DISCONTINUOUS PETROV-GALERKIN (DPG) METHOD WITH OPTIMAL TEST FUNCTIONS

Progress Report

Three DPG Punchlines

Leszek Demkowicz
ICES, The University of Texas at Austin

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Collaboration:

Portland State: J. Gopalakrishnan

ICES: T. Bui-Thanh, O. Ghattas, B. Moser, T. Ellis, J. Zitelli

Argonne: N. Roberts

Boeing: D. Young

Basque U: D. Pardo

C.U. of Hong-Kong: W. Qiu

KAUST: V. Calo

Livermoore: J. Bramwell

Los Alamos: M. Shaskov

Rice: J. Chan

Sandia: P.N. Bochev, K.J. Peterson, D. Ridzal and Ch. M. Siefert

C.U. Chile: I. Muga, N. Heuer

U. Helsinki: A. Niemi

U. Nevada: J. Li

U. Tel-Aviv I. Harari

Three DPG Punchlines

1: DPG Method is a Ritz method. It supports adaptivity with no preasymptotic behavior.

2: You can control the norm in which you want to converge.

3: DPG is easy to code.

Punchline 1

DPG Method is a Ritz method. It supports adaptivity with no preasymptotic behavior.

Primal DPG Method for Maxwell equations.

Assume

$$J_S^{\mathsf{imp}} = n \times H^{imp}$$

and look for the unknown surface current on the skeleton also in the same form.

$$\begin{cases} E \in H(\mathsf{curl}, \Omega),\ n \times E = n \times E^{\mathsf{imp}} \text{ on } \Gamma_1 \\ \hat{h} \in \mathsf{tr}_{\Gamma_h} H(\mathsf{curl}, \Omega),\ n \times \hat{h} = n \times (-i\omega H^{imp}) \text{ on } \Gamma_2 \\ (\frac{1}{\mu} \boldsymbol{\nabla} \times E, \boldsymbol{\nabla}_h \times F) + ((-\omega^2 \epsilon + i\omega\sigma)E, F) + \langle n \times \hat{h}, F \rangle_{\Gamma_h} = -i\omega(J^{\mathsf{imp}}, F) \\ \hfill \forall F \in H(\mathsf{curl}, \Omega_h)\,. \end{cases}$$

FE discretization for curl-curl problem

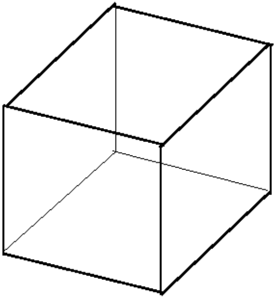
Hexahedral meshes
$H(\text{curl})$ element for electric field E:

$$(\mathcal{P}^{p-1} \otimes \mathcal{P}^{p} \otimes \mathcal{P}^{p}) \times (\mathcal{P}^{p} \otimes \mathcal{P}^{p-1} \otimes \mathcal{P}^{p}) \times (\mathcal{P}^{p} \otimes \mathcal{P}^{p} \otimes \mathcal{P}^{p-1})$$

and trace of the same element for flux (surface current) $\hat{h}$.
Same element for the enriched space but with order $p + \Delta p$.
In reported experiments: $p = 2$, $\Delta p = 2$.

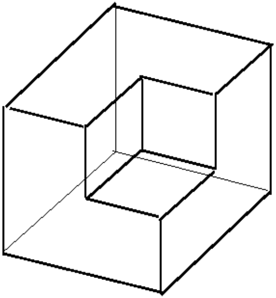
A 3D Maxwell example

Take a cube $(0,2)^3$



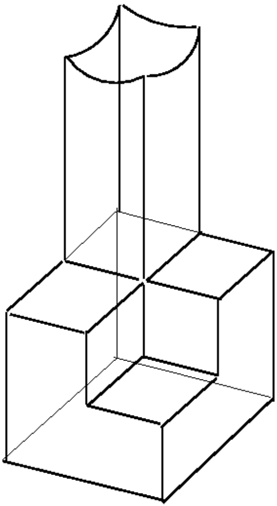
Fichera corner

Divide it into eight smaller cubes and remove one:



Fichera corner microwave

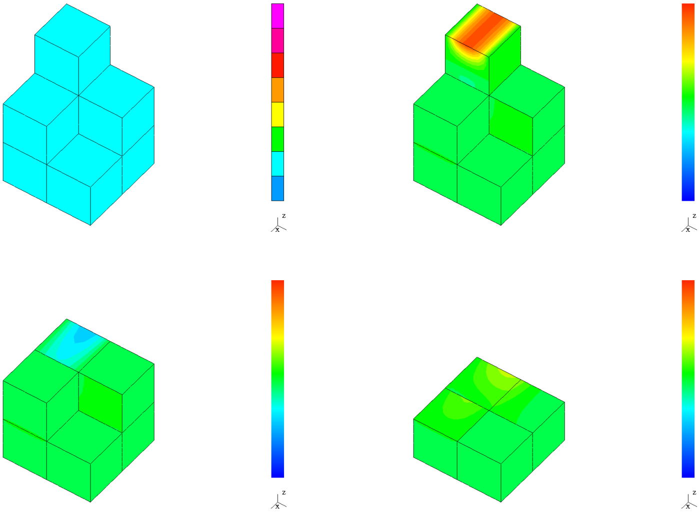
Attach a waveguide:



$\epsilon = \mu = 1, \sigma = 0$
$\omega = 5$(1.6 wavelengths in the cube)

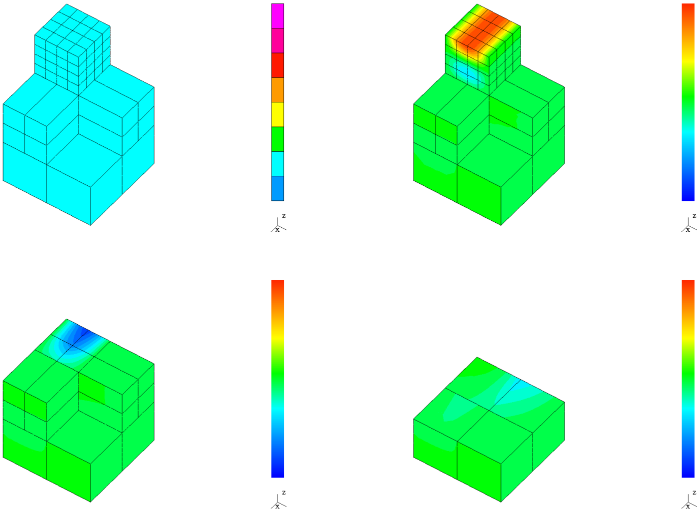
Cut the waveguide and use the lowest propagating mode for BC along the cut.

Fichera corner microwave, $p = 2$.



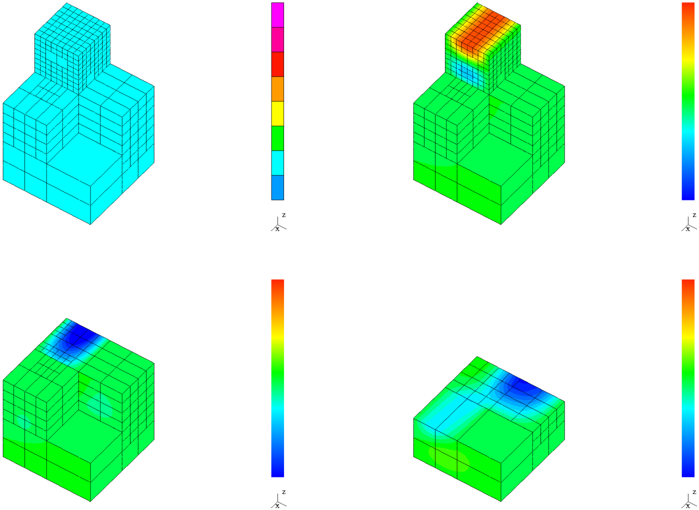
Initial mesh and real part of E_1

Fichera corner microwave, $p = 2$.



