Space-Time Navier-Stokes with Entropy Variables

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Consider the DPG Navier-Stokes derivation from previously with primitive variables:

$$\left(\frac{1}{\mu}\mathbb{D}, \mathbb{S}\right) + (2\boldsymbol{u}, \nabla\cdot\mathbb{S}) - \left(\frac{2}{3}\boldsymbol{u}, \nabla \operatorname{tr}\mathbb{S}\right) - \left\langle \frac{4}{3}\hat{\boldsymbol{u}}, \mathbb{S}\boldsymbol{n}_x \right\rangle = 0 \tag{1a}$$

$$\left(\frac{Pr}{C_p\mu}\boldsymbol{q}, \boldsymbol{\tau}\right) - (T, \nabla\cdot\boldsymbol{\tau}) + \left\langle \hat{T}, \tau_n \right\rangle = 0 \tag{1b}$$

$$-\left(\begin{pmatrix} \rho\boldsymbol{u} \\ \rho \end{pmatrix}, \nabla_{xt} v_c\right) + \left\langle \hat{t}_c, v_c \right\rangle = (f_c, v_c) \tag{1c}$$

$$-\left(\begin{pmatrix} \rho\boldsymbol{u}\otimes\boldsymbol{u} + \rho RT\boldsymbol{I} - \mathbb{D} \\ \rho\boldsymbol{u} \end{pmatrix}, \nabla_{xt}\boldsymbol{v}_m\right) + \left\langle \hat{\boldsymbol{t}}_m, \boldsymbol{v}_m \right\rangle = (\boldsymbol{f}_m, \boldsymbol{v}_m) \tag{1d}$$

$$-\left(\begin{pmatrix} \rho\boldsymbol{u}\left(C_vT + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) + \boldsymbol{u}\rho RT + \boldsymbol{q} - \boldsymbol{u}\cdot\mathbb{D} \\ \rho\left(C_vT + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) \end{pmatrix}, \nabla_{xt} v_e\right) + \left\langle \hat{t}_e, v_e \right\rangle = (f_e, v_e)\,, \tag{1e}$$

where

$$\begin{aligned}
\hat{\boldsymbol{u}} &= \operatorname{tr}(\boldsymbol{u}) \\
\hat{T} &= \operatorname{tr}(T) \\
\hat{t}_c &= \operatorname{tr}(\rho\boldsymbol{u})\cdot\boldsymbol{n}_x + \operatorname{tr}(\rho)\, n_t \\
\hat{\boldsymbol{t}}_m &= \operatorname{tr}(\rho\boldsymbol{u}\otimes\boldsymbol{u} + \rho RT\boldsymbol{I} - \mathbb{D})\cdot\boldsymbol{n}_x + \operatorname{tr}(\rho\boldsymbol{u})\, n_t \\
\hat{t}_e &= \operatorname{tr}\left(\rho\boldsymbol{u}\left(C_vT + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) + \boldsymbol{u}\rho RT + \boldsymbol{q} - \boldsymbol{u}\cdot\mathbb{D}\right)\cdot\boldsymbol{n}_x \\
&\quad + \operatorname{tr}\left(\rho\left(C_vT + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\right) n_t\,.
\end{aligned}$$

Now define primitive fluxes for continuity, momentum, and energy equations:

$$\begin{aligned}
\boldsymbol{F}_c^p &:= \rho\boldsymbol{u} \\
\mathbb{F}_m^p &:= \rho\boldsymbol{u}\otimes\boldsymbol{u} + \rho RT\boldsymbol{I} \\
\boldsymbol{F}_e^p &:= \rho\boldsymbol{u}\left(C_vT + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) + \boldsymbol{u}\rho RT
\end{aligned}$$

Our bilinear form is then simplified:

$$\left(\frac{1}{\mu}\mathbb{D},\mathbb{S}\right)+(2\boldsymbol{u},\nabla\cdot\mathbb{S})-\left(\frac{2}{3}\boldsymbol{u},\nabla\operatorname{tr}\mathbb{S}\right)-\left\langle\frac{4}{3}\hat{\boldsymbol{u}},\mathbb{S}\boldsymbol{n}_x\right\rangle=0 \tag{2a}$$

$$\left(\frac{Pr}{C_p\mu}\boldsymbol{q},\boldsymbol{\tau}\right)-(T,\nabla\cdot\boldsymbol{\tau})+\left\langle\hat{T},\tau_n\right\rangle=0 \tag{2b}$$

