Bubble Sort
(What are Algorithms and How to Analyze Algorithms)

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Bubble Sort

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Sorting

The sorting problem:

Input: A sequence A of n integers $a_1\ a_2\ \cdots\ a_n$.

Output: A permutation $a'_1\ a'_2\ \ldots\ a'_n$ of A *s.t.*
$a'_1 \le a'_2 \le \cdots \le a'_n$ (non-decreasing order).

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$$3\ 1\ 4\ 2 \Longrightarrow 1\ 2\ 3\ 4$$

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- unrealistic: `sort` instruction

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- unrealistic: `sort` instruction
- realistic: arithmetic, data movement, and control

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Correctness!

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Finiteness: termination

Effectiveness: RAM (Random-Access Machine) model

- unrealistic: `sort` instruction
- realistic: arithmetic, data movement, and control
- CAS[1] for sort: compare and swap if out-of-order

[1] Forget about that CAS in computer architecture.

Inversions

$$A = a_1, a_2, \ldots, a_n.$$

If $i < j$ and $a_i > a_j$, then (a_i, a_j) is an **inversion**.

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$$\begin{aligned} A \text{ has no adjacent inversions } &\implies \forall i \in [1, n-1] : a_i \le a_{i+1} \\ &\implies A \text{ is sorted.} \end{aligned}$$

$$\boxed{A \text{ is sorted } \iff A \text{ has no adjacent inversions.}}$$

Bubble Sort

Bubble Sort

Basic idea: to eliminate all adjacent inversions

```
procedure BubbleSortOverview(A : a_1 a_2 ··· a_n)
    repeat
        Pick any i
        if a_i > a_{i+1} then                                  ▷ CAS
            Swap(a_i, a_{i+1})
    until no adjacent inversions
```

Bubble Sort

Basic idea: to eliminate all adjacent inversions

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procedure BubbleSortOverview(A : a_1 a_2 ··· a_n)
    repeat
        Pick any i                              ▷ Definiteness!
        if a_i > a_{i+1} then                   ▷ CAS
            Swap(a_i, a_{i+1})
    until no adjacent inversions     ▷          Definiteness!
```

Bubble Sort

Basic idea: to eliminate all adjacent inversions

---

procedure BUBBLESORTOVERVIEW$(A : a_1 \; a_2 \; \cdots \; a_n)$

  repeat

    Pick any i ▷ **Definiteness!**

    if $a_i > a_{i+1}$ **then** ▷ CAS

      SWAP(a_i, a_{i+1})

  until no adjacent inversions ▷ **Finiteness! Definiteness!**

Bubble Sort: Definiteness

```
procedure BubbleSort(A : a_1 a_2 ··· a_n)
    repeat
        [ ]
        for i ← 1 : n − 1 do                    ▷ Pick i
            if a_i > a_{i+1} then
                Swap(a_i, a_{i+1})
                [ ]
    until [ ]
```

Bubble Sort: Definiteness

procedure BUBBLESORT($A : a_1\ a_2\ \cdots\ a_n$)

  repeat

    $\boxed{\text{swapped} \gets \text{false}}$

    for $i \gets 1 : n - 1$ **do** ▷ Pick i

      if $a_i > a_{i+1}$ **then**

        SWAP(a_i, a_{i+1})

        $\boxed{\phantom{\text{swapped} \gets \text{true}}}$

  until $\boxed{\phantom{\text{swapped} = \text{false}}}$

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procedure BubbleSort($A : a_1\ a_2\ \cdots\ a_n$)
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            if $a_i > a_{i+1}$ **then**
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                $\boxed{\text{swapped} \leftarrow \text{true}}$
    until $\boxed{\phantom{\text{swapped} = \text{false}}}$

Bubble Sort: Definiteness

