Minimum Spanning Trees

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Nanjing University

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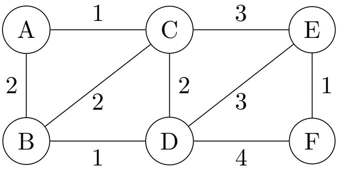
$G = (V, E)$: connected, undirected, weighted graph $(w(e))$



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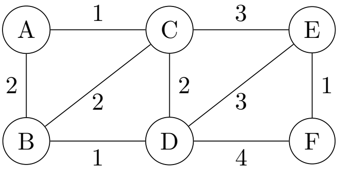

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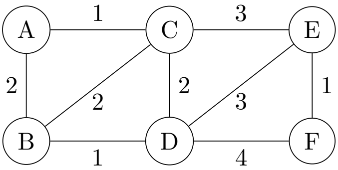

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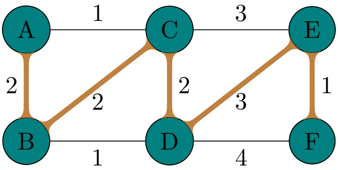

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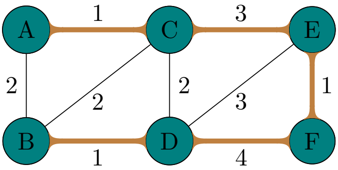

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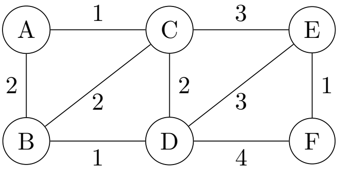
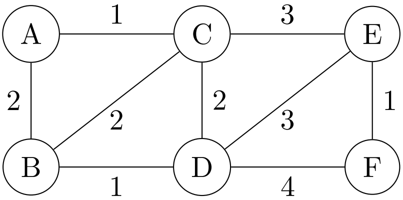


Spanning tree $T = (V, E' \subseteq E)$: connected, acyclic ($\Rightarrow n - 1$ edges)

$$w(T) = \sum_{e \in E'} w(e)$$

Minimum Spanning Tree

$$\text{MST:} \quad \arg\min_{T} \; w(T)$$



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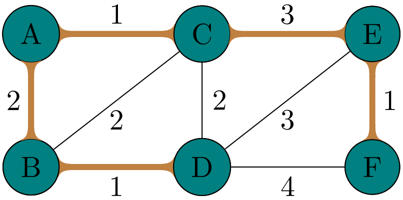

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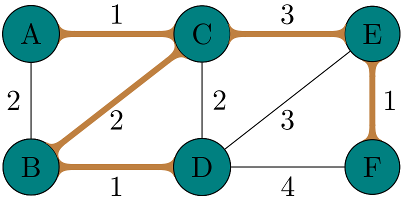

A Wrong Algorithm

Wrong divide-and-conquer algorithm for MST

Input: $G = (V, E, w)$

Divide: $V = (S, V \setminus S); \; ||S| - |V \setminus S|| \leq 1$

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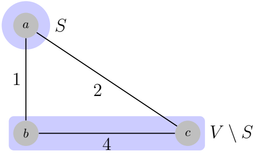
Input: $G = (V, E, w)$

Divide: $V = (S, V \setminus S)$; $||S| - |V \setminus S|| \leq 1$ (Cut)



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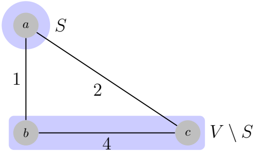

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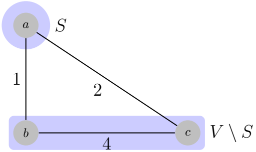
Combine: $T_1 + T_2 + \{e\}$: e is a **lightest** edge across $(S, V \setminus S)$



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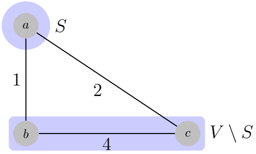

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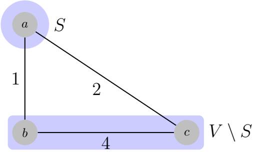

The edges bc and ad do **not** belong to any MST.

What if:

Invariant: Manages a set of edges X which is a subset of **some** MST.

