Space-Time Navier-Stokes with Entropy Variables

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Nonlinear Forms

Primitive Variables

Consider the DPG Navier-Stokes derivation from previously with primitive variables:

$$\left(\frac{1}{\mu}\mathbb{D},\mathbb{S}\right)+(2\boldsymbol{u},\nabla\cdot\mathbb{S})-\left(\frac{2}{3}\boldsymbol{u},\nabla\operatorname{tr}\mathbb{S}\right)-\left\langle\frac{4}{3}\hat{\boldsymbol{u}},\mathbb{S}\boldsymbol{n}_x\right\rangle=0 \tag{1a}$$

$$\left(\frac{Pr}{C_p\mu}\boldsymbol{q},\boldsymbol{\tau}\right)-(T,\nabla\cdot\boldsymbol{\tau})+\left\langle\hat{T},\tau_n\right\rangle=0 \tag{1b}$$

$$-\left(\begin{pmatrix}\rho\boldsymbol{u}\\ \rho\end{pmatrix},\nabla_{xt}v_c\right)+\langle\hat{t}_c,v_c\rangle=(f_c,v_c) \tag{1c}$$

$$-\left(\begin{pmatrix}\rho\boldsymbol{u}\otimes\boldsymbol{u}+\rho RT\boldsymbol{I}-\mathbb{D}\\ \rho\boldsymbol{u}\end{pmatrix},\nabla_{xt}\boldsymbol{v}_m\right)+\langle\hat{\boldsymbol{t}}_m,\boldsymbol{v}_m\rangle=(\boldsymbol{f}_m,\boldsymbol{v}_m) \tag{1d}$$

$$-\left(\begin{pmatrix}\rho\boldsymbol{u}\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)+\boldsymbol{u}\rho RT+\boldsymbol{q}-\boldsymbol{u}\cdot\mathbb{D}\\ \rho\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\end{pmatrix},\nabla_{xt}v_e\right)+\langle\hat{t}_e,v_e\rangle=(f_e,v_e)\,, \tag{1e}$$

where

$$\begin{aligned}
\hat{\boldsymbol{u}}&=\operatorname{tr}(\boldsymbol{u})\\
\hat{T}&=\operatorname{tr}(T)\\
\hat{t}_c&=\operatorname{tr}(\rho\boldsymbol{u})\cdot\boldsymbol{n}_x+\operatorname{tr}(\rho)\,n_t\\
\hat{\boldsymbol{t}}_m&=\operatorname{tr}(\rho\boldsymbol{u}\otimes\boldsymbol{u}+\rho RT\boldsymbol{I}-\mathbb{D})\cdot\boldsymbol{n}_x+\operatorname{tr}(\rho\boldsymbol{u})\,n_t\\
\hat{t}_e&=\operatorname{tr}\left(\rho\boldsymbol{u}\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)+\boldsymbol{u}\rho RT+\boldsymbol{q}-\boldsymbol{u}\cdot\mathbb{D}\right)\cdot\boldsymbol{n}_x\\
&\quad+\operatorname{tr}\left(\rho\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\right)n_t\,.
\end{aligned}$$

Now define primitive fluxes for continuity, momentum, and energy equations:

$$\begin{aligned}
\boldsymbol{F}_c^p&:=\rho\boldsymbol{u}\\
\mathbb{F}_m^p&:=\rho\boldsymbol{u}\otimes\boldsymbol{u}+\rho RT\boldsymbol{I}\\
\boldsymbol{F}_e^p&:=\rho\boldsymbol{u}\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)+\boldsymbol{u}\rho RT
\end{aligned}$$

Our bilinear form is then simplified:

$$\left(\frac{1}{\mu}\mathbb{D},\mathbb{S}\right)+(2\boldsymbol{u},\nabla\cdot\mathbb{S})-\left(\frac{2}{3}\boldsymbol{u},\nabla\operatorname{tr}\mathbb{S}\right)-\left\langle\frac{4}{3}\hat{\boldsymbol{u}},\mathbb{S}\boldsymbol{n}_x\right\rangle=0 \tag{2a}$$

$$\left(\frac{Pr}{C_p\mu}\boldsymbol{q},\boldsymbol{\tau}\right)-(T,\nabla\cdot\boldsymbol{\tau})+\left\langle\hat{T},\tau_n\right\rangle=0 \tag{2b}$$

$$-\left(\begin{pmatrix}\boldsymbol{F}_c^p\\ \rho\end{pmatrix},\nabla_{xt}v_c\right)+\langle\hat{t}_c,v_c\rangle=(f_c,v_c) \tag{2c}$$

