05: Transmission of random signals in linear system

EE21: Random Signal Processing

March 2016

Zhigang SU
Professor
Sino-European Institute of Aviation Engineering
Civil Aviation University of China

Outline

System

Given a stochastic process $X(t)$, we assign according to some rule to each of its samples $X(t, \xi_i)$ a function $Y(t, \xi_i)$. We have thus created another process

$$Y(t) = T[X(t)]$$

whose samples are the function $Y(t, \xi_i)$.

The process $Y(t)$ so formed can be considered as the output of a system (transformation) with input the process $X(t)$.

The system is completely specified in terms of the operator T, that is, the rule of correspondence between the samples of the input $X(t)$ and the output $Y(t)$.

Deterministic and Stochastic System

The system is deterministic if it operates only on the variable t treating ξ as a parameter.

This means that if two samples $X(t, \xi_1)$ and $X(t, \xi_2)$ of the input are identical in t, then the corresponding samples $Y(t, \xi_1)$ and $Y(t, \xi_2)$ of the output are also identical in t.

The system is called stochastic if T operates on both variables t and ξ.

This means that there exist two outcomes ξ_1 and ξ_2 such that $X(t, \xi_1) = X(t, \xi_2)$ identically in t but $Y(t, \xi_1) \neq Y(t, \xi_2)$.

These classifications are based on the terminal properties of the system.

If the system is specified in terms of physical elements or by an equation, then it is deterministic (stochastic) if the elements or the coefficients of the defining equations are deterministic (stochastic).

Outline

Memoryless Systems

Definition

A system is called memoryless if its output is given by

$$Y(t) = g[X(t)]$$

where $g(X)$ is a function of X. Thus, at a given time $t = t_1$, the output $Y(t_1)$ depends only on $X(t_1)$ and not on any other past or future values of $X(t)$.

From this it follows that the first-order density $f_Y(y;t)$ of $Y(t)$ can be expressed in terms of the corresponding density $f_X(x;t)$ of $X(t)$,

$$E\{Y(t)\} = \int_{-\infty}^{\infty} g(x) f_X(x;t) dx$$

Similarly, since $Y(t_1) = g[X(t_l)]$ and $Y(t_2) = g[X(t_2)]$, the second-order density $f_Y(y_1, y_2; t_1, t_2)$ of $Y(t)$ can be determined in terms of the corresponding density $f_X(x_l, x_2; t_1, t_2)$ of $X(t)$.

Furthermore,

$$E\{Y(t_1)Y(t_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1)g(x_2)f_X(x_1, x_2; t_1, t_2)dx_1dx_2$$

The n^{th}-order density $f_Y(y_1, \dots, y_n; t_1, \dots, t_n)$ of $Y(t)$ can be determined from the corresponding density of $X(t)$, where the underlying transformation is the system

$$Y(t_1) = g[X(t_1)], \cdots, Y(t_n) = g[X(t_n)] \tag{1}$$

Stationary

Theorem

Suppose that the input to a memoryless system is an SSS process $X(T)$. We shall show that the resulting output $Y(t)$ is also SSS.

Proof: To determine the n^{th}-order density of $Y(t)$, we solve the system

$$Y(t_1) = g[X(t_1)], \cdots, Y(t_n) = g[X(t_n)] \quad (2)$$

If this system has a unique solution, then

$$f_Y(y_1, \ldots, y_n; t_1, \ldots, t_n) = \frac{f_X(x_1, \ldots, x_n; t_1, \ldots, t_n)}{|g'(x_1) \cdots g'(x_n)|} \quad (3)$$

From the stationarity of $X(t)$ it follows that the numerator in (3) is invariant to a shift of the time origin. And since the denominator does not depend on t, we conclude that the left side does not change if t_i is replaced by $t_i + c$.

Hence $Y(t)$ is SSS.

Notes

1. If $X(t)$ is stationary of order N. then $Y(t)$ is stationary of order N.
2. If $X(t)$ is stationary in an interval, then $Y(t)$ is stationary in the same interval.
3. If $X(t)$ is WSS stationary, then $Y(t)$ might not be stationary in any sense.

Square-law detector

A square-law detector is a memory-less system whose output equals

$$Y(t) = X^2(t)$$

We shall determine its first- and second-order densities. If $y > 0$, then the system $y = x^2$ has the two solutions $\pm\sqrt{y}$. Furthermore. $y'(x) = \pm 2\sqrt{y}$; hence

$$f_Y(y;t) = \frac{1}{2\sqrt{y}}[f_X(\sqrt{y};t) + f_X(-\sqrt{y};t)]$$

If $y_1 > 0$ and $y_2 > 0$, then the system

$$y_1 = x_1^2 \qquad y_2 = x_2^2$$

has the four solutions $(\pm\sqrt{y_1}, \pm\sqrt{y_2})$. Furthermore, its Jacobian equals $\pm 4\sqrt{y_1 y_2}$; hence

$$f_Y(y_1, y_2; t_1, t_2) = \frac{1}{4\sqrt{y_1 y_2}} \sum f_X(\pm\sqrt{y_1}, \pm\sqrt{y_2}; t_1, t_2)$$

where the summation has four terms.

Note that, if $X(t)$ is SSS, then

$$f_X(x;t) = f_X(x)$$

is independent of t and

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; \tau)$$

depends only on $\tau = t_1 - t_2$.

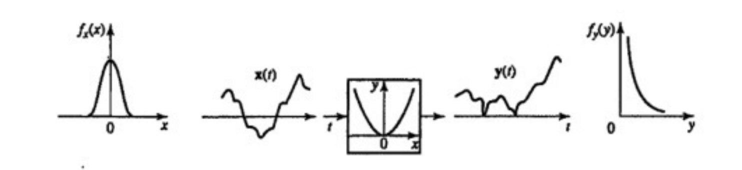
Hence $f_Y(y)$ is independent of t and $f_Y(y_1, y_2; \tau)$ depends only on $\tau = t_1 - t_2$.

Example

Suppose that $X(t)$ is a normal stationary process with zero mean and autocorrelation $R_X(\tau)$. In this case, $f_X(x)$ is normal with variance $R_X(0)$.

If $Y(t) = X^2(t)$ (as shown in Fig), then $E\{Y(t)\} = R_X(0)$ and

$$f_Y(y) = \frac{1}{\sqrt{2\pi R_X(0) y}} e^{-y/2R_X(0)} U(y)$$



We shall show that

$$R_Y(\tau) = R_X^2(0) + 2R_X^2(\tau) \tag{4}$$

Proof.

