Bubble Sort

(A Taste of Algorithms: Definition, Design, and Analysis)

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Bubble Sort

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1. The Sorting Problem
2. Bubble Sort
3. Analysis of Bubble Sort

Algorithms

What is an algorithm?

Algorithms

What is an algorithm? What is computation?

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Correctness!

Algorithms

What is an algorithm? What is computation?

Input $\longrightarrow$ **Algorithm** (sequence of operations) $\longrightarrow$ Output

Correctness!

Definiteness: precisely defined operations

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What is an algorithm? What is computation?

Input → **Algorithm** (sequence of operations) → Output

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Definiteness: precisely defined operations

Finiteness: termination

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What is an algorithm? What is computation?

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Effectiveness: a reasonable model; basic operations

Algorithms

What is an algorithm? What is computation?

$$\text{Input} \longrightarrow \boxed{\begin{array}{c}\text{Algorithm}\\ \text{(sequence of operations)}\end{array}} \longrightarrow \text{Output}$$

Correctness!

Definiteness: precisely defined operations

Finiteness: termination

Effectiveness: a reasonable model; basic operations

- for sorting: compare, swap

Sorting

The sorting problem:

Input: A sequence of n integers A: $a_1\ a_2\ \cdots\ a_n$.

Output: A permutation $a'_1\ a'_2\ \ldots\ a'_n$ of A $s.t.$ $a'_1 \leq a'_2 \leq \cdots \leq a'_n$ (non-decreasing order).

$$3 \quad 1 \quad 4 \quad 2 \quad \Longrightarrow 1 \quad 2 \quad 3 \quad 4$$

Inversions

$$A = a_1 \quad a_2 \quad \ldots \quad a_n.$$

If $i < j$ and $a_i > a_j$, then (a_i, a_j) is an **inversion**.

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$$3 \quad 1 \quad 4 \quad 2$$

#inversions $= 3$
#adjacent inversions $= 2$

$$4 \quad 3 \quad 2 \quad 1$$

#inversions $= 3 + 2 + 1 = 6$
#adjacent inversions $= 3$

Inversions

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If $i < j$ and $a_i > a_j$, then (a_i, a_j) is an **inversion**.

Adjacent inversion: (a_i, a_{i+1})

3 1 4 2

#inversions = 3
#adjacent inversions = 2

4 3 2 1

#inversions = 3 + 2 + 1 = 6
#adjacent inversions = 3

1 2 3 4

#inversions = 0
#adjacent inversions = 0

Inversions

$\boxed{\text{Theorem: } A \text{ is sorted} \iff A \text{ has no adjacent inversions.}}$

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Inversions

$\boxed{\text{Theorem: } A \text{ is sorted } \iff A \text{ has no adjacent inversions.}}$

$$\begin{aligned} A \text{ is sorted } &\implies A \text{ has no inversions} \\ &\implies A \text{ has no adjacent inversions.} \end{aligned}$$

$$A \text{ has no adjacent inversions } \implies \forall i \in [1, n-1] : a_i \leq a_{i+1}$$

Inversions

Theorem: A is sorted $\iff$ A has no adjacent inversions.

$$\begin{aligned} A \text{ is sorted } &\implies A \text{ has no inversions} \\ &\implies A \text{ has no adjacent inversions.} \end{aligned}$$

$$\begin{aligned} A \text{ has no adjacent inversions } &\implies \forall i \in [1, n-1] : a_i \le a_{i+1} \\ &\implies A \text{ is sorted.} \end{aligned}$$

Bubble Sort

Bubble Sort: Basic Idea

Basic idea: to eliminate all adjacent inversions

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```
repeat
    pick any i
    if a_i > a_{i+1} then
        SWAP(a_i, a_{i+1})
until no adjacent inversions
```

Bubble Sort: Basic Idea

Basic idea: to eliminate all adjacent inversions

