SOME CONCRETE ASPECTS OF HILBERT'S 17TH PROBLEM

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This paper is dedicated to the memory of Raphael M. Robinson and Olga Taussky-Todd.

1. INTRODUCTION

Hilbert's 17th Problem asks whether a real positive semidefinite (psd) polynomial in several variables must be a sum of squares of rational functions. This paper gives a survey of the literature on two closely related questions: What can be said about a psd polynomial that is not a sum of squares of *polynomials*? How can one write a given psd polynomial as a sum of squares of rational functions? These questions go back to Hilbert himself, and his interest in them predated the 1900 Paris Congress.

This paper began as a presentation to the Séminaire Delon–Dickmann–Gondard at the Université Paris VII in January 1995. I gave a summary of the history of the answers to the first question above, and a detailed exposition of my recent contribution towards understanding the second question above for positive definite forms. The paper on which I based the second part has appeared in print [72], and has also been discussed in a recent *Monthly* article [64]. For these reasons, this paper mainly addresses the first question. An earlier, unrefereed, version of this manuscript appeared [73] in the seminar's annual preprint collection.

Sadly, two mathematicians influential to the development of this subject have passed away recently: Raphael M. Robinson (1911–1995) and Olga Taussky-Todd (1906–1995). The reader will see below the vital contributions made by Professors Robinson and Taussky.

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2. Notations

Let $H_d(K^n)$ denote the set of homogeneous forms of degree d in n variables ("n-ary d-ics") with coefficients from the field K. By identifying $p \in H_d(K^n)$ with the $N = \binom{n+d-1}{n-1}$-tuple of its coefficients, we see that $H_d(K^n) \approx K^N$. Suppose m is an even integer. A form $p \in H_m(\mathbb{R}^n)$ is called *positive semidefinite* or *psd* if $p(x_1, \ldots, x_n) \geq 0$ for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Following [14], we denote the set of psd forms in $H_m(\mathbb{R}^n)$ by $P_{n,m}$. Since $P_{n,m}$ is closed under addition and closed under multiplication by positive scalars, it is a convex cone. In fact, $P_{n,m}$ is a *closed* convex cone: if $p_n \to p$ coefficientwise, and each p_n is psd, then so is p. A psd form is called *positive definite* or *pd* if $p(x_1, \ldots, x_n) = 0$ implies $x_j = 0$ for $1 \leq j \leq n$. It is not difficult to see that the pd forms constitute the interior of the cone $P_{n,m}$.

A form $p \in H_m(\mathbb{R}^n)$ is called *a sum of squares* or *sos* if it can be written as a sum of squares of polynomials. It is easy to show that if $p \in H_m(\mathbb{R}^n)$ and $p = \sum_k h_k^2$ with $h_k \in \mathbb{R}[x_1, \ldots, x_n]$, then each $h_k \in H_{m/2}(\mathbb{R}^n)$. Again following [14], we denote the set of sos forms in $H_m(\mathbb{R}^n)$ by $\Sigma_{n,m}$. It is easy to see that $\Sigma_{n,m}$ is a convex cone; less obvious is the fact that it is closed. This was first proved by R. M. Robinson [77]. Finally, we note the inclusion $\Sigma_{n,m} \subseteq P_{n,m}$ and define $\Delta_{n,m} = P_{n,m} \setminus \Sigma_{n,m}$. If $p \in \Delta_{n,m}$, then p may be construed as lying in $\Delta_{n_1,m}$ for $n_1 \geq n$; for even $m_1 \geq m$, it is easy to show that $x_1^{m_1-m} p \in \Delta_{n,m_1}$.

By setting $x_n = 1$, any $p \in H_d(K^n)$ can be dehomogenized into a polynomial over K in $n-1$ variables, of degree $\leq d$. And any polynomial $f(x_1, \ldots, x_n)$ over K of degree d can be homogenized into a form $p \in H_e(K^{n+1})$ with $e \geq d$, by adding a new variable y, and defining

$$p(x_1, \ldots, x_n, y) = y^e f(x_1/y, \ldots, x_n/y).$$

The properties of being psd and sos are inherited under dehomogenization, and conversely, are preserved when a polynomial is homogenized into a form of even degree. However, the property of being positive definite is not preserved upon homogenization. For example, $f(x,y) = x^2 + (1-xy)^2$ takes only positive values for real (x,y), but its homogenized form $p(x,y,z) = x^2 z^2 + (z^2 - xy)^2$ has non-trivial zeros at $(1,0,0)$ and $(0,1,0)$, corresponding to f's "zeros at infinity".

Many examples presented here as forms orginally appeared in the literature as non-homogeneous polynomials. When we have made such a change, it will be noted explicitly.

3. Hilbert's 17th Problem

"It is a truth universally acknowledged, that a mathematical object that is nonnegative in all orderings must be in want of a representation as a sum of squares." – after Jane Austen

a. Before 1900. It was well-known by the late 19th century that $P_{n,m} = \Sigma_{n,m}$ when $m = 2$ or $n = 2$. This is easy for $m = 2$: any psd n-ary quadratic form p can be diagonalized as sum of $\mathrm{rank}(p) \leq n$ squares of linear forms. If $p(x,y) \in P_{2,m}$, then $f(t) = p(t,1) \geq 0$ for all real t, so the roots of f are either real (with even

multiplicity) or appear in complex conjugate pairs, and the leading coefficient of f is positive. Thus, we have the factorization

$$f(t) = c^2 \prod_{j=1}^{r} (t - t_j)^{2a_j} \prod_{k=1}^{s} \big(t - (\alpha_k + i\beta_k)\big) \prod_{k=1}^{s} \big(t - (\alpha_k - i\beta_k)\big)$$

$$= P(t)^2 \big(Q(t) + iR(t)\big)\big(Q(t) - iR(t)\big) = \big(P(t)Q(t)\big)^2 + \big(P(t)R(t)\big)^2.$$

It follows (upon homogenizing f) that p is also a sum of two polynomial squares.

Suppose in the foregoing that the $2s$ complex roots of f are distinct. Then there are 2^{s-1} ways to allocate the pairs of conjugate linear factors into an unordered pair $\{Q + iR, Q - iR\}$ of complex conjugate polynomials. Since $Q + iR$ is always monic, $\deg Q > \deg R$. We obtain in this way 2^{s-1} different (c.f. [19, p. 109]) representations $f = g^2 + h^2$ in which $\deg h < \frac{1}{2} \deg f$. For example, with $s = 3$,

$$t^6 + 1 = (t^3)^2 + 1^2 = (t^3 - 2t)^2 + (2t^2 - 1)^2 = \big(t^3 - \tfrac{1}{2}t \pm \tfrac{\sqrt{3}}{2}\big)^2 + \big(t^2 \mp \tfrac{\sqrt{3}}{2}t - \tfrac{1}{2}\big)^2.$$

In 1888, the 26-year old David Hilbert proved two remarkable results in one paper, [38]. First, he showed that $\Sigma_{3,4} = P_{3,4}$; in fact, he showed that every $p \in P_{3,4}$ can be written as the sum of three squares of quadratic forms. (For an elementary proof, with "three" replaced by "five", see [15]; modern expositions of Hilbert's proof have been given by Cassels (in [67, pp. 89–93]) and Swan [82].) Hilbert's second result is that the preceding are the *only* cases for which $\Delta_{n,m} = \emptyset$. That is, if $n \geq 3$ and $m \geq 6$ or $n \geq 4$ and $m \geq 4$, then there exist forms $p \in P_{n,m}$ that are not sos. These can be derived as noted above from forms in $\Delta_{3,6}$ and $\Delta_{4,4}$. Hilbert's proofs used the techniques of 19th century algebraic geometry. Since we shall later describe shorter and simpler examples, we present only a sketch of Hilbert's construction of $p \in \Delta_{3,6}$. The final step in the proof is a critical observation that embodies one of the essential principles of real algebra: if $p = \sum_k h_k^2$ and $p(u) = 0$ for some $u \in \mathbb{R}^n$, then $0 = \sum_k h_k^2(u)$, hence $h_k(u) = 0$ for all k.

