Ultra-Weak DPG for Compressible Navier-Stokes

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January 11, 2016

Nonlinear Forms

Primitive Variables

Consider the DPG Navier-Stokes derivation from previously with primitive variables:

$$\left(\frac{1}{\mu}\mathbb{D},\mathbb{S}\right)+(\boldsymbol{u},\nabla\cdot\mathbb{S})-\langle\hat{\boldsymbol{u}},\mathbb{S}\boldsymbol{n}_x\rangle=0 \tag{1a}$$

$$\left(\frac{Pr}{C_p\mu}\boldsymbol{q},\boldsymbol{\tau}\right)-(T,\nabla\cdot\boldsymbol{\tau})+\left\langle\hat{T},\tau_n\right\rangle=0 \tag{1b}$$

$$-\left(\begin{pmatrix}\rho\boldsymbol{u}\\ \rho\end{pmatrix},\nabla_{xt}v_c\right)+\left\langle\hat{t}_c,v_c\right\rangle=(f_c,v_c) \tag{1c}$$

$$-\left(\begin{pmatrix}\rho\boldsymbol{u}\otimes\boldsymbol{u}+\rho RT\boldsymbol{I}-\left(\mathbb{D}+\mathbb{D}^T-\frac{2}{3}\operatorname{tr}(\mathbb{D})\boldsymbol{I}\right)\\ \rho\boldsymbol{u}\end{pmatrix},\nabla_{xt}\boldsymbol{v}_m\right)+\left\langle\hat{\boldsymbol{t}}_m,\boldsymbol{v}_m\right\rangle=(\boldsymbol{f}_m,\boldsymbol{v}_m) \tag{1d}$$

$$-\left(\begin{pmatrix}\rho\boldsymbol{u}\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)+\boldsymbol{u}\rho RT+\boldsymbol{q}-\boldsymbol{u}\cdot\left(\mathbb{D}+\mathbb{D}^T-\frac{2}{3}\operatorname{tr}(\mathbb{D})\boldsymbol{I}\right)\\ \rho\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\end{pmatrix},\nabla_{xt}v_e\right)+\left\langle\hat{t}_e,v_e\right\rangle=(f_e,v_e)\,, \tag{1e}$$

where

$$\begin{aligned}
\hat{\boldsymbol{u}}&=\operatorname{tr}(\boldsymbol{u})\\
\hat{T}&=\operatorname{tr}(T)\\
\hat{t}_c&=\operatorname{tr}(\rho\boldsymbol{u})\cdot\boldsymbol{n}_x+\operatorname{tr}(\rho)\,n_t\\
\hat{\boldsymbol{t}}_m&=\operatorname{tr}\left(\rho\boldsymbol{u}\otimes\boldsymbol{u}+\rho RT\boldsymbol{I}-\left(\mathbb{D}+\mathbb{D}^T-\frac{2}{3}\operatorname{tr}(\mathbb{D})\boldsymbol{I}\right)\right)\cdot\boldsymbol{n}_x+\operatorname{tr}(\rho\boldsymbol{u})\,n_t\\
\hat{t}_e&=\operatorname{tr}\left(\rho\boldsymbol{u}\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)+\boldsymbol{u}\rho RT+\boldsymbol{q}-\boldsymbol{u}\cdot\left(\mathbb{D}+\mathbb{D}^T-\frac{2}{3}\operatorname{tr}(\mathbb{D})\boldsymbol{I}\right)\right)\cdot\boldsymbol{n}_x\\
&\quad+\operatorname{tr}\left(\rho\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\right)n_t\,.
\end{aligned}$$

Now define primitive fluxes for continuity, momentum, and energy equations:

$$\begin{aligned}
\boldsymbol{F}_c^p&:=\rho\boldsymbol{u}\\
\mathbb{F}_m^p&:=\rho\boldsymbol{u}\otimes\boldsymbol{u}+\rho RT\boldsymbol{I}\\
\boldsymbol{F}_e^p&:=\rho\boldsymbol{u}\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)+\boldsymbol{u}\rho RT
\end{aligned}$$