$$-\left(\begin{pmatrix}\mathbb{F}_m^p-\mathbb{D}\\ \rho\boldsymbol{u}\end{pmatrix},\nabla_{xt}\boldsymbol{v}_m\right)+\langle\hat{\boldsymbol{t}}_m,\boldsymbol{v}_m\rangle=(\boldsymbol{f}_m,\boldsymbol{v}_m) \tag{2d}$$

$$-\left(\begin{pmatrix}\boldsymbol{F}_e^p+\boldsymbol{q}-\boldsymbol{u}\cdot\mathbb{D}\\ \rho\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\end{pmatrix},\nabla_{xt}v_e\right)+\langle\hat{t}_e,v_e\rangle=(f_e,v_e)\ , \tag{2e}$$

Conservation Variables

Now we wish to do a change of variables to conservation variables:

$$\begin{aligned}\rho&=\rho\\ \boldsymbol{m}&=\rho\boldsymbol{u}\\ E&=\rho\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\end{aligned}$$

We can define new fluxes in conservation variables:

$$\begin{aligned}\boldsymbol{F}_c^c&=\boldsymbol{m}\\ \mathbb{F}_m^c&=\frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\rho}+(\gamma-1)\left(E-\frac{\boldsymbol{m}\cdot\boldsymbol{m}}{2\rho}\right)\boldsymbol{I}\\ \boldsymbol{F}_e^c&=\frac{\boldsymbol{m}}{\rho}E+(\gamma-1)\left(E-\frac{\boldsymbol{m}\cdot\boldsymbol{m}}{2\rho}\right)\frac{\boldsymbol{m}}{\rho}\end{aligned}$$

and our new bilinear form is

$$\left(\frac{1}{\mu}\mathbb{D},\mathbb{S}\right)+\left(2\frac{\boldsymbol{m}}{\rho},\nabla\cdot\mathbb{S}\right)-\left(\frac{2}{3}\frac{\boldsymbol{m}}{\rho},\nabla\operatorname{tr}\mathbb{S}\right)-\left\langle\frac{4}{3}\hat{\boldsymbol{u}},\mathbb{S}\boldsymbol{n}_x\right\rangle=0 \tag{3a}$$

$$\left(\frac{Pr}{C_p\mu}\boldsymbol{q},\boldsymbol{\tau}\right)-\left(\frac{E-\frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m}}{C_v\rho},\nabla\cdot\boldsymbol{\tau}\right)+\left\langle\hat{T},\tau_n\right\rangle=0 \tag{3b}$$

$$-\left(\begin{pmatrix}\boldsymbol{F}_c^c\\ \rho\end{pmatrix},\nabla_{xt}v_c\right)+\langle\hat{t}_c,v_c\rangle=(f_c,v_c) \tag{3c}$$

$$-\left(\begin{pmatrix}\mathbb{F}_m^c-\mathbb{D}\\ \boldsymbol{m}\end{pmatrix},\nabla_{xt}\boldsymbol{v}_m\right)+\langle\hat{\boldsymbol{t}}_m,\boldsymbol{v}_m\rangle=(\boldsymbol{f}_m,\boldsymbol{v}_m) \tag{3d}$$

$$-\left(\begin{pmatrix}\boldsymbol{F}_e^c+\boldsymbol{q}-\frac{\boldsymbol{m}}{\rho}\cdot\mathbb{D}\\ E\end{pmatrix},\nabla_{xt}v_e\right)+\langle\hat{t}_e,v_e\rangle=(f_e,v_e)\ , \tag{3e}$$

Entropy Variables

Now we wish to do a change of variables to entropy variables:

$$
\begin{aligned}
V_c &= \frac{-E + (E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m})\left(\gamma + 1 - \ln\left[\frac{(\gamma-1)(E-\frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m})}{\rho^\gamma}\right]\right)}{E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m}}\\
\boldsymbol{V}_m &= \frac{\boldsymbol{m}}{E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m}}\\
V_e &= \frac{-\rho}{E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m}}
\end{aligned}
$$

with reverse mapping:

$$
\begin{aligned}
\rho &= -\alpha V_e\\
\boldsymbol{m} &= \alpha \boldsymbol{V}_m\\
E &= \alpha\left(1 - \frac{1}{2V_e}\boldsymbol{V}_m\cdot\boldsymbol{V}_m\right)
\end{aligned}
$$

where

$$
\alpha(V_c, \boldsymbol{V}_m, V_e) = \left[\frac{\gamma-1}{-(V_e)^\gamma}\right]^{\frac{1}{\gamma-1}}\exp\left[\frac{-\gamma + V_c - \frac{1}{2V_e}\boldsymbol{V}_m\cdot\boldsymbol{V}_m}{\gamma-1}\right]
$$

