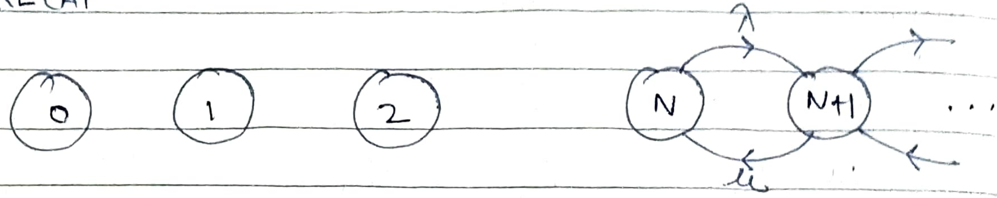
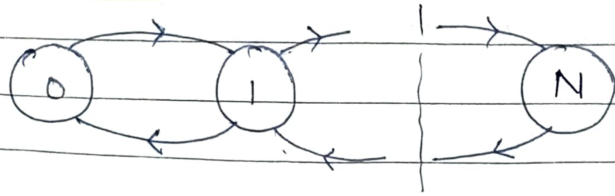
QUEUING THEORY

RECAP



M/M/1/N



A/B/C/D/E

max # customers in the system.

$$\lambda P_{N-1} = \mu P_N$$

$$\lambda P_{N-2} = \mu P_{N-2} \quad \cdots \quad \lambda P_0 = \mu P_1$$

$$P_n = \rho^n P_0 \qquad \rho = \frac{\lambda}{\mu} \qquad n = 0, 1, 2, \ldots N$$

$$\sum_{n=0}^{N} P_n = 1$$

$$P_0 \left(1 + \rho + \rho^2 + \ldots \rho^N\right) = 1$$

$$P_0 \left(\frac{1-\rho^{N+1}}{1-\rho}\right) = 1 \Rightarrow P_0 = \frac{1-\rho}{1-\rho^{N+1}}$$

blocking propability $P_N = \dfrac{1-\rho}{1-\rho^{N+1}} \rho^N$ $(\rho > 1?)$

= Pr.(when queue is full)

for M/M/1 $\rho = 2$ $N = 2$

$U = 1 - P_0 = \rho$ $\qquad P_0 = \dfrac{1-2}{1-4} = \dfrac{1}{3}$ $\qquad U = 1 - \dfrac{1}{3} = \dfrac{2}{3}$

M/M/1/N

$\rho = 1 \Rightarrow P_0 = \frac{1}{N+1} \quad P_n = \frac{1}{N+1}$ } equiprobable

(1) $N = 5 \Rightarrow \quad P_0 = \frac{1-0.1}{1-0.1^5} = \frac{0.9}{0.99999} \approx 0.9 \quad P_N = 9\times10^{-6}$

$\rho = 0.1$

$\rho = 0.5 \Rightarrow \quad P_0 = 0.508 \quad P_N = 0.016$

$\rho = 1 \Rightarrow \quad P_0 = 1/6 = 0.167 \quad P_N = 0.167$

$\rho = 2 \Rightarrow \quad P_0 = 0.016 \quad P_N = 0.508$

$\rho = 5 \Rightarrow \quad P_0 = 0.00026 \quad P_N = 0.8$

(2) $\rho = 0.9$

$N = 1 \quad P_N = 0.47$

$N = 2 \quad P_N = 0.3$

$N = 20 \quad P_N = 0.014$

$N = 100 \quad P_N = 2.7\times10^{-6}$

$L = E(n) = 0.P_0 + 1.P_1 + 2.P_2 + \dots N.P_N$

$= P_0 \left(\rho + 2\rho^2 + 3\rho^3 + \dots N\rho^N \right)$

$\rho L = P_0 \left(\rho^2 + 2\rho^3 + \dots (N-1)\rho^N + N\rho^{N+1} \right)$

$$L - \rho L = P_0 \left(\rho + \rho^2 + \rho^3 + \dots \rho^N - N\rho^{N+1} \right)$$

$$L(1-\rho) = P_0 \left\{ \frac{\rho(1-\rho^{N+1})}{1-\rho} - N\rho^{N+1} \right\}$$

