TRIBHUVAN UNIVERSITY

Institution of Science and Technology

Bachelor Level/First Year/Second Semester/Science — Full Marks: 80

Computer Science and Information Technology [MTH. 163] — Pass Marks: 32

(Mathematics II) — Time: 3 hrs.

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

MODEL QUESTIONS-ANWERS

Group 'A'

Attempt any three questions: **(3×10 = 30)**

1. What is pivot position? Apply elementary row operation to transform the following matrix first into echelon form and then into reduced echelon form:

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}.$$

Solution: Definition (Pivot Position)

A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A pivot column is a column of A that contains a pivot position.

Problem Part:

Here,

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

Interchange rows 1 and 3. (We could have interchanged rows 1 and 2 instead.)

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \quad \begin{bmatrix} \text{Applying} \\ R_2 \to R_2 - R_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \begin{bmatrix} \text{Applying} \\ R_3 \to 3R_3 - 2R_2 \end{bmatrix}$$

$$\sim\begin{bmatrix}3 & -9 & 12 & -9 & 0 & -9\\ 0 & 2 & -4 & 4 & 0 & -14\\ 0 & 0 & 0 & 0 & 1 & 4\end{bmatrix} \quad \begin{bmatrix}\text{Applying}\\ R_1\to R_1-6R_3\\ R_2\to R_2-2R_3\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & -3 & 4 & -3 & 0 & -3\\ 0 & 1 & -2 & 2 & 0 & -7\\ 0 & 0 & 0 & 0 & 1 & 4\end{bmatrix} \quad \begin{bmatrix}\text{Applying}\\ R_1\to \frac{1}{3}R_1\\ R_2\to \frac{1}{2}R_2\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 0 & -2 & 3 & 0 & -24\\ 0 & 1 & -2 & 2 & 0 & -7\\ 0 & 0 & 0 & 0 & 1 & 4\end{bmatrix} \quad \begin{bmatrix}\text{Applying}\\ R_1\to R_1+3R_2\end{bmatrix}$$

This is the reduced echelon form of the original matrix.

2. Define linear transformation with an example. Check the following transformation is linear or not? T: $\mathcal{R}^2 \to \mathcal{R}^3$ be defined by T(x, y) = (x, 2y). Also, let T(x, y) = (3x + y, 5x + 7y, x + 3y). Show that T is a one– to–one linear transformation. Does T maps $\mathcal{R}^2$ onto $\mathcal{R}^3$? [3+ 2+5]

Solution: Definition (Matrix Transformation):

For each $x \in \mathbb{R}^n$, T(x) is computed as $Ax \in \mathbb{R}^m$, where A is m×n matrix behaves as transformation operator.

Example: Let

$$A = \begin{bmatrix}1 & -3\\ 3 & 5\\ -1 & 7\end{bmatrix},\ u = \begin{bmatrix}2\\ -1\end{bmatrix}.$$

Then,

$$T(u) = Au = \begin{bmatrix}1 & -3\\ 3 & 5\\ -1 & 7\end{bmatrix}\begin{bmatrix}2\\ -1\end{bmatrix} = \begin{bmatrix}2+3\\ 6-5\\ -2-7\end{bmatrix} = \begin{bmatrix}5\\ 1\\ -9\end{bmatrix}.$$

Problem Part: Let

$$T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2).$$

Then

$$T(x) = \begin{bmatrix}3x_1 + x_2\\ 5x_1 + 7x_2\\ x_1 + 3x_2\end{bmatrix} = \begin{bmatrix}3 & 1\\ 5 & 7\\ 1 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = Ax \text{ (say)}.$$

(i) Since we have, T is one–to–one linear transformation if and only if the columns of A are linearly independent.

Here, A = 3×2 matrix in which one column is not a multiple of another. This means the columns of A are linearly independent. Therefore T is one–to–one linear transformation.

(ii) Since we have T is onto if and only if the columns of A span $\mathbb{R}^3$.

Clearly A has only 2 columns. So, it has at most two pivot positions. This means A does not span $\mathbb{R}^3$. Therefore, T is not onto.

3. **Find the LU factorization of** $\begin{bmatrix} 0 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$.

Note: The factorization of given matrix is not possible because the pivot point is zero i.e. $a_{11} = 0$. *(Here the matrix has correction at the position of* a_{11} *with 2. For reference see –*

(i) Mathematics II (Revised Edition 2076) by Binod Pd Dhakal and et al, Example 17, Page 66, KEC Publication)

(ii) Linear Algebra and its Applications (3rd Edition 2011), by David C Lay, Example 2, Page 161, Pearson Publication.

Q. **Find the LU factorization of** $\begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$.

Solution: Let,

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix} \quad \begin{bmatrix} \text{Applying} \\ R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 + 3R_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \quad \begin{bmatrix} \text{Applying} \\ R_3 \to R_3 + 3R_2 \\ R_4 \to R_4 - 4R_2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \quad \begin{bmatrix} \text{Applying} \\ R_4 \to R_4 - 2R_3 \end{bmatrix}$$

$$= U.$$

Here, U has pivot values 2 in first column, 3 in second column, third column has no pivot value, 2 in fourth column and 5 in fifth column.