This is Hilbert's method to construct a sextic polynomial $F(x,y) \geq 0$ that is not a sum of squares of polynomials. Let $\phi(x,y)$ and $\psi(x,y)$ be two real cubic polynomials with no non-constant common factor, and with common zeros at $\{P_1, \ldots, P_9\} \subset \mathbb{R}^2$. (By Bezout's Theorem, nine is the maximum number of common zeros of two cubics, even in $\mathbb{C}^2$.) It is (or used to be) well-known that any cubic $h(x,y)$ that vanishes at eight of the P_j's must vanish at the ninth. Choose a quadratic polynomial $f(x,y) \neq 0$ that vanishes at P_1, P_2, P_3, P_4 and P_5 and a quartic polynomial $g(x,y) \neq 0$ that vanishes at P_1, P_2, P_3, P_4 and P_5, and is singular at P_6, P_7 and P_8. (Such curves exist by constant-counting arguments: there are 5 conditions on f and $\binom{4}{2} = 6$ coefficients in a quadratic, and $5 + 3 \cdot 3 = 14$ conditions on g and $\binom{6}{2} = 15$ coefficients in a quartic.) It can then be shown that there exists λ so that

$$F(x,y) := \phi^2(x,y) + \psi^2(x,y) + \lambda f(x,y)g(x,y) \geq 0$$

for all real (x,y), and that $F(P_j) = 0$ for $1 \leq j \leq 8$, but $F(P_9) > 0$. If $F = \sum_k h_k^2$, then each h_k is a cubic and $h_k(P_j) = 0$ for $1 \leq j \leq 8$, hence $h_k(P_9) = 0$ for all k, contradicting $\sum_k h_k^2(P_9) = F(P_9) > 0$.

The most complete modern exposition of this method seems to be in Gel'fand–Vilenkin [29, pp. 232–235], which also established the connection between forms in $\Delta_{n,m}$ and the Hamburger moment problem in $n-1$ variables. For more on this connection, see [71] and the references contained within. Robinson [77] made a judicious choice of ϕ and ψ that greatly simplified Hilbert's methods (see §4b), and cited an unpublished example of Ellison using the original construction. Ellison also generalized a key step of Hilbert's construction in [28, p. 668].

The earliest published reference to [38] seems to be [46], by Hilbert's close friend Adolf Hurwitz. Hurwitz proves the arithmetic-geometric inequality by showing that for even m, the form $x_1^m + \cdots + x_m^m - mx_1\cdots x_m$ is a sum of squares of forms. He remarks in a footnote (p. 507): "Die Möglichkeit einer solchen Darstellung ist freilich nicht von vornherein klar. Es giebt nämlich, wie Herr Hilbert gezeigt hat, positive Formen, welche *nicht* als Summen von Formenquadraten darstellbar sind." The Hurwitz construction, which can also be found in [35, p. 55], is simplified somewhat in [69].

In 1893, Hilbert [39] generalized his earlier result on $P_{3,4}$; his proof seems to be non-constructive, and lacks a modern exposition. Suppose $p \in P_{3,m}$ with $m \geq 6$. Then there exist $p_1 \in P_{3,m-4}$ and $h_{1k} \in H_{m-2}(\mathbb{R}^3)$ so that $pp_1 = h_{11}^2 + h_{12}^2 + h_{13}^2$. If $m = 6$ or 8, then p_1 is already known to be the sum of three squares of forms, and hence (as Landau later noted [51]), the four-square identity implies that $pp_1^2 = (pp_1)p_1$ is the sum of four squares of forms. If $m \geq 10$, then the argument can be applied to p_1, so that there exists $p_2 \in P_{3,m-8}$ with $p_1p_2 = h_{21}^2 + h_{22}^2 + h_{23}^2$. Thus, if $m = 10$ or 12, then $p(p_1p_2)^2 = (pp_1)(p_1p_2)p_2$ is the sum of four squares of forms, etc. An easy induction allows us to conclude that there exists $q \in P_{3,t}$ with $t = \lfloor \frac{(m-2)^2}{8} \rfloor$ so that pq^2 is the sum of four squares of forms. It follows that p is the sum of four squares of rational functions with psd denominator q.

b. Hilbert's "Hilbert's 17th Problem". In his 1900 Address to the International Congress of Mathematicians in Paris [41], Hilbert posed a generalization of his results as the 17th Problem: Must every psd form p be a sum of squares of rational functions? We quote from the contemporary English translation [6, p. 24] of Hilbert's paper:

> "A rational integral function or form in any number of variables with real coefficients such that it becomes negative for no real values of these variables, is said to be *definite*. The system of all definite forms is invariant with respect to the operations of addition and multiplication, but the quotient of two definite forms—in case it should be an integral function of the variables—is also a definite form. The square of any form is evidently always a definite form. But since, as I have shown ([38]), not every definite form can be compounded by addition from squares of forms, the question arises—which I have answered affirmatively for ternary forms ([39])—whether every definite form may not be expressed as a quotient of sums of squares of forms. At the same time it is desirable, for certain questions as to the possibility of certain geometrical constructions, to know whether the coefficients of the forms to be used in the expression may always be taken from the realm of rationality given by the coefficients of the form represented ([40, §38])."

A more precise modern formulation is as follows: suppose $p \in P_{n,m} \cap H_m(K^n)$

where $K \subseteq \mathbb{R}$. Is it true that there exist positive $\lambda_k \in K$ and $h_k \in K(x_1, \ldots, x_n)$ so that $p = \sum_k \lambda_k h_k^2$? Upon clearing denominators, we get an equivalent formulation: do there exist positive $\lambda_k \in K$, $q \in H_r(K^n)$ (for some r) and $g_k \in H_{m/2+r}(K^n)$ so that $pq^2 = \sum_k \lambda_k g_k^2$?

The permission to use positive weights λ_k from K is not explicitly mentioned by Hilbert, and might seem peculiar to those unfamiliar with real algebra, but it is essential. For example, let $K = \mathbb{Q}(\sqrt{2})$, $(n, m) = (1, 0)$ and $p(x) = \sqrt{2}$, which is trivially psd. If we could write $p(x) = \sum_k h_k^2(x)$, with $h_k \in K[x]$, then each h_k would be a constant. Thus, $h_k = \alpha_k + \beta_k \sqrt{2}$, with $\alpha_k, \beta_k \in \mathbb{Q}$, so $\sqrt{2} = \sum_k (\alpha_k + \beta_k \sqrt{2})^2$. This implies $-\sqrt{2} = \sum_k (\alpha_k - \beta_k \sqrt{2})^2$, a contradiction to the order in $\mathbb{R}$. Initiates will recognize that $\sqrt{2}$ is negative in one ordering of K, and so is not a sum of squares in K.

Lam [49, pp. 16–18] discusses three aspects of Hilbert's work that might have motivated a study of formally real fields and ordered fields: the 17th Problem, some foundational questions in geometry and the study of totally positive elements in number fields. Modern discussions of the geometric roots of Hilbert's 17th Problem have been made by Prestel [66] and Delzell [23]. The Hilbert-Landau-Siegel Theorem states that if x is an element in a number field F that is positive in each embedding of F into $\mathbb{R}$, then x is a sum of four squares. Elements that are not totally positive, such as $\sqrt{2}$ above, are negative in at least one embedding into $\mathbb{R}$, and so cannot be sums of squares at all.

c. After 1920. In 1927, Emil Artin [1] used the Artin-Schreier theory of real closed fields to answer Hilbert's 17th Problem in the affirmative, under the additional hypothesis that the field K has a unique order. (This hypothesis is satisfied by $\mathbb{R}$.) He solved it for all subfields of $\mathbb{R}$ under the addition "weight" hypothesis mentioned in the last paragraph. However, Artin's proof gives no information about any specific representation of a particular form $p \in P_{n,m}$ as a sum of squares of rational functions.

Among the many generalizations of the 17th Problem, we mention one in detail. In 1981, Becker [2, 3] gave necessary and sufficient conditions for a rational function p over a formally real field to be a sum of $2k$-th powers of rational functions. For such functions over $\mathbb{R}$, the criterion is, roughly speaking, that p must be psd, its degree must be a multiple of $2k$ and all real zeros must have "$2k$-th order". A concrete application of this theorem [3, p. 144] is that for all $k \geq 1$, there exist positive $\lambda_{jk} \in \mathbb{Q}$ and f_{jk}, g_{jk} in $\mathbb{Q}[t]$ so that

$$B(t) := \frac{1 + t^2}{2 + t^2} = \sum_j \lambda_{jk} \left(\frac{f_{jk}(t)}{g_{jk}(t)} \right)^{2k}.$$

As with Artin's result, one does not obtain an explicit representation of $B(t)$ as a sum of $2k$-th powers of rational functions. These are not hard to find for small k, and there was some interest in finding them for all k. Powers has recently written an excellent paper [64] on the history of this, the "Champagne Problem".

We shall give an expression for $B(t)$ as a sum of $2k$-th powers (as above) at the end of §7; however the coefficients and polynomials have real coefficients, rather than the rational ones requested by Becker. One can deduce from recent work of

Becker–Powers [4] that there is a representation of $B(t)$ as a sum of $2k$-th powers in which each g_{jk} is positive definite. Schmid has also recently shown [79] that if f and g are positive definite polynomials in one variable with the same degree, then $(f/g)(t)$ can be written as above, but where f_{jk} and g_{jk} are positive definite polynomials of the *same* degree.

