Discrete mathematics course
Chapter 2: Graph theory

Anh Tuan GIANG

ICTLab, ICT Department
University of Science and Technology of Hanoi-USTH

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Definition

Definition 1.1.

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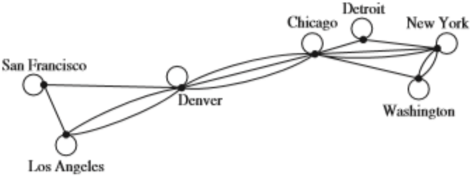

FIGURE 3 **A Computer Network with Diagnostic Links.**

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A directed graph (or digraph) (V, E) consists of a nonempty set of vertices V and a set of directed edges(or arcs) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.

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FIGURE 4 A Communications Network with One-Way Communications Links.

Influence Graphs

- In studies of group behavior it is observed that certain people can influence the thinking of others. A directed graph called an influence graph can be used to model this behavior. Each person of the group is represented by a vertex. There is a directed edge from vertex a to vertex b when the person represented by vertex a can influence the person represented by vertex b. This graph does not contain loops and it does not contain multiple directed edges.

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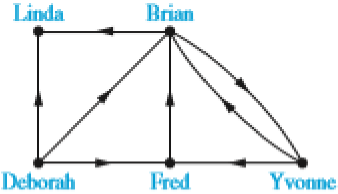

FIGURE 7 An Influence Graph.

Call Graphs

- Graphs can be used to model telephone calls made in a network, such as a long distance telephone network. In particular, a directed multigraph can be used to model calls where each telephone number is represented by a vertex and each telephone call is represented by a directed edge. The edge representing a call starts at the telephone number from which the call was made and ends at the telephone number to which the call was made.

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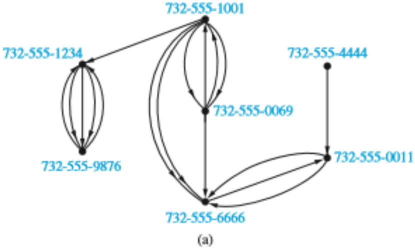

FIGURE 8 **A Call Graph.**

Module Dependency Graphs

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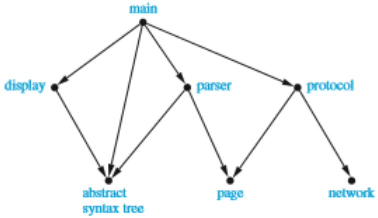

FIGURE 9 A Module Dependency Graph.

Precedence Graphs

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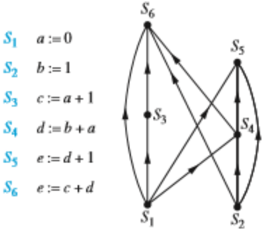

FIGURE 10 **A Precedence Graph.**

Single-elimination tournaments

- A tournament where each contestant is eliminated after one loss is called a single-elimination tournament.

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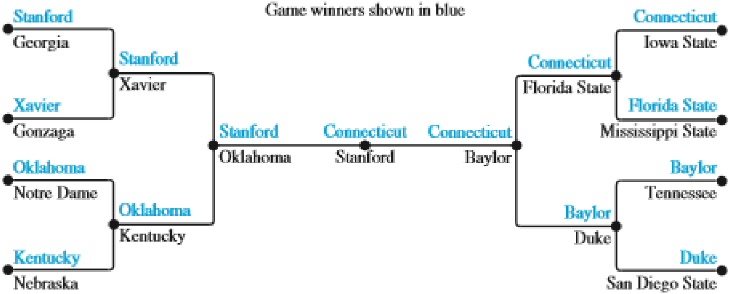

FIGURE 14 A Single-Elimination Tournament.

Basic terminology

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Two vertices u and v in an undirected graph G are called adjacent (or neighbors) in G if u and v are endpoints of an edge e of G. Such an edge e is called incident with the vertices u and v and e is said to connect u and v.

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Definition 2.2.

The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the neighborhood of v. If A is a subset of V, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A. So, $N(A) = \cup_{v \in A} N(v)$.

Basic terminology: continue ...

Definition 2.3.

The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by $deg(v)$.

Basic terminology: continue ...

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- What are the degrees and what are the neighborhoods of the vertices in the graphs G and H displayed in Figure below?

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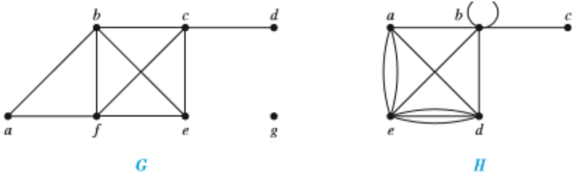

FIGURE 1 The Undirected Graphs G and H.

Basic terminology: continue ...

