II. Grover's algorithm

- algorithm "searching an unsorted database" with $N = 2^n$ elements in $O(\sqrt{N})$ time — rather: find x s.t. $f(x) = 1$

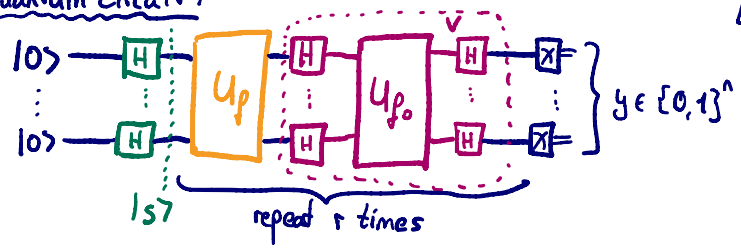
  → classical alg. needs on average $\frac{N}{2} = O(N)$ time

- goal: find ω, given an oracle U_f with $f: \{0,1\}^n \to \{0,1\}$, $f(x) = \begin{cases} 1, & \text{if } x = \omega \\ 0, & \text{else} \end{cases}$, $f_0(x) = \begin{cases} 0, & \text{if } x = 00\ldots0 \\ 1, & \text{else} \end{cases}$

  phase oracle: $U_f|x\rangle = (-1)^{f(x)}|x\rangle$ ⇒ U_f: $|\omega\rangle \to -|\omega\rangle$, $|x\rangle \to |x\rangle \ \forall x \neq \omega$ ⇒ U_{f_0}: $|0\rangle^{\otimes n} \to |0\rangle^{\otimes n}$, $|x\rangle \to -|x\rangle \ \forall x \neq 00\ldots0$

  ↳ $U_f = \mathbb{1} - 2|\omega\rangle\langle\omega|$ ↳ $U_{f_0} = 2|0\rangle\langle 0|^{\otimes n} - \mathbb{1}$

- quantum circuit:



Claim: $y = \omega$ (with high prob.)

Proof: Let us define the uniform superposition state $|s\rangle := H^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle$

and $V := H^{\otimes n} \cdot U_{f_0} \cdot H^{\otimes n} = H^{\otimes n} \cdot 2|0\rangle\langle 0|^{\otimes n} \cdot H^{\otimes n} - H^{\otimes n} \cdot H^{\otimes n} = 2|s\rangle\langle s| - \mathbb{1}$

⇒ Grover's algorithm carries out the operation $(V \cdot U_f)^r$ on the state $|s\rangle$.

Let Σ be the plane spanned by $|s\rangle$ and $|\omega\rangle$ and let $|\omega^\perp\rangle$ be the state orthogonal to $|\omega\rangle$ in Σ: $|\omega^\perp\rangle := \frac{1}{\sqrt{2^n - 1}} \sum_{x \neq \omega} |x\rangle$

⇒ $|s\rangle = \sqrt{\frac{2^n - 1}{2^n}}|\omega^\perp\rangle + \frac{1}{\sqrt{2^n}}|\omega\rangle =: \cos\frac{\theta}{2}|\omega^\perp\rangle + \sin\frac{\theta}{2}|\omega\rangle$

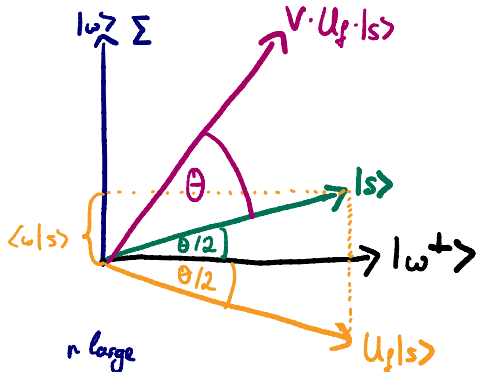
define θ s.t. $\sin\frac{\theta}{2} = \frac{1}{\sqrt{2^n}}$ ⇒ $\theta = 2 \cdot \arcsin\frac{1}{\sqrt{2^n}}$

protocol:

1.) Prepare $|s\rangle$

2.) Apply $U_f = \mathbb{1} - 2|\omega\rangle\langle\omega|$ → reflection at $|\omega^\perp\rangle$

3.) Apply $V = 2|s\rangle\langle s| - \mathbb{1}$ → reflection at $|s\rangle$



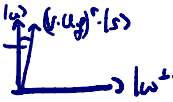
⇒ $V \cdot U_f$ corresponds to a rotation by an angle θ

⇒ after r applications of 2) & 3), the state is rotated by $r \cdot \theta$

↳ choose r, s.t. $r \cdot \theta + \frac{\theta}{2} \overset{!}{\approx} \frac{\pi}{2}$ ⇒ $r = \frac{\pi}{2\theta} - \frac{1}{2} = \frac{\pi}{4 \cdot \arcsin\frac{1}{\sqrt{2^n}}} - \frac{1}{2} \overset{r \text{ large}}{\approx} \frac{\pi}{4}\sqrt{2^n} = O(\sqrt{N})$

⇒ after r calls to the oracle, the final meas. will result in state $|\omega\rangle$ with min. probability

$$p(\omega) \geq 1 - \sin^2\frac{\theta}{2} = 1 - \frac{1}{2^n}$$

(if 

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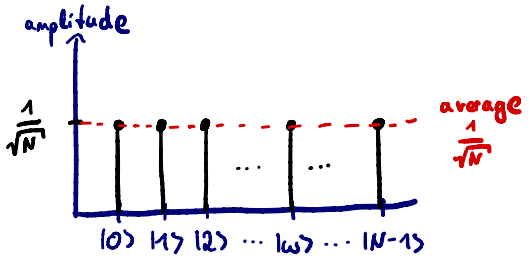
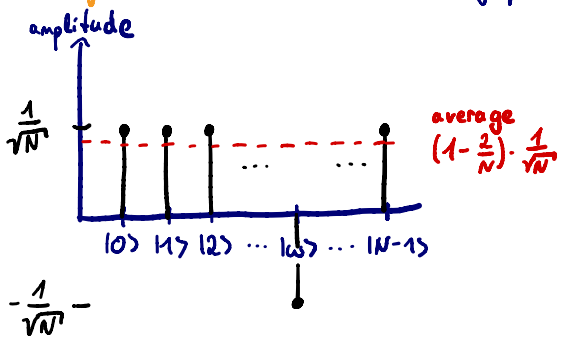
Amplitude amplification

The general idea behind Grover's algorithm is amplitude amplification. Let us have a look at the amplitudes at each step in Grover's algorithm:

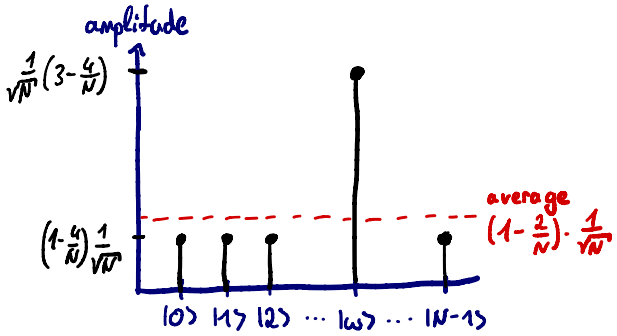
1.) $|s\rangle := H^{\otimes n}|0\rangle^{\otimes n}$



2.) $U_f|s\rangle = (\mathbb{1} - 2|\omega\rangle\langle\omega|)|s\rangle$ → flip amplitude of $|\omega\rangle$