There are many expositions of Artin's proof in the literature, e.g., [5, 47, 49, 50]. Ribenboim [74] and Pfister [61] wrote surveys on Hilbert's 17th Problem in the 1970s; two more recent surveys are by Gondard [30] and Scheiderer [78]. The deep connections of Hilbert's 17th Problem with logic were initiated by A. Robinson [75, 76] in the mid-1950's; Delzell has written [25] a recent detailed history of logicians' interest in Hilbert's 17th Problem.

The spectacular development of real algebra and real algebraic geometry is well-known (see, e.g., [5]) and will not be further discussed here. Rajwade [67] contains detailed expositions of much of the material discussed and alluded to here. Lam has written two wonderful expository articles on real algebra: [49, 50]. In 1982, he was awarded the Steele Prize by the AMS, in part for [49] ([50] had not yet appeared). Taussky wrote two survey articles ([83, 84]) on sums of squares in algebra. The first one was particularly influential in calling attention to the ubiquitous role of sums of squares in algebra, and was awarded the Ford Prize by the MAA in 1971. Olga was always supportive and encouraging to all of us interested in sums of squares, and, as a direct link to Hilbert, embodied the intellectual continuity of mathematics.

4. EXAMPLES FROM THE 1960S AND 1970S

For reasons that may be more psychological than mathematical, it took nearly 80 years for explicit forms in $\Delta_{n,m}$ to appear in the literature, and when they appeared, they were much simpler than Hilbert's original examples. Interestingly, different authors constructed substantially different examples.

a. Examples of Motzkin. The way the first one arose is described in the introduction to Theodore S. Motzkin's collected works [57, pp. xvi–xvii]: "During many of his years at UCLA, Motzkin conducted seminars that were very exciting to the students and faculty members who participated in them. Some of Motzkin's most beautiful and important work made its first appearance here. ... [D]uring a seminar on inequalities, a colleague presented Artin's solution of Hilbert's 17th [P]roblem, ... Motzkin wondered out loud what would happen if the classical inequalities of the type $f(x_1, \ldots, x_n) \geq 0$ (such as the arithmetic-geometric inequality, when suitably formulated) were proved by expressing f in the form $f = \sum_{i=1}^m \rho_i^2$, and in particular if the ρ_i would turn out to be polynomials. At the next meeting of the seminar he carried out this program and presented for the first time the now celebrated Motzkin polynomial ... Although some results of the seminar were published in the proceedings of a symposium at Dayton, Ohio [56], the polynomial was still not as widely known as it became after O. Taussky-Todd mentioned its existence to A. Pfister who, along with J. W. S. Cassels and W. J. Ellison, did further work in this area [see [8]]."

Motzkin proved [56, p. 217] that for $n \geq 3$,

$$(t_1^2 + \cdots + t_{n-1}^2 - nu^2)t_1^2 \cdots t_{n-1}^2 + u^{2n} \in \Delta_{n,2n}.$$

The form in the special case $n = 3$ was denoted S' in [15] and M in [68]. We give the proof for $n = 3$; the general case is similar. Rename variables, and let

$$M(x, y, z) = (x^2 + y^2 - 3z^2)x^2y^2 + z^6 = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2.$$

The fact that M is psd follows from the arithmetic-geometric inequality $\frac{a+b+c}{3} \geq (abc)^{1/3}$, applied to $(a, b, c) = (x^4y^2, x^2y^4, z^6)$. If M were sos, then the equation $M(x, y, z) = \sum_k h_k^2(x, y, z)$ would hold for suitable $h_k \in H_3(\mathbb{R}^3)$. Write out M as a ternary sextic, using all potential monomials:

$$0x^6 \;+\; 0x^5y \;+ 1x^4y^2 \;+ 0x^3y^3 \;+ 1x^2y^4 \;+ 0xy^5 \;+\; 0y^6$$
$$+\; 0x^5z \;+\; 0x^4yz + 0x^3y^2z + 0x^2y^3z + 0xy^4z + \;0y^5z$$
$$+\; 0x^4z^2 \;+ 0x^3yz^2 - 3x^2y^2z^2 + 0xy^3z^2 + 0y^4z^2$$
$$+\; 0x^3z^3 \;+ 0x^2yz^3 + 0xy^2z^3 + 0y^3z^3$$
$$+\; 0x^2z^4 \;+ 0xyz^4 + 0y^2z^4$$
$$+\; 0xz^5 \;+ 0yz^5$$
$$+\; 1z^6.$$

Now write out $h_k(x, y, z)$, utilizing the same geometric scheme:

$$A_k x^3 \;+\; B_k x^2y \;+ C_k xy^2 \;+\; D_k y^3$$
$$+\; E_k x^2z \;+ F_k xyz + G_k y^2z$$
$$+\; H_k xz^2 \;+ I_k yz^2$$
$$+\; J_k z^3.$$

Since the coefficient of x^6 in M is 0, the corresponding coefficient in $\sum_k h_k^2$, $\sum_k A_k^2$, also equals 0. Thus, $A_k = 0$ for all k. This can also be seen directly: $M(1, 0, 0) = 0$ and $h_k(1, 0, 0) = A_k$. Now look at the coefficient of x^4z^2 in $\sum_k h_k^2$: it is $\sum_k(E_k^2 + 2A_kH_k)$. Since $A_k = 0$ and the coefficient of x^4z^2 in M is 0, it follows that $E_k = 0$ for all k as well. Continuing down the xz edge, we compare the coefficients of x^2z^4 in $\sum_k h_k^2$ and M: $\sum_k 2E_kJ_k + H_k^2 = 0$. Since $E_k = 0$, it follows that $H_k = 0$. (These also follow from M vanishing to 5th order at $(1,0,0)$ in the direction of $(0,0,1)$.) A similar argument, applied to the coefficients of y^6, y^4z^2 and y^2z^4, shows that $D_k = G_k = I_k = 0$.

At this point there are two paths to our conclusion. We have already reduced our task to drawing a contradiction from the equation

$$x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2 = \sum_k (B_k x^2y + C_k xy^2 + F_k xyz + J_k z^3)^2.$$

Since $M(1, \pm 1, \pm 1) = 0$, we have $h_k(1, \pm 1, \pm 1) = 0$, hence

$$B_k + C_k + F_k + J_k = B_k + C_k - F_k - J_k = -B_k + C_k - F_k + J_k = -B_k + C_k + F_k - J_k = 0.$$

Thus, $B_k = C_k = F_k = J_k = 0$, so $h_k(x, y, z) = 0$, contradicting $\sum h_k^2 = M$.

It is more telling to consider the coefficient of $x^2 y^2 z^2$ in M and $\sum_k h_k^2$; the contradiction is immediate from $-3 = \sum_k F_k^2$! This second argument is more powerful. Let $N(x, y, z) = M(x, y, z) + x^2 y^2 z^2$. Then N is evidently psd. If $N = \sum_k \bar{h}_k^2$, then, as before, it is easy to show that each $\bar{h}_k$ can only use the same monomials as h_k. However, the zeros of N are just $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, and these imply only that $A_k = D_k = J_k = 0$, information we already know. On the other hand, a consideration of the coefficient of $x^2 y^2 z^2$ in N and $\sum_k \bar{h}_k^2$ gives the contradiction: $-2 = \sum_k F_k^2$.

By Hilbert's 1893 Theorem, M must be a sum of $2^{3-1} = 4$ squares of rational functions. Such an explicit representation follows from the identity

$$(x^2 + y^2)^2 M(x, y, z) = x^2 y^2 (x^2 + y^2 + z^2)(x^2 + y^2 - 2z^2)^2 + (x^2 - y^2)^2 z^6.$$

Hilbert's Theorem was generalized (using entirely different methods) in a celebrated 1967 paper of Pfister [60] (see also [67, § 5]): every $p \in P_{n,m}$ is the sum of at most 2^{n-1} squares of rational functions. In 1971, Cassels–Ellison–Pfister [8] proved that M cannot be written as a sum of three squares of rational functions. Their methods were extended to a family of ternary sextics by Christie [21] in 1976. We proved in 1980 [20] that every form that is the sum of two squares of rational functions is in fact the sum of two squares of forms, and asked whether this is true for sums of three squares (p. 254). Very recently, Leep–Starr [53] showed that the answer to this question is "no". Let

$$F(x, y, z) = x^4 y^2 + x^2 y^4 + z^6 + 8x^2 y^2 z^2,$$
$$G(x, y, z) = 3x^4 y^2 - 2x^3 y^3 + \tfrac{1}{2} x^2 y^4 + 4x^3 y^2 z + x^2 y^3 z$$
$$+ 10x^2 y^2 z^2 + 2xy^2 z^3 - 8x^2 y z^3 + 4xy z^4 + 8z^6.$$

Then F and G are each the sums of three squares of rational functions, but F is not a sum of three squares of forms and G is not even sos.

b. Examples of Robinson. In the late 1960s at Berkeley, Raphael M. Robinson [77, p. 264] saw "an unpublished example of a ternary sextic worked out recently by W. J. Ellison using Hilbert's method [see §3a]. It is, as would be expected, very complicated. After seeing this, I discovered that an astonishing simplification would be possible by dropping some unnecessary assumptions made by Hilbert." He adds in a footnote: "When I submitted this paper for publication, I did not think that any such example had ever appeared in print. However, shortly thereafter, T. S. Motzkin called my attention to the fact that he had published a counterexample for the case of ternary sextics in 1967. I have added an Appendix which discusses Motzkin's result."

