

Space-Time DPG: Designing a Method for Parallel CFD

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DPG Summary

Overview of Features

- Robust for singularly perturbed problems
- Stable in the preasymptotic regime
- Designed for adaptive mesh refinement
- Ideal for high-performance computing

DPG is a minimum residual method:

$$u_h = \arg \min_{w_h \in U_h} \frac{1}{2} \|Bw_h - l\|_{V'}^2$$
$$\Updownarrow$$

$$b(u_h, R_V^{-1} B \delta u_h) = l(R_V^{-1} B \delta u_h) \quad \forall \delta u_h \in U_h$$

where $v_{\delta u_h} := R_V^{-1} B \delta u_h$ are the **optimal test functions**.

Heat Equation

Simplest Nontrivial Space-Time Problem

Equation is elliptic in space, but hyperbolic in time (parabolic in space-time).

$$\frac{\partial u}{\partial t} - \epsilon \Delta u = f$$

This is really just a composite of Fourier's law and conservation of energy.

$$\sigma - \epsilon \nabla u = 0$$

$$\frac{\partial u}{\partial t} - \nabla \cdot \sigma = f$$

We can rewrite this in terms of a space-time divergence.

$$\begin{aligned} \frac{1}{\epsilon} \sigma - \nabla u &= 0 \\ \nabla_{xt} \cdot \begin{pmatrix} -\sigma \\ u \end{pmatrix} &= f \end{aligned}$$

Heat Equation

DPG Formulation

Multiply by test function and integrate by parts over space-time element K.

$$\begin{aligned} \left(\frac{1}{\epsilon} \boldsymbol{\sigma}, \boldsymbol{\tau} \right) + (u, \nabla \cdot \boldsymbol{\tau}) - \langle \hat{u}, \boldsymbol{\tau} \cdot \mathbf{n}_x \rangle &= 0 \\ - \left(\begin{pmatrix} -\boldsymbol{\sigma} \\ u \end{pmatrix}, \nabla_{xt} v \right) + \langle \hat{t}, v \rangle &= f \end{aligned}$$

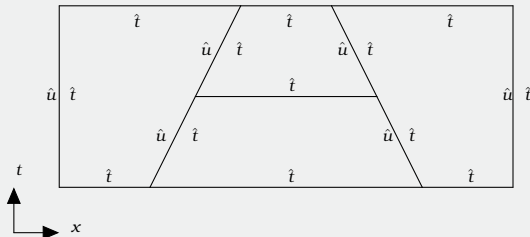
where

$$\hat{u} := \text{tr}(u)$$

$$\hat{t} := \text{tr}(-\boldsymbol{\sigma}) \cdot \mathbf{n}_x + \text{tr}(u) \cdot n_t$$

- Trace \hat{u} defined on spatial boundaries
- Flux \hat{t} defined on all boundaries

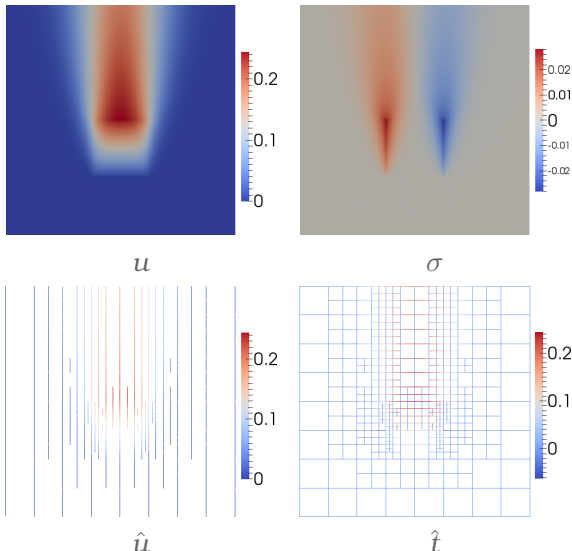
Support of Trace Variables



Heat equation

Pulsed Source Problem

- Initial condition $u = 0$.
- Apply unit source $x \in [3/8, 5/8]$, $t \in [1/4, 1/2]$.
- Should see no temporal diffusion.
- Space-time adaptivity picks up areas of rapid change.



Strong Form

The compressible Navier-Stokes equations are

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ \rho \mathbf{u} \\ \rho e_0 \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho \mathbf{u} \\ \rho \mathbf{u} \otimes \mathbf{u} + p \mathbf{I} - \mathbb{D} \\ \rho \mathbf{u} e_0 + \mathbf{u} p + \mathbf{q} - \mathbf{u} \cdot \mathbb{D} \end{bmatrix} = \begin{bmatrix} f_c \\ \mathbf{f}_m \\ f_e \end{bmatrix},$$

where

$$\mathbb{D} = 2\mu \mathbf{S}^* = 2\mu \left[\frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) - \frac{1}{3} \nabla \cdot \mathbf{u} \mathbf{I} \right],$$

$$\mathbf{q} = -C_p \frac{\mu}{Pr} \nabla T,$$

and (assuming an ideal gas EOS)

$$p = \rho RT.$$

First Order Space-Time Form

Writing this in space-time in terms of ρ , \mathbf{u} , T , \mathbb{D} , and \mathbf{q} :