The random variables $X(t+\tau)$ and $X(t)$ are jointly normal with zero mean. Let $Z(t) = \alpha_1 X(t+\tau) + \alpha_2 X(t)$, hence

$$\Phi_Z(\omega) = \exp\{j\mu_z\omega - \sigma_Z^2\omega^2/2\}$$

Because $\mu_Z = \alpha_1\mu_1 + \alpha_2\mu_2 = 0$ and

$$\sigma_Z^2 = \alpha_1^2\sigma_1^2 + 2r\alpha_1\alpha_2\sigma_1\sigma_2 + \alpha_2^2\sigma_2^2$$

therefore with $\omega = 1$,

$$\Phi_Z(\omega = 1) = \Phi(\alpha_1, \alpha_2) = \exp\{-A\}$$

where $A = (\alpha_1^2\sigma_1^2 + 2r\alpha_1\alpha_2\sigma_1\sigma_2 + \alpha_2^2\sigma_2^2)/2$. Expanding the exponential and using the linearity of expected values, we obtain the series

$$\Phi(\alpha_1, \alpha_2) = \sum_{n=0}^{\infty} \frac{1}{m!} \begin{pmatrix} n \\ k \end{pmatrix} E\{X^k(t+\tau)X^{n-k}(t)\}\alpha_1^k\alpha_2^{n-k} \qquad (5)$$

Proof.

To calculate $E\{X^2(t+\tau)X^2(t)\}$, we shall equate the coefficient

$$\frac{1}{4!}\left(\begin{array}{c}4\\2\end{array}\right)E\{X^2(t+\tau)X^2(t)\}$$

of $\alpha_1^2\alpha_2^2$ in (5) with the corresponding coefficient of the expansion of e^{-A}. In this expansion, the factor $\alpha_1^2\alpha_2^2$ appear only in the terms

$$\frac{A^2}{2}=\frac{1}{8}(\alpha_1^2\sigma_1^2+2r\alpha_1\alpha_2\sigma_1\sigma_2+\alpha_2^2\sigma_2^2)^2$$

Hence

$$\frac{1}{4!}\left(\begin{array}{c}4\\2\end{array}\right)E\{X^2(t+\tau)X^2(t)\}=\frac{1}{8}(2\sigma_1^2\sigma_2^2+4r^2\sigma_1^2\sigma_2^2)$$

Proof.

and

$$\begin{aligned} & E\{X^2(t+\tau)X^2(t)\} \\ = \quad & E\{X^2(t+\tau)\}E\{X^2(t)\} + 2E^2\{X(t+\tau)X(t)\} \end{aligned}$$

and

$$R_Y(\tau) = R_X^2(0) + 2R_X^2(\tau)$$

results.

$\square$

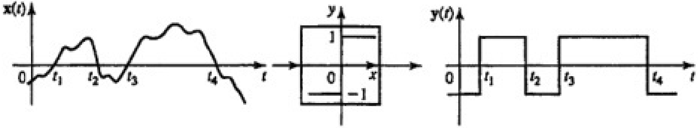
Note in particular that

$$E\{Y^2(t)\} = R_Y(0) = 3R_X^2(0) \qquad \sigma_Y^2 = 2R_X^2(0)$$

Hard limiter

Consider a memoryless system with

$$g(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \tag{6}$$



Its output $y(t)$ takes the values ± 1 and

$$\begin{aligned} P\{Y(t) = 1)\} &= P\{X(t) > 0\} = 1 - F_X(0) \\ P\{Y(t) = -1\} &= P\{X(t) < 0\} = F_X(0) \end{aligned}$$

Hence

$$E\{Y(t)\} = 1 \times P\{Y(t) = 1\} - 1 \times P\{Y(t) = -1\} = 1 - 2F_X(0)$$

The product $Y(t+\tau)Y(t)$ equals 1 if $X(t+\tau)X(t) > 0$ and it equals -1 otherwise.

Hence

$$R_Y(\tau) = P\{X(t+\tau)X(t) > 0\} - P\{X(t+\tau)X(t) < 0\} \quad (7)$$

Thus, in the probability plane of the random variables $X(t+\tau)$ and $X(t)$, $R_Y(\tau)$ equals the masses in the first and third quadrants minus the masses in the second and fourth quadrants.

Example

We shall show that if $X(t)$ is a normal stationary process with zero mean, then the autocorrelation of the output of a hard limiter equals

$$R_Y(\tau) = \frac{2}{\pi} \arcsin \frac{R_X(\tau)}{R_X(0)} \tag{8}$$

This result is known as the *arcsine law*.

Proof.

The random variables $X(t+\tau)$ and $X(t)$ are jointly normal with zero mean, variance $R_X(0)$, and correlation coefficient $R_X(\tau)/R_X(0)$. Hence,

$$
\begin{aligned}
P[X(t+\tau)X(t)>0] &= \frac{1}{2}+\frac{1}{\pi}\arctan\frac{R_X(\tau)/R_X(0)}{\sqrt{1-(R_X(\tau)/R_X(0))^2}} \\
&= \frac{1}{2}+\frac{1}{\pi}\arctan\frac{R_X(\tau)}{\sqrt{R_X^2(0)-R_X^2(\tau)}} \\
&= \frac{1}{2}+\frac{1}{\pi}\arcsin\frac{R_X(\tau)}{R_X(0)} \\
P[X(t+\tau)X(t)>0] &= \frac{1}{2}-\frac{1}{\pi}\arcsin\frac{R_X(\tau)}{R_X(0)}
\end{aligned}
$$

Proof.

Inserting in

$$R_Y(\tau) = P\{X(t+\tau)X(t) > 0\} - P\{X(t+\tau)X(t) < 0\}$$

we obtain

$$R_Y(\tau) = \frac{1}{2} + \frac{\alpha}{\pi} - \left(\frac{1}{2} - \frac{\alpha}{\pi}\right) = \frac{2\alpha}{\pi} = \frac{2}{\pi} \arcsin \frac{R_X(\tau)}{R_X(0)}$$

and (8) follows. □

Bussgang's Theorem

Example

Using Price's theorem. we shall show that if the input to a memoryless system $Y = g(X)$ is a zero-mean normal process $X(t)$, the cross-correlation of $X(t)$ with the resulting output $Y(t) = g[X(t)]$ is proportional to $R_{XX}(\tau)$:

$$R_{XY}(\tau) = KR_{XX}(\tau) \quad \text{where} \quad K = E\{g'[X(t)]\} \tag{9}$$

Proof.

