Bubble Sort

(A Taste of Algorithms: Definition, Design, and Analysis)

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Bubble Sort

1. The Sorting Problem
2. Bubble Sort
3. Analysis of Bubble Sort

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Algorithms

What is an algorithm?

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Correctness!

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Definiteness: precisely defined operations

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Definiteness: precisely defined operations

Finiteness: termination

Algorithms

What is an algorithm? What is computation?

$$\xrightarrow{\text{Input}} \boxed{\begin{array}{c}\text{Algorithm}\\ \text{(sequence of operations)}\end{array}} \xrightarrow{\text{Output}}$$

Correctness!

Definiteness:	precisely defined operations
Finiteness:	termination
Effectiveness:	a reasonable model; basic operations

Algorithms

What is an algorithm? What is computation?

Input → Algorithm (sequence of operations) → Output

Correctness!

Definiteness: precisely defined operations

Finiteness: termination

Effectiveness: a reasonable model; basic operations

- for sorting: compare, swap

Sorting

The sorting problem:

Input: A sequence of n integers A: $a_1\ a_2\ \cdots\ a_n$.

Output: A permutation A': $a'_1\ a'_2\ \ldots\ a'_n$ of A *s.t.* $a'_1 \le a'_2 \le \cdots \le a'_n$ (non-decreasing order).

$$3 \quad 1 \quad 4 \quad 2 \quad \Longrightarrow 1 \quad 2 \quad 3 \quad 4$$

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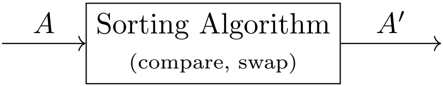

Inversions

$$A = a_1 \quad a_2 \quad \cdots \quad a_i \quad \cdots \quad a_j \quad \cdots \quad a_n.$$

If $i < j$ and $a_i > a_j$, then (a_i, a_j) is an **inversion**.

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#inversions = 3
#adjacent inversions = 2

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Adjacent inversion: (a_i, a_{i+1})

3 1 4 2

$\#\text{inversions} = 3$
$\#\text{adjacent inversions} = 2$

4 3 2 1

$\#\text{inversions} = 3 + 2 + 1 = 6$
$\#\text{adjacent inversions} = 3$

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3 1 4 2

#inversions = 3
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4 3 2 1

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1 2 3 4

#inversions = 0
#adjacent inversions = 0

Inversions

Theorem: A is sorted $\iff$ A has no adjacent inversions.

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$$A \text{ is sorted } \implies A \text{ has no adjacent inversions.}$$

$$A \text{ has no adjacent inversions } \implies \forall i \in [1, n-1] : a_i \leq a_{i+1}$$

Inversions

Theorem: A is sorted $\iff$ A has no adjacent inversions.

$$A \text{ is sorted } \implies A \text{ has no adjacent inversions.}$$

$$\begin{aligned} A \text{ has no adjacent inversions } &\implies \forall i \in [1, n-1] : a_i \le a_{i+1} \\ &\implies A \text{ is sorted.} \end{aligned}$$

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Bubble Sort: Basic Idea

Basic idea: to eliminate all adjacent inversions

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repeat
  pick any i
  if $a_i > a_{i+1}$ **then**
    SWAP(a_i, a_{i+1})
until no adjacent inversions

Bubble Sort: Basic Idea

Basic idea: to eliminate all adjacent inversions

```
repeat
    pick any i                                ▷ Definiteness!
    if a_i > a_{i+1} then
        SWAP(a_i, a_{i+1})
until no adjacent inversions                  ▷ Definiteness!
```

Bubble Sort: Basic Idea

Basic idea: to eliminate all adjacent inversions

repeat
    pick any i ▷ **Definiteness!**
    if $a_i > a_{i+1}$ **then**
        SWAP(a_i, a_{i+1})
until no adjacent inversions ▷ **Finiteness! Definiteness!**