The Generic MST Algorithm

Overview: Grow the MST one edge at a time

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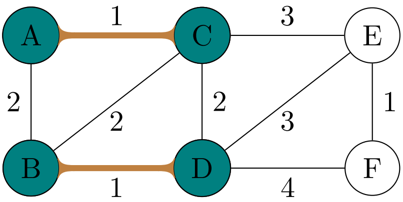
How to find a safe e for X in each iteration?

The Cut Property

Given that X is part of some MST:

Then, $X + \{e\}$ is also a part of some MST.



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Given that X is part of some MST:

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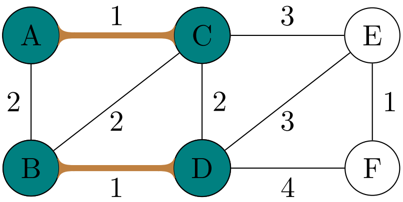

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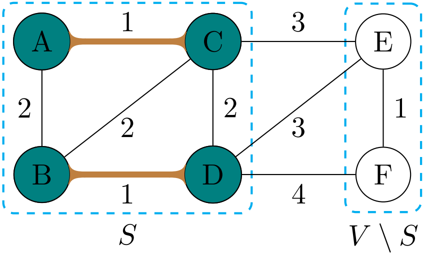

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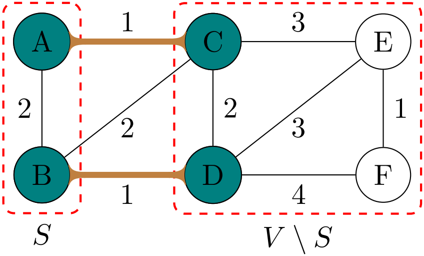

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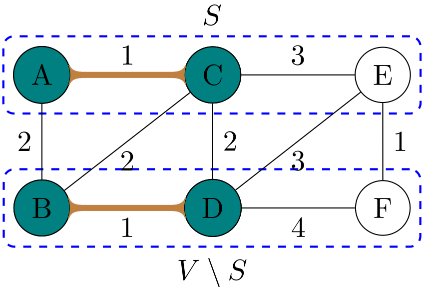

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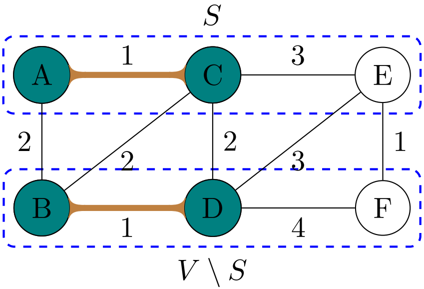

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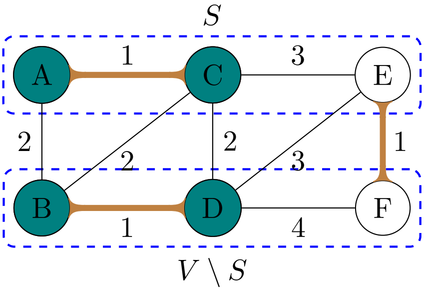

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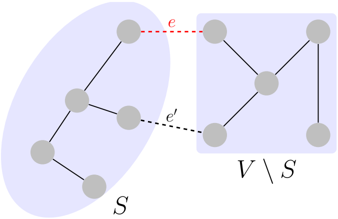

Basic idea: $e \notin T \Rightarrow e \in T'$.

- $T + \{e\}$ to construct a cycle C
- $\exists e' \in C$ such that e' across the cut; $w(e') \geq w(e)$
- $T' = T + \{e\} - \{e'\}$
- $w(T') \leq w(T) \Rightarrow w(T') = w(T) \Rightarrow T'$ is an MST

