interior ray limit as $p \to 1-0$ equals $S(\omega)$ for almost all ω_0. Thus

$$\mathbf{T}_n \ge 0 \quad n = 0 \to \infty \Rightarrow S(\omega) \ge 0 \quad \text{almost everywhere (a.e.)} \qquad (18)$$

Proof.

More interestingly, subject to the additional constraint, known as the Paley-Wiener criterion

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln S(\omega) d\omega > -\infty \tag{19}$$

every $\mathbf{T}_k$, $k = 0 \to \infty$, must be positive definite.

This follows from (15). In fact, if some $\mathbf{T}_k$ is singular, then there exists a nontrivial vector $\mathbf{a}$ such $\mathbf{T}_k \boldsymbol{a} = 0$ and, from (15),

$$\mathbf{a}^H \boldsymbol{T}_k \boldsymbol{a} = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\omega) \left| \sum_{m=0}^{k} a_m e^{jm\omega} \right|^2 d\omega = 0$$

Proof.

Since $S(\omega) \geq 0$, $a.e.$, this expression gives

$$S(\omega)\left|\sum_{m=0}^{k} a_m e^{jm\omega}\right|^2 = 0 \qquad a.e.$$

and $\sum_{m=0}^{k} a_m e^{-jm\omega} \neq 0$, $a.e.$, implies

$$S(\omega) = 0 \qquad a.e.$$

and

$$\int_{-\pi}^{\pi} \ln S(\omega) d\omega = -\infty$$

contradicting (19). Hence subject to (19), every

$$\mathbf{T}_k > 0 \qquad k = 0 \rightarrow \infty \tag{20}$$

$\square$

Example

If $R[m] = a^{|m|}$, then

$$\begin{aligned} S(z) &= \sum_{m=-\infty}^{-1} a^{-m} z^{-m} + \sum_{m=0}^{\infty} a^{m} z^{-m} = \frac{az}{1-az} + \frac{z}{z-a} \\ &= \frac{a^{-1}-a}{(a^{-1}+a)-(z^{-1}+z)} \end{aligned}$$

Hence $S(\omega) = S(e^{j\omega}) = \frac{a^{-1}-a}{a^{-1}+a-2\cos\omega}$

Example

Proceeding as in the analog case, we can show that the process

$$X[n] = \sum_i c_i e^{j\omega_i n}$$

is WSS iff the coefficients c_i, are uncorrelated with zero mean. In this case,

$$R[m] = \sum_i \sigma_i^2 e^{j\beta_i |m|} \qquad S(\omega) = 2\pi \sum_i \sigma_i^2 \delta(\omega - \beta_i) \qquad |\omega| < \pi \tag{21}$$

where $\sigma_i^2 = E\{c_i^2\}$, $\omega_i = 2\pi k_i + \beta_i$, and $|\beta_i| < \pi$.