classmate

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EXPECTATION

H.W. find out. binomial distribution geometric distribution.

prove that. $E(ax+b) = aEx + b$.

$Y = aX + b$

$Y(\omega) = aX(\omega) + b$

RECAP

- probability axioms
- derived few important results
- Conditional probability
- random variables (discrete & continuous)
- pmf, pdf, cdf

(i) $0 \leq p(E) \leq 1$

(ii) $p(\Omega) = 1$

(iii) if $E_1 \cap E_2 = \phi$ $\quad E_1, E_2 \in \mathcal{F}$

$p(E_1 \cup E_2) = p(E_1) + p(E_2)$

$p(A|B) = \dfrac{p(A \cap B)}{p(B)}$ if $p(B) > 0$

gaussian distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad x \in (-\infty, \infty)$$

where, μ = mean, σ = standard deviation.

~~if~~ $\mu = 0$ $\sigma = 1$ is called normal distribution

EXPECTATION OF RV

X

e.g. i toss a coin n times, how many heads we expect?
N/2.

$$\Omega = \{\underbrace{HHH\ldots H}_{n}, \underbrace{HTH\ldots H}_{n}, \ldots, \underbrace{TTT\ldots T}_{n}\} \quad 2^n \text{ outcomes}$$

$X : \Omega \to \mathbb{R}$ $\qquad X(\Omega) = \{0, 1, 2, \ldots n\}$

$X(\omega)$ = # heads in 'ω' $\forall\, \Omega$
↳ sample

K experiments $\qquad$ 1 = experiment is tossing coin n times

k_1 times $\quad$ 0 H $\quad$ ~~E(x)~~ $\omega_0 \in \Omega$
k_2 times $\quad$ 1 H $\quad$ ω_1
$\vdots$ $\qquad\qquad\qquad \vdots$ $\qquad k_1 + k_2 + \ldots k_{n+1} = K$.

k_{n+1} times $\quad$ n H $\quad$ ω_n

$$E(x) = \frac{0.K_1 + 1.K_2 + \dots + n.K_{n+1}}{K}$$

sample avg.

$$\lim_{K\to\infty} = 0 . \lim_{K\to\infty} \frac{K_1}{K} + 1 . \lim_{K\to\infty} \frac{K_2}{K} + \dots n . \lim_{K\to\infty} \frac{K_{n+1}}{K}$$

$$\lim_{K\to\infty} \frac{K_1}{K} = p(X=0)$$

$$\lim_{K\to\infty} \frac{K_i}{K} = p(X=i-1) \qquad i = \{1 \dots n+1\}$$

$$EX = \; = \sum_{x \in X(\Omega)} x \, p(X=x)$$ expectation for DRV.

$$X(\omega) = (x_1, \dots x_k)$$

$$EX = \int_{x : p(x) > 0} x . f_X(x) \, dx$$ expectation of CRV

$p: x \to x + dx$

$= f_X(x) \, dx$

HW. find out binomial distribution
geometric distribution.

prove that $E(aX+b) = aEx + b$

$$Y = aX + b$$

$$Y(\omega) = aX(\omega) + b$$

$$p(Y=y) = p(\{\omega \mid Y(\omega) = y\})$$

$$= p(\{\omega \mid aX(\omega) + b = y\})$$

bernoulli RV

coin toss $p(H) = p$

$X = 1 \quad p$

$\;\;\, = 0 \quad \underbrace{1-p}_{\text{prob.}}$

if H, $X = 1$
if T, $X = 0$.

$E = 1 \cdot p + 0 \cdot (1-p) = p.$

$Y = 5x + 3$

if H $Y = 5 \cdot 1 + 3 = 8$

if T $Y = 5 \cdot 0 + 3 = 3.$

$EY = 8p + 3(1-p)$

E is linear operator.

$= 5p + 3.$

binomial RV (n, p)

X = # successes in n independent bernolli trials.

$X_i = 1$ if ith bernoulli trial is success,
$\;\;\;\; = 0$ otherwise.

$X = \sum_i X_i$

$EY = \sum_i EX_i = \sum_i p = n \cdot p.$

HW $E(g(x)) = \sum_x g(x)\, p(X = x)$

poisson RV

$X \in \{0, 1, 2, \dots, \infty\}$

$$p(X=k) = \frac{e^{-\lambda}\lambda^k}{k!} \qquad \sum_k p(X=k) = 1$$

$$EX = \sum_{k=0}^{\infty} k \cdot \frac{e^{-\lambda}\lambda^k}{k!} = \sum_{k=1}^{\infty} \frac{e^{-\lambda}\lambda^k}{(k-1)!}$$

$$= \sum_{k'=0}^{\infty} \frac{e^{-\lambda}\lambda^{k'+1}}{k'!}$$

$$= \lambda e^{-\lambda} \underbrace{\sum_{k'=0}^{\infty} \frac{\lambda^{k'}}{k'!}}_{e^{\lambda}} = \lambda$$

VARIANCE

how much spread the distribution values are.

$$E(X-EX)^2 \xrightarrow{\text{prove}} EX^2 - [\underset{\mu}{EX}]^2$$

M_1 - fist moment.

mean (μ) is called the first moment of RV.

