

Discontinuous Petrov-Galerkin (DPG) Method

With Optimal Test Functions

Progress Report

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- 1 DPG Method is a Ritz method. It supports adaptivity with no preasymptotic behavior.
- 2 You can control the norm in which you want to converge.
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Assume

$$J_S^{\text{imp}} = n \times H^{\text{imp}}$$

and look for the unknown surface current on the skeleton also in the same form.

$$\left\{ \begin{array}{l} E \in H(\text{curl}, \Omega), \quad n \times E = n \times E^{\text{imp}} \text{ on } \Gamma_1 \\ \hat{h} \in \text{tr}_{\Gamma_h} H(\text{curl}, \Omega), \quad n \times \hat{h} = n \times (-i\omega H^{\text{imp}}) \text{ on } \Gamma_2 \\ (\frac{1}{\mu} \nabla \times E, \nabla_h \times F) + ((-\omega^2 \epsilon + i\omega \sigma) E, F) + \langle n \times \hat{h}, F \rangle_{\Gamma_h} = -i\omega (J^{\text{imp}}, F) \\ \forall F \in H(\text{curl}, \Omega_h). \end{array} \right.$$

Hexahedral meshes

$H(\text{curl})$ element for electric field E :

$$(\mathcal{P}^{p-1} \otimes \mathcal{P}^p \otimes \mathcal{P}^p) \times (\mathcal{P}^p \otimes \mathcal{P}^{p-1} \otimes \mathcal{P}^p) \times (\mathcal{P}^p \otimes \mathcal{P}^p \otimes \mathcal{P}^{p-1})$$

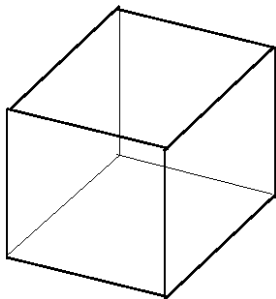
and trace of the same element for flux (surface current) \hat{h} .

Same element for the enriched space but with order $p + \Delta p$.

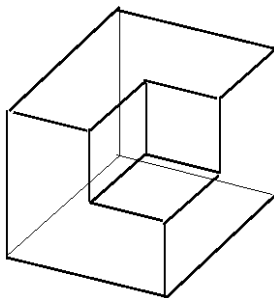
In reported experiments: $p = 2$, $\Delta p = 2$.

A 3D Maxwell example

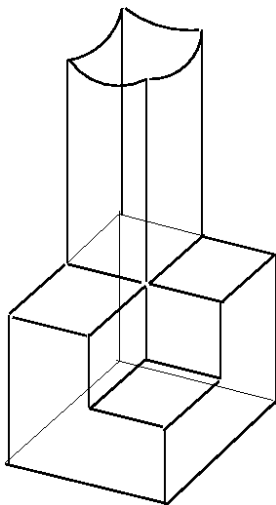
Take a cube $(0, 2)^3$



Divide it into eight smaller cubes and remove one:



Attach a waveguide:

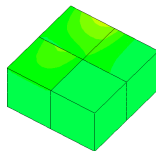
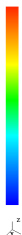
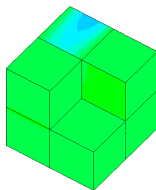
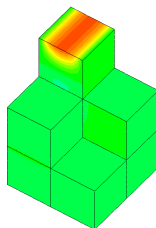
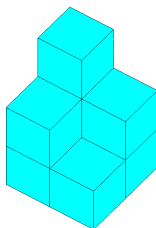


$$\epsilon = \mu = 1, \sigma = 0$$

$$\omega = 5(1.6 \text{ wavelengths in the cube})$$

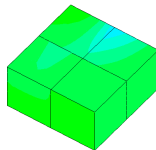
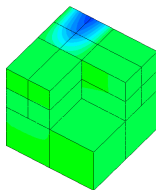
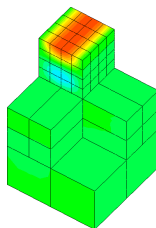
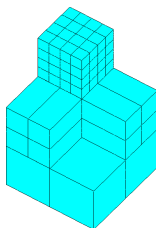
Cut the waveguide and use the lowest propagating mode for BC along the cut.

Fichera corner microwave, $p = 2$.



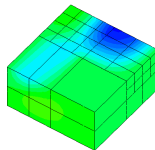
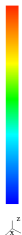
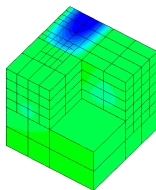
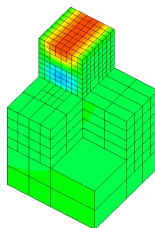
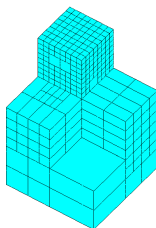
Initial mesh and real part of E_1

Fichera corner microwave, $p = 2$.



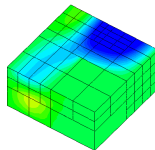
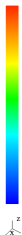
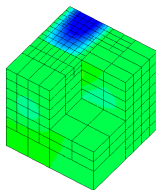
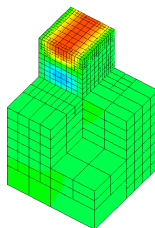
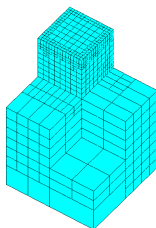
Mesh and real part of E_1 after two refinements

Fichera corner microwave, $p = 2$.



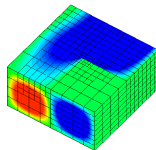
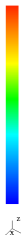
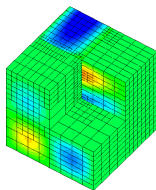
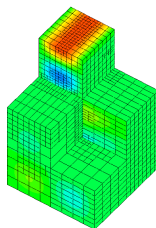
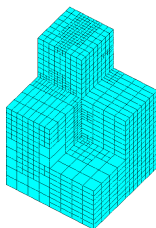
Mesh and real part of E_1 after four refinements

Fichera corner microwave, $p = 2$.



Mesh and real part of E_1 after six refinements

Fichera corner microwave, $p = 2$.



Mesh and real part of E_1 after eight refinements

From Ph.D. Dissertation of Jesse Chan: Compressible Navier-Stokes Equations: Carter's flat plate problem^{1 2}



$$M_{\infty} = 3, \text{Re}_L = 1000, \text{Pr} = 0.72, \gamma = 1.4, \theta_{\infty} = 390^{\circ}[\text{R}]$$

¹ L.D., J.T. Oden, W. Rachowicz, "A New Finite Element Method for Solving Compressible Navier-Stokes Equations Based on an Operator Splitting Method and h/p Adaptivity," *Comput. Methods Appl. Mech. Engrg.*, **84**, 275-326, 1990.

² J. Chan, L.D., R. Moser, "A DPG method for steady viscous compressible flow," *Computers and Fluids*, to appear.

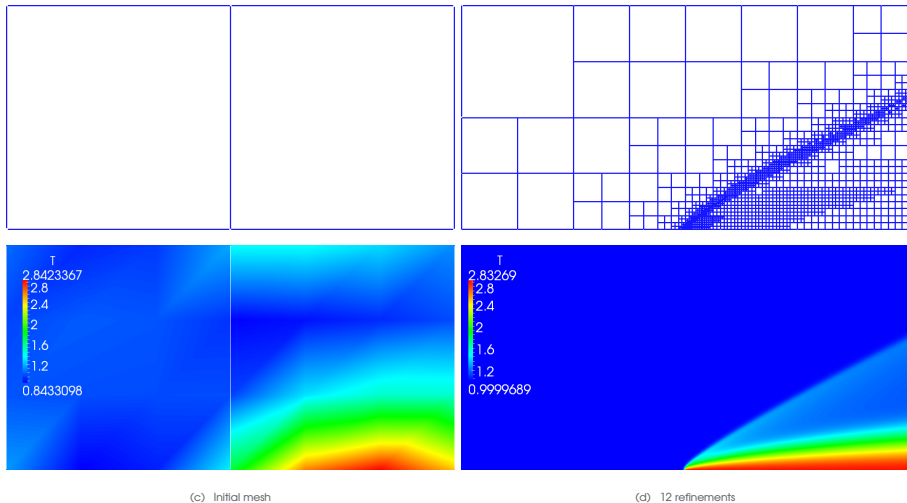


Figure : $Re = 1000$, initial and final meshes.

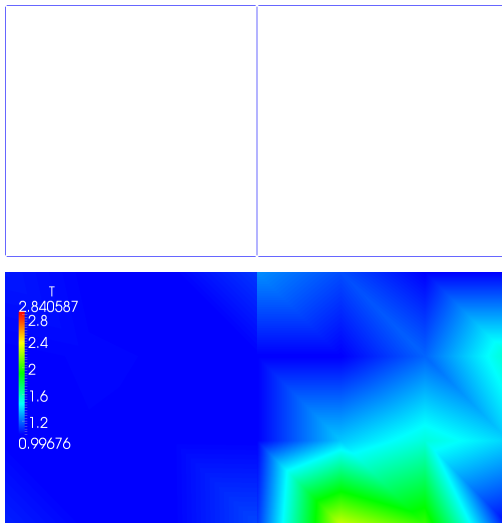


Figure : $Re = 10,000$, $p = 2$, initial mesh.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

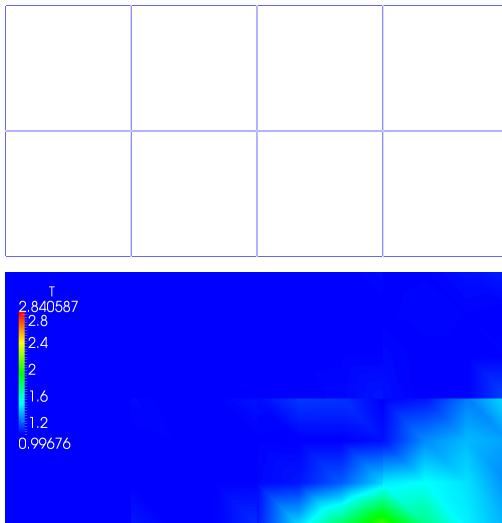


Figure : $Re = 10,000$, $p = 2$, refinement number 1.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