```
repeat
    pick any i                          ▷ Definiteness!
    if a_i > a_{i+1} then
        Swap(a_i, a_{i+1})
until no adjacent inversions            ▷ Definiteness!
```

Bubble Sort: Basic Idea

Basic idea: to eliminate all adjacent inversions

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repeat
    pick any i                                  ▷ Definiteness!
    if a_i > a_{i+1} then
        SWAP(a_i, a_{i+1})
until no adjacent inversions                    ▷ Finiteness! Definiteness!
```

Bubble Sort: Definiteness

procedure BUBBLESORT($A : a_1\ a_2\ \cdots\ a_n$)
  repeat
    $\boxed{}$
    for $\boxed{}$ **do** ▷ Pick i
      if $a_i > a_{i+1}$ **then**
        SWAP(a_i, a_{i+1})
        $\boxed{}$
  until $\boxed{\text{no adjacent inversions}}$

Bubble Sort: Definiteness

```
procedure BubbleSort(A : a_1 a_2 ··· a_n)
    repeat
        [                  ]
        for [i ← 1 : n − 1] do                        ▷ Pick i
            if a_i > a_{i+1} then
                Swap(a_i, a_{i+1})
                [                  ]
    until [no adjacent inversions]
```

Bubble Sort: Definiteness

procedure BUBBLESORT($A : a_1\ a_2\ \cdots\ a_n$)
  repeat
    $\boxed{\text{swapped}}$
    for $\boxed{i \leftarrow 1 : n-1}$ **do** ▷ Pick i
      if $a_i > a_{i+1}$ **then**
        SWAP(a_i, a_{i+1})
        $\boxed{}$
  until $\boxed{\text{no adjacent inversions}}$

Bubble Sort: Definiteness

---

procedure BUBBLESORT($A : a_1\ a_2\ \cdots\ a_n$)
    repeat
        $\boxed{\text{swapped} \leftarrow \text{false}}$
        for $\boxed{i \leftarrow 1 : n - 1}$ **do** ▷ Pick i
            if $a_i > a_{i+1}$ **then**
                SWAP(a_i, a_{i+1})
                $\boxed{\phantom{\text{swapped} \leftarrow \text{true}}}$
    until $\boxed{\text{no adjacent inversions}}$

Bubble Sort: Definiteness

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procedure BubbleSort(A : a_1 a_2 ··· a_n)
    repeat
        [swapped ← false]
        for [i ← 1 : n − 1] do                     ▷ Pick i
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                [swapped ← true]
    until [no adjacent inversions]
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Bubble Sort: Definiteness

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procedure BubbleSort(A : a_1 a_2 ··· a_n)
    repeat
        [swapped ← false]
        for [i ← 1 : n − 1] do                    ▷ Pick i
            if a_i > a_{i+1} then
                Swap(a_i, a_{i+1})
                [swapped ← true]
    until [swapped = false]
```

Bubble Sort: Example

2

4

1

3

Bubble Sort: Example

2	2
4	4
1	1
3	3

Bubble Sort: Example

2	2	2
4	4	4
1	1	3
3	3	1

Bubble Sort: Example

2	2	2	2
4	4	4	4
1	1	3	3
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2	2	2	2	4
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Bubble Sort: Example

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Bubbled Up

Bubble Sort: Example

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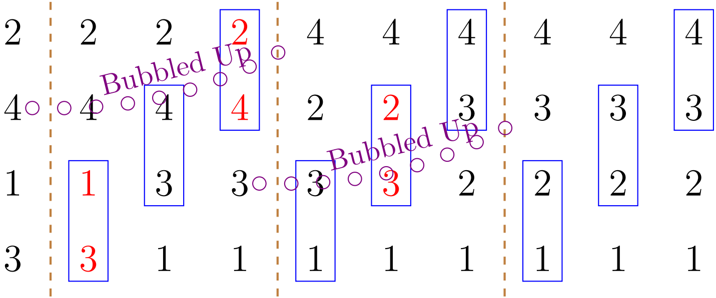
Bubbled Up

Bubbled Up

Bubble Sort: Example



After each "**for**" loop, one more element is bubbled up to its final position.

Bubble Sort: Finiteness