Bubble Sort: Basic Idea

Basic idea: to eliminate all adjacent inversions

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repeat
  pick any i ▷ **Definiteness!**
  if $a_i > a_{i+1}$ **then**
    $\textsc{Swap}(a_i, a_{i+1})$
until no adjacent inversions ▷ **Finiteness! Definiteness!**

$$\text{Finiteness} \implies \text{Correctness}.$$

$\boxed{\text{Theorem: } A \text{ is sorted} \iff A \text{ has no adjacent inversions.}}$

Bubble Sort: Definiteness

```
procedure BubbleSort(A : a_1 a_2 ··· a_n)
    repeat
        
        for            do                         ▷ Pick i
            if a_i > a_{i+1} then
                Swap(a_i, a_{i+1})
                
    until no adjacent inversions
```

Bubble Sort: Definiteness

procedure BubbleSort($A : a_1\ a_2\ \cdots\ a_n$)

  repeat

    for $\boxed{i \leftarrow 1 : n-1}$ **do** ▷ Pick i

      if $a_i > a_{i+1}$ **then**

        Swap(a_i, a_{i+1})

  until $\boxed{\text{no adjacent inversions}}$

Bubble Sort: Definiteness

procedure BubbleSort($A : a_1\ a_2\ \cdots\ a_n$)
  repeat
    swapped
    for $\boxed{i \leftarrow 1 : n-1}$ **do** ▷ Pick i
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  until no adjacent inversions

Bubble Sort: Definiteness

```
procedure BubbleSort(A : a_1 a_2 ··· a_n)
    repeat
        [swapped ← false]
        for [i ← 1 : n − 1] do                    ▷ Pick i
            if a_i > a_{i+1} then
                Swap(a_i, a_{i+1})
                [          ]
    until [no adjacent inversions]
```

Bubble Sort: Definiteness

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procedure BubbleSort(A : a_1 a_2 ··· a_n)
    repeat
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        for [i ← 1 : n − 1] do                          ▷ Pick i
            if a_i > a_{i+1} then
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    until [no adjacent inversions]
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Bubble Sort: Definiteness

procedure BubbleSort($A : a_1\ a_2\ \cdots\ a_n$)
    repeat
        $\boxed{\text{swapped} \gets \text{false}}$
        for $\boxed{i \gets 1 : n-1}$ **do** ▷ Pick i
            if $a_i > a_{i+1}$ **then**
                Swap(a_i, a_{i+1})
                $\boxed{\text{swapped} \gets \text{true}}$
    until $\boxed{\text{swapped} = \text{false}}$

Bubble Sort: Example

2

4

1

3

Bubble Sort: Example

2	2
4	4
1	1
3	3

Bubble Sort: Example

2	2	2
4	4	4
1	1	3
3	3	1

Bubble Sort: Example

2	2	2	2
4	4	4	4
1	1	3	3
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Bubble Sort: Example

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1	1	3	3	3	3	2	2	2
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Bubble Sort: Example

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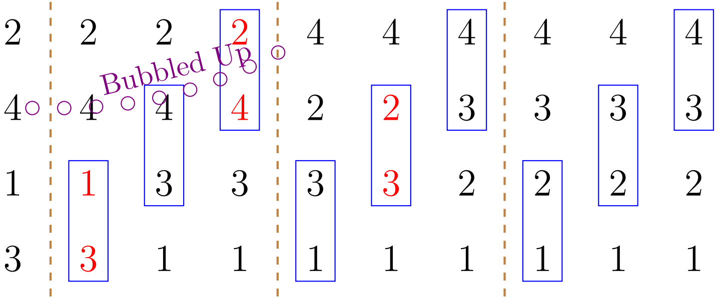