Robinson chose specific cubics for Hilbert's construction: $\phi(x, y) = x^3 - x$ and $\psi(x, y) = y^3 - y$. The nine common zeros of ϕ and ψ, $\{P_j\}$, are the square array $\{-1, 0, 1\}^2$, and f and g are also not hard to find. Where Hilbert had argued that some λ makes $\phi^2 + \psi^2 + \lambda f g$ psd, Robinson was able to choose a specific value for λ, and derived the psd form

$$R(x, y, z) = x^6 + y^6 + z^6 - (x^4 y^2 + x^2 y^4 + x^4 z^2 + x^2 z^4 + y^4 z^2 + y^2 z^4) + 3x^2 y^2 z^2.$$

(Robinson primarily discussed $R(x, y, 1)$; the notation R was introduced in [14].)

Robinson proved that R is psd by writing $(x^2 + y^2)R(x, y, 1)$ as a sum of squares of polynomials. The inequality $R \geq 0$ had actually appeared in Motzkin [56, §3.4], and is a special case of an inequality due to Schur (for a proof, see [35, p. 64]):

$$u^r(u-v)(u-w) + v^r(v-u)(v-w) + w^r(w-u)(w-v) \geq 0 \ \text{ if } r, u, v, w \geq 0.$$

(Take $r = 1$ and $(u, v, w) = (x^2, y^2, z^2)$ to obtain R.) For much more on Schur's inequalities and related sextic forms, see [18].

It is easy to see that $R = 0$ on the set

$$\mathcal{Z} := \{(1, \pm 1, \pm 1), (1, \pm 1, 0), (1, 0, \pm 1), (0, 1, \pm 1)\}.$$

If $R = \sum_k h_k^2$, where each h_k is a ternary cubic, then h_k vanishes on $\mathcal{Z}$. This imposes ten linearly independent equations on the ten coefficients of h_k, that together imply that $h_k = 0$. As before, this contradicts $R = \sum_k h_k^2$. We shall make repeated reference in the remainder of the paper to the Robinson form R and its zero-set $\mathcal{Z}$.

Robinson also gave the first explicit example in $\Delta_{4,4}$:

$$f(x, y, z, w) = x^2(x - w)^2 + y^2(y - w)^2 + z^2(z - w)^2 + 2xyz(x + y + z - 2w).$$

The proof that $f \in \Delta_{4,4}$ is not quite as simple as the proof for $R \in \Delta_{3,6}$, and f has been replaced as the exemplary element of $\Delta_{4,4}$ by Q (see §4c). In the Appendix of [77], Robinson gives a method for generalizing Motzkin's example: if f is a real polynomial in n variables with degree $d < 2n$ that is not sos, then neither is

$$g(x_1, \ldots, x_n) := x_1^2 \cdots x_n^2 f(x_1, \ldots, x_n) + 1.$$

(Of course, g is not necessarily psd!) When $n = 2$ and $f(x_1, x_2) = x_1^2 + x_2^2 - 3$, this construction produces $M(x_1, x_2, 1)$.

c. Examples of Choi–Lam. In 1973, Man-Duen Choi was trying to classify positive linear mappings—mappings between matrix algebras that preserve the cone of positive semidefinite matrices. In the real case, this reduces to the cone of psd biquadratic forms—quartic forms that are quadratic forms in two different sets of variables. Choi learned of a paper [48, p. 14] by an electrical engineer, purporting to show that every psd biquadratic form is sos. A recent paper of Calderón [7] had covered some low-dimensional cases and convinced Choi that the result could not be extended. He tried to find the flaw in the proof, and, in doing so, constructed a counterexample (in [11]). He writes [12]: "Without Koga's false proof, I would not have dared construct a counterexample. Actually, I had been haunted by Hilbert's non-constructive example [in [29]] when I was a graduate student."

Choi's counterexample was

$$F(x_1, x_2, x_3; y_1, y_2, y_3) := x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2 + 2(x_1^2 y_2^2 + x_2^2 y_3^2 + x_3^2 y_1^2)$$
$$-2x_1 x_2 y_1 y_2 - 2x_1 x_3 y_1 y_3 - 2x_2 x_3 y_2 y_3.$$

Choi also specialized F in [11] to give some other forms in $\Delta_{4,4}$ and $\Delta_{3,6}$.

Choi had a lectureship in Berkeley from 1973–1976, and started working with Tsit-Yuen Lam, who had already written extensively on quadratic forms. The following year, Choi and Lam wrote the first two papers devoted to a systematic study of this subject: [14] and [15]. They made monomial substitutions in $B :=$ $F - (x_1^2 y_2^2 + x_2^2 y_3^2 + x_3^2 y_1^2)$, which they proved to lie in $\Delta_{6,6}$, and gave two more simple explicit elements of $\Delta_{4,4}$ and $\Delta_{3,6}$:

$$Q(x, y, z, w) := B(x, w, z, y, z, w) = x^2 y^2 + x^2 z^2 + y^2 z^2 + w^4 - 4wzyz,$$
$$S(x, y, z) := B(yz, xz, xy, x, y, z) = x^4 y^2 + y^4 z^2 + z^4 x^2 - 3x^2 y^2 z^2.$$

In each case, the fact that these forms are psd is immediate from the arithmetic-geometric inequality, and the fact that these forms are not sos follows in a manner similar to that shown in §4a for M. Choi and Lam called this approach the "term-inspection method"; it was later generalized [19] as the "Gram matrix method", which is described in §5b.

Choi and Lam constructed a number of other examples of psd forms that are not sos. One is a symmetric quaternary quartic: $\sum x_i^2 x_j^2 + \sum x_i^2 x_j x_k - 2x_1 x_2 x_3 x_4$. Another arises from making the substitution $x_1 \to x_i^2$ in a quadratic form previously studied by A. Horn:

$$H(x_1, \ldots, x_5) = (x_1^2 + \cdots + x_5^2)^2 - 4(x_1^2 x_2^2 + \cdots + x_4^2 x_5^2 + x_5^2 x_1^2).$$

(In this, the "Horn form", $x_i^2 x_j^2$ has coefficient ∓ 2 in H depending on whether or not i and j are adjacent in the set $\{1, 2, 3, 4, 5\}$, viewed cyclically.)

Let $C \subset \mathbb{R}^t$ be a closed convex cone. An element $x \in C$ is called *extremal* if $x = y_1 + y_2$, $y_i \in C$, implies that $y_i = \lambda_i x$ for some $\lambda_i > 0$. Every element in a closed convex cone C for which $C \cap -C$ is the origin can be written as a finite sum of extremal elements. Choi and Lam studied extremal elements in the convex cones $P_{n,m}$ and $\Sigma_{n,m}$. (As noted earlier, we can identify a form $p \in H_m(\mathbb{R}^n)$ with the N-tuple of its coefficients, and so view $\Sigma_{n,m}$ and $P_{n,m}$ as subsets of $\mathbb{R}^N$.) Calderón had also used extremality in studying psd biquadratic forms in [7].

Extremality has the following interpretations in the cones $P_{n,m}$ and $\Sigma_{n.m}$. The form $p \in P_{n,m}$ is extremal if and only if $p = q_1 + q_2$ with $q_j \in P_{n,m}$ implies that $q_j = \lambda_j p$ for constants λ_j satisfying $\lambda_1 + \lambda_2 = 1$. In particular, if $p(x_1, \ldots, x_n) \ge q(x_1, \ldots, x_n) \ge 0$ for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$, then q is a scalar multiple of p. If $p \in \Sigma_{n,m}$ is extremal, then $p = h^2$ for some $h \in H_{m/2}(\mathbb{R}^n)$, but this is not sufficient; for example, $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$ is not extremal in $\Sigma_{2,4}$. Some sufficient conditions are discussed in [13]; it is *not* necessarily true that if h^2 is extremal in $\Sigma_{n,m}$, then it is also extremal in $P_{n,m}$.