Our bilinear form is then simplified:

$$\left(\frac{1}{\mu}\mathbb{D},\mathbb{S}\right)+(\boldsymbol{u},\nabla\cdot\mathbb{S})-\langle\hat{\boldsymbol{u}},\mathbb{S}\boldsymbol{n}_x\rangle=0 \tag{2a}$$

$$\left(\frac{Pr}{C_p\mu}\boldsymbol{q},\boldsymbol{\tau}\right)-(T,\nabla\cdot\boldsymbol{\tau})+\left\langle\hat{T},\tau_n\right\rangle=0 \tag{2b}$$

$$-\left(\begin{pmatrix}\boldsymbol{F}_c^p\\ \rho\end{pmatrix},\nabla_{xt}v_c\right)+\langle\hat{t}_c,v_c\rangle=(f_c,v_c) \tag{2c}$$

$$-\left(\begin{pmatrix}\mathbb{F}_m^p-\left(\mathbb{D}+\mathbb{D}^T-\frac{2}{3}\operatorname{tr}(\mathbb{D})\boldsymbol{I}\right)\\ \rho\boldsymbol{u}\end{pmatrix},\nabla_{xt}\boldsymbol{v}_m\right)+\langle\hat{\boldsymbol{t}}_m,\boldsymbol{v}_m\rangle=(\boldsymbol{f}_m,\boldsymbol{v}_m) \tag{2d}$$

$$-\left(\begin{pmatrix}\boldsymbol{F}_e^p+\boldsymbol{q}-\boldsymbol{u}\cdot\left(\mathbb{D}+\mathbb{D}^T-\frac{2}{3}\operatorname{tr}(\mathbb{D})\boldsymbol{I}\right)\\ \rho\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\end{pmatrix},\nabla_{xt}v_e\right)+\langle\hat{t}_e,v_e\rangle=(f_e,v_e)\ , \tag{2e}$$

Linearization

We define the set of linear variables: $L:=\left\{\hat{\boldsymbol{u}},\hat{T},\hat{t}_c,\hat{\boldsymbol{t}}_m,\hat{t}_e\right\}$. Let U be the set of variables involved in nonlinear interactions. We apply a linearization $U\approx\tilde{U}+\Delta U$ and solve

$$R_{,U}(\tilde{U})\Delta U+R(L)=-R(\tilde{U})\,,$$

where

$$\begin{aligned}R(L)=&-\langle\hat{\boldsymbol{u}},\mathbb{S}\boldsymbol{n}_x\rangle+\left\langle\hat{T},\tau_n\right\rangle+\langle\hat{t}_c,v_c\rangle+\langle\hat{\boldsymbol{t}}_m,v_m\rangle+\langle\hat{t}_e,v_e\rangle\\&-(f_c,v_c)-(\boldsymbol{f}_m,\boldsymbol{v}_m)-(f_e,v_e)\end{aligned}$$

The set of nonlinear variables is $U^p:=\{\rho,\boldsymbol{u},T,\mathbb{D},\boldsymbol{q}\}$. Then $R_{,U^p}(\tilde{U}^p)\Delta U^p$ is

$$\begin{aligned}&\left(\frac{1}{\mu}\Delta\mathbb{D},\mathbb{S}\right)+(\Delta\boldsymbol{u},\nabla\cdot\mathbb{S})\\&+\left(\frac{Pr}{C_p\mu}\Delta\boldsymbol{q},\boldsymbol{\tau}\right)-(\Delta T,\nabla\cdot\boldsymbol{\tau})\\&-\left(\begin{pmatrix}\boldsymbol{F}_{c,U^p}^p\Delta U^p\\ \Delta\rho\end{pmatrix},\nabla_{xt}v_c\right)\\&-\left(\begin{pmatrix}\mathbb{F}_{m,U^p}^p\Delta U^p-\left(\Delta\mathbb{D}+(\Delta\mathbb{D})^T-\frac{2}{3}\operatorname{tr}(\Delta\mathbb{D})\boldsymbol{I}\right)\\ \Delta\rho\tilde{\boldsymbol{u}}+\tilde{\rho}\Delta\boldsymbol{u}\end{pmatrix},\nabla_{xt}\boldsymbol{v}_m\right)\\&-\left(\begin{pmatrix}\boldsymbol{F}_{e,U^p}^p\Delta U^p+\boldsymbol{q}-\Delta\boldsymbol{u}\cdot\left(\tilde{\mathbb{D}}+(\tilde{\mathbb{D}})^T-\frac{2}{3}\operatorname{tr}(\tilde{\mathbb{D}})\boldsymbol{I}\right)-\tilde{\boldsymbol{u}}\cdot\left(\Delta\mathbb{D}+(\Delta\mathbb{D})^T-\frac{2}{3}\operatorname{tr}(\Delta\mathbb{D})\boldsymbol{I}\right)\\ C_v\Delta\rho\tilde{T}+C_v\tilde{\rho}\Delta T+\frac{1}{2}\left(\Delta\rho\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}}+\tilde{\rho}\Delta\boldsymbol{u}\cdot\tilde{\boldsymbol{u}}+\tilde{\rho}\tilde{\boldsymbol{u}}\cdot\Delta\boldsymbol{u}\right)\end{pmatrix},\nabla_{xt}v_e\right)\end{aligned}$$