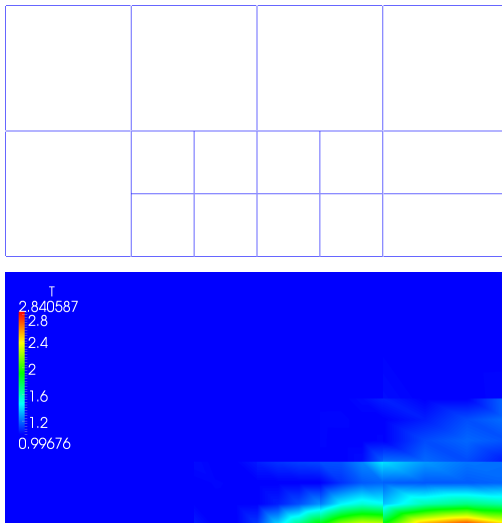


Figure : $Re = 10,000$, $p = 2$, refinement number 2.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

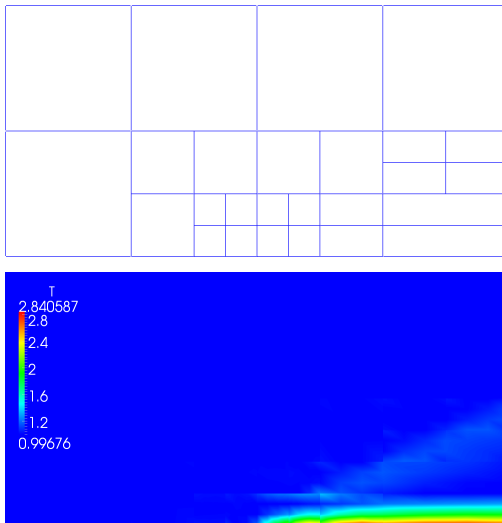


Figure : $Re = 10,000$, $p = 2$, refinement number 3.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

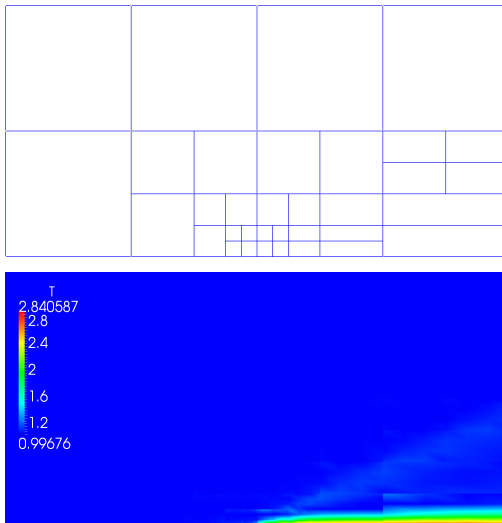


Figure : $Re = 10,000$, $p = 2$, refinement number 4.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

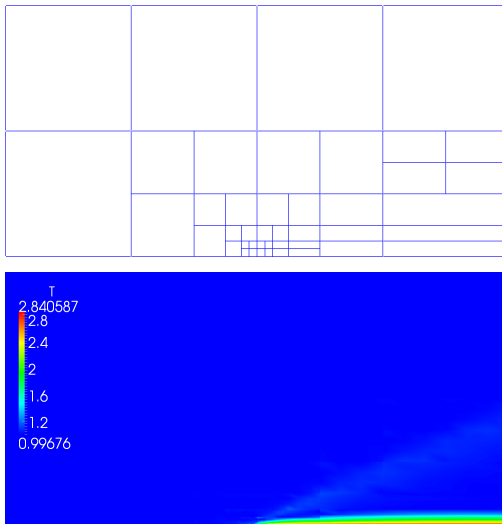


Figure : $Re = 10,000$, $p = 2$, refinement number 5.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

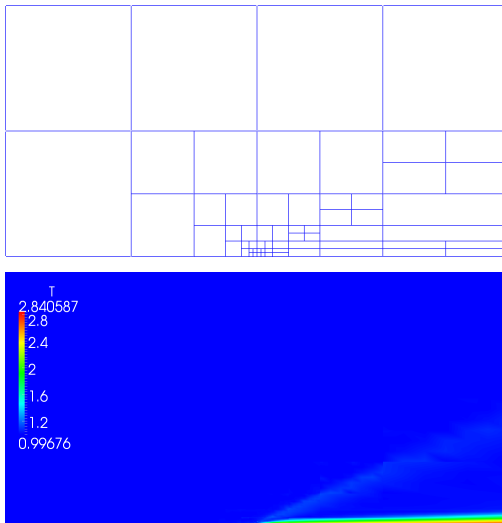


Figure : $Re = 10,000$, $p = 2$, refinement number 6.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

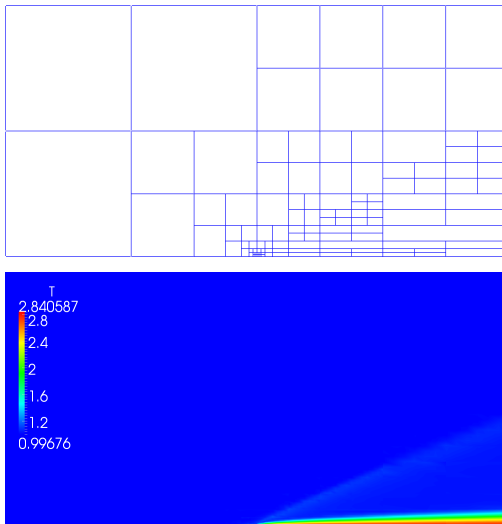


Figure : $Re = 10,000$, $p = 2$, refinement number 7.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

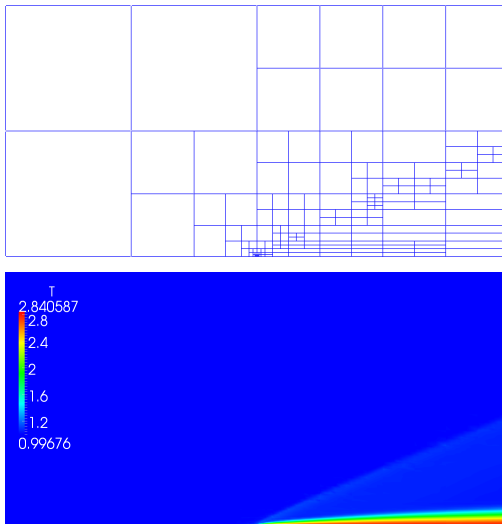


Figure : $Re = 10,000$, $p = 2$, refinement number 8.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

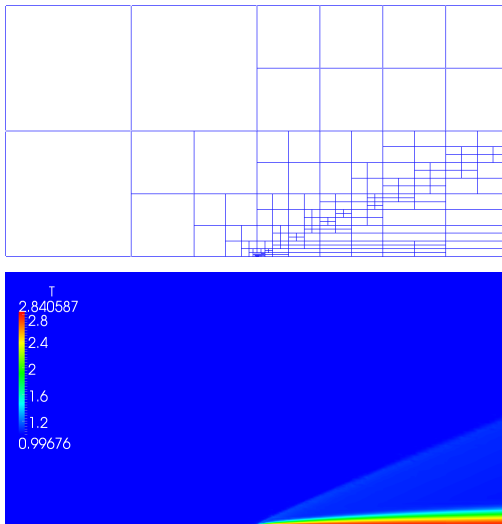


Figure : $Re = 10,000$, $p = 2$, refinement number 9.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

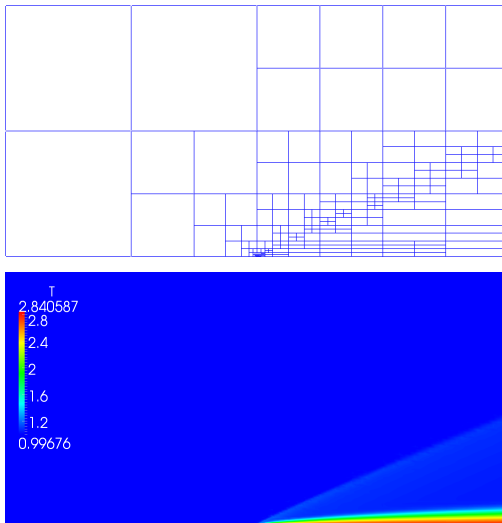


Figure : $Re = 10,000$, $p = 2$, refinement number 10.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

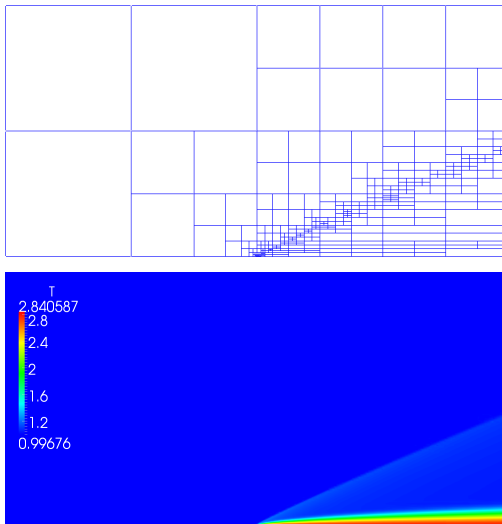


Figure : $Re = 10,000$, $p = 2$, refinement number 11.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

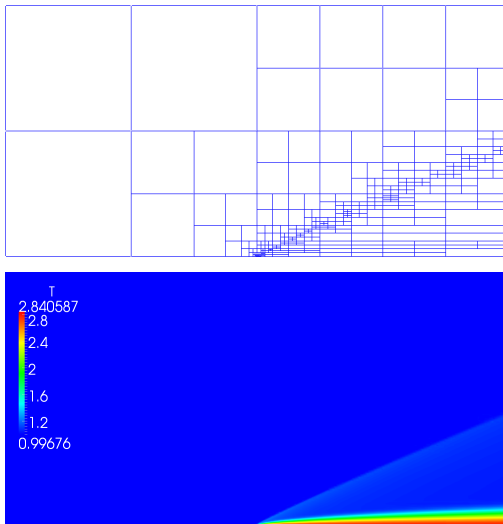


Figure : $Re = 10,000$, $p = 2$, refinement number 12.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