Perhaps the most significant results in [14] and [15] were the proofs that the forms M, R, S and Q are all extremal elements in their respective $P_{n,m}$'s. It is remarkable that these early examples, chosen for their simplicity, are each extremal. The basic argument of the proofs is straightforward. If $p(u) = 0$ for some $u \in \mathbb{R}^n$, then $\frac{\partial p}{\partial x_j}(u) = 0$ for all j, since p is psd. If $p \ge q \ge 0$ as above, then $q(u) = 0$ and so $\frac{\partial q}{\partial x_j}(u) = 0$ as well. For example, setting $p = R \in P_{3.6}$, we find that the zero-set $\mathcal{Z}$ imposes $3 \cdot 10$ linear equations on the $\binom{3+6-1}{3-1} = 28$ coefficients of q.

Miraculously, this linear system has rank 27, and the only ternary sextics whose derivatives vanish on $\mathcal{Z}$ are the multiples of R.

These paragraphs do not completely describe the contents of [14] and [15]; many of the ideas in these papers have yet to be fully developed.

My entry into the subject came in late 1976. I was studying the two-dimensional Hamburger moment problem as it applied to an embedding problem in functional analysis which had earlier arisen in my thesis—see [71, pp. 117–120] for details. I found a reference to the abstract of [77] and was immediately captivated. My Duke colleague Leonard Carlitz gave me his copy of [77] and I wrote to Prof. Robinson, who directed me to the then-new [14] and [15]. I first met Lam at the 1977 AMS-MAA Winter Meetings in St. Louis, and showed him a counterexample to a minor conjecture in the preprint of [15]. I visited him at his Berkeley home that summer and had the shortest four-hour conversation of my life. I enjoyed a post-doctoral year at Berkeley in 1978–1979, and Choi, Lam and I have worked together (with occasional fourth authors) ever since.

d. Examples of Lax–Lax and Schmüdgen. Two other forms in $\Delta_{n,m}$ were discovered independently in the 1970's. Anneli and Peter Lax [52] showed that

$$A(x_1, x_2, x_3, x_4, x_5) := \sum_{i=1}^{5} \prod_{j \neq i} (x_i - x_j),$$

which appeared on the 1971 International Mathematical Olympiad, is psd and not sos. They observe that A is a polynomial in the $x_i - x_j$'s, so it is "really" a form in four variables. (Olympiad contestants were only asked to prove that A is psd!)

Konrad Schmüdgen [80], following the program of Gel'fand–Vilenkin, produced a sextic polynomial that homogenizes to a form in $\Delta_{3,6}$:

$$q(x,y,z) = 200(x^3 - 4xz^2)^2 + 200(y^3 - 4yz^2)^2 + (y^2 - x^2)x(x+2z)(x^2 - 2xz + 2y^2 - 8z^2).$$

The proof that q is psd involves decomposing $\mathbb{R}^3$ into ten regions; the proof that q is not sos involves the eight zeros of q, as predicted by Hilbert's original argument.

5. Some later developments

a. Zeros of psd forms and multiforms. My first collaboration with Choi and Lam, [16], was largely concerned with the number of zeros of psd forms. Recall that R has the ten zeros of $\mathcal{Z}$. We showed that if $p \in P_{3,6}$ and p has more than ten zeros, viewed projectively, then p is divisible by the square of an indefinite form and p is the sum of three squares of ternary cubics. If $p \in P_{3,6}$ has exactly ten zeros, then it cannot be sos. Moreover, if $p \in P_{3,m}$ has more than $m^2/4$ zeros, then it is either not sos or is divisible by the square of an indefinite form. If $p \in P_{4,4}$ has more than eleven zeros, then it has infinitely many zeros, and is the sum of six squares of quaternary quadratics.

Not every set of ten points (counted projectively) in $\mathbb{R}^3$ can be the zero-set of some form $p \in P_{3,6}$, but $\mathcal{Z}$ is not the only possibility. Here is a previously

unpublished example. For the real parameter $a \geq 0$, let

$$f_a(x, y, z) = a^4(x^6 + y^6 + z^6)$$
$$+(1 - 2a^6)(x^4 y^2 + y^4 z^2 + z^4 x^2) + (a^8 - 2a^2)(x^2 y^4 + y^2 z^4 + z^2 x^4)$$
$$-3(1 - 2a^2 + a^4 - 2a^6 + a^8)x^2 y^2 z^2.$$

Then $f_0 = S$ (see §4c), $f_1 = R$, and it can be shown that $f_a \in \Delta_{3,6}$ for $0 < a < 1$ with the following ten zeros: $\{(1, \pm 1, \pm 1), (\pm a, 1, 0), (0, \pm a, 1), (1, 0, \pm a)\}$.

A *multiform* of type $(n_1, \ldots, n_r; m_1, \ldots, m_r)$ is a form in $\sum n_k$ variables in r blocks—$\{x_{11}, \ldots, x_{1,n_1}; \ldots; x_{r,1}, \ldots, x_{r,n_r}\}$—so that for fixed k, each term is homogeneous of degree m_k in the $x_{k,j}$'s. It is shown in [16] that a psd multiform of type $(n_1, \ldots, n_r; m_1, \ldots, m_r)$ must be sos if and only if its type is $(2, n; m, 2)$ (or $(n, 2; 2, m)$). The counterexamples were closely based on Q and S. The fact that a psd multiform of type $(2, n; m, 2)$ is sos was already known: it is the assertion that an n-ary quadratic form $\sum_{ij} f_{ij}(y_1, y_2)x_i x_j$ $(f_{ij} \in H_m(\mathbb{R}^2))$ that is psd for every fixed (y_1, y_2) is a sum of squares of forms that are $\mathbb{R}[y_1, y_2]$-linear in the x_i's. This had been proved by Djoković, Jakubovič, Popov, and Rosenblum–Rovnyak in somewhat different contexts (see [16] for details and references). Calderón [7] had also proved it in the special case $m = 2$.

b. The Gram matrix method. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we shall write $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $|\alpha| = \sum_k \alpha_k$. Suppose

$$f(x_1, \ldots, x_n) = \sum_{|\alpha|=d} c(\alpha) x^\alpha \in H_d(\mathbb{R}^n),$$

and let $C(f) = conv(\{\alpha : c(\alpha) \neq 0\}) \subset \mathbb{R}^n$ denote the *Newton polytope* of f; $C(f)$ is a subset of the simplex whose vertices are $d e_k$. It was proved in [68] that, if $p = \sum_k h_k^2$, then $\frac{1}{2}C(p) \supseteq C(h_k)$. In the Motzkin example, $C(M)$ is the triangle whose vertices are $(4, 2, 0), (2, 4, 0), (0, 0, 6)$, so if $M = \sum_k h_k^2$, the monomials in h_k must come from the lattice points contained in the triangle with vertices $(2, 1, 0)$, $(1, 2, 0)$ and $(0, 0, 3)$. This triangle has one non-vertex lattice point, $(1,1,1)$, and the corresponding monomials are $x^2 y$, xy^2, z^3 and xyz. In this way, the first part of the term-inspection method can be automated.

The rest of the term-inspection method is formalized in [19] into the "Gram matrix" method. Suppose $p(x) = \sum_\alpha a_\alpha x^\alpha \in P_{n,2d}$ and $p(x) = \sum_{k=1}^t h_k^2(x)$, where $h_k(x) = \sum_\beta u_\beta^{(k)} x^\beta \in H_d(\mathbb{R}^n)$ for $1 \leq k \leq t$. Let $U_\beta = (u_\beta^{(1)}, \ldots, u_\beta^{(t)}) \in \mathbb{R}^t$. Then

$$\sum_{|\alpha|=2d} a_\alpha x^\alpha = p(x) = \sum_{k=1}^t h_k^2(x) = \sum_{k=1}^t \left(\sum_{|\beta|=d} u_\beta^{(k)} x^\beta \right) \left(\sum_{|\beta'|=d} u_{\beta'}^{(k)} x^{\beta'} \right)$$
$$= \sum_{|\beta|, |\beta'|=d} (U_\beta \cdot U_{\beta'}) x^{\beta + \beta'}.$$

By comparing the coefficients of x^α in p and $\sum_k h_k^2$, we see that

$$a_\alpha = \sum_{\beta + \beta' = \alpha} U_\beta \cdot U_{\beta'}.$$

Conversely, if there exist vectors $\{U_\beta\} \subset \mathbb{R}^t$ satisfying these equations for all α, then we can write p as a sum of t squares by using the coordinates of the U_β's as the coefficients of the h_k's. The dot product matrix $(U_\beta \cdot U_{\beta'})$ is called the *Gram matrix* associated to the expression $p = \sum_k h_k^2$. In order to state the main result of the Gram matrix method, we first recall that a symmetric matrix can serve as the set of dot products of vectors in $\mathbb{R}^t$ if and only if the corresponding quadratic form is psd with rank at most t. We also define the *length* of an sos form p to be the smallest number of forms $\{h_k\}$ required to write $p = \sum h_k^2$.