Theorem 2.4 (Handshaking theorem).

Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} deg(v).$$

(Note that this applies even if multiple edges and loops are present.)

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Question: how many edges are there in a graph with 10 vertices each of degree six?

Basic terminology: continue ...

Theorem 2.5.

An undirected graph has an even number of vertices of odd degree.

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Proof.

Let V_1 and V_2 be the set of vertices of even degree and the set of vertices of odd degree, respectively, in an undirected graph $G = (V, E)$ with m edges. Then

$$2m = \sum_{v \in V} deg(v) = \sum_{v \in V_1} deg(v) + \sum_{v \in V_2} deg(v)$$

□

Basic terminology: continue ...

Definition 2.6.

When (u, v) is an edge of the graph G with directed edges, u is said to be adjacent to v and v is said to be adjacent from u. The vertex u is called the initial vertex of (u, v), and v is called the terminal or end vertex of (u, v). The initial vertex and terminal vertex of a loop are the same.

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Definition 2.7.

In a graph with directed edges the in-degree of a vertex v, denoted by $deg^{-}(v)$, is the number of edges with v as their terminal vertex. The out-degree of v, denoted by $deg^{+}(v)$, is the number of edges with v as their initial vertex.

Basic terminology: continue ...

Question: find the in-degree and out-degree of each vertex in the graph G below

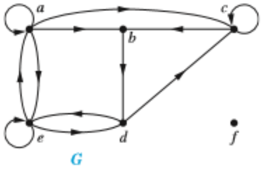

FIGURE 2 The Directed Graph G.

Basic terminology: continue ...

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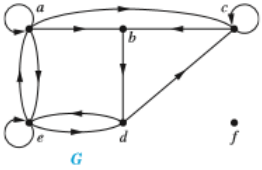

FIGURE 2 The Directed Graph G.

Theorem 2.8.

Let $G = (V, E)$ ne a graph with directed edges. Then

$$\sum_{v \in V} deg^{-}(v) = \sum_{v \in V} deg^{+}(v) = |E|$$

Basic terminology: continue ...

Definition 2.9.

A simple graph G is called bipartite if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a bipartition of the vertex set V of G.

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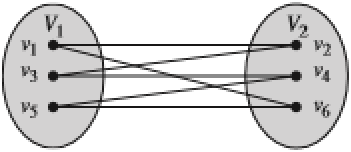

FIGURE 7 Showing That C_6 Is Bipartite.

Basic terminology: continue ...

Question: are the graphs G and H displayed in Figure below bipartite?

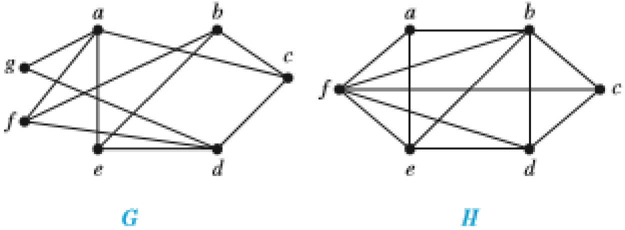

FIGURE 8 The Undirected Graphs G and H.

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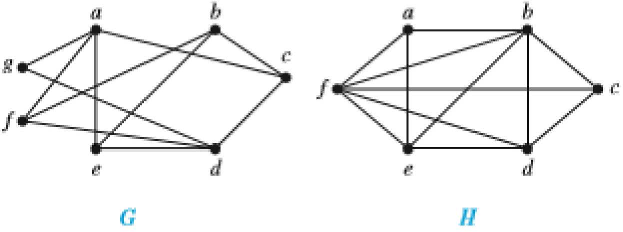

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Theorem 2.10.

A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Basic terminology: continue ...

Marriages on an Island: suppose that there are m men and n women on an island. Each person has a list of members of the opposite gender acceptable as a spouse. We construct a bipartite graph $G = (V_1, V_2)$ where V_1 is the set of men and V_2 is the set of women so that there is an edge between a man and a woman if they find each other acceptable as a spouse. A matching in this graph consists of a set of edges, where each pair of endpoints of an edge is a husband-wife pair. A maximum matching is a largest possible set of married couples, and a complete matching of V_1 is a set of married couples where every man is married, but possibly not all women.

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Theorem 2.11 (Hallḿarriage theorem).

The bipartite graph $G = (V, E)$ with bipartition (V_1, V_2) has a complete matching from V_1 to V_2 if and only if $|N(A)| \geq |A|$ for all subsets A of V_1.

Basic terminology: continue ...

Definition 2.12.

A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a proper subgraph of G if $H = G$.

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Let $G = (V, E)$ be a simple graph. The subgraph induced by a subset W of the vertex set V is the graph (W, F), where the edge set F contains an edge in E if and only if both end points of this edge are in W.