```
procedure BubbleSort(A : a_1 a_2 ··· a_n)
    repeat
        swapped ← false
        for i ← 1 : n − 1 do                    ▷ Pick i
            if a_i > a_{i+1} then
                Swap(a_i, a_{i+1})
                swapped ← true
    until swapped = false                        ▷ No swaps
```

The inner "**for**" loops:

1) $\exists$ loop : no swaps $\implies$ swapped $=$ false $\implies$ terminates

Bubble Sort: Finiteness

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procedure BubbleSort(A : a_1 a_2 ··· a_n)
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                swapped ← true
    until swapped = false               ▷ No swaps
```

The inner "**for**" loops:

1) $\exists$ loop : no swaps $\Longrightarrow$ swapped = false $\Longrightarrow$ terminates

2) $\forall$ loop : has swaps **Impossible!**

Bubble Sort: Finiteness

$\boxed{\text{Fact: total \#inversions is finite.}}$

[1] Not on #adjacent inversions! Think about it.

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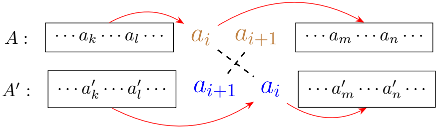
Effects of $\textsc{Swap}(a_i, a_{i+1})$ on #inversions[1]:



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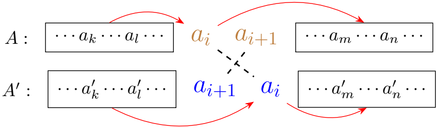

$$-1 : (a_i, a_{i+1})$$

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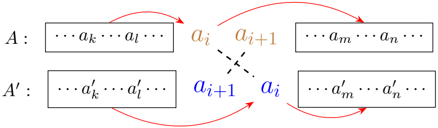

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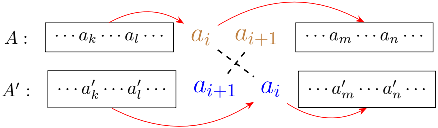

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$$(a_k, a_l), (a_m, a_n), (a_k, a_i), (a_i, a_m)$$

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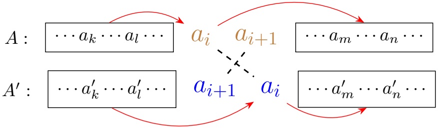

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Bubble Sort: Correctness

Finiteness

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$$\textbf{Finiteness} \implies \exists \text{ loop} : \text{no swaps}$$

Bubble Sort: Correctness

$$
\begin{aligned}
\textbf{Finiteness} \quad &\Longrightarrow \exists \text{ loop} : \text{ no swaps} \\
&\Longrightarrow A \text{ has no adjacent inversions any more}
\end{aligned}
$$

Bubble Sort: Correctness

$$
\textbf{Finiteness} \implies \exists \text{ loop} : \text{no swaps}
$$
$$
\implies A \text{ has no adjacent inversions any more}
$$
$$
\implies A \text{ is already sorted.}
$$

Optimizing Bubble Sort (I)[2]

After each "**for**" loop, one more element is bubbled up to its final position.

procedure BUBBLESORT($A : a_1\ a_2\ \cdots\ a_n$)
    $n \leftarrow \text{len}(A)$
    repeat
        swapped $\leftarrow$ false
        for $i \leftarrow 1 : n - 1$ **do**
            if $a_i > a_{i+1}$ **then**
                SWAP(a_i, a_{i+1})
                swapped $\leftarrow$ true
        $n \leftarrow n - 1$ ▷ One maximal bubbles up
    until swapped = false

[2] See Appendix for "Optimizing Bubble Sort (II)".