The following theorems are proved in [19, p. 106]:

(1) Suppose $p(x) = \sum_\alpha a_\alpha x^\alpha$, and let $V = (v_{\beta\beta'})$ be a real symmetric matrix. The following statements are equivalent:

(A) p is a sum of squares and V is a Gram matrix associated to p (with respect to some expression $p = \sum_k h_k^2$);

(B) V is psd and $\displaystyle\sum_{\beta+\beta'=\alpha} v_{\beta\beta'} = a_\alpha$ for all α.

(2) If p is a sum of squares, then the length of p is equal to the minimum rank of V, where V ranges over all Gram matrices associated to p.

Very recently, Powers–Wörmann [65] constructed an algorithm that implements the Gram matrix method.

c. Generalizations of M and S. The quadrinomial property of the counterexamples M and S is best possible. It was proved in [68] that any psd form that is not sos has at least four terms. Moreover, if $p \in \Delta_{n,m}$ has four terms, then it is extremal in $P_{n,m}$ if and only if, after a linear scaling of the variables $x_j \to \sigma_j x_j$, we have $p(x) = x^{2a} + x^{2b} + x^{2c} - 3x^{2d}$, where $a, b, c \in \mathbb{Z}_+^n$ have the following geometric property: if T is the triangle with vertices a, b, c, then $T \cap \mathbb{Z}^n$ must equal $\{a, b, c, d\}$, where $d = \frac{1}{3}(a + b + c)$ is the median of T. This implies the extremality of the forms M (with $a = (2,1,0)$, $b = (1,2,0)$, $c = (0,0,3)$, $d = (1,1,1)$) and S (with $a = (2,1,0)$, $b = (0,2,1)$, $c = (1,0,2)$, $d = (1,1,1)$).

An *agiform* (see [70]) is a form derived by making even monomial substitutions into the arithmetic-geometric inequality. Suppose $a_i \in (2\mathbb{Z})_+^n$ and λ_i are positive reals, $\sum_i \lambda_i = 1$, so that $\sum_i \lambda_i a_i = b \in \mathbb{Z}^n$. Then the agiform $\sum_i \lambda_i x^{a_i} - x^b$ is psd as a consequence of the arithmetic-geometric inequality. Necessary and sufficient conditions are given in [70] for an agiform to be sos, and necessary and sufficient conditions are given for an agiform to be an extremal psd form. These conditions depend heavily on the combinatorial structure of the lattice points contained in the simplex with vertices a_i. This paper also contains six explicit families of extremal psd forms in n variables, two each generalizing M, S and Q; three of these families were defined in [15] and proved there to be psd but not sos.

One consequence of the sos characterization of agiforms is that, if $p(x_1, \ldots, x_n)$ is an agiform and $r \geq n$, then $p(x_1^r, \ldots, x_n^r)$ is sos, so p is a sum of squares of forms in the variables $x_k^{1/r}$. This property is not true for all psd forms. If $H \in \Delta_{5,4}$ denotes the Horn form (see §4c), then it can be shown that for every $r \geq 1$, $H(x_1^r, \ldots, x_5^r)$ is not sos. A related question involves taking odd powers of psd forms. Stengle [81] proved in 1979 that for $m \geq 1$, every odd power of

$$x_1^{2m+1} x_3^{2m+1} + \left(x_2^2 x_3^{2m-1} - x_1^{2m+1} - x_1 x_3^{2m}\right)^2$$

is psd and not sos. It can be shown that this property also holds for the odd powers of $S(x_1, x_2, x_3)$ and $M(x_1, x_2, x_3)$.

d. Symmetric Examples. One obstacle to understanding the geometry of $P_{n,m}$ and $\Sigma_{n,m}$ is that these cones lie in R^N for $N = \binom{n+m-1}{n-1}$; if $\Delta_{n,m} \neq \emptyset$, then $N \geq 28$. One way to overcome this obstacle is to take sections of lower dimension. This suggests a study of even symmetric forms. Every psd even symmetric form of degree 2 or 4 is sos, so the simplest "interesting" case is $m = 6$; psd and sos even symmetric n-ary sextics were analyzed in [17]. Such a form p can be written

$$p(x_1, \ldots, x_n) = \alpha \sum_{i=1}^{n} x_i^6 + \beta \sum_{i \neq j} x_i^4 x_j^2 + \gamma \sum_{i < j < k} x_i^2 x_j^2 x_k^2.$$

It is more convenient to express p in terms of a different basis: write

$$p(x_1, \ldots, x_n) = a \sum_{i=1}^{n} x_i^6 + b \left(\sum_{i=1}^{n} x_i^4 \right) \left(\sum_{i=1}^{n} x_i^2 \right) + c \left(\sum_{i=1}^{n} x_i^2 \right)^3.$$

(These two expressions for p are related by $\alpha = a + b + c$, $\beta = b + 3c$, $\gamma = 6c$.) Let $p^*(t) = a + bt + ct^2$ and let $v^{(k)}$ denote any n-tuple whose coordinates consist of k 1's and $n - k$ 0's. Note that $p(v^{(k)}) = ak + bk^2 + ck^3 = kp^*(k)$, hence an immediate necessary condition that p be psd is that $p^*(k) \geq 0$ for $k = 1, 2, \ldots, n$. This is also a sufficient condition: p is psd if and only if it is nonnegative on the "test set" $\{v^{(1)}, \ldots, v^{(n)}\}$. The necessary and sufficient condition that p be sos is that $p^*(t) \geq 0$ for $t \in \{1\} \cup [2, n]$. (There is no obvious interpretation of $p^*(t)$ in this case; one might imagine that $p^*(v^{(t)})$ represents "evaluating" p at an "n-tuple" which has a non-integral t number of coordinates equal to 1, and the rest equal to 0.) For the Robinson form, a direct computation using the change-of-basis formulas given above shows that $R^*(t) = \frac{1}{2}(2 - t)(3 - t)$; since $R^*(t) < 0$ for $2 < t < 3$, this gives another proof that R is not sos.

William Harris [36] studied even symmetric octics ($m = 8$) and even symmetric ternary forms ($n = 3$). One surprising result is that every psd even symmetric ternary octic is sos. Harris gives test sets that determine whether an even n-ary symmetric octic or ternary decic ($m = 10$) is psd, and a list of all the extremal even ternary symmetric octics, as well as many new examples in $\Delta_{3,10}$ and $\Delta_{4,8}$.

We mentioned [18] earlier, in the context of showing that R is psd. This paper contains an extensive discussion of symmetric ternary forms. The possible sides of a triangle can be parameterized $(a, b, c) = (x^2 + y^2, x^2 + z^2, y^2 + z^2)$ in view of the triangle inequality, so any symmetric polynomial inequality satisfied by the sides a, b, c of an arbitrary triangle can be interpreted as a psd even ternary symmetric polynomial and vice versa. Harris [36] gives all symmetric polynomial inequalities of degree at most five satisfied by the sides of an arbitrary triangle.

6. Pólya's Theorem

In 1928, George Pólya [63] (see also [35, pp. 57–59]) gave an explicit solution to Hilbert's 17th Problem for even positive definite forms $p \in P_{n,2d}$; that is, for those

positive definite forms p that can be written $p(x_1, \ldots, x_n) = f(x_1^2, \ldots, x_n^2)$ for some $f \in H_d(\mathbb{R}^n)$.

Suppose $f(y_1, \ldots, y_n) > 0$ for $y \in \Delta_n := \{\sum_j y_j = 1, \ y_j \geq 0, \ 1 \leq j \leq n\}$. Pólya constructs a sequence of polynomials $\{f_t\}$ that converges uniformly to f on the compact set Δ_n as $t \to \infty$; hence, for $t \geq t_0 := t_0(f)$ and $y \in \Delta_n$, we have $f_t(y) \geq 0$. Elementary combinatorial manipulations give (for positive integers r)

$$\left(\sum_{i=1}^n y_i\right)^r f(y_1, \ldots, y_n) = r!(r+d)^d \sum_{|j|=r+d} \frac{f_{r+d}\left(\frac{j_1}{r+d}, \ldots, \frac{j_n}{r+d}\right)}{j_1! \cdots j_n!} y^j.$$

Since $\left(\frac{j_1}{r+d}, \ldots, \frac{j_n}{r+d}\right) \in \Delta_n$, the form $(\sum y_i)^r f(y_1, \ldots, y_n)$ has positive coefficients when $r \geq t_0(f) - d$.

Another way of viewing this result is that any form f that is positive on Δ_n can be written as the quotient of two polynomials with positive coefficients, where the denominator is a power of $\sum_i y_i$. Without this specification on the denominator, this result had been proved by Poincaré [62] in 1883 (the date is wrong in [35]) for $n = 2$ and by Meissner [55] in 1911 for $n = 3$.