$$\mathbb{D} - \mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + \frac{2\mu}{3} \nabla \cdot \mathbf{u} \mathbf{I} = 0$$

$$\mathbf{q} + C_p \frac{\mu}{Pr} \nabla T = 0$$

$$\nabla_{xt} \cdot \begin{pmatrix} \rho \mathbf{u} \\ \rho \end{pmatrix} = f_c$$

$$\nabla_{xt} \cdot \begin{pmatrix} \rho \mathbf{u} \otimes \mathbf{u} + \rho RT \mathbf{I} - \mathbb{D} \\ \rho \mathbf{u} \end{pmatrix} = \mathbf{f}_m$$

$$\nabla_{xt} \cdot \begin{pmatrix} \rho \mathbf{u} (C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u}) + \mathbf{u} \rho RT + \mathbf{q} - \mathbf{u} \cdot \mathbb{D} \\ \rho (C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u}) \end{pmatrix} = f_e.$$

DPG Formulation

Multiplying by test functions and integrating by parts:

$$\begin{aligned}
 (\mathbb{D}, \mathbb{S}) + (2\mu \mathbf{u}, \nabla \cdot \mathbb{S}) - \left(\frac{2\mu}{3} \mathbf{u}, \nabla \operatorname{tr} \mathbb{S} \right) - \langle 2\mu \hat{\mathbf{u}}, \mathbb{S} \mathbf{n}_x \rangle + \left\langle \frac{2\mu}{3} \hat{\mathbf{u}}, \mathbb{S} \mathbf{n}_x \right\rangle &= 0 \\
 (\mathbf{q}, \boldsymbol{\tau}) - \left(C_p \frac{\mu}{Pr} T, \nabla \cdot \boldsymbol{\tau} \right) + \left\langle C_p \frac{\mu}{Pr} \hat{T}, \boldsymbol{\tau}_n \right\rangle &= 0 \\
 - \left(\left(\begin{array}{c} \rho \mathbf{u} \\ \rho \end{array} \right), \nabla_{xt} v_c \right) + \langle \hat{t}_c, v_c \rangle &= (f_c, v_c) \\
 - \left(\left(\begin{array}{c} \rho \mathbf{u} \otimes \mathbf{u} + \rho RT \mathbf{I} - \mathbb{D} \\ \rho \mathbf{u} \end{array} \right), \nabla_{xt} \mathbf{v}_m \right) + \langle \hat{\mathbf{t}}_m, \mathbf{v}_m \rangle &= (\mathbf{f}_m, \mathbf{v}_m) \\
 - \left(\left(\begin{array}{c} \rho \mathbf{u} (C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u}) + \mathbf{u} \rho RT + \mathbf{q} - \mathbf{u} \cdot \mathbb{D} \\ \rho (C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u}) \end{array} \right), \nabla_{xt} v_e \right) + \langle \hat{t}_e, v_e \rangle &= (f_e, v_e),
 \end{aligned}$$

where $\hat{\mathbf{u}}$ and \hat{T} are spatial traces and \hat{t}_c , $\hat{\mathbf{t}}_m$, and \hat{t}_e are fluxes.

Spatial traces and fluxes are defined as follows:

$$\hat{\mathbf{u}} = \text{tr}(\mathbf{u})$$

$$\hat{T} = \text{tr}(T)$$

$$\hat{t}_c = \text{tr}(\rho \mathbf{u}) \cdot \mathbf{n}_x + \text{tr}(\rho) n_t$$

$$\hat{t}_m = \text{tr}(\rho \mathbf{u} \otimes \mathbf{u} + \rho RT \mathbf{I} - \mathbb{D}) \cdot \mathbf{n}_x + \text{tr}(\rho \mathbf{u}) n_t$$

$$\begin{aligned} \hat{t}_e = & \text{tr} \left(\rho \mathbf{u} \left(C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right) + \mathbf{u} \rho RT + \mathbf{q} - \mathbf{u} \cdot \mathbb{D} \right) \cdot \mathbf{n}_x \\ & + \text{tr} \left(\rho \left(C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right) \right) n_t. \end{aligned}$$

Linearization

Fluxes, traces, and \mathbf{q} are linear in the above bilinear form, but we need to linearize in ρ , \mathbf{u} , T , and \mathbb{D} (Jacobian and residual not shown here).

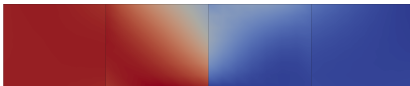
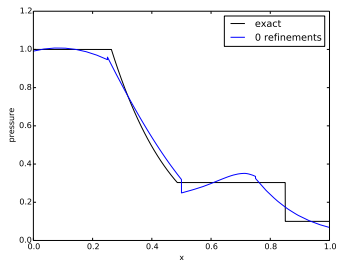
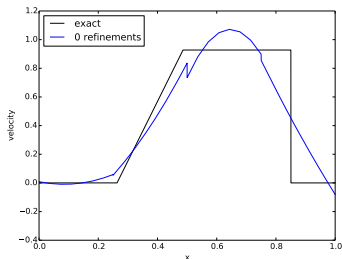
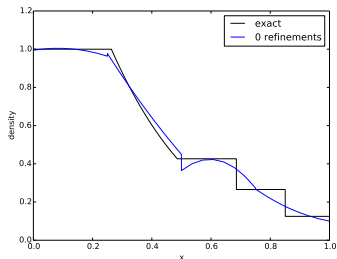
Test Norm

