Deriving the ANOVA table for the three factor model with normal errors

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Consider the three factor model with normal errors, i.e., $Y_{ijk} = \mu + \tau_i + \theta_j + \eta_k + (\tau\theta)_{ij} + (\theta\eta)_{jk} + \epsilon_{ijk}$, $i = 1, \ldots, a$, $j = 1, \ldots, b$, $k = 1, \ldots, c$, with constraints $\sum_i \tau_i = 0$, $\sum_j \theta_j = 0$, $\sum_k \eta_k = 0$, $\sum_i (\tau\theta)_{ij} = 0$, $\sum_j (\tau\theta)_{ij} = 0$, $\sum_j (\theta\eta)_{jk} = 0$, and $\sum_k (\theta\eta)_{jk} = 0$. Develop a suitable sequence of nested hypotheses, giving the relevant sums of squares. Complete the ANOVA table.

$$\mathbf{Y} = \mathbf{X}\beta + \epsilon$$

Without loss of generality, $i = 1, 2$, $j = 1, 2$, and $k = 1, 2$. Therefore,

$$\mathbf{Y} = \begin{bmatrix} Y_{111} \\ Y_{112} \\ Y_{121} \\ Y_{122} \\ Y_{211} \\ Y_{212} \\ Y_{221} \\ Y_{222} \end{bmatrix}$$

and β is

$$\beta = \begin{bmatrix} \mu \\ \tau_1 \\ \tau_1 \\ \theta_1 \\ \theta_2 \\ \eta_1 \\ \eta_2 \\ (\tau\theta)_{11} \\ (\tau\theta)_{12} \\ (\tau\theta)_{21} \\ (\tau\theta)_{22} \\ (\theta\eta)_{11} \\ (\theta\eta)_{12} \\ (\theta\eta)_{21} \\ (\theta\eta)_{22} \end{bmatrix}$$

and $\mathbf{X}$ is

1	1	0	1	0	1	0	1	0	0	0	1	0	0	0
1	1	0	1	0	0	1	1	0	0	0	0	1	0	0
1	1	0	0	1	1	0	0	1	0	0	0	0	1	0
1	1	0	0	1	0	1	0	1	0	0	0	0	0	1
1	0	1	1	0	1	0	0	0	1	0	1	0	0	0
1	0	1	1	0	0	1	0	0	1	0	0	1	0	0
1	0	1	0	1	1	0	0	0	0	1	0	0	1	0
1	0	1	0	1	0	1	0	0	0	1	0	0	0	1

so $\mathbf{X}'$ is

1	1	1	1	1	1	1	1
1	1	1	1	0	0	0	0
0	0	0	0	1	1	1	1
1	1	0	0	1	1	0	0
0	0	1	1	0	0	1	1
1	0	1	0	1	0	1	0
0	1	0	1	0	1	0	1
1	1	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	1	1	0	0
0	0	0	0	0	0	1	1
1	0	0	0	1	0	0	0
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	1	0	0	0	1

therefore $\mathbf{X}'\mathbf{X}$ is

8	4	4	4	4	4	4	2	2	2	2	2	2	2	2
4	4	0	2	2	2	2	2	2	0	0	1	1	1	1
4	0	4	2	2	2	2	0	0	2	2	1	1	1	1
4	2	2	4	0	2	2	2	0	2	0	2	2	0	0
4	2	2	0	4	2	2	0	2	0	2	0	0	2	2
4	2	2	2	2	4	0	1	1	1	1	2	0	2	0
4	2	2	2	2	0	4	1	1	1	1	0	2	0	2
2	2	0	2	0	1	1	2	0	0	0	1	1	0	0
2	2	0	0	2	1	1	0	2	0	0	0	0	1	1
2	0	2	2	0	1	1	0	0	2	0	1	1	0	0
2	0	2	0	2	1	1	0	0	0	2	0	0	1	1
2	1	1	2	0	2	0	1	0	1	0	2	0	0	0
2	1	1	2	0	0	2	1	0	1	0	0	2	0	0
2	1	1	0	2	2	0	0	1	0	1	0	0	2	0
2	1	1	0	2	0	2	0	1	0	1	0	0	0	2

