Bubble Sort

(What are Algorithms and How to Analyze Algorithms)

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Sorting

The sorting problem:

Input: A sequence A of n integers $a_1\ a_2\ \cdots\ a_n$.

Output: A permutation $a'_1\ a'_2\ \ldots\ a'_n$ of A *s.t.* $a'_1 \leq a'_2 \leq \cdots \leq a'_n$ (non-decreasing order).

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$$3\ 1\ 4\ 2 \Longrightarrow 1\ 2\ 3\ 4$$

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- RAM (Random-Access Machine) model
- unrealistic: `sort` instruction
- realistic: arithmetic, data movement, and control
- CAS for sort: compare and swap if out-of-order

Inversions

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If $i < j$ and $a_i > a_j$, then (a_i, a_j) is an **inversion**.

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$$A \text{ has no adjacent inversions} \implies \forall i \in [1, n-1] : a_i \leq a_{i+1}$$
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$$\boxed{A \text{ is sorted} \iff A \text{ has no adjacent inversions.}}$$

Bubble Sort

Bubble Sort

Basic idea: to eliminate all adjacent inversions

```
1: procedure BubbleSortOverview(A : a_1 a_2 ··· a_n)
2:     repeat
3:         Pick any i
4:         if a_i > a_{i+1} then                    ▷ CAS
5:             Swap(a_i, a_{i+1})