The test norm is

$$\begin{aligned}
 & \|\nabla \mathbf{v}_m + \nabla v_e \otimes \tilde{\mathbf{u}}\|^2 + \|\nabla v_e\|^2 \\
 & + \left\| -\tilde{\mathbf{u}} \cdot \nabla v_c - \frac{\partial v_c}{\partial t} - \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} : \nabla \mathbf{v}_m - R\tilde{T}\nabla \cdot \mathbf{v}_m - \tilde{\mathbf{u}} \cdot \frac{\partial \mathbf{v}_m}{\partial t} \right. \\
 & \quad \left. - C_v \tilde{T} \tilde{\mathbf{u}} \cdot \nabla v_e - \frac{1}{2} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} \tilde{\mathbf{u}} \cdot \nabla v_e - R\tilde{T} \tilde{\mathbf{u}} \nabla v_e - C_v \tilde{T} \frac{\partial v_e}{\partial t} - \frac{1}{2} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} \frac{\partial v_e}{\partial t} \right\|^2 \\
 & + \left\| -\tilde{\rho} \nabla v_c - \tilde{\rho} \tilde{\mathbf{u}} \cdot \nabla \mathbf{v}_m - \tilde{\rho} \nabla \mathbf{v}_m \cdot \tilde{\mathbf{u}} - \tilde{\rho} \frac{\partial \mathbf{v}_m}{\partial t} - C_v \tilde{\rho} \tilde{T} \nabla v_e - \frac{1}{2} \tilde{\rho} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} \nabla v_e \right. \\
 & \quad \left. - \frac{1}{2} \tilde{\rho} \tilde{\mathbf{u}} \cdot \nabla v_e \tilde{\mathbf{u}} - \frac{1}{2} \tilde{\rho} \nabla v_e \cdot \tilde{\mathbf{u}} \tilde{\mathbf{u}} - R\tilde{\rho} \tilde{T} \nabla v_e + \tilde{\mathbb{D}} \cdot \nabla v_e - \frac{1}{2} \tilde{\rho} \tilde{\mathbf{u}} \frac{\partial v_e}{\partial t} - \frac{1}{2} \tilde{\rho} \tilde{\mathbf{u}} \frac{\partial v_e}{\partial t} \right\|^2 \\
 & + \left\| -R\tilde{\rho} \nabla \cdot \mathbf{v}_m - C_v \tilde{\rho} \tilde{\mathbf{u}} \nabla v_e - R\tilde{\rho} \tilde{\mathbf{u}} \nabla v_e - C_v \tilde{\rho} \frac{\partial v_e}{\partial t} \right\|^2 \\
 & + \left\| \frac{1}{\mu} \mathbb{S} \right\|^2 + \left\| \frac{Pr}{C_p \mu} \boldsymbol{\tau} \right\|^2 + \|v_c\|^2 + \|\mathbf{v}_m\|^2 + \|v_e\|^2 .
 \end{aligned}$$

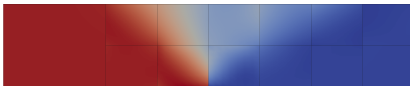
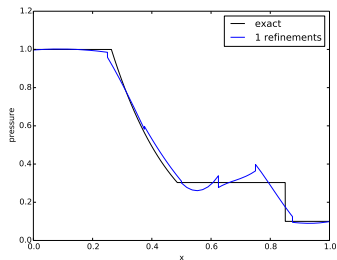
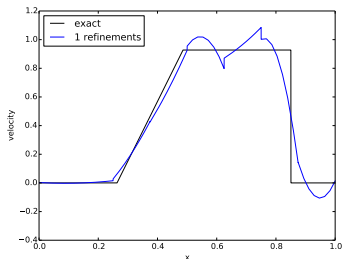
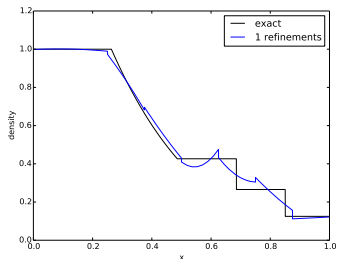
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