```
procedure BubbleSort(A : a_1 a_2 ··· a_n)
    repeat
        [swapped ← false]
        for i ← 1 : n − 1 do                        ▷ Pick i
            if a_i > a_{i+1} then
                Swap(a_i, a_{i+1})
                [swapped ← true]
    until [swapped = false]
```

Bubble Sort: Example

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procedure BubbleSort(A : a_1 a_2 ··· a_n)
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2

4

1

3

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2	2
4	4
1	1
3	3

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2	2	2
4	4	$\boxed{4}$
1	$\boxed{1}$	$\boxed{3}$
3	$\boxed{3}$	1

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2	2	2	$\boxed{2}$	4	4	$\boxed{4}$
4	4	$\boxed{4}$	$\boxed{4}$	2	$\boxed{2}$	$\boxed{3}$
1	$\boxed{1}$	$\boxed{3}$	3	$\boxed{3}$	$\boxed{3}$	2
3	$\boxed{3}$	1	1	$\boxed{1}$	1	1

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Bubble Sort: Finiteness

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The inner "**for**" loops:

1) $\exists$ loop : no swaps $\implies$ swapped = false $\implies$ terminates

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The inner "**for**" loops:

1) $\exists$ loop : no swaps $\implies$ swapped = false $\implies$ terminates

2) $\forall$ loop : has swaps **Impossible!**

Bubble Sort: Finiteness

Effects of $\textsc{Swap}(a_i, a_{i+1})$ on adjacent inversions:

$$
\begin{array}{cccccccc}
5 & 3 & 1 & 6 & 7 & 2 & 4 & 8 \\
5 & 3 & 1 & 6 & 2 & 7 & 4 & 8
\end{array}
$$

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$-1 : (7, 2)$

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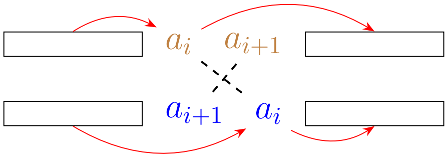
Effects of $\text{Swap}(a_i, a_{i+1})$ on adjacent inversions:

$$\begin{array}{cccccccc} 5 & 3 & 1 & 6 & 7 & 2 & 4 & 8 \\ 5 & 3 & 1 & 6 & 2 & 7 & 4 & 8 \end{array}$$

$-1 : (7, 2)$ $\qquad$ $+2 : (6, 2), (7, 4)$

Bubble Sort: Finiteness

Effects of $\text{SWAP}(a_i, a_{i+1})$ on #inversions:



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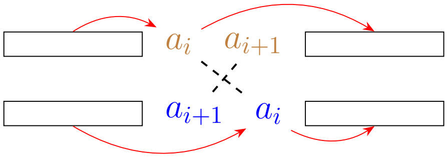

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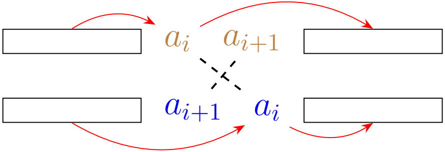

$-1 : (a_i, a_{i+1})$ $+0$

Bubble Sort: Finiteness

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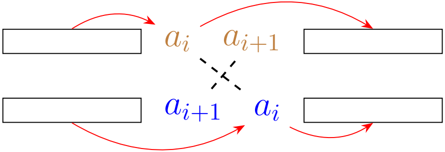

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Total #inversions is finite.

Bubble Sort: Correctness

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Finiteness

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$\textbf{Finiteness} \implies \exists \text{ loop} : \text{no swaps}$

Bubble Sort: Correctness

$$\boxed{A \text{ is sorted} \iff A \text{ has no adjacent inversions.}}$$

$$\begin{aligned} \textbf{Finiteness} \quad &\implies \exists \text{ loop} : \text{ no swaps} \\ &\implies A \text{ has no adjacent inversions any more} \end{aligned}$$

Bubble Sort: Correctness

$$\boxed{A \text{ is sorted} \iff A \text{ has no adjacent inversions.}}$$

$$\textbf{Finiteness} \implies \exists \text{ loop : no swaps}$$
$$\implies A \text{ has no adjacent inversions any more}$$
$$\implies A \text{ is already sorted.}$$

Bubble Sort: Correctness

```
procedure BubbleSort(A : a_1 a_2 ··· a_n)
    repeat
        swapped ← false
        for i ← 1 : n − 1 do                    ▷ Loop invariant?
            if a_i > a_{i+1} then                ▷ CAS
                Swap(a_i, a_{i+1})
                swapped ← true
    until swapped = false
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Loop invariant:

Before the k-th ($k \geq 1$) "**for**" loop, $a_{n-(k-1)} \cdots a_n$

(1) consists of the largest $(k-1)$ elements

(2) in sorted order.

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Correctness: Initialization + Maintenance + Termination

Optimizing Bubble Sort

```
procedure BubbleSort(A : a_1 a_2 ··· a_n)
    n ← len(A)
    repeat
        swapped ← false
        for i ← 1 : n − 1 do
            if a_i > a_{i+1} then
                Swap(a_i, a_{i+1})
                swapped ← true
        n ← n − 1                          ▷ One maximal bubbles up
    until swapped = false
```

Optimizing Bubble Sort

3 1 4 2 5 6 7

1 3 2 4 5 6 7

procedure BUBBLESORT($A : a_1\ a_2\ \cdots\ a_n$)
  repeat
    swapped $\leftarrow$ false
    lsp $\leftarrow$ 0 ▷ lsp: the last swap position
    for $i \leftarrow 1 : n - 1$ **do**
      if $a_i > a_{i+1}$ **then**
        SWAP(a_i, a_{i+1})
        swapped $\leftarrow$ true
        lsp $\leftarrow$ i ▷ Update lsp
    $n \leftarrow$ lsp ▷ Elements after lsp are sorted
  until swapped = false