3.) $V \cdot U_f \cdot |s\rangle = (2|s\rangle\langle s| - \mathbb{1}) \cdot U_f \cdot |s\rangle$ → reflect amplitudes about the average amplitude



as for $|\psi\rangle := \sum_i \alpha_i |i\rangle$ $V|\psi\rangle$ yields:

$$(2|s\rangle\langle s| - \mathbb{1})\cdot|\psi\rangle = 2\cdot\frac{1}{N}\cdot\sum_j |j\rangle\cdot\sum_k\langle k|\cdot\sum_i \alpha_i|i\rangle - \sum_i \alpha_i|i\rangle$$

$$= 2\cdot\frac{\sum_k \alpha_k}{N}\cdot\sum_j|j\rangle - \sum_j \alpha_j|j\rangle$$

$$= \sum_j \underbrace{\left(2\cdot\langle\alpha\rangle - \alpha_j\right)}_{\text{reflection of } \alpha_j \text{ about average } \langle\alpha\rangle}|j\rangle$$

(→ if $\alpha_j = \langle\alpha\rangle + \Delta$ then $\alpha_j' = 2\langle\alpha\rangle - \alpha_j = \langle\alpha\rangle - \Delta$)

⇒ by repeating step 2) & 3), the amplitude of $|\omega\rangle$ will increase further ⇒ amplitude amplification!

Multiple marked elements

When we have M marked elements ω_i, we define the winning state as

$$|\omega\rangle := \frac{1}{\sqrt{M}}\sum_{i=1}^{M}|\omega_i\rangle \qquad \rightarrow \qquad |\omega^\perp\rangle := \frac{1}{\sqrt{N-M}}\cdot\sum_{x\notin\{\omega_1,\dots,\omega_M\}}|x\rangle$$

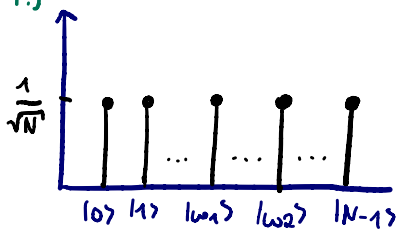
$$\Rightarrow |s\rangle = \frac{\sqrt{N-M}}{\sqrt{N}}\cdot|\omega^\perp\rangle + \sqrt{\frac{M}{N}}\cdot|\omega\rangle =: \cos\frac{\theta}{2}|\omega^\perp\rangle + \sin\frac{\theta}{2}|\omega\rangle$$

$$\hookrightarrow \sin\frac{\theta}{2} = \sqrt{\frac{M}{N}} \Rightarrow \text{angle becomes larger!}$$

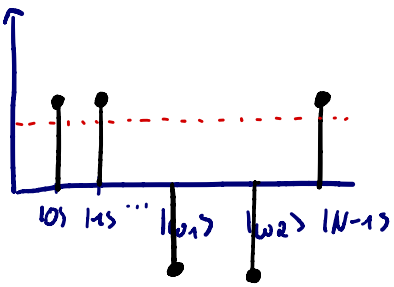
$$\Rightarrow r = \frac{\pi}{4\cdot\arcsin\left(\sqrt{\frac{M}{N}}\right)} - \frac{1}{2} = \mathcal{O}\left(\sqrt{\frac{N}{M}}\right)$$

→ we can see this speedup also when looking at the amplitudes:

1.)

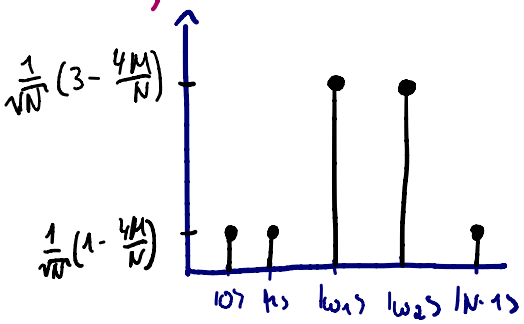


2.)



average $(1-\frac{2M}{N})\cdot\frac{1}{\sqrt{N}}$

3.)



⇒ goes faster