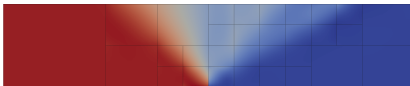
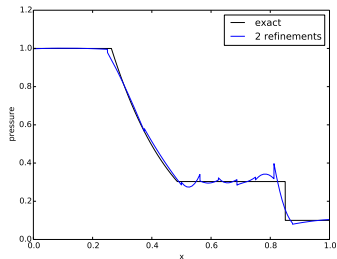
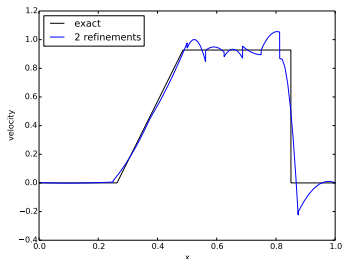
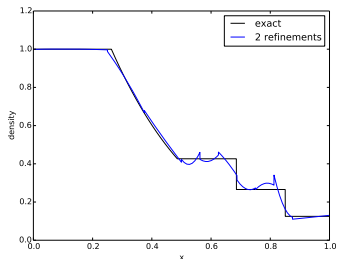
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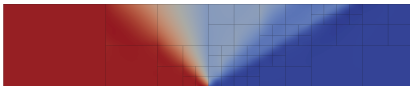
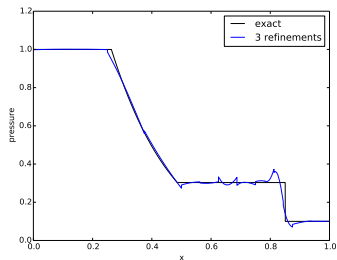
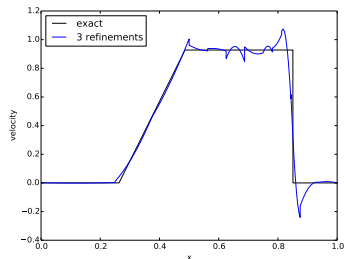
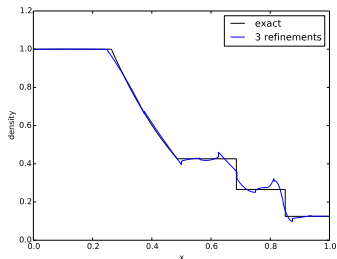
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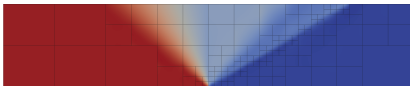
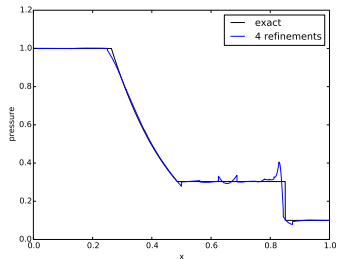
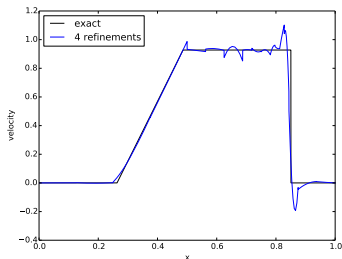
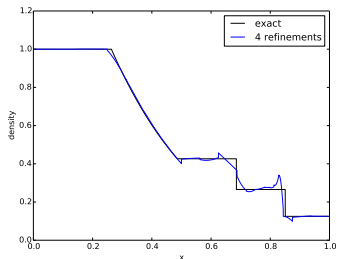
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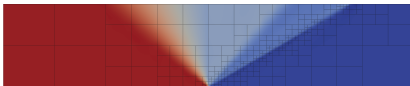
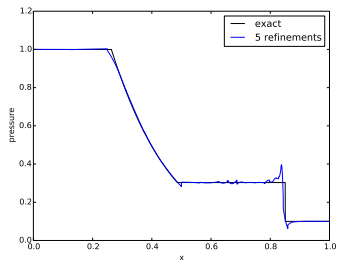
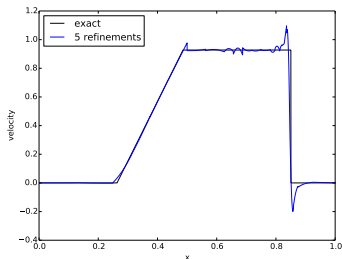
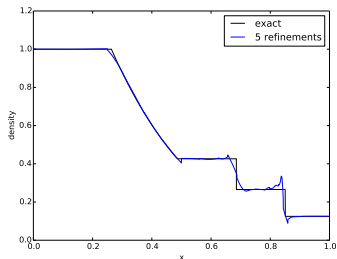
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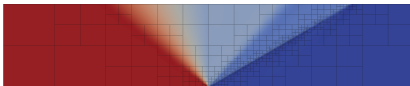
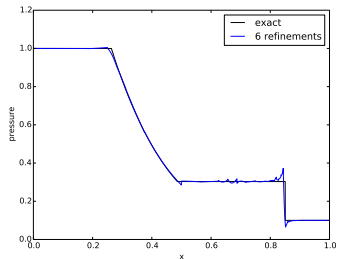
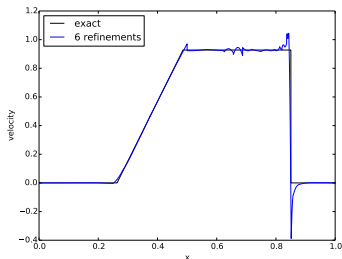
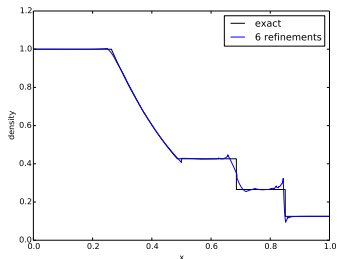
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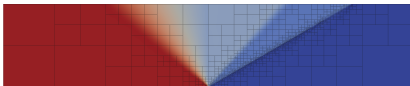
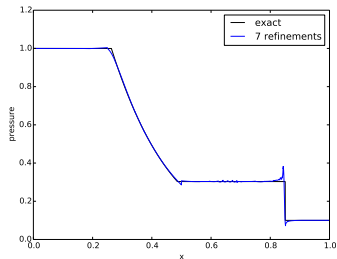
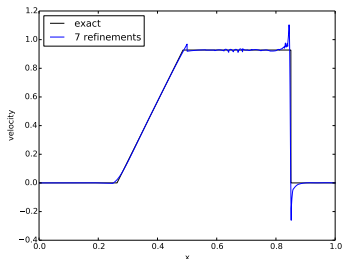
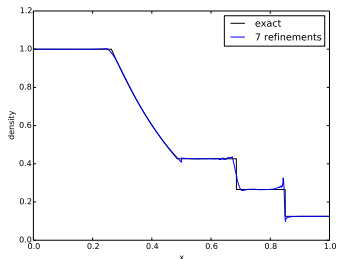
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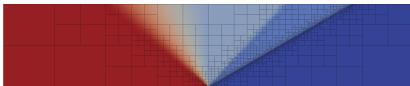