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Definition 2.14.

The union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

Representing Graphs

- One way to represent a graph without multiple edges is to list all the edges of this graph. Another way to represent a graph with no multiple edges is to use adjacency lists, which specify the vertices that are adjacent to each vertex of the graph.

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- Adjacency lists
- Adjacency matrices: suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Suppose that the vertices of G are listed arbitrarily as $v_1, v_2, \ldots, v_n$. The adjacency matrix A (or A_G) of G, with respect to this listing of the vertices, is the $n \times n$ zeroone matrix with 1 as its (i, j)th entry when v_i and v_j are adjacent, and 0 as its (i, j)th entry when they are not adjacent. In other words, if its adjacency matrix is $A = [a_{ij}]$, then

$$a_{ij} = \begin{cases} 1 & \text{if } v_i, v_j \text{ is an edge of } G, \\ 0 & \text{otherwise} \end{cases}$$

Representing Graphs

- Trade-off between adjacency lists and adjacency matrices?

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- Another common way to represent graphs is to use incidence matrices. Let $G = (V, E)$ be an undirected graph. Suppose that $v_1, v_2, ..., v_n$ are the vertices and $e_1, e_2, ..., e_m$ are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix $M = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise} \end{cases}$$

Isomorphism of Graphs

Definition 3.1.

The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function f is called an isomorphism. Two simple graphs that are not isomorphic are called nonisomorphic.

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Example: show that the graphs $G = (V, E)$ and $H = (W, F)$, displayed in Figure below, are isomorphic.

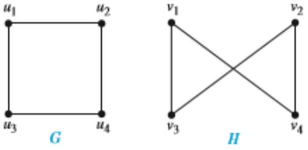

FIGURE 8 The Graphs G and H.

Paths

Definition 4.1.

Let n be a non-negative integer and G an undirected graph. A path of length n from u to v in G is a sequence of n edges $e_1, \ldots, e_n$ of G for which there exists a sequence $x_0 = u, x_1, \ldots, x_{n1}, x_n = v$ of vertices such that e_i has, for $i = 1, \ldots, n$, the end points x_{i1} and x_i. When the graph is simple, we denote this path by its vertex sequence $x_0, x_1, \ldots, x_n$ and ends at the same vertex, that is, if $u = v$, and has length greater than zero. The path or circuit is said to pass through the vertices $x_1, x_2, \ldots, x_{n1}$ or traverse the edges $e_1, e_2, \ldots, e_n$. A path or circuit is simple if it does not contain the same edge more than once.

Paths

Definition 4.2.

Let n be a non-negative integer and G a directed graph. A path of length n from u to v in G is a sequence of edges $e_1, e_2, \ldots, e_n$ of G such that e_1 is associated with (x_0, x_1), e_2 is associated with (x_1, x_2), and so on, with e_n associated with (x_{n1}, x_n), where $x_0 = u$ and $x_n = v$. When there are no multiple edges in the directed graph, this path is denoted by its vertex sequence $x_0, x_1, x_2, \ldots, x_n$. A path of length greater than zero that begins and ends at the same vertex is called a circuit or cycle.A path or circuit is called simple if it does not contain the same edge more than once.

Connectedness in Undirected Graphs

Definition 4.3.

An undirected graph is called connected if there is a path between every pair of distinct vertices of the graph. An undirected graph that is not connected is called disconnected. We say that we disconnect a graph when we remove vertices or edges, or both, to produce a disconnected subgraph.

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Theorem 4.4.

There is a simple path between every pair of distinct vertices of a connected undirected graph.

Connectedness in Directed Graphs

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Definition 4.6.

A directed graph is weakly connected if there is a path between every two vertices in the underlying undirected graph.

Counting Paths Between Vertices

Theorem 4.7.

Let G be a graph with adjacency matrix A with respect to the ordering $v_1, v_2, \ldots, v_n$ of the vertices of the graph (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length r from v_i to v_j, where r is a positive integer, equals the (i, j)th entry of A^r.

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How many paths of length four are there from a to d in the simple graph G in Figure below?

FIGURE 8 The Graph G.

Euler circuits and paths

Definition 5.1.

An Euler circuit in a graph G is a simple circuit containing every edge of G. An Euler path in G is a simple path containing every edge of G.

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Question: Which graphs below have Euler circuit, path?

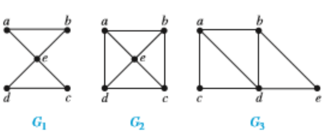

FIGURE 3 The Undirected Graphs G_1, G_2, and G_3.

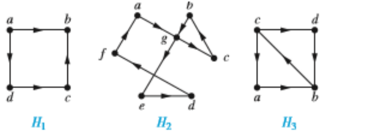

FIGURE 4 The Directed Graphs H_1, H_2, and H_3.