The normal equations are

$$\mathbf{X}'\mathbf{X}\beta = \mathbf{X}'\mathbf{Y}$$

Therefore, applying the matrix multiplication above, we have

$$8\mu + 4\sum_i \tau_i + 4\sum_j \theta_j + 4\sum_k \eta_k + 2\sum_j (\tau\theta)_{1j} + 2\sum_j (\tau\theta)_{2j} + 2\sum_k (\theta\eta)_{1k} + 2\sum_k (\theta\eta)_{2k} = Y_{\ldots}$$

$$4\mu + 4\tau_i + 2\sum_j \theta_j + 2\sum_k \eta_k + 2\sum_j (\tau\theta)_{ij} + \sum_k (\theta\eta)_{1k} + \sum_k (\theta\eta)_{2k} = Y_{i..}$$

$$4\mu + 2\sum_i \tau_i + 4\theta_j + 2\sum_k \eta_k + 2\sum_i (\tau\theta)_{ij} + 2\sum_k (\theta\eta)_{jk} = Y_{.j.}$$

$$4\mu + 2\sum_i \tau_i + 2\sum_j \theta_j + 4\eta_k + \sum_j (\tau\theta)_{1j} + \sum_j (\tau\theta)_{2j} + 2\sum_j (\theta\eta)_{jk} = Y_{..k}$$

$$2\mu + 2\tau_i + 2\theta_j + \sum_k \eta_k + 2(\tau\theta)_{ij} + \sum_k (\theta\eta)_{jk} = Y_{ij.}$$

$$2\mu + \sum_i \tau_i + 2\theta_j + 2\eta_k + \sum_i (\tau\theta)_{ij} + 2(\theta\eta)_{jk} = Y_{.jk}$$

From the problem, we are given constraints

$$\sum_i \tau_i = 0$$

$$\sum_j \theta_j = 0$$

$$\sum_k \eta_k = 0$$

$$\sum_i (\tau\theta)_{ij} = 0$$

$$\sum_j (\tau\theta)_{ij} = 0$$

$$\sum_j (\theta\eta)_{jk} = 0$$

$$\sum_k (\theta\eta)_{jk} = 0$$

So, entering all our constraints into our normal equations we get our LSE, under the constraints. Thus, the LSE for μ is

$$8\mu + 4\sum_i \tau_i + 4\sum_j \theta_j + 4\sum_k \eta_k + 2\sum_j (\tau\theta)_{1j} + 2\sum_j (\tau\theta)_{2j} + 2\sum_k (\theta\eta)_{1k} + 2\sum_k (\theta\eta)_{2k} =$$

$$8\mu + 4(0) + 4(0) + 4(0) + 2(0) + 2(0) + 2(0) + 2(0) =$$

$$8\mu = Y_{...}$$

$$\implies \hat{\mu} = \frac{1}{8} Y_{...} = \overline{Y}_{...}$$

The LSE for τ_i is

$$4\mu + 4\tau_i + 2\sum_j \theta_j + 2\sum_k \eta_k + 2\sum_j (\tau\theta)_{ij} + \sum_k (\theta\eta)_{1k} + \sum_k (\theta\eta)_{2k} =$$

$$4\mu + 4\tau_i + 2(0) + 2(0) + 2(0) + 0 + 0 =$$

$$4\mu + 4\tau_i = Y_{i..}$$

$$\implies \hat{\tau}_i = \frac{1}{4}Y_{i..} - \hat{\mu} = \overline{Y}_{i..} - \hat{\mu} = \overline{Y}_{i..} - \overline{Y}_{...}$$