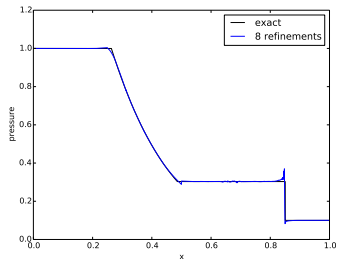
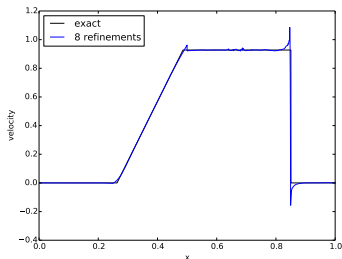
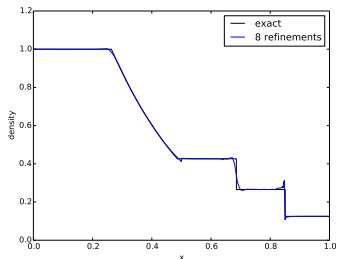
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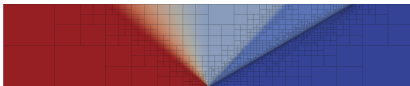
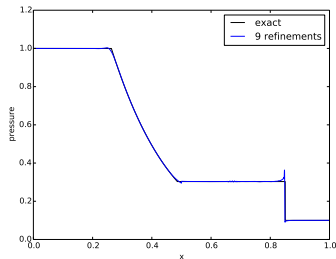
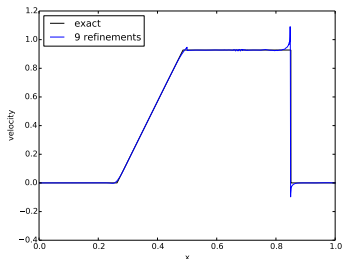
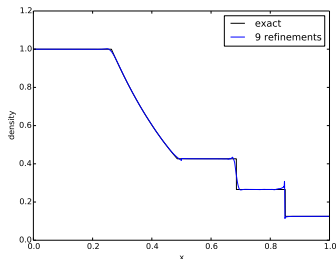
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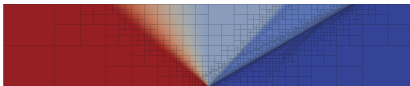
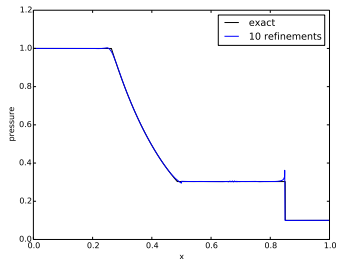
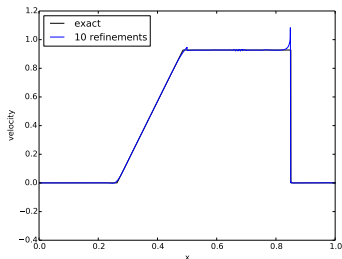
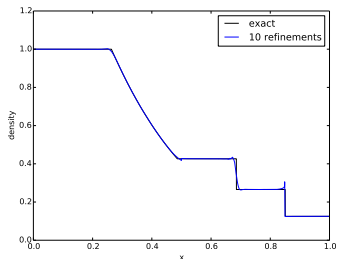
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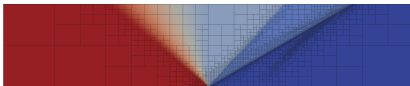
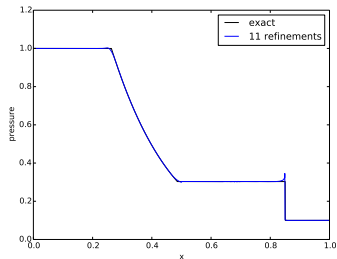
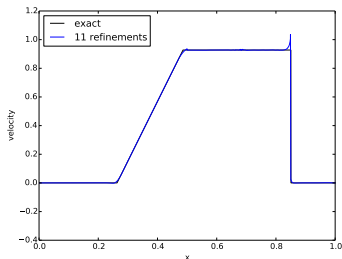
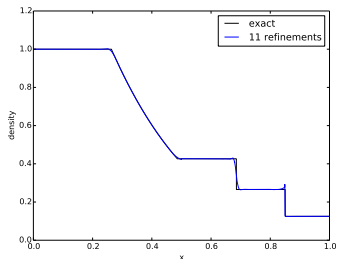
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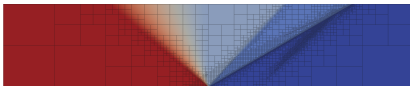
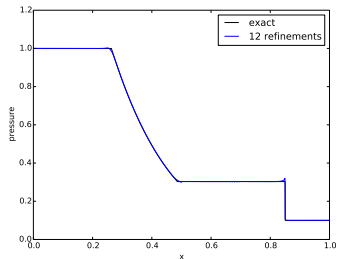
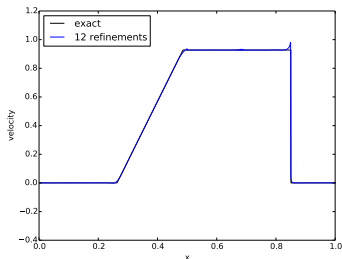
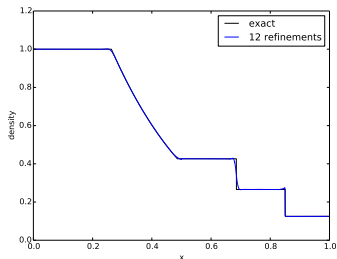
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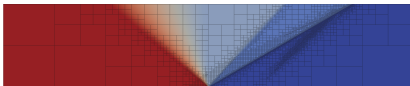
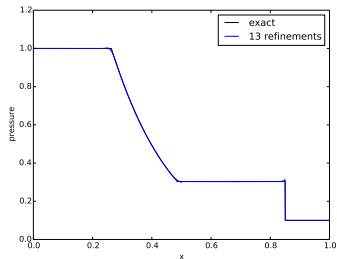
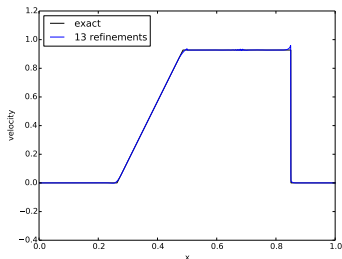
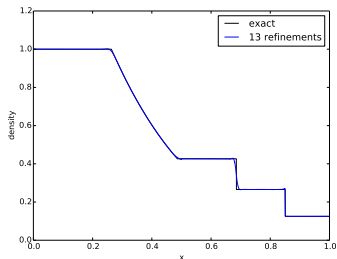
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