Euler circuits and paths

Theorem 5.2.

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Theorem 5.3.

A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

Hamilton circuits and paths

Definition 5.4.

A simple path in a graph G that passes through every vertex exactly once is called a Hamilton path, and a simple circuit in a graph G that passes through every vertex exactly once is called a Hamilton circuit. That is, the simple path $x_0, x_1, ..., x_{n1}, x_n$ in the graph $G = (V, E)$ is a Hamilton path if $V = \{x_0, x_1, ..., x_{n1}, x_n\}$ and $x_i \neq x_j$ for $0 \leq i < j \leq n$, and the simple circuit $x_0, x_1, ..., x_{n1}, x_n, x_0$ (with $n > 0$) is a Hamilton circuit if $x_0, x_1, ..., x_{n1}, x_n$ is a Hamilton path.

Hamilton circuits and paths

Theorem 5.5 (Dirac theorem).

If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $n \div 2$, then G has a Hamilton circuit.

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Theorem 5.6 (Ore theorem).

If G is a simple graph with n vertices with $n \geq 3$ such that $deg(u) + deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.

Dijkstra Algorithm

ALGORITHM 1 Dijkstra's Algorithm.

procedure *Dijkstra*(G: weighted connected simple graph, with
    all weights positive)
{G has vertices $a = v_0, v_1, \ldots, v_n = z$ and lengths $w(v_i, v_j)$
    where $w(v_i, v_j) = \infty$ if $\{v_i, v_j\}$ is not an edge in G}
for $i := 1$ **to** n
    $L(v_i) := \infty$
$L(a) := 0$
$S := \emptyset$
{the labels are now initialized so that the label of a is 0 and all
    other labels are ∞, and S is the empty set}
while $z \notin S$
    $u :=$ a vertex not in S with $L(u)$ minimal
    $S := S \cup \{u\}$
    for all vertices v not in S
        if $L(u) + w(u, v) < L(v)$ **then** $L(v) := L(u) + w(u, v)$
        {this adds a vertex to S with minimal label and updates the
        labels of vertices not in S}
return $L(z)$ {$L(z) =$ length of a shortest path from a to z}

Dijkstra Algorithm Example

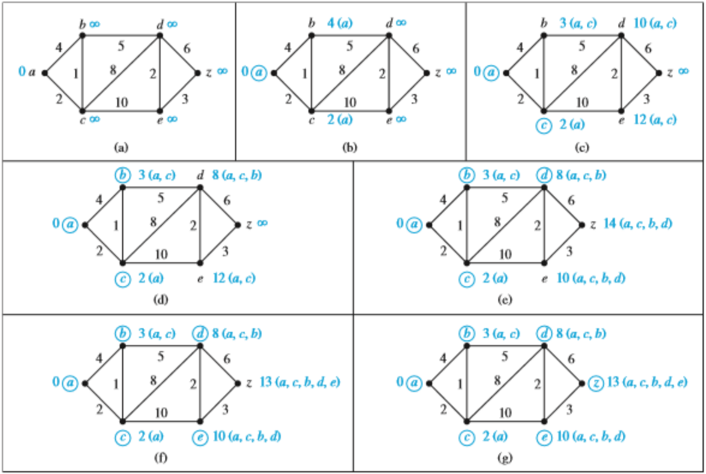

FIGURE 4 Using Dijkstra's Algorithm to Find a Shortest Path from *a* to *z*.

Definition

Definition 7.1.

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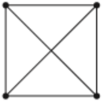

FIGURE 2 The Graph K_4.

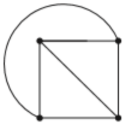

FIGURE 3 K_4 Drawn with No Crossings.

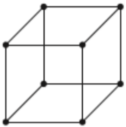

FIGURE 4 The Graph Q_3.

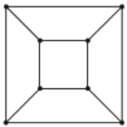

FIGURE 5 A Planar Representation of Q_3.

Euler formula

Theorem 7.2.

Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then

$$r = e - v + 2$$

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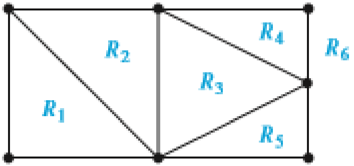

FIGURE 8 The Regions of the Planar Representation of a Graph.

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Corollary 7.5.

If a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

Definition

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A coloring of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

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Definition 8.2.

The chromatic number of a graph is the least number of colors needed for a coloring of this graph. The chromatic number of a graph G is denoted by $\chi(G)$.

Definition

Theorem 8.3 (Four color theorem).

The chromatic number of a planar graph is no greater than four.

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Frequency Assignments Television: channels 2 through 13 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel. How can the assignment of channels be modeled by graph coloring?