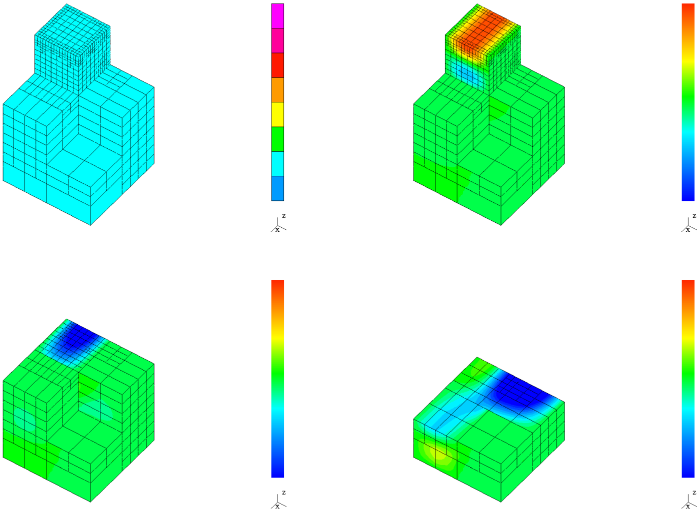
Mesh and real part of E_1 after two refinements

Fichera corner microwave, $p = 2$.



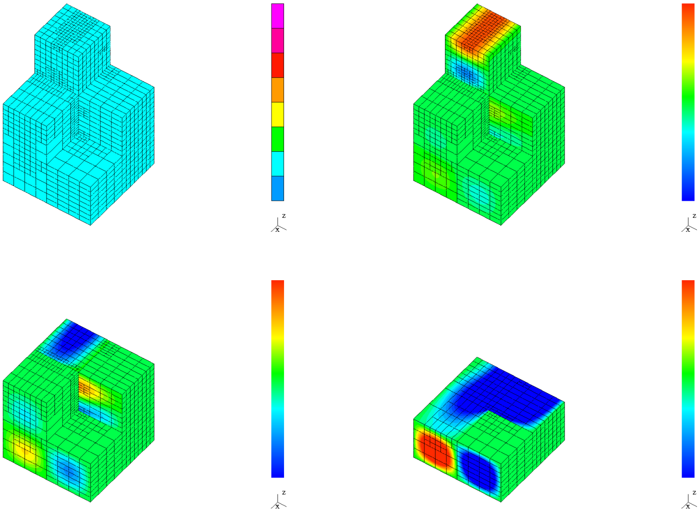
Mesh and real part of E_1 after four refinements

Fichera corner microwave, $p = 2$.



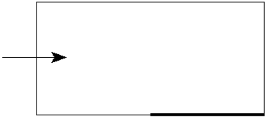
Mesh and real part of E_1 after six refinements

Fichera corner microwave, $p = 2$.



Mesh and real part of E_1 after eight refinements

From Ph.D. Dissertation of Jesse Chan: Compressible Navier-Stokes Equations: Carter's flat plate problem *

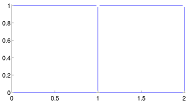


$$\mathsf{M}_\infty = 3, \mathsf{Re}_L = 1000, \mathsf{Pr} = 0.72, \gamma = 1.4, \theta_\infty = 390^o[\mathsf{R}]$$

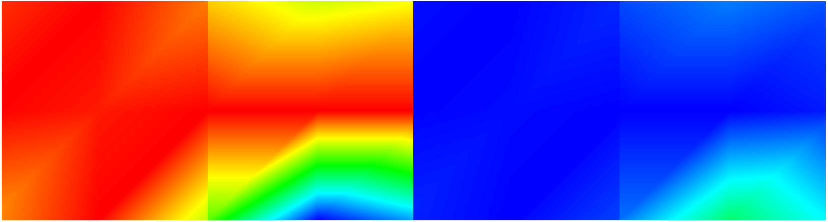
* L.D., J.T. Oden, W. Rachowicz, "A New Finite Element Method for Solving Compressible Navier-Stokes Equations Based on an Operator Splitting Method and hp Adaptivity,", *Comput. Methods Appl. Mech. Engrg.*, **84**, 275-326, 1990.

Extrapolation to Compressible NS Equations

Initial Mesh ($p = 2$):

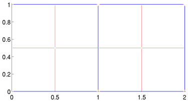


Horizontal velocity and temperature

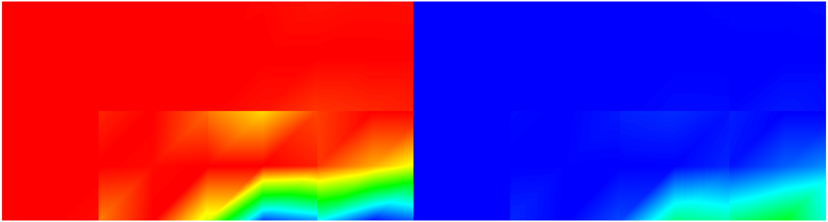


Extrapolation to Compressible NS Equations

Mesh 1:

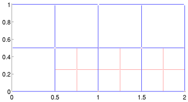


Horizontal velocity and temperature

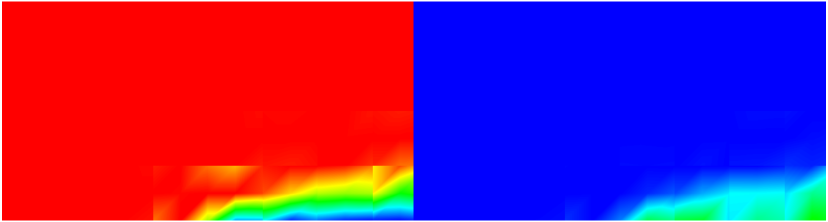


Extrapolation to Compressible NS Equations

Mesh 2:

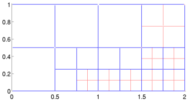


Horizontal velocity and temperature

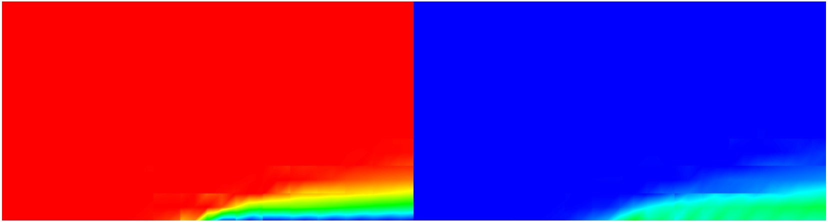


Extrapolation to Compressible NS Equations

Mesh 3:

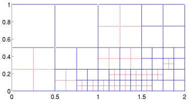


Horizontal velocity and temperature

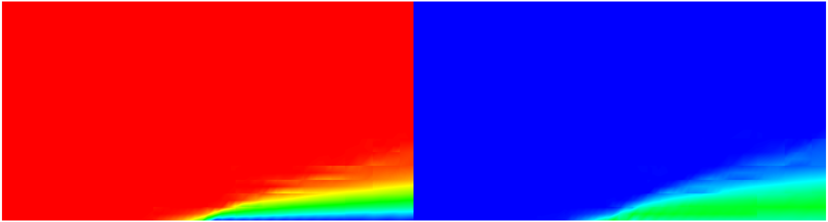


Extrapolation to Compressible NS Equations

Mesh 4:

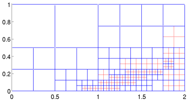


Horizontal velocity and temperature

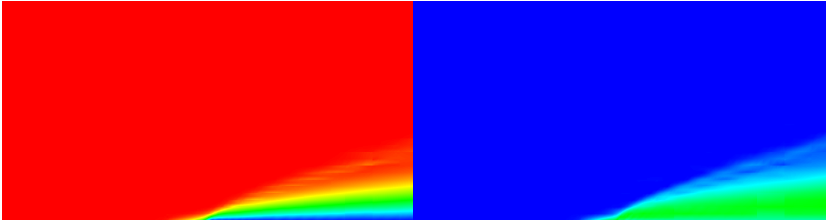


Extrapolation to Compressible NS Equations

Mesh 5:

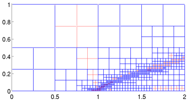


Horizontal velocity and temperature

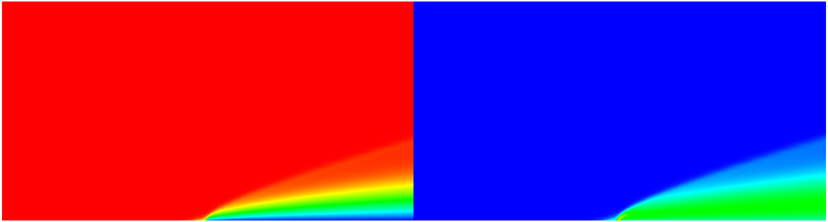


Extrapolation to Compressible NS Equations

Mesh 7:

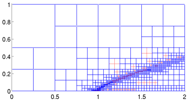


Horizontal velocity and temperature

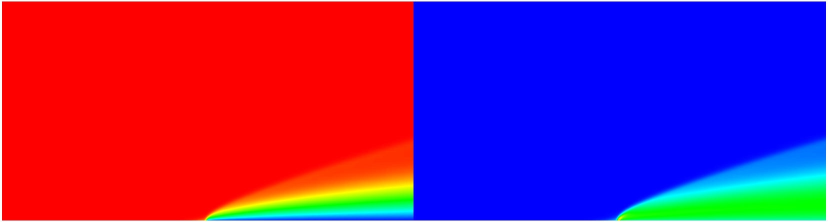


Extrapolation to Compressible NS Equations

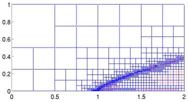
Mesh 8:



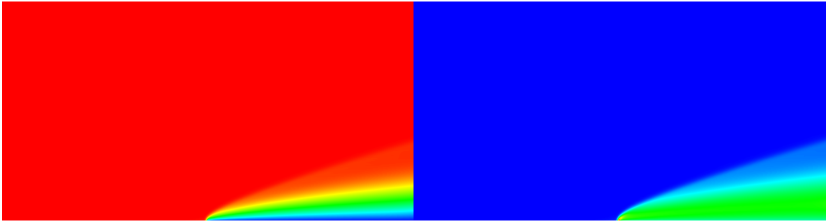
Horizontal velocity and temperature



Mesh 9:

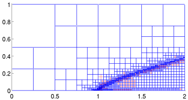


Horizontal velocity and temperature

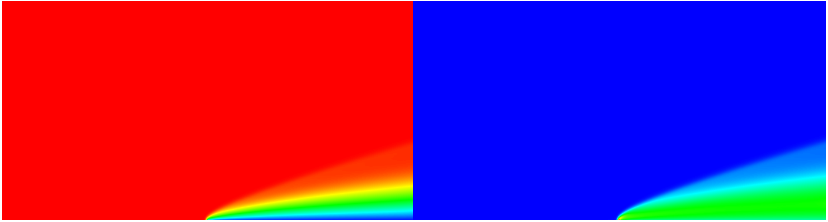


Extrapolation to Compressible NS Equations

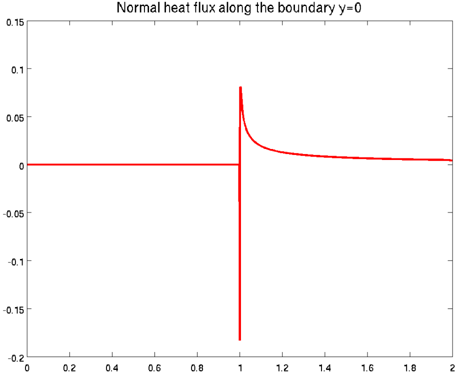
Mesh 10:



Horizontal velocity and temperature



Extrapolation to Compresible NS Equations



Heat flux along the plate

Punchline 2

You can control the norm in which you want to converge.

The simplest singular perturbation problem: reaction-dominated diffusion [†]

[†] L.D. and I. Harari, "Primal DPG Method for Reaction dominated Diffusion", in preparation.

The simplest singular perturbation problem: Reaction-dominated diffusion

$$\left\{ \begin{array}{rll} u & = 0 & \text{on } \Gamma \\ -\epsilon^2 \Delta u + c(x) u & = f & \text{in } \Omega \end{array} \right.$$

where $0 < c_0 \leq c(x) \leq c_1$.
Standard variational formulation:

$$\left\{ \begin{array}{l} u \in H^1(\Omega) \\ \\ \epsilon^2 (\boldsymbol{\nabla} u, \boldsymbol{\nabla} v) + (cu, v) = (f, v) \quad v \in H^1(\Omega) \end{array} \right.$$

Standard Galerkin method delivers the best approximation error in the energy norm:

$$\|u\|^2_{\epsilon^k} := \epsilon^k \|\boldsymbol{\nabla} u\|^2 + \|c^{1/2} u\|^2, \quad k = 2$$

Convergence in "balanced" norm

Fact: Under favorable regularity conditions, the solution is *uniformly* bounded in data f in a "balanced" norm [‡] :

$$\|u\|_\epsilon^2 := \epsilon\|\nabla u\|^2 + \|c^{1/2}u\|^2$$

i.e.

$$\|u\|_\epsilon \lesssim \|f\|_{\text{appropriate}}$$

Question: Can we select the test norm in such a way that the DPG method will be *robust* in the balanced norm ?

$$\|u-u_h\|_\epsilon + \|\hat{t}-\hat{t}_h\|_? \lesssim \inf_{w_h}\|u-w_h\|_\epsilon + \inf_{\hat{r}_h}\|\hat{t}-\hat{r}_h\|_?$$

[‡] R. Lin and M. Stynes, "A balanced finite element method for singularly perturbed reaction-diffusion problems", *SIAM J. Numer. Anal.*, **50(5):** 2729–2743, 2012.