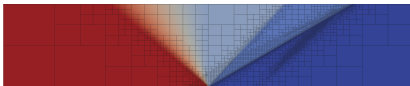
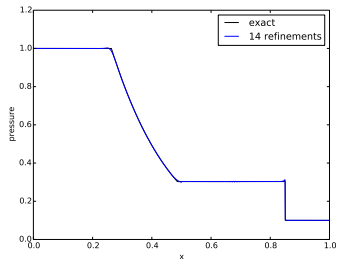
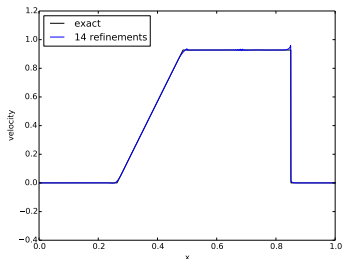
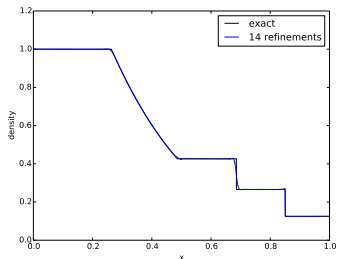
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



Compressible Navier-Stokes

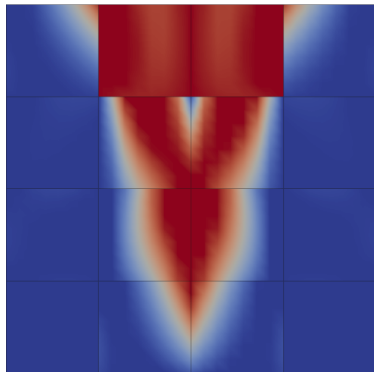
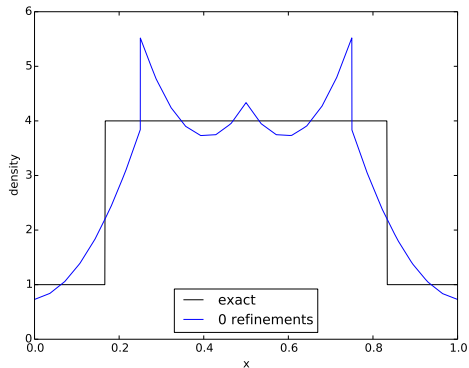
Sod Shock Tube with $\mu = 10^{-5}$



