Minimum Spanning Trees

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1. The MST Problem
2. The Generic MST Algorithm
3. Kruskal's and Prim's Algorithms

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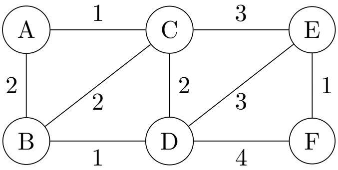
Minimum Spanning Tree

$G = (V, E)$: connected, undirected, weighted graph $(w(e))$



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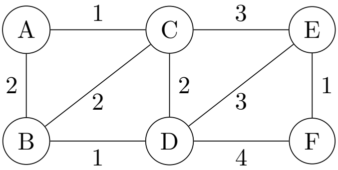

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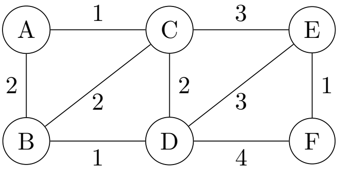

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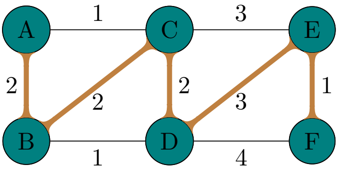

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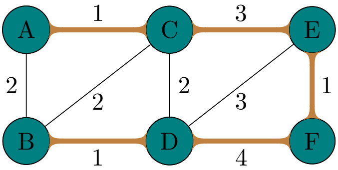

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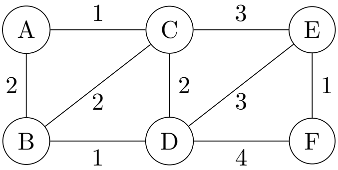

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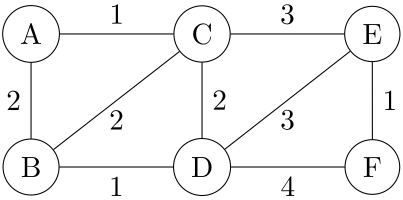
$$w(T) = \sum_{e \in E'} w(e)$$

Minimum Spanning Tree

$$\text{MST:} \quad \arg\min_{T} \; w(T)$$



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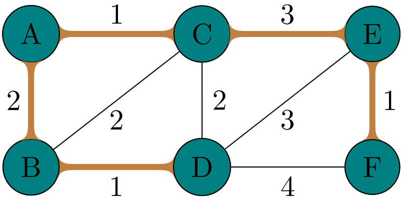

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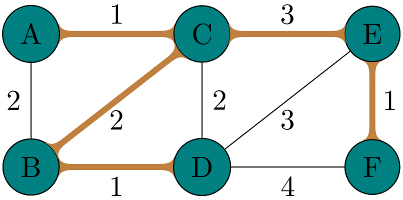
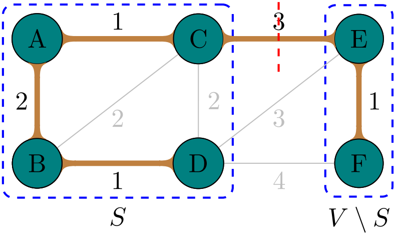


A Simple Property



Cut: $V = (S, V \setminus S)$

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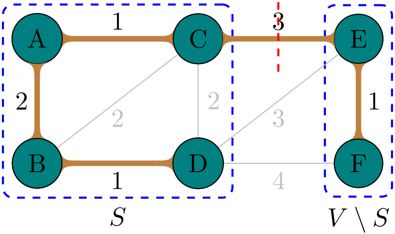

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1. MST in each connected component
2. ce: a **lightest** edge across cut

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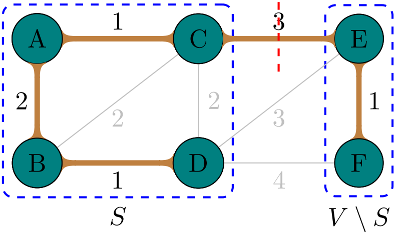

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Copy&Paste Argument; Exchange Argument