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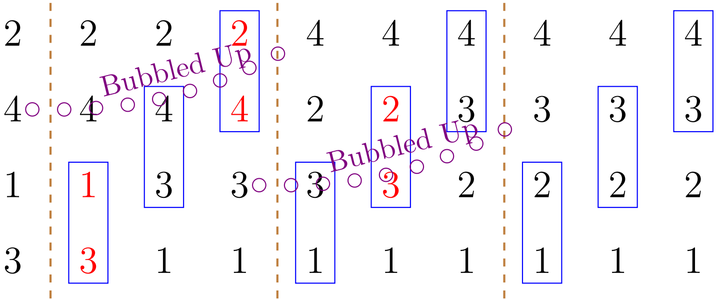

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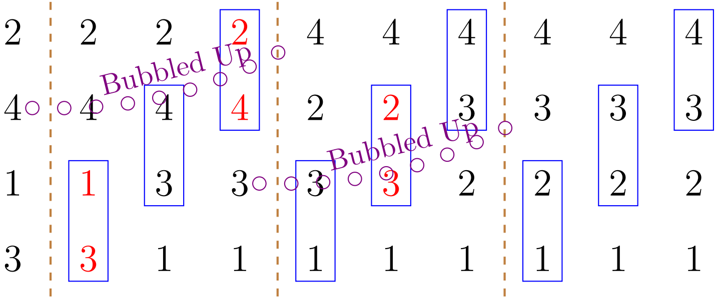

After each "**for**" loop, one more element is bubbled up to its final position.

Optimizing Bubble Sort (I)[1]

After each "**for**" loop, one more element is bubbled up to its final position.

---

procedure $\textsc{BubbleSort}(A : a_1\ a_2\ \cdots\ a_n)$
    $n \leftarrow \text{len}(A)$
    repeat
        swapped $\leftarrow$ false
        for $i \leftarrow 1 : n-1$ **do**
            if $a_i > a_{i+1}$ **then**
                $\textsc{Swap}(a_i, a_{i+1})$
                swapped $\leftarrow$ true
        $n \leftarrow n - 1$ $\triangleright$ One maximal bubbles up
    until swapped $=$ false

---

[1] See Appendix for "Optimizing Bubble Sort (II)".

Bubble Sort: Finiteness

procedure BUBBLESORT($A : a_1\ a_2\ \cdots\ a_n$)
  repeat
    swapped $\leftarrow$ false
    for $i \leftarrow 1 : n-1$ **do** ▷ Pick i
      if $a_i > a_{i+1}$ **then**
        SWAP(a_i, a_{i+1})
        swapped $\leftarrow$ true
  until swapped $=$ false ▷ No swaps

The inner "**for**" loops:

1) $\exists$ loop : no swaps $\implies$ swapped $=$ false $\implies$ terminates

Bubble Sort: Finiteness

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procedure BubbleSort(A : a_1 a_2 ··· a_n)
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                swapped ← true
    until swapped = false                   ▷ No swaps
```

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Bubble Sort: Finiteness

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Bubble Sort: Finiteness

$\boxed{\text{Fact: total \#inversions is finite.}}$

[2] Not on #adjacent inversions! Think about it.

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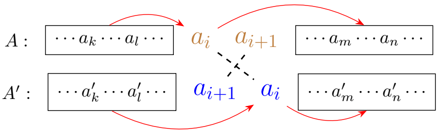
Effects of $\textsc{Swap}(a_i, a_{i+1})$ on #inversions[2]:



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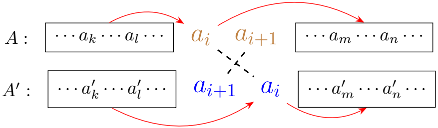

$$-1 : (a_i, a_{i+1})$$

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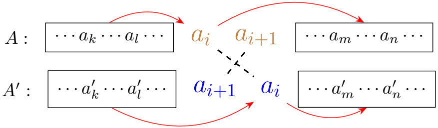

$$-1 : (a_i, a_{i+1})$$

$+0$: relative order between any other two elements does not change!

$$\langle a_k, a_l \rangle, \langle a_m, a_n \rangle, \langle a_k, a_i \rangle, \langle a_i, a_m \rangle$$

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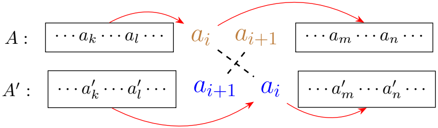

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$\boxed{\text{Lemma: } \textsc{Swap}(a_i, a_{i+1}) \implies -1 \text{ inversion}}$

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Time Complexity of Bubble Sort

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- Different inputs $\implies$ $|C|$ and $|S|$ vary:
  - Best-case, worst-case, and average-case analysis

Best-case and Worst-case Analysis

Best-case: Worst-case:

$$|C| = (\qquad);$$

$$|S| = (\qquad).$$

Best-case and Worst-case Analysis

Best-case: $1 \quad 2 \cdots n$

$$
\begin{aligned}
|C| &= (\underbrace{\qquad\qquad\qquad}_{\substack{\text{Best-case:}\\ \text{non-decreasingly sorted}}} \quad \underbrace{\qquad\qquad\qquad}_{\text{Worst-case:}});\\
|S| &= (\qquad\qquad\qquad\qquad\qquad\qquad).
\end{aligned}
$$

Best-case and Worst-case Analysis

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$$
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|C| &= \Big(\underbrace{\min : n-1,}_{\text{Best-case: non-decreasingly sorted}} \qquad \underbrace{\qquad\qquad}_{\text{Worst-case:}} \Big); \\
|S| &= \Big(\min : 0, \qquad\qquad \Big).
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|S| &= \big(\, \min : 0, \qquad\qquad\qquad\qquad \big).
\end{aligned}
$$

Worst-case: $n \quad n-1 \cdots 1$

Best-case and Worst-case Analysis

Best-case: $1 \quad 2 \cdots n$

$$
\begin{aligned}
|C| &= (\underbrace{\min : n-1,}_{\substack{\text{Best-case:} \\ \text{non-decreasingly sorted}}} \quad \underbrace{\max : \tfrac{n^2-n}{2}}_{\substack{\text{Worst-case:} \\ \text{non-increasingly sorted}}});\\
|S| &= (\min : 0, \quad \max : \tfrac{n^2-n}{2}).
\end{aligned}
$$

Worst-case: $n \quad n-1 \cdots 1$

$|S|$: #Swaps (Average-case Analysis)[3]

Assumptions on inputs:

1. The input is a random permutation ("average input")

[3] An exercise: what is $|C|$ (#Comparisions) in average?

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1. The input is a random permutation ("average input")
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$$\boxed{\text{Lemma: } \textsc{Swap}(a_i, a_{i+1}) \implies -1 \text{ inversion}}$$

$$|S| = \mathbb{E}(\#\text{inversions})$$

[3] An exercise: what is $|C|$ (#Comparisions) in average?

$|S|$: #Swaps (Average Analysis)

$$I_{ij} = \begin{cases} 1 & (a_i, a_j) \text{ is an inversion} \\ 0 & \text{o.w.} \end{cases}$$

$$X = \sum_{1 \le i < n} \sum_{i < j \le n} I_{ij} \qquad (\#\text{inversions})$$

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$$\mathbb{E}(X) = \mathbb{E}(\sum_{i} \sum_{j > i} I_{ij})$$

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$$\mathbb{E}(I_{ij}) = \mathbb{P}\{I_{ij} = 1\}$$