Upon replacing y_i by x_i^2, Δ_n becomes the unit sphere, and Pólya's Theorem implies that if p is positive definite and even, then for sufficiently large r,

$$\left(\sum_{i=1}^n x_i^2\right)^r p(x_1, \ldots, x_n)$$

is a sum of monomials with positive coefficients. Since each monomial in the product uses only even exponents, it follows that $\left(\sum x_i^2\right)^r p(x_1, \ldots, x_n)$ is a *prima facie* sum of squares of monomials. And since the coefficients in the product evidently come from the same field as the coefficients of p, Pólya's Theorem gives a concrete, constructive solution to Hilbert's 17th Problem in the special case that p is even and positive definite. Pólya remarks on the significance of his result [63, p. 144]: "Es kann schliesslich bemerkt werden, dass die Darstellung einigermassen in Zusammenhang mit einer Fragestellung von Hilbert steht, die kürzlich durch E. Artin mit tiefgehenden Mittleln gelöst wurde."

In 1940, Habicht [31] (see also [35, pp. 300–304]) used Pólya's Theorem to prove directly that a (not necessarily even) positive definite form is a sum of squares of rational functions. The denominators in the representation are positive definite, but are no longer easy to compute; in particular, they are no longer necessarily powers of $\sum_i x_i^2$. The coefficients, however, are still in the original field. (It was erroneously stated in [23, p. 90] that Habicht proved that the denominators are powers of $\sum_i x_i^2$; I thank Delzell for informing me of this error.) A key step in the proof of Habicht's Theorem is the observation that, if $p \in P_{n,2d}$ is positive definite, then the following positive definite form of degree $2^n d$ is even:

$$q(x_1, \ldots, x_n) := \prod_{\epsilon_k = \pm 1} p(x_1, \epsilon_2 x_2, \ldots, \epsilon_n x_n).$$

Motzkin and Straus [58] partially generalized Pólya's Theorem to power series in several variables, and discussed some related algebraic questions. Catlin and

D'Angelo [9, 10] have recently generalized Pólya's Theorem (with denominator information) to polynomials in several complex variables. Handelman [32, 33] solved a related question. Suppose a polynomial p in several variables has nonnegative coefficients. For which f does there always exist an r so that $p^r f$ has nonnegative coefficients? Recently, De Loera and Santos [54] have turned Pólya's Theorem into an explicit algorithm, and made quantitative estimates for $t_0(f)$.

The restriction to positive definite forms is necessary. There exist positive semi-definite forms p that have the remarkable property that, in any representation $p = \sum_k \phi_k^2$, where $\phi_k = f_k/g_k$ is a rational function, each g_k must have a specified non-trivial zero. The existence of these so-called "bad points" insures that $p(\sum x_i^2)^r$ can never be a sum of squares of forms for *any* r. Habicht's Theorem implies that no positive definite form can have a bad point. Bad points were first noted by E. G. Straus in an unpublished 1956 letter to G. Kreisel. An extensive history of this topic can be found in Delzell's thesis [22], and in his [25, 26]. An example from [22] is $D[w, x, y, z] := w^2 S(x, y, z) + z^8$, where $S \in \Delta_{3,6}$ is defined in §4c. It is easy to show that D has a bad point at $(w, x, y, z) = (1, 0, 0, 0)$. Suppose $q \in H_d(\mathbb{R}^4)$ and $q(1, 0, 0, 0) = c \neq 0$. Then the the component of $q^2 D$ with highest degree in w is $c^2 w^{2d+2} S(x, y, z)$. Suppose now that $q^2 D = \sum h_k^2$ is a sum of squares of forms. Then the degree of w in h_k is $\leq d + 1$, and if we denote the component of w^{d+1} in h_k by $\bar{h}_k(x, y, z) w^{d+1}$, then by comparing the component of w^{2d+2} on both sides, we conclude that $c^2 S(x, y, z) = \sum_k \bar{h}_k^2(x, y, z)$, so S is sos, a contradiction.

7. Uniform denominators in Hilbert's 17th Problem

The final section of this paper is devoted to a sketch of the proof of the main theorem in [72]: if $p \in P_{n,m}$ is positive definite, then for sufficiently large r, $(\sum x_i^2)^r p$ is a sum of squares. That is, Pólya's conclusion about the shape of the denominator holds under Habicht's weaker hypothesis on p. Moreover, if $p \in H_m(K^n)$ is positive definite, then for sufficiently large r, $(\sum x_i^2)^r p$ is a positive linear combination over K of a set of $(2r + m)$-th powers of linear forms with rational coefficients, in which the linear forms depend only on the parameters m, r, n and not the form p. (Ellison [28] showed in 1969 that for all (n, m), $m \geq 4$, there are forms in $\Sigma_{n,m}$ that are not a sum of powers of linear forms, so the conclusion about $(\sum x_i^2)^r p$ is stronger than that it is sos.) Each component of the proof is, or could have been, familiar to Hilbert. For much more on this subject, see [64, 71, 72]. The construction will be concrete enough to give an explicit representation for Becker's $B(t)$ (see §3c) as a sum of $2k$-th powers over $\mathbb{R}$, but, unfortunately, not over $\mathbb{Q}$.

Write $G_n(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$. We shall say that an algebraic identity

$$G_n^s(x_1, \ldots, x_n) = (x_1^2 + \cdots + x_n^2)^s = \sum_{k=1}^{N} \lambda_k (\alpha_{k1} x_1 + \cdots + \alpha_{kn} x_n)^{2s},$$

where $0 < \lambda_k$ and $\alpha_{kj} \in \mathbb{R}$, is a *Hilbert Identity*. As part of his solution of Waring's Problem, Hilbert [42] proved that Hilbert Identities exist for every n and s, with the additional algebraic property that $\lambda_k, \alpha_{kj} \in \mathbb{Q}$. We shall call these *rational* Hilbert Identities. There are no known families of *explicit* rational Hilbert Identities for arbitrary n, s, although they are not hard to find for $s = 1, 2, 3$; see [71, §8, 9].

Hausdorff [37] gives explicit Hilbert Identities for all (n, s), using the roots of the Hermite polynomials; these are not, in general, rational. A simpler non-rational family based on trigonometric identities will be used below for $n = 2$.

Hilbert actually only showed the existence of rational Hilbert Identities for $n = 5$, but his method applies for all n. He also showed that rational Hilbert identities exist with $N \leq \binom{n+2s-1}{n-1}$. Nathanson [59, pp. 75–85] gives a clear modern exposition of [42], including a discussion of Hausdorff's construction. An alternative, self-contained proof of the existence of rational Hilbert Identities, together with some generalizations, is contained in [72, pp. 95–96].

The key to the proof is the differentiation of both sides of a (rational) Hilbert Identity. To be specific, if $h \in H_d(\mathbb{R}^n)$, define the associated d-th order differential operator $h(D)$ by replacing each appearance of x_j by $\frac{\partial}{\partial x_j}$; thus, $G_n(D) = \sum_j \frac{\partial^2}{\partial x_j^2} = \Delta$, the Laplacian. Fortunately, there already were known formulas to compute the effect of $h(D)$ on both sides of a Hilbert Identity.

In the 19th century, Sylvester and Clifford developed the method of "contravariant differentiation". As one consequence, if $h \in H_d(K^n)$ and $d \leq m$, then

$$h(D) \sum_k \lambda_k (\alpha_{k1} x_1 + \cdots + \alpha_{kn} x_n)^m$$

$$= (m)_d \sum_k \lambda_k h(\alpha_{k1}, \ldots, \alpha_{kn})(\alpha_{k1} x_1 + \cdots + \alpha_{kn} x_n)^{m-d}.$$

(Here, $(a)_t$ denotes the falling factorial $a(a-1)\cdots(a-(t-1))$.) This identity is not hard to prove. It suffices to consider a single m-th power, and by linearity, it suffices to consider $h(x) = x^i$. But then $h(D)$ is a product of successive $\frac{\partial}{\partial x_j}$'s, and the formula is immediate by the Chain Rule.

