DPG for Engineers

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Abstract

We discuss and explain the discontinuous Petrov-Galerkin finite element method with an engineering audience in mind.

1 Motivation and Introductory Concepts

The discontinuous Petrov-Galerkin finite element method is a promising new framework for automated computing of a broad class of partial differential equations.

1.1 A Framework for Automated Computing

1.2 Mathematical Fundamentals

1.2.1 Banach, Hilbert, and Sobolev Spaces

Consider a bounded domain, $\Omega \in \mathbb{R}^d$ and a vector space of functions defined on Ω. A Banach space is a complete normed vector space. An example of which is $L_1(\Omega)$ which is the set of all functions defined on Ω such that the L_1 norm $\|u\|_{L_1(\Omega)} := \int_\Omega |u|$ is finite. A Hilbert space is a Banach space where the norm is defined through an inner product. An example of which is $L_2(\Omega)$ for which the norm is $\|u\|_{L_2(\Omega)} := (u,u)^{\frac{1}{2}}_{L_2(\Omega)} := \left(\int_\Omega u^2\right)^{\frac{1}{2}}$. Finally, a Sobolev space is a Hilbert space where the inner product contains derivatives of the function. A common Sobolev space is $H^1(\Omega)$ where $\|u\|^2_{H^1(\Omega)} := \|u\|^2_{L_2(\Omega)} + \|\nabla u\|^2_{L_2(\Omega)}$.

1.2.2 Variational Formulations, Dual Spaces, and Riesz Maps

Consider a Banach space U. The dual space, U' is the set of all linear functionals on U such that the pairing $\langle u, w\rangle_{U\times U'} \in \mathbb{R}$ for all $u \in U$ and $w \in U'$. Any PDE problem can abstractly be written as: find solution $u \in U$ such that

$$Bu = l$$

where $B : U \to V'$ maps functions to some dual space and $l \in U'$. By the calculus of variations, this is equivalent to the problem: find solution $u \in U$ such that

$$\langle Bu, v\rangle = \langle l, v\rangle$$

for all functions $v \in V$ and v is called a test function. This defines a bilinear form $b(u, v) = f(v)$ where u is often called the trial function. Now let U and V be Hilbert rather than merely Banach. The Riesz representation theorem defines a bijective, isometric mapping from every member in V to a corresponding member in the dual space V'. We denote this mapping $R_V : V \to V'$. Thus, we can take our operator equation representing the strong form of our PDE

$$Bu - l = 0 \quad \in V'$$

and map it back to V via the inverse Riesz map:

$$R_V^{-1}(Bu - l) = 0 \quad \in V.$$

1.3 Ritz Methods and Least Squares

1.4 SUPG – Or Stability Through the Appropriate Test Space

2 Development of a DPG Method for the Poisson Equation

3 DPG for Well Posed PDEs