A Wrong Divide & Conquer Algorithm

Input: $G = (V, E, w)$

Divide: $V = (S, V \setminus S);\ ||S| - |V \setminus S|| \le 1$

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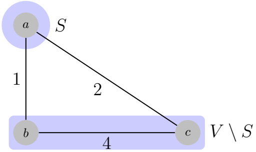

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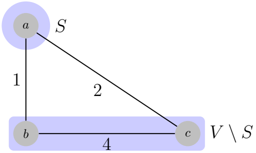
Combine: $T_1 + T_2 + \{e\}$: e is a lightest edge across $(S, V \setminus S)$



A Wrong Algorithm

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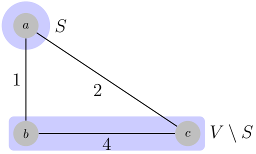

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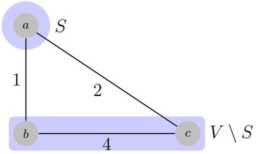

The edges bc and ad do **not** belong to any MST.

What if:

Invariant: Manages a set of edges X which is a subset of **some** MST.

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Overview: Grow the MST one edge at a time

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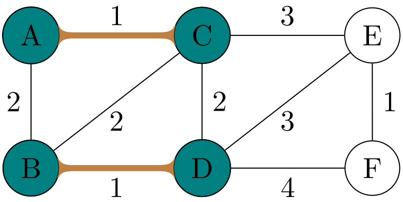
How to find a safe e for X in each iteration?

The Cut Property

Given that X is part of some MST T:

Then, $X + \{e\}$ is also a part of some MST T'.



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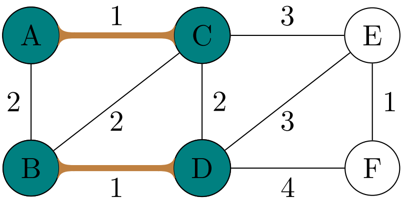

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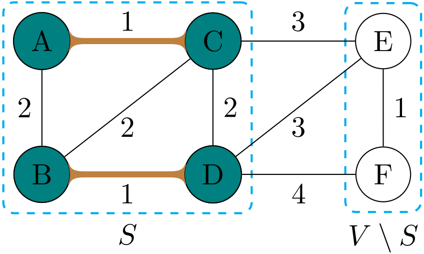

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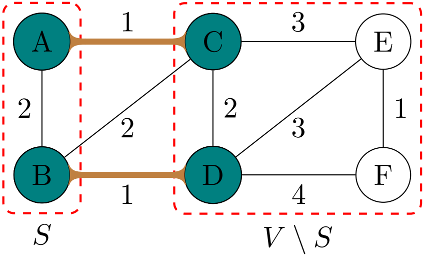

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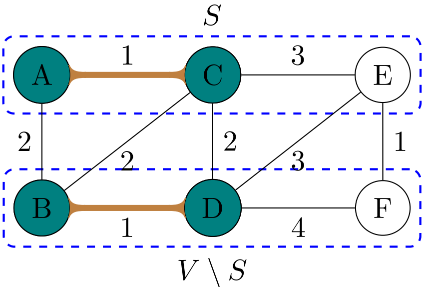

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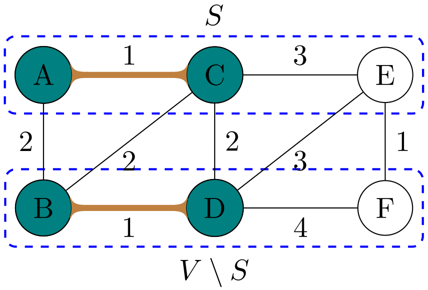

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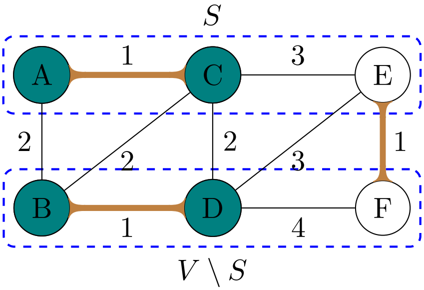

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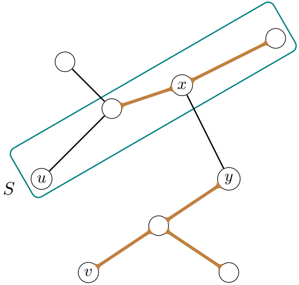