For a specific τ, the random variables $X = X(t)$ and $Z = X(t+\tau)$ are jointly normal with zero mean and covariance $\mu = E\{XZ\} = R_{XX}(\tau)$. With

$$I = E\{(Zg(X)\} = E\{X(t+\tau)Y(t)\} = R_{XY}(\tau)$$

it follows that

$$\frac{\partial I}{\partial \mu} = E\left\{\frac{\partial^2[zg(x)]}{\partial x \partial z}\right\} = E\{g'[x(t)]\} = K \tag{10}$$

If $\mu = 0$, the random variables $X(t+\tau)$ and $X(t)$ are independent; hence $I = 0$. Integrating (10)with respect to μ, we obtain

$$I = K\mu$$

and (9) results. □

Special case: Hard limiter

Suppose that $g(x) = \mathrm{sgn} x$. In this case, $g'(x) = 2\delta(x)$, hence

$$K = E\{2\delta(x)\} = 2\int_{-\infty}^{\infty} \delta(x) f(x) dx = 2f(0)$$

where

$$f(x) = \frac{1}{\sqrt{2\pi R_{XX}(0)}} \exp\left(-\frac{x^2}{2R_{XX}(0)}\right)$$

is the first-order density of $X(t)$. Inserting into (9), we obtain

$$R_{XY}(\tau) = R_{XX}(\tau)\sqrt{\frac{2}{\pi R_{XX}(0)}} \qquad Y(t) = \mathrm{sgn}\ X(t) \quad (11)$$

Special case: Limiter

Suppose next that $y(t)$ is the output of a limiter

$$g(x) = \begin{cases} x & |x| < c \\ c & |x| > c \end{cases}$$

$$g'(x) = \begin{cases} 1 & |x| < c \\ 0 & |x| > c \end{cases}$$

In this case,

$$K = \int_{-c}^{c} f(x)dx = 2G\left(\frac{c}{\sqrt{R_{xx}(0)}}\right) - 1 \qquad (12)$$

Outline

1. **Systems with Stochastic Inputs**
   - Memoryless Systems
   - Linear Time-invariant Systems
   - General moments
2. **Vector Processes and Multiterminal Systems**
3. **The Effect of LTI Systems on WSS Processes**

Linear Systems

Definition

The notation

$$y(t) = L[x(t)] \tag{13}$$

will indicate that $y(t)$ is the output of a linear system with input $x(t)$, This means that

$$L[a_1 x_1(t) + a_2 x_2(t)] = a_1 L[x_1(t)] + a_2 L[x_2(t)] \tag{14}$$

for any a_1, a_2, $x_1(t)$, $x_2(t)$.

This is the familiar definition of linearity and it also holds if the coefficients a_1 and a_2 are random variables because, as we have assumed, the system is deterministic, that is, it operates only on the variable t.

Note If a system is specified by its internal structure or by a differential equation, then (14) holds only if $y(t)$ is the zero-state response. The response due to the initial conditions (zero-input response) will not be considered.

Time-invariant System

Definition

A system is called time-invariant if its response to $x(t+c)$ equals $y(t+c)$.

We shall assume throughout that all linear systems under consideration are time-invariant.

It is well known that the output of a linear system is a convolution

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\alpha)h(\alpha)d\alpha \tag{15}$$

where

$$h(t) = L[\delta(t)]$$

is its impulse response.

If $X(t)$ is a normal process, then $Y(t)$ is also a normal process. This is an extension of the familiar property of linear transformations of normal random variables and can be justified if we approximate the integral in (15) by a sum:

$$Y(t_i) \simeq \sum_k X(t_i - \alpha_k) h(\alpha_k) \Delta(\alpha)$$

If $X(t)$ is SSS, then $Y(t)$ is also SSS.

Indeed, since $Y(t+c) = L[X(t+c)]$ for every c, we conclude that if the processes $X(t)$ and $X(t+c)$ have the same statistical properties, so do the processes $Y(t)$ and $Y(t+c)$.

If $X(t)$ is WSS, the processes $X(t)$ and $Y(t)$ are jointly WSS.

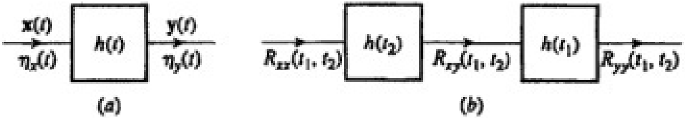
Fundamental theorem

For any linear system

$$E\{L[X(t)]\} = L[E\{X(t)\}] \tag{16}$$

In other words, the mean $\mu_Y(t)$ of the output $Y(t)$ equals the response of the system to the mean $\mu_X(t)$ of the input

$$\mu_Y(t) = L[\mu_X(t)] \tag{17}$$



This is a simple extension of the linearity of expected values to arbitrary linear operators. In the context of (15) it can be deduced if we write the integral as a limit of a sum. This yields

$$E\{Y(t)\} = \int_{\infty}^{\infty} E\{X(t-\alpha)\}h(\alpha)d\alpha = \mu_X(t) * h(t) \tag{18}$$

Frequency interpretation

At the i^{th} trial the input to our system is a function $X(t, \xi_i)$ yielding as output the function $Y(t, \xi_i) = L[X(t, \xi_i)]$. For large n.

$$E\{Y(t)\} \simeq \frac{Y(t, \xi_1) + \cdots + Y(t, \xi_n)}{n} = \frac{L[X(t, \xi_1) + \cdots + X(t, \xi_n)]}{n}$$

From the linearity of the system it follows that the last term above equals

$$L\left[\frac{x(t, \xi_1) + \cdots + x(t, \xi_n)}{n}\right]$$

This agrees with

$$E\{L[X(t)]\} = L[E\{X(t)\}]$$

because the fraction is nearly equal to $E\{X(t)\}$.

Notes

1. From (17) it follows that if

$$X^c(t) = X(t) - \mu_X(t) \qquad Y^c(t) = Y(t) - \mu_Y(t)$$

then

$$L[X^c(t)] = L[X(t)] - L[\mu_X(t)] = Y^c(t) \tag{19}$$

Thus the response of a linear system to the centered input $X^c(t)$ equals the centered output $Y^c(t)$.

2. Suppose that

$$X(t) = f(t) + \nu(t) \qquad E\{\nu(t)\} = 0$$

In this case, $E\{X(t)\} = f(t)$; hence

$$\mu_Y(t) = f(t) * h(t)$$

Thus, if $X(t)$ is the sum of a deterministic signal $f(t)$ and a random component $\nu(t)$, then for the determination of the mean of the output we can ignore $\nu(t)$ provided that the system is linear and $E\{\nu(t)\} = 0$.

