Discrete mathematics course
Chapter 4: Boolean algebra

Anh Tuan GIANG

ICTLab, ICT Department
University of Science and Technology of Hanoi-USTH

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A short introduction

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  - Complement: $\bar{0} = 1, \bar{1} = 0$
  - Or (sum): $0 + 1 = 1$
  - And (product): $1.0 = 0$

Examples

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- Example 2: translate $1.0 + \overline{(0+1)} = 0$, the equality found in Example 1, into a logical equivalence.
- Example 3: translate the logical equivalence $(T \wedge T) \vee \neq F \equiv T$ into an identity in Boolean algebra.

Definition

Definition 1.1.

Let $B = 0, 1$. Then $B_n = (x1, x2, ..., xn)|x_i \in Bfor 1 \leq i \leq n$ is the set of all possible n-tuples of 0s and 1s. The variable x is called a Boolean variable if it assumes values only from B, that is, if its only possible values are 0 and 1. A function from B_n to B is called a Boolean function of degree n.

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- Representing the second example in graphical illustration by using n-cube.
- How many different Boolean functions of degree n are there?

Boolean algebra identities

TABLE 5 **Boolean Identities.**

Identity	*Name*
$\overline{\overline{x}} = x$	Law of the double complement
$x + x = x$ $x \cdot x = x$	Idempotent laws
$x + 0 = x$ $x \cdot 1 = x$	Identity laws
$x + 1 = 1$ $x \cdot 0 = 0$	Domination laws
$x + y = y + x$ $xy = yx$	Commutative laws
$x + (y + z) = (x + y) + z$ $x(yz) = (xy)z$	Associative laws
$x + yz = (x + y)(x + z)$ $x(y + z) = xy + xz$	Distributive laws
$\overline{(xy)} = \overline{x} + \overline{y}$ $\overline{(x + y)} = \overline{x}\,\overline{y}$	De Morgan's laws
$x + xy = x$ $x(x + y) = x$	Absorption laws
$x + \overline{x} = 1$	Unit property
$x\overline{x} = 0$	Zero property

Boolean algebra identities

- Show that the distributive law $x(y+z) = xy + xz$ is valid.

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- Translate the distributive law $x + yz = (x+y)(x+z)$ into a logical equivalence.
- Prove the absorption law $x(x+y) = x$.

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The **dual** of a Boolean expression is obtained by interchanging Boolean sums and Boolean products and interchanging 0s and 1s.

- Find the duals of $x(y+0)$ and $\bar{x}.1+(\bar{y}+z)$.
- Construct an identity from the absorption law $x(x+y)=x$ by taking duals.

The Abstract Definition of a Boolean Algebra

Definition 1.3.

A Boolean algebra is a set B with two binary operations $\vee$ and $\wedge$, elements 0 and 1, and an unary operation $^{-}$such that these properties hold for all $x, y, andz$ in B

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$\left.\begin{array}{l} x \vee 0 = x \\ x \wedge 1 = x \end{array}\right\}$	Identity laws
$\left.\begin{array}{l} x \vee \overline{x} = 1 \\ x \wedge \overline{x} = 0 \end{array}\right\}$	Complement laws
$\left.\begin{array}{l} (x \vee y) \vee z = x \vee (y \vee z) \\ (x \wedge y) \wedge z = x \wedge (y \wedge z) \end{array}\right\}$	Associative laws
$\left.\begin{array}{l} x \vee y = y \vee x \\ x \wedge y = y \wedge x \end{array}\right\}$	Commutative laws
$\left.\begin{array}{l} x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \\ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \end{array}\right\}$	Distributive laws

Sum-of-products expansions

Example: find Boolean expressions that represent the functions $F(x, y, z)$ and $G(x, y, z)$, which are given in Table below.

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TABLE 1

x	y	z	F	G
1	1	1	0	0
1	1	0	0	1
1	0	1	1	0
1	0	0	0	0
0	1	1	0	0
0	1	0	0	1
0	0	1	0	0
0	0	0	0	0

Sum-of-products expansions

Definition 2.1.

A literal is a Boolean variable or its complement. A minterm of the Boolean variables $x_1, x_2, ..., x_n$ is a Boolean product $y_1 y_2 ... y_n$, where $y_i = x_i$ or $y_i = \bar{x}_i$. Hence, a minterm is a product of n literals, with one literal for each variable.

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Example: Find a minterm that equals 1 if $x_1 = x_3 = 0$ and $x_2 = x_4 = x_5 = 1$, and equals 0 otherwise.

Sum-of-products expansions

Example: Find the sum-of-products expansion for the function $F(x, y, z) = (x + y)z$.