The LSE for θ_j is

$$4\mu + 2\sum_i \tau_i + 4\theta_j + 2\sum_k \eta_k + 2\sum_i (\tau\theta)_{ij} + 2\sum_k (\theta\eta)_{jk} =$$

$$4\mu + 2(0) + 4\theta_j + 2(0) + 2(0) + 2(0) =$$

$$4\mu + 4\theta_j = Y_{.j.}$$

$$\implies \hat{\theta}_j = \frac{1}{4}Y_{.j.} - \hat{\mu} = \overline{Y}_{.j.} - \overline{Y}_{...}$$

The LSE for η_k is

$$4\mu + 2\sum_i \tau_i + 2\sum_j \theta_j + 4\eta_k + \sum_j (\tau\theta)_{1j} + \sum_j (\tau\theta)_{2j} + 2\sum_j (\theta\eta)_{jk} = Y_{..k}$$

$$4\mu + 2(0) + 2(0) + 4\eta_k + 0 + 0 + 2(0) =$$

$$4\mu + 4\eta_k = Y_{..k}$$

$$\implies \hat{\eta_k} = \frac{1}{4}Y_{..k} - \hat{\mu} = \overline{Y}_{..k} - \overline{Y}_{...}$$

The LSE for $(\tau\theta)_{ij}$ is

$$2\mu + 2\tau_i + 2\theta_j + \sum_k \eta_k + 2(\tau\theta)_{ij} + \sum_k (\theta\eta)_{jk} =$$

$$2\mu + 2\tau_i + 2\theta_j + 0 + 2(\tau\theta)_{ij} + 0 =$$

$$2\mu + 2\tau_i + 2\theta_j + 2(\tau\theta)_{ij} = Y_{ij.}$$

$$\implies (\hat{\tau\theta})_{ij} = \frac{1}{2}Y_{ij.} - (\hat{\mu} + \hat{\tau}_i + \hat{\theta}_j)$$

$$\hat{\mu} + \hat{\tau}_i + \hat{\theta}_j = \overline{Y}_{...} + \overline{Y}_{i..} - \overline{Y}_{...} + \overline{Y}_{.j.} - \overline{Y}_{...} = \overline{Y}_{i..} + \overline{Y}_{.j.} - \overline{Y}_{...}$$

Therefore,

$$(\hat{\tau\theta})_{ij} = \overline{Y}_{ij.} - \overline{Y}_{i..} - \overline{Y}_{.j.} + \overline{Y}_{...}$$

The LSE for $(\theta\eta)_{jk}$

$$2\mu + \sum_i \tau_i + 2\theta_j + 2\eta_k + \sum_i (\tau\theta)_{ij} + 2(\theta\eta)_{jk} =$$

$$2\mu + 0 + 2\theta_j + 2\eta_k + 0 + 2(\theta\eta)_{jk} =$$

$$2\mu + 2\theta_j + 2\eta_k + 2(\theta\eta)_{jk} = Y_{.jk}$$

$$\implies (\hat{\theta\eta})_{jk} = \frac{1}{2}Y_{.jk} - (\hat{\mu} + \hat{\theta}_j + \hat{\eta}_k)$$

$$\hat{\mu} + \hat{\theta}_j + \hat{\eta}_k = \overline{Y}_{...} + \overline{Y}_{.j.} - \overline{Y}_{...} + \overline{Y}_{..k} - \overline{Y}_{...} = \overline{Y}_{.j.} + \overline{Y}_{..k} - \overline{Y}_{...}$$

Therefore,

$$(\hat{\theta\eta})_{jk} = \frac{1}{2}Y_{.jk} - (\hat{\mu} + \hat{\theta}_j + \hat{\eta}_k) = \overline{Y}_{.jk} - \overline{Y}_{.j.} - \overline{Y}_{..k} + \overline{Y}_{...}$$