We can define new fluxes in entropy variables:

$$
\begin{aligned}
\boldsymbol{F}_c^e &= \alpha\boldsymbol{V}_m\\
\mathbb{F}_m^e &= -\alpha\frac{\boldsymbol{V}_m\otimes\boldsymbol{V}_m}{V_e} + \alpha(\gamma-1)\boldsymbol{I}\\
\boldsymbol{F}_e^e &= -\alpha\frac{\boldsymbol{V}_m}{V_e}\left(1 - \frac{1}{2V_e}\boldsymbol{V}_m\cdot\boldsymbol{V}_m\right) - \alpha(\gamma-1)\frac{\boldsymbol{V}_m}{V_e}
\end{aligned}
$$

and our new bilinear form is

$$
\left(\frac{1}{\mu}\mathbb{D}, \mathbb{S}\right) - \left(2\frac{\boldsymbol{V}_m}{V_e}, \nabla\cdot\mathbb{S}\right) + \left(\frac{2}{3}\frac{\boldsymbol{V}_m}{V_e}, \nabla\,\mathrm{tr}\,\mathbb{S}\right) - \left\langle\frac{4}{3}\hat{\boldsymbol{u}}, \mathbb{S}\boldsymbol{n}_x\right\rangle = 0 \tag{4a}
$$

$$
\left(\frac{Pr}{C_p\mu}\boldsymbol{q}, \boldsymbol{\tau}\right) + \left(\frac{1}{C_v V_e}, \nabla\cdot\boldsymbol{\tau}\right) + \left\langle\hat{T}, \tau_n\right\rangle = 0 \tag{4b}
$$

$$
-\left(\begin{pmatrix}\boldsymbol{F}_c^e\\ -\alpha V_e\end{pmatrix}, \nabla_{xt}v_c\right) + \left\langle\hat{t}_c, v_c\right\rangle = (f_c, v_c) \tag{4c}
$$

$$
-\left(\begin{pmatrix}\mathbb{F}_m^e - \mathbb{D}\\ \alpha\boldsymbol{V}_m\end{pmatrix}, \nabla_{xt}\boldsymbol{v}_m\right) + \left\langle\hat{\boldsymbol{t}}_m, \boldsymbol{v}_m\right\rangle = (\boldsymbol{f}_m, \boldsymbol{v}_m) \tag{4d}
$$

$$
-\left(\begin{pmatrix}\boldsymbol{F}_e^e + \boldsymbol{q} + \frac{\boldsymbol{V}_m}{V_e}\cdot\mathbb{D}\\ \alpha\left(1 - \frac{1}{2V_e}\boldsymbol{V}_m\cdot\boldsymbol{V}_m\right)\end{pmatrix}, \nabla_{xt}v_e\right) + \left\langle\hat{t}_e, v_e\right\rangle = (f_e, v_e)\ , \tag{4e}
$$

Linearization

For each change of variables, we maintain the same linear variables: $L := \left\{\boldsymbol{q}, \hat{\boldsymbol{u}}, \hat{e}, \hat{t}_c, \hat{\boldsymbol{t}}_m, \hat{t}_e\right\}$. Let U be the set of variables involved in nonlinear interactions. We apply a linearization $U \approx \tilde{U} + \Delta U$ and solve

$$R_{,U}(\tilde{U})\Delta U + R(L) = -R(\tilde{U})\,,$$

where

$$\begin{aligned} R(L) = {} & \left(\frac{Pr}{C_p\mu}\boldsymbol{q}, \boldsymbol{\tau}\right) - (\boldsymbol{q}, \nabla v_e) - \left\langle \frac{4}{3}\hat{\boldsymbol{u}}, \mathbb{S}\boldsymbol{n}_x \right\rangle + \left\langle \hat{T}, \tau_n \right\rangle + \left\langle \hat{t}_c, v_c \right\rangle + \left\langle \hat{\boldsymbol{t}}_m, v_m \right\rangle + \left\langle \hat{t}_e, v_e \right\rangle \\ & - (f_c, v_c) - (\boldsymbol{f}_m, \boldsymbol{v}_m) - (f_e, v_e) \end{aligned}$$

