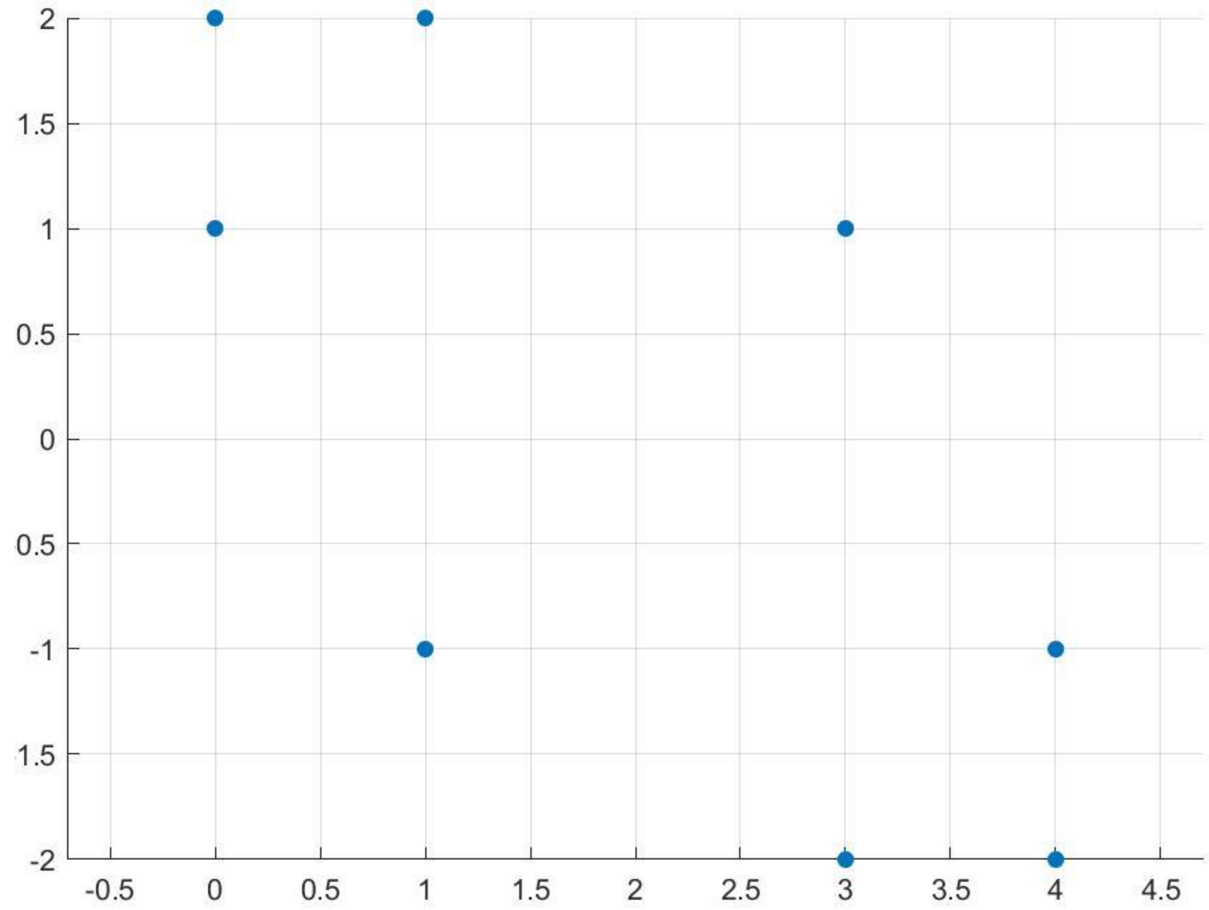
EXERCICE 2

Let the set of points

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\}\subset\mathbb{R}^2$$

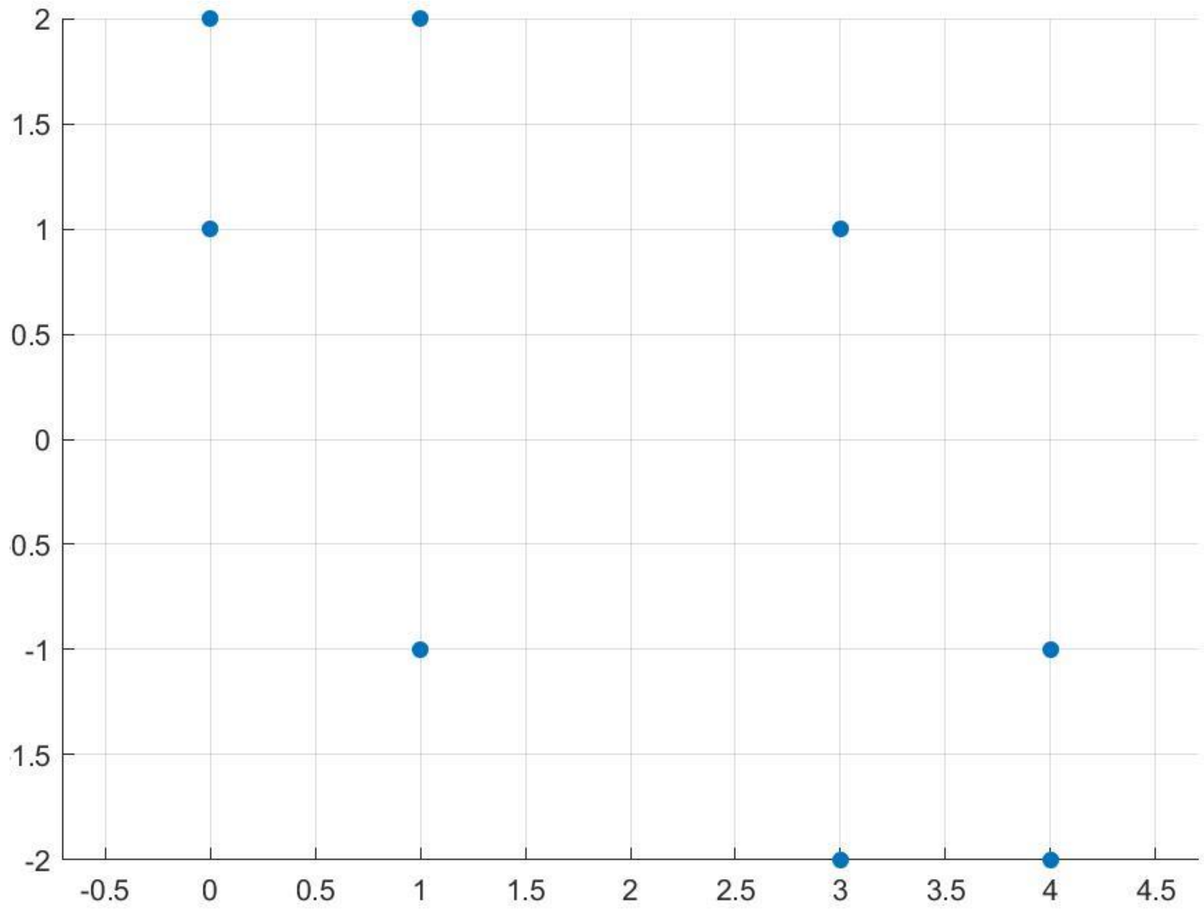
be driven from a Gaussian distribution. Estimate the parameters of such Gaussian distribution.



EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

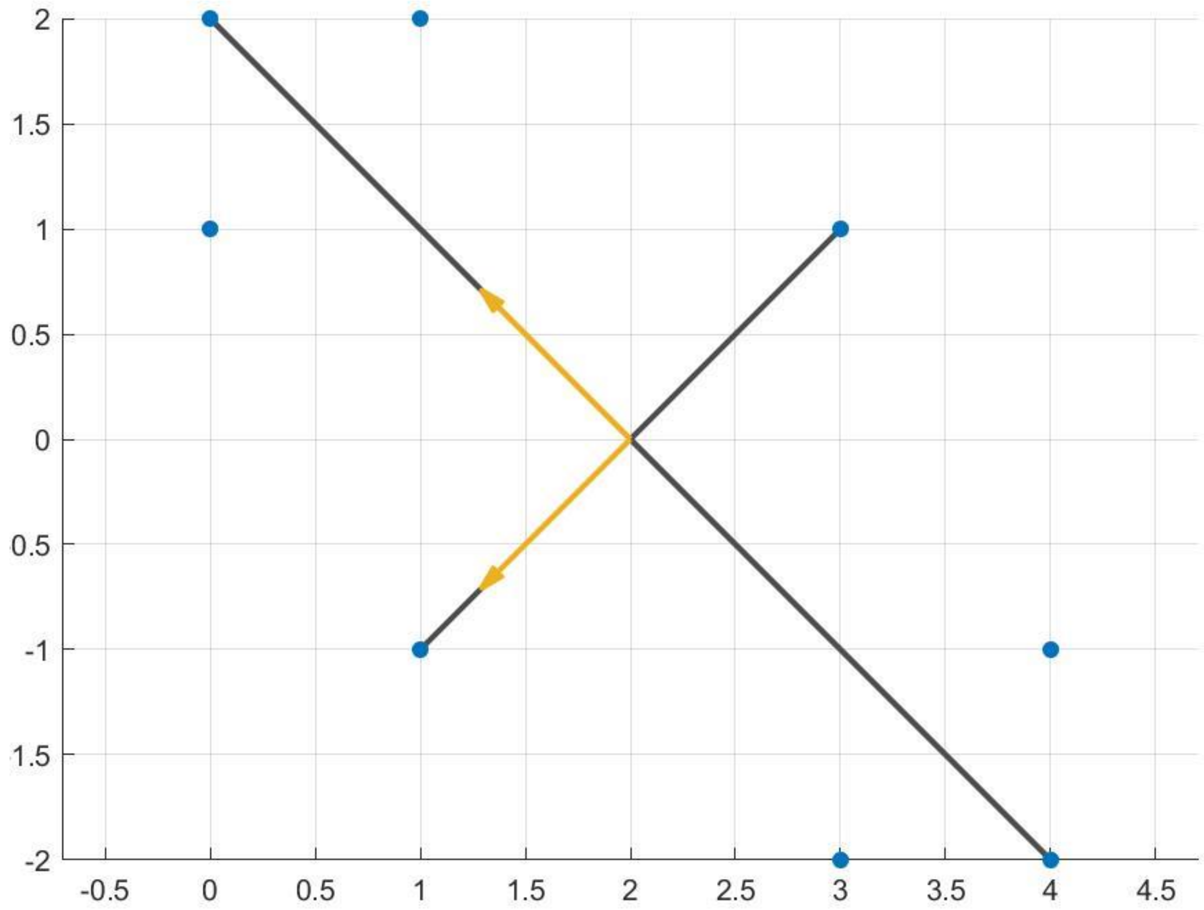
(a) Draw the set of points. Intuitively, draw the principal directions of the covariance matrix given by the Gaussian model.