Optimizing Bubble Sort

3 1 4 2 5 6 7

1 3 2 4 5 6 7

procedure BUBBLESORT($A : a_1\ a_2\ \cdots\ a_n$)
  repeat
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    $n \leftarrow$ lsp ▷ Elements after lsp are sorted
  until swapped = false

Optimizing Bubble Sort

3	1	4	2	5	6	7
1	3	2	4	5	6	7

procedure BUBBLESORT($A : a_1\ a_2\ \cdots\ a_n$)
    repeat
        lsp $\leftarrow 0$ $\triangleright$ lsp: the last swap position
        for $i \leftarrow 1 : n-1$ **do**
            if $a_i > a_{i+1}$ **then**
                SWAP(a_i, a_{i+1})
                lsp $\leftarrow$ i $\triangleright$ Update lsp
        $n \leftarrow$ lsp $\triangleright$ Elements after lsp are sorted
    until $n = 0$

Bubble Sort

1. Sorting
2. Bubble Sort
3. Analysis

Time Complexity of Bubble Sort

- Finiteness is NOT enough $\Longrightarrow$ Quantitative finiteness

Time Complexity of Bubble Sort

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- Time on real computers varies $\implies$ #Ops on RAM model:

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$|P|$: #Passes (the "**for**" loops)

$|C|$: #Comparisons ($\textbf{if } a_{i-1} > a_i$)

$|S|$: #Swaps ($\textsc{Swap}(a_{i-1}, a_i)$)

Time Complexity of Bubble Sort

- Finiteness is NOT enough $\implies$ Quantitative finiteness
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$$|P| : \text{\#Passes} \qquad \text{(the "\textbf{for}" loops)}$$
$$|C| : \text{\#Comparisons} \qquad (\textbf{if } a_{i-1} > a_i)$$
$$|S| : \text{\#Swaps} \qquad (\textsc{Swap}(a_{i-1}, a_i))$$

- Different inputs $\implies$ different execution time:

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- Different inputs $\implies$ different execution time:
  - Best-case, worst-case, and average-case analysis

Best-case and Worst-case Analysis

Best-case: Worst-case:

$$
\begin{aligned}
|P| &= (\qquad\qquad\qquad\qquad\qquad);\\
|C| &= (\qquad\qquad\qquad\qquad\qquad\quad);\\
|S| &= (\qquad\qquad\qquad\qquad\qquad\quad).
\end{aligned}
$$

Best-case and Worst-case Analysis

Best-case: 1 2 3 4 5 6 7 8

$$\underbrace{\qquad}_{\substack{\text{Best-case:} \\ \text{ascendingly sorted}}} \qquad \underbrace{\qquad}_{\text{Worst-case:}}$$

$$\begin{aligned} |P| &= (\qquad\qquad\qquad\qquad); \\ |C| &= (\qquad\qquad\qquad\qquad\qquad); \\ |S| &= (\qquad\qquad\qquad\qquad\qquad). \end{aligned}$$

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Best-case: 1 2 3 4 5 6 7 8

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& \qquad\qquad \underline{\text{Best-case:}} \qquad\qquad \underline{\text{Worst-case:}} \\
& \qquad\qquad \text{ascendingly sorted} \\
|P| &= (\min : 1, \qquad\qquad\qquad\qquad); \\
|C| &= (\min : n - 1, \qquad\qquad\qquad\qquad); \\
|S| &= (\min : 0, \qquad\qquad\qquad\qquad).
\end{aligned}$$

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$$\frac{\text{Best-case:}}{\text{ascendingly sorted}} \qquad \frac{\text{Worst-case:}}{\text{descendingly sorted}}$$

$$|P| = (\min : 1, \qquad);$$

$$|C| = (\min : n - 1, \qquad);$$

$$|S| = (\min : 0, \qquad).$$

Worst-case: 8 7 6 5 4 3 2 1

Best-case and Worst-case Analysis

Best-case: 1 2 3 4 5 6 7 8

$$
\begin{aligned}
& \qquad\qquad \underbrace{}_{}\frac{\text{Best-case:}}{\text{ascendingly sorted}} \qquad \frac{\text{Worst-case:}}{\text{descendingly sorted}} \\
|P| &= (\min : 1, \qquad\quad \max : n); \\
|C| &= (\min : n-1, \quad \max : \tfrac{n^2-n}{2}); \\
|S| &= (\min : 0, \qquad\quad \max : \tfrac{n^2-n}{2}).
\end{aligned}
$$

Worst-case: 8 7 6 5 4 3 2 1

$|S|$: #Swaps (Average Analysis)

Assumptions on inputs:

1. The input is a random permutation
2. All numbers are distinct

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Question: What is the expected #inversions?

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I_{ij} : indicator of inversion (a_i, a_j)

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$$X = \sum_j \sum_{i<j} I_{ij}$$

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$$E(X) = \sum_j \sum_{i<j} \frac{1}{2} = \binom{n}{2} \cdot \frac{1}{2} = \frac{n(n-1)}{4}$$