Bubble Sort

Time Complexity of Bubble Sort

- Finiteness is NOT enough $\Longrightarrow$ Quantitative finiteness

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$$|C| : \text{\#Comparisons} \qquad (\textbf{if } a_i > a_{i+1})$$

$$|S| : \text{\#Swaps} \qquad (\textsc{Swap}(a_i, a_{i+1}))$$

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- Different inputs $\implies$ $|C|$ and $|S|$ vary:

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- Different inputs $\implies$ $|C|$ and $|S|$ vary:
  - Best-case, worst-case, and average-case analysis

Best-case and Worst-case Analysis

Best-case: Worst-case:

$$
\begin{aligned}
|C| &= (\qquad\qquad\qquad\qquad);\\
|S| &= (\qquad\qquad\qquad\qquad).
\end{aligned}
$$

Best-case and Worst-case Analysis

Best-case: $1 \quad 2 \cdots n$

$$|C| = \Big(\underbrace{\qquad}_{\text{Best-case: non-decreasingly sorted}} \qquad \underbrace{\qquad}_{\text{Worst-case:}} \Big);$$

$$|S| = \Big(\qquad\qquad \Big).$$

Best-case and Worst-case Analysis

Best-case: $1 \quad 2 \cdots n$

$$
\begin{aligned}
& \qquad \overline{\text{Best-case:}} \qquad\qquad \overline{\text{Worst-case:}} \\
& \qquad \text{non-decreasingly sorted} \\
|C| &= (\min : n-1, \qquad\qquad\qquad);\\
|S| &= (\min : 0, \qquad\qquad\qquad).
\end{aligned}
$$

Best-case and Worst-case Analysis

Best-case: $1 \quad 2 \cdots n$

$$
\begin{aligned}
|C| &= (\,\min : n-1, \qquad\qquad\qquad\qquad);\\
|S| &= (\,\min : 0, \qquad\qquad\qquad\qquad\quad).
\end{aligned}
$$

$\frac{\text{Best-case:}}{\text{non-decreasingly sorted}}$ $\frac{\text{Worst-case:}}{\text{non-increasingly sorted}}$

Worst-case: $n \quad n-1 \cdots 1$

Best-case and Worst-case Analysis

Best-case: $1 \quad 2 \cdots n$

$$\begin{aligned}
& \quad \underline{\text{Best-case:}} \qquad\qquad \underline{\text{Worst-case:}} \\
& \text{non-decreasingly sorted} \quad \text{non-increasingly sorted} \\
|C| &= (\min : n-1, \qquad \max : \tfrac{n^2-n}{2}); \\
|S| &= (\min : 0, \qquad \max : \tfrac{n^2-n}{2}).
\end{aligned}$$

Worst-case: $n \quad n-1 \cdots 1$

$$\#\text{inversions} = (n-1) + (n-2) + \cdots + 1 = \frac{n^2-n}{2}.$$

$|S|$: #Swaps (Average-case Analysis)[3]

Assumptions on inputs:

1. The input is a random permutation ("average input")

[3] An exercise: what is $|C|$ (#Comparisions) in average?

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$$|S| = \mathbb{E}(\#\text{inversions})$$

[3] An exercise: what is $|C|$ (#Comparisions) in average?

$|S|$: #Swaps (Average Analysis)

$$I_{ij} = \begin{cases} 1 & (a_i, a_j) \text{ is an inversion} \\ 0 & \text{o.w.} \end{cases}$$

$$X = \sum_{1 \le i < n} \sum_{i < j \le n} I_{ij} \qquad \text{(\#inversions)}$$

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$$\mathbb{E}(I_{ij}) = \mathbb{P}\{I_{ij} = 1\} = \frac{1}{2} \qquad (a_i \neq a_j; \text{half: } a_i < a_j, \text{half: } a_i > a_j)$$