Output Autocorrelation

Theorem

(a)

$$R_{XY}(t_l, t_2) = L_2[R_{XX}(t_1, t_2)] \tag{20}$$

In the notation just established, L_2 means that the system operates on the variable t_2, treating t_1 as a parameter. In the context of (15) this means that

$$R_{XY}(t_l, t_2) = \int_{-\infty}^{\infty} R_{XX}(t_1, t_2 - \alpha) h(\alpha) d\alpha \tag{21}$$

(b)

$$R_{YY}(t_l, t_2) = L_1[R_{XY}(t_1, t_2)] \tag{22}$$

In this case, the system operates on t_1.

$$R_{YY}(t_l, t_2) = \int_{-\infty}^{\infty} R_{XY}(t_1 - \alpha, t_2) h(\alpha) d\alpha \tag{23}$$

Proof.

Multiplying (13) by $X(t_1)$ and using (14), we obtain

$$X(t_1)Y(t) = L_1[X(t_1)X(t)]$$

and (20) follows with $t = t_2$. The proof of (22) is similar: We multiply (13) by $Y(t_2)$ and use (16). This yields

$$E\{Y(t)Y(t_2)\} = L_1[E\{X(t)X(t_2)\}]$$

and (22) follows with $t = t_1$. □

If $R_{XX}(t_1, t_2)$ is the input to the given system and the system operates on t_2, the output equals $R_{XY}(t_1, t_2)$.

If $R_{XY}(t_1, t_2)$ is the input and the system operates on t_1, the output equals $R_{YY}(t_1, t_2)$.

Inserting (21) into (23). we obtain

$$R_{YY}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{XX}(t_1 - \alpha, t_2 - \beta) h(\alpha) h(\beta) d\alpha d\beta$$

This expresses $R_{YY}(t_1, t_2)$ directly in terms of $R_{XX}(t_1, t_2)$. However, conceptually and operationally, it is preferable to find first $R_{XY}(t_1, t_2)$.

Example

A stationary process $\nu(t)$ with autocorrelation $R_{\nu\nu}(\tau) = q\delta(\tau)$ (white noise) is applied at $t = 0$ to a linear system with

$$h(t) = e^{-ct}U(t)$$

We shall show that the autocorrelation of the resulting output $Y(t)$ equals

$$R_{YY}(t_1, t_2) = \frac{q}{2c}(1 - e^{-2ct_1})e^{-c|t_2 - t_1|} \tag{24}$$

for $0 < t_1 < t_2$.

Proof.

We can use the preceding results if we assume that the input to the system is the process

$$x(t) = \nu(t)U(t)$$

With this assumption, all correlations are 0 if $t_l < 0$ or $t_2 < 0$. For $t_l > 0$ and $t_2 > 0$,

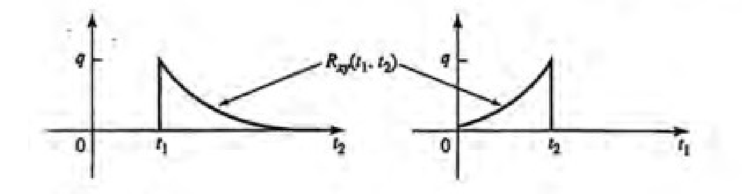
$$R_{XX}(t_1, t_2) = E\{\nu(t_1)\nu(t_2)\} = q\delta(t_1 - t_2)$$

As we see from $R_{XY}(t_1, t_2)$ equals the response of the system to $q\delta(t_1 - t_2)$ considered as a function of t_2.

Since $\delta(t_1 - t_2) = \delta(t_2 - t_1)$ and $L[\delta(t_1 - t_2)] = h(t_2 - t_1)$ (time invariance), we conclude that

$$R_{XY}(t_1, t_2) = qh(t_2 - t_1) = qe^{-c(t_2 - t_1)}U(t_2 - t_1)U(t_1)$$

Proof.



In Fig., we show $R_{XY}(t_1, t_2)$ as a function of t_l and t_2. Inserting into (23), we obtain

$$R_{XY}(t_1, t_2) = q \int_0^{t_1} e^{t_1 - \alpha - t_2} e^{-c\alpha} d\alpha \quad t_1 < t_2$$

and (24) results. □

Note that

$$E\{Y^2(t)\} = R_{YY}(t, t) = \frac{q}{2c}(1 - e^{-2ct}) = q \int_0^t h^2(\alpha) d\alpha$$

Corollary

The autocovariance $C_{YY}(t_1, t_2)$ of $Y(t)$ is the autocorrelation of the process $Y^c(t) = Y(t) - \mu_Y(t)$ and, as we see from (19), $Y^c(t)$ equals $L[X^c(t)]$. Applying (21) and (23) to the centered processes $X^c(t)$ and $Y^c(t)$. we obtain

$$C_{XY}(t_1, t_2) = C_{XX}(t_1, t_2) * h(t_2) \quad (25)$$

$$C_{YY}(t_1, t_2) = C_{XY}(t_1, t_2) * h(t_1) \quad (26)$$

where the convolutions are in t_1 and t_2, respectively.

Complex processes

The preceding results can be readily extended to complex processes and to systems with complex-valued $h(t)$. Reasoning as in the real case, we obtain

$$R_{XY}(t_1, t_2) = R_{XX}(t_1, t_2) * h^*(t_2) \quad (27)$$

$$R_{YY}(t_1, t_2) = R_{XY}(t_1, t_2) * h(t_1) \quad (28)$$

Response to white noise

Theorem

If the input to a linear system is white noise with autocorrelation

$$R_{XX}(t_1, t_2) = q(t_1)\delta(t_1 - t_2)$$

then

$$E\{|Y(t)|^2\} = q(t) * |h(t)|^2 = \int_{-\infty}^{\infty} q(t-\alpha)|h(\alpha)|^2 d\alpha \qquad (29)$$

Proof.

From (27) and (28) it follows that

$$R_{XY}(t_1,t_2) = q(t_1)\delta(t_1-t_2) * h^*(t_2) = q(t_1)h^*(t_2-t_1) \quad (30)$$

$$R_{YY}(t_1,t_2) = \int_{-\infty}^{\infty} q(t_1-\alpha)h^*[t_2-(t_1-\alpha)]h(\alpha)d\alpha \quad (31)$$

and with $t_1 = t_2 = t$. (29) results. $\square$

Special cases

(a) If $X(t)$ is stationary white noise, then $q(t) = q$ and (29) yields

$$E\{Y(t)^2\} = qE \quad \text{where} \quad E = \int_{-\infty}^{\infty} |h(t)|^2 dt$$

is the energy of $h(t)$.