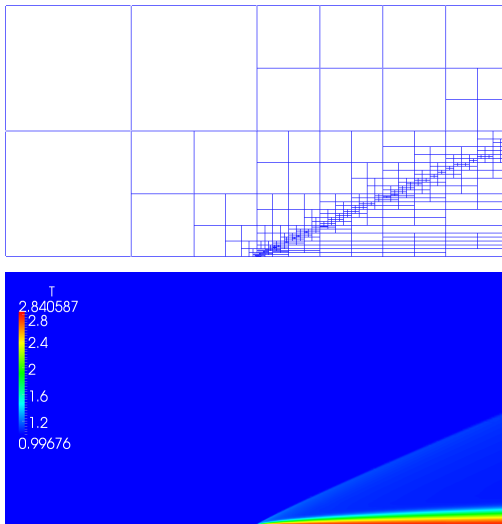


Figure : $Re = 10,000$, $p = 2$, refinement number 13.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

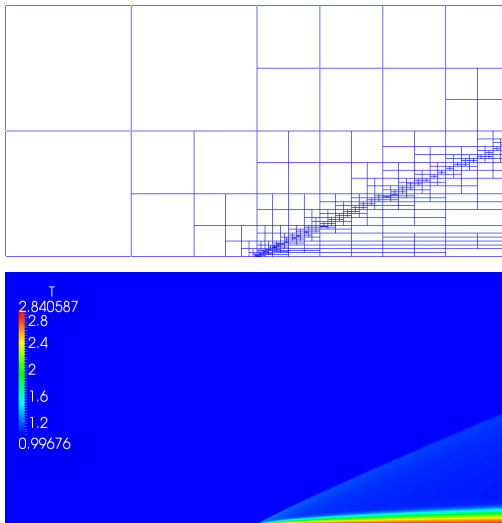


Figure : $Re = 10,000$, $p = 2$, refinement number 14.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

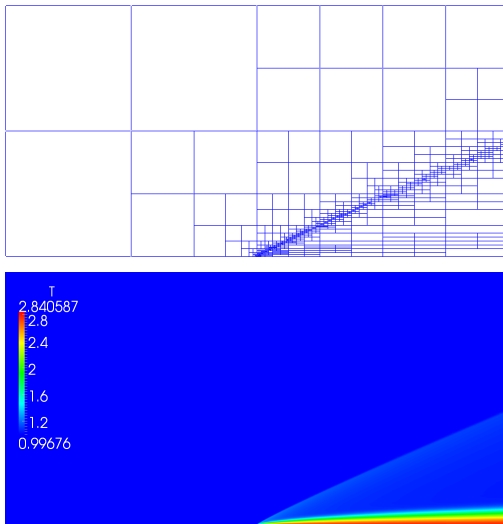


Figure : $Re = 10,000$, $p = 2$, refinement number 15.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

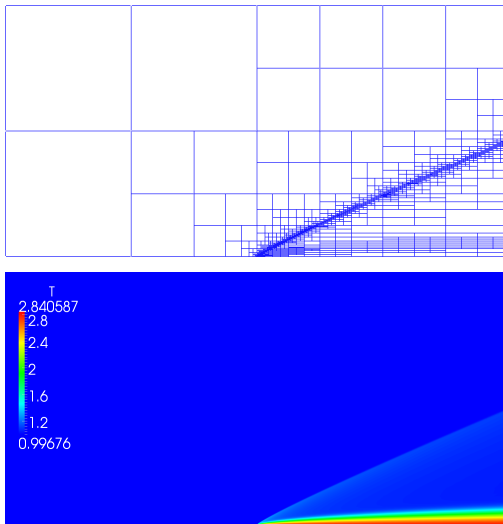


Figure : $Re = 10,000$, $p = 2$, refinement number 16.

From Ph.D. Dissertation of Jesse Chan: Compressible Navier-Stokes Equations: Holden ramp problem^{3 4}

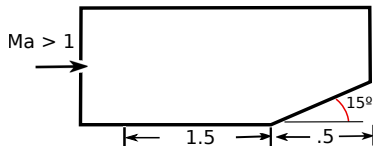


Figure : $M_\infty = 6$, $Re_{plate} = 33,936$, $Pr = 0.72$, $\gamma = 1.4$, $\theta_\infty = 390^\circ[R]$

³L.D., J.T. Oden, W. Rachowicz, "A New Finite Element Method for Solving Compressible Navier-Stokes Equations Based on an Operator Splitting Method and h/p Adaptivity," *Comput. Methods Appl. Mech. Engrg.*, **84**, 275-326, 1990.

⁴J. Chan, L.D., R. Moser, "A DPG method for steady viscous compressible flow," *Computers and Fluids*, to appear.

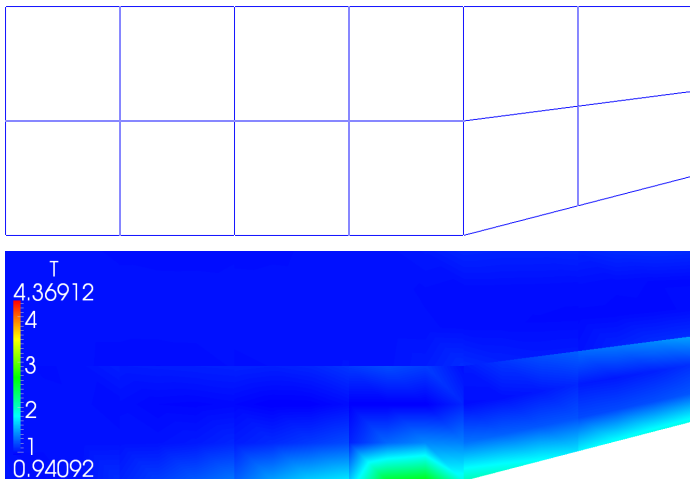


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, initial mesh.

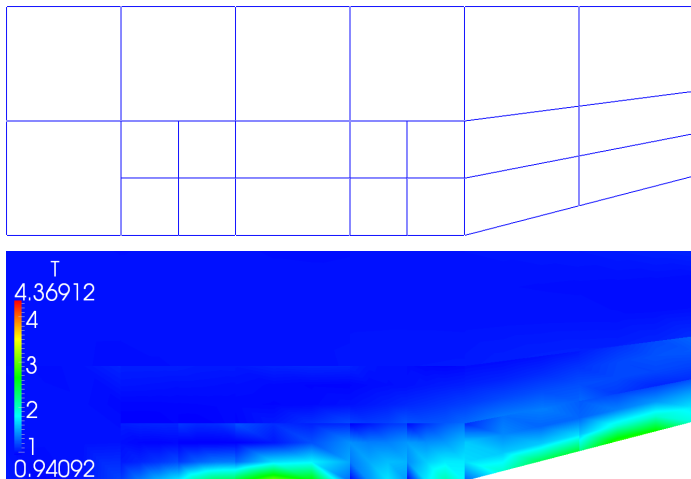


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 1.

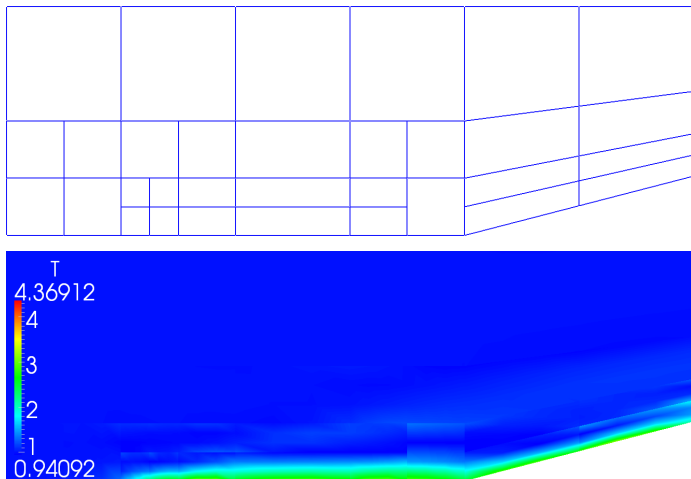


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 2.

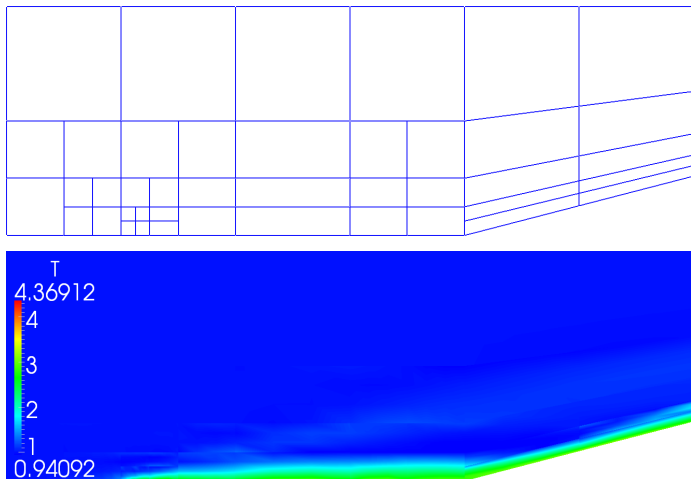


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 3.

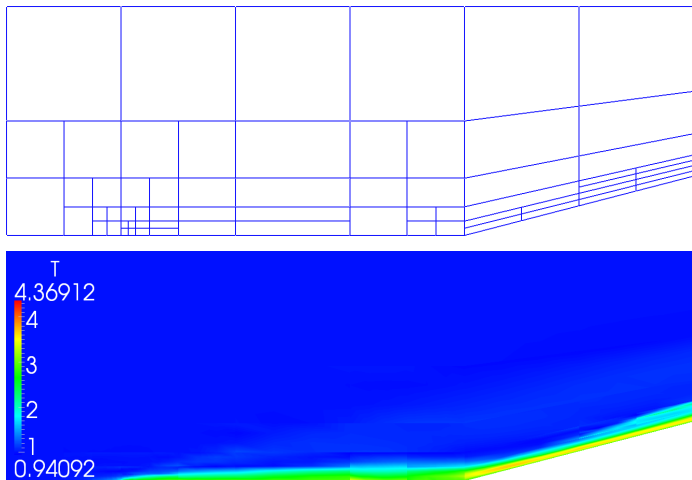


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 4.