A bit of history: Optimal test functions of Barret and Morton §

For each $w \in U_h$, determine the corresponding v_w that solves the auxiliary variational problem:

$$\left\{ \begin{array}{l} v_w \in H_0^1(\Omega) \\ \underbrace{\epsilon^2(\nabla \delta u, \nabla v_w) + (c\,\delta u, v_w)}_{\text{the bilinear form we have}} = \underbrace{\epsilon(\nabla \delta u, w) + (c\,\delta u, w)}_{\text{the bilinear form we want}} \quad \forall \delta u \in H_0^1(\Omega) \end{array} \right.$$

With the optimal test functions, the Galerkin orthogonality for the original form changes into Galerkin orthogonality in the desired, "balanced" norm:

$$\epsilon^2(\nabla(u-u_h), \nabla v_w) + (c\,(u-u_h), v_w) = 0 \quad \Longrightarrow \quad \epsilon(\nabla(u-u_h), \nabla v_u) + (c\,(u-u_h), w) = 0$$

Consequently, the PG solution delivers the best approximation error in the desired norm.

§

- J.W. Barret and K. W. Morton, "Approximate Symmetrization and Petrov-Galerkin Methods for Diffusion-Convection Problems", *Comp. Meth. Appl. Mech and Engng.*, **46**, 97 (1984).
- L. D. and J. T. Oden, "An Adaptive Characteristic Petrov-Galerkin Finite Element Method for Convection-Dominated Linear and Nonlinear Parabolic Problems in One Space Variable", *Journal of Computational Physics*, **68(1):** 188–273, 1986.

Constructing optimal test norm [¶]

Theorem

Let v_u be the Barret-Morton optimal test function corresponding to u. Let $\|v_u\|_V$ be a test norm such that

$$\|v_u\|_V \lesssim \|u\|_\epsilon$$

Then

$$\|u - u_h\|_\epsilon \lesssim \|u - u_h\|_E = \inf_{w_h \in U_h} \|u - w_h\|_E \leq \textit{BAE estimate}$$

Proof:

$$\begin{aligned}
\|u\|_\epsilon^2 &= \epsilon(\boldsymbol{\nabla} u, \boldsymbol{\nabla} u) + (cu, u) = \epsilon^2(\boldsymbol{\nabla} u, \boldsymbol{\nabla} v_u) + (cu, v_u) \\
&= b((u,\hat{t}), v_u) \leq \frac{b((u,\hat{t}),v_u)}{\|v_u\|_V} \|v_u\|_V \\
&\leq \sup_v \frac{b((u,\hat{t}),v_u)}{\|v\|_V} \|v_u\|_V = \|(u,\hat{t})\|_E \, \|v_u\|_V \\
&\lesssim \|(u,\hat{t})\|_E \, \|u\|_\epsilon
\end{aligned}$$

¶ L. D., M. Heuer, "Robust DPG Method for Convection-Dominated Diffusion Problems," *SIAM J. Num. Anal*, **51:** 2514–2537, 2013.

Constructing optimal test norm

The point: Construction of the optimal test norm is reduced to the stability (robustness) analysis for the Barret-Morton test functions.