(b)If $h(t)$ is of short duration relative to the variations of $q(t)$, then

$$E\{Y(t)^2\} \simeq q \int_{-\infty}^{\infty} |h(\alpha)|^2 d\alpha = Eq(t) \tag{32}$$

This relationship justifies the term average intensity used to describe the function $q(t)$.

(c) If $R_{\nu\nu}(\tau) = q\delta(\tau)$ and $\nu(t)$ is applied to the system at $t = 0$, then $q(t) = qU(t)$ and (29) yields

$$E\{Y(t)^2\} \simeq q \int_{-\infty}^{0} |h(\alpha)|^2 d\alpha$$

Example

The integral

$$Y(t) = \int_0^t \nu(\alpha) d\alpha$$

can be considered as the output of a linear system with input $X(t) = \nu(t)U(t)$ and impulse response $h(t) = U(t)$. If, therefore, $\nu(t)$ is white noise with average intensity $q(t)$, then $X(t)$ is white noise with average intensity $q(t)U(t)$ and (29) yields

$$E\{Y(t)^2\} = q(t)U(t) * U(t) = \int_0^t q(\alpha) d\alpha$$

Differentiators

A differentiator is a linear system whose output is the derivative of the input

$$L[X(t)] = X'(t)$$

We can, therefore, use the preceding results to find the mean and the autocorrelation of $X'(t)$.

From (17) it follows that

$$\mu_{X'}(t) = L[\mu_X(t)] = \mu'_X(t) \tag{33}$$

Similarly [see (20)]

$$R_{XX'}(t_1, t_2) = L_2[R_{XX}(t_1, t_2)] = \frac{\partial R_{XX}(t_1, t_2)}{\partial t_2} \tag{34}$$

because, in this case, L_2 means differentiation with respect to t_2. Finally,

$$R_{X'X'}(t_1, t_2) = L_1[R_{XX'}(t_1, t_2)] = \frac{\partial R_{XX'}(t_1, t_2)}{\partial t_1} \tag{35}$$

Combining, we obtain

$$R_{X'X'}(t_1, t_2) = \frac{\partial^2 R_{XX}(t_1, t_2)}{\partial t_1 \partial t_2} \tag{36}$$

Stationary processes

If $X(t)$ is WSS, then $\mu_X(t)$ is constant; hence

$$E\{X'(t)\} = 0 \tag{37}$$

Furthermore, since $R_{XX}(t_1, t_2) = R_{XX}(\tau)$, we conclude with $\tau = t_1 - t_2$ that

$$\frac{\partial R_{XX}(t_1 - t_2)}{\partial t_2} = -\frac{dR_{XX}(\tau)}{d\tau} \qquad \frac{\partial^2 R_{XX}(t_1 - t_2)}{\partial t_1 \partial t_2} = -\frac{d^2 R_{XX}(\tau)}{d\tau^2}$$

Hence

$$R_{XX'}(\tau) = -R'_{XX}(\tau) \qquad R_{X'X'}(\tau) = -R''_{XX}(\tau) \tag{38}$$

Outline

1. **Systems with Stochastic Inputs**
   - Memoryless Systems
   - Linear Time-invariant Systems
   - General moments
2. **Vector Processes and Multiterminal Systems**
3. **The Effect of LTI Systems on WSS Processes**

General moments

The moments of any order of the output $Y(t)$ of a linear system can be expressed in terms of the corresponding moments of the input $X(t)$. As an illustration. we shall determine the third-order moment

$$R_{YYY}(t_1, t_2, t_3) = E\{Y_1(t)Y_2(t)Y_3(t)\}$$

of $Y(t)$ in terms of the third-order moment $R_{XXX}(t_l, t_2, t_3)$ of $X(t)$. Proceeding as in (20), we obtain

$$\begin{aligned} & E\{X(t_1)X(t_2)Y(t_3)\} = L_3[E\{x(t_1)x(t_2)x(t_3)\}] \\ = & \int_{-\infty}^{\infty} R_{XXX}(t_1, t_2, t_3 - \gamma)h(\gamma)d\gamma \end{aligned} \tag{39}$$

$$\begin{aligned} & E\{X(t_1)Y(t_2)Y(t_3)\} = L_2[E\{X(t_1)X(t_2)Y(t_3)\}] \\ = & \int_{-\infty}^{\infty} R_{XXY}(t_1, t_2 - \beta, t_3)h(\beta)d\beta \end{aligned} \tag{40}$$

$$E\{Y(t_1)Y(t_2)Y(t_3)\} = L_1[E\{X(t_1)Y(t_2)Y(t_3)\}]$$

$$= \int_{-\infty}^{\infty} R_{XYY}(t_1 - \alpha, t_2, t_3)h(\alpha)d\alpha \quad (41)$$

Note that for the evaluation of $R_{YYY}(t_1, t_2, t_3)$ for specific times t_1, t_2, and t_3. the function $R_{XXX}(t_1, t_2, t_3)$ must be known for every t_1, t_2, and t_3.

Outline

Notations

We consider now systems with n inputs $X_i(t)$ and r outputs $Y_j(t)$.

As a preparation, We introduce the notion of autocorrelation and cross-correlation for vector processes starting with a review of the standard matrix notation.

The expression $\boldsymbol{A} = [a_{ij}]$ will mean a matrix with elements a_{ij}.

The notation

$$\boldsymbol{A}^T = [a_{ji}] \quad \boldsymbol{A}^* = [a_{ij}^*] \quad \boldsymbol{A}^H = [a_{ji}^*]$$

will mean the transpose, the conjugate, and the conjugate transpose of $\boldsymbol{A}$.

A column vector will be identified by $\boldsymbol{A} = [a_i]$.

Whether $\boldsymbol{A}$ is a vector or a general matrix will be understood from the context.

If $\boldsymbol{A} = [a_i]$ and $\boldsymbol{B} = [b_j]$ are two vectors with m elements each, the product $\boldsymbol{A}^T\boldsymbol{B} = a_1b_1 + \cdots + a_mb_m$ is a number, and the product $\boldsymbol{A}\boldsymbol{B}^T = [a_ib_j]$ is an $m \times m$ matrix with elements a_ib_j.