The entries of column of pivot value and to be reduced value which are determine the row reduction of A to U are,

$$\begin{bmatrix} 2 \\ -4 \\ 2 \\ -6 \end{bmatrix} \quad \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad [5]$$

$$\begin{array}{cccc} \div 2 & \div 3 & \div 2 & \div 5 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \begin{bmatrix} 1 \\ -2 \\ 1 \\ -3 \end{bmatrix} & \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \end{bmatrix} & [1] \end{array}$$

Therefore L is

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

Thus,

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

$= LU.$

4. Find a least square solution of the inconsistent system Ax = b for

$$\mathbf{A} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}.$$

Solution: Let,

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

Here,

$$A^T A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$$

and,

$$A^T b = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}.$$

Since we have the set of least squares solutions of $Ax = b$ coincides with the non–empty set of solutions of $A^T Ax = A^T b$.

Therefore,

$$\hat{x} = (A^T A)^{-1} (A^T b) \qquad \text{...... (i)}$$

Here,

$$|A^TA| = \begin{vmatrix} 17 & 1 \\ 1 & 5 \end{vmatrix} = 84 \neq 0.$$

Then,

$$(A^TA)^{-1} = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}.$$

Therefore (i) becomes,

$$\hat{x} = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}\begin{bmatrix} 19 \\ 11 \end{bmatrix} = \frac{1}{84}\begin{bmatrix} 84 \\ 168 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Group 'B'

Attempt any ten questions: **(10×5 = 50)**

5. Compute u + v, u – 2v and 2u + v where $u = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

Solution: Let,

$$u = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, v = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}.$$

Now, $$u + v = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix},$$

$$u - 2v = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix}$$

and $$2u + v = 2\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \\ 7 \end{bmatrix}.$$

6. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, **and define T:** $\mathcal{R}^2 \to \mathcal{R}^2$ **be defined by T(x) = Ax, find the image under T of** $u = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ **and** $v = \begin{bmatrix} a \\ b \end{bmatrix}$.

Solution: Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and the transformation T: $\mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x) = Ax$.

Also, let

$$u = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \text{ and } v = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Then

$$T(u) = Au = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}.$$

and

$$T(v) = Av = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2a \\ 2b \end{bmatrix}.$$

Thus, the images of u and v under T are $\begin{bmatrix} 2 \\ -6 \end{bmatrix}$ and $\begin{bmatrix} 2a \\ 2b \end{bmatrix}$.

7. Let $A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}$. What value(s) of k, if any, will make AB = BA?

Solution: Let $A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}$.

Here,

$$AB = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix} = \begin{bmatrix} 8+15 & -10+5k \\ -12+3 & 15+k \end{bmatrix} = \begin{bmatrix} 23 & -10+5k \\ -9 & 15+k \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 8+15 & 20-5 \\ 6-3k & 15+k \end{bmatrix} = \begin{bmatrix} 23 & 15 \\ 6-3k & 15+k \end{bmatrix}$$

And, suppose AB = BA. That is,

$$\begin{bmatrix} 23 & -10+5k \\ -9 & 15+k \end{bmatrix} = \begin{bmatrix} 23 & 15 \\ 6-3k & 15+k \end{bmatrix}$$

This, implies,

$$-10 + 5k = 15 \Rightarrow k = 5.$$

$$-9 = 6 - 3k \Rightarrow k = 5 \text{ (which is same as above).}$$

Thus, at k = 5, we get AB = BA.

8. Compute det A, where $A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$.

Solution: Here,

$$\det(A) = \begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} \quad \left[\because \text{Taking out the common factor 2 from } R_1\right]$$

$$= 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -24 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & -2 \end{vmatrix} \quad \left[\because \text{Applying } R_2 \to R_2 - 3R_1,\ R_3 \to R_3 + 3R_1,\ R_4 \to R_4 - R_1\right]$$

$$= 2\begin{vmatrix} 3 & -4 & -24 \\ -12 & 10 & 10 \\ 0 & -3 & -2 \end{vmatrix} \quad [\because \text{Computing from 1}]$$

$$= 2\begin{vmatrix} 3 & -4 & -24 \\ 0 & -6 & -86 \\ 0 & -3 & -2 \end{vmatrix} \quad \left[\because \begin{matrix}\text{Applying} \\ R_2 \to R_2 + 4R_1\end{matrix}\right]$$

$$= 2 \times 3\begin{vmatrix} -6 & -86 \\ -3 & -2 \end{vmatrix} \quad [\because \text{Computing from 3}]$$

$$= 6(-12 - 258)$$

$$= -1620$$

Thus, $\det(A) = \begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = -1620.$

9. Let H be the set of all vectors of the form $\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix}$. Show that H is a subspace of $\mathcal{R}^3$.

Solution: Let H be the set of all vectors of the form $\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix}$.

That is, $H = \left\{\begin{bmatrix} 2t \\ 0 \\ -t \end{bmatrix}\right\} = \{tv\}$ where, $v = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} \in R^3$

So, $H = \text{Span}\{v\}$ where $v \in R^3$. Therefore, H is subspace of R^3.

