Minimum Spanning Trees

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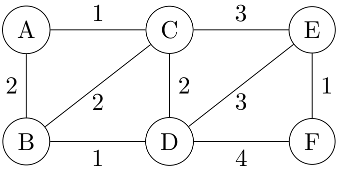
Minimum Spanning Trees

1. The MST Problem
2. The Generic MST Algorithm
3. Kruskal's and Prim's Algorithms

Minimum Spanning Trees

Minimum Spanning Tree

$G = (V, E)$: connected, undirected, weighted graph $(w(e))$



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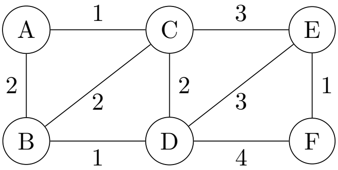

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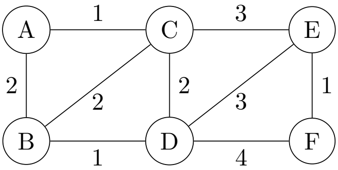

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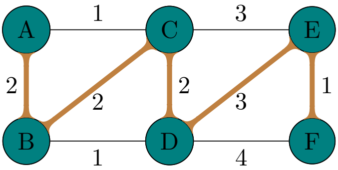

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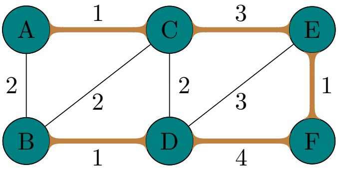

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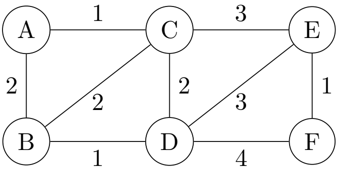

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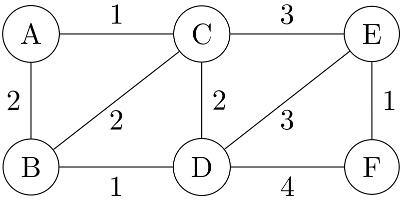
$$w(T) = \sum_{e \in E'} w(e)$$

Minimum Spanning Tree

$$\text{MST:} \quad \arg\min_{T} \; w(T)$$



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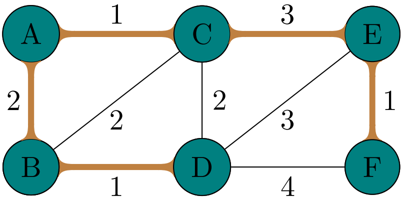

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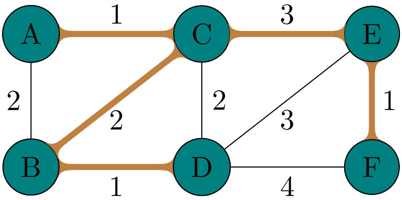

A Wrong Algorithm

Wrong divide-and-conquer algorithm for MST

Input: $G = (V, E, w)$

Divide: $V = (S, V \setminus S); \; ||S| - |V \setminus S|| \leq 1$

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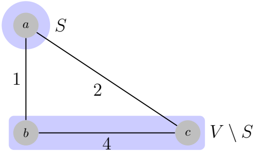
Input: $G = (V, E, w)$

Divide: $V = (S, V \setminus S)$; $||S| - |V \setminus S|| \leq 1$ (Cut)



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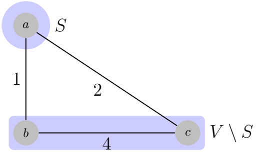

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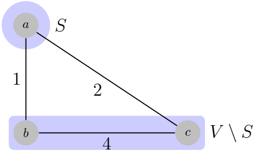
Combine: $T_1 + T_2 + \{e\}$: e is a **lightest** edge across $(S, V \setminus S)$



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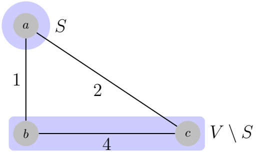

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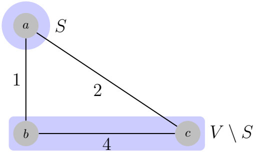

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What if:

Invariant: Manages a set of edges X which is a subset of **some** MST.

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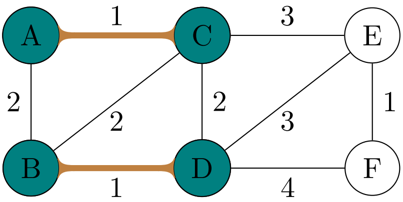
How to find a safe e for X in each iteration?

The Cut Property

Given that X is part of some MST T:

Then, $X + \{e\}$ is also a part of some MST T'.



