Space-Time DPG: Designing a Method for Parallel CFD

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Overview of Features

- Robust for singularly perturbed problems
- Stable in the preasymptotic regime
- Designed for adaptive mesh refinement

DPG is a minimum residual method:

$$u_h = \underset{w_h \in U_h}{\arg\min} \frac{1}{2} \|Bw_h - l\|_{V'}^2$$

$$\Updownarrow$$

$$b(u_h, R_V^{-1} B\delta u_h) = l(R_V^{-1} B\delta u_h) \quad \forall \delta u_h \in U_h$$

where $v_{\delta u_h} := R_V^{-1} B\delta u_h$ are the **optimal test functions**.

Heat Equation

Simplest Nontrivial Space-Time Problem

Equation is elliptic in space, but hyperbolic in time.

$$\frac{\partial u}{\partial t} - \epsilon \Delta u = f$$

This is really just a composite of Fourier's law and conservation of energy.

$$\begin{aligned} \boldsymbol{\sigma} - \epsilon \nabla u &= 0 \\ \frac{\partial u}{\partial t} - \nabla \cdot \boldsymbol{\sigma} &= f \end{aligned}$$

We can rewrite this in terms of a space-time divergence.

$$\begin{aligned} \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= 0 \\ \nabla_{xt} \cdot \begin{pmatrix} -\boldsymbol{\sigma} \\ u \end{pmatrix} &= f \end{aligned}$$

DPG Formulation

Multiply by test function and integrate by parts over space-time element K.

$$
\left(\frac{1}{\epsilon}\boldsymbol{\sigma},\boldsymbol{\tau}\right)+(u,\nabla\cdot\boldsymbol{\tau})-\langle\hat{u},\boldsymbol{\tau}\cdot\boldsymbol{n}_x\rangle=0
$$

$$
-\left(\begin{pmatrix}-\boldsymbol{\sigma}\\ u\end{pmatrix},\nabla_{xt}v\right)+\langle\hat{t},v\rangle=f
$$

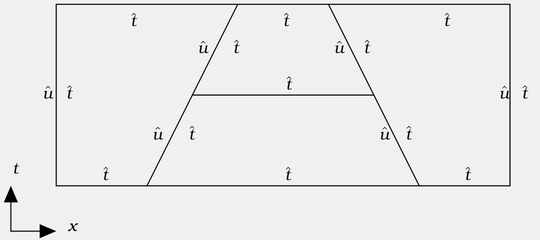
where

$$\hat{u}:=\operatorname{tr}(u)$$

$$\hat{t}:=\operatorname{tr}(-\boldsymbol{\sigma})\cdot\boldsymbol{n}_x+\operatorname{tr}(u)\cdot n_t$$

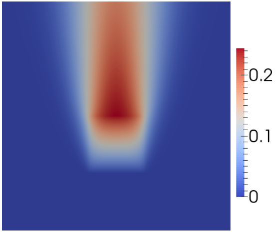
- Trace $\hat{u}$ defined on spatial boundaries
- Flux $\hat{t}$ defined on all boundaries

Support of Trace Variables



Heat equation

Pulsed Source Problem



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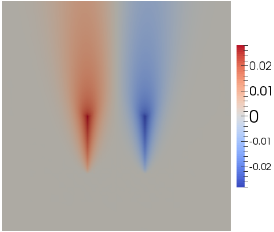

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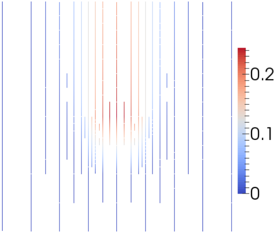
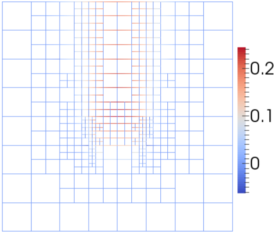


$\hat{u}$



$\hat{t}$

Compressible Navier-Stokes

Strong Form

The compressible Navier-Stokes equations are

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \rho\boldsymbol{u} \\ \rho e_0\end{bmatrix} + \nabla\cdot\begin{bmatrix}\rho\boldsymbol{u} \\ \rho\boldsymbol{u}\otimes\boldsymbol{u} + p\boldsymbol{I} - \mathbb{D} \\ \rho\boldsymbol{u}e_0 + \boldsymbol{u}p + \boldsymbol{q} - \boldsymbol{u}\cdot\mathbb{D}\end{bmatrix} = \begin{bmatrix}f_c \\ \boldsymbol{f}_m \\ f_e\end{bmatrix},$$

where

$$\mathbb{D} = 2\mu\boldsymbol{S}^* = 2\mu\left[\frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right) - \frac{1}{3}\nabla\cdot\boldsymbol{u}\boldsymbol{I}\right],$$

$$\boldsymbol{q} = -C_p\frac{\mu}{Pr}\nabla T,$$

and (assuming an ideal gas EOS)