10. Find basis and the dimension of the subspace

$$\mathbf{H} = \left\{\begin{bmatrix} s - 2t \\ s + t \\ 3t \end{bmatrix}, s, t \in \mathcal{R}\right\}.$$

Solution: Let,

$$H = \left\{\begin{bmatrix} s - 2t \\ s + t \\ 3t \end{bmatrix}, s, t \in \mathcal{R}\right\}$$

Here,

$$\begin{bmatrix} s - 2t \\ s + t \\ 3t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} = sv_1 + tv_2$$

where, $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$

which shows that H is linear combination of v_1, v_2. Clearly, $v_1 \neq 0$, v_2 is not a multiple of v_1. So, by Spanning Set Theorem, $\{v_1, v_2\}$ span H and since it is linearly independent. So, it is a basis for H and dim H = 2.

11. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

Solution: Given, $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

Here, $\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 \\ 0 & -21 \end{bmatrix}$ $R_2 \to 2R_2 - 3R_1$

This shows the eigenvalues for A are $\lambda = 2, -21$

At $\lambda = 2$,

$$Ax = \lambda x \Rightarrow \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x_1 + 3x_2 \\ 3x_1 - 6x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3x_2 \\ 3x_1 - 8x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This implies $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

At $\lambda = -21$,

$$Ax = \lambda x \Rightarrow \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -21\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x_1 + 3x_2 \\ 3x_1 - 6x_2 \end{bmatrix} = \begin{bmatrix} -21x_1 \\ -21x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 23x_1 + 3x_2 \\ 3x_1 + 15x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 23x_1 + 3x_2 \\ x_1 + 5x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives, $x_1 + 5x_2 = 0 \Rightarrow x_1 = -5x_2$.

$23x_1 + 3x_2 = 0 \Rightarrow -102x_2 = 0 \Rightarrow x_2 = 0$.

Therefore, $x_1 = 0$.

This implies $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Thus the eigenvalues are 2, and -21 and eigenvector is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ at both points.

12. Define orthogonal set. Show that $\{u_1, u_2, u_3\}$ is an orthogonal set, where $u_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $u_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$.

Solution: Definition (Orthogonal Set)

A set of vectors $\{u_1, u_2, \ldots, u_p\}$ in $\mathbb{R}^n$, is said to be an orthogonal set if $u_i \cdot u_j = 0$ for $i \neq j$ for $i, j = 1, 2, \ldots, p$.

Problem Part:

Let $u_1 = (3, 1, 1), u_2 = (-1, 2, 1), u_3 = \left(-\frac{1}{2}, -2, \frac{7}{2}\right)$.

Here, $u_1 \cdot u_2 = (3, 1, 1) \cdot (-1, 2, 1) = -3 + 2 + 1 = 0$.

$$u_2 \cdot u_3 = (-1, 2, 1) \cdot \left(-\frac{1}{2}, -2, \frac{7}{2}\right) = \frac{1}{2} - 4 + \frac{7}{2} = 0.$$

$$u_1 \cdot u_3 = (3, 1, 1) \cdot \left(-\frac{1}{2}, -2, \frac{7}{2}\right) = -\frac{3}{2} - 2 + \frac{7}{2} = 0.$$

Therefore, $\{u_1, u_2, u_3\}$ is an orthogonal set.

Also, $\|u_1\| = u_1 \cdot u_1 = (3, 1, 1) \cdot (3, 1, 1) = 9 + 1 + 1 = 11 \neq 0,$

$\|u_2\| = u_2 \cdot u_2 = (-1, 2, 1) \cdot (-1, 2, 1) = 1 + 4 + 1 = 6 \neq 0,$

$$\|u_3\| = u_3 \cdot u_3 = \left(-\frac{1}{2}, -2, \frac{7}{2}\right) \cdot \left(-\frac{1}{2}, -2, \frac{7}{2}\right)$$

$$= \frac{1}{4} + 4 + \frac{49}{4} = \frac{64}{4} = 16 \neq 0.$$

So (u_1, u_2, u_3) is a set of non–zero vectors.

Since every orthogonal set of non–zero vectors is a basis for the subspace of the space.

Here $\{u_1, u_2, u_3\}$ is an orthogonal set of vectors, so $\{u_1, u_2, u_3\}$ is a basis for $\mathcal{R}^3$ and therefore, is an orthogonal basis for $\mathcal{R}^3$.

13. Let W = Span $\{x_1, x_2\}$, where $x_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Construct an orthogonal basis $\{v_1, v_2\}$ for W.

Solution: Let W = Span $\{x_1, x_2\}$ where $x_1 = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = (3, 6, 0)$ and $x_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = (1, 2, 2)$.

Let $\quad v_1 = x_1 = (3, 6, 0)$

Let, $\quad v_2 = x_2 - \left(\dfrac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$

$$= x_2 - \left(\frac{x_2 \cdot x_1}{x_1 \cdot x_1}\right) x_1 \qquad [\because v_1 = x_1]$$

$$= (1, 2, 2) - \left(\frac{(1, 2, 2) \cdot (3, 6, 0)}{(3, 6, 0) \cdot (3, 6, 0)}\right)(3, 6, 0)$$

$$= (1, 2, 2) - \left(\frac{3 + 12 + 0}{9 + 36 + 0}\right)(3, 6, 0) = (1, 2, 2) - \left(\frac{1}{3}\right)(3, 6, 0)$$

$$= (0, 0, 2)$$

Here $\{v_1, v_2\}$ be an orthogonal basis for W.