Finally, $\hat{\beta}$

$$\hat{\beta} = \begin{bmatrix} \overline{Y}_{...} \\ \overline{Y}_{i..} - \overline{Y}_{...} \\ \overline{Y}_{.j.} - \overline{Y}_{...} \\ \overline{Y}_{..k} - \overline{Y}_{...} \\ \overline{Y}_{ij.} - \overline{Y}_{i..} - \overline{Y}_{.j.} + \overline{Y}_{...} \\ \overline{Y}_{.jk} - \overline{Y}_{.j.} - \overline{Y}_{..k} + \overline{Y}_{...} \end{bmatrix}$$

Consider a sequence of nested hypotheses, $S_{H_0} \supset S_{H_1} \supset S_{H_2} \supset S_{H_3} \supset S_{H_4} \supset S_{H_5} \supset S_{H_6}$.

$$S_{H_0} = \{Y_{ijk} = \mu + \tau_i + \theta_j + \eta_k + (\tau\theta)_{ij} + (\theta\eta)_{jk} + \epsilon_{ijk}\}$$

$$S_{H_1} = \{Y_{ijk} = \mu + \tau_i + \theta_j + \eta_k + (\tau\theta)_{ij} + \epsilon_{ijk}\}$$

$$S_{H_2} = \{Y_{ijk} = \mu + \tau_i + \theta_j + \eta_k + \epsilon_{ijk}\}$$

$$S_{H_3} = \{Y_{ijk} = \mu + \tau_i + \theta_j + \epsilon_{ijk}\}$$

$$S_{H_4} = \{Y_{ijk} = \mu + \tau_i + \epsilon_{ijk}\}$$

$$S_{H_5} = \{Y_{ijk} = \mu + \epsilon_{ijk}\}$$

$$S_{H_6} = \{Y_{ijk} = \epsilon_{ijk}\}$$

The vectors of fitted values under the models are as follows.

Model 1

$$\hat{y}_{H_0} = \{\hat{\mu} + \hat{\tau}_i + \hat{\theta}_j + \hat{\eta}_k + (\hat{\tau\theta})_{ij} + (\hat{\theta\eta})_{jk}\}$$

where

$$\hat{\mu}+\hat{\tau}_i+\hat{\theta}_j+\hat{\eta}_k+(\hat{\tau\theta})_{ij}+(\hat{\theta\eta})_{jk} = \overline{Y}_{...}+\overline{Y}_{i..}-\overline{Y}_{...}+\overline{Y}_{.j.}-\overline{Y}_{...}+\overline{Y}_{..k}-\overline{Y}_{...}+\overline{Y}_{ij.}-\overline{Y}_{i..}-\overline{Y}_{.j.}+\overline{Y}_{...}+\overline{Y}_{.jk}-\overline{Y}_{.j.}-\overline{Y}_{..k}+\overline{Y}_{...} =$$

$$\overline{Y}_{ij.} + \overline{Y}_{.jk} - \overline{Y}_{.j.} = \hat{y}_{H_0}$$

Model 2

$$\hat{y}_{H_1} = \{\hat{\mu} + \hat{\tau}_i + \hat{\theta}_j + \hat{\eta}_k + (\hat{\tau\theta})_{ij}\}$$

where

$$\hat{\mu} + \hat{\tau}_i + \hat{\theta}_j + \hat{\eta}_k + (\hat{\tau\theta})_{ij} = \overline{Y}_{...} + \overline{Y}_{i..} - \overline{Y}_{...} + \overline{Y}_{.j.} - \overline{Y}_{...} + \overline{Y}_{..k} - \overline{Y}_{...} + \overline{Y}_{ij.} - \overline{Y}_{i..} - \overline{Y}_{.j.} + \overline{Y}_{...} =$$

$$\overline{Y}_{ij.} + \overline{Y}_{..k} - \overline{Y}_{...} = \hat{y}_{H_1}$$

Model 3

$$\hat{y}_{H_2} = \{\hat{\mu} + \hat{\tau}_i + \hat{\theta}_j + \hat{\eta}_k\}$$

where

$$\hat{\mu} + \hat{\tau}_i + \hat{\theta}_j + \hat{\eta}_k = \overline{Y}_{...} + \overline{Y}_{i..} - \overline{Y}_{...} + \overline{Y}_{.j.} - \overline{Y}_{...} + \overline{Y}_{..k} - \overline{Y}_{...} =$$

$$\overline{Y}_{i..} + \overline{Y}_{.j.} + \overline{Y}_{..k} - 2\overline{Y}_{...} = \hat{y}_{H_2}$$

