Minimum Spanning Trees

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Definition (MST)

Given:

- connected, undirected, weighted graph $G = (V, E)$
- edge weight $w(e)$

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MST: Mimimize $w(T)$ over all possible STs

Minimum Spanning Tree

MST Example

A Wrong Algorithm

Wrong divide-and-conquer algorithm for MST

Input: $G = (V, E, w)$

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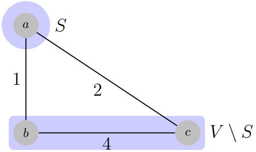
Input: $G = (V, E, w)$

Divide: $V = (S, V \setminus S)$; $||S| - |V \setminus S|| \leq 1$ (Cut)



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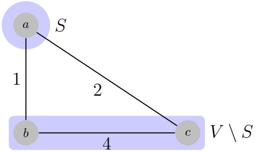
Conquer: T_1: an MST of S; T_2: an MST of $V \setminus S$



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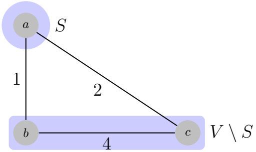
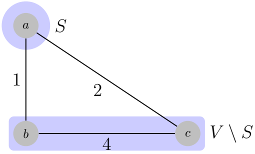
Combine: $T_1 + T_2 + \{e\}$: e is a **lightest** edge across $(S, V \setminus S)$



A Wrong Algorithm

What is wrong?



The edges *bc* and *ad* do **not** belong to any MST.

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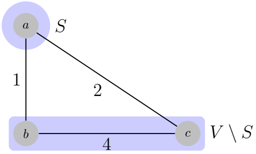

The edges bc and ad do **not** belong to any MST.

What if:

Invariant: Manages a set of edges X which is a subset of **some** MST.

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The Generic MST Algorithm

Overview: Grow the MST one edge at a time

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The Generic MST Algorithm

Overview: Grow the MST one edge at a time

State: Manage a set of edges X

Invariant: Prior to each iteration, X is a subset of some MST

Iteration: Pick a **safe** edge e s.t.
$X \cup \{e\}$ is also a subset of some MST

The Generic MST Algorithm

Proof.

Initialization:

Maintenance:

Termination:

□

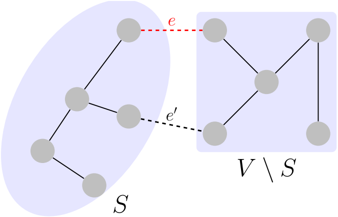
The Cut Property

Cut Property

- Graph $G = (V, E)$; X is part of an MST.
- A cut $(S, V \setminus S)$ **respecting** X (X does not cross $(S, V \setminus S)$)
- Let e be a lightest edge across $(S, V \setminus S)$

Then, $X + \{e\}$ is part of some MST T.

The Cut Property



Proof.

Basic idea: $e \notin T \Rightarrow e \in T'$.

- $T + \{e\}$ to construct a cycle C
- $\exists e' \in C$ such that e' across the cut; $w(e') \geq w(e)$
- $T' = T + \{e\} - \{e'\}$
- $w(T') \leq w(T) \Rightarrow w(T') = w(T) \Rightarrow T'$ is an MST

□

Minimum Spanning Trees

1. The MST Problem
2. The Generic MST Algorithm
3. The Algorithms of Kruskal and Prim