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1: **procedure** BUBBLESORT($A : a_1\ a_2\ \cdots\ a_n$)
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3: [ ]
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Bubble Sort

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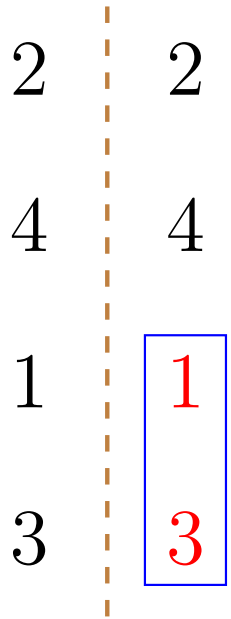

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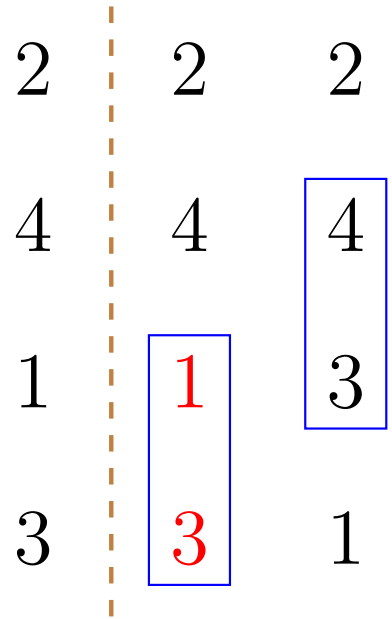

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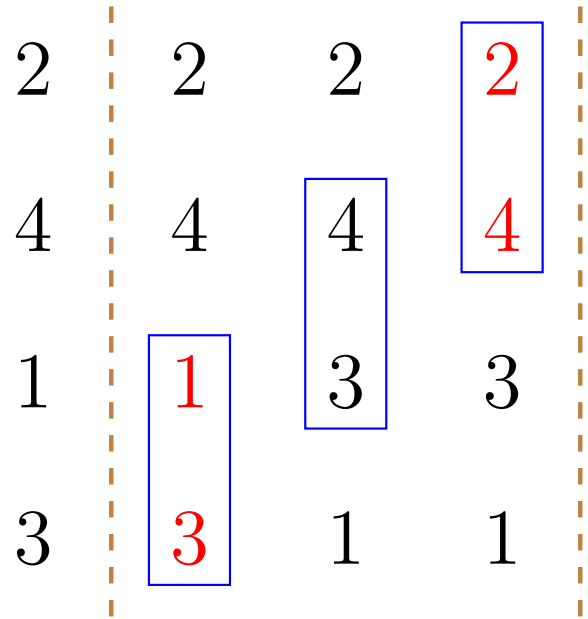

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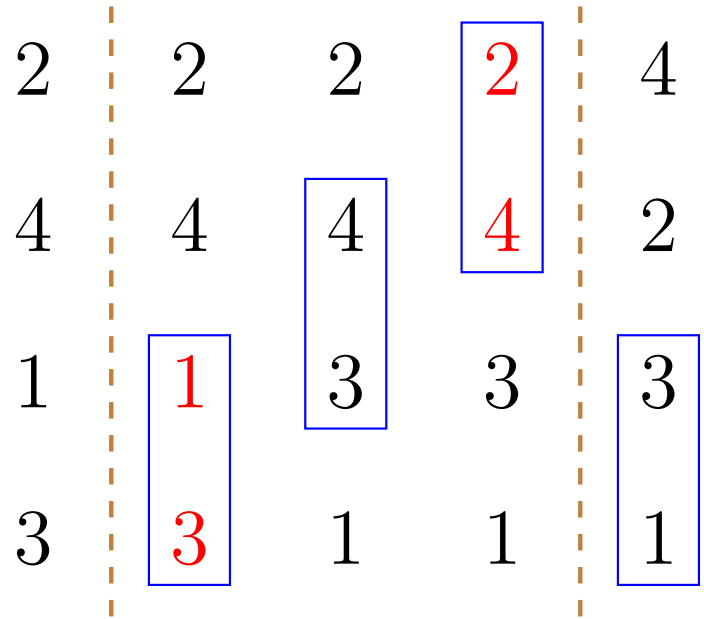