Model 4

$$\hat{y}_{H_3} = \{\hat{\mu} + \hat{\tau}_i + \hat{\theta}_j\}$$

where

$$\hat{\mu} + \hat{\tau}_i + \hat{\theta}_j = \overline{Y}_{...} + \overline{Y}_{i..} - \overline{Y}_{...} + \overline{Y}_{.j.} - \overline{Y}_{...} =$$

$$\overline{Y}_{i..} + \overline{Y}_{.j.} - \overline{Y}_{...} = \hat{y}_{H_3}$$

Model 5

$$\hat{y}_{H_4} = \{\hat{\mu} + \hat{\tau}_i\}$$

where

$$\hat{\mu} + \hat{\tau}_i = \overline{Y}_{...} + \overline{Y}_{i..} - \overline{Y}_{...} = \overline{Y}_{i..} = \hat{y}_{H_4}$$

Model 6

$$\hat{y}_{H_5} = \{\hat{\mu}\}$$

where

$$\hat{\mu} = \overline{Y}_{...}$$

Finally, **Model 7**

$$\hat{y}_{H_6} = \{\mathbf{0}\}$$

Therefore, our $\hat{y}_{H_l}$ vector is

$$\hat{y}_{H_l} = \begin{bmatrix} \overline{Y}_{ij.} + \overline{Y}_{.jk} - \overline{Y}_{.j.} \\ \overline{Y}_{ij.} + \overline{Y}_{..k} - \overline{Y}_{...} \\ \overline{Y}_{i..} + \overline{Y}_{.j.} + \overline{Y}_{..k} - 2\overline{Y}_{...} \\ \overline{Y}_{i..} + \overline{Y}_{.j.} - \overline{Y}_{...} \\ \overline{Y}_{i..} \\ \overline{Y}_{...} \\ \mathbf{0} \end{bmatrix}$$

Where $l = 0, 1, \ldots, 6$.

The reductions in the model sum of squares due to H_l are:

$$Q(H_0) = y\hat{_{H_0}}' y\hat{_{H_0}} = \sum_i \sum_j \sum_k \left(\overline{Y}_{ij.} + \overline{Y}_{.jk} - \overline{Y}_{.j.}\right)^2$$

$$Q(H_1) = y\hat{_{H_1}}' y\hat{_{H_1}} = \sum_i \sum_j \sum_k \left(\overline{Y}_{ij.} + \overline{Y}_{..k} - \overline{Y}_{...}\right)^2$$

$$Q(H_2) = y\hat{_{H_2}}' y\hat{_{H_2}} = \sum_i \sum_j \sum_k \left(\overline{Y}_{i..} + \overline{Y}_{.j.} + \overline{Y}_{..k} - 2\overline{Y}_{...}\right)^2$$

$$Q(H_3) = y\hat{_{H_3}}' y\hat{_{H_3}} = \sum_i \sum_j \sum_k \left(\overline{Y}_{i..} + \overline{Y}_{.j.} - \overline{Y}_{...}\right)^2 = 2 \sum_i \sum_j \left(\overline{Y}_{i..} + \overline{Y}_{.j.} - \overline{Y}_{...}\right)^2$$

$$Q(H_4) = y\hat{_{H_4}}' y\hat{_{H_4}} = \sum_i \sum_j \sum_k \left(\overline{Y}_{i..}\right)^2 = 4 \sum_i \left(\overline{Y}_{i..}\right)^2$$

$$Q(H_5) = \hat{y_{H_5}}'\hat{y_{H_5}} = \sum_i \sum_j \sum_k \left(\overline{Y}_{\dots}\right)^2 = 8\left(\overline{Y}_{\dots}\right)^2$$