EXERCICE 2

$$\left\{\binom{0}{2},\binom{0}{1},\binom{1}{2},\binom{1}{-1},\binom{3}{1},\binom{3}{-2},\binom{4}{-1},\binom{4}{-2}\right\} \subset \mathbb{R}^2$$

(a) Draw the set of points. Intuitively, draw the principal directions of the covariance matrix given by the Gaussian model.



EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

(b) Compute the mean and the covariance matrix of the given set of points.

MEAN

$$\boldsymbol{\mu} = \frac{1}{N}\sum_{n=1}^{N} \boldsymbol{x}^{(n)} = ?$$

COVARIANCE

$$\boldsymbol{\Sigma} = \frac{1}{N-1}(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^{\mathrm{T}} = ?$$

This is the **unbiased** version of the sample covariance.
And it is what the Python function `np.cov` computes.

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

(b) Compute the mean and the covariance matrix of the given set of points.

MEAN

$$\boldsymbol{\mu} = \frac{1}{N}\sum_{n=1}^{N} \boldsymbol{x}^{(n)} = \frac{1}{8}\left[\begin{pmatrix}0\\2\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}3\\1\end{pmatrix}+\begin{pmatrix}3\\-2\end{pmatrix}+\begin{pmatrix}4\\-1\end{pmatrix}+\begin{pmatrix}4\\-2\end{pmatrix}\right] = \frac{1}{8}\begin{pmatrix}16\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$$

COVARIANCE

$\boldsymbol{\Sigma} = \frac{1}{N-1}(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^{\mathrm{T}}$ where

$$\boldsymbol{X} - \boldsymbol{\mu} = \begin{pmatrix}0 & 0 & 1 & 1 & 3 & 3 & 4 & 4\\ 2 & 1 & 2 & -1 & 1 & -2 & -1 & -2\end{pmatrix} - \begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}-2 & -2 & -1 & -1 & 1 & 1 & 2 & 2\\ 2 & 1 & 2 & -1 & 1 & -2 & -1 & -2\end{pmatrix}$$

$$\Rightarrow \boldsymbol{\Sigma} = \frac{1}{N-1}(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^{\mathrm{T}} = \frac{1}{7}\begin{pmatrix}20 & -14\\ -14 & 20\end{pmatrix}$$

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

(c) Compute the principal directions of the covariance matrix. Draw the resulting new coordinate system and check if they correspond to the directions you sketched in (1).

Principal directions of $\mathbf{\Sigma}$ = eigenvectors of $\mathbf{\Sigma} \Longrightarrow$ We have to diagonalize $\mathbf{\Sigma}$

To diagonalize $\mathbf{\Sigma}$:

1. Compute the characteristic polynomial $p_{\mathbf{\Sigma}}(\lambda)$.
2. Compute the eigenvalues (which are the roots of $p_{\mathbf{\Sigma}}(\lambda)$).
3. Compute the eigenvectors (which are the basis of the null space of $\mathbf{\Sigma} - \lambda \mathrm{Id}$ for each eigenvalue λ).

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\}\subset\mathbb{R}^2$$

(c) Compute the principal directions of the covariance matrix. Draw the resulting new coordinate system and check if they correspond to the directions you sketched in (1).

To diagonalize $\mathbf{\Sigma}$:

1. Compute the characteristic polynomial $p_{\mathbf{\Sigma}}(\lambda) = \det(\mathbf{\Sigma} - \lambda \mathrm{Id})$.

$$\mathbf{\Sigma} = \frac{1}{7}\begin{pmatrix}20 & -14\\-14 & 20\end{pmatrix} \Rightarrow p_{\mathbf{\Sigma}}(\lambda) = ?$$

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

(c) Compute the principal directions of the covariance matrix. Draw the resulting new coordinate system and check if they correspond to the directions you sketched in (1).

To diagonalize $\mathbf{\Sigma}$:

1. Compute the characteristic polynomial $p_{\mathbf{\Sigma}}(\lambda) = \det(\mathbf{\Sigma} - \lambda \mathrm{Id})$.

$$\mathbf{\Sigma} = \frac{1}{7}\begin{pmatrix}20 & -14\\-14 & 20\end{pmatrix} \Rightarrow p_{\mathbf{\Sigma}}(\lambda) = \det\begin{pmatrix}\frac{20}{7}-\lambda & -2\\-2 & \frac{20}{7}-\lambda\end{pmatrix} = \left(\lambda^2 - \frac{40}{7}\lambda + \frac{204}{49}\right)$$

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

(c) Compute the principal directions of the covariance matrix. Draw the resulting new coordinate system and check if they correspond to the directions you sketched in (1).

To diagonalize $\mathbf{\Sigma}$:

2. Compute the eigenvalues (which are the roots of $p_{\mathbf{\Sigma}}(\lambda)$).

$$p_{\mathbf{\Sigma}}(\lambda) = \lambda^2 - \frac{40}{7}\lambda + \frac{204}{49} \Longrightarrow \begin{cases} \lambda_1 = ? \\ \lambda_2 = ? \end{cases}$$

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

(c) Compute the principal directions of the covariance matrix. Draw the resulting new coordinate system and check if they correspond to the directions you sketched in (1).

To diagonalize $\mathbf{\Sigma}$:

2. Compute the eigenvalues (which are the roots of $p_{\mathbf{\Sigma}}(\lambda)$).

$$p_{\mathbf{\Sigma}}(\lambda) = \lambda^2 - \frac{40}{7}\lambda + \frac{204}{49} = \left(\lambda - \frac{6}{7}\right)\left(\lambda - \frac{34}{7}\right) \Longrightarrow \begin{cases} \lambda_1 = \dfrac{34}{7} \\ \lambda_2 = \dfrac{6}{7} \end{cases}$$

$$\lambda = \frac{40 \pm \sqrt{(-40)^2 - 4 \cdot 204}}{2}$$

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

(c) Compute the principal directions of the covariance matrix. Draw the resulting new coordinate system and check if they correspond to the directions you sketched in (1).

To diagonalize $\mathbf{\Sigma}$:

3. Compute the eigenvectors (which are the basis of the null space of $\mathbf{\Sigma} - \lambda\mathrm{Id}$ for each eigenvalue λ).

$$\boxed{\lambda_1 = \frac{34}{7}} \quad \boldsymbol{v}^{(1)} = \mathrm{Ker}\left(\mathbf{\Sigma} - \frac{34}{7}\mathrm{Id}\right) = ?$$

$$\boxed{\lambda_2 = \frac{6}{7}} \quad \boldsymbol{v}^{(2)} = \mathrm{Ker}\left(\mathbf{\Sigma} - \frac{6}{7}\mathrm{Id}\right) = ?$$

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

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To diagonalize $\mathbf{\Sigma}$:

3. Compute the eigenvectors (which are the basis of the null space of $\mathbf{\Sigma} - \lambda \mathrm{Id}$ for each eigenvalue λ).