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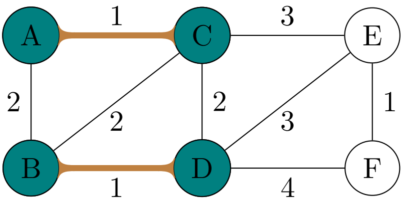

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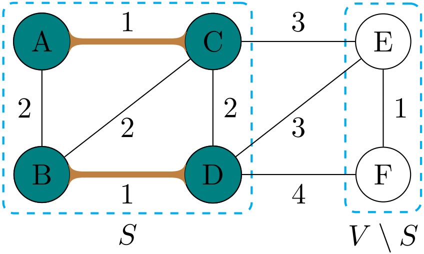

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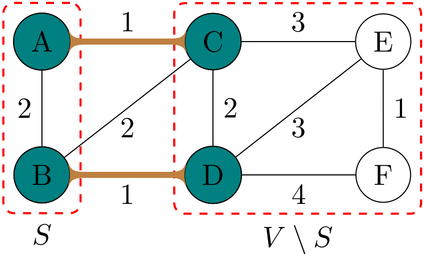

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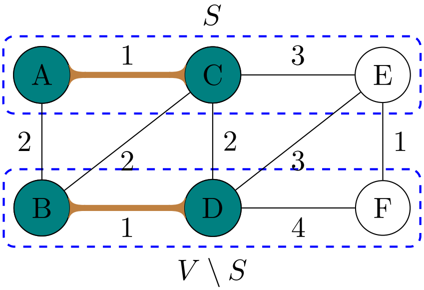

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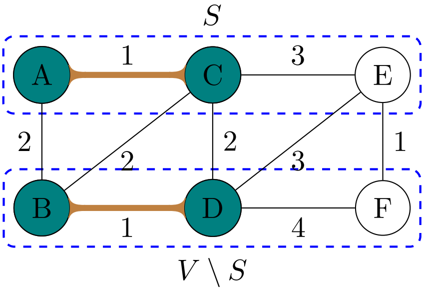

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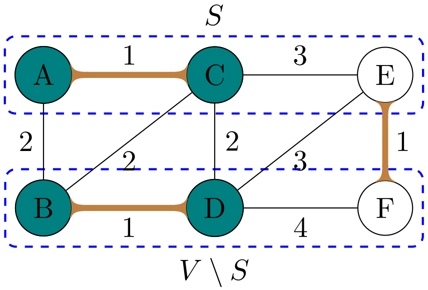

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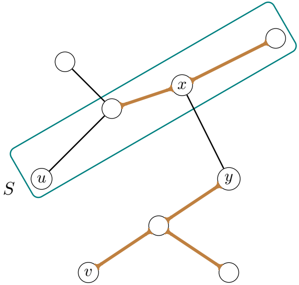

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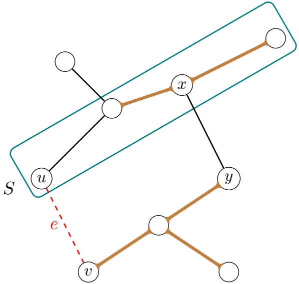

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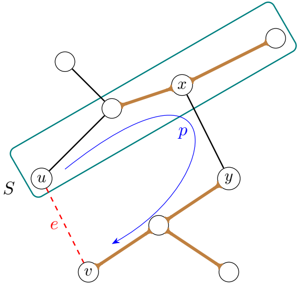

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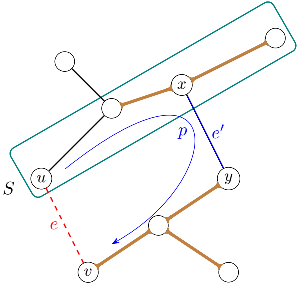

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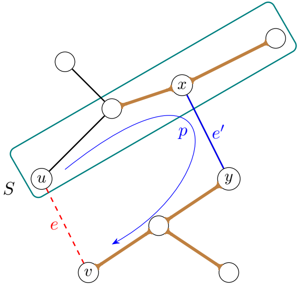

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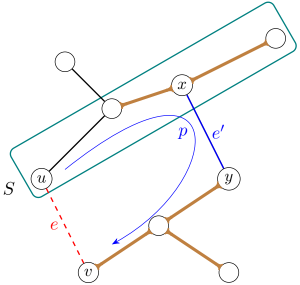

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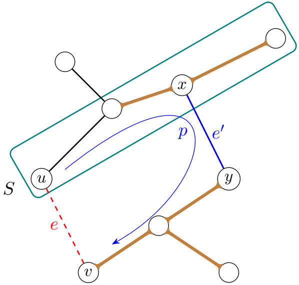

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- $w(T') \leq w(T) \Rightarrow T'$ is an MST
- $e' \notin X$(**respect**) $\Rightarrow X + \{e\} \subseteq T'$

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Kruskal's Algorithm

sort (non-descreasingly) the edges E

$X = \emptyset$
for $e \in E$ in non-descreasing order
  if $X \cup \{e\}$ does not produce cycle
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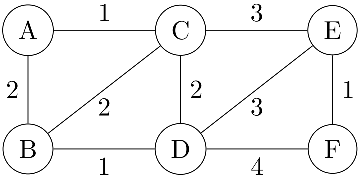

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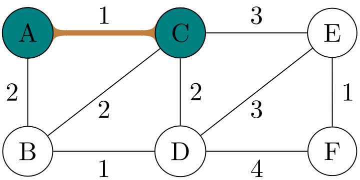