$$Q(H_6) = \hat{y_{H_6}}'\hat{y_{H_6}} = \sum_i \sum_j \sum_k \left(\mathbf{0}\right)^2 = \mathbf{0}$$

$$? = SST = Q(H_0|H_1) + Q(H_1|H_2) + \cdots + SSE_{H_0}$$

$$Q(H_0|H_1) = Q(H_0) - Q(H_1)$$

$$Q(H_1|H_2) = Q(H_1) - Q(H_2)$$

$$Q(H_2|H_3) = Q(H_2) - Q(H_3)$$

$$Q(H_3|H_4) = Q(H_3) - Q(H_4)$$

$$Q(H_4|H_5) = Q(H_4) - Q(H_5)$$

$$Q(H_5|H_6) = Q(H_5) - Q(H_6)$$

$$\therefore SST = Q(H_0) - Q(H_6) + SSE_{H_0}$$

Where

$$Q(H_6) = \mathbf{0}$$

So,

$$SST = Q(H_0) + SSE_{H_0}$$

$$\implies SSE_{H_0} = SST - Q(H_0)$$

and

$$SST_c = Q(H_4|H_5) + Q(H_3|H_4) + Q(H_2|H_3) + Q(H_1|H_2) + Q(H_0|H_1) + SST - Q(H_0) =$$

$$Q(H_0) - Q(H_5) + SST - Q(H_0) = SST - Q(H_5)$$

Source	**SS**	**df**	**MS**
τ	$Q(H_4\|H_5)$	1	$Q(H_4\|H_5)/1$
θ	$Q(H_3\|H_4)$	1	$Q(H_3\|H_4)/1$
η	$Q(H_2\|H_3)$	1	$Q(H_2\|H_3)/1$
$(\tau\theta)$	$Q(H_1\|H_2)$	1	$Q(H_1\|H_2)/1$
$(\theta\eta)$	$Q(H_0\|H_1)$	1	$Q(H_0\|H_1)/1$
$Error_{H_0}$	$SST - Q(H_0)$	2	$(SST - Q(H_0))/2$
Tot	SST_c	7	

Where

$$Q(H_l|H_{l+1}) = (y\hat{_{H_l}} - y\hat{_{H_{l+1}}})'(y\hat{_{H_l}} - y\hat{_{H_{l+1}}})$$

So,

(**1**)

$$Q(H_4|H_5) = (y\hat{_{H_4}} - y\hat{_{H_5}})'(y\hat{_{H_4}} - y\hat{_{H_5}}) =$$

$$\sum_i \sum_j \sum_k \left(\overline{Y}_{i..} - \overline{Y}_{...}\right)^2 = 4 \sum_i \left(\overline{Y}_{i..} - \overline{Y}_{...}\right)^2$$

Estimating 1 parameter with constraint on mean, therefore d.f. $a - 1 = 2 - 1 = 1$.

(**2**)

$$Q(H_3|H_4) = (y\hat{_{H_3}} - y\hat{_{H_4}})'(y\hat{_{H_3}} - y\hat{_{H_4}}) =$$

$$\sum_i \sum_j \sum_k \left(\overline{Y}_{i..} + \overline{Y}_{.j.} - \overline{Y}_{...} - \overline{Y}_{i..}\right)^2 = 4 \sum_j \left(\overline{Y}_{.j.} - \overline{Y}_{...}\right)^2$$

Estimating 1 parameter with constraint on mean, therefore d.f. $b - 1 = 2 - 1 = 1$.

(**3**)

$$Q(H_2|H_3) = (y\hat{_{H_2}} - y\hat{_{H_3}})'(y\hat{_{H_2}} - y\hat{_{H_3}}) =$$

$$\sum_i \sum_j \sum_k \left(\overline{Y}_{i..} + \overline{Y}_{.j.} + \overline{Y}_{..k} - 2\overline{Y}_{...} - \overline{Y}_{i..} - \overline{Y}_{.j.} + \overline{Y}_{...}\right)^2 = 4 \sum_k \left(\overline{Y}_{..k} - \overline{Y}_{...}\right)^2$$

Estimating 1 parameter with constraint on mean, therefore d.f. $c - 1 = 2 - 1 = 1$.