$$\boxed{\lambda_1 = \frac{34}{7}} \quad \boldsymbol{v}^{(1)} = \mathrm{Ker}\begin{pmatrix}20-34 & -14\\ -14 & 20-34\end{pmatrix} = \mathrm{Ker}\begin{pmatrix}-14 & -14\\ -14 & -14\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} \Rightarrow \boldsymbol{v}^{(1)} := \frac{\boldsymbol{v}^{(1)}}{\left|\boldsymbol{v}^{(1)}\right|} = \frac{\sqrt{2}}{2}\begin{pmatrix}-1\\1\end{pmatrix}$$

$$\mathrm{Ker}(\boldsymbol{A}) = \mathrm{Ker}(\alpha\boldsymbol{A}), \forall\alpha \in \mathbb{R}$$

$$\begin{pmatrix}\frac{20}{7} & \frac{-14}{7}\\ \frac{-14}{7} & \frac{20}{7}\end{pmatrix} - \begin{pmatrix}\frac{34}{7} & 0\\ 0 & \frac{34}{7}\end{pmatrix} = \frac{1}{7}\left[\begin{pmatrix}20 & -14\\ -14 & 20\end{pmatrix} - \begin{pmatrix}34 & 0\\ 0 & 34\end{pmatrix}\right]$$

$$\begin{pmatrix}-14 & -14\\ -14 & -14\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

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To diagonalize $\boldsymbol{\Sigma}$:

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$$\boxed{\lambda_1 = \frac{34}{7}} \quad \boldsymbol{v}^{(1)} = \mathrm{Ker}\begin{pmatrix}20-34 & -14\\ -14 & 20-34\end{pmatrix} = \mathrm{Ker}\begin{pmatrix}-14 & -14\\ -14 & -14\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} \Rightarrow \boldsymbol{v}^{(1)} := \frac{\boldsymbol{v}^{(1)}}{|\boldsymbol{v}^{(1)}|} = \frac{\sqrt{2}}{2}\begin{pmatrix}-1\\1\end{pmatrix}$$

$$\boxed{\lambda_2 = \frac{6}{7}} \quad \boldsymbol{v}^{(2)} = \mathrm{Ker}\begin{pmatrix}20-6 & -14\\ -14 & 20-6\end{pmatrix} = \mathrm{Ker}\begin{pmatrix}14 & -14\\ -14 & 14\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \Rightarrow \boldsymbol{v}^{(2)} := \frac{\boldsymbol{v}^{(2)}}{|\boldsymbol{v}^{(2)}|} = \frac{\sqrt{2}}{2}\begin{pmatrix}1\\1\end{pmatrix}$$

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

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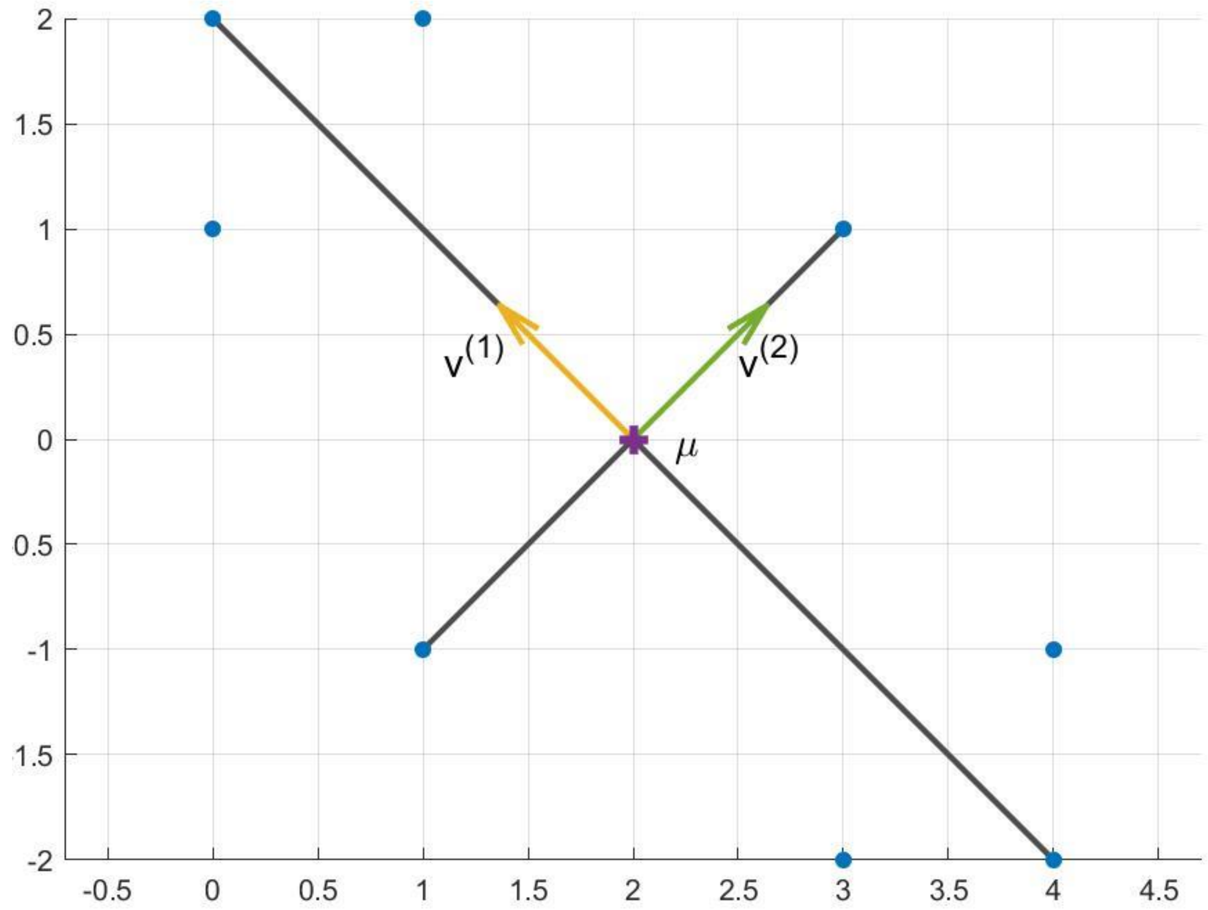


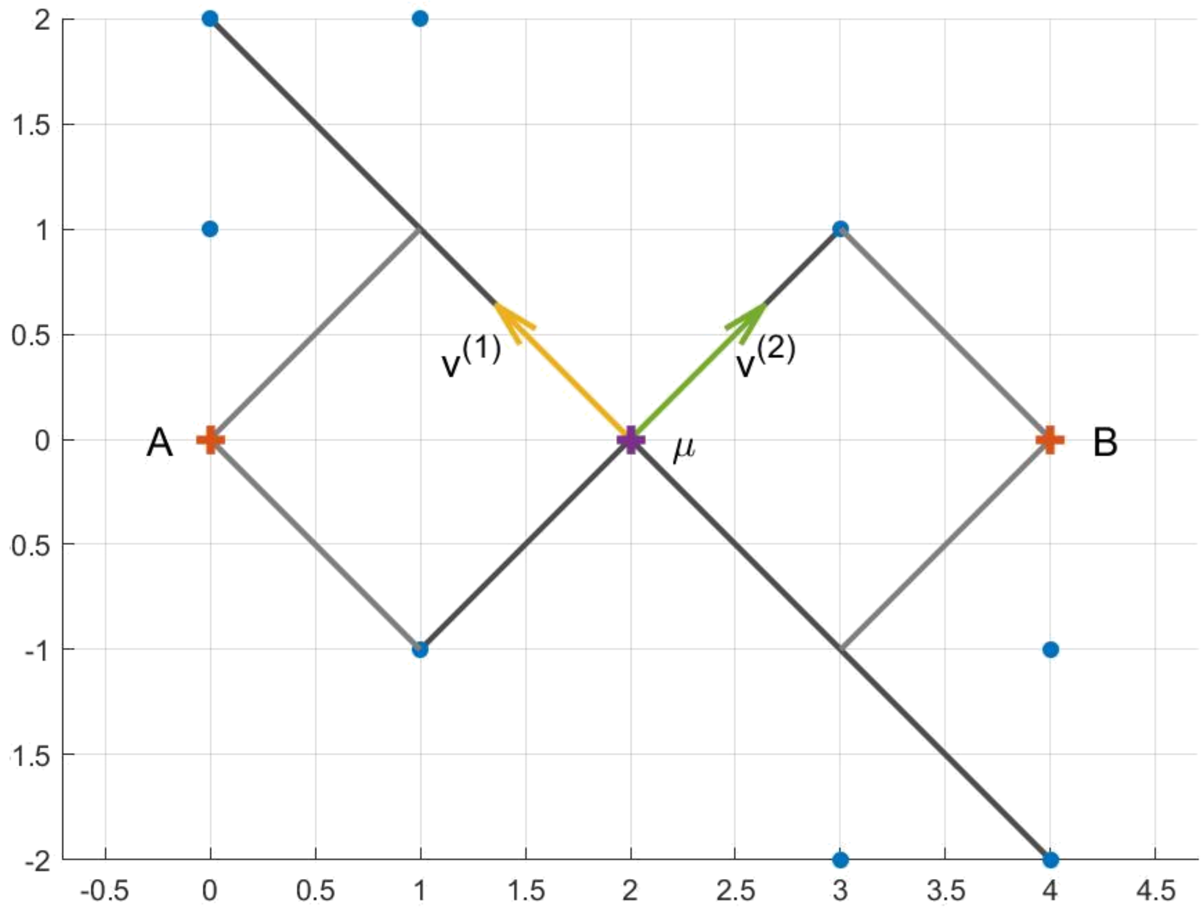
$$\boldsymbol{v}^{(1)} = \frac{\sqrt{2}}{2}\begin{pmatrix}-1\\1\end{pmatrix}$$

$$\boldsymbol{v}^{(2)} = \frac{\sqrt{2}}{2}\begin{pmatrix}1\\1\end{pmatrix}$$

EXERCICE 2

$$\left\{\binom{0}{2},\binom{0}{1},\binom{1}{2},\binom{1}{-1},\binom{3}{1},\binom{3}{-2},\binom{4}{-1},\binom{4}{-2}\right\}\subset\mathbb{R}^2$$

(d) Compute the coordinates of the points $\mathrm{A}=\binom{0}{0}$ and $\mathrm{B}=\binom{4}{0}$ in the system of coordinates adapted to the Gaussian.