$$p = \rho RT.$$

Compressible Navier-Stokes

First Order Space-Time Form

Writing this in space-time in terms of ρ, $\boldsymbol{u}$, T, $\mathbb{D}$, and $\boldsymbol{q}$:

$$
\begin{aligned}
\mathbb{D} - \mu\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right) + \frac{2\mu}{3}\nabla \cdot \boldsymbol{u}\boldsymbol{I} &= 0\\
\boldsymbol{q} + C_p \frac{\mu}{Pr}\nabla T &= 0\\
\nabla_{xt} \cdot \begin{pmatrix} \rho \boldsymbol{u} \\ \rho \end{pmatrix} &= f_c\\
\nabla_{xt} \cdot \begin{pmatrix} \rho \boldsymbol{u} \otimes \boldsymbol{u} + \rho RT\boldsymbol{I} - \mathbb{D} \\ \rho \boldsymbol{u} \end{pmatrix} &= \boldsymbol{f}_m\\
\nabla_{xt} \cdot \begin{pmatrix} \rho \boldsymbol{u}\left(C_v T + \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}\right) + \boldsymbol{u}\rho RT + \boldsymbol{q} - \boldsymbol{u} \cdot \mathbb{D} \\ \rho\left(C_v T + \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}\right) \end{pmatrix} &= f_e\,.
\end{aligned}
$$

Compressible Navier-Stokes

DPG Formulation

Multiplying by test functions and integrating by parts:

$$
\begin{aligned}
(\mathbb{D},\mathbb{S}) + (2\mu\boldsymbol{u},\nabla\cdot\mathbb{S}) - \left(\frac{2\mu}{3}\boldsymbol{u},\nabla\operatorname{tr}\mathbb{S}\right) - \langle 2\mu\hat{\boldsymbol{u}},\mathbb{S}\boldsymbol{n}_x\rangle + \left\langle \frac{2\mu}{3}\hat{\boldsymbol{u}},\mathbb{S}\boldsymbol{n}_x\right\rangle &= 0\\
(\boldsymbol{q},\boldsymbol{\tau}) - \left(C_p\frac{\mu}{Pr}T,\nabla\cdot\boldsymbol{\tau}\right) + \left\langle C_p\frac{\mu}{Pr}\hat{T},\tau_n\right\rangle &= 0\\
-\left(\begin{pmatrix}\rho\boldsymbol{u}\\ \rho\end{pmatrix},\nabla_{xt}v_c\right) + \langle\hat{t}_c,v_c\rangle &= (f_c,v_c)\\
-\left(\begin{pmatrix}\rho\boldsymbol{u}\otimes\boldsymbol{u}+\rho RT\boldsymbol{I}-\mathbb{D}\\ \rho\boldsymbol{u}\end{pmatrix},\nabla_{xt}\boldsymbol{v}_m\right) + \langle\hat{\boldsymbol{t}}_m,\boldsymbol{v}_m\rangle &= (\boldsymbol{f}_m,\boldsymbol{v}_m)\\
-\left(\begin{pmatrix}\rho\boldsymbol{u}\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)+\boldsymbol{u}\rho RT+\boldsymbol{q}-\boldsymbol{u}\cdot\mathbb{D}\\ \rho\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\end{pmatrix},\nabla_{xt}v_e\right) + \langle\hat{t}_e,v_e\rangle &= (f_e,v_e)\ ,
\end{aligned}
$$

where $\hat{u}$ and $\hat{T}$ are spatial traces and $\hat{t}_c$, $\hat{\boldsymbol{t}}_m$, and $\hat{t}_e$ are fluxes.

Compressible Navier-Stokes

Flux and Trace Variables

Spatial traces and fluxes are defined as follows:

$$
\begin{aligned}
\hat{\boldsymbol{u}} &= \operatorname{tr}(\boldsymbol{u}) \\
\hat{T} &= \operatorname{tr}(T) \\
\hat{t}_c &= \operatorname{tr}(\rho \boldsymbol{u}) \cdot \boldsymbol{n}_x + \operatorname{tr}(\rho)\, n_t \\
\hat{\boldsymbol{t}}_m &= \operatorname{tr}(\rho \boldsymbol{u} \otimes \boldsymbol{u} + \rho R T \boldsymbol{I} - \mathbb{D}) \cdot \boldsymbol{n}_x + \operatorname{tr}(\rho \boldsymbol{u})\, n_t \\
\hat{t}_e &= \operatorname{tr}\left(\rho \boldsymbol{u}\left(C_v T + \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}\right) + \boldsymbol{u} \rho R T + \boldsymbol{q} - \boldsymbol{u} \cdot \mathbb{D}\right) \cdot \boldsymbol{n}_x \\
&\quad + \operatorname{tr}\left(\rho\left(C_v T + \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}\right)\right) n_t \,.
\end{aligned}
$$

Linearization

Fluxes, traces, and $\boldsymbol{q}$ are linear in the above bilinear form, but we need to linearize in ρ, $\boldsymbol{u}$, T, and $\mathbb{D}$ (Jacobian and residual not shown here).

Compressible Navier-Stokes

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