A vector process $\boldsymbol{X}(t) = [X_i(t)]$ is a vector, the components of which are stochastic processes.

The mean

$$\boldsymbol{\mu}(t) = E\{X(t)\} = [\mu_i(t)]$$

of $\boldsymbol{X}(t)$ is a vector with components $\mu_i(t) = E\{X_i(t)\}$.

The autocorrelation $\boldsymbol{R}(t_1 t_2)$ or $\boldsymbol{R}_{XX}(t_1, t_2)$ of a vector process $\boldsymbol{X}(t)$ is an $m \times m$ matrix

$$\boldsymbol{R}(t_1, t_2) = E\{\boldsymbol{X}(t_1)\boldsymbol{X}^H(t_2)\} \quad (42)$$

with elements $E\{X_i(t_1)X_j^*(t_2)\}$.

We define similarly the cross-correlation matrix

$$R_{XY}(t_1, t_2) = E\{\boldsymbol{X}(t_1)\boldsymbol{Y}^H(t_2)\} \quad (43)$$

of the vector processes

$$\boldsymbol{X}(t) = [X_i(t)] \;\; i = 1, \cdots, m \;\; \boldsymbol{Y}(t) = [Y_j(t)] \;\; j = 1, \cdots, r \quad (44)$$

A multiterminal system with m inputs $X_i(t)$ and r outputs $Y_j(t)$ is a rule for assigning to an m vector $\boldsymbol{X}(t)$ an r vector $\boldsymbol{Y}(t)$.

If the system is linear and time-invariant, it is specified in terms of its impulse response matrix. This is an $r \times m$ matrix

$$\boldsymbol{H}(t) = [h_{ji}(t)] \quad i = 1, \cdots, m \quad j = 1, \cdots, r \tag{45}$$

defined as: Its component $h_{ji}(t)$ is the response of the j^{th} output when the i^{th} input equals $\delta(t)$ and all other inputs equal 0.

From this and the linearity of the system, it follows that the response $Y_j(t)$ of the j^{th} output to an arbitrary input $\boldsymbol{X}(t) = [X_i(t)]$ equals

$$Y_j(t) = \int_{-\infty}^{\infty} h_{j1}(\alpha)x_1(t-\alpha)d\alpha + \cdots + \int_{-\infty}^{\infty} h_{jm}(\alpha)x_m(t-\alpha)d\alpha$$

Hence

$$\boldsymbol{Y}(t) = \int_{-\infty}^{\infty} \boldsymbol{H}(\alpha)\boldsymbol{X}(t-\alpha)d\alpha \tag{46}$$

In this material, $\boldsymbol{X}(t)$ and $\boldsymbol{Y}(t)$ are column vectors and $\boldsymbol{H}(t)$ is an $r \times m$ matrix. We shall use this relationship to determine the autocorrelation $\boldsymbol{R}_{yy}(t_1, t_2)$ of $\boldsymbol{Y}(t)$.

Premultiplying the conjugate transpose of (46) by $\boldsymbol{X}(t_l)$ and setting $t = t_2$, we obtain

$$\boldsymbol{X}(t_1)\boldsymbol{Y}^H(t_2) = \int_{-\infty}^{\infty} \boldsymbol{X}(t_1)\boldsymbol{X}^H(t_2 - \alpha)\boldsymbol{H}^H(\alpha)d\alpha$$

Hence

$$\boldsymbol{R}_{XY}(t_1, t_2) = \int_{-\infty}^{\infty} \boldsymbol{R}_{XX}(t_1, t_2 - \alpha)\boldsymbol{H}^H(\alpha)d\alpha \tag{47}$$

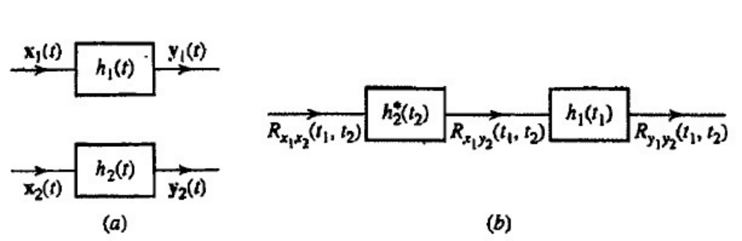
Postmultiplying (46) by $\boldsymbol{Y}^H(t_2)$ and setting $t = t_1$, we obtain

$$\boldsymbol{R}_{YY}(t_1, t_2) = \int_{-\infty}^{\infty} \boldsymbol{H}(\alpha)\boldsymbol{R}_{XY}(t_1 - \alpha, t_2)d\alpha \tag{48}$$

as in (27) and (28).

These results can be used to express the cross-correlation of the outputs of several scalar systems in terms of the cross-correlation of their inputs.

Example



In Fig. we show two systems with inputs $X_1(t)$and $X_2(t)$ and outputs

$$Y_1(t_1, t_2) = \int_{-\infty}^{\infty} h_1(\alpha) X_1(t-\alpha) d\alpha \quad (49)$$

$$Y_2(t_1, t_2) = \int_{-\infty}^{\infty} h_2(\alpha) X_2(t-\alpha) d\alpha \quad (50)$$

Example

These signals can be considered as the components of the output vector $\boldsymbol{Y}^T(t) = [Y_1(t), Y_2(t)]$ of a 2×2 system with input vector $\boldsymbol{X}^T(t) = [X_1(t), X_2(t)]$ and impulse response matrix

$$H(t) = \begin{bmatrix} h_1(t) & 0 \\ 0 & h_2(t) \end{bmatrix}$$

Inserting into (47)-(48), we obtain

$$R_{X_1Y_2}(t_1, t_2) = \int_{-\infty}^{\infty} R_{X_1X_2}(t_1, t_2 - \alpha) h_2^*(\alpha) d\alpha \tag{51}$$

$$R_{Y_1Y_2}(t_1, t_2) = \int_{-\infty}^{\infty} h_1(\alpha) R_{X_1Y_2}(t_1 - \alpha, t_2) d\alpha \tag{52}$$

Thus, to find $R_{X_1Y_2}(t_1, t_2)$, we use $R_{X_1X_2}(t_1, t_2)$ as the input to the conjugate $h_2^*(t)$ of $h_2(t)$, operating on the variable t_2. To find $R_{Y_1Y_2}(t_1, t_2)$, we use $R_{X_1Y_2}(t_1, t_2)$ as the input to $h_1(t)$ operating on the variable t_l.