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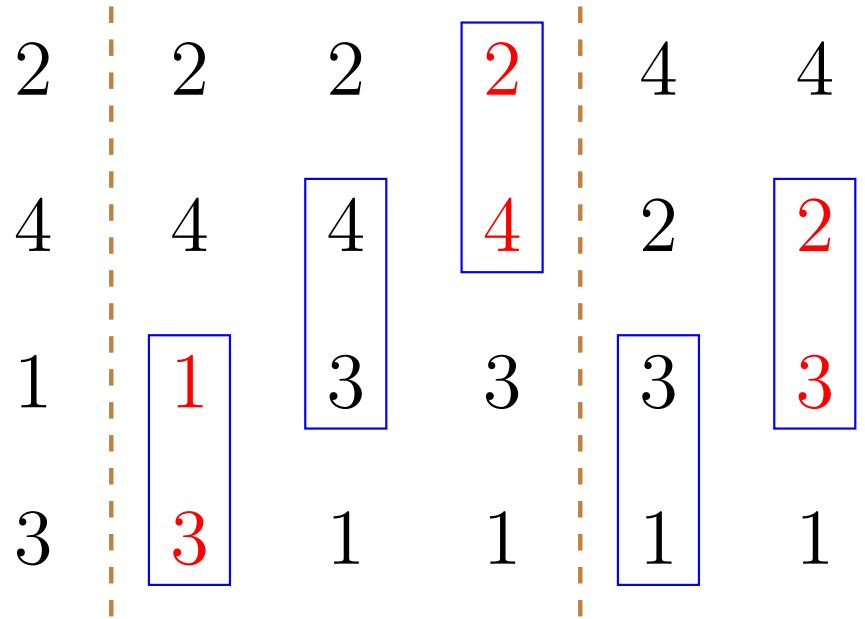

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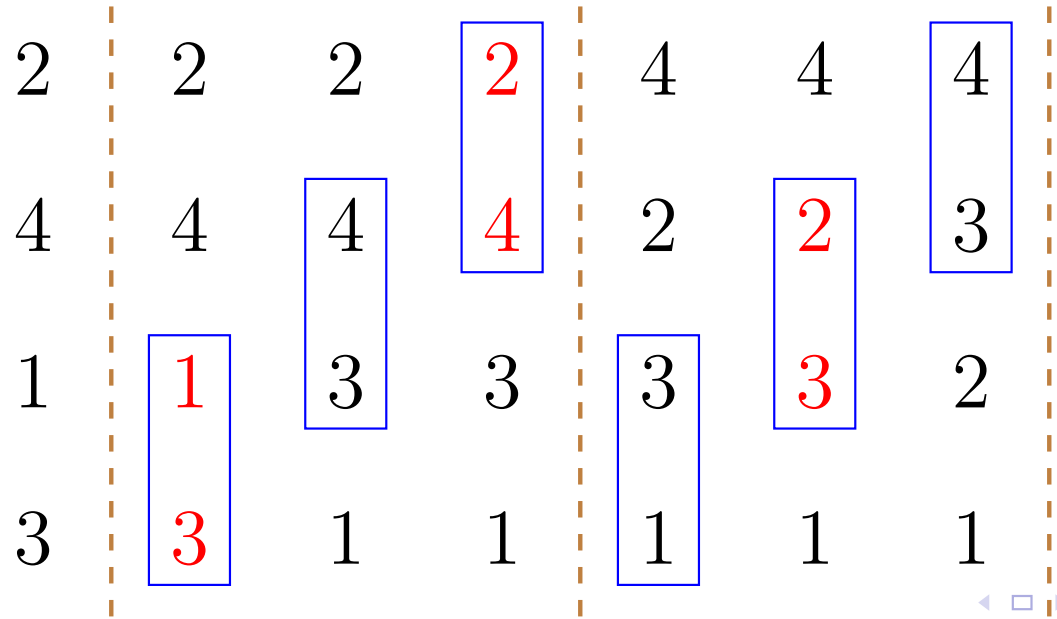

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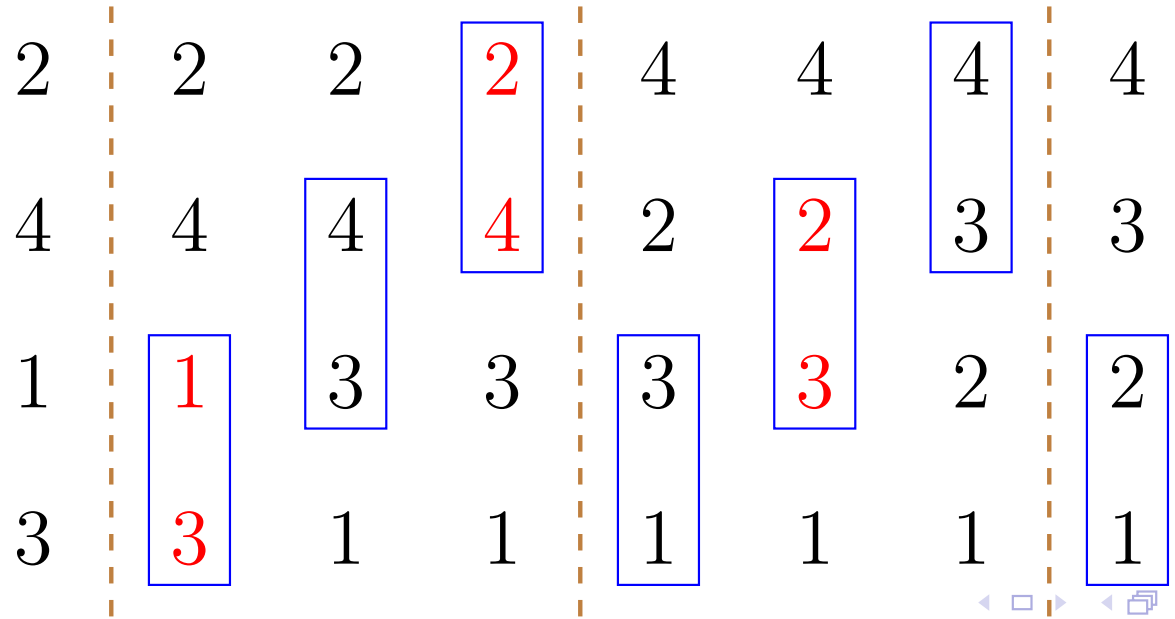

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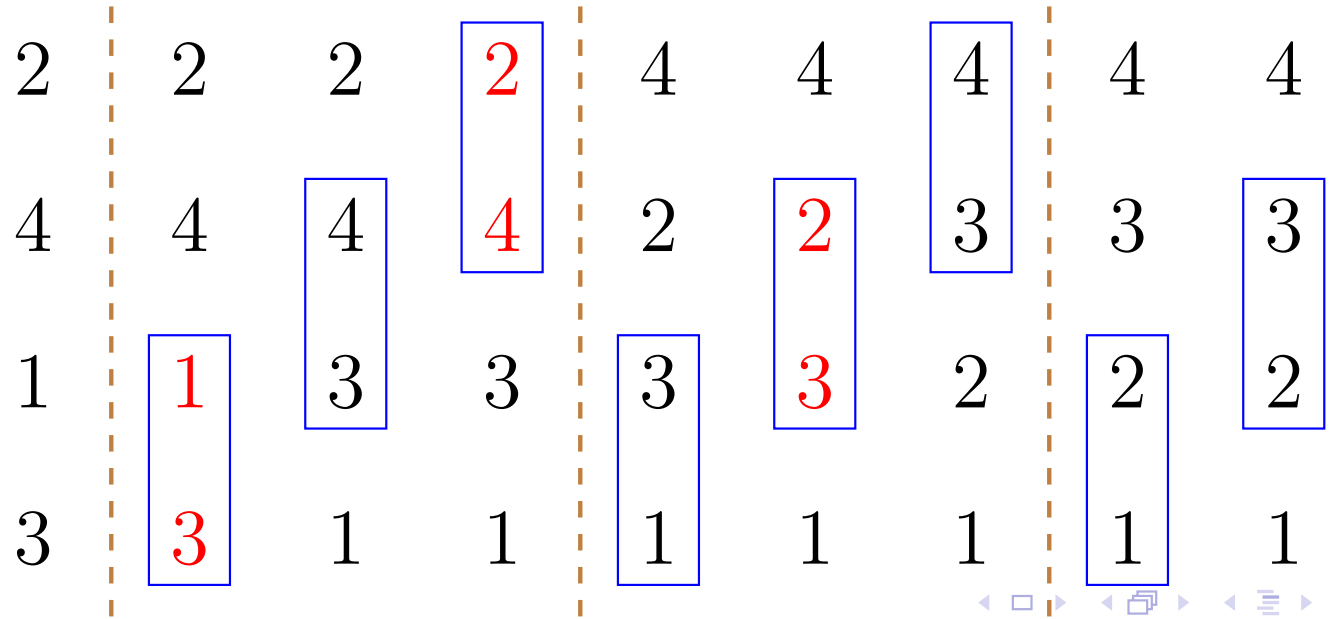

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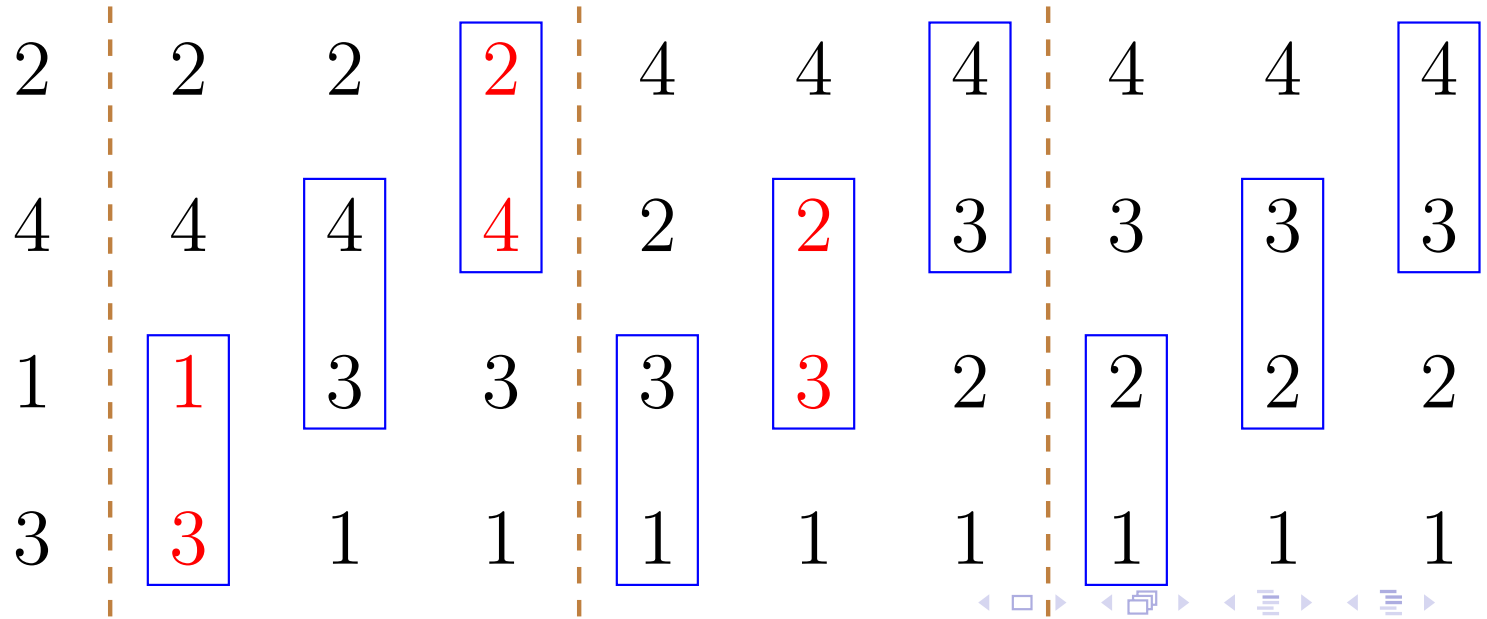

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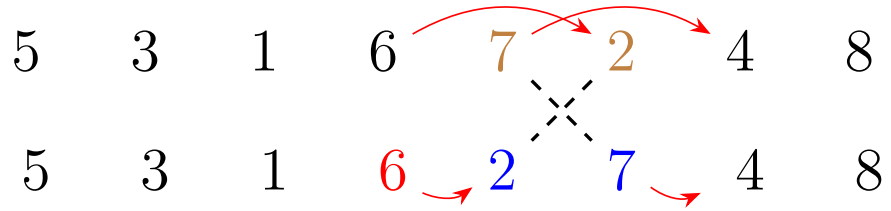
Bubble Sort

Effects of $\textsc{Swap}(a_i, a_{i+1})$ on adjacent inversions:



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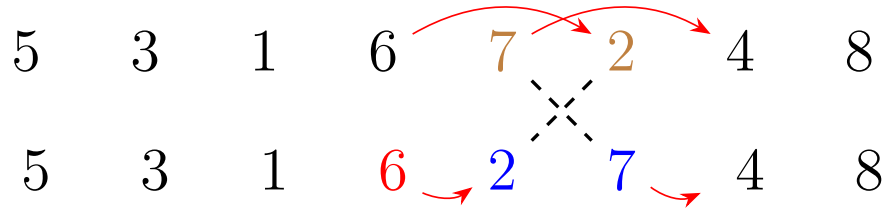

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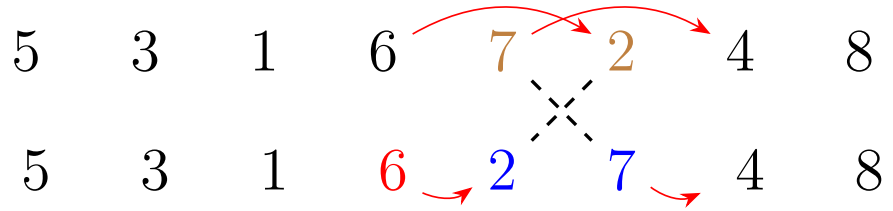

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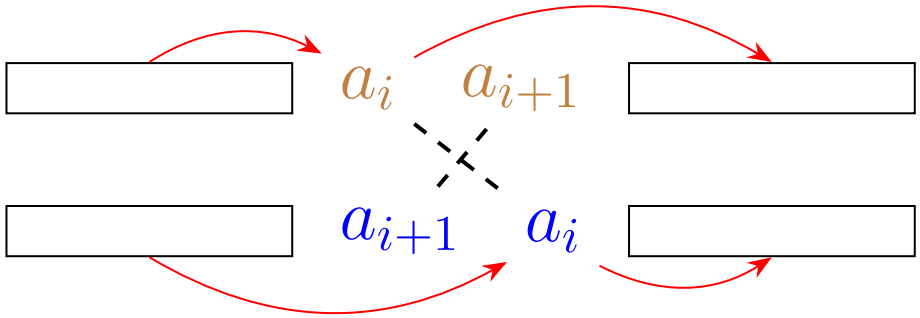

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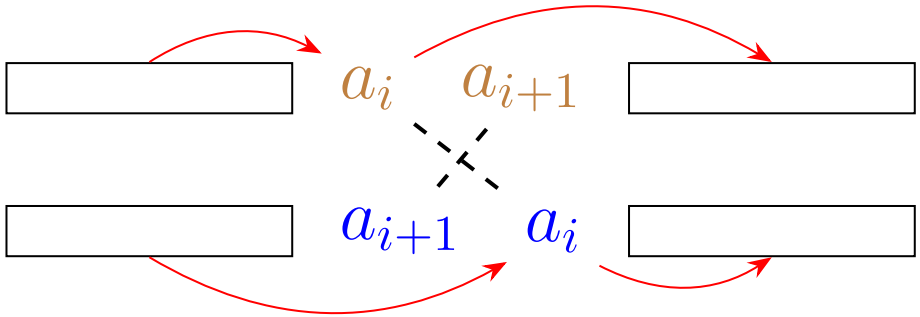

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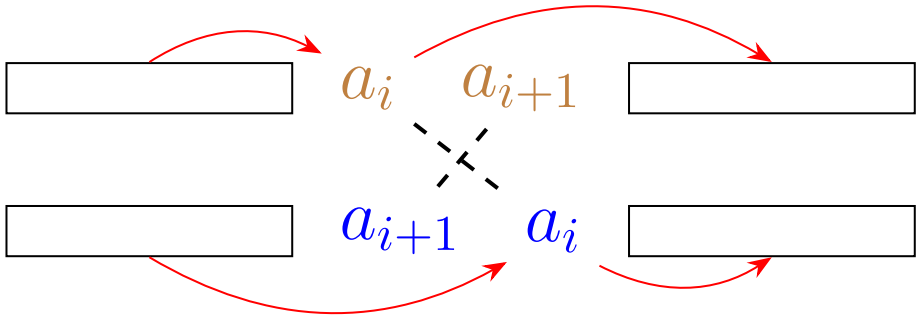

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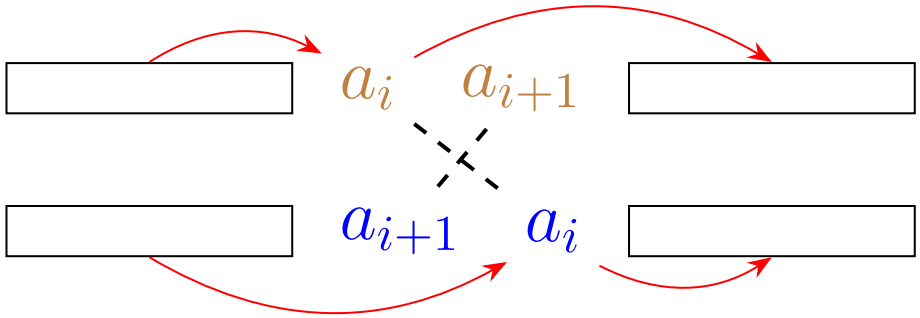

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Total #inversions is finite.

Optimizing Bubble Sort