14. Let * be defined on $\mathcal{Q}^+$ by $a * b = \dfrac{ab}{2}$. Then show that $\mathcal{Q}^+$ forms a group.

Solution: Given that * is defined on $\mathcal{Q}^+$ by $a * b = \dfrac{ab}{2}$.

Closure: For all $a, b \in \mathbb{Q}^+$, $a * b = \dfrac{ab}{2} \in \mathbb{Q}^+$.

That is, the elements of $\mathbb{Q}^+$ are closed under '*'.

Associativity: For all a, b, c $\in \mathbb{Q}^+$,

$$(a * b) * c = \frac{ab}{2} * c = \frac{abc}{4}.$$

Again, $a * (b * c) = a * \frac{bc}{2} = \frac{abc}{4}.$

Thus, $*$ is associative

Existence of Identity: For all, $a \in Q^+$;

$$a * 2 = \frac{a \cdot 2}{2} = a \quad \text{and} \quad 2 * a = \frac{2a}{2} = a$$

Hence 2 is the identity element for $*$.

Existence of Inverse:

Finally $a * b = 2 = b * a$, where b is inverse of a under $*$.

$$\frac{ab}{2} = 2 = \frac{ba}{2} \Rightarrow b = \frac{4}{a} \in Q^+$$

Therefore, $\frac{4}{a}$ is an inverse for a. Hence Q^+ is a group under the binary operation $*$.

Thus, $*$ satisfies all conditions for group under addition, so Q^+ is a group under addition.

15. **Define ring with an example. Compute the product in the given ring (12)(16) in $\mathbb{Z}_{15}$.**

Solution: Definition of Ring:

A non–empty set R together with two binary operator + and • denoted by <R, +, •> is called ring if the following conditions are satisfied:

(i) Closure: For all a, b $\in$ R then $(a + b) \in R$.

(ii) Commutative: For all a, b $\in$ R then $a + b = b + a$.

(iii) Associativity: For all a, b, c $\in$ R then $(a + b) + c = a + (b + c)$.

(iv) Existence of identity element: For all $a \in R$ there exists an element $0 \in R$ such that $a + 0 = 0 + a = a$.

(v) Existence of inverse element: For all $a \in R$ there exists $a' \in R$ such that $a + a' = 0 = a' + a$.

(vi) Closure: For all a, b $\in$ R then $ab \in R$.

(vii) Associativity: For all a, b, c $\in$ R then $(ab)c = a(bc)$.

(viii) Distributive: For all a, b, c $\in$ R,

$a(b + c) = ab + ac$ (left distributive)

$(a + b)c = ac + bc$ (right distributive).

Example: A set of real number R is a ring.

Problem Part:

Since, $(12)(16) = 192$ and $\frac{192}{15} = 12\frac{12}{15}$.

Therefore, $(12)(16) = 12$ in $\mathbb{Z}_{15}$.

TRIBHUVAN UNIVERSITY

Institution of Science and Technology

Bachelor Level/First Year/Second Semester/Science
Computer Science and Information Technology [MTH. 163]
(Mathematics II)

Full Marks: 80
Pass Marks: 32
Time: 3 hrs.

Candidates are required to give their answers in their own words as far as practicable.

The figures in the margin indicate full marks.

TU QUESTIONS-ANSWERS 2075

Group 'A'

Attempt any three questions: **(3×10 = 30)**

1. **When a system of linear equations is consistent and inconsistent? Give an example for each. Test the consistency and solve: $x + y + z = 4$, $x + 2y + 2z = 2$, $2x + 2y + z = 5$.** **[2+1+7]**

Solution:

Definition (Consistent and Inconsistent System)

A system of linear equations is called consistent if it has solution (that may be one solution or infinitely many solutions) and called inconsistent if it has no solution.

Example: Consider a system

$$x_1 - 2x_2 = -1$$
$$x_1 - 3x_2 = -3$$

This system has solution (3, 2). So, the system is consistent and has unique solution.

Consider a system

$$x_1 - x_2 = 1$$
$$-3x_1 + 3x_2 = -3$$

Here second equation is the thrice time multiple of first. So, the system has infinite solutions and is consistent.

Consider a system

$$x_1 + 2x_2 = -1$$
$$x_1 + 2x_2 = 2$$

Here the equations represent parallel lines. So, the system has no solution. So, the system is inconsistent.

Problem Part:

Solution: Given system is,

$$x + y + z = 4$$
$$x + 2y + 2z = 2$$
$$2x + 2y + z = 5$$

The matrix notation of the system is,

$$\begin{bmatrix} 1 & 1 & 1 & : & 4 \\ 1 & 2 & 2 & : & 2 \\ 2 & 2 & 1 & : & 5 \end{bmatrix}$$

Apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 4 \\ 0 & 1 & 1 & : & -2 \\ 0 & 0 & -1 & : & -3 \end{bmatrix}$$

Apply $R_3 \to -R_3$ then the above matrix reduces to

$$\sim \begin{bmatrix} 1 & 1 & 1 & : & 4 \\ 0 & 1 & 1 & : & -2 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$ is triangular form.