(**4**)

$$Q(H_1|H_2) = (y\hat{_{H_1}} - y\hat{_{H_2}})'(y\hat{_{H_1}} - y\hat{_{H_2}}) =$$

$$\sum_i \sum_j \sum_k \left(\overline{Y}_{ij.} + \overline{Y}_{..k} - \overline{Y}_{...} - \overline{Y}_{i..} - \overline{Y}_{.j.} - \overline{Y}_{..k} + 2\overline{Y}_{...}\right)^2 = 2 \sum_i \sum_j \left(\overline{Y}_{ij.} - \overline{Y}_{i..} - \overline{Y}_{.j.} + \overline{Y}_{...}\right)^2$$

Estimating 1 parameter with constraint on 2 means, therefore d.f. $(a - 1)(b - 1) = (2 - 1)(2 - 1) = 1$

(**5**)

$$Q(H_0|H_1) = (y\hat{_{H_0}} - y\hat{_{H_1}})'(y\hat{_{H_0}} - y\hat{_{H_1}}) =$$

$$\sum_i \sum_j \sum_k \left(\overline{Y}_{ij.} + \overline{Y}_{.jk} - \overline{Y}_{.j.} - \overline{Y}_{ij.} - \overline{Y}_{..k} + \overline{Y}_{...}\right)^2 = 2\sum_j \sum_k \left(\overline{Y}_{.jk} - \overline{Y}_{ij.} - \overline{Y}_{..k} + \overline{Y}_{...}\right)^2$$

Estimating 1 parameter with constraint on 2 means, therefore d.f. $(b-1)(c-1) = (2-1)(2-1) = 1$

(**6**)

$$Q(H_5|H_6) = (y\hat{_{H_5}} - y\hat{_{H_6}})'(y\hat{_{H_5}} - y\hat{_{H_6}}) =$$

$$\sum_i \sum_j \sum_k \left(\overline{Y}_{...} - \mathbf{0}\right)^2 = 8\left(\overline{Y}_{...}\right)^2$$

$$SST = (Y)'(Y) =$$

$$\sum_i \sum_j \sum_k (Y_{ijk})^2$$

and

$$Q(H_0) = y\hat{_{H_0}}'y\hat{_{H_0}} = \sum_i \sum_j \sum_k \left(\overline{Y}_{ij.} + \overline{Y}_{.jk} - \overline{Y}_{.j.}\right)^2$$

Therefore,

$$SSE_{H_0} = SST - Q(H_0) = \sum_i \sum_j \sum_k (Y_{ijk})^2 - \sum_i \sum_j \sum_k \left(\overline{Y}_{ij.} + \overline{Y}_{.jk} - \overline{Y}_{.j.}\right)^2 =$$

$$\sum_i \sum_j \sum_k \left(Y_{ijk} - \overline{Y}_{ij.} - \overline{Y}_{.jk} + \overline{Y}_{.j.}\right)^2$$

Estimating 1 parameter with constraint on 3 means so d.f. $b(a-1)(c-1) = 2(2-1)(2-1) = 2$. Also, we know d.f. of total is $N-1 = abc-1 = 8-1 = 7$ so solving for SSE d.f. by summing the remaining d.f.'s leaves 2 as only possible choice.

$$SST_c = SST - Q(H_5) = \sum_i \sum_j \sum_k (Y_{ijk})^2 - \sum_i \sum_j \sum_k \left(\overline{Y}_{...}\right)^2 =$$

$$\sum_i \sum_j \sum_k \left(Y_{ijk} - \overline{Y}_{...}\right)^2$$

Total d.f. $N-1 = abc-1 = 8-1 = 7$, due to constraint on grand mean.