Lemma

Let

$$\|v\|_V^2 := \epsilon^3 \|\nabla v\|^3 + \|c^{1/2} v\|^2$$

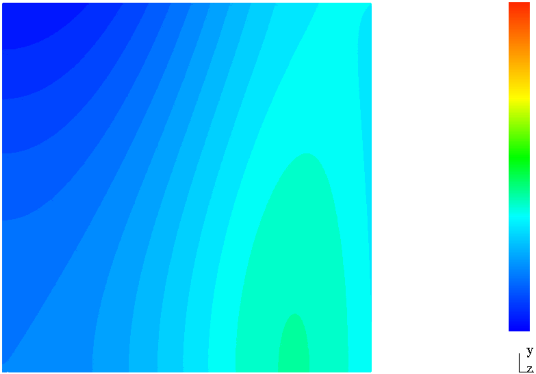
Then

$$\|v_u\| \lesssim \|u\|_\epsilon$$

In order to avoid boundary layers in the optimal test functions (that we cannot resolve using simple enriched space) we scale the reaction term with a mesh-dependent factor:

$$\|v\|_{V,mod}^2 := \epsilon^3 \|\nabla v\|^3 + \min\{1, \frac{\epsilon^3}{h^2}\} \|c^{1/2} v\|^2$$

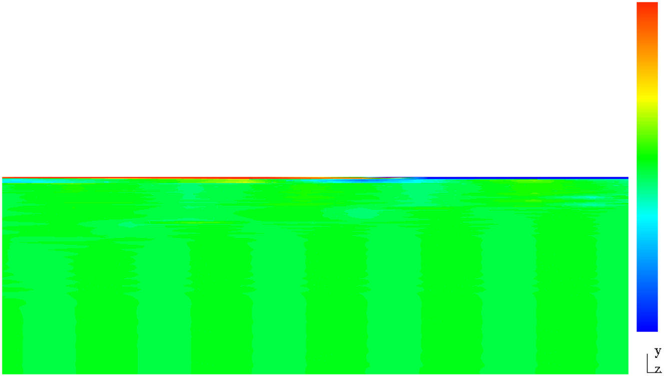
Manufactured solution of Lin and Stynes, $\epsilon = 10^{-1}$



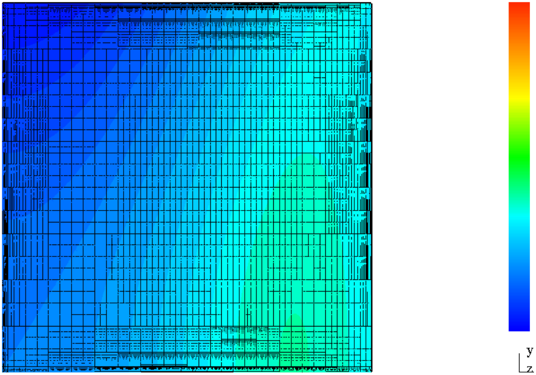
The functions exhibits strong boundary layers invisible in this scale.

Range: (-0.6,0.6)

Manufactured solution of Lin and Stynes, $\epsilon = 10^{-1}$

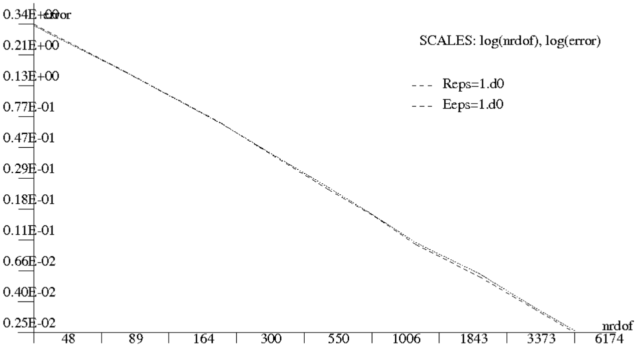


Zoom on the north boundary layer.



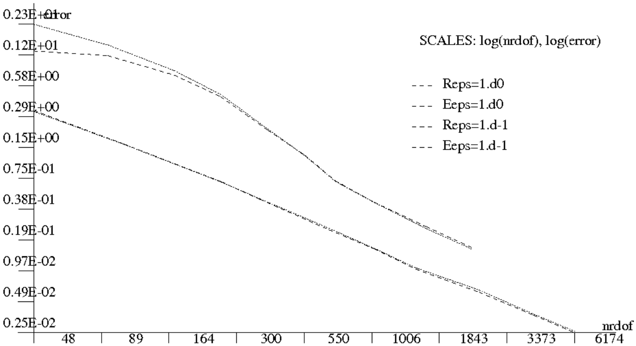
Optimal h-adaptive mesh and numerical solution for $\epsilon = 10^{-1}$