Minimum Spanning Trees

Kruskal's Algorithm

```
sort (non-descreasingly) the edges E

X = ∅
for e ∈ E in non-descreasing order
  if X ∪ {e} does not produce cycle
     X ← X ∪ {e}
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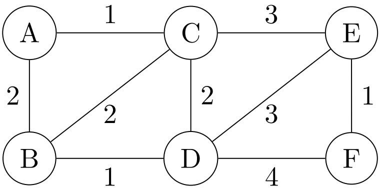

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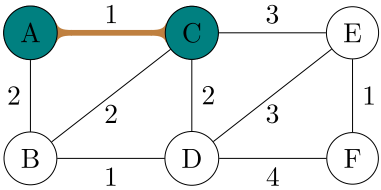

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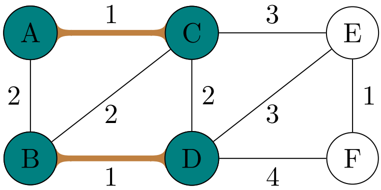

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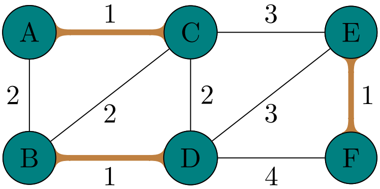

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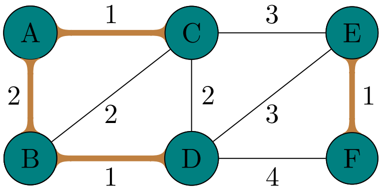

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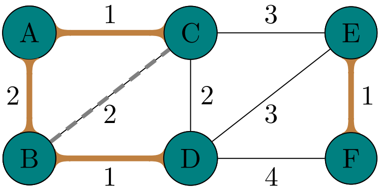

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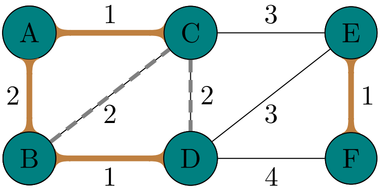

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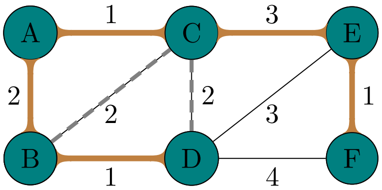

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State: forest $\triangleq$ a collection of connected components

Ops: on connected components

- cycle detection
- union two CCs

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Using the **disjoint-set** data structure.

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$X = \emptyset$
$S = \{s\}$ `// pick any` $s \in V$
$R = V \setminus S$
while $R \neq \emptyset$
  $e = (u, v) \leftarrow$ `a lightest edge across` (S, R)
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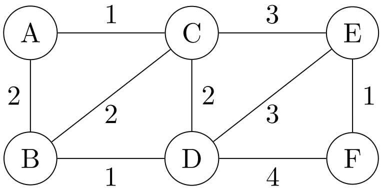

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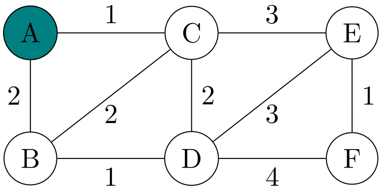

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S = {s}  // pick any s ∈ V
R = V \ S
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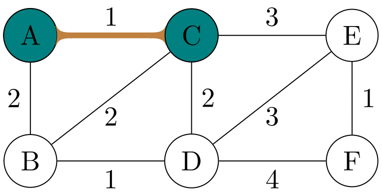

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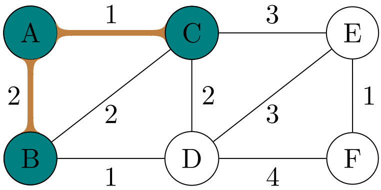

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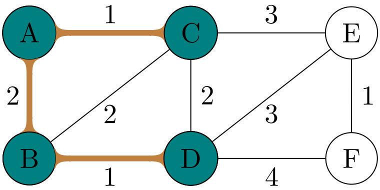

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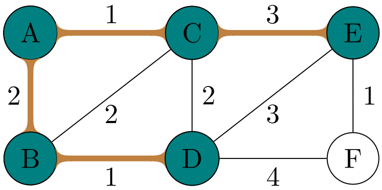

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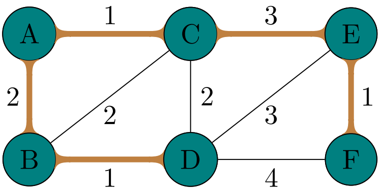

Prim's Algorithm

State: a growing tree (CC)

Op: identifying a lightest edge

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Using the **priority-queue (min-heap)** data structure.