Primitive Variables

The set of nonlinear variables is $U^p := \{\rho, \boldsymbol{u}, T, \mathbb{D}\}$. Then $R_{,U^p}(\tilde{U}^p)\Delta U^p$ is

$$\begin{aligned} & \left(\frac{1}{\mu}\Delta\mathbb{D}, \mathbb{S}\right) + (2\Delta\boldsymbol{u}, \nabla\cdot\mathbb{S}) - \left(\frac{2}{3}\Delta\boldsymbol{u}, \nabla \operatorname{tr}\mathbb{S}\right) \\ & - (\Delta T, \nabla\cdot\boldsymbol{\tau}) \\ & - \left(\begin{pmatrix} \boldsymbol{F}^p_{c,U^p}\Delta U^p \\ \Delta\rho \end{pmatrix}, \nabla_{xt} v_c\right) \\ & - \left(\begin{pmatrix} \mathbb{F}^p_{m,U^p}\Delta U^p - \Delta\mathbb{D} \\ \Delta\rho\tilde{\boldsymbol{u}} + \tilde{\rho}\Delta\boldsymbol{u} \end{pmatrix}, \nabla_{xt}\boldsymbol{v}_m\right) \\ & - \left(\begin{pmatrix} \boldsymbol{F}^p_{e,U^p}\Delta U^p - \Delta\boldsymbol{u}\cdot\tilde{\mathbb{D}} - \tilde{\boldsymbol{u}}\cdot\Delta\mathbb{D} \\ C_v\Delta\rho\tilde{T} + C_v\tilde{\rho}\Delta T + \frac{1}{2}\left(\Delta\rho\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}} + \tilde{\rho}\Delta\boldsymbol{u}\cdot\tilde{\boldsymbol{u}} + \tilde{\rho}\tilde{\boldsymbol{u}}\cdot\Delta\boldsymbol{u}\right) \end{pmatrix}, \nabla_{xt} v_e\right) \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{F}^p_{c,U^p}\Delta U^p &:= \Delta\rho\tilde{\boldsymbol{u}} + \tilde{\rho}\Delta\boldsymbol{u} \\ \mathbb{F}^p_{m,U^p} &:= \Delta\rho\tilde{\boldsymbol{u}}\otimes\tilde{\boldsymbol{u}} + \tilde{\rho}\Delta\boldsymbol{u}\otimes\tilde{\boldsymbol{u}} + \tilde{\rho}\tilde{\boldsymbol{u}}\otimes\Delta\boldsymbol{u} + R\left(\Delta\rho\tilde{T} + \tilde{\rho}\Delta T\right)\boldsymbol{I} \\ \boldsymbol{F}^p_{e,U^p} &:= C_v\Delta\rho\tilde{\boldsymbol{u}}\tilde{T} + C_v\tilde{\rho}\Delta\boldsymbol{u}\tilde{T} + C_v\tilde{\rho}\tilde{\boldsymbol{u}}\Delta T \\ &\quad + \frac{1}{2}\Delta\rho\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}} + \frac{1}{2}\tilde{\rho}\Delta\boldsymbol{u}\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}} + \frac{1}{2}\tilde{\rho}\tilde{\boldsymbol{u}}\Delta\boldsymbol{u}\cdot\tilde{\boldsymbol{u}} + \frac{1}{2}\tilde{\rho}\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}\cdot\Delta\boldsymbol{u} \\ &\quad + R\Delta\boldsymbol{u}\tilde{\rho}\tilde{T} + R\tilde{\boldsymbol{u}}\Delta\rho\tilde{T} + R\tilde{\boldsymbol{u}}\tilde{\rho}\Delta T \end{aligned}$$

and $R(\tilde{U}^p)$ is

$$\begin{aligned} & \left(\frac{1}{\mu}\tilde{\mathbb{D}}, \mathbb{S}\right) + (2\tilde{\boldsymbol{u}}, \nabla\cdot\mathbb{S}) - \left(\frac{2}{3}\tilde{\boldsymbol{u}}, \nabla \operatorname{tr}\mathbb{S}\right) \\ & - \left(\tilde{T}, \nabla\cdot\boldsymbol{\tau}\right) \\ & - \left(\begin{pmatrix} \boldsymbol{F}^p_c(\tilde{U}^p) \\ \tilde{\rho} \end{pmatrix}, \nabla_{xt} v_c\right) \\ & - \left(\begin{pmatrix} \mathbb{F}^p_m(\tilde{U}^p) - \tilde{\mathbb{D}} \\ \tilde{\rho}\tilde{\boldsymbol{u}} \end{pmatrix}, \nabla_{xt}\boldsymbol{v}_m\right) \\ & - \left(\begin{pmatrix} \boldsymbol{F}^p_e(\tilde{U}^p) - \tilde{\boldsymbol{u}}\cdot\tilde{\mathbb{D}} \\ \tilde{\rho}\left(C_v\tilde{T} + \frac{1}{2}\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}}\right) \end{pmatrix}, \nabla_{xt} v_e\right) \end{aligned}$$