Bubble Sort

(A Taste of Algorithms: Definition, Design, and Analysis)

Hengfeng Wei

Institute of Computer Software
Nanjing University

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Bubble Sort

Bubble Sort

Sorting

The sorting problem:

Input: A sequence of n integers A: $a_1\ a_2\ \cdots\ a_n$.

Output: A sorted (non-decreasing order).

$$3 \quad 1 \quad 4 \quad 2 \quad \Longrightarrow 1 \quad 2 \quad 3 \quad 4$$

Algorithms

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Correctness!

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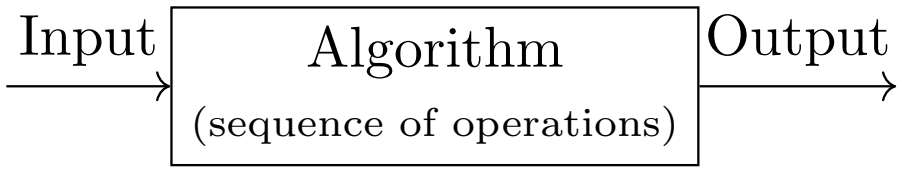
- unrealistic: `sort` operation
- realistic: arithmetic, data movement, and control

Algorithms



Correctness!

Definiteness: precisely defined operations

Finiteness: termination

Effectiveness: a reasonable model; basic operations

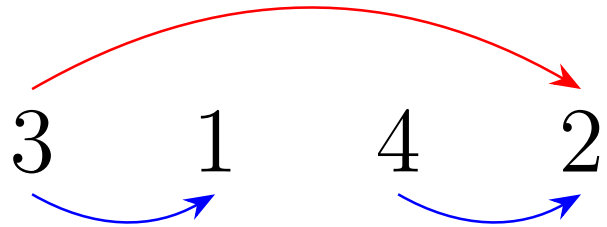
- unrealistic: `sort` operation
- realistic: arithmetic, data movement, and control
- CAS[1] for sort: compare and swap if out-of-order

[1] Forget about that CAS in computer architecture.

Inversions

$$A = a_1 \quad a_2 \quad \dots \quad a_n.$$

If $i < j$ and $a_i > a_j$, then (a_i, a_j) is an **inversion**.



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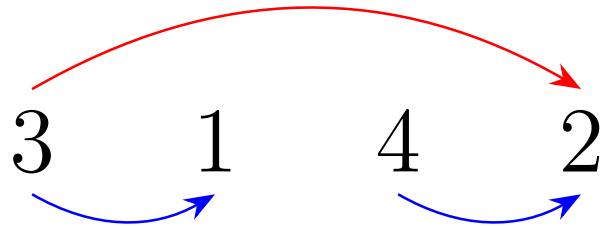

Adjacent inversion: $j = i + 1$

Inversions

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Basic idea: to eliminate all adjacent inversions

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1: **procedure** BUBBLESORT($A : a_1\ a_2\ \cdots\ a_n$)
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6: **until** no adjacent inversions ▷ **Finiteness! Definiteness!**

Bubble Sort: Definiteness

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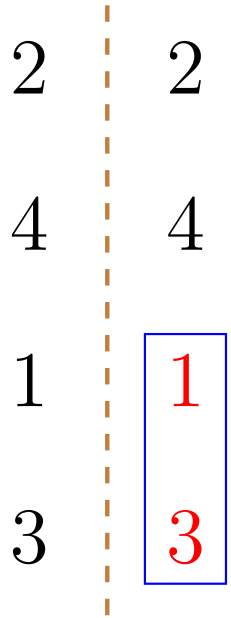

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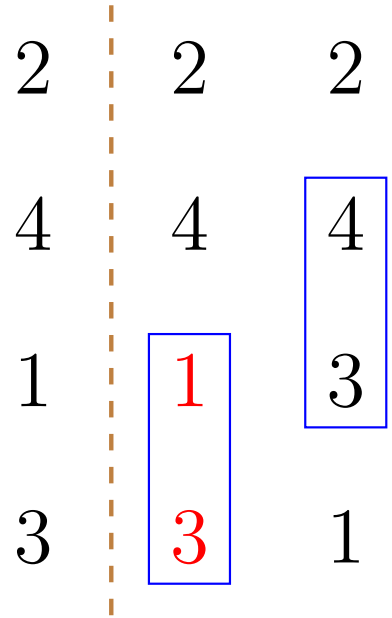