Outline

Consider an LTI system whose input is a sample function of a WSS random process $X(t)$, i.e., a signal chosen by a probabilistic experiment from the ensemble that constitutes the random process $X(t)$; more simply, we say that the input is the random process $X(t)$.

The WSS input is characterized by its mean and its autocovariance or (equivalently) autocorrelation function.

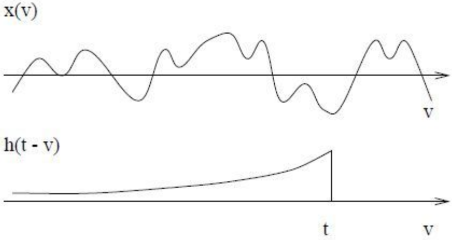
Among other considerations, we are interested in knowing when the output process $Y(t)$ i.e., the ensemble of signals obtained as responses to the signals in the input ensemble will itself be WSS, and want to determine its mean and autocovariance or autocorrelation functions, as well as its cross-correlation with the input process.

For an LTI system whose impulse response is $h(t)$, the output $Y(t)$ is given by the convolution

$$Y(t) = \int_{-\infty}^{+\infty} h(v)X(t-v)dv = \int_{-\infty}^{+\infty} X(v)h(t-v)dv \qquad (53)$$

for any specific input $X(t)$ for which the convolution is well-defined.

The convolution is well-defined if, for instance, the input $X(t)$ is bounded and the system is bounded-input bounded-output (BIBO) stable, i.e. $h(t)$ is absolutely integrable.



Rather than requiring that every sample function of our input process be bounded, it will suffice for our convolution computations below to assume that $E[X^2(t)] = R_{XX}(0)$ is finite. With this assumption, and also assuming that the system is BIBO stable, we ensure that $Y(t)$ is a well-defined random process, and that the formal manipulations we carry out below — for instance, interchanging expectation and convolution — can all be justified more rigorously by methods that are beyond our scope here.

In fact, the results we obtain can also be applied, when properly interpreted, to cases where the input process does not have a bounded second moment, e.g., when $X(t)$ is so-called CT white noise, for which $R_{XX}(\tau) = \delta(\tau)$. The results can also be applied to a system that is not BIBO stable, as long as it has a well-defined frequency response $H(j\omega)$, as in the case of an ideal lowpass filter, for example.

We can use the convolution relationship (53) to deduce the first- and secondorder properties of $Y(t)$.

What we shall establish is that $Y(t)$ is itself WSS, and that $X(t)$ and $Y(t)$ are in fact jointly WSS.

We will also develop relationships for the autocorrelation of the output and the cross-correlation between input and output.

Mean

First, consider the mean value of the output. Taking the expected value of both sides of (53), we find

$$
\begin{aligned}
E[Y(t)] &= E\{\int_{-\infty}^{+\infty} h(v)x(t-v)dv\} && (54)\\
&= \int_{-\infty}^{+\infty} h(v)E[x(t-v)]dv && (55)\\
&= \int_{-\infty}^{+\infty} h(v)\mu_x dv && (56)\\
&= \mu_x \int_{-\infty}^{+\infty} h(v)dv && (57)\\
&= H(j0)\mu_x = \mu_y && (58)
\end{aligned}
$$

In other words, the mean of the output process is constant, and equals the mean of the input scaled by the the DC gain of the system.

This is also what the response of the system would be if its input were held constant at the value μ_x.

The preceding result and the linearity of the system also allow us to conclude that applying the zero-mean WSS process $X(t) - \mu_X$ to the input of the stable LTI system would result in the zero-mean process $Y(t) - \mu_Y$ at the output.

This fact will be useful below in converting results that are derived for correlation functions into results that hold for covariance functions.

Cross-correlation

Next consider the cross-correlation between output and input:

$$E[Y(t+\tau)X(t)] = E\{[\int_{-\infty}^{+\infty} h(v)X(t+\tau-v)dv]X(t)\}$$
$$= \int_{-\infty}^{+\infty} h(v)E\{X(t+\tau-v)X(t)\}dv \quad (59)$$

Since $X(t)$ is WSS, $E\{X(t+\tau-v)X(t)\} = R_{XX}(\tau-v)$, so

$$E[Y(t+\tau)X(t)] = \int_{-\infty}^{+\infty} h(v)R_{XX}(\tau-v)dv$$
$$= h(\tau) * R_{XX}(\tau)$$
$$= R_{YX}(\tau) \quad (60)$$

Note that the cross-correlation depends only on the lag τ between the sampling instants of the output and input processes, not on both τ and the absolute time location t.

Also, this cross-correlation between the output and input is deterministically related to the autocorrelation of the input, and can be viewed as the signal that would result if the system input were the autocorrelation function, as indicated in Figure

$$R_{xx}(\tau) \longrightarrow \boxed{h(\tau)} \longrightarrow R_{yx}(\tau)$$

We can also conclude that

$$R_{XY}(\tau) = R_{YX}(-\tau) = R_{XX}(-\tau) * h(-\tau) = R_{XX}(\tau) * h(-\tau). \quad (61)$$

where the second equality follows from Eqn. (60) and the fact that time-reversing the two functions in a convolution results in time-reversal of the result.

The above relations can also be expressed in terms of covariance functions, rather than in terms of correlation functions. For this, simply consider the case where the input to the system is the zero-mean WSS process $X(t) - \mu_X$, with corresponding zero-mean output $Y(t) - \mu_Y$.

Since the correlation function for $X(t) - \mu_X$ is the same as the covariance function for $X(t)$, i.e., since

$$R_{X(t)-\mu_X, X(t)-\mu_X}(\tau) = C_{XX}(\tau), \tag{62}$$

the results above hold unchanged when every correlation function is replaced by the corresponding covariance function.

We therefore have, for instance, that

$$C_{YX}(\tau) = h(\tau) * C_{XX}(\tau), \tag{63}$$

Autocorrelation

Next we consider the autocorrelation of the output $Y(t)$:

$$
\begin{aligned}
E\{Y(t+\tau)Y(t)\} &= E\{[\int_{-\infty}^{+\infty} h(v)X(t+\tau-v)dv]Y(t)\} \\
&= \int_{-\infty}^{+\infty} h(v)\,\underbrace{E\{X(t+\tau-v)Y(t)\}}_{R_{XY}(\tau-v)}\,dv \\
&= \int_{-\infty}^{+\infty} h(v)R_{XY}(\tau-v)dv \\
&= h(\tau) * R_{XY}(\tau) \\
&= R_{YY}(\tau). \qquad (64)
\end{aligned}
$$

Note that the autocorrelation of the output depends only on τ, and not on both τ and t.