EXERCICE 2

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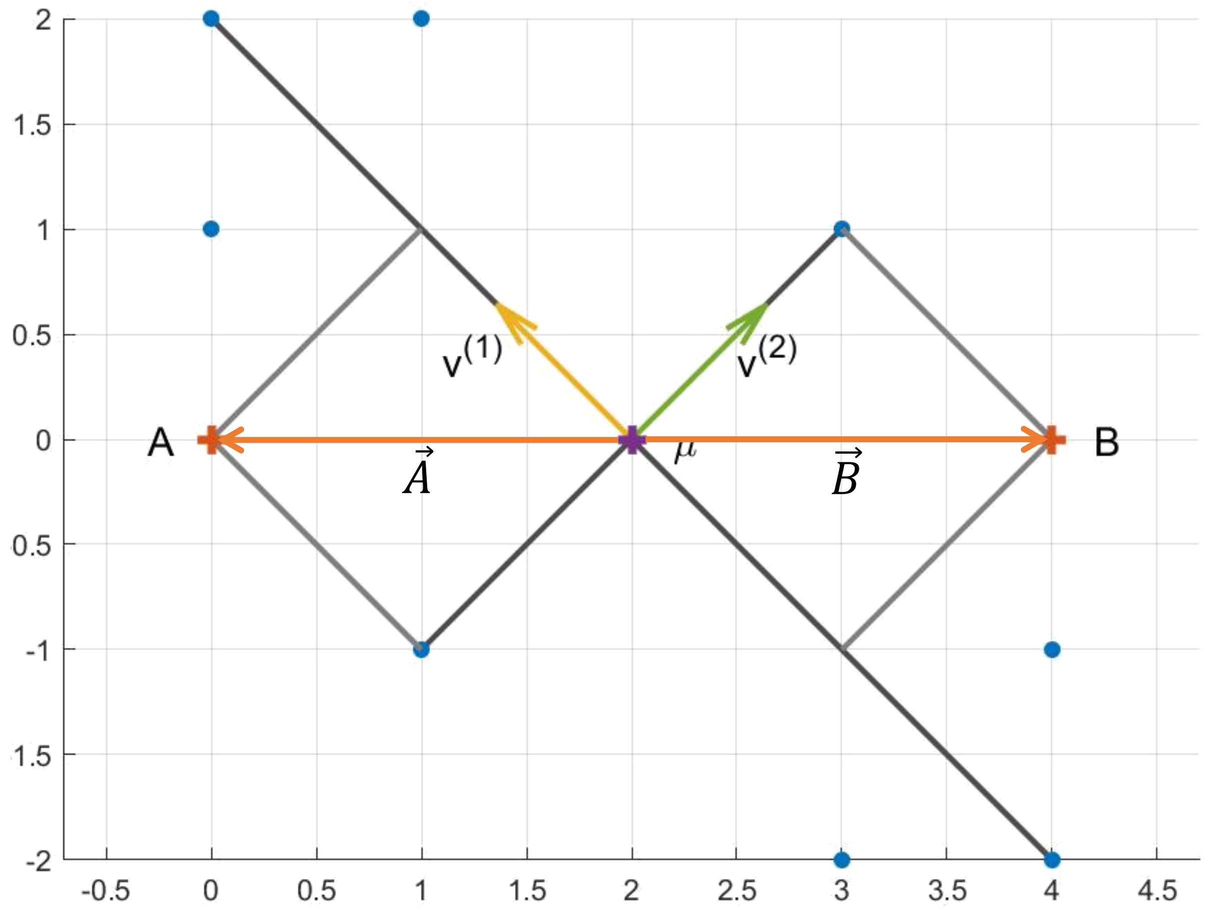

$$\vec{A} = A - \boldsymbol{\mu} = \binom{0}{0} - \binom{2}{0} = \binom{-2}{0} \longrightarrow \vec{A}_{<v>}?$$

$$\vec{B} = B - \boldsymbol{\mu} = \binom{4}{0} - \binom{2}{0} = \binom{2}{0} \longrightarrow \vec{B}_{<v>}?$$

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

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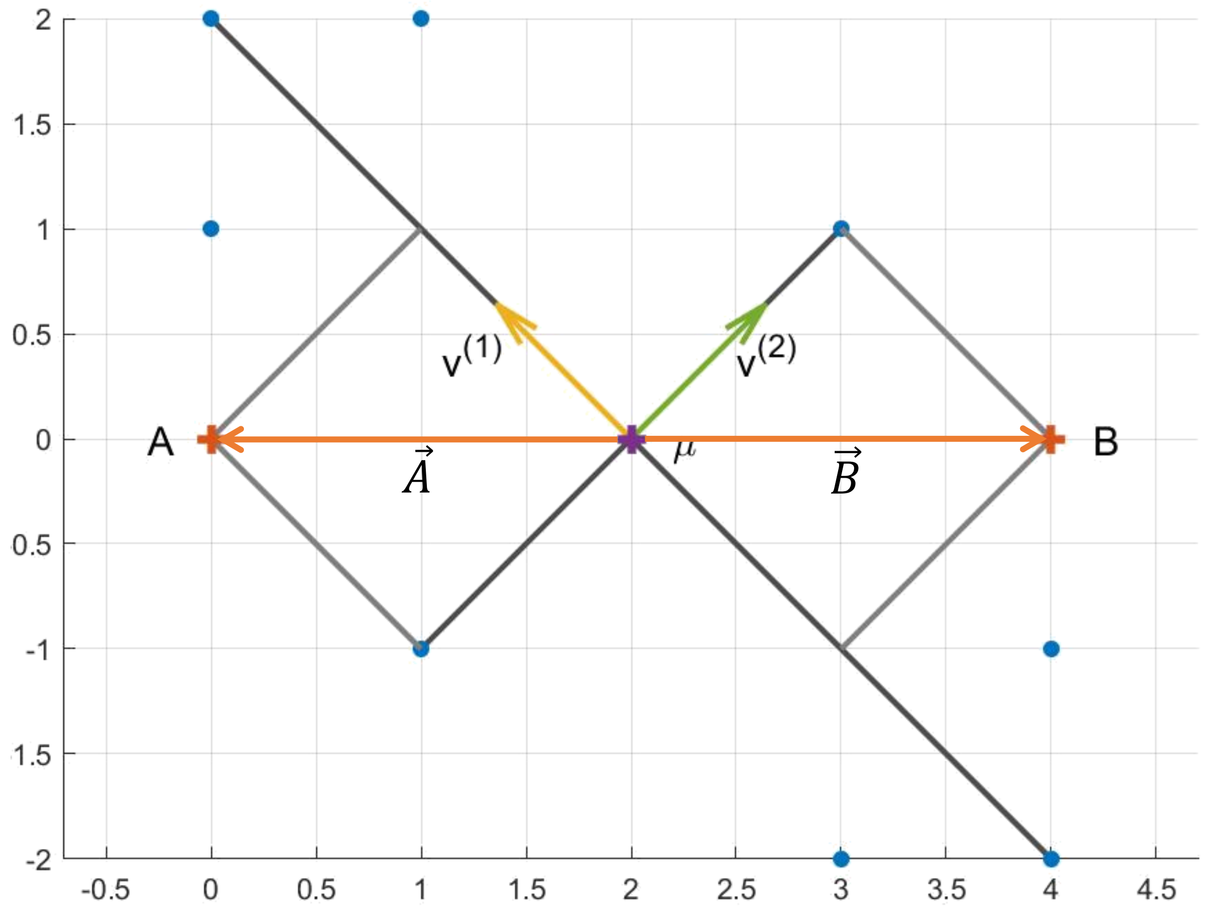

$$A_{<\nu>} = {}_{<\varepsilon>}\mathrm{M}_{<\nu>}\vec{A} = \frac{\sqrt{2}}{2}\begin{pmatrix}-1 & 1\\1 & 1\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}-2\\0\end{pmatrix} = \begin{pmatrix}\sqrt{2}\\-\sqrt{2}\end{pmatrix}$$

$$B_{<\nu>} = {}_{<\varepsilon>}\mathrm{M}_{<\nu>}\vec{B} = \frac{\sqrt{2}}{2}\begin{pmatrix}-1 & 1\\1 & 1\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}-\sqrt{2}\\\sqrt{2}\end{pmatrix}$$

$\boldsymbol{v}^{(1)}$ and $\boldsymbol{v}^{(2)}$ orthonormal

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

(e) Write the PDF of the resulting Gaussian model in the new coordinate system and the standard coordinate system.