where

$$\begin{aligned}\boldsymbol{F}_{c,U^p}^p\Delta U^p&:=\Delta\rho\tilde{\boldsymbol{u}}+\tilde{\rho}\Delta\boldsymbol{u}\\ \mathbb{F}_{m,U^p}^p&:=\Delta\rho\tilde{\boldsymbol{u}}\otimes\tilde{\boldsymbol{u}}+\tilde{\rho}\Delta\boldsymbol{u}\otimes\tilde{\boldsymbol{u}}+\tilde{\rho}\tilde{\boldsymbol{u}}\otimes\Delta\boldsymbol{u}+R\left(\Delta\rho\tilde{T}+\tilde{\rho}\Delta T\right)\boldsymbol{I}\\ \boldsymbol{F}_{e,U^p}^p&:=C_v\Delta\rho\tilde{\boldsymbol{u}}\tilde{T}+C_v\tilde{\rho}\Delta\boldsymbol{u}\tilde{T}+C_v\tilde{\rho}\tilde{\boldsymbol{u}}\Delta T\\&\quad+\frac{1}{2}\Delta\rho\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}}+\frac{1}{2}\tilde{\rho}\Delta\boldsymbol{u}\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}}+\frac{1}{2}\tilde{\rho}\tilde{\boldsymbol{u}}\Delta\boldsymbol{u}\cdot\tilde{\boldsymbol{u}}+\frac{1}{2}\tilde{\rho}\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}\cdot\Delta\boldsymbol{u}\\&\quad+R\Delta\boldsymbol{u}\tilde{\rho}\tilde{T}+R\tilde{\boldsymbol{u}}\Delta\rho\tilde{T}+R\tilde{\boldsymbol{u}}\tilde{\rho}\Delta T\end{aligned}$$

and $R(\tilde{U}^p)$ is

$$\begin{aligned}
&\left(\frac{1}{\mu}\tilde{\mathbb{D}},\mathbb{S}\right)+(\tilde{\boldsymbol{u}},\nabla\cdot\mathbb{S})\\
&+\left(\frac{Pr}{C_p\mu}\tilde{\boldsymbol{q}},\boldsymbol{\tau}\right)-\left(\tilde{T},\nabla\cdot\boldsymbol{\tau}\right)\\
&-\left(\left(\begin{array}{c}\boldsymbol{F}_c^p(\tilde{U}^p)\\ \tilde{\rho}\end{array}\right),\nabla_{xt}v_c\right)\\
&-\left(\left(\begin{array}{c}\mathbb{F}_m^p(\tilde{U}^p)-\left(\tilde{\mathbb{D}}+(\tilde{\mathbb{D}})^T-\frac{2}{3}\operatorname{tr}(\tilde{\mathbb{D}})\boldsymbol{I}\right)\\ \tilde{\rho}\tilde{\boldsymbol{u}}\end{array}\right),\nabla_{xt}\boldsymbol{v}_m\right)\\
&-\left(\left(\begin{array}{c}\boldsymbol{F}_e^p(\tilde{U}^p)-\tilde{\boldsymbol{u}}\cdot\left(\tilde{\mathbb{D}}+(\tilde{\mathbb{D}})^T-\frac{2}{3}\operatorname{tr}(\tilde{\mathbb{D}})\boldsymbol{I}\right)\\ \tilde{\rho}\left(C_v\tilde{T}+\frac{1}{2}\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}}\right)\end{array}\right),\nabla_{xt}v_e\right)
\end{aligned}$$

Primitive Variables

For the sake of simplifying notation, we drop the Δ notation from before. Any values from the previous solution are denoted with a $\tilde{}$ notation while current values lack this. In the primitive variable formulation, $\Sigma=\{\mathbb{D},\boldsymbol{q}\}$, $U=\{\rho,u_x,u_y,T\}$, $\Psi=\{\mathbb{S},\boldsymbol{\tau}\}$, and $V=\{v_c,v_x,v_y,v_e\}$. We have the following definitions:

$$M^*\Psi+K^*\nabla V=\left(\begin{array}{c}M_{\mathbb{D}}^*\mathbb{S}\\ M_{\boldsymbol{q}}^*\boldsymbol{\tau}\end{array}\right)+\left(\begin{array}{c}K_{\mathbb{D}}^*\nabla V\\ K_{\boldsymbol{q}}^*\nabla V\end{array}\right)$$