Compressible Navier-Stokes

Noh Implosion with $\mu = 10^{-3}$

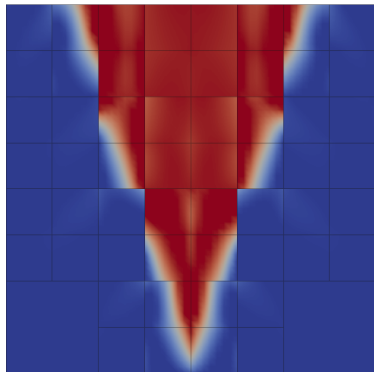
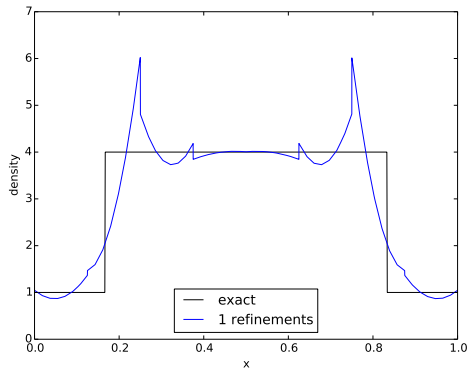
Infinitely strong shock propagation.



Compressible Navier-Stokes

Noh Implosion with $\mu = 10^{-3}$

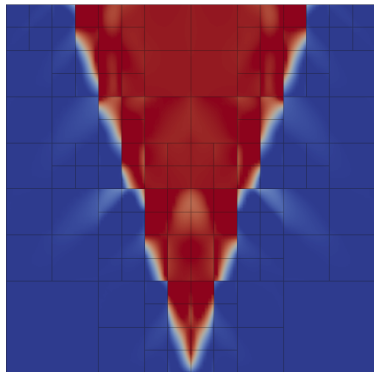
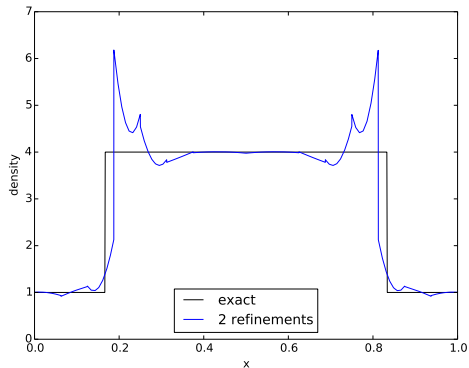
Infinitely strong shock propagation.



Compressible Navier-Stokes

Noh Implosion with $\mu = 10^{-3}$

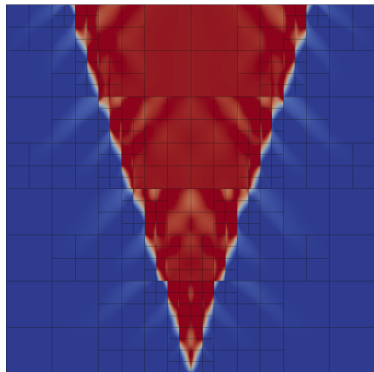
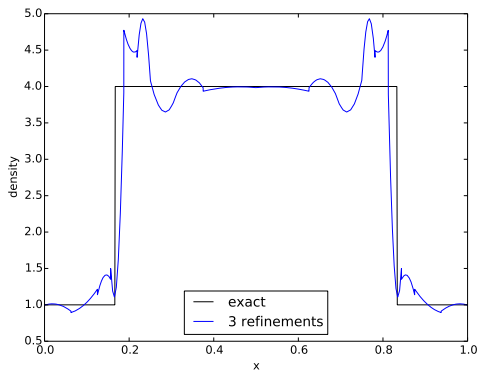
Infinitely strong shock propagation.



Compressible Navier-Stokes

Noh Implosion with $\mu = 10^{-3}$

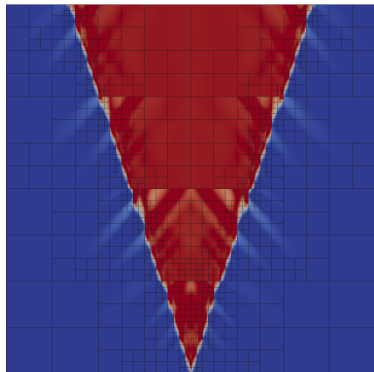
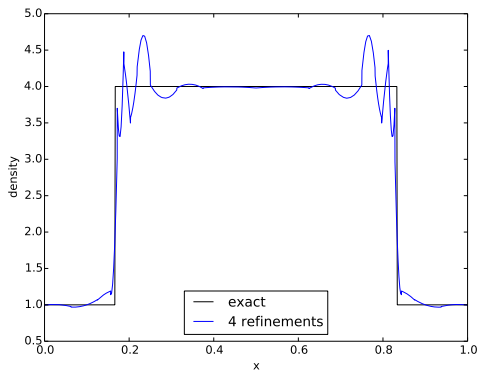
Infinitely strong shock propagation.