Gaussian PDF :

$$f(\boldsymbol{x}) = (2\pi)^{-\frac{\mathrm{k}}{2}} \det(\boldsymbol{\Sigma})^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right]$$

- Standard coordinate system $\left\{\boldsymbol{o} = \begin{pmatrix}0\\0\end{pmatrix}; \boldsymbol{e}^{(1)} = \begin{pmatrix}1\\0\end{pmatrix}, \boldsymbol{e}^{(2)} = \begin{pmatrix}0\\1\end{pmatrix}\right\}$:

$$\left.\begin{aligned}\boldsymbol{\mu} &= \begin{pmatrix}2\\0\end{pmatrix}\\ \boldsymbol{\Sigma} &= \frac{1}{7}\begin{pmatrix}20 & -14\\-14 & 20\end{pmatrix}\end{aligned}\right\} f(\boldsymbol{x}) = ?$$

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

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$$\left.\begin{aligned}\boldsymbol{\mu} &= \begin{pmatrix}2\\0\end{pmatrix}\\ \boldsymbol{\Sigma} &= \frac{1}{7}\begin{pmatrix}20 & -14\\ -14 & 20\end{pmatrix}\end{aligned}\right\} f(\boldsymbol{x}) = (2\pi)^{-1}\cdot\det\left(\frac{1}{7}\begin{pmatrix}20 & -14\\ -14 & 20\end{pmatrix}\right)^{-\frac{1}{2}}\cdot\exp\left[-\frac{1}{2}\left(\boldsymbol{x}-\begin{pmatrix}2\\0\end{pmatrix}\right)^{\mathrm{T}}\begin{pmatrix}\frac{20}{7} & \frac{-14}{7}\\ \frac{-14}{7} & \frac{20}{7}\end{pmatrix}^{-1}\left(\boldsymbol{x}-\begin{pmatrix}2\\0\end{pmatrix}\right)\right]$$

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

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Gaussian PDF :

$$f(\boldsymbol{x}) = (2\pi)^{-\frac{k}{2}} \det(\boldsymbol{\Sigma})^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right]$$

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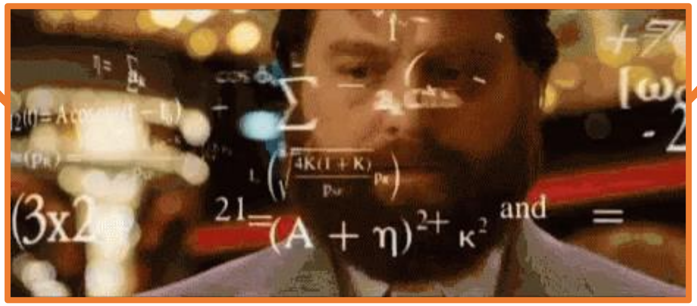
$$\left.\begin{aligned}\boldsymbol{\mu} &= \begin{pmatrix}2\\0\end{pmatrix}\\ \boldsymbol{\Sigma} &= \frac{1}{7}\begin{pmatrix}20 & -14\\-14 & 20\end{pmatrix}\end{aligned}\right\} f(\boldsymbol{x}) = (2\pi)^{-1}\cdot\det\left(\frac{1}{7}\begin{pmatrix}20 & -14\\-14 & 20\end{pmatrix}\right)^{-\frac{1}{2}}\cdot\exp\left[-\frac{1}{2}\left(\boldsymbol{x}-\begin{pmatrix}2\\0\end{pmatrix}\right)^{\mathrm{T}}\begin{pmatrix}\frac{20}{7} & \frac{-14}{7}\\ \frac{-14}{7} & \frac{20}{7}\end{pmatrix}^{-1}\left(\boldsymbol{x}-\begin{pmatrix}2\\0\end{pmatrix}\right)\right]$$



EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

(e) Write the PDF of the resulting Gaussian model in the new coordinate system and the standard coordinate system.

Gaussian PDF :

$$f(\boldsymbol{x}) = (2\pi)^{-\frac{k}{2}} \det(\boldsymbol{\Sigma})^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right]$$

- <u>New coordinate system</u> $\left\{\boldsymbol{o} = \boldsymbol{\mu} = \begin{pmatrix}2\\0\end{pmatrix}; \boldsymbol{v}^{(1)} = \frac{\sqrt{2}}{2}\begin{pmatrix}-1\\1\end{pmatrix}, \boldsymbol{v}^{(2)} = \frac{\sqrt{2}}{2}\begin{pmatrix}1\\1\end{pmatrix}\right\}$:

$$\left.\begin{aligned}\boldsymbol{\mu}_{<\mathcal{V}>} &= \begin{pmatrix}0\\0\end{pmatrix}\\ \boldsymbol{\Sigma}_{<\mathcal{V}>} &= \frac{1}{7}\begin{pmatrix}34 & 0\\0 & 6\end{pmatrix}\end{aligned}\right\} \quad f(\boldsymbol{x}) = ?$$

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

(e) Write the PDF of the resulting Gaussian model in the new coordinate system and the standard coordinate system.

Gaussian PDF :

$$f(\boldsymbol{x}) = (2\pi)^{-\frac{k}{2}} \det(\boldsymbol{\Sigma})^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right]$$

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$$\left.\begin{aligned}\boldsymbol{\mu}_{<\mathcal{V}>} &= \begin{pmatrix}0\\0\end{pmatrix}\\ \boldsymbol{\Sigma}_{<\mathcal{V}>} &= \frac{1}{7}\begin{pmatrix}34 & 0\\0 & 6\end{pmatrix}\end{aligned}\right\} \quad f(\boldsymbol{x}) = (2\pi)^{-1}\cdot\det\left(\frac{1}{7}\begin{pmatrix}34 & 0\\0 & 6\end{pmatrix}\right)^{-\frac{1}{2}}\cdot\exp\left[-\frac{1}{2}\left(\boldsymbol{x}-\begin{pmatrix}0\\0\end{pmatrix}\right)^{\mathrm{T}}\begin{pmatrix}\frac{34}{7} & 0\\0 & \frac{6}{7}\end{pmatrix}^{-1}\left(\boldsymbol{x}-\begin{pmatrix}0\\0\end{pmatrix}\right)\right]$$

$$\det\begin{pmatrix}\frac{34}{7} & 0\\0 & \frac{6}{7}\end{pmatrix} = \lambda_1\cdot\lambda_2 = \frac{34\cdot 6}{7\cdot 7}$$

$$\begin{pmatrix}\frac{34}{7} & 0\\0 & \frac{6}{7}\end{pmatrix}^{-1} = \begin{pmatrix}\frac{34}{7}^{-1} & 0\\0 & \frac{6}{7}^{-1}\end{pmatrix} = \begin{pmatrix}\frac{7}{34} & 0\\0 & \frac{7}{6}\end{pmatrix}$$

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

(e) Write the PDF of the resulting Gaussian model in the new coordinate system and the standard coordinate system.

Gaussian PDF :

$$f(\boldsymbol{x}) = (2\pi)^{-\frac{k}{2}} \det(\boldsymbol{\Sigma})^{-\frac{1}{2}} \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right]$$

- New coordinate system $\left\{\boldsymbol{o} = \boldsymbol{\mu} = \begin{pmatrix}2\\0\end{pmatrix}; \boldsymbol{v}^{(1)} = \frac{\sqrt{2}}{2}\begin{pmatrix}-1\\1\end{pmatrix}, \boldsymbol{v}^{(2)} = \frac{\sqrt{2}}{2}\begin{pmatrix}1\\1\end{pmatrix}\right\}$:

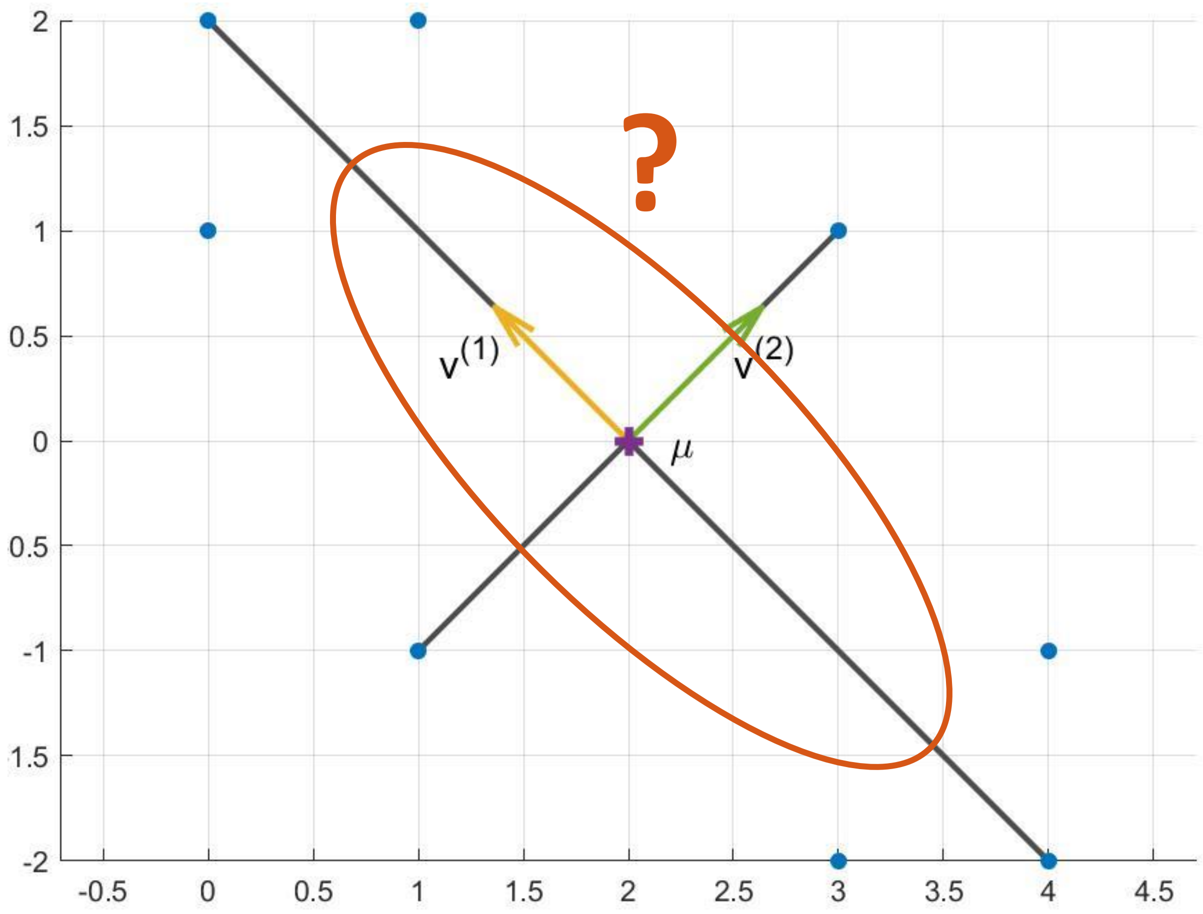
$$\left.\begin{aligned}\boldsymbol{\mu}_{<\mathcal{V}>} &= \begin{pmatrix}0\\0\end{pmatrix}\\ \boldsymbol{\Sigma}_{<\mathcal{V}>} &= \frac{1}{7}\begin{pmatrix}34 & 0\\0 & 6\end{pmatrix}\end{aligned}\right\} \quad f(\boldsymbol{x}) = (2\pi)^{-1}\cdot\frac{7}{2\sqrt{51}}\cdot\exp\left[-\frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\begin{pmatrix}\frac{7}{34} & 0\\0 & \frac{7}{6}\end{pmatrix}\boldsymbol{x}\right]$$

Should be written in the new coordinate system.

EXERCICE 2

$$\left\{\binom{0}{2},\binom{0}{1},\binom{1}{2},\binom{1}{-1},\binom{3}{1},\binom{3}{-2},\binom{4}{-1},\binom{4}{-2}\right\} \subset \mathbb{R}^2$$

(f) Draw the ellipses defined by the points with Mahalanobis distances equal to 1 and 2 (without computing it).



REMINDER

Equation of an ellipse

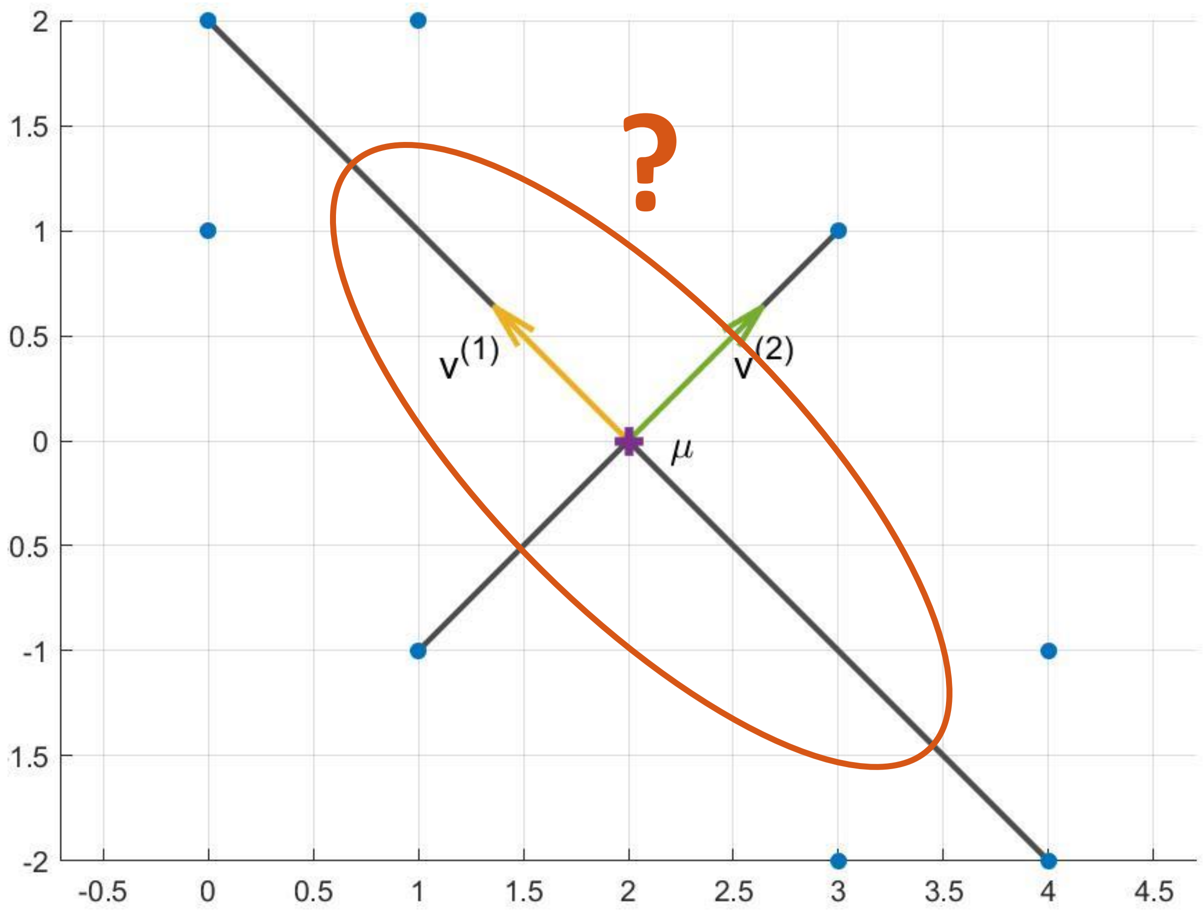
$$k^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} = d_{\mathrm{Mah}}^2$$

$a, b, k?$

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

(f) Draw the ellipses defined by the points with Mahalanobis distances equal to 1 and 2 (without computing it).



REMINDER
Equation of an ellipse

$$k^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} = d^2_{\mathrm{Mah}}$$

$\rightarrow a, b, k?$

$$d^2_{\mathrm{Mah}} = (\boldsymbol{x} - \boldsymbol{m})^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{m})$$

$$= \left(\begin{pmatrix}x_1\\x_2\end{pmatrix} - \begin{pmatrix}0\\0\end{pmatrix}\right)^{\mathrm{T}} \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix}^{-1} \left(\begin{pmatrix}x_1\\x_2\end{pmatrix} - \begin{pmatrix}0\\0\end{pmatrix}\right)$$

$$= \begin{pmatrix}x_1 & x_2\end{pmatrix} \begin{pmatrix}\frac{1}{\lambda_1} & 0\\0 & \frac{1}{\lambda_2}\end{pmatrix} \begin{pmatrix}x_1\\x_2\end{pmatrix} = \frac{x_1^2}{\lambda_1} + \frac{x_2^2}{\lambda_2}$$

