QUEUING THEORY

RECAP

- memoryless random variables
- exponential (λ) | poisson (λt) | geometric

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{oth.} \end{cases} \qquad p(X=n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

(i) two successive arrivals independent

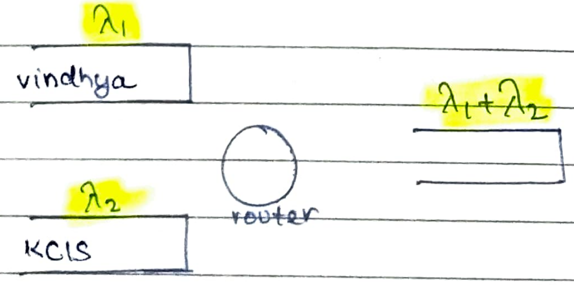
$\frac{o(\Delta t)}{\Delta t} \to 0$ as $\Delta t \to 0$

(ii) p(exactly one arrival in Δt) $= \lambda \Delta t + o(\Delta t)$

(iii) p(more than one arrival in Δt) $= o(\Delta t)$

then the process is a poisson process with parameter λt.

$$p(n \text{ arrivals in } t \text{ time units}) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$



$$p(K \text{ arrivals in } t \text{ time units}) = e^{-(\lambda_1+\lambda_2)t} \frac{((\lambda_1+\lambda_2)t)^K}{K!}$$

$= E_1$ K in first queue + 0 in second queue

E_2 K-1 in first queue + 1 in second queue

⋮

E_{K+1} 0 in first queue + K in second queue.

$$= \sum_{K_1=0}^{K} p(\overbrace{K_1 \text{ packets in queue1}}^{A} \;\&\; \overbrace{K-K_1 \text{ packets in queue2}}^{B})$$

independent

$$= \sum_{K_1=0}^{K} e^{-\lambda_1 t} \frac{(\lambda_1 t)^{K_1}}{K_1!} \times e^{-\lambda_2 t} \frac{(\lambda_2 t)^{K-K_1}}{(K-K_1)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)t}}{K!} \left\{ \underbrace{\sum_{K_1=0}^{K} \frac{K!}{K_1!(K-K_1)!} \times (\lambda_1 t)^{K_1} (\lambda_2 t)^{K-K_1}} \right\}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)t}}{K!} ((\lambda_1+\lambda_2)t)^K$$

$=$ poisson $(\lambda_1+\lambda_2)$

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poisson RV $X(\lambda)$ $E_X = \lambda$

exponential RV $X(\mu)$ $E_X = 1/\mu$

$p(T \geq t) = p(\text{no packets in } t) = e^{-\lambda t}$

next packet arrival time.

$= 1 - p(T \leq t)$ $\therefore$ $p(T \leq t) = 1 - e^{-\lambda t}$

exponential.

$\lambda = 5$ packets / second

$p(5 \text{ packets in } 2 \text{ seconds}) = e^{-10} \times \frac{10^5}{5!} = 0.03$

$p(5 \text{ packets in } 1 \text{ second}) = e^{-5} \times \frac{5^5}{5!} = 0.17$

$p(10 \text{ packets in } 1 \text{ second}) = e^{-5} \times \frac{5^{10}}{10!} = 0.0183$

A / B / C / D / E or A / B / C

A - arrival process (mean λ)

B - service rate (μ)

C - # servers

D - max. no of customers, or packets in the system.

E - population, max. no. of customer will ever need the service

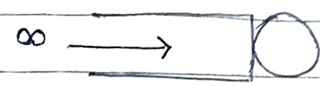
M/M/1

M - memoryless (poisson) λ

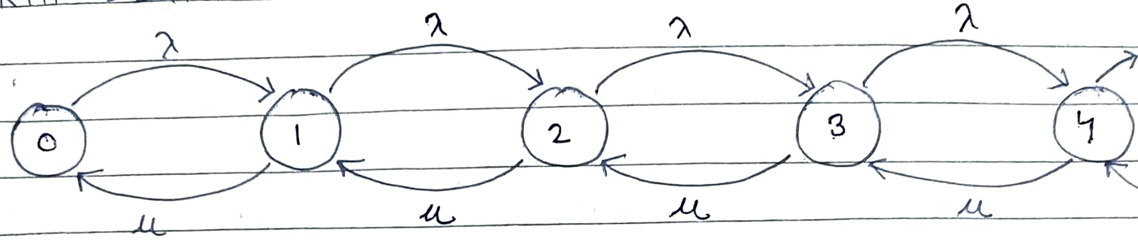
M - service process memoryless (μ) exponential

1 - one server

∞ - buffer, ∞ - population size.