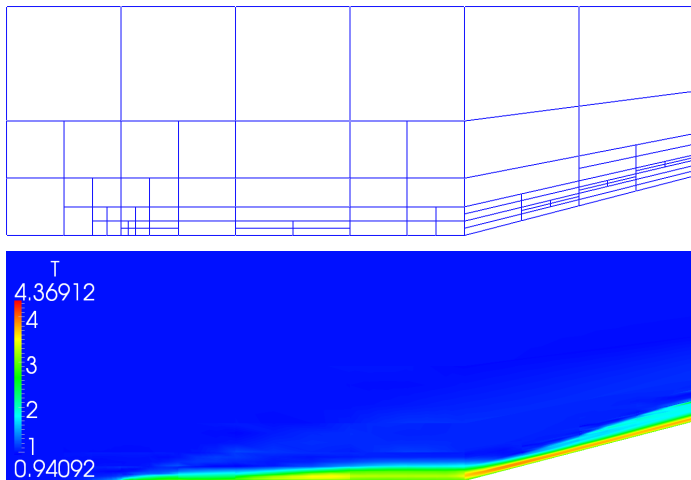


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 5.

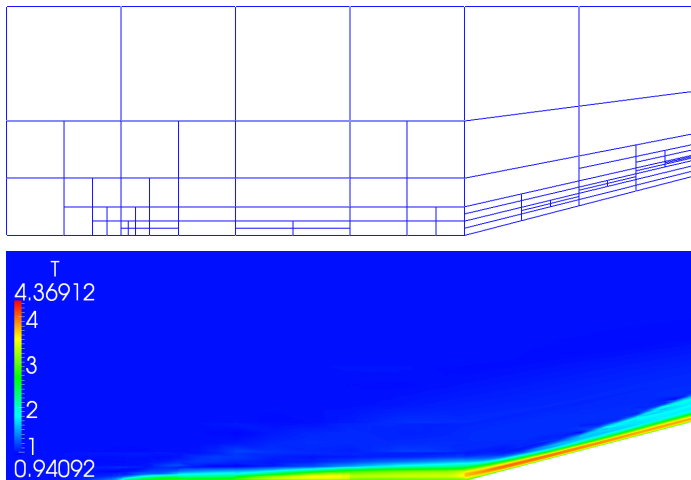


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 6.

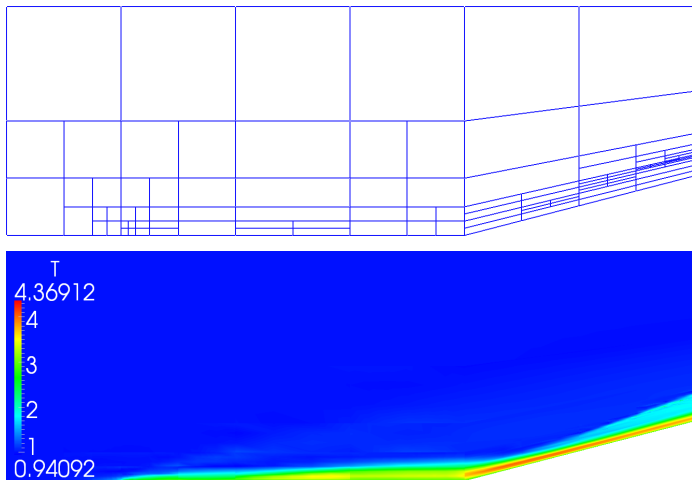


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 7.

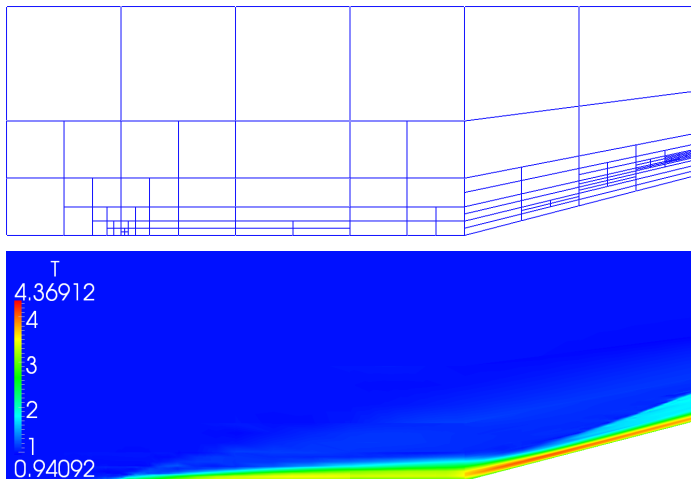


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 8.

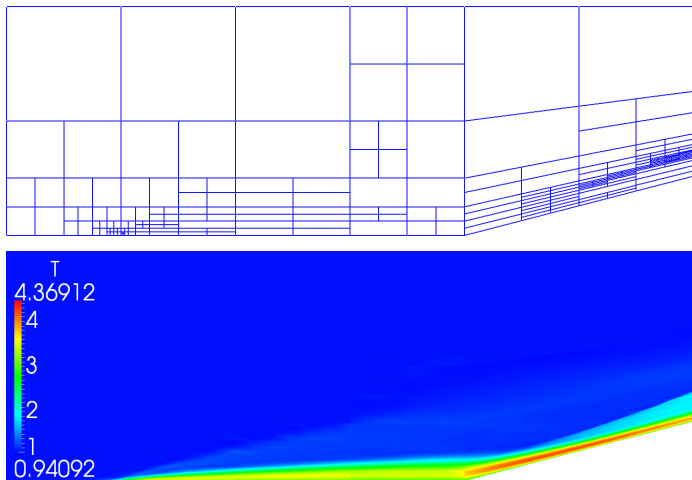


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 9.

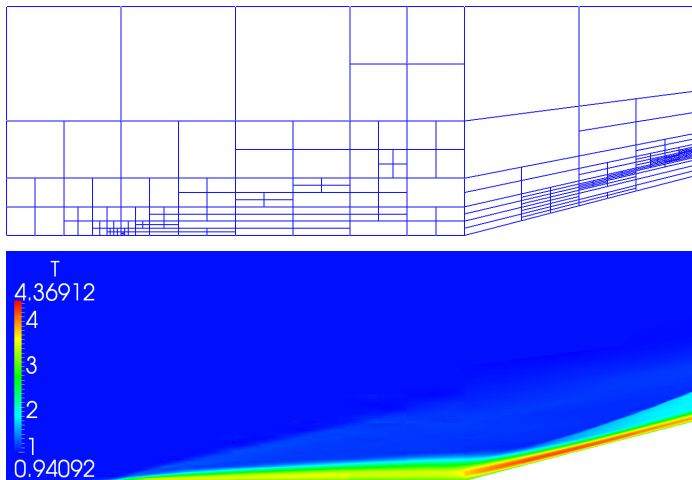


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 10.

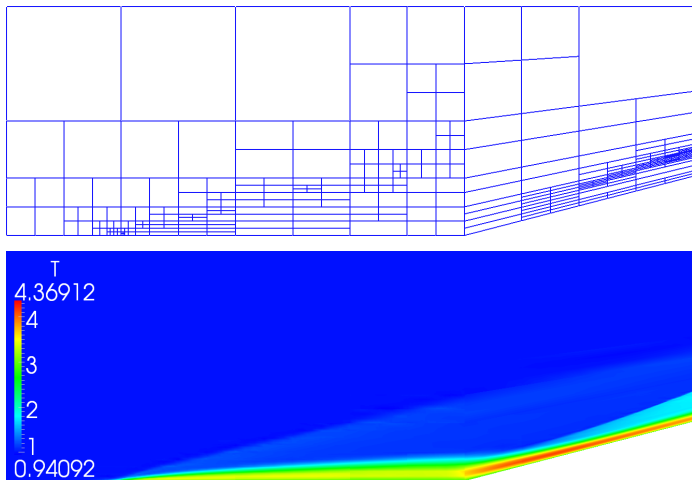


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 11.

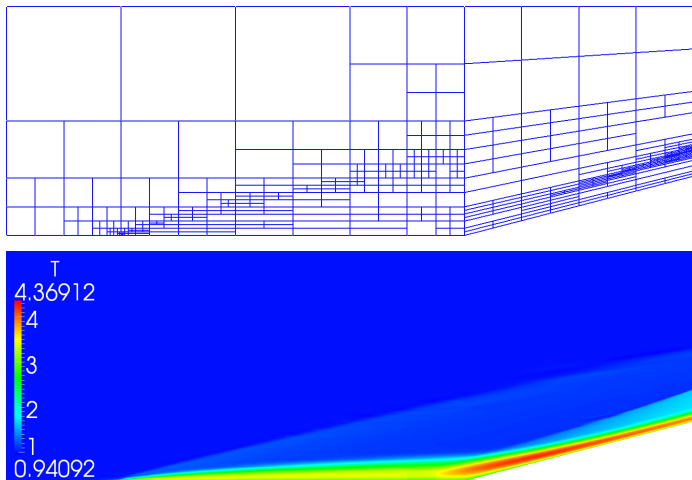


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 12.

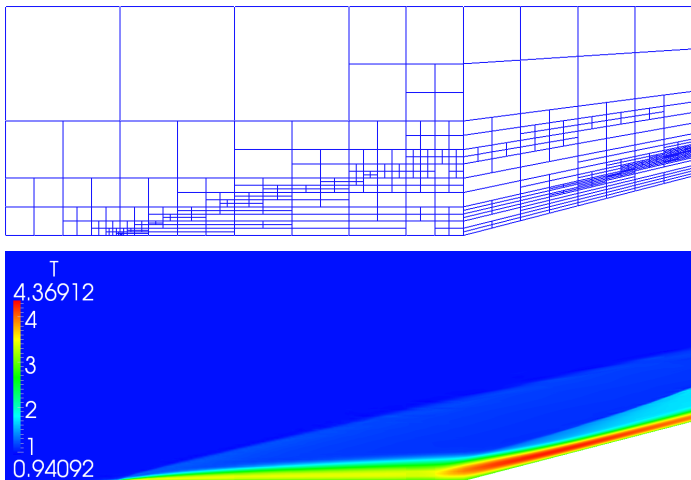


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 13.