EX^2 ~~variance~~ is called as the second moment.

$\therefore$ moment generating functions

$$Ee^{tX} = 1 + tM_1 + \frac{t^2}{2!}M_2 + \dots$$

EX^i is called the ith moment.

MEMORYLESS RV

exponential RV $f_X(x) = \lambda e^{-\lambda x}$ if $x \geq 0$

$= 0$ o.w.

cdf $F_X(x) = 1 - e^{-\lambda x}$ $x \geq 0$

$F_X(x) = p(X \leq x)$

$p(X > t \mid X > t_0)$

$p(A \mid B) = \dfrac{p(A \cap B)}{p(B)}$ $\dfrac{p(X > t)}{p(X > t_0)}$

$p(A^c) = 1 - p(A)$ $= \dfrac{e^{-\lambda t}}{e^{-\lambda t_0}}$

$= \dfrac{1 - (1 - e^{-\lambda t})}{1 - (1 - e^{-\lambda t_0})}$

$p(X > t - t_0)$

$= e^{\lambda(t_0 - t)}$

$= e^{-\lambda(t - t_0)}$

$\therefore p(X > t \mid X > t_0) = p(X > t - t_0)$

markov property

game play learning.

physics engine

X: time at which 1st packet arrives

$t_0 = 100 \quad t = 105 \qquad t_0' = 5\,sec$

$t' = 10\,sec.$

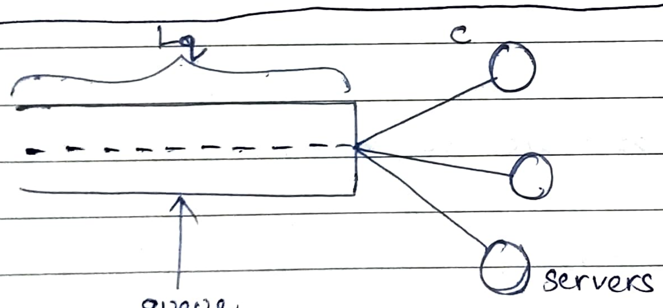
doesnt matter.

$$p(X > t \mid X > t_0) = p(X > t_0' \mid X > t_0')$$

memory less.

$$p(X_{t_{n+1}} = X_{t_n} = X_{t_n}, \dots X_{t_0} = x_{t_0})$$

$$= p(X_{t_{n+1}} = x_{t_n} \mid X_{t_n} = x_{t_n})$$ ← markov



Ls: average or expected packet / customers / requests at servers.

Lq: –||– in the queue.

time spents in the system.

L = avg. or expected in the system.

$$\boxed{L = L_q + L_s} \qquad \omega = \omega_s + \omega_q$$

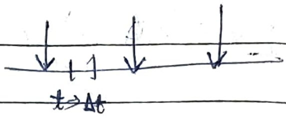
average rate of arrival λ

$\lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} \to 0$

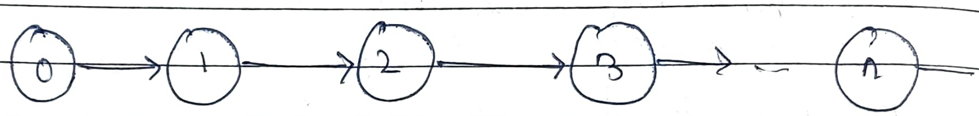
(i) from $t \to t + \Delta t$

$p(\text{one arrival}) = \lambda \Delta t + o(\Delta t)$

(ii) $p(\text{more than one arrival})$



(iii) # arrivals in non-overlapping intervals are indepent of each other



state n: till time t, n packets arrived.

$q_k(t)$ = prob. that there are k arrivals till time t.

$k \geq 1 \qquad q_k(t+\Delta t) = q_{k-1}(t) \times \lambda \Delta t$
$\qquad + q_k(t)(1 - \lambda \Delta t)$

$q_k(t+\Delta t) - q_k(t) = \lambda \Delta t (q_{k-1} - q_k)$

$\frac{q_k(t+\Delta t) - q_k(t)}{\Delta t} = \lambda q_{k-1} - \lambda q_k$ —— (1)

for $k = 0$.

$q_0(t+\Delta t) = q_0(1 - \lambda \Delta t)$

$$\frac{q_0(t+\Delta t) - q_0(t)}{\Delta t} = -\lambda q_0(t) \quad \text{--- (2).}$$

$\lim_{\Delta t \to 0}$ in (2). $\frac{dq_0(t)}{dt} = -\lambda q_0(t)$

$$q_0(t) = e^{-\lambda t} \times c$$

$$q_0(0) = 1 \Rightarrow c = 1$$

$$q_0(t) = e^{-\lambda t}$$

$\lim_{\Delta t \to 0}$ in eq. (1).

$$\frac{dq_k(t)}{dt} = \lambda q_{k-1}(t) - \lambda q_k(t)$$

for $k = 1$

$$\frac{dq_1(t)}{dt} = \lambda q_0(t) - \lambda q_1(t)$$

$$\frac{dq_1(t)}{dt} = -\lambda q_1(t) + \lambda e^{-\lambda t}$$

$$q_1(t) = (\lambda t) e^{-\lambda t}$$

$\vdots$

$$q_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

poisson distribution.

(memory less)