The equation form of the matrix notation is,

$$x + y + z = 4 \qquad \ldots \text{(i)}$$
$$y + z = -2 \qquad \ldots \text{(ii)}$$
$$z = 3 \qquad \ldots \text{(iii)}$$

From (iii), we get $z = 3$.
then (ii) gives, $y = -5$
And, (i) gives, $x = 6$
Thus, the solution of the given linear system is $(x, y, z) = (6, -5, \overset{3}{6})$.

2. What is the condition of a matrix to have an inverse? Find the inverse of the matrix, $\begin{pmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{pmatrix}$. **[2+8]**

Solution: If A is an invertible matrix then there is a matrix C such that

$$AC = I = CA.$$

In such case, C is called inverse of A and write as $C = A^{-1}$.

Condition for existence of inverse of a matrix.

Let A be a given matrix then the inverse of A i.e. A^{-1} exists if det (A) is non-zero.

Problem Part:

$$\text{Let } A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$$

$$\text{Here, det}(A) = \begin{vmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{vmatrix}$$
$$= 1(25 + 24) + 2(-5 - 30) - 1(4 - 25)$$
$$= 49 - 70 + 21$$
$$= 0$$

So, the inverse of A does not exist.

3. **Define linearly independent set of vectors with an example. Show that the vectors (1, – 4, 3), (0, 3, 1) and (3, –5, 4) are linearly independent. Do they form a basis? Justify.** [2+5+3]

Solution: Definition (Linearly independent and dependent vectors)

An indexed set of vectors $\{v_1, v_2, \ldots v_p\}$ in V is said to be linearly independent if the vector equation.

$$c_1v_1 + c_2v_2 + \ldots + c_pv_p = 0$$

has only the trivial solution, i.e. $c_1 = 0, c_2 = 0, \ldots c_p = 0$.

For otherwise, the vectors are linearly dependent.

Example: The set {(1, 0, 0), (0, 1, 0), (0, 0, 1)} is linearly independent vectors.

Problem Part:

Given vectors are (1, –4, 3), (0, 3, 1) and (3, –5, 4).

Here we have to show that v_1, v_2, v_3 are linearly independent and they span $\mathbb{R}^3$.

For linearly independent, Ax = 0

$$\begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 3 & -5 & 4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 7 & -5 & 0 \end{bmatrix} \text{ [Applying } R_3 \to R_3 - 3R_1]$$

$$\sim \begin{bmatrix} 1 & -4 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & -22 & 0 \end{bmatrix} \text{ [Applying } R_2 \to 3R_3 - 7R_2]$$

No basic variable so having a trivial solution. Thus v_1, v_2, v_3 are linearly independent.

For v_1, v_2, v_3 span R^3

$$A = \begin{bmatrix} 1 & -4 & 3 \\ 0 & 3 & 1 \\ 3 & -5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & -22 \end{bmatrix}$$

Each row has pivot so, column of A span R^3. So, $\{v_1, v_2, v_3\}$ span R^3. Thus $\{v_1, v_2, v_3\}$ is basic for R^3.

4. **Find a least square solution of Ax = b for** $A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 5 \\ 7 \\ 3 \end{bmatrix}$.

Solution: Let,

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 5 \\ 7 \\ 3 \end{bmatrix}.$$

Here,

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 5 & 0 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 10 \\ 8 & 20 & 26 \\ 10 & 26 & 38 \end{bmatrix}$$

and,

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 5 & 0 & 2 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \\ 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 18 \\ 30 \\ 38 \end{bmatrix}.$$

Since we have the set of least squares solutions of $Ax = b$ coincides with the non–empty set of solutions of $A^TAx = A^Tb$.

Therefore,

$$\hat{x} = (A^TA)^{-1}(A^Tb) \qquad \text{...... (i)}$$

Here,

$$|A^TA| = \begin{vmatrix} 4 & 8 & 10 \\ 8 & 20 & 26 \\ 10 & 26 & 38 \end{vmatrix} = 84 \neq 0.$$

Then,

$$(A^TA)^{-1} = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}.$$

Therefore (i) becomes,

$$\hat{x} = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}\begin{bmatrix} 19 \\ 11 \end{bmatrix} = \frac{1}{84}\begin{bmatrix} 84 \\ 168 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Group 'B'

Attempt any ten questions: **(10×5 = 50)**

5. Change into reduced echelon form of the matrix:

$$\begin{bmatrix} 0 & 3 & -6 \\ 3 & -7 & 8 \\ 3 & -9 & 12 \end{bmatrix}.$$

Solution: Here,

$$\begin{bmatrix} 0 & 3 & -6 \\ 3 & -7 & 8 \\ 3 & -9 & 12 \end{bmatrix}$$

Interchange rows 1 and 3. (We could have interchanged rows 1 and 2 instead.)