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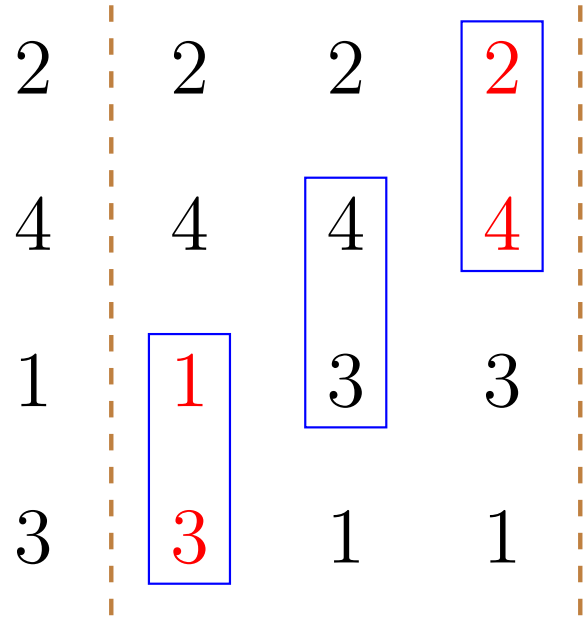

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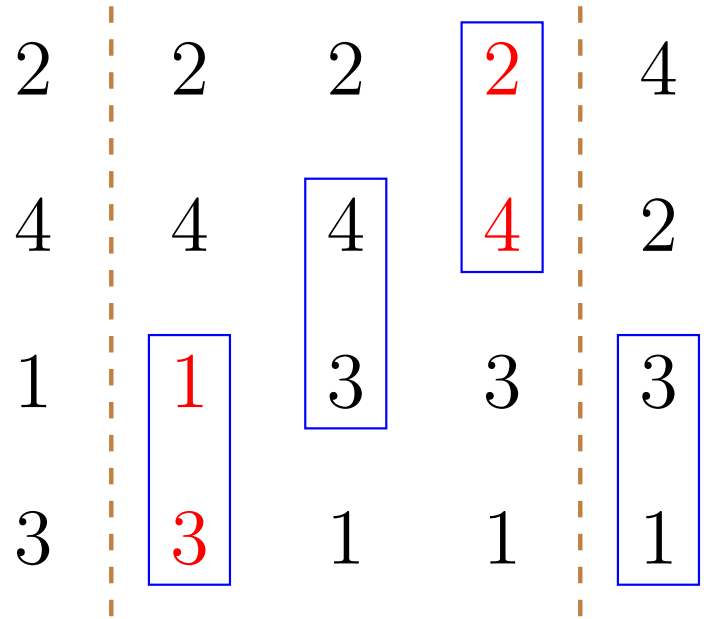

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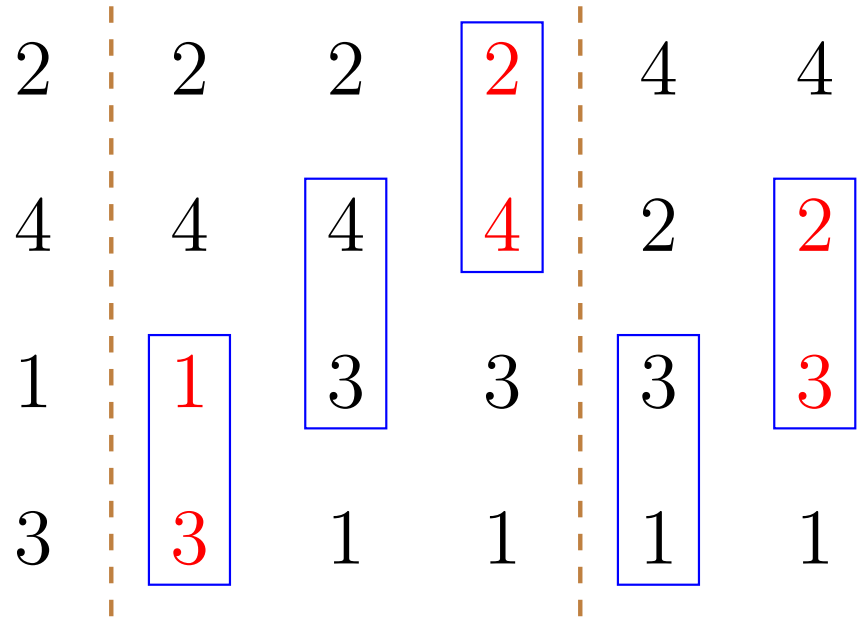