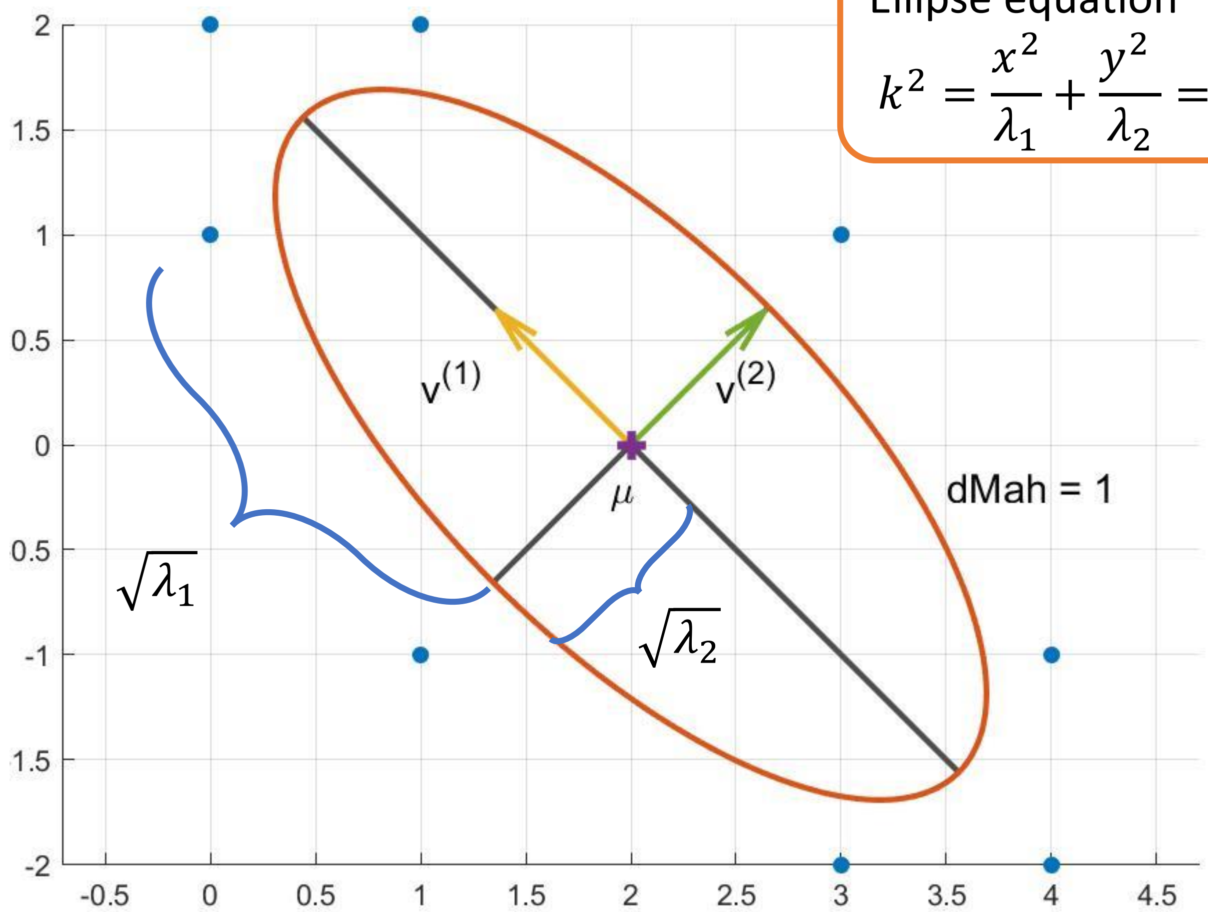
EXERCICE 2

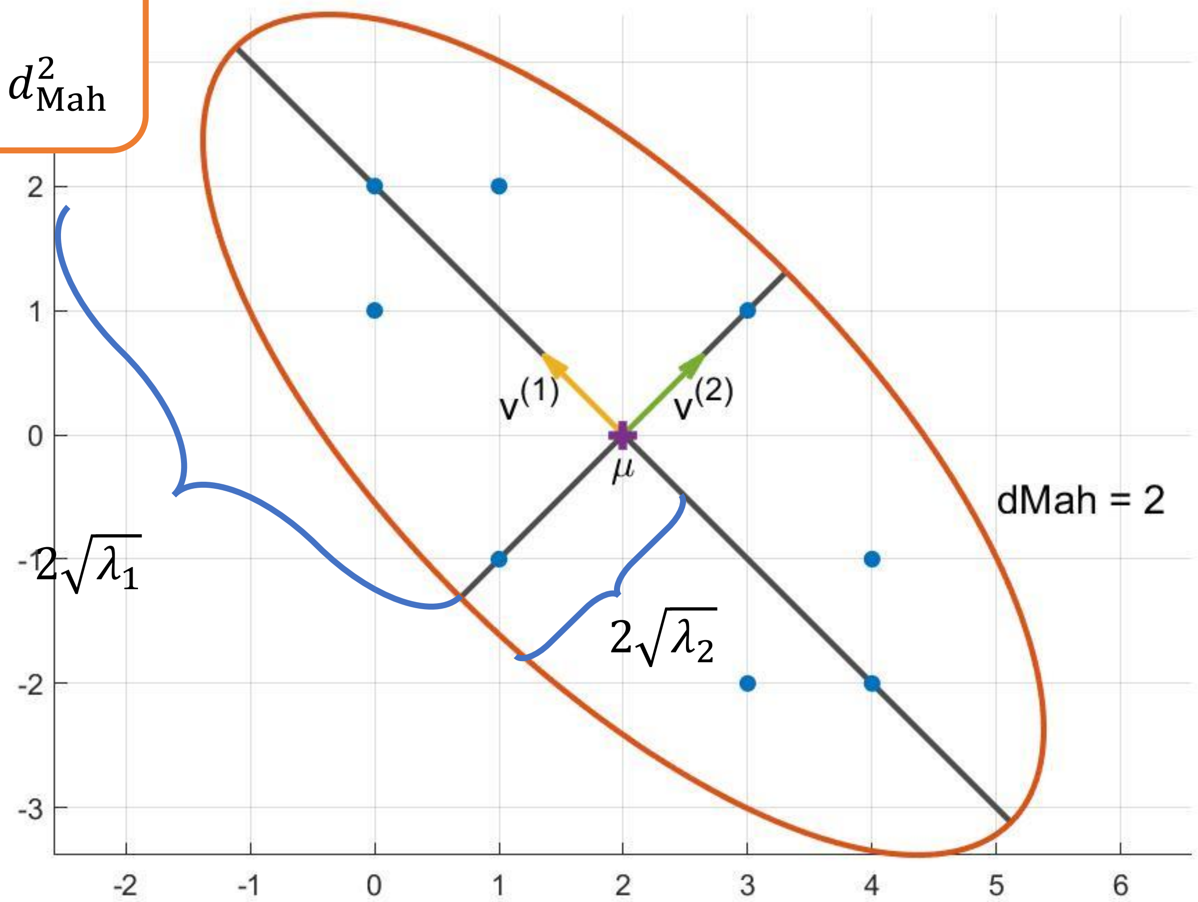
$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\}\subset\mathbb{R}^2$$

(f) Draw the ellipses defined by the points with Mahalanobis distances equal to 1 and 2 (without computing it).

Ellipse equation

$$k^2=\frac{x^2}{\lambda_1}+\frac{y^2}{\lambda_2}=d^2_{\mathrm{Mah}}$$





EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

(g) Write the equation of the ellipse defined by the points at Mahalanobis distance 3 in the new coordinate system and the cartesian coordinate system.

Mahalanobis distance: $$d(\boldsymbol{x}) = \sqrt{(\boldsymbol{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}$$

- <u>Standard coordinate system</u> $\left\{\boldsymbol{o} = \begin{pmatrix}0\\0\end{pmatrix}; \boldsymbol{e}^{(1)} = \begin{pmatrix}1\\0\end{pmatrix}, \boldsymbol{e}^{(2)} = \begin{pmatrix}0\\1\end{pmatrix}\right\}$:

$$\boldsymbol{\mu} = \begin{pmatrix}2\\0\end{pmatrix}$$

$$\boldsymbol{\Sigma} = \frac{1}{7}\begin{pmatrix}20 & -14\\-14 & 20\end{pmatrix}$$

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\}\subset\mathbb{R}^2$$

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$$\left.\begin{aligned}\boldsymbol{\mu} &= \begin{pmatrix}2\\0\end{pmatrix}\\ \boldsymbol{\Sigma} &= \frac{1}{7}\begin{pmatrix}20 & -14\\-14 & 20\end{pmatrix}\end{aligned}\right\} d(\boldsymbol{x}) = \sqrt{\left(\boldsymbol{x}-\begin{pmatrix}2\\0\end{pmatrix}\right)^{\mathrm{T}}\begin{pmatrix}\frac{20}{7} & \frac{-14}{7}\\ \frac{-14}{7} & \frac{20}{7}\end{pmatrix}^{-1}\left(\boldsymbol{x}-\begin{pmatrix}2\\0\end{pmatrix}\right)}$$

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

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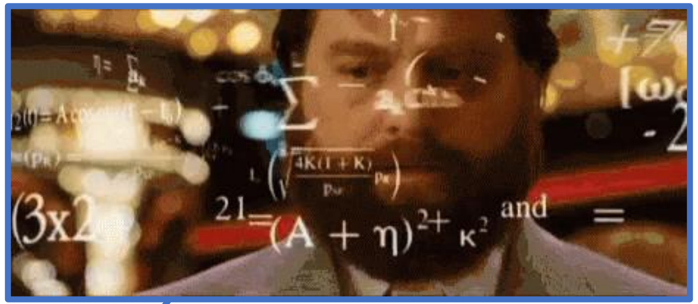
$$\left.\begin{array}{l}\boldsymbol{\mu} = \begin{pmatrix}2\\0\end{pmatrix}\\[2ex] \boldsymbol{\Sigma} = \dfrac{1}{7}\begin{pmatrix}20 & -14\\-14 & 20\end{pmatrix}\end{array}\right\} d(\boldsymbol{x}) = \sqrt{\left(\boldsymbol{x}-\begin{pmatrix}2\\0\end{pmatrix}\right)^{\mathrm{T}}\begin{pmatrix}\frac{20}{7} & \frac{-14}{7}\\ \frac{-14}{7} & \frac{20}{7}\end{pmatrix}^{-1}\left(\boldsymbol{x}-\begin{pmatrix}2\\0\end{pmatrix}\right)}$$



$$\Longrightarrow 3^2 = d(\boldsymbol{x})^2 = \left(\boldsymbol{x}-\begin{pmatrix}2\\0\end{pmatrix}\right)^{\mathrm{T}}\begin{pmatrix}\frac{20}{7} & \frac{-14}{7}\\ \frac{-14}{7} & \frac{20}{7}\end{pmatrix}^{-1}\left(\boldsymbol{x}-\begin{pmatrix}2\\0\end{pmatrix}\right) = \cdots\cdots\cdots$$

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\}\subset\mathbb{R}^2$$

(g) Write the equation of the ellipse defined by the points at Mahalanobis distance 3 in the new coordinate system and the cartesian coordinate system.

