Bubble Sort

(What are Algorithms and How to Analyze Algorithms)

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Bubble Sort

Bubble Sort

1. Sorting
2. Bubble Sort
3. Analysis

Sorting

The sorting problem:

Input: A sequence A of n integers $a_1\ a_2\ \cdots\ a_n$.

Output: A permutation $a'_1\ a'_2\ \ldots\ a'_n$ of A *s.t.*
$a'_1 \leq a'_2 \leq \cdots \leq a'_n$ (non-decreasing order).

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$$3 \; 1 \; 4 \; 2 \Longrightarrow 1 \; 2 \; 3 \; 4$$

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- RAM (Random-Access Machine) model
- unrealistic: `sort` instruction
- realistic: arithmetic, data movement, and control
- CAS for sort: compare and swap if out-of-order

Inversions

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$$\boxed{A \text{ is sorted } \iff A \text{ has no adjacent inversions.}}$$

Bubble Sort

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Bubble Sort

Basic idea: to eliminate all adjacent inversions

```
procedure BubbleSortOverview(A : a_1 a_2 ··· a_n)
    repeat
        Pick any i
        if a_i > a_{i+1} then                ▷ CAS
            Swap(a_i, a_{i+1})
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Definiteness!

Finiteness!

Bubble Sort

```
procedure BubbleSort(A : a_1 a_2 ··· a_n)
    repeat
        [        ]
        for i ← 1 : n − 1 do                                  ▷ Pick i
            if a_i > a_{i+1} then
                Swap(a_i, a_{i+1})
                [        ]
    until [        ]
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Bubble Sort

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procedure BubbleSort(A : a_1 a_2 ··· a_n)
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Bubble Sort

---

procedure $\textsc{BubbleSort}(A : a_1\ a_2\ \cdots\ a_n)$
    repeat
        $\boxed{\text{swapped} \leftarrow \text{false}}$
        for $i \leftarrow 1 : n-1$ **do** ▷ Pick i
            if $a_i > a_{i+1}$ **then**
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The inner "**for**" loops:

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1) $\exists$ loop : no swaps $\implies$ swapped = false $\implies$ terminates

2) $\forall$ loop : has swaps **Impossible!**

Bubble Sort

Effects of $\textsc{Swap}(a_i, a_{i+1})$ on adjacent inversions:

$$\begin{array}{cccccccc} 5 & 3 & 1 & 6 & 7 & 2 & 4 & 8 \\ 5 & 3 & 1 & 6 & 2 & 7 & 4 & 8 \end{array}$$

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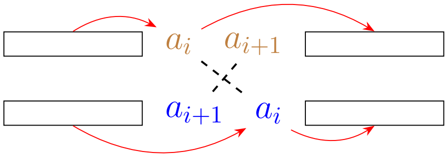
Effects of $\textsc{Swap}(a_i, a_{i+1})$ on #inversions:



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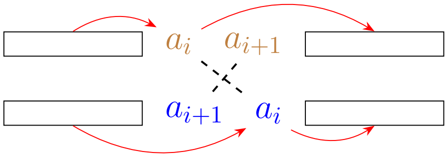

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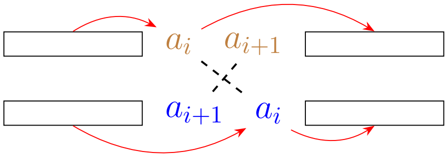

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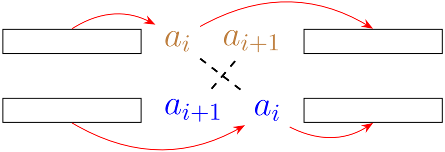

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Total #inversions is finite.

Optimizing Bubble Sort

```
procedure BubbleSort(A : a_1 a_2 ··· a_n)
    n ← len(A)
    repeat
        swapped ← false
        for i ← 1 : n − 1 do
            if a_i > a_{i+1} then
                Swap(a_i, a_{i+1})
                swapped ← true
        n ← n − 1                    ▷ One maximal bubbles up
    until swapped = false
```

Optimizing Bubble Sort

```
procedure BubbleSort(A : a_1 a_2 ··· a_n)
    repeat
        swapped ← false
        lsp ← 0                         ▷ lsp: the last swap position
        for i ← 1 : n − 1 do
            if a_i > a_{i+1} then
                Swap(a_i, a_{i+1})
                swapped ← true
                lsp ← i                 ▷ Update lsp
        n ← lsp                         ▷ Elements after lsp are sorted
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                lsp ← i                      ▷ Update lsp
        n ← lsp                              ▷ Elements after lsp are sorted
    until n = 0
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Time Complexity of Bubble Sort

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$$|P| : \#\text{Passes} \qquad \text{(the "}\textbf{for}\text{" loops)}$$

$$|C| : \#\text{Comparisons} \qquad (\textbf{if } a_{i-1} > a_i)$$

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- Different inputs $\Longrightarrow$ different execution time:
  - Best-case, worst-case, and average-case analysis

Best-case and Worst-case Analysis

Best-case: Worst-case:

$$
\begin{aligned}
|P| &= (\qquad\qquad\qquad\qquad); \\
|C| &= (\qquad\qquad\qquad\qquad\quad); \\
|S| &= (\qquad\qquad\qquad\qquad\quad).
\end{aligned}
$$

Best-case and Worst-case Analysis

Best-case: 1 2 3 4 5 6 7 8

$$\underline{\text{Best-case:}} \qquad \underline{\text{Worst-case:}}$$
ascendingly sorted

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Best-case: 1 2 3 4 5 6 7 8

$$\underbrace{\text{Best-case:}}_{\text{ascendingly sorted}} \qquad \underbrace{\text{Worst-case:}}$$

$$\begin{aligned}
|P| &= (\min : 1, \qquad\qquad\qquad); \\
|C| &= (\min : n - 1, \qquad\qquad\qquad); \\
|S| &= (\min : 0, \qquad\qquad\qquad).
\end{aligned}$$

Best-case and Worst-case Analysis

Best-case: 1 2 3 4 5 6 7 8

$$
\begin{aligned}
|P| &= (\,\min : 1, \quad \overset{\text{Best-case:}}{\text{ascendingly sorted}} \qquad \overset{\text{Worst-case:}}{\text{descendingly sorted}} \quad); \\
|C| &= (\,\min : n-1, \qquad\qquad); \\
|S| &= (\,\min : 0, \qquad\qquad).
\end{aligned}
$$

Worst-case: 8 7 6 5 4 3 2 1

Best-case and Worst-case Analysis

Best-case: 1 2 3 4 5 6 7 8

$$
\begin{aligned}
& & \underline{\text{Best-case:}} & & \underline{\text{Worst-case:}} \\
& & \text{ascendingly sorted} & & \text{descendingly sorted} \\
|P| & = (& \min : 1, & & \max : n); \\
|C| & = (& \min : n-1, & & \max : \tfrac{n^2-n}{2}); \\
|S| & = (& \min : 0, & & \max : \tfrac{n^2-n}{2}).
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Assumptions on inputs:

1. The input is a random permutation
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Question: What is the expected #inversions?

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$$X = \sum_{j} \sum_{i<j} I_{ij}$$

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$$E(X) = \sum_j \sum_{i<j} \frac{1}{2} = \binom{n}{2} \cdot \frac{1}{2} = \frac{n(n-1)}{4}$$