It is somewhat more difficult to evaluate $h(D)G_n^s$. Each differentiation reduces the exponent of G_n by at most one, so G_n^{s-d} divides $h(D)G_n^s$. This suggests the notation

$$h(D)G_n^s = \Phi_s(h)G_n^{s-d},$$

where $\Phi_s(h)$ has degree $2s - d - 2(s - d) = d$. Thus Φ_s is a linear map from $H_d(\mathbb{R}^n)$ to itself. An explicit formula for Φ_s follows from Hobson's Theorem: if $h \in H_d(\mathbb{R}^n)$ and F is any "sufficiently" differentiable function of one variable, then

$$h(D)F(G_n) = \sum_{k \geq 0} \frac{2^d}{2^{2k} k!} \Delta^k(h) F^{(d-k)}(G_n).$$

The right-hand side is a finite sum; if $k > d/2$, then $\Delta^k(h) = 0$. Hobson's Theorem is proved in [43, 44], see also [45]; it was lauded by Hardy [34] as an "elegant theorem in formal differentiation". Now, set $F(t) = t^s$, so $F^{(j)}(t) = (s)_j t^{s-j}$:

$$h(D)G_n^s = \sum_{k \geq 0} \frac{(s)_{d-k}}{2^{2k-d}k!} \Delta^k(h) G_n^{s-d+k} = \left(\sum_{k \geq 0} \frac{(s)_{d-k}}{2^{2k-d}k!} \Delta^k(h) G_n^k \right) G_n^{s-d}.$$

Thus,

$$\Phi_s(h) = \sum_{k \geq 0} \frac{(s)_{d-k}}{2^{2k-d}k!} \Delta^k(h) G_n^k.$$

Putting this all together, we see that $h(D)$ applied to a Hilbert Identity gives:

$$h(D)G_n^s = h(D)\left(\sum_{k=1}^{N} \lambda_k (\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^{2s}\right);$$

$$\Phi_s(h)G_n^{s-d} = (2s)_d \sum_{k=1}^{N} \lambda_k h(\alpha_{k1}, \ldots, \alpha_{kn})(\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^{2s-d}.$$

Now set $r = s - d$. We see that $\Phi_s(h)G_n^r$ is a linear combination of $(2r + d)$-th powers of linear forms. If h happens to be psd with coefficients in K, so d is even, then $\Phi_s(h)G_n^r$ is a positive linear combination over K of $(2r + d)$-th powers of linear forms, each of which is, *per se* a square. Thus $\Phi_s(h)$ can be written as a sum of squares of rational functions with denominator $G_n^{r/2}$. The proof will be complete if we can show that, if p is positive definite, then $h = \Phi_s^{-1}(p)$ is also positive definite for sufficiently large s.

The formula given below for $\Phi_s^{-1}(p)$ is apparently new to [72], but is well within the grasp of Hobson's techniques. If $s > d$, then

$$\Phi_s^{-1}(p) = \frac{1}{(s)_d 2^d} \sum_{\ell \geq 0} \frac{(-1)^\ell}{2^{2\ell}\ell!(\frac{n}{2} + s - 1)_\ell} \Delta^\ell(p)G_n^\ell$$

$$= \frac{1}{(s)_d 2^d}\left(p - \frac{\Delta(p)G_n}{2(n + 2s - 2)} + \frac{\Delta^2(p)G_n^2}{8(n + 2s - 2)(n + 2s - 4)} - \cdots\right).$$

This sum is also finite; if $\ell > m/2$, then $\Delta^\ell(p) = 0$. If $p \in H_d(K^n)$, then it is not hard to see that

$$\lim_{s \to \infty}(s)_d 2^d \Phi_s^{-1}(p) = p.$$

Thus, if p is positive definite, then so is $\Phi_s^{-1}(p)$, for sufficiently large s.

The preceding can be made quantitative. If p is positive definite, let

$$\epsilon(p) := \frac{\inf\{p(u) : u \in S^{n-1}\}}{\sup\{p(u) : u \in S^{n-1}\}}$$

measure how "close" p is to having a zero. After some pleasant estimates omitted here, including one comparing the L_∞ norms of p and Δp on S^{n-1}, it can be shown that if $p \in P_{n,m}$ is positive definite and $s \geq \frac{nm(m-1)}{(4\log 2)\epsilon(p)} - \frac{n-m}{2}$, then $\Phi_s^{-1}(p) \in P_{n,m}$.

Theorem. *Suppose $p \in H_m(K^n)$ is positive definite. If $r \geq \frac{nm(m-1)}{(4\log 2)\epsilon(p)} - \frac{n+m}{2}$, then pG_n^r is a nonnegative K-linear combination of a set of $(m + 2r)$-th powers of linear forms in $\mathbb{Q}[x_1, \ldots, x_n]$. This set depends only on n, m and r.*

The last sentence above is based on the fact that the linear forms come from the original rational Hilbert Identities. Interestingly, the analysis of Pólya's Theorem in [54] also shows a dependence of $t_0(f)$ on $\epsilon(f)^{-1}$.

Let $P_{n,m}^{(\epsilon)}$ be the set of $p \in P_{n,m}$ so that $\epsilon(p) \geq \epsilon$; observe that $P_{n,m} = \bigcup_{\epsilon > 0} P_{n,m}^{(\epsilon)}$. For each $\epsilon > 0$, the Theorem implies that if $p \in P_{n,m}^{(\epsilon)}$, $r \geq \frac{nm(m-1)}{(4\log 2)\epsilon} - \frac{n+m}{2}$ is even

and $G_n^{m+r} = \sum \lambda_k (\alpha_k \cdot x)^{2m+2r}$, then after applying $\left(\Phi_{r+m}^{-1}(p)\right)(D)$ and clearing fractions, we obtain

$$p(x_1, \ldots, x_n) = \sum_k \lambda_k(p) \left(\frac{(\alpha_{k1}x_1 + \cdots + \alpha_{kn}x_n)^{m/2+r}}{(x_1^2 + \cdots + x_n^2)^{r/2}} \right)^2,$$

where $\lambda_k(p) \geq 0$ is linear in p.

There has been considerable interest in the representations of p as a sum of squares of rational functions with continuous dependence on p, such as the one given above; see [24, 27]. Such a formula cannot hold over all of $P_{n,m}$. It is not hard to show that if $p \in P_{n,m}$ is not positive definite, then pG_n^r cannot be written as a sum of $(2r+m)$-th powers of linear forms over $\mathbb{R}$. The interested reader is referred to [72] for details.

The Theorem also gives concrete information about sums of $2k$-th powers of rational functions. The following Corollary (without the specification of the denominators) can be given an abstract proof using Becker's methods.

Corollary. *If $p \in K[x_1, \ldots, x_n]$ is a positive definite form of degree $m = 2kt$, then p is a nonnegative K-linear combination of $2k$-th powers of rational functions in $\mathbb{Q}[x_1, \ldots, x_n]$ whose denominators are powers of G_n. If p and q are positive definite forms in $K[x_1, \ldots, x_n]$ and $\deg p - \deg q$ is a multiple of $2k$, then p/q is a nonnegative K-linear combination of $2k$-th powers of rational functions whose numerators are in $\mathbb{Q}[x_1, \ldots, x_n]$ and whose denominators are products of powers of G_n and q.*

We conclude this paper with a sketch of an explicit formula for $B(t)$ as a sum of $2k$-th powers. We start with the simple observation that

$$B(t) = \frac{1+t^2}{2+t^2} = \frac{(1+t^2)(2+t^2)^{2k-1}}{(2+t^2)^{2k}};$$

hence if we can write $(1+t^2)(2+t^2)^{k-1}$ and $(2+t^2)^k$ as sums of $2k$-th powers of linear polynomials, then their product is a sum of $2k$-th powers of quadratics, each of which can then be divided by $(2+t^2)$ to give $B(t)$ as a sum of $2k$-th powers of rational functions.

There is an explicit family of Hilbert Identities for $n = 2$. If $v \geq s+1$, then

$$(x^2 + y^2)^s = \frac{2^{2s}}{v\binom{2s}{s}} \sum_{j=0}^{v-1} \left(\cos\left(\tfrac{j\pi}{v}\right)x + \sin\left(\tfrac{j\pi}{v}\right)y \right)^{2s}.$$

For a proof, see [71, p.124].) By taking $s = k$, $v = k+2$, $x = \sqrt{2}$, $y = t$ and applying the explicit formula for Φ_s^{-1} given above, we obtain after several pages of computation a formula for $B(t)$. Let $L_j(x, y) = (\cos\frac{j\pi}{k+2})x + (\sin\frac{j\pi}{k+2})y$ and $\lambda_j = 3k - (k+1)\cos(\frac{2j\pi}{k+2})$ for $0 \leq j \leq k+1$. Then

$$B(t) = \frac{1+t^2}{2+t^2} = \frac{2^{4k-2}}{k(k+2)^2\binom{2k}{k}^2} \sum_{i=0}^{k+1}\sum_{j=0}^{k+1} \lambda_j \left(\frac{L_i(\sqrt{2}, t)L_j(\sqrt{2}, t)}{2+t^2} \right)^{2k}.$$

Although this gives $B(t)$ as a sum of $2k$-th powers in $\mathbb{R}(t)$, the summands are not in $\mathbb{Q}(t)$. Such a representation cannot yet be found by these methods, because there is no known explicit infinite family of rational Hilbert Identities for $n = 2$.

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