Derivation of Space-Time DPG for the Euler Equations

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The 1D Euler equations are

$$\frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho u \\ \rho e + \frac{1}{2}\rho u^2 \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \rho u \\ \rho u^2 + p \\ (\rho e + \frac{1}{2}\rho u^2 + p)u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

If we define

$$\nabla := \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial t} \end{bmatrix}$$

then we can rewrite the 1D Euler equations as a system.

$$\nabla \cdot \begin{bmatrix} \rho u \\ \rho \end{bmatrix} = 0$$

$$\nabla \cdot \begin{bmatrix} \rho u^2 + p \\ \rho u \end{bmatrix} = 0$$

$$\nabla \cdot \begin{bmatrix} (\rho e + \frac{1}{2}\rho u^2 + p)u \\ \rho e + \frac{1}{2}\rho u^2 \end{bmatrix} = 0$$

Multiplying each equation by v_m (for mass), v_x (for x-momentum), and v_e (for energy), and integrating by parts over spacetime Q:

$$\int_{\partial Q} v_m \begin{bmatrix} \rho u \\ \rho \end{bmatrix} \cdot \mathbf{n} - \int_Q \begin{bmatrix} \rho u \\ \rho \end{bmatrix} \cdot \nabla v_m = 0$$

$$\int_{\partial Q} v_x \begin{bmatrix} \rho u^2 + p \\ \rho u \end{bmatrix} \cdot \mathbf{n} - \int_Q \begin{bmatrix} \rho u^2 + p \\ \rho u \end{bmatrix} \cdot \nabla v_x = 0$$

$$\int_{\partial Q} v_e \begin{bmatrix} (\rho e + \frac{1}{2}\rho u^2 + p)u \\ \rho e + \frac{1}{2}\rho u^2 \end{bmatrix} \cdot \mathbf{n} - \int_Q \begin{bmatrix} (\rho e + \frac{1}{2}\rho u^2 + p)u \\ \rho e + \frac{1}{2}\rho u^2 \end{bmatrix} \cdot \nabla v_e = 0$$

Now identify the fluxes

$$\hat{F}_m := \begin{bmatrix} \rho u \\ \rho \end{bmatrix} \cdot \mathbf{n}$$

$$\hat{F}_x := \begin{bmatrix} \rho u^2 + p \\ \rho u \end{bmatrix} \cdot \mathbf{n}$$

$$\hat{F}_e := \begin{bmatrix} (\rho e + \frac{1}{2}\rho u^2 + p)u \\ \rho e + \frac{1}{2}\rho u^2 \end{bmatrix} \cdot \mathbf{n}$$

Assuming an ideal gas equation of state, linearize the volume terms.

$$F_m(\mathbf{U}) := \begin{bmatrix} \rho u \\ \rho \end{bmatrix} \approx \begin{bmatrix} u & \rho & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\rho \\ \Delta u \\ \Delta e \end{bmatrix}$$

$$F_x(\mathbf{U}) := \begin{bmatrix} \rho u^2 + \rho(\gamma-1)e \\ \rho u \end{bmatrix} \approx \begin{bmatrix} u^2 + (\gamma-1)e & 2\rho u & \rho(\gamma-1) \\ u & \rho & 0 \end{bmatrix} \begin{bmatrix} \Delta\rho \\ \Delta u \\ \Delta e \end{bmatrix}$$

$$F_e(\mathbf{U}) := \begin{bmatrix} (\rho e + \frac{1}{2}\rho u^2 + \rho(\gamma-1)e)u \\ \rho e + \frac{1}{2}\rho u^2 \end{bmatrix} \approx \begin{bmatrix} \gamma e u + \frac{1}{2}u^3 & \rho\gamma e + \frac{3}{2}\rho u^2 & \rho\gamma u \\ e + \frac{1}{2}u^2 & \rho u & \rho \end{bmatrix} \begin{bmatrix} \Delta\rho \\ \Delta u \\ \Delta e \end{bmatrix}$$

Finally, our space-time DPG formulation of the Euler equations is as follows.

Given background flow quantities ρ, u, $e \in L^2(Q)$, find $\Delta\rho$, Δu, $\Delta e \in L^2(Q)$, and $\hat{F}_m$, $\hat{F}_x$, $\hat{F}_e \in H^{-\frac{1}{2}}$ such that

$$\int_{\partial Q} \hat{F}_m v_m - \int_Q \begin{bmatrix} u & \rho & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\rho \\ \Delta u \\ \Delta e \end{bmatrix} \cdot \nabla v_m = \int_Q \begin{bmatrix} \rho u \\ \rho \end{bmatrix} \cdot \nabla v_m$$

$$\int_{\partial Q} \hat{F}_x v_x - \int_Q \begin{bmatrix} u^2 + (\gamma-1)e & 2\rho u & \rho(\gamma-1) \\ u & \rho & 0 \end{bmatrix} \begin{bmatrix} \Delta\rho \\ \Delta u \\ \Delta e \end{bmatrix} \cdot \nabla v_x = \int_Q \begin{bmatrix} \rho u^2 + \rho(\gamma-1)e \\ \rho u \end{bmatrix} \cdot \nabla v_x$$

$$\int_{\partial Q} \hat{F}_e v_e - \int_Q \begin{bmatrix} \gamma e u + \frac{1}{2}u^3 & \rho\gamma e + \frac{3}{2}\rho u^2 & \rho\gamma u \\ e + \frac{1}{2}u^2 & \rho u & \rho \end{bmatrix} \begin{bmatrix} \Delta\rho \\ \Delta u \\ \Delta e \end{bmatrix} \cdot \nabla v_e = \int_Q \begin{bmatrix} (\rho e + \frac{1}{2}\rho u^2 + \rho(\gamma-1)e)u \\ \rho e + \frac{1}{2}\rho u^2 \end{bmatrix} \cdot \nabla v_e$$

for all v_m, v_x, and v_e.

As a first cut, we will use the graph norm.