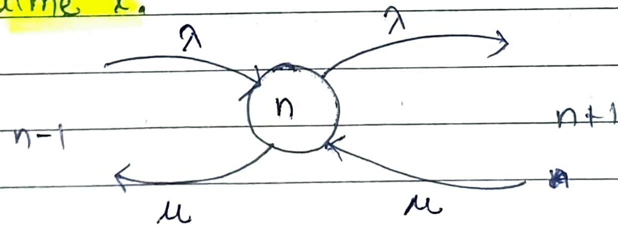


BIRTH - DEATH PROCESS



load factor $\rho = \frac{\lambda}{\mu} < 1$

$P_n(t)$ = probability of n customers in the system at time t.



$P_n(t + \Delta t)$ = no new arrival & no new departure & you were in state n at time t.

OR one new arrival & you were in state n-1 at time t

OR one new departure & you were in state n+1 at time t.

$$P_n(t+\Delta t) = (1-\lambda\Delta t)(1-\mu\Delta t)\, P_n(t) + \lambda\Delta t\, P_{n-1}(t) + \mu\Delta t\, P_{n+1}(t)$$

$$\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -(\lambda+\mu) P_n(t) + \lambda\mu(\Delta t) P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t)$$

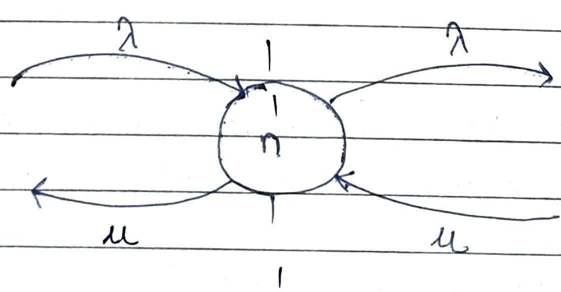
taking $\lim \Delta t \to 0$,

$$\frac{dP_n(t)}{dt} = -(\lambda+\mu) P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t)$$

for steady state $\frac{dP_n(t)}{dt} = 0$.

$$\Rightarrow (\lambda+\mu) P_n(t) = \lambda P_{n-1}(t) + \mu P_{n+1}(t)$$

$$\Rightarrow (\lambda+\mu) P_n = \lambda P_{n-1} + \mu P_{n+1}$$

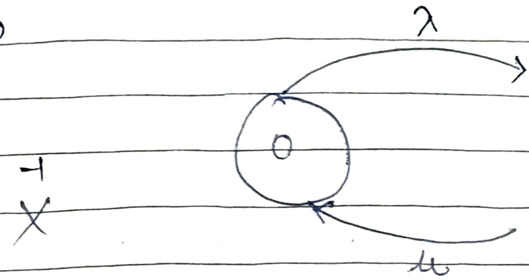


calculating flow

$$\lambda P_{n-1} + \mu P_{n+1} = (\lambda+\mu) P_n$$

entry exit

for $n = 0$



$\lambda P_0 = \mu P_1$ M/M/1

$\Rightarrow P_1 = \frac{\lambda}{\mu} P_0$

$\Rightarrow P_1 = \rho P_0$

$\cancel{\lambda P_0} + \mu P_2 = (\lambda + \cancel{\mu}) P_1$

$P_2 = \frac{\lambda P_1}{\mu} = \rho P_1 = \rho^2 P_0$

$P_n = \rho^n P_0$

$\sum_{n=0}^{\infty} P_n = 1$

$P_0 + \rho P_1 + \rho^2 P_2 + \dots = 1$ $P_0\left(\frac{1}{1-\rho}\right) = 1$

$P_0 = 1 - \rho$

Utilization $U = 1 - P_0 = \rho$ M/M/1

	$\rho = 0.1$	$\rho = 0.5$	$\rho = 0.9$
$p(0 \le \ \le 3)$	0.9999	0.9375	0.34
$p(4 \le \ )$			
$p(12 \ge \ )$	10^{-12}		0.2835

$P_0 + P_1 + P_2 + P_3$

$= 0.9 + 0.09 + 0.009 + 0.0009$

$1 - (P_0 + P_1 + \dots P_{11})$

$= 1 - \underbrace{0.999\dots}_{12 \text{ times}}$

$\rho = 0.9 \qquad \frac{0.9}{1-0.9} = 9 = L$

$W = \frac{9}{18} = 0.5 \qquad W_{ser} = \frac{1}{20} = 0.05$

$W_q = 0.45$

$\rho = \frac{\lambda}{\mu} = \frac{19}{20} = 0.95$

$L = \frac{0.95}{0.05} = 19$

$W = 1 \qquad W_q = \frac{19}{20} \times 60 = 57 sec$

$\rho = 0.2 \qquad L = \frac{0.2}{0.8} = 0.25$

$W = \frac{0.25}{2} = 0.125$

$W_q = \frac{1}{8} - \frac{1}{10} \quad 0.025$