Lin/Stynes example, $\epsilon = 1$



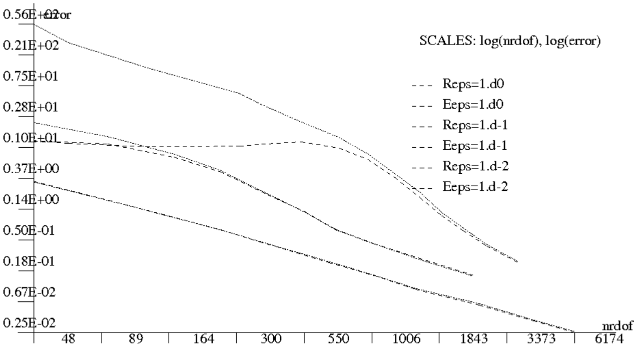
Residual and "balanced" error of u for h-adaptive solution, $p = 2$

Lin/Stynes example, $\epsilon = 10^0, 10^{-1}$



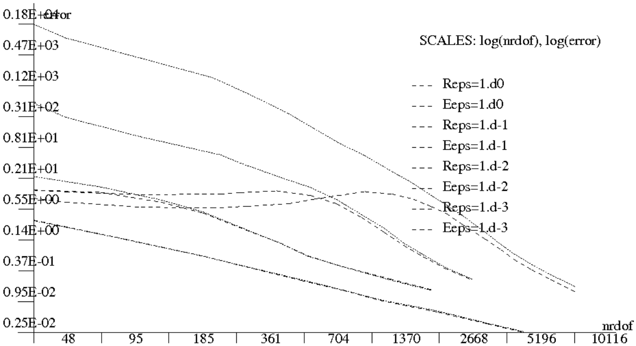
Residual and "balanced" error of u for h-adaptive solution, $p = 2$

Lin/Stynes example, $\epsilon = 10^0, 10^{-1}, 10^{-2}$.



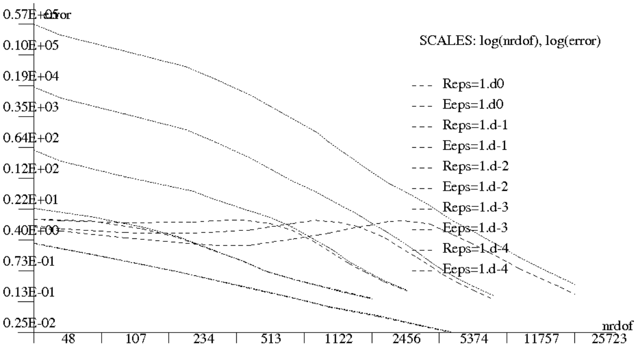
Residual and “balanced” error of u for h-adaptive solution, $p = 2$

Lin/Stynes example, $\epsilon = 10^0, 10^{-1}, 10^{-2}, 10^{-3}$.



Residual and “balanced” error of u for h-adaptive solution, $p = 2$

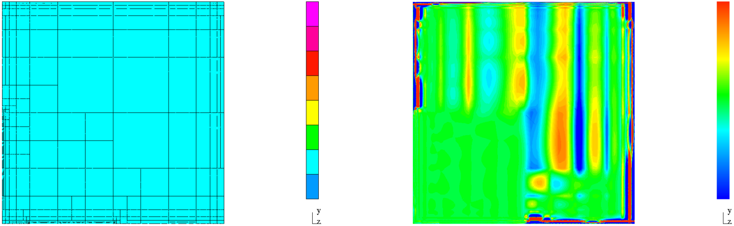
Lin/Stynes example, $\epsilon = 10^0, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$.



Residual and "balanced" error of u for h-adaptive solution, $p = 2$

Other tricks we can play: zooming on the solution

Question: Can we select the test norm in such a way that the DPG method would deliver high accuracy in a preselected subdomain, e.g. $(0, \frac{1}{2})^2 \subset (0,1)^2$?
Answer: Yes!



Optimal mesh and the corresponding pointwise error (range $(-0.001 - 0.001)$.

Punchline 3

DPG is easy to code.

Other Applications

- Wave propagation problems (sonars, full wave form inversion in geomechanics, cloaking)

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- Elasticity, shells (volumetric, shear, membrane locking)

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- Wave propagation problems (sonars, full wave form inversion in geomechanics, cloaking)
- Elasticity, shells (volumetric, shear, membrane locking)
- Metamaterials

Thank You !

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