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$$\mathbb{E}(X) = \sum_i \sum_{i<j} \frac{1}{2} = \binom{n}{2} \cdot \frac{1}{2} = \frac{n(n-1)}{4} = O(n^2)$$

Faster Algorithms

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$$\ldots \text{and faster: } O(n \lg n) \to O(n)?$$

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— *Donald E. Knuth*

$$\text{faster: } O(n^2) \to O(n \lg n)?$$

$$\ldots\text{and faster: } O(n \lg n) \to O(n)?$$

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4 Appendix

Bubble Sort: Correctness

Finiteness

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$$\textbf{Finiteness} \implies \exists \text{ loop} : \text{ no swaps}$$

Bubble Sort: Correctness

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$$\implies A \text{ has no adjacent inversions any more}$$

Bubble Sort: Correctness

$$\begin{aligned}
\textcolor{blue}{\textbf{Finiteness}} \quad &\Longrightarrow \exists \text{ loop} : \text{ no swaps} \\
&\Longrightarrow A \text{ has no adjacent inversions any more} \\
&\Longrightarrow A \text{ is already sorted.}
\end{aligned}$$

Bubble Sort: Correctness

procedure BUBBLESORT($A : a_1\ a_2\ \cdots\ a_n$)
  repeat
    swapped $\leftarrow$ false
    for $i \leftarrow 1 : n-1$ **do** ▷ **Loop invariant?**
      if $a_i > a_{i+1}$ **then** ▷ CAS
        SWAP(a_i, a_{i+1})
        swapped $\leftarrow$ true
  until swapped = false

Loop invariant:

Before the k-th ($k \geq 1$) "**for**" loop, $a_{n-(k-1)} \cdots a_n$
(1) consists of the largest $(k-1)$ elements
(2) in sorted order.

Correctness: Initialization + Maintenance + Termination

Bubble Sort: Finiteness

Idea: well-founded relation over $\mathbf{N}$

Effects of $\textsc{Swap}(a_i, a_{i+1})$ on adjacent inversions:

5 3 1 6 7 2 4 8

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Bubble Sort: Finiteness

Idea: well-founded relation over $\mathbf{N}$

Effects of $\textsc{Swap}(a_i, a_{i+1})$ on adjacent inversions:

$$5 \quad 3 \quad 1 \quad 6 \quad 7 \quad 2 \quad 4 \quad 8$$

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$-1 : (7, 2)$

Bubble Sort: Finiteness

Idea: well-founded relation over $\mathbf{N}$

Effects of $\textsc{Swap}(a_i, a_{i+1})$ on adjacent inversions:

5 3 1 6 7 2 4 8

5 3 1 6 2 7 4 8

$-1 : (7, 2)$ $+2 : (6, 2), (7, 4)$

Optimizing Bubble Sort (II)

Idea: After each "**for**" loop, all elements after "lsp" are settled.

procedure BUBBLESORT($A : a_1\ a_2\ \cdots\ a_n$)
    repeat
        swapped $\leftarrow$ false
        lsp $\leftarrow$ 0 ▷ lsp: the last swap position
        for $i \leftarrow 1 : n-1$ **do**
            if $a_i > a_{i+1}$ **then**
                SWAP(a_i, a_{i+1})
                swapped $\leftarrow$ true
                lsp $\leftarrow$ i ▷ Update lsp
        $n \leftarrow$ lsp ▷ Elements after lsp are sorted
    until swapped = false

Optimizing Bubble Sort (II)

Idea: After each "**for**" loop, all elements after "lsp" are settled.

```
procedure BubbleSort(A : a_1 a_2 ··· a_n)
    repeat
        swapped ← false
        lsp ← 0                                  ▷ lsp: the last swap position
        for i ← 1 : n − 1 do
            if a_i > a_{i+1} then
                Swap(a_i, a_{i+1})
                swapped ← true
                lsp ← i                          ▷ Update lsp
        n ← lsp                                  ▷ Elements after lsp are sorted
    until swapped = false
```

Optimizing Bubble Sort (II)

Idea: After each "**for**" loop, all elements after "lsp" are settled.

procedure BubbleSort($A : a_1\ a_2\ \cdots\ a_n$)
    repeat
        lsp $\leftarrow 0$ ▷ lsp: the last swap position
        for $i \leftarrow 1 : n-1$ **do**
            if $a_i > a_{i+1}$ **then**
                Swap(a_i, a_{i+1})
                lsp $\leftarrow$ i ▷ Update lsp
        $n \leftarrow$ lsp ▷ Elements after lsp are sorted
    until lsp $= 0$