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$$\mathbb{E}(X) = \sum_i \sum_{i<j} \frac{1}{2} = \binom{n}{2} \cdot \frac{1}{2} = \frac{n(n-1)}{4}$$

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$$X = \sum_{1 \le i < n} \sum_{i < j \le n} I_{ij} \qquad (\#\text{inversions})$$

$$\mathbb{E}(X) = \mathbb{E}(\sum_i \sum_{j>i} I_{ij}) = \sum_i \sum_{j>i} \mathbb{E}(I_{ij}) \quad (\text{linearity of expectation})$$

$$\mathbb{E}(I_{ij}) = \mathbb{P}\{I_{ij} = 1\} = \frac{1}{2} \qquad (a_i \neq a_j; \text{half: } a_i < a_j, \text{half: } a_i > a_j)$$

$$\mathbb{E}(X) = \sum_i \sum_{i<j} \frac{1}{2} = \binom{n}{2} \cdot \frac{1}{2} = \frac{n(n-1)}{4} = O(n^2)$$

Faster Algorithms

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Bubble Sort: Correctness

```
procedure BubbleSort(A : a_1 a_2 ··· a_n)
    repeat
        swapped ← false
        for i ← 1 : n − 1 do                ▷ Loop invariant?
            if a_i > a_{i+1} then           ▷ CAS
                Swap(a_i, a_{i+1})
                swapped ← true
    until swapped = false
```

Loop invariant:

Before the k-th ($k \geq 1$) "**for**" loop, $a_{n-(k-1)} \cdots a_n$
(1) consists of the largest $(k-1)$ elements
(2) in sorted order.

Correctness: Initialization + Maintenance + Termination

Bubble Sort: Finiteness

Idea: well-founded relation over $\mathbf{N}$

Effects of $\textsc{Swap}(a_i, a_{i+1})$ on adjacent inversions:

5 3 1 6 7 2 4 8

5 3 1 6 2 7 4 8

Bubble Sort: Finiteness

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$$
\begin{array}{cccccccc}
5 & 3 & 1 & 6 & 7 & 2 & 4 & 8 \\
5 & 3 & 1 & 6 & 2 & 7 & 4 & 8
\end{array}
$$

$-1 : (7, 2)$

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$$\begin{array}{cccccccc} 5 & 3 & 1 & 6 & 7 & 2 & 4 & 8 \\ 5 & 3 & 1 & 6 & 2 & 7 & 4 & 8 \end{array}$$

$-1 : (7, 2)$ $+2 : (6, 2), (7, 4)$

Optimizing Bubble Sort (II)

Idea: After each "**for**" loop, all elements after "lsp" are settled.

```
procedure BubbleSort(A : a_1 a_2 ··· a_n)
    repeat
        swapped ← false
        lsp ← 0                                  ▷ lsp: the last swap position
        for i ← 1 : n − 1 do
            if a_i > a_{i+1} then
                Swap(a_i, a_{i+1})
                swapped ← true
                lsp ← i                          ▷ Update lsp
        n ← lsp                                  ▷ Elements after lsp are sorted
    until swapped = false
```

Optimizing Bubble Sort (II)

Idea: After each "**for**" loop, all elements after "lsp" are settled.

procedure BubbleSort($A : a_1 \ a_2 \ \cdots \ a_n$)
    repeat
        swapped $\leftarrow$ false
        lsp $\leftarrow 0$ ▷ lsp: the last swap position
        for $i \leftarrow 1 : n - 1$ **do**
            if $a_i > a_{i+1}$ **then**
                Swap(a_i, a_{i+1})
                swapped $\leftarrow$ true
                lsp $\leftarrow$ i ▷ Update lsp
        $n \leftarrow$ lsp ▷ Elements after lsp are sorted
    until swapped $=$ false

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                Swap(a_i, a_{i+1})
                lsp $\leftarrow$ i ▷ Update lsp
        $n \leftarrow$ lsp ▷ Elements after lsp are sorted
    until lsp $= 0$