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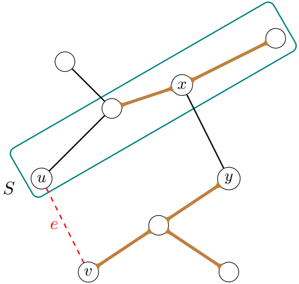

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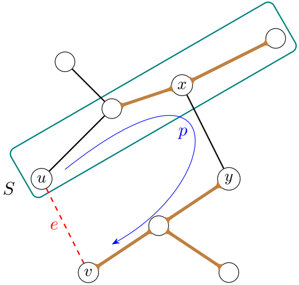

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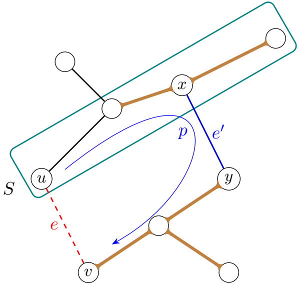

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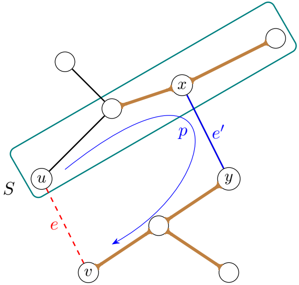

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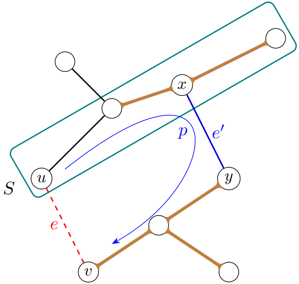

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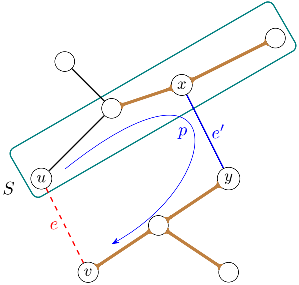

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- $w(T') \leq w(T) \Rightarrow T'$ is an MST
- $e' \notin X$(**respect**) $\Rightarrow X + \{e\} \subseteq T'$

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Kruskal's Algorithm

sort (non-descreasingly) the edges E

$X = \emptyset$
for $e \in E$ in non-descreasing order
  if $X \cup \{e\}$ does not produce cycle
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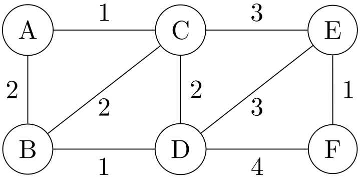

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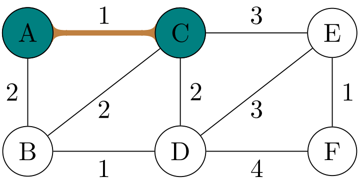

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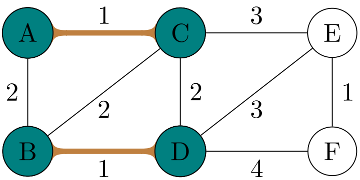

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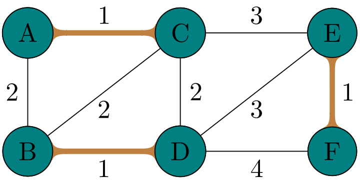

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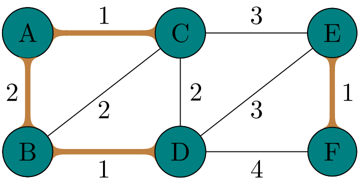

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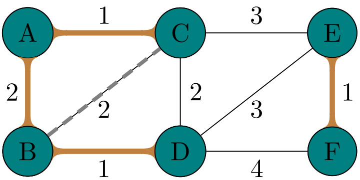

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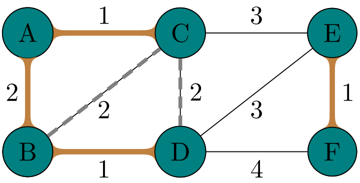

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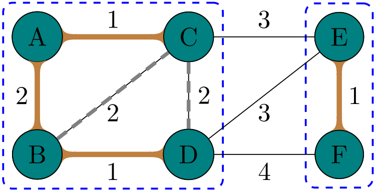