Compressible Navier-Stokes

Noh Implosion with $\mu = 10^{-3}$

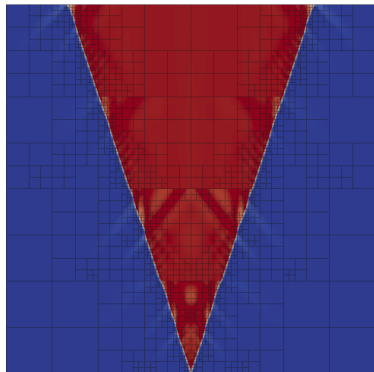
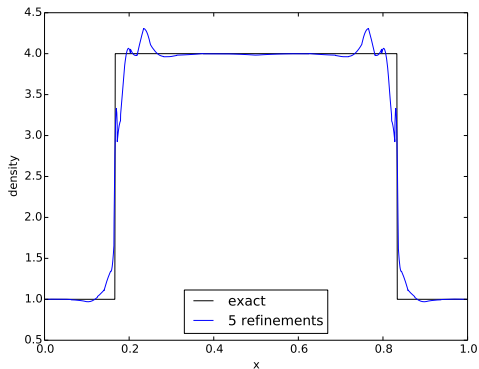
Infinitely strong shock propagation.



Compressible Navier-Stokes

Noh Implosion with $\mu = 10^{-3}$

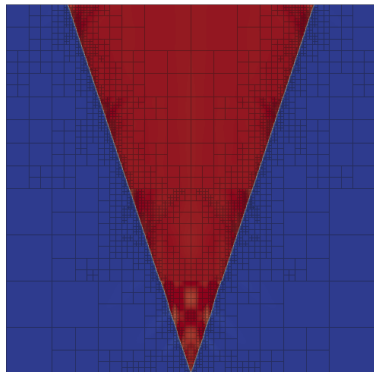
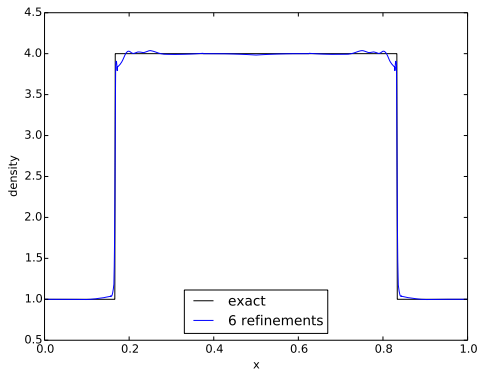
Infinitely strong shock propagation.



Compressible Navier-Stokes

Noh Implosion with $\mu = 10^{-3}$

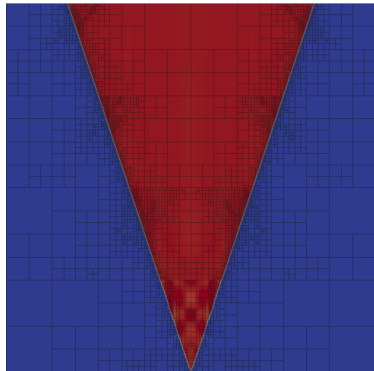
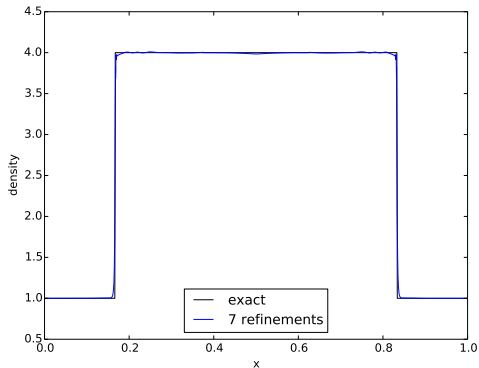
Infinitely strong shock propagation.



Compressible Navier-Stokes

Noh Implosion with $\mu = 10^{-3}$

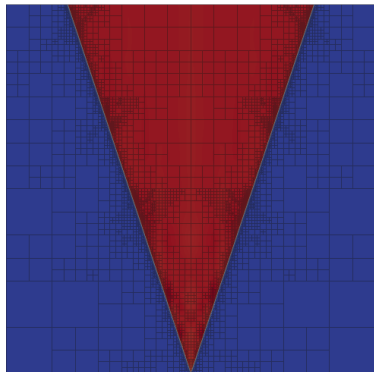
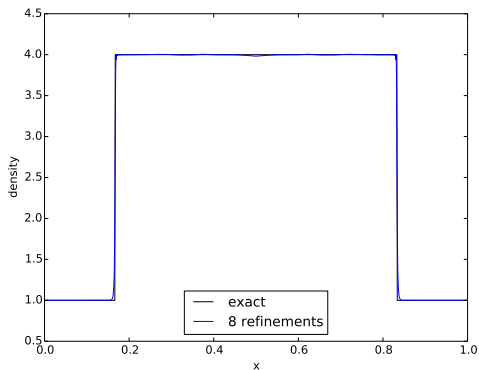
Infinitely strong shock propagation.



Compressible Navier-Stokes

Noh Implosion with $\mu = 10^{-3}$

Infinitely strong shock propagation.



Conclusions

- A problem in conservation form can be transformed to a space-time divergence.
- Fluxes change character from spatial to temporal boundaries.
- Traces are only defined on spatial boundaries.

Future Work

- Proof of robustness for convection-diffusion.
- Analysis of robust test norms.
- Time slabs will reduce the simulation cost.
- Two and three dimensions for more realistic problems.
- Incompressible Navier-Stokes.
- Iterative solvers for parallel scalability.