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}}$$

$$L(1-\rho) = \rho - N\rho^{N+1} \left(\frac{1-\rho}{1-\rho^{N+1}} \right)$$

$$= \rho - \frac{N\rho^{N+1}}{1-\rho^{N+1}} + \frac{N\rho^{N+2}}{1-\rho^{N+1}} \quad ??$$

$$L = \frac{\rho}{1-\rho} \left(1 - (N+1) P_N \right)$$

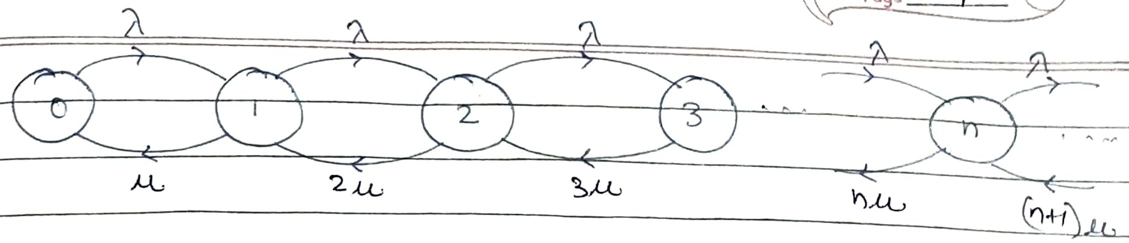
for $\rho = 1$, $L = \frac{1+2+\dots N}{N+1} = \frac{N}{2}$

M/M/∞ QUEUING SYSTEM — ∞ servers
∞ customers
∞ population.

$L = L_q + L_s$ $\qquad$ $L = \lambda \omega$

$$L = \frac{\rho}{1-\rho} \left(1 - (N+1) P_N \right)$$

$$\omega = \frac{1}{\mu - \lambda} \left(1 - (N+1) P_N \right)$$



o server1

o server2

⋮

o server n

$$\lambda P_{n-1} = n\mu P_n$$

$$P_n = \left(\frac{\rho^n}{n!}\right) P_0$$

$$\sum_{n=0}^{\infty} P_n = 1 \qquad P_0\left(1 + \sum_{n=1}^{\infty} \frac{\rho^n}{n!}\right) = 1$$

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\rho^n}{n!}} = e^{-\rho}$$

$$\underline{\underline{P_0 = e^{-\rho}}}$$

$$\underline{\underline{P_n = \left(\frac{\rho^n}{n!}\right) e^{-\rho}}}$$

$$L = \sum_{n=0}^{\infty} nP_n = \sum_{n=1}^{\infty} \frac{n}{n!} \rho^n e^{-\rho}$$

$$= \rho \sum_{n'=0}^{\infty} \frac{\rho^{n'}}{n'!} e^{-\rho}$$

$$= \rho\, e^{\rho} \cdot e^{-\rho} = \rho$$

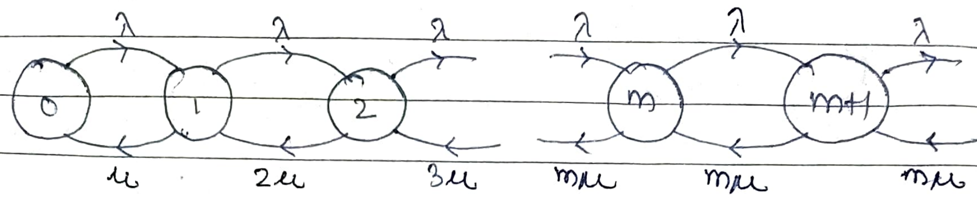
$$\therefore \underline{L = \rho}$$

memoryless service

M/M/m QUEUING SYSTEM

m servers

memoryless arrival



$\lambda P_{n-1} = n\mu P_n \qquad n \leq m$

$\qquad\qquad m\mu P_n \qquad \text{if } n > m$

$$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right) P_0 = \frac{\rho}{n!} P_0 \qquad \text{if } n \leq m$$

$$= \frac{\rho}{m!\,(m)^{n-m}} P_0 \qquad \text{if } n > m$$

$$\sum_{n=0}^{\infty} P_n = 1$$

$$\Rightarrow P_0 = \frac{1}{1 + \sum_{n=1}^{m-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{m!}\left(\frac{\lambda}{\mu}\right)^m \sum_{n=m}^{\infty} \frac{1}{m^{n-m}}\left(\frac{\lambda}{\mu}\right)^{n-m}}$$

$\rho = \frac{\lambda}{m\mu}$ in M/M/m

$$L_q = \sum_{n=m+1}^{\infty} (n-m) P_n \qquad P_m$$

$$= \sum_{n=m+1}^{\infty} (n-m) \cdot \frac{1}{m!} \cdot \frac{1}{m^{n-m}} \cdot \left(\frac{\rho}{m}\right)^n \cdot P_0$$

$$L_q = \frac{\rho}{1-\rho} \cdot \left(\frac{\rho^m}{m!\,(1-\rho)} \times P_0\right)$$

$= P_m + P_{m+1} + \dots$

prob. (queuing)

after queue starts filling in, the system starts behaving similar to M/M/1 with service rate mu.

M/M/m/m QUEUING SYSTEM

memoryless arrival — memoryless service — m servers — max # customers / packets in system

P_0 P_n.

M/G/1 — general service process

M/D/1 — deterministic service process.