$$-\left(\begin{array}{c}\mathcal{F}^*\\ C^*\end{array}\right)\cdot\nabla_{xt}V+G^*\nabla\Psi=-\left(\begin{array}{c}\boldsymbol{F}_c^*\cdot\nabla V+\boldsymbol{C}_c^*\cdot V_{,t}\\ \boldsymbol{F}_m^*\cdot\nabla V+\boldsymbol{C}_m^*\cdot V_{,t}\\ \boldsymbol{F}_e^*\cdot\nabla V+\boldsymbol{C}_e^*\cdot V_{,t}\end{array}\right)+\left(\begin{array}{c}\boldsymbol{G}_c^*\nabla\Psi\\ \boldsymbol{G}_m^*\nabla\Psi\\ \boldsymbol{G}_e^*\nabla\Psi\end{array}\right)$$

$$
\begin{aligned}
M_{\mathbb{D}}^{*}\mathbb{S} &= \frac{1}{\mu}\mathbb{S} \\
M_{\boldsymbol{q}}^{*}\boldsymbol{\tau} &= \frac{Pr}{C_p\mu}\boldsymbol{\tau} \\
K_{\mathbb{D}}^{*}\nabla V &= \nabla\boldsymbol{v}_m + (\nabla\boldsymbol{v}_m)^T - \frac{2}{3}\nabla\cdot\boldsymbol{v}_m\boldsymbol{I} + \tilde{\boldsymbol{u}}\otimes\nabla v_e + (\tilde{\boldsymbol{u}}\otimes\nabla v_e)^T - \frac{2}{3}\tilde{\boldsymbol{u}}\cdot\nabla v_e\boldsymbol{I} \\
K_{\boldsymbol{q}}^{*}\nabla V &= -\nabla v_e \\
\boldsymbol{F}_c^{*}\cdot\nabla V &= \tilde{\boldsymbol{u}}\cdot\nabla v_c + \tilde{\boldsymbol{u}}\otimes\tilde{\boldsymbol{u}} : \nabla\boldsymbol{v}_m + R\tilde{T}\nabla\cdot\boldsymbol{v}_m + C_v\tilde{T}\tilde{\boldsymbol{u}}\cdot\nabla v_e + \frac{1}{2}\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}\cdot\nabla v_e + R\tilde{T}\tilde{\boldsymbol{u}}\cdot\nabla v_e \\
\boldsymbol{C}_c^{*}\cdot V_{,t} &= v_{c,t} + \tilde{\boldsymbol{u}}\cdot\boldsymbol{v}_{m,t} + (C_v\tilde{T} + \frac{1}{2}\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}})v_{e,t} \\
\boldsymbol{F}_m\cdot\nabla\boldsymbol{v}_m &= \tilde{\rho}\nabla v_c + (\nabla\boldsymbol{v}_m + (\nabla\boldsymbol{v}_m)^T)\tilde{\rho}\tilde{\boldsymbol{u}} + C_v\tilde{T}\tilde{\rho}\nabla v_e + \frac{1}{2}\tilde{\rho}\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}}\nabla v_e + \tilde{\rho}\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}\cdot\nabla v_e + R\tilde{T}\tilde{\rho}\nabla v_e \\
&\quad - \tilde{\mathbb{D}}\nabla v_e - (\tilde{\mathbb{D}})^T\nabla v_e + \frac{2}{3}\operatorname{tr}(\tilde{\mathbb{D}})\nabla v_e \\
\boldsymbol{C}_m^{*}\cdot V_{,t} &= \tilde{\rho}\boldsymbol{v}_{m,t} + \tilde{\rho}\tilde{\boldsymbol{u}}v_{e,t} \\
\boldsymbol{F}_e^{*}\cdot\nabla V &= R\tilde{\rho}\nabla\cdot\boldsymbol{v}_m + C_v\tilde{\rho}\tilde{\boldsymbol{u}}\cdot\nabla v_e + R\tilde{\rho}\tilde{\boldsymbol{u}}\cdot\nabla v_e \\
\boldsymbol{C}_e^{*}\cdot V_{,t} &= C_v\tilde{\rho}v_{e,t} \\
\boldsymbol{G}_c^{*}\nabla\Psi &= 0 \\
\boldsymbol{G}_m^{*}\nabla\Psi &= \nabla\cdot\mathbb{S} \\
\boldsymbol{G}_e^{*}\nabla\Psi &= -\nabla\cdot\boldsymbol{\tau}
\end{aligned}
$$