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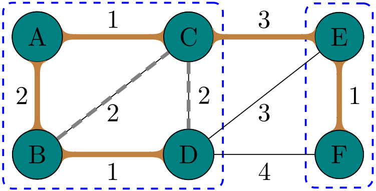

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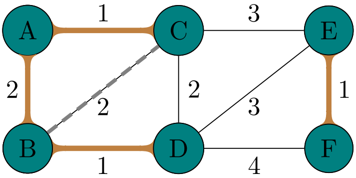

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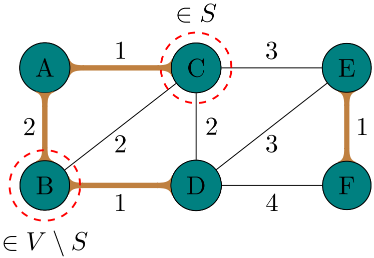

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State: forest $\triangleq$ a collection of connected components

Ops: on connected components

- cycle detection
- union two CCs

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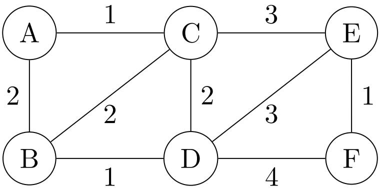
Using the **disjoint-set** data structure.

Prim's Algorithm

$X = \emptyset$
$S = \{s\}$ `// pick any` $s \in V$
$R = V \setminus S$
while $R \neq \emptyset$
  $e = (u, v) \leftarrow$ `a lightest edge across` (S, R)
  $X \leftarrow X \cup \{e\}$
  $S \leftarrow S \cup \{u\} \quad R \leftarrow R \setminus \{v\}$



Prim's Algorithm

$X = \emptyset$
$S = \{s\}$ `// pick any` $s \in V$
$R = V \setminus S$
while $R \neq \emptyset$
  $e = (u, v) \leftarrow$ `a lightest edge across` (S, R)
  $X \leftarrow X \cup \{e\}$
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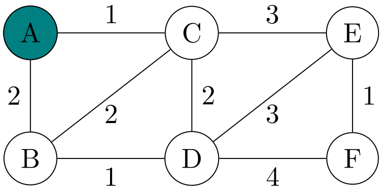
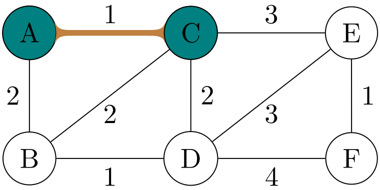


Prim's Algorithm

```
X = ∅
S = {s} // pick any s ∈ V
R = V \ S
while R ≠ ∅
   e = (u, v) ← a lightest edge across (S, R)
   X ← X ∪ {e}
   S ← S ∪ {u}   R ← R \ {v}
```



Prim's Algorithm

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$R = V \setminus S$
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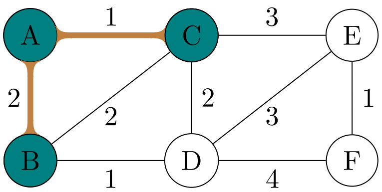

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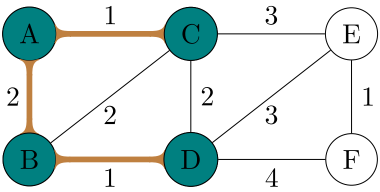

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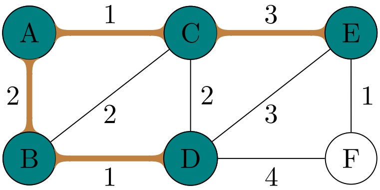

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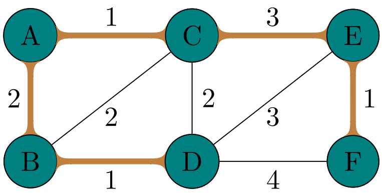

Prim's Algorithm

State: a growing tree (CC)

Op: identifying a lightest edge

Prim's Algorithm

State: a growing tree (CC)

Op: identifying a lightest edge

Using the **priority-queue (min-heap)** data structure.