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- <u>New coordinate system</u> $\left\{\boldsymbol{o}=\boldsymbol{\mu}=\begin{pmatrix}2\\0\end{pmatrix};\boldsymbol{v}^{(1)}=\frac{\sqrt{2}}{2}\begin{pmatrix}-1\\1\end{pmatrix},\boldsymbol{v}^{(2)}=\frac{\sqrt{2}}{2}\begin{pmatrix}1\\1\end{pmatrix}\right\}$:

$$\left.\begin{array}{l}\boldsymbol{\mu}_{<\mathcal{V}>}=\begin{pmatrix}0\\0\end{pmatrix}\\[2ex]\boldsymbol{\Sigma}_{<\mathcal{V}>}=\frac{1}{7}\begin{pmatrix}34&0\\0&6\end{pmatrix}\end{array}\right\}\quad d(\boldsymbol{x})=\sqrt{\left(\boldsymbol{x}-\begin{pmatrix}0\\0\end{pmatrix}\right)^{\mathrm{T}}\begin{pmatrix}\frac{34}{7}&0\\0&\frac{6}{7}\end{pmatrix}^{-1}\left(\boldsymbol{x}-\begin{pmatrix}0\\0\end{pmatrix}\right)}$$

$$\begin{pmatrix}\frac{34}{7}&0\\0&\frac{6}{7}\end{pmatrix}^{-1}=\begin{pmatrix}\frac{34}{7}^{-1}&0\\0&\frac{6}{7}^{-1}\end{pmatrix}=\begin{pmatrix}\frac{7}{34}&0\\0&\frac{7}{6}\end{pmatrix}$$

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

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$$\left.\begin{aligned}\boldsymbol{\mu}_{<\mathcal{V}>} &= \begin{pmatrix}0\\0\end{pmatrix}\\ \boldsymbol{\Sigma}_{<\mathcal{V}>} &= \frac{1}{7}\begin{pmatrix}34 & 0\\0 & 6\end{pmatrix}\end{aligned}\right\} \quad d(\boldsymbol{x}) = \sqrt{\left(\boldsymbol{x}-\begin{pmatrix}0\\0\end{pmatrix}\right)^{\mathrm{T}}\begin{pmatrix}\frac{7}{34} & 0\\0 & \frac{7}{6}\end{pmatrix}\left(\boldsymbol{x}-\begin{pmatrix}0\\0\end{pmatrix}\right)}$$

$$\Longrightarrow 3^2 = d(\boldsymbol{x})^2 = \boldsymbol{x}^{\mathrm{T}}\begin{pmatrix}\frac{7}{34} & 0\\0 & \frac{7}{6}\end{pmatrix}\boldsymbol{x} = \begin{pmatrix}x_1 & x_2\end{pmatrix}\begin{pmatrix}\frac{7}{34} & 0\\0 & \frac{7}{6}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \frac{7\cdot x_1^2}{34}+\frac{7\cdot x_2^2}{6} \quad\Longrightarrow\quad 9 = \frac{x_1^2}{34/7}+\frac{x_2^2}{6/7}$$

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\} \subset \mathbb{R}^2$$

(g) Write the equation of the ellipse defined by the points at Mahalanobis distance 3 in the new coordinate system and the cartesian coordinate system.

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$$\left.\begin{array}{l}\boldsymbol{\mu}_{<\mathcal{V}>} = \begin{pmatrix}0\\0\end{pmatrix}\\[2ex] \boldsymbol{\Sigma}_{<\mathcal{V}>} = \frac{1}{7}\begin{pmatrix}34 & 0\\0 & 6\end{pmatrix}\end{array}\right\} \quad d(\boldsymbol{x}) = \sqrt{\left(\boldsymbol{x}-\begin{pmatrix}0\\0\end{pmatrix}\right)^{\mathrm{T}}\begin{pmatrix}\frac{7}{34} & 0\\0 & \frac{7}{6}\end{pmatrix}\left(\boldsymbol{x}-\begin{pmatrix}0\\0\end{pmatrix}\right)}$$

$$9 = \frac{x_1^2}{34/7} + \frac{x_2^2}{6/7}$$

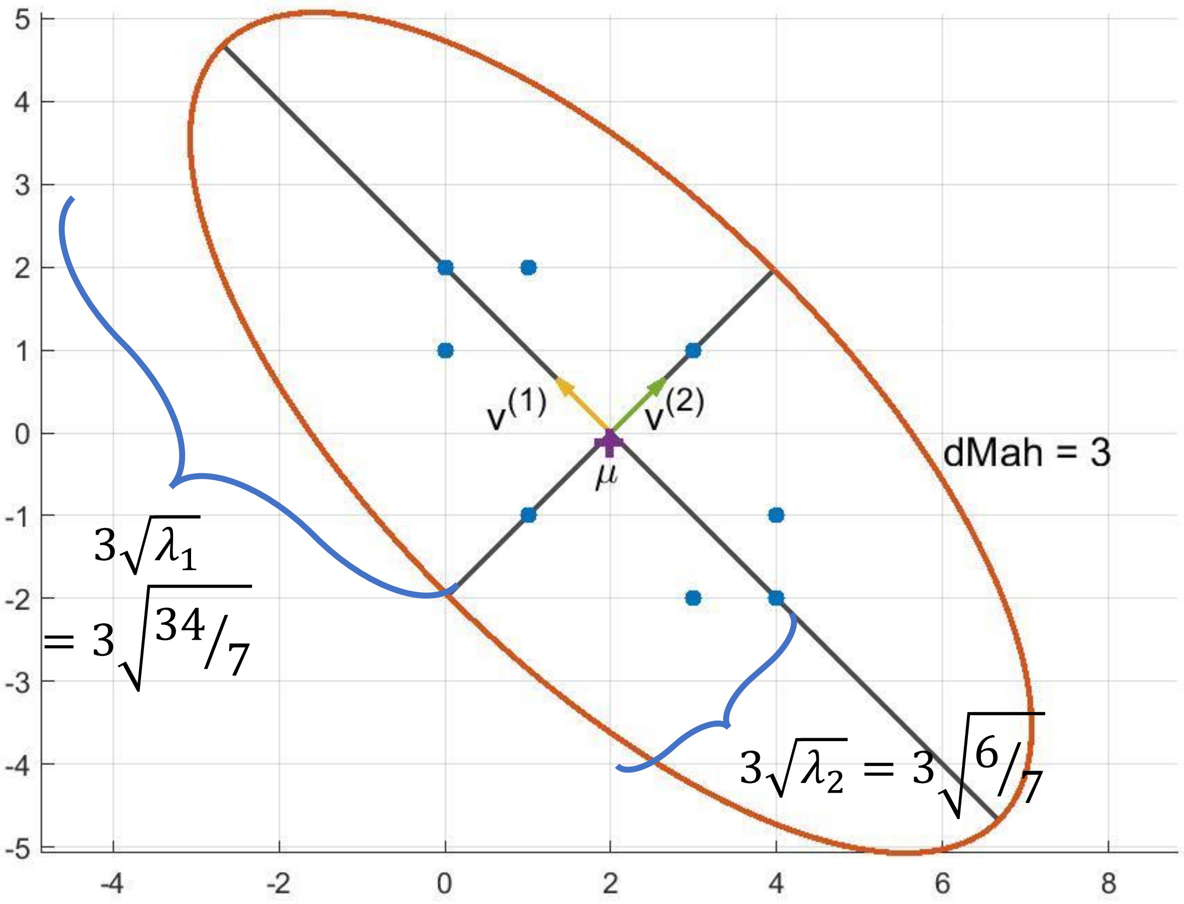
$$k^2 = \frac{x_1^2}{\lambda_1} + \frac{x_2^2}{\lambda_2}$$

Ellipse of Mahalanobis distance equal to k

EXERCICE 2

$$\left\{\begin{pmatrix}0\\2\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}3\\1\end{pmatrix},\begin{pmatrix}3\\-2\end{pmatrix},\begin{pmatrix}4\\-1\end{pmatrix},\begin{pmatrix}4\\-2\end{pmatrix}\right\}\subset\mathbb{R}^2$$

(g) Write the equation of the ellipse defined by the points at Mahalanobis distance 3 in the new coordinate system and the cartesian coordinate system.



Ellipse equation

$$9 = \frac{x_1^2}{34/7} + \frac{x_2^2}{6/7}$$