```
1: procedure BubbleSort(A : a_1 a_2 ··· a_n)
2:     n ← len(A)
3:     repeat
4:         swapped ← false
5:         for i ← 1 : n − 1 do
6:             if a_i > a_{i+1} then
7:                 Swap(a_i, a_{i+1})
8:                 swapped ← true
9:         n ← n − 1                          ▷ One maximal bubbles up
10:    until swapped = false
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9:                 lsp ← i                  ▷ Update lsp
10:        n ← lsp                          ▷ Elements after lsp are sorted
11:    until swapped = false
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9:     until n = 0
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Time Complexity of Bubble Sort

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- Different inputs $\Longrightarrow$ different execution time:
  - Best-case, worst-case, and average-case analysis

Best-case and Worst-case Analysis

	Best-case:	Worst-case:
$\lvert P\rvert$	$=($	$);$
$\lvert C\rvert$	$=($	$);$
$\lvert S\rvert$	$=($	$).$

Best-case and Worst-case Analysis

Best-case: 1 2 3 4 5 6 7 8

	Best-case: ascendingly sorted	Worst-case:
$\lvert P\rvert$	$= ($	$);$
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Best-case and Worst-case Analysis

Best-case: 1 2 3 4 5 6 7 8

	Best-case: ascendingly sorted	Worst-case:
$\|P\|$	$= (\min: 1,$	$);$
$\|C\|$	$= (\min: n-1,$	$);$
$\|S\|$	$= (\min: 0,$	$).$

Best-case and Worst-case Analysis

Best-case: 1 2 3 4 5 6 7 8

	Best-case: ascendingly sorted	Worst-case: descendingly sorted
$\lvert P\rvert$	$= (\min: 1,$	$);$
$\lvert C\rvert$	$= (\min: n-1,$	$);$
$\lvert S\rvert$	$= (\min: 0,$	$).$

Worst-case: 8 7 6 5 4 3 2 1

Best-case and Worst-case Analysis

Best-case: 1 2 3 4 5 6 7 8

	Best-case: ascendingly sorted	Worst-case: descendingly sorted
$\|P\|$	$= (\min: 1,$	$\max: n);$
$\|C\|$	$= (\min: n-1,$	$\max: \frac{n^2-n}{2});$
$\|S\|$	$= (\min: 0,$	$\max: \frac{n^2-n}{2}).$

Worst-case: 8 7 6 5 4 3 2 1

$|S|$: #Swaps (Average Analysis)

Assumptions on inputs:

1. The input is a random permutation
2. All numbers are distinct

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Question: What is the expected #inversions?

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I_{ij} : indicator of inversion (a_i, a_j)

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$$E(X) = \sum_j \sum_{i<j} \frac{1}{2} = \binom{n}{2} \cdot \frac{1}{2} = \frac{n(n-1)}{4}$$