$$\sim \begin{bmatrix} 3 & -9 & 12 \\ 3 & -7 & 8 \\ 0 & 3 & -6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & -9 & 12 \\ 0 & 2 & -4 \\ 0 & 3 & -6 \end{bmatrix} \qquad [\text{Applying } R_2 \to R_2 - R_1]$$

$$\sim \begin{bmatrix} 3 & -9 & 12 \\ 0 & 2 & -4 \\ 0 & 0 & 0 \end{bmatrix} \quad [\text{Applying } R_3 \to R_3 - R_2]$$

$$\sim \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \text{Applying } R_1 \to \frac{1}{3} R_1 \\ R_2 \to \frac{1}{2} R_2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad [\text{Applying } R_1 \to R_1 + 2R_2]$$

This is the reduced echelon form of the original matrix.

6. **Define linear transformation with an example. Is a transformation T: $\mathcal{R}^2 \to \mathcal{R}^3$ defined by T(x, y) = (3x + y, 5x + 7y, x + 3y) linear? Justify.** **[2 + 3]**

Solution:

Definition (linear transformation)

A transformation $T : U \to V$ is linear if for $u, v \in U$ and for any scalar a

(i) $T(u + v) = T(u) + T(v)$

(ii) $T(au) = aT(u)$

Alternatively,

A transformation $T : U \to V$ is called linear if for any $u, v \in U$ and for any two scalars a, b,

$$T(au + bv) = aT(u) + bT(v)$$

Problem Part:

Let $T : R^2 \to R^3$ is defined by

$$T(x, y) = (3x + y, 5x + 7y, x + 3y)$$

Let $u = (x_1, y_1)$ and $v = (x_2, y_2)$ then $u, v \in R^2$

Also, let a, b are two scalars,

$$\begin{aligned}
\text{Here, } T(au + bv) &= T(a(x_1, y_1) + b(x_2, y_2)) \\
&= T(ax_1 + bx_2, ay_1 + by_2) \\
&= (3(ax_1 + bx_2) + (ay_1 + by_2), 5(ax_1 + bx_2) \\
&\qquad + 7(ay_1 + by_2), (ax_1 + bx_2) + 3(ay_1 + by_2)) \\
&= a(3x_1 + y_1, 5x_1 + 7y_1, x_1 + 3y_1) + b(3x_2 + y_2, 5x_2 \\
&\qquad + 7y_2, x_2 + 3y_2) \\
&= aT(x_1, y_1) + bT(x_2, y_2) \\
&= aT(u) + bT(v)
\end{aligned}$$

This means T is linear.

7. Let $A = \begin{bmatrix} -1 & -2 \\ 5 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 2 \\ k & -1 \end{bmatrix}$. What value(s) of k, if any, will make AB = BA?

Solution: Let $A = \begin{bmatrix} -1 & -2 \\ 5 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 2 \\ k & -1 \end{bmatrix}$.

Here,

$$AB = \begin{bmatrix} -1 & -2 \\ 5 & 9 \end{bmatrix}\begin{bmatrix} 9 & 2 \\ k & -1 \end{bmatrix} = \begin{bmatrix} -9-2k & 0 \\ 45+9k & 1 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 9 & 2 \\ k & -1 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 5 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k-5 & -2k-9 \end{bmatrix}$$

And, suppose AB = BA. That is,

$$\begin{bmatrix} -9-2k & 0 \\ 45+9k & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k-5 & -2k-9 \end{bmatrix}.$$

This, implies,

$$-9-2k = 1 \Rightarrow k = -5.$$

$$45+9k = -k-5 \Rightarrow k = -5 \text{ (which is same as above).}$$

Thus, at $k = -5$, we get AB = BA.

8. Define determinant. Evaluate without expanding $\begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix}$.

Solution: Definition (Determinant)

For $n \geq 2$, the determinant of an n×n matrix $A = [a_{ij}]$ of n terms of the form

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}).$$

Problem Part:

Here,

$$\begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 3 & -3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{vmatrix} \quad [\text{Applying } R_3 \to R_3 - 3R_2]$$

$= (1)(1)(3)$ [Multiple leading diagonal entries]

$= 3$

9. Define subspace of a vector space. Let $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix}, s, t \in R \right\}$. Show that H is a subspace of $\mathcal{R}^3$.

Solution: Definition (Vector Subspace)

Let V be a vector space over the field $\mathbb{K}$. Then a non-empty subset W of V is called a subspace of V if W satisfies the conditions:

(i) $w_1 + w_2 \in W$ for all $w_1, w_2 \in W$.

(ii) $aw \in W$ for all $w \in W$, $a \in \mathbb{K}$.

(iii) $0 \in W$.

Problem Part:

Let $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix}, s, t \in R \right\}$.

(i) Taking, $0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$, is an zero element in W.

(ii) For all $\alpha, \beta \in \mathbb{R}$ and $w_1 = \begin{bmatrix} s_1 \\ t_1 \\ 0 \end{bmatrix}$, $w_2 = \begin{bmatrix} s_2 \\ t_2 \\ 0 \end{bmatrix} \in W$ then

$$\alpha w_1 + \beta w_2 = \begin{bmatrix} \alpha s_1 + \beta s_2 \\ \alpha t_1 + \beta t_2 \\ 0 \end{bmatrix} \in W.$$

Hence, W is a subspace of V.