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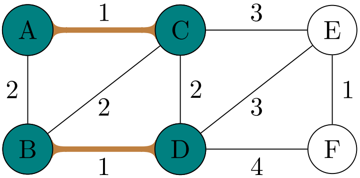

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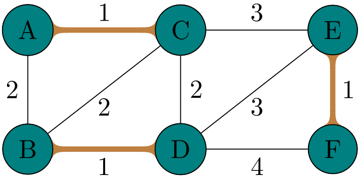

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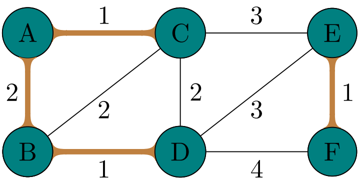

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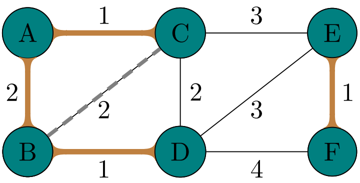

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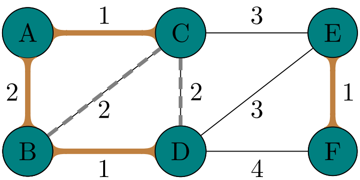

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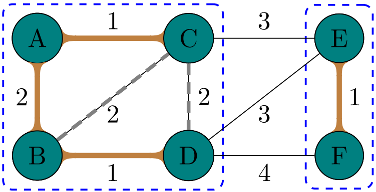

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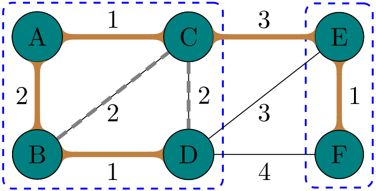

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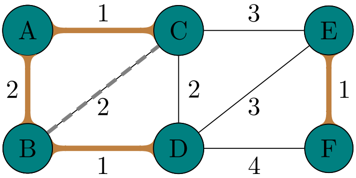

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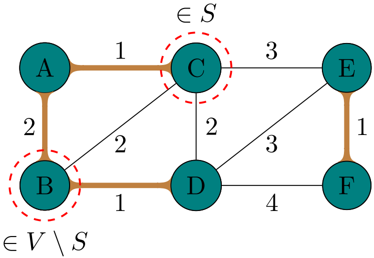

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State: forest $\triangleq$ a collection of connected components

Ops: on connected components

- cycle detection
- union two CCs

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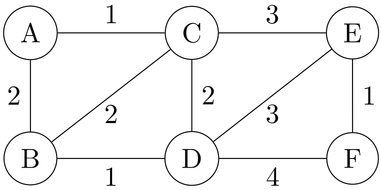
Using the **disjoint-set** data structure.

Prim's Algorithm

$X = \emptyset$
$S = \{s\}$ `// pick any` $s \in V$
$R = V \setminus S$
while $R \neq \emptyset$
  $e = (u, v) \leftarrow$ `a lightest edge across` (S, R)
  $X \leftarrow X \cup \{e\}$
  $S \leftarrow S \cup \{u\} \quad R \leftarrow R \setminus \{v\}$



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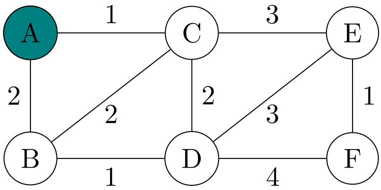

Prim's Algorithm

$X = \emptyset$
$S = \{s\}$ // pick any $s \in V$
$R = V \setminus S$
while $R \neq \emptyset$
    $e = (u, v) \leftarrow$ a lightest edge across (S, R)
    $X \leftarrow X \cup \{e\}$
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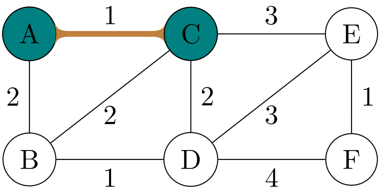

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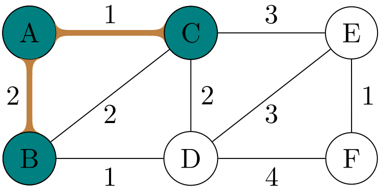

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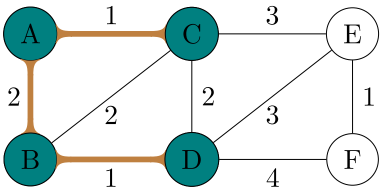

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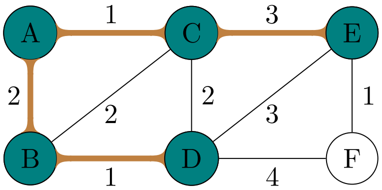

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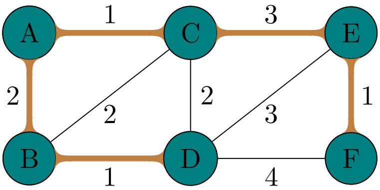

Prim's Algorithm

State: a growing tree (CC)

Op: identifying a lightest edge

Prim's Algorithm

State: a growing tree (CC)

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Using the **priority-queue (min-heap)** data structure.