Sum-of-products expansions

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TABLE 2

x	y	z	$x+y$	$\bar{z}$	$(x+y)\bar{z}$
1	1	1	1	0	0
1	1	0	1	1	1
1	0	1	1	0	0
1	0	0	1	1	1
0	1	1	1	0	0
0	1	0	1	1	1
0	0	1	0	0	0
0	0	0	0	1	0

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  - $xy = \overline{\bar{x} + \bar{y}}$
- NAND, NOR

Basic types of logic gates

- Inverter

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- AND

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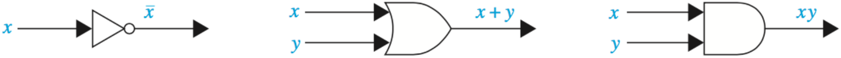

(a) Inverter (b) OR gate (c) AND gate

FIGURE 1 **Basic Types of Gates.**

Basic types of logic gates

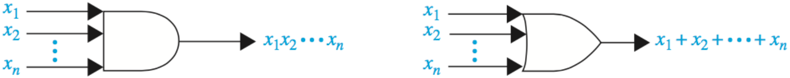

FIGURE 2 **Gates with *n* Inputs.**

Examples

Example: construct circuit that produces the output $xy + \bar{x}y$.

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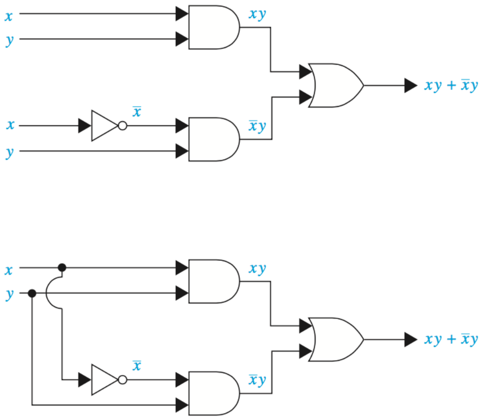

FIGURE 3 **Two Ways to Draw the Same Circuit.**

Examples

Example: construct circuit that produces the output:.

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- $(x + y)\bar{x}$

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Example: construct circuit that produces the output:.

- $(x + y)\bar{x}$
- $\bar{x}\overline{(y + \bar{z})}$
- $(x + y + z)(\bar{x}\bar{y}\bar{z})$

Examples

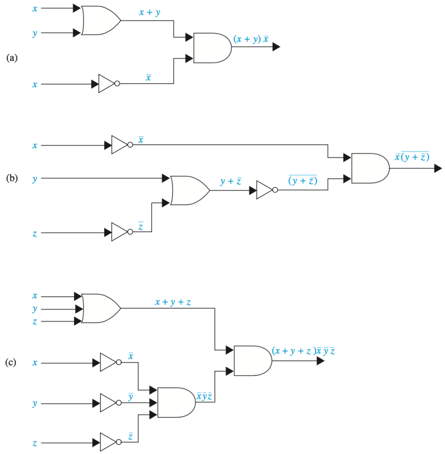

FIGURE 4 **Circuits that Produce the Outputs Specified in Example 1.**

Examples

- A committee of three individuals decides issues for an organization. Each individual votes either yes or no for each proposal that arises. A proposal is passed if it receives at least two yes votes. Design a circuit that determines whether a proposal passes.

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- Sometimes light fixtures are controlled by more than one switch. Circuits need to be designed so that flipping any one of the switches for the fixture turns the light on when it is off and turns the light off when it is on. Design circuits that accomplish this when there are two switches and when there are three switches.

Introduction

Question: showing the output of the two circuits below

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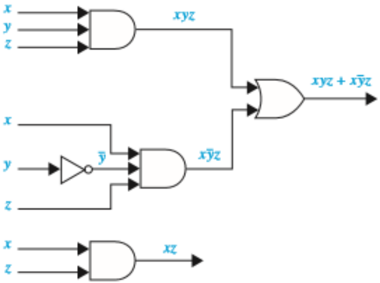

FIGURE 1 Two Circuits with the Same Output.

Karnaugh Maps

Definition 4.1.

We circle blocks of cells in the K-map that represent minterms that can be combined and then find the corresponding sum-of-products. The goal is to identify the largest possible blocks, and to cover all the 1s with the fewest blocks using the largest blocks first and always using the largest possible blocks.

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	y	$\bar{y}$
x	xy	$x\bar{y}$
$\bar{x}$	$\bar{x}y$	$\bar{x}\bar{y}$

FIGURE 2
K-maps in Two Variables.

Karnaugh Maps examples

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- $xy + \bar{x}y$
- $x\bar{y} + \bar{x}y$
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Now, using K-maps to simplify the sum-of-products expansions given in above Example.