10. Find the dimension of the null space and column space of

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$$

Solution: Let,

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 13 & 26 & -26 \end{pmatrix} \quad \begin{pmatrix} \text{Applying} \\ R_2 \to 3R_2 + R_1 \\ R_3 \to 3R_3 + 2R_1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 2 & -2 \end{pmatrix} \quad \begin{pmatrix} \text{Applying} \\ R_2 \to \frac{1}{5} R_2 \text{ and } R_3 \to \frac{1}{13} R_3 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} \text{Applying} \\ R_3 \to R_3 - R_2 \end{pmatrix}$$

In the echelon form of A, there are three free variables x_2, x_4 and x_5. So, the dimension of Nul A is 3. Also, it has two pivot columns that is first and third column, so dim Col A is 2.

11. Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$.

Solution:

Let $A = \begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$

Let λ be a scalar value such that

$$\det(A - \lambda I) = 0$$

i.e. $\begin{vmatrix} 6-\lambda & 3 & -8 \\ 0 & -2-\lambda & 0 \\ 1 & 0 & -3-\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (6-\lambda)(2-\lambda)(-3-\lambda) + 1[3(-3-\lambda)] = 0$

$\Rightarrow \quad (3+\lambda)[(6-\lambda)(2-\lambda)+3] = 0$

$\Rightarrow \quad (3+\lambda)(\lambda^2 - 8\lambda + 15) = 0$

$\Rightarrow \quad (3+\lambda)(\lambda-5)(\lambda-3) = 0$

This gives, $\lambda = 3, 5, -3$

So, the eigen values corresponding to A are $\lambda = 3, 5, -3$.

And the eigen vector X corresponding to A at the eigen values λ is,

$$Ax = \lambda x$$

$\Rightarrow \begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ .... (1)

At $\lambda = 3$,

$6x_1 + 3x_2 - 8x_3 = 3x_1$

$-2x_2 = 3x_2$

$x_1 - 3x_3 = 3x_3$

These implies

$3x_1 + 3x_2 - 8x_3 = 0$ .... (2)

$5x_2 = 0$ .... (3)

$x_1 - 6x_3 = 0$ .... (4)

From (3), $x_2 = 0$

From (4) $x_1 = 6x_3$

From (2), $x_3 = \frac{1}{8}(3x_1 + 3x_2) = \frac{18x_3}{8}$

$\Rightarrow \quad 10x_3 = 0$

$\Rightarrow \quad x_3 = 0$

Then, $x_1 = 0$

Thus, $x = (x_1, x_2, x_3) = (0, 0, 0)$ be eigen vector of at $\lambda = 3$.

Next at $\lambda = 5$,

$$6x_1 + 3x_2 - 8x_3 = 5x_1$$
$$-2x_2 = 5x_2$$
$$x_1 - 3x_3 = 5x_3$$

These implies

$$x_1 + 3x_2 - 8x_3 = 0 \quad \text{......... (5)}$$
$$7x_2 = 0 \quad \text{......... (6)}$$
$$x_1 - 8x_3 = 0 \quad \text{......... (7)}$$

From (6), $x_2 = 0$

From (7), $x_1 = 8x_3$

From (5), $8x_3 = x_1 + 3x_2 = 8x_3$

$\Rightarrow \quad 0 = 0$

That is x_3 is free.

Therefore, $x = (8x_3, 0, x_3)$ be the eigen vector of A at $\lambda = 5$.

Next $\lambda = -3$

$$6x_1 + 3x_2 - 8x_3 = -3x_1$$
$$-2x_2 = -3x_2$$
$$-3x_3 = -3x_3$$

This implies,

$$9x_1 + 3x_2 - 8x_3 = 0 \quad \text{......... (8)}$$
$$x_2 = 0$$
$$x_1 = 0$$

And (i) gives $x_3 = 0$

Therefore, $x = (x_1, x_2, x_3) = (0, 0, 0)$ be eigen vector of A at $\lambda = -3$.

12. Find the LU factorization of the matrix $\begin{pmatrix} 2 & 5 \\ 6 & -7 \end{pmatrix}$.

Solution: Let,

$$A = \begin{pmatrix} 2 & 5 \\ 6 & -7 \end{pmatrix} \sim \begin{pmatrix} 2 & 5 \\ 0 & -22 \end{pmatrix} \quad \begin{bmatrix} \text{Applying} \\ R_2 \to R_2 - 3R_1 \end{bmatrix}$$
$$= U.$$

Here, U has pivot values 2 in first column, –22 in second column.

The entries of column of pivot value and to be reduced value which are determine the row reduction of A to U are,

$$\begin{bmatrix} 2 \\ 6 \end{bmatrix} \qquad [-22]$$
$$\div 2 \qquad \div(-22)$$
$$\downarrow \qquad \downarrow$$
$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad [1]$$

Therefore L is

$$L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$$

Thus,

$$A = \begin{pmatrix} 2 & 5 \\ 6 & -7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 0 & -22 \end{pmatrix} = LU.$$

13. Define group. Show that the set of all integers Z forms group under addition operation.

Solution: Definition (Group)

Let $(G, *)$ be a binary structure then G is said to be a group with the binary operation $*$ if the following conditions are satisfied.