Putting this together with the earlier results, we conclude that $X(t)$ and $Y(t)$ are jointly WSS, as claimed.

The corresponding result for covariances is

$$C_{YY}(\tau) = h(\tau) * C_{XY}(\tau), \tag{65}$$

Combining (64) with (61), we find that

$$R_{YY}(\tau) = R_{XX}(\tau) * \underbrace{h(\tau) * h(-\tau)}_{h(\tau)*h(-\tau) \triangleq \bar{R}_{hh}(\tau)} = R_{XX}(\tau) * \bar{R}_{hh}(\tau) \tag{66}$$

The function $\bar{R}_{hh}(\tau)$ is typically referred to as the deterministic autocorrelation function of $h(t)$, and is given by

$$\bar{R}_{hh}(\tau) = h(\tau) * h(-\tau) = \int_{-\infty}^{+\infty} h(t+\tau)h(t)dt. \tag{67}$$

For the covariance function version of (66), we have

$$C_{YY}(\tau) = C_{XX}(\tau) * h(\tau) * h(-\tau) = C_{XX}(\tau) * \bar{R}_{hh}(\tau) \tag{68}$$

Note that the deterministic correlation function of h(t) is still what we use, even when relating the covariances of the input and output. Only the means of the input and output processes get adjusted in arriving at the present result; the impulse response is untouched.

The correlation relations in Eqns. (60), (61), (64) and (66), as well as their covariance counterparts, are very powerful, and we will make considerable use of them.

Of equal importance are their statements in the Fourier and Laplace transform domains. Denoting the Fourier and Laplace transforms of the correlation function $R_{XX}(\tau)$ by $S_{XX}(j\omega)$ and $S_{XX}(s)$ respectively, and similarly for the other correlation functions of interest, we have:

$$S_{YX}(j\omega) = S_{XX}(j\omega)H(j\omega) \quad (69)$$

$$S_{YY}(j\omega) = S_{XX}(j\omega)|H(j\omega)|^2 \quad (70)$$

$$S_{YX}(s) = S_{XX}(s)H(s) \quad (71)$$

$$S_{YY}(j\omega) = S_{XX}(s)H(s)H(-s). \quad (72)$$

We can denote the Fourier and Laplace transforms of the covariance function $C_{XX}(\tau)$ by $D_{XX}(j\omega)$ and $D_{XX}(s)$ respectively, and similarly for the other covariance functions of interest, and then write the same sorts of relationships as above.

Exactly parallel results hold in the DT case.

Consider a stable discrete-time LTI system whose impulse response is $h[n]$ and whose input is the WSS random process $X[n]$. Then, as in the continuous-time case, we can conclude that the output process $Y[n]$ is jointly WSS with the input process $X[n]$, and

$$\mu_Y = \mu_X \sum_{-\infty}^{\infty} h[n] \quad (73)$$

$$R_{YX}[m] = h[m] * R_{XX}[m] \quad (74)$$

$$R_{YY}[m] = R_{XX}[m] * \bar{R}_{hh}[m] \quad (75)$$

where $\bar{R}_{hh}[m]$ is the deterministic autocorrelation function of $h[m]$, defined as

$$\bar{R}_{hh}[m] = \sum_{-\infty}^{+\infty} h[n+m]h[n]. \quad (76)$$

The corresponding Fourier and Z-transform statements of these relationships are:

$$\begin{aligned}
\mu_Y &= H(e^{j0})\mu_X && (77)\\
S_{YX}(e^{j\Omega}) &= S_{XX}(e^{j\Omega})H(e^{j\Omega}) && (78)\\
S_{yy}(e^{j\omega}) &= S_{XX}(e^{j\omega})|H(e^{j\omega})|^2 && (79)\\
\mu_Y &= H(1)\mu_X && (80)\\
S_{YX}(z) &= S_{XX}(z)H(z) && (81)\\
S_{YY}(z) &= S_{XX}(z)H(z)H(\frac{1}{z}). && (82)
\end{aligned}$$

All of these expressions can also be rewritten for covariances and their transforms.

The Fourier transform of the autocorrelation function is termed the power spectral density (PSD) of the process.

Example

Suppose the input $X(t)$ to a CT LTI system is a random telegraph wave, with changes in sign at times that correspond to the arrivals in a Poisson process with rate λ, i.e.,

$$P(k \text{ switches in an interval of length } T) = \frac{(\lambda T)^k e^{-\lambda T}}{k!}. \quad (83)$$

Then, assuming $X(0)$ takes the values ± 1 with equal probabilities, we can determine that the process $X(t)$ has zero mean and correlation function

$$R_{XX}(\tau) = e^{-2\lambda|\tau|},$$

so it is WSS (for $t \geq 0$).

If we determine the cross-correlation $R_{YX}(\tau)$ with the output $Y(t)$ and then use the relation

$$R_{YX}(\tau) = R_{XX}(\tau) * h(\tau) \quad (84)$$

we can obtain the system impulse response $h(\tau)$.

If $S_{YX}(s)$, $S_{XX}(s)$ and $H(s)$ denote the associated Laplace transforms, then

$$H(s) = \frac{S_{YX}(s)}{S_{XX}(s)}. \tag{85}$$

Note that $S_{XX}(s)$ is a rather well-behaved function of the complex variable s in this case, whereas any particular sample function of the process $X(t)$ would not have such a well-behaved transform. The same comment applies to $S_{YX}(s)$.

Example

Suppose that we know the autocorrelation function $R_{XX}[m]$ of the input $X[n]$ to a DT LTI system, but do not have access to $X[n]$ and therefore cannot determine the cross-correlation $R_{YX}[m]$ with the output $Y[n]$, but can determine the output autocorrelation $R_{YY}[m]$. For example, if

$$R_{XX}[m] = \delta[m] \tag{86}$$

and we determine $R_{YY}[m]$ to be $R_{YY}[m] = (\frac{1}{2})^{|m|}$,then

$$R_{YY}[m] = (\frac{1}{2})^{|m|} = \bar{R}_{hh}[m] = h[m] * h[-m]. \tag{87}$$

Equivalently, $H(z)H(z^{-1})$ can be obtained from the Z-transform $S_{YY}[z]$ of $R_{YY}[m]$.

Additional assumptions or constraints, for instance on the stability and causality of the system and its inverse, may allow one to recover $H(z)$ from knowledge of $H(z)H(z^{-1})$.