$$-\left(\begin{pmatrix}\boldsymbol{F}_c^p\\ \rho\end{pmatrix},\nabla_{xt}v_c\right)+\langle\hat{t}_c,v_c\rangle=(f_c,v_c) \tag{2c}$$

$$-\left(\begin{pmatrix}\mathbb{F}_m^p-\mathbb{D}\\ \rho\boldsymbol{u}\end{pmatrix},\nabla_{xt}\boldsymbol{v}_m\right)+\langle\hat{\boldsymbol{t}}_m,\boldsymbol{v}_m\rangle=(\boldsymbol{f}_m,\boldsymbol{v}_m) \tag{2d}$$

$$-\left(\begin{pmatrix}\boldsymbol{F}_e^p+\boldsymbol{q}-\boldsymbol{u}\cdot\mathbb{D}\\ \rho\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\end{pmatrix},\nabla_{xt}v_e\right)+\langle\hat{t}_e,v_e\rangle=(f_e,v_e)\ , \tag{2e}$$

Now we wish to do a change of variables to conservation variables:

$$\begin{aligned}\rho&=\rho\\ \boldsymbol{m}&=\rho\boldsymbol{u}\\ E&=\rho\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\end{aligned}$$

We can define new fluxes in conservation variables:

$$\begin{aligned}\boldsymbol{F}_c^c&=\boldsymbol{m}\\ \mathbb{F}_m^c&=\frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\rho}+(\gamma-1)\left(E-\frac{\boldsymbol{m}\cdot\boldsymbol{m}}{2\rho}\right)\boldsymbol{I}\\ \boldsymbol{F}_e^c&=\frac{\boldsymbol{m}}{\rho}E+(\gamma-1)\left(E-\frac{\boldsymbol{m}\cdot\boldsymbol{m}}{2\rho}\right)\frac{\boldsymbol{m}}{\rho}\end{aligned}$$

and our new bilinear form is

$$\left(\frac{1}{\mu}\mathbb{D},\mathbb{S}\right)+\left(2\frac{\boldsymbol{m}}{\rho},\nabla\cdot\mathbb{S}\right)-\left(\frac{2}{3}\frac{\boldsymbol{m}}{\rho},\nabla\operatorname{tr}\mathbb{S}\right)-\left\langle\frac{4}{3}\hat{\boldsymbol{u}},\mathbb{S}\boldsymbol{n}_x\right\rangle=0 \tag{3a}$$

$$\left(\frac{Pr}{C_p\mu}\boldsymbol{q},\boldsymbol{\tau}\right)-\left(\frac{E-\frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m}}{C_v\rho},\nabla\cdot\boldsymbol{\tau}\right)+\left\langle\hat{T},\tau_n\right\rangle=0 \tag{3b}$$

$$-\left(\begin{pmatrix}\boldsymbol{F}_c^c\\ \rho\end{pmatrix},\nabla_{xt}v_c\right)+\langle\hat{t}_c,v_c\rangle=(f_c,v_c) \tag{3c}$$

$$-\left(\begin{pmatrix}\mathbb{F}_m^c-\mathbb{D}\\ \boldsymbol{m}\end{pmatrix},\nabla_{xt}\boldsymbol{v}_m\right)+\langle\hat{\boldsymbol{t}}_m,\boldsymbol{v}_m\rangle=(\boldsymbol{f}_m,\boldsymbol{v}_m) \tag{3d}$$

$$-\left(\begin{pmatrix}\boldsymbol{F}_e^c+\boldsymbol{q}-\frac{\boldsymbol{m}}{\rho}\cdot\mathbb{D}\\ E\end{pmatrix},\nabla_{xt}v_e\right)+\langle\hat{t}_e,v_e\rangle=(f_e,v_e)\ , \tag{3e}$$

Now we wish to do a change of variables to entropy variables:

$$
\begin{aligned}
V_c &= \frac{-E + (E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m})\left(\gamma + 1 - \ln\left[\frac{(\gamma-1)(E-\frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m})}{\rho^{\gamma}}\right]\right)}{E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m}} \\
\boldsymbol{V}_m &= \frac{\boldsymbol{m}}{E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m}} \\
V_e &= \frac{-\rho}{E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m}}
\end{aligned}
$$

with reverse mapping:

$$
\begin{aligned}
\rho &= -\left(E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m}\right)V_e \\
\boldsymbol{m} &= \left(E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m}\right)\boldsymbol{V}_m \\
E &= \left(E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m}\right)\left(1 - \frac{1}{2V_e}\boldsymbol{V}_m\cdot\boldsymbol{V}_m\right)
\end{aligned}
$$