1. Closure: For all $a, b \in G$ then $a*b \in G$.
2. Associativity: For all $a, b, c \in G$ then $(a * b) * c = a * (b * c)$.
3. Existence of identity element: For any $a \in G$ there exist an element $e \in G$ such that, $a * e = e * a = a$.
4. Existence of inverse element: For any $a \in G$ there exists $a' \in G$ such that $a * a' = e = a' * a$.

where * is additive or multiplicative operation.

Problem Part:

Let $\mathbb{Z}$ be a set of all integers.

1. Closure: For all $a, b \in \mathbb{Z}$ then $a + b$ is again an integer, so $(a + b) \in \mathbb{Z}$.
2. Associativity: For all $a, b, c \in \mathbb{Z}$ then $(a + b) + c = a + (b + c)$.
3. Existence of identity element: For any $a \in \mathbb{Z}$ there exist $0 \in \mathbb{Z}$ such that, $a + 0 = 0 + a = a$.
4. Existence of inverse element: For any $a \in \mathbb{Z}$ there exist $(-a) \in \mathbb{Z}$ such that $a + (-a) = 0 = (-a) + a$.

This means $\mathbb{Z}$ is group under addition.

15. Define ring with an example. Compute the product in the given ring $(-3, 5)(2, -4)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{11}$.

Solution: Definition of Ring:

A non–empty set R together with two binary operator + and • denoted by <R, +, •> is called ring if the following conditions are satisfied:

(i) Closure: For all a, b $\in$ R then $(a + b) \in R$.

(ii) Commutative: For all a, b $\in$ R then $a + b = b + a$.

(iii) Associativity: For all a, b, c $\in$ R then $(a + b) + c = a + (b + c)$.

(iv) Existence of identity element: For all $a \in R$ there exists an element $0 \in R$ such that $a + 0 = 0 + a = a$.

(v) Existence of inverse element: For all $a \in R$ there exists $a' \in R$ such that $a + a' = 0 = a' + a$.

(vi) Closure: For all a, b $\in$ R then $ab \in R$.

(vii) Associativity: For all a, b, c $\in$ R then $(ab)c = a(bc)$.

(viii) Distributive: For all a, b, c $\in$ R,

$$a(b + c) = ab + ac \quad \text{(left distributive)}$$
$$(a + b)c = ac + bc \quad \text{(right distributive).}$$

Example: A set of real number R is a ring.

Problem Part:

Since in Z_4, $-3 = 1$ and in Z_{11}, $-4 = 7$.

So, $(-3, 5)(2, -4) = (1, 5)(2, 7) = (2, 2)$.

15. State and prove the Pythagorean Theorem of two vectors and verify this for u = (1, –1) and v = (1, 1).

Solution: The Pythagorean Theorem

Two vectors u and v are orthogonal if and only if

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

Proof:

First suppose that u and v are orthogonal. Therefore,

$$u \bullet v = 0 \quad \text{.... (i)}$$

Since $\|u\|^2 = u \bullet u$. So,

$$\begin{aligned}
\|u + v\|^2 &= (u + v) \bullet (u + v) \\
&= u \bullet (u + v) + v \bullet (u + v) \\
&= u \bullet u + u \bullet v + v \bullet u + v \bullet v \\
&= \|u\|^2 + 0 + 0 + \|v\|^2 \quad \text{(using (i)]} \\
&= \|u\|^2 + \|v\|^2
\end{aligned}$$

Conversely, suppose that

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

$$\Rightarrow (u + v) \bullet (u + v) = \|u\|^2 + \|v\|^2$$

$$\Rightarrow u \bullet u + u \bullet v + v \bullet u + v \bullet v = \|u\|^2 + \|v\|^2$$

$$\Rightarrow \|u\|^2 + u \bullet v + v \bullet u + \|v\|^2 = \|u\|^2 + \|v\|^2$$

$$\Rightarrow \quad u \bullet v + v \bullet u = 0$$

$$\Rightarrow \quad 2u \bullet v = 0$$

$$\Rightarrow \quad u \bullet v = 0.$$

This means the vectors u and v are orthogonal.

Problem Part:

Let $u = (1, -1)$ and $v = (1, 1)$. Then

$$\|u + v\|^2 = \|(1, -1) + (1, 1)\|^2 = \|(2, 0)\|^2 = \left(\sqrt{4 + 0}\right)^2 = 4.$$

And,

$$\|u\|^2 = \|(1, -1)\|^2 = \left(\sqrt{1 + 1}\right)^2 = 2.$$

$$\|v\|^2 = \|(1, 1)\|^2 = \left(\sqrt{1 + 1}\right)^2 = 2.$$

Thus, $\quad \|u + v\|^2 = \|u\|^2 + \|v\|^2.$

This means u and v verifies the Pythagorean Theorem.