Robustness for transient problems

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Assume that boundary conditions are applied on the boundary $\Gamma_0 \subset \Gamma$. Recall that, for the ultra-weak variational formulation

$$b\left(\left(u, \widehat{u}\right), v\right) = \left(u, A_h^* v\right)_{L^2(\Omega)} + \langle \widehat{u}, [\![v]\!] \rangle_{\Gamma_h \setminus \Gamma_0}$$

we can recover

$$\|u\|_{L^2(\Omega)}^2 = b(u, v^*)$$

for conforming v^* satisfying the adjoint equation

$$\begin{aligned} A^* v^* &= u \\ v^* &= 0 \text{ on } \Gamma_h \setminus \Gamma_0. \end{aligned}$$

Together, these give necessary conditions on the test norm $\|\cdot\|_V$ such that we have L^2 robustness (this gives robustness in the variable u; for the first order formulation, conditions for σ must also be shown).

$$\|u\|_{L^2(\Omega)}^2 = b(u, v^*) \leq \frac{b(u, v^*)}{\|v^*\|_V} \|v^*\|_V \leq \|u\|_E \|v^*\|_V$$

Thus, showing $\|v^*\|_V \lesssim \|u\|_{L^2(\Omega)}$ gives the result that $\|u\|_{L^2(\Omega)} \lesssim \|u\|_E$.

1 Reaction-diffusion

Consider reaction diffusion

$$\begin{aligned} \frac{\partial u}{\partial t} + u - \epsilon \Delta u &= f \\ u &= 0 \text{ on } \Gamma_1 \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_2 \\ u(t=0) &= u_0. \end{aligned}$$

The adjoint equation satisfies

$$\begin{aligned} -\frac{\partial v}{\partial t} + v - \epsilon \Delta v &= u \\ v &= 0 \text{ on } \Gamma_1 \\ \frac{\partial v}{\partial n} &= 0 \text{ on } \Gamma_2 \\ v(t=T) &= 0. \end{aligned}$$

(The boundary conditions can be derived by taking the ultra-weak formulation and choosing boundary conditions such that the temporal flux and spatial flux terms $\langle \widehat{u}, [\![\tau_n]\!] \rangle_{\Gamma_1}$ and $\left\langle \widehat{f}_n, [\![v]\!] \right\rangle_{\Gamma_2}$ are zero.)

We can then derive that the test norm

$$\|v\|_V^2 = \left\|\frac{\partial v}{\partial t}\right\|^2 + \|v\|^2 + \epsilon \|\nabla v\|^2$$

provides the necessary bound $\|v^*\|_V \lesssim \|u\|_{L^2(\Omega)}$.

To see, this we multiply the adjoint equation by two terms as follows:

1. Multiply by v and integrate over $\Omega \times [0,T] = Q$ to get

$$-\int_Q \frac{\partial v}{\partial t} v + \int_Q v^2 + \epsilon \int_Q |\nabla v|^2 - \epsilon \int_0^T \int_\Gamma \frac{\partial v}{\partial n} v = \int_Q uv.$$

Noting that either $v = 0$ or $\frac{\partial v}{\partial n} = 0$ on the boundary removes the integral over Γ. Next, we can factor the first term and use Young's inequality to get

$$-\int_0^T \frac{\partial}{\partial t} \int_\Omega v^2 + \|v\|_Q^2 + \epsilon \|\nabla v\|_Q^2 \leq \frac{1}{2} \|u\|_Q^2 + \frac{1}{2} \|v\|_Q^2$$

Integrating by parts the first term gives

$$-\int_\Omega v^2 \Big|_0^T + \frac{1}{2} \|v\|_Q^2 + \epsilon \|\nabla v\|_Q^2 \leq \frac{1}{2} \|u\|_Q^2$$

Using boundary condition $v = 0$ at $t = T$ gives

$$\frac{1}{2} \|v\|_Q^2 + \epsilon \|\nabla v\|_Q^2 \leq \int_\Omega v(t=0)^2 + \frac{1}{2} \|v\|_Q^2 + \epsilon \|\nabla v\|_Q^2 \leq \frac{1}{2} \|u\|_Q^2.$$

2. Multiply by $-\frac{\partial v}{\partial t}$ and integrate over Q. Young's inequality changes the right hand side to

$$\int_Q \frac{\partial v}{\partial t}^2 - \int_Q v \frac{\partial v}{\partial t} + \epsilon \int_Q \Delta v \frac{\partial v}{\partial t} = \int_Q -u \frac{\partial v}{\partial t} \leq \frac{1}{2} \|u\|_Q^2 + \frac{1}{2} \left\|\frac{\partial v}{\partial t}\right\|_Q^2.$$

The term $\int_Q v \frac{\partial v}{\partial t}$ can be reduced to the positive contribution $\int_\Omega v(t=0)^2$ as above. We can then take the Laplacian term, integrate by parts in space to get

$$\int_Q \Delta v \frac{\partial v}{\partial t} = \int_0^T \int_\Omega \Delta v \frac{\partial v}{\partial t} = \int_0^T \int_\Gamma \frac{\partial v}{\partial t} \frac{\partial v}{\partial n} - \int_0^T \int_\Omega \nabla \left(\frac{\partial v}{\partial t}\right) \nabla v.$$

Since either $v = 0$ or $\frac{\partial v}{\partial n} = 0$ on Γ, the first term disappears. The second term can be bounded by noting

$$-\int_0^T \int_\Omega \nabla \left(\frac{\partial v}{\partial t}\right) \nabla v = -\int_0^T \frac{\partial}{\partial t} \int_\Omega |\nabla v|^2 = -\int_\Omega |\nabla v|^2 \Big|_0^T.$$

Since $v = 0$ at $t = T$, $\nabla v = 0$ at $t = T$ as well, and we are left with the positive contribution $\int_\Omega |\nabla v(t=0)|^2$. Then,

$$\frac{1}{2} \left\|\frac{\partial v}{\partial t}\right\|_Q^2 \leq \frac{1}{2} \|u\|_Q.$$

Together, these two show that, under test norm

$$\|v\|_V^2 = \left\|\frac{\partial v}{\partial t}\right\|^2 + \|v\|^2 + \epsilon \|\nabla v\|^2,$$

the adjoint equation v^* satisfies

$$\|v^*\|_V \lesssim \|u\|_{L^2(\Omega)}$$

and thus the DPG energy norm robustly bounds the L^2 norm from above

$$\|u\|_{L^2(\Omega)} \lesssim \|u\|_E.$$

2 Convection-diffusion

Truman, your turn :).