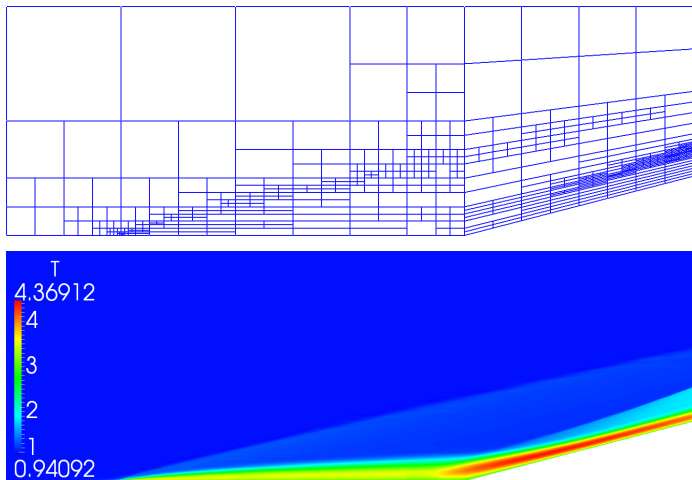


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 14.

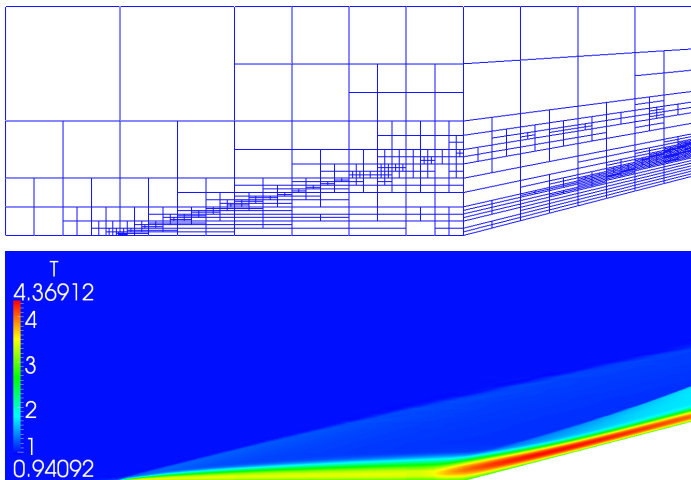


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 15.

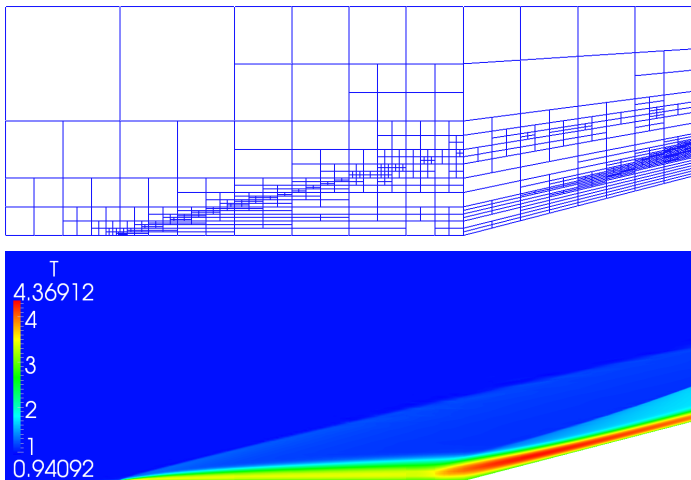


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 16.

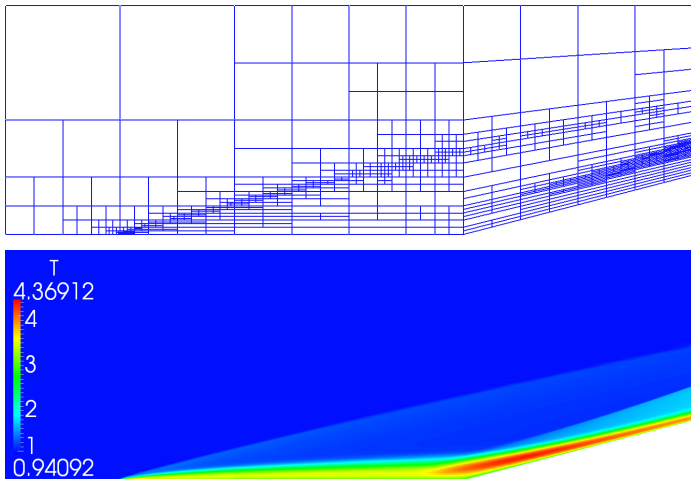


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 17.

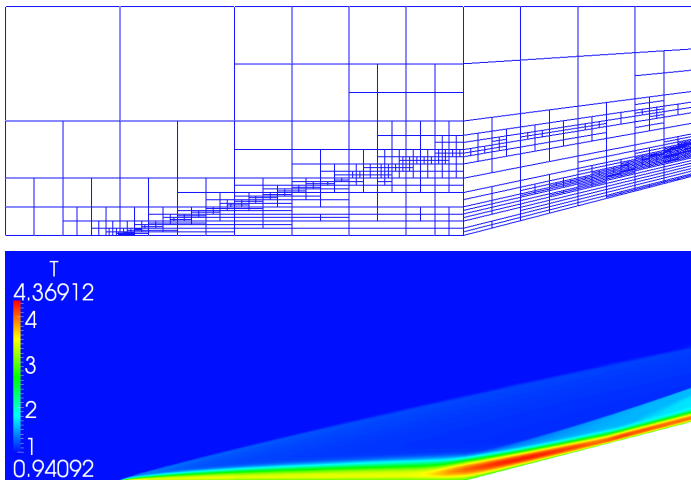


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 18.

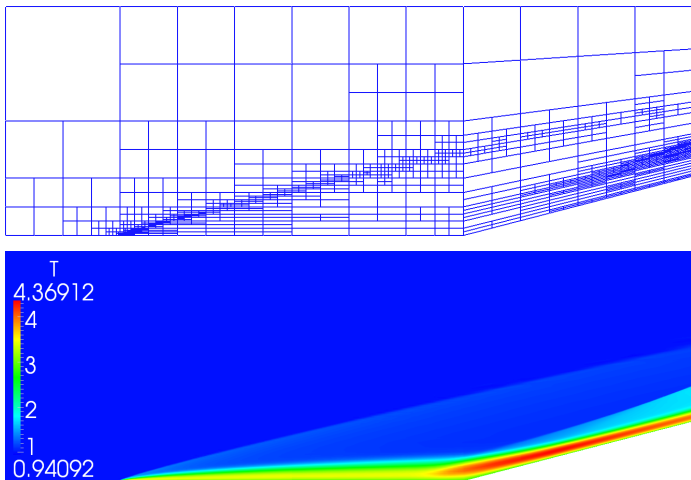


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 19.

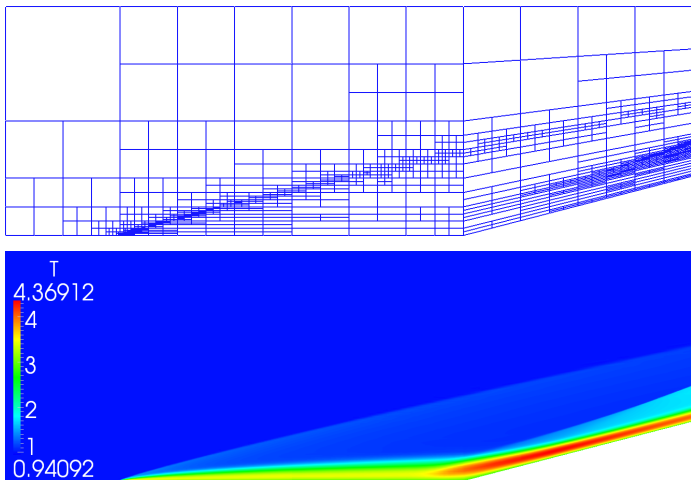


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 20.

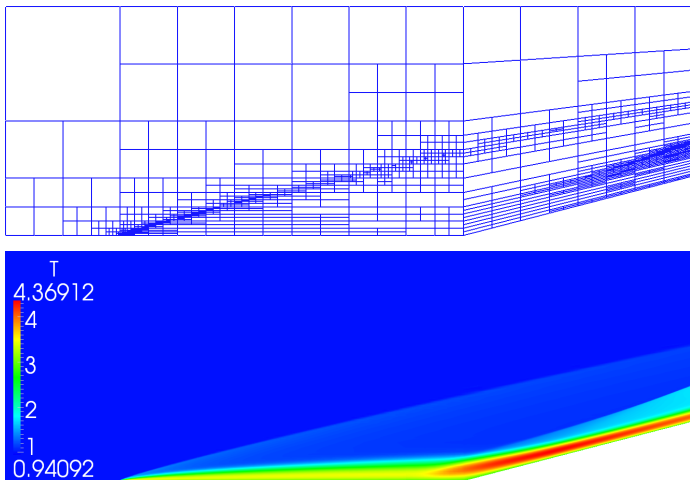


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 21.

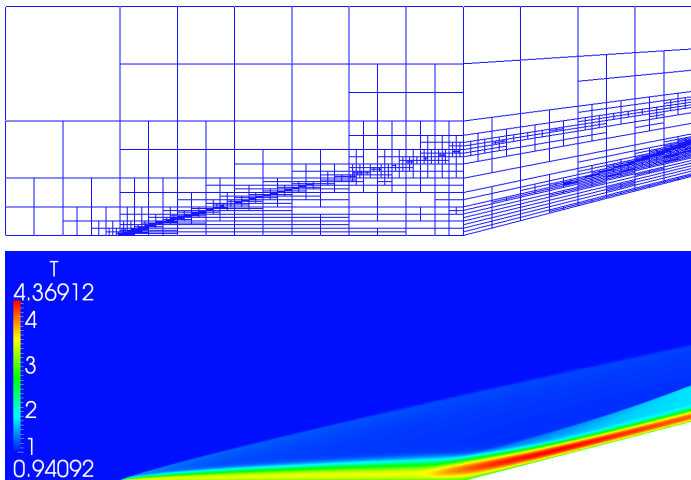


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 22.

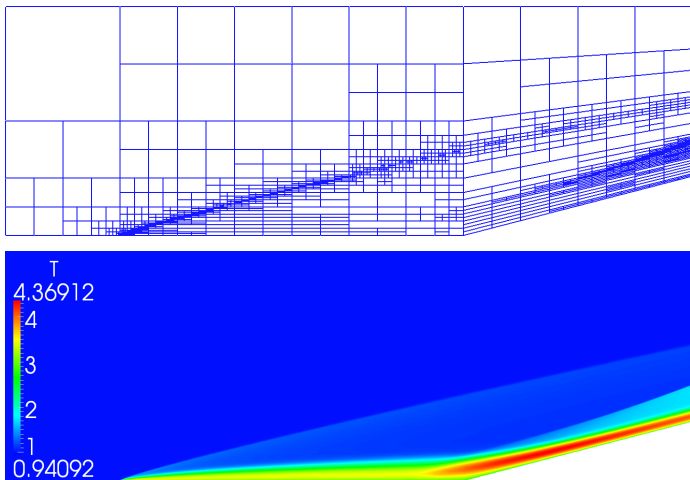


Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 23.

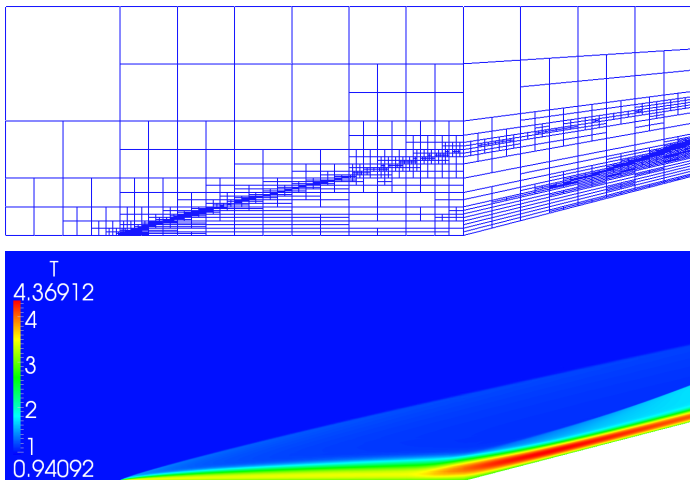


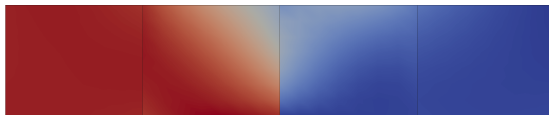
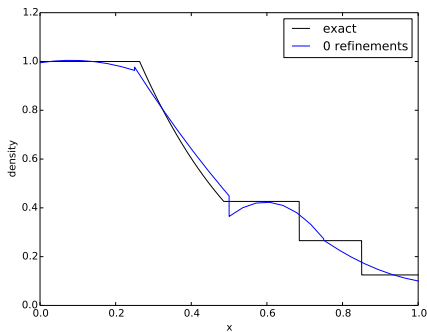
Figure : Mach 6, $Re \approx 34,000$, $p = 2$, refinement number 24.

**From Ph.D. Proposal of Truman Ellis:
Space-Time Compressible Navier-Stokes Equations:
Sod Shock Tube Problem⁵**

⁵G.A. Sod, "A Survey of Several Finite Difference Methods for Systems of Nonlinear Hyperbolic Conservation Laws", *J. Comp. Methods Phys.*, **27**, 1-31, 1978.

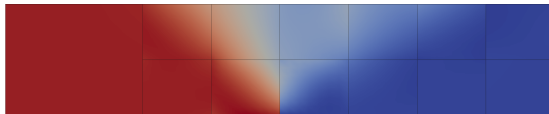
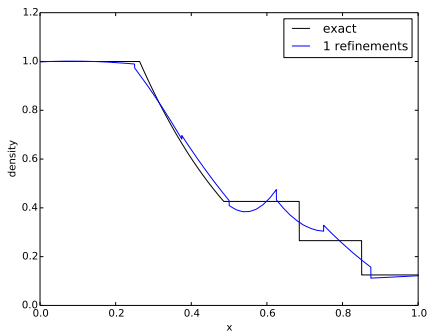
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



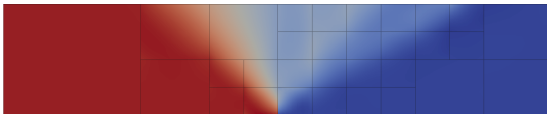
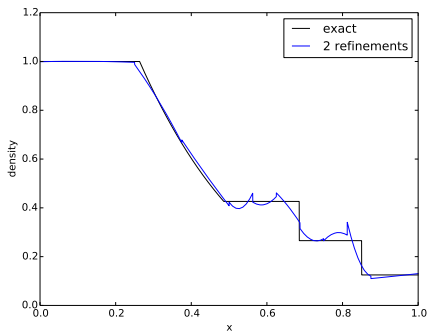
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



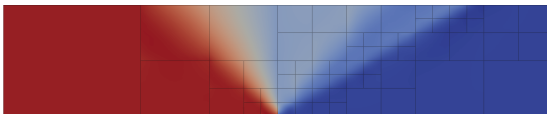
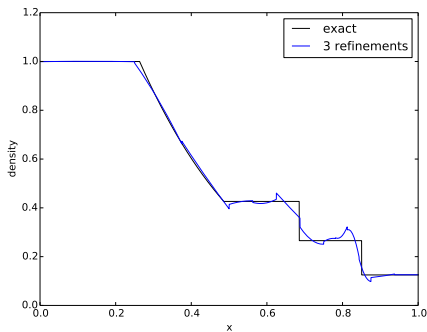
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



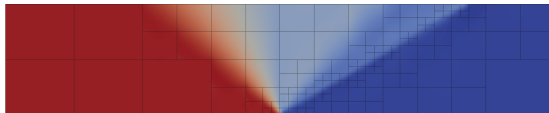
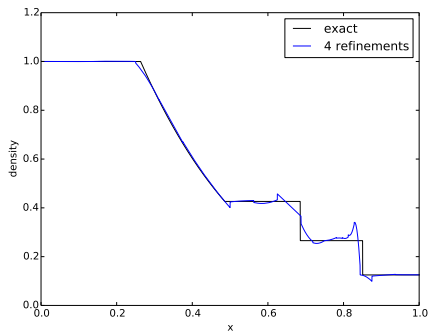
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



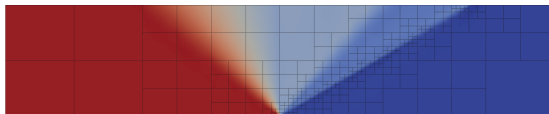
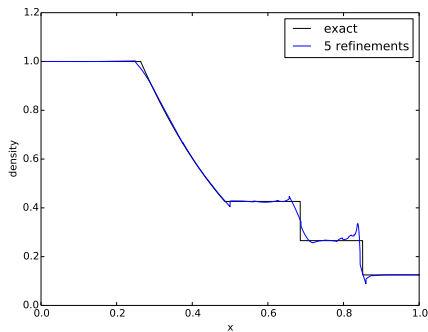
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



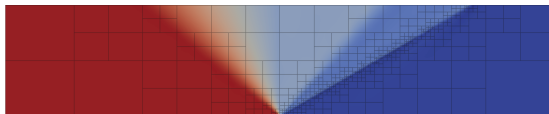
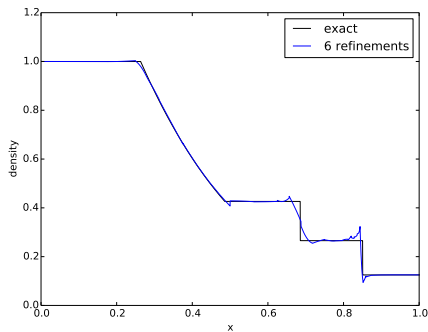
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



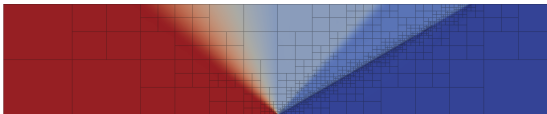
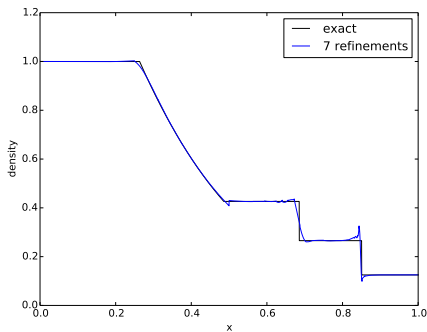
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



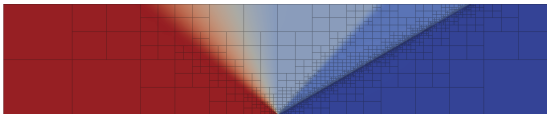
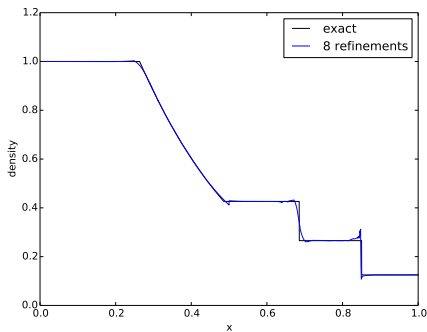
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



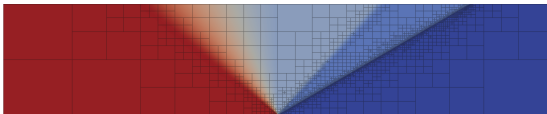
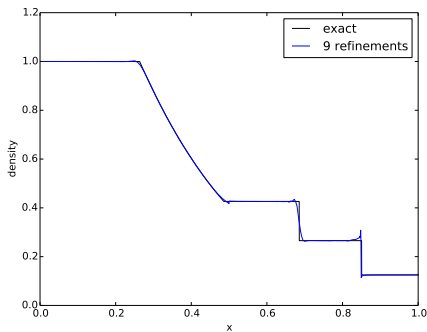
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



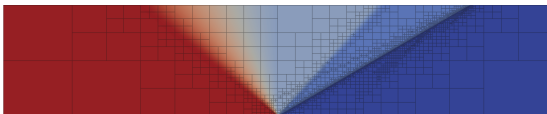
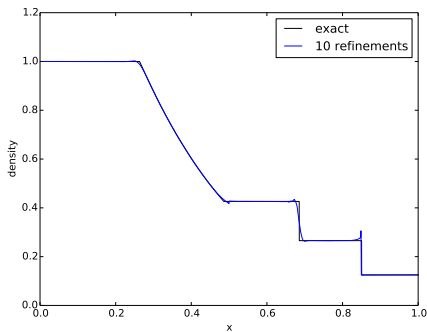
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



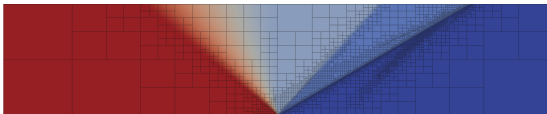
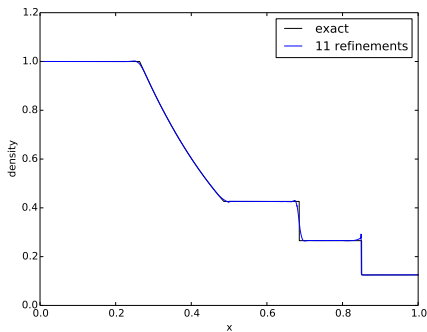
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



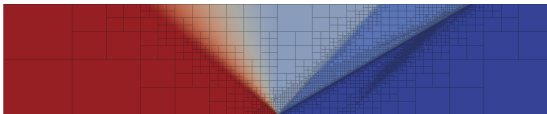
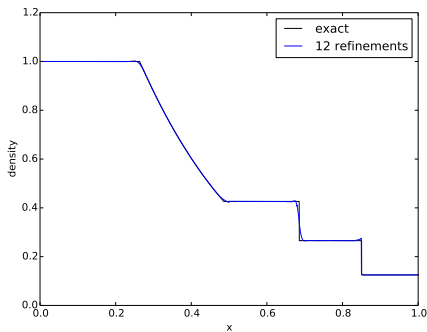
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



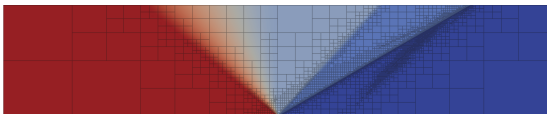
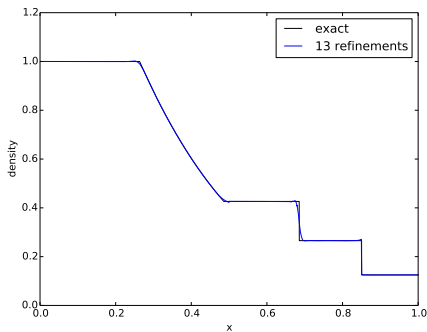
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



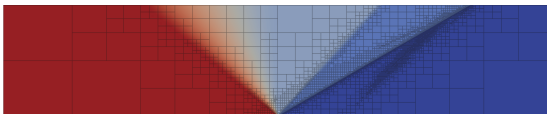
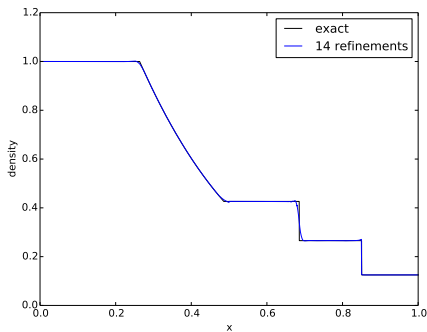
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

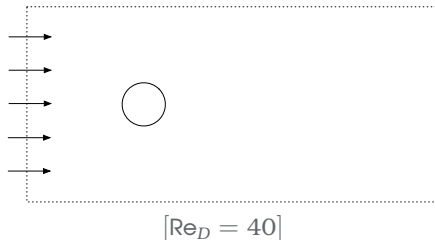


Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

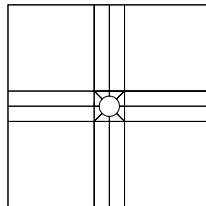


From Ph.D. Dissertation of Nathan V. Roberts: Incompressible Navier-Stokes: Flow Past a Cylinder⁶

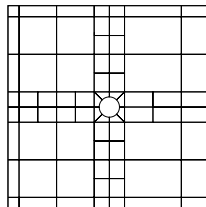
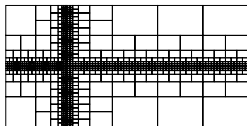


⁶L.S.G. Kovaszny, "Hot-wire investigation of the wake behind cylinders at low Reynolds numbers." *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, **198**(1053):174-190, 1949.

Domain: 240×120 cylinder diameters. Begin with a $p = 3$ preliminary mesh that simply captures geometry (detail):

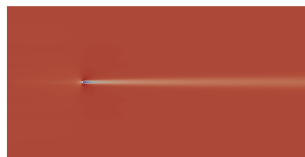
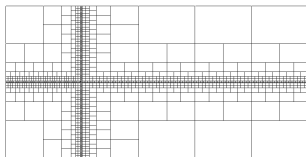


Perform initial anisotropic refinements to get aspect ratios below 2:

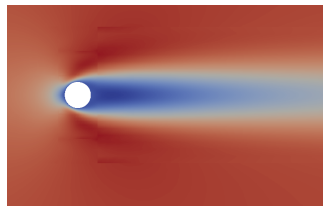
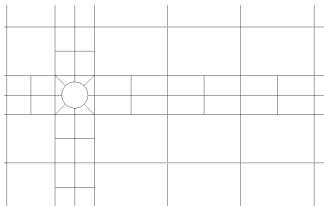


Incompressible Navier-Stokes

Initial Mesh ($p = 3$), and horizontal velocity solution:

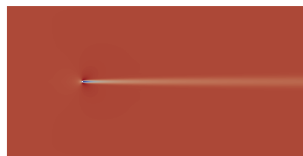
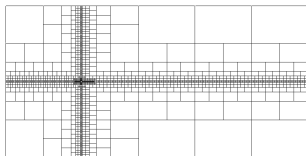


Mesh detail, and horizontal velocity solution

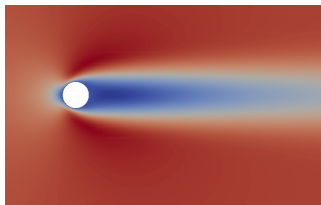
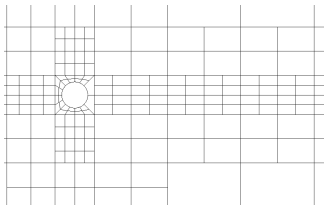


Relative energy error of solution: 9.76×10^{-2} .

Mesh after 1 h -refinement, and horizontal velocity solution:

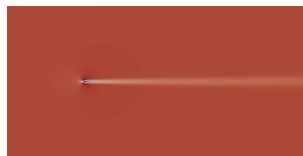
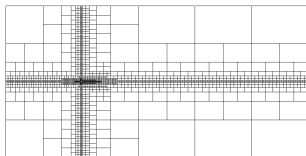


Mesh detail, and horizontal velocity solution

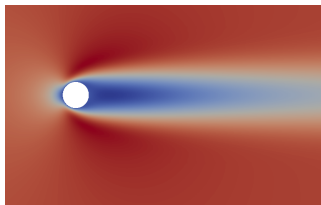
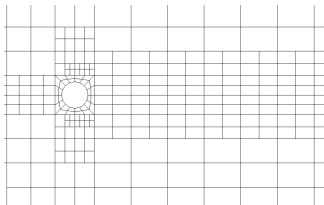


Relative energy error of solution: 1.81×10^{-2} .

Mesh after 2 h -refinements, and horizontal velocity solution:

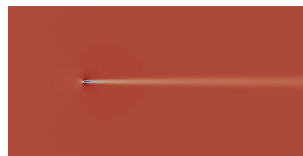
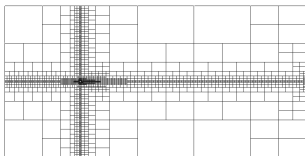


Mesh detail, and horizontal velocity solution

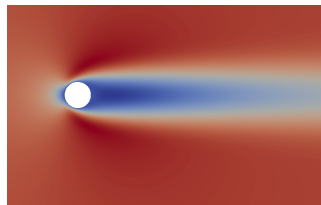
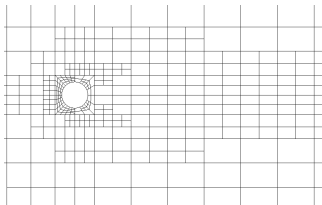


Relative energy error of solution: 3.26×10^{-3} .

Mesh after 3 h -refinements, and horizontal velocity solution:

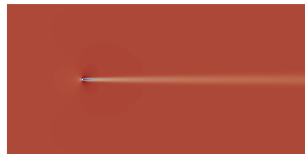
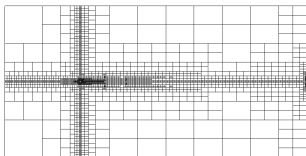


Mesh detail, and horizontal velocity solution

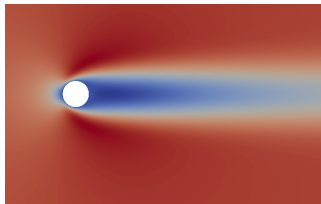
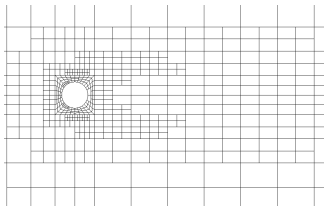


Relative energy error of solution: 1.29×10^{-3} .

Mesh after 4 h -refinements, and horizontal velocity solution:



Mesh detail, and horizontal velocity solution



Relative energy error of solution: 5.74×10^{-4} .

Chan, Ellis, and Roberts results were computed using Camellia.⁷

Camellia:

- is easy to use (Stokes cavity flow h -adaptive driver takes ≈ 130 lines of code)
- supports MPI, h - and p - adaptivity
- built atop Trilinos
- supports 1-3D spatial meshes (1D and 3D just added)
- space-time meshes coming soon
- open source release planned for 2015

⁷ Nathan V. Roberts. "Camellia: A software framework for discontinuous Petrov-Galerkin methods." *Computers & Mathematics with Applications*, 2014 (In review).

Three DPG Punchlines

- 1 DPG Method is a Ritz method. It supports adaptivity with no preasymptotic behavior.
- 2 You can control the norm in which you want to converge.
- 3 DPG is easy to code.

The simplest singular perturbation problem: reaction-dominated diffusion⁸

⁸L.D. and I. Harari, “Primal DPG Method for Reaction dominated Diffusion”, in preparation.

The simplest singular perturbation problem:

Reaction-dominated diffusion

$$\begin{cases} u = 0 & \text{on } \Gamma \\ -\epsilon^2 \Delta u + c(x)u = f & \text{in } \Omega \end{cases}$$

where $0 < c_0 \leq c(x) \leq c_1$.

Standard variational formulation:

$$\begin{cases} u \in H^1(\Omega) \\ \epsilon^2 (\nabla u, \nabla v) + (cu, v) = (f, v) \quad v \in H^1(\Omega) \end{cases}$$

Standard Galerkin method delivers the best approximation error in the energy norm:

$$\|u\|_{\epsilon^k}^2 := \epsilon^k \|\nabla u\|^2 + \|c^{1/2}u\|^2, \quad k = 2$$

Fact: Under favorable regularity conditions, the solution is *uniformly* bounded in data f in a “balanced” norm⁹:

$$\|u\|_{\epsilon}^2 := \epsilon \|\nabla u\|^2 + \|c^{1/2}u\|^2$$

i.e.

$$\|u\|_{\epsilon} \lesssim \|f\|_{\text{appropriate}}$$

Question: Can we select the test norm in such a way that the DPG method will be *robust* in the balanced norm?

$$\|u - u_h\|_{\epsilon} + \|\hat{t} - \hat{t}_h\|_? \lesssim \inf_{w_h} \|u - w_h\|_{\epsilon} + \inf_{\hat{r}_h} \|\hat{t} - \hat{r}_h\|_?$$

⁹R. Lin and M. Stynes, “A balanced finite element method for singularly perturbed reaction-diffusion problems”, *SIAM J. Numer. Anal.*, **50**(5): 2729–2743, 2012.

A bit of history:

Optimal test functions of Barret and Morton^{10 11}

For each $w \in U_h$, determine the corresponding v_w that solves the auxiliary variational problem:

$$\begin{cases} v_w \in H_0^1(\Omega) \\ \underbrace{\epsilon^2(\nabla \delta u, \nabla v_w) + (c \delta u, v_w)}_{\text{the bilinear form we have}} = \underbrace{\epsilon(\nabla \delta u, w) + (c \delta u, w)}_{\text{the bilinear form we want}} \quad \forall \delta u \in H_0^1(\Omega) \end{cases}$$

With the optimal test functions, the Galerkin orthogonality for the original form changes into Galerkin orthogonality in the desired, “balanced” norm:

$$\epsilon^2(\nabla(u - u_h), \nabla v_w) + (c(u - u_h), v_w) = 0 \quad \implies \quad \epsilon(\nabla(u - u_h), \nabla v_u) + (c(u - u_h), w) = 0$$

Consequently, the PG solution delivers the best approximation error in the desired norm.

¹⁰ J.W. Barret and K. W. Morton, “Approximate Symmetrization and Petrov-Galerkin Methods for Diffusion-Convection Problems”, *Comp. Meth. Appl. Mech and Engng.*, **46**, 97 (1984).

¹¹ L. D. and J. T. Oden, “An Adaptive Characteristic Petrov-Galerkin Finite Element Method for Convection-Dominated Linear and Nonlinear Parabolic Problems in One Space Variable”, *Journal of Computational Physics*, **68**(1): 188–273, 1986.

Theorem

Let v_u be the Barret-Morton optimal test function corresponding to u . Let $\|v_u\|_V$ be a test norm such that

$$\|v_u\|_V \lesssim \|u\|_\epsilon$$

Then

$$\|u - u_h\|_\epsilon \lesssim \|u - u_h\|_E = \inf_{w_h \in U_h} \|u - w_h\|_E \leq \text{BAE estimate}$$

Proof:

$$\begin{aligned} \|u\|_\epsilon^2 &= \epsilon(\nabla u, \nabla u) + (cu, u) = \epsilon^2(\nabla u, \nabla v_u) + (cu, v_u) \\ &= b((u, \hat{t}), v_u) \leq \frac{b((u, \hat{t}), v_u)}{\|v_u\|_V} \|v_u\|_V \\ &\leq \sup_v \frac{b((u, \hat{t}), v_u)}{\|v\|_V} \|v_u\|_V = \|(u, \hat{t})\|_E \|v_u\|_V \\ &\lesssim \|(u, \hat{t})\|_E \|u\|_\epsilon \end{aligned}$$

¹²L. D., M. Heuer, "Robust DPG Method for Convection-Dominated Diffusion Problems," *SIAM J. Num. Anal.* **51**: 2514–2537, 2013.

The point: Construction of the optimal test norm is reduced to the stability (robustness) analysis for the Barret-Morton test functions.

Lemma

Let

$$\|v\|_V^2 := \epsilon^3 \|\nabla v\|^3 + \|c^{1/2}v\|^2$$

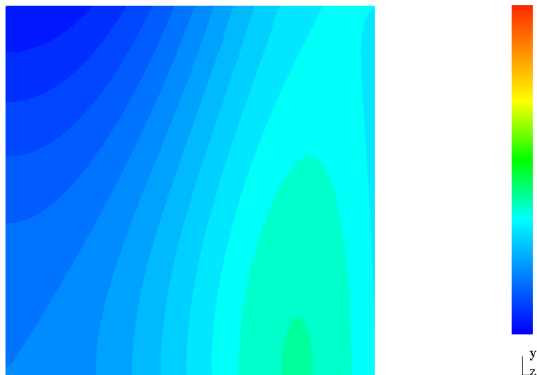
Then

$$\|v_u\| \lesssim \|u\|_\epsilon$$

In order to avoid boundary layers in the optimal test functions (that we cannot resolve using simple enriched space) we scale the reaction term with a mesh-dependent factor:

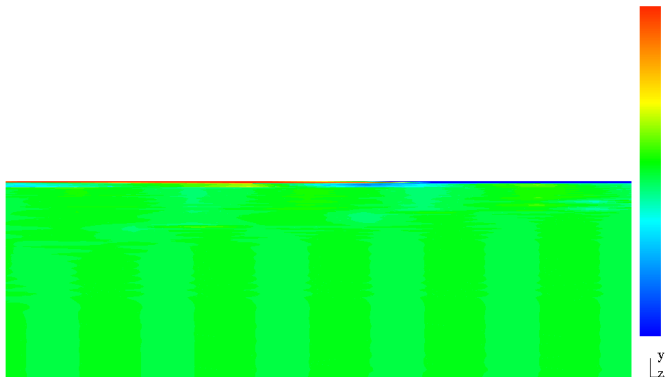
$$\|v\|_{V,mod}^2 := \epsilon^3 \|\nabla v\|^3 + \min\left\{1, \frac{\epsilon^3}{h^2}\right\} \|c^{1/2}v\|^2$$

Manufactured solution of Lin and Stynes, $\epsilon = 10^{-1}$



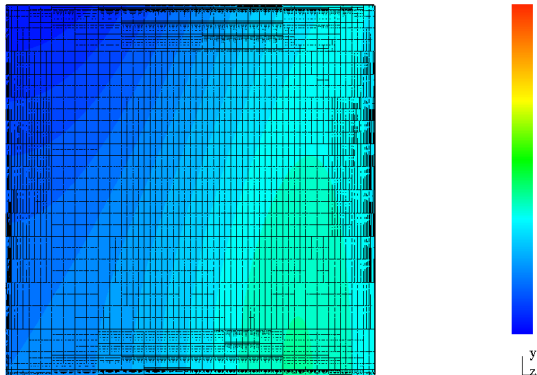
The function exhibits strong boundary layers invisible in this scale.

Range: $(-0.6, 0.6)$

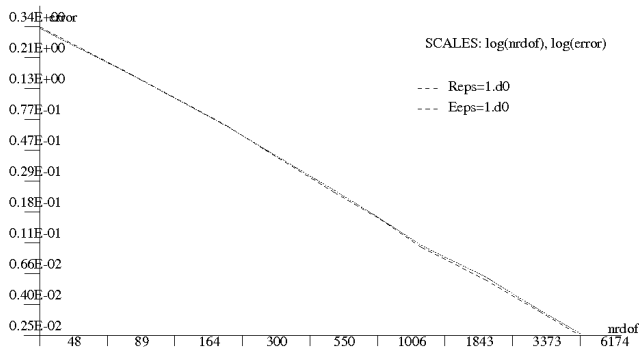


Zoom on the north boundary layer.

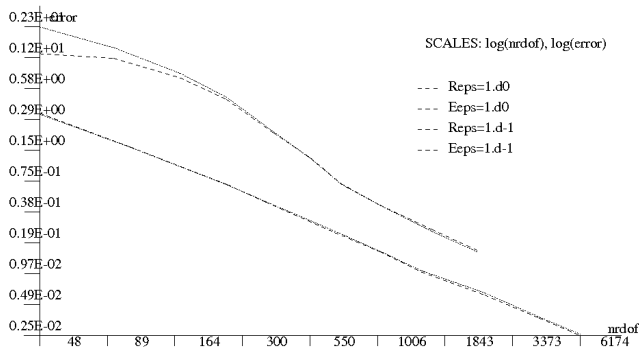
Optimal mesh for $\epsilon = 10^{-1}$



Optimal h -adaptive mesh and numerical solution for $\epsilon = 10^{-1}$

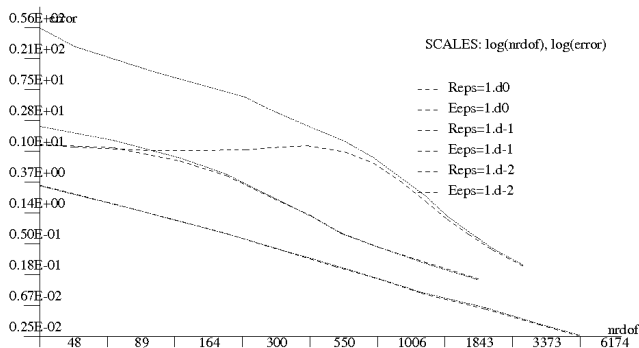


Residual and “balanced” error of u for h -adaptive solution, $p = 2$