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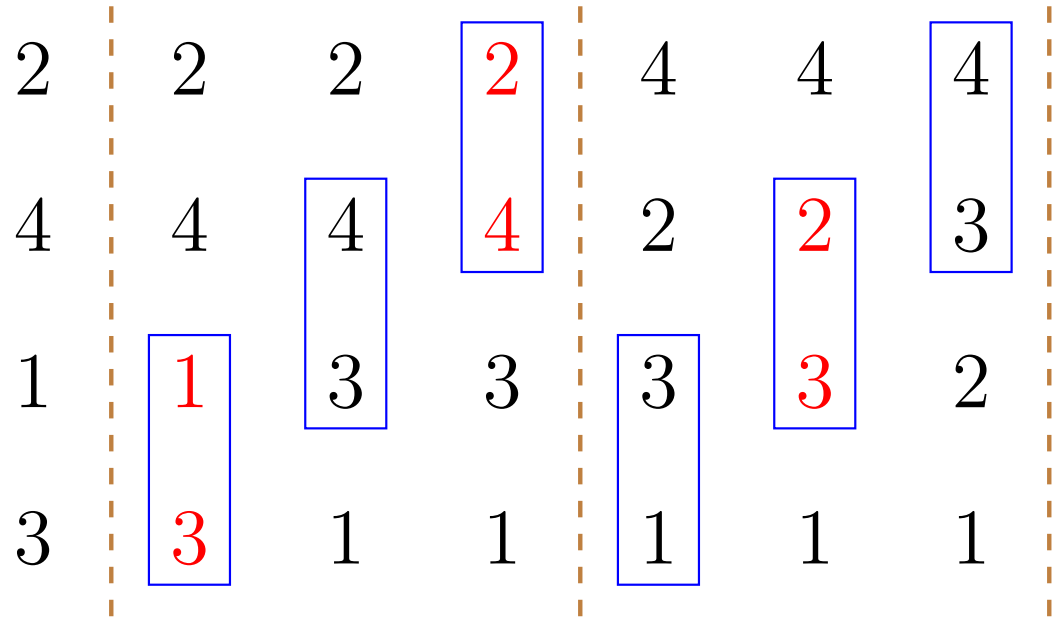

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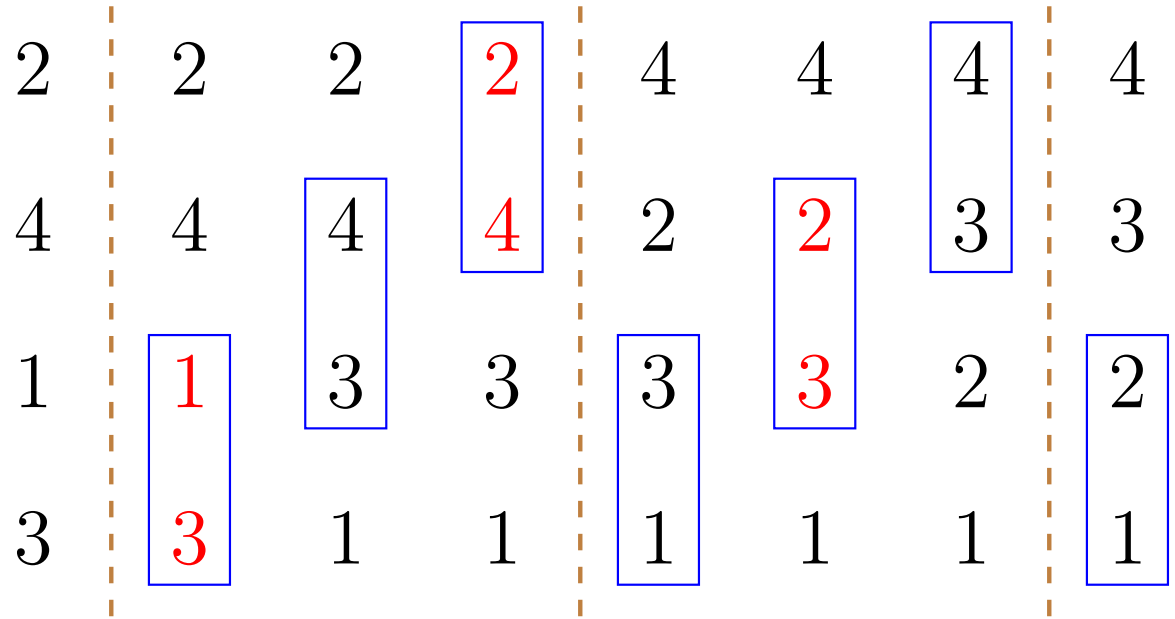

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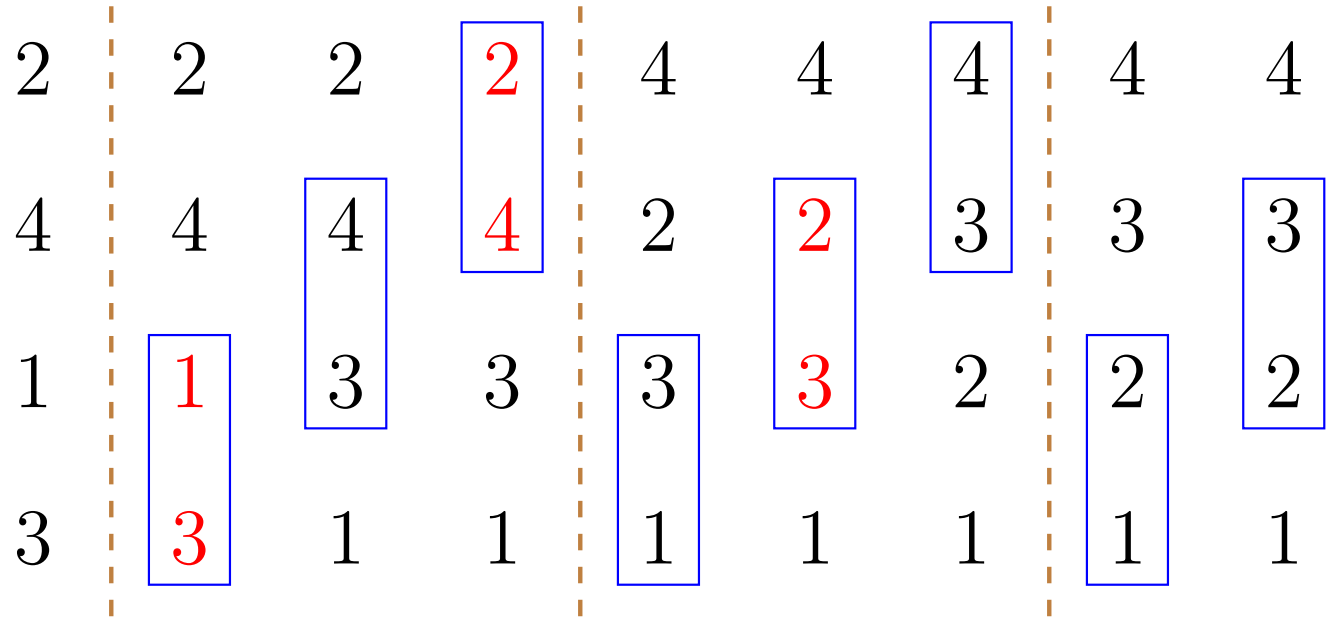

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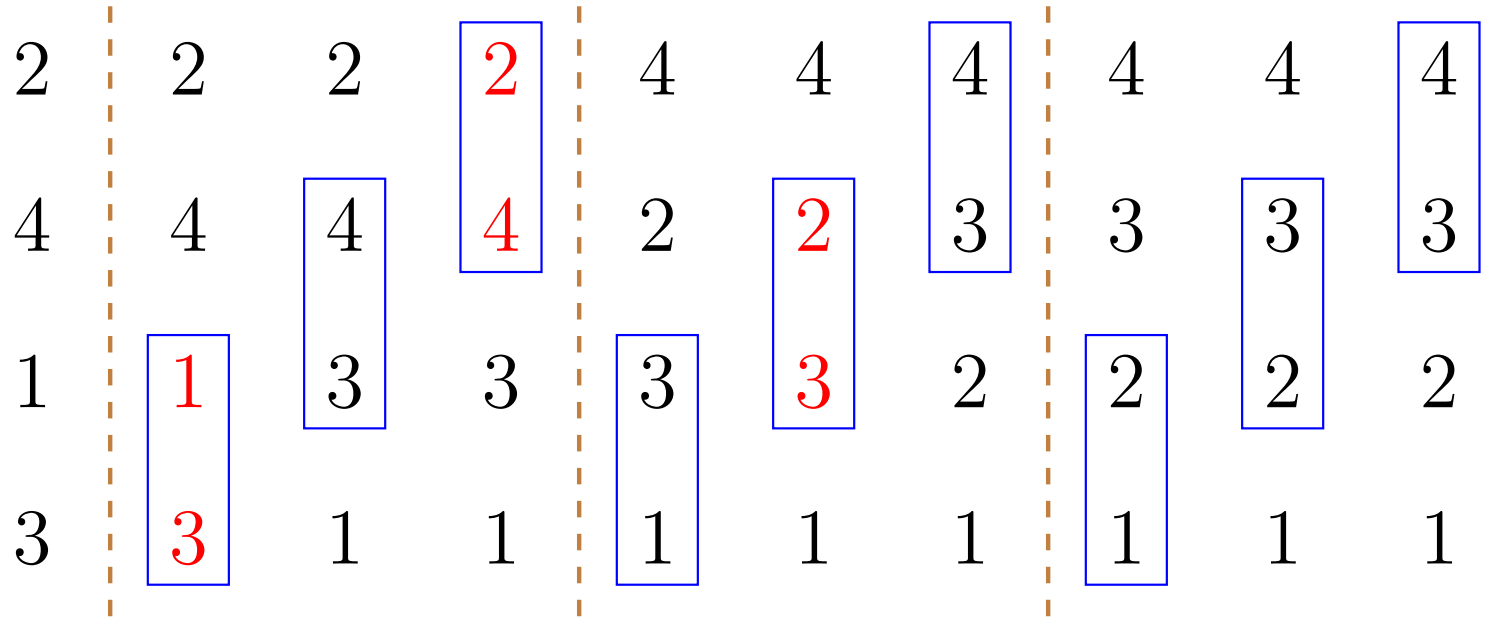

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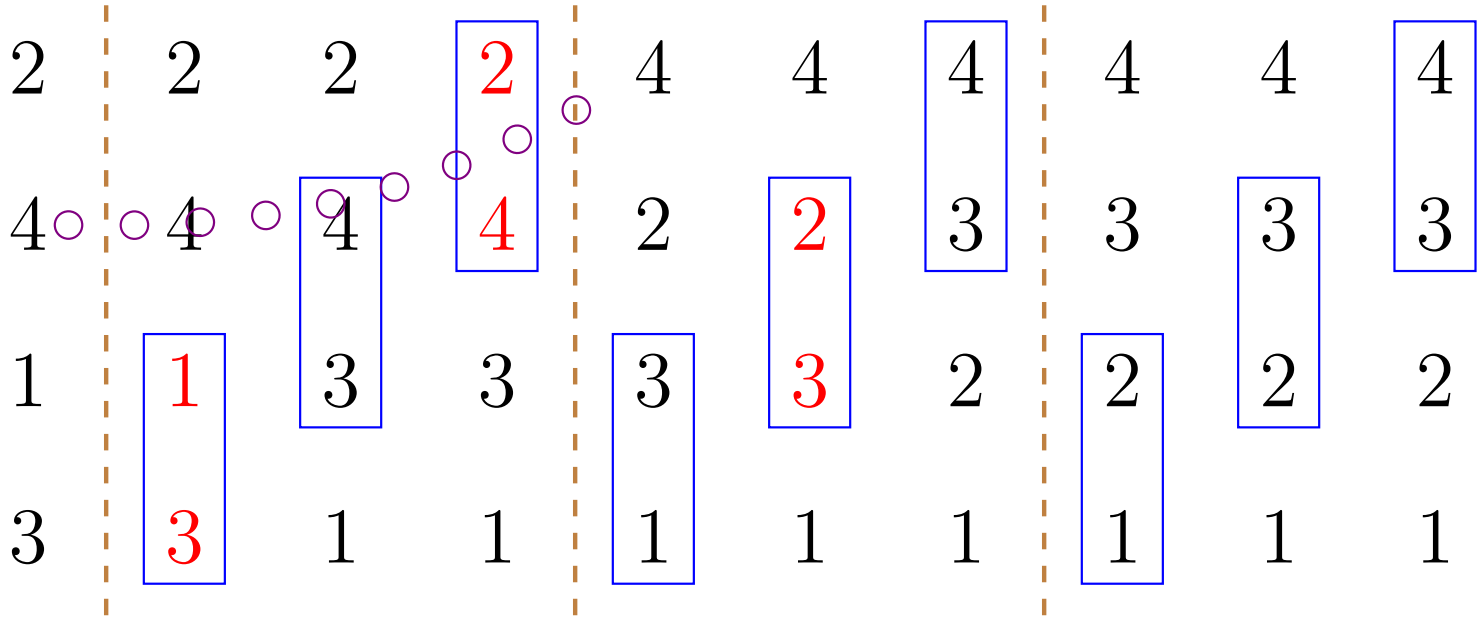

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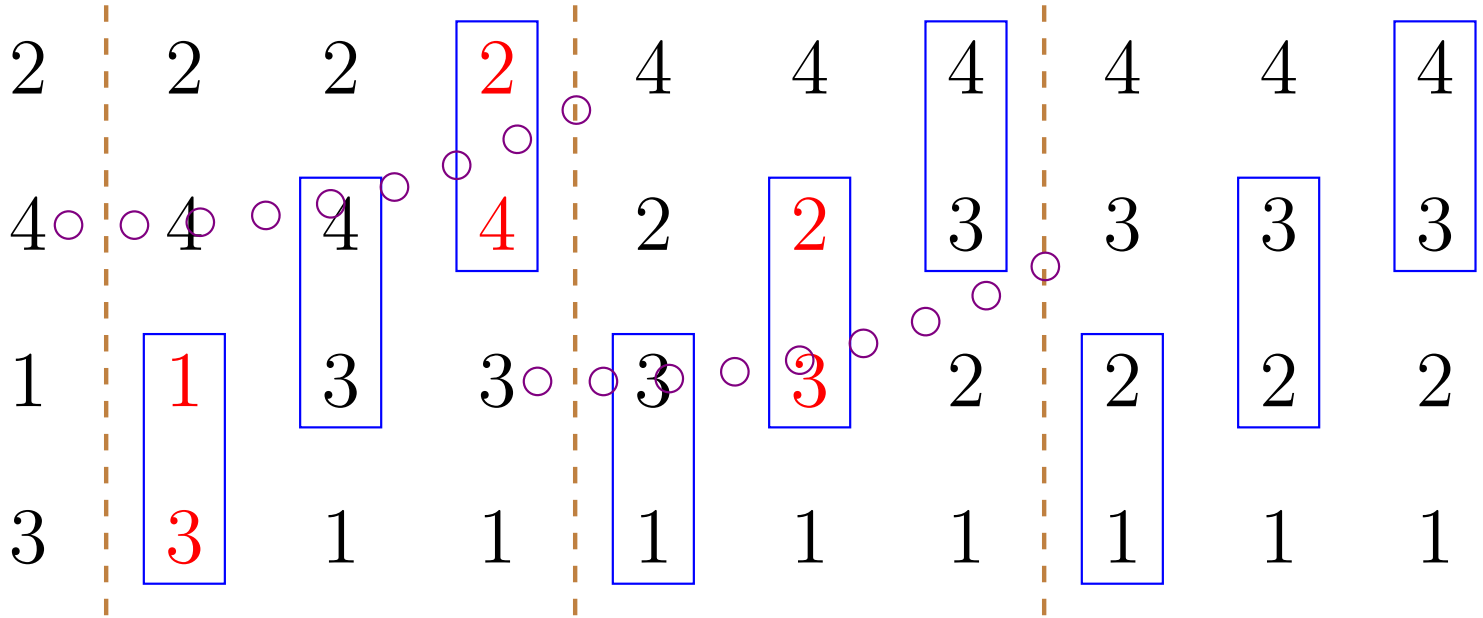

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The inner "**for**" loops:

1) $\exists$ loop : no swaps $\Longrightarrow$ swapped = false $\Longrightarrow$ terminates

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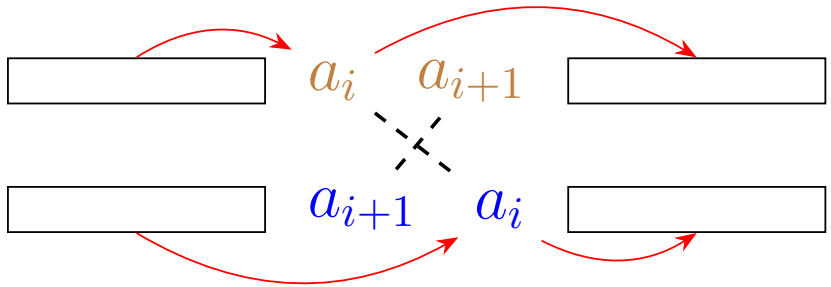

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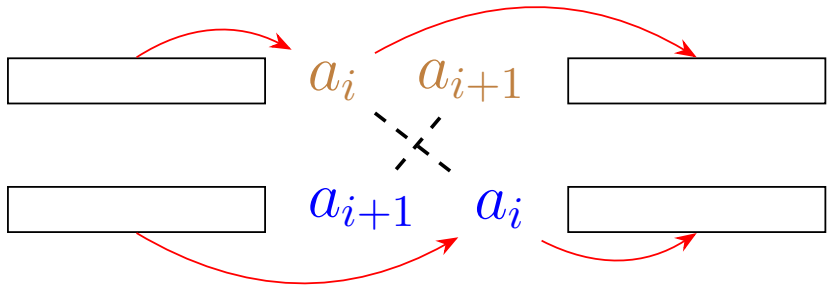

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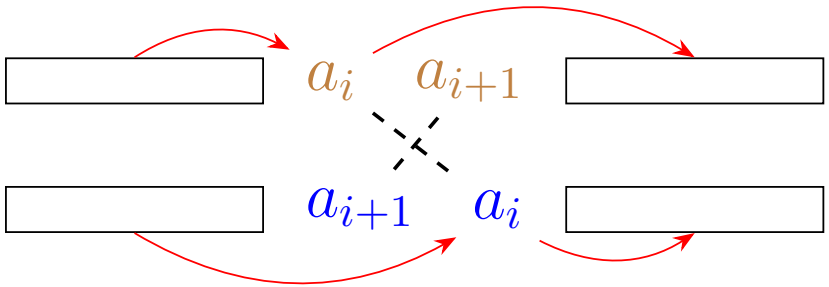

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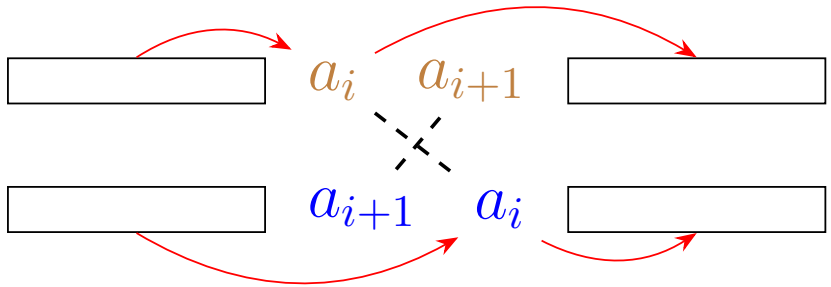

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Finiteness $\Longrightarrow$ $\exists$ loop : no swaps

$\Longrightarrow$ A has no adjacent inversions any more

$\Longrightarrow$ A is already sorted.

Optimizing Bubble Sort (I)[2]

Idea: After each "**for**" loop, one more element is settled.

```
1: procedure BubbleSort(A : a_1 a_2 ··· a_n)
2:     n ← len(A)
3:     repeat
4:         swapped ← false
5:         for i ← 1 : n − 1 do
6:             if a_i > a_{i+1} then
7:                 Swap(a_i, a_{i+1})
8:                 swapped ← true
9:         n ← n − 1                          ▷ One maximal bubbles up
10:    until swapped = false
```

[2] See Appendix for "Optimizing Bubble Sort (II)".

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Time Complexity of Bubble Sort

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  - Best-case, worst-case, and average-case analysis

Best-case and Worst-case Analysis

	Best-case:	Worst-case:
$\lvert C \rvert$	$= ($	$);$
$\lvert S \rvert$	$= ($	$).$

Best-case and Worst-case Analysis

Best-case: 1 2 3 4 5 6 7 8

Best-case: ascendingly sorted	Worst-case:
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	Best-case: ascendingly sorted	Worst-case:
$\|C\|$	$= ($ min : $n-1,$	$);$
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Best-case and Worst-case Analysis

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	Best-case: ascendingly sorted	Worst-case: descendingly sorted
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Worst-case: 8 7 6 5 4 3 2 1

Best-case and Worst-case Analysis

Best-case: 1 2 3 4 5 6 7 8

	Best-case: ascendingly sorted	Worst-case: descendingly sorted
$\|C\|$	$= (\min: n-1,$	$\max: \frac{n^2-n}{2});$
$\|S\|$	$= (\min: 0,$	$\max: \frac{n^2-n}{2}).$

Worst-case: 8 7 6 5 4 3 2 1

$$\#\text{inversions} = (n-1) + (n-2) + \cdots + 1 = \frac{n^2-n}{2}.$$

$|S|$: #Swaps (Average Analysis)

Assumptions on inputs:

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$$|S| = \mathbb{E}(\#\text{inversions})$$

$|S|$: #Swaps (Average Analysis)

$$I_{ij} = \begin{cases} 1 & (a_i, a_j) \text{ is an inversion} \\ 0 & \text{o.w.} \end{cases}$$

$$X = \sum_{j} \sum_{i<j} I_{ij} \qquad (\#\text{inversions})$$

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Faster Algorithms

It took a good deal of work to analyze the bubble sort; and although [...], the results are disappointing since they tell us that the bubble sort isn't really very good at all.

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4 Appendix

Bubble Sort: Correctness

```
1: procedure BubbleSort(A : a_1 a_2 ··· a_n)
2:     repeat
3:         swapped ← false
4:         for i ← 1 : n − 1 do        ▷ Loop invariant?
5:             if a_i > a_{i+1} then                ▷ CAS
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7:                 swapped ← true
8:     until swapped = false
```

Loop invariant:

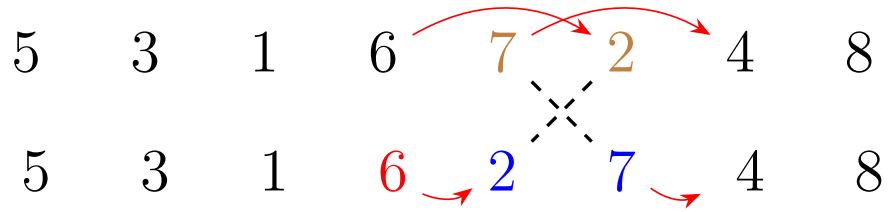
Before the k-th ($k \geq 1$) "**for**" loop, $a_{n-(k-1)} \cdots a_n$
(1) consists of the largest $(k-1)$ elements
(2) in sorted order.

Correctness: Initialization + Maintenance + Termination

Bubble Sort: Finiteness

Idea: well-founded relation over $\mathbf{N}$

Effects of $\textsc{Swap}(a_i, a_{i+1})$ on adjacent inversions:



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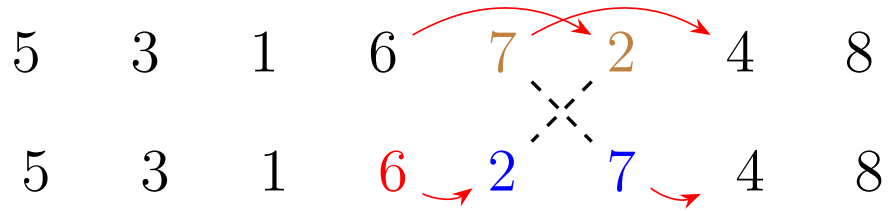

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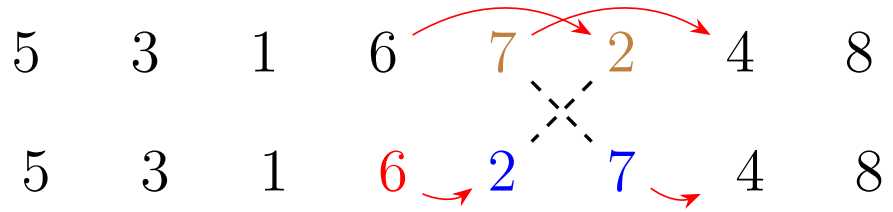

$-1 : (7, 2)$ $+2 : (6, 2), (7, 4)$

Optimizing Bubble Sort (II)

Idea: After each "**for**" loop, all elements after "lsp" are settled.

```
1:  procedure BubbleSort(A : a_1 a_2 ··· a_n)
2:      repeat
3:          swapped ← false
4:          lsp ← 0                          ▷ lsp: the last swap position
5:          for i ← 1 : n − 1 do
6:              if a_i > a_{i+1} then
7:                  Swap(a_i, a_{i+1})
8:                  swapped ← true
9:                  lsp ← i                  ▷ Update lsp
10:         n ← lsp                          ▷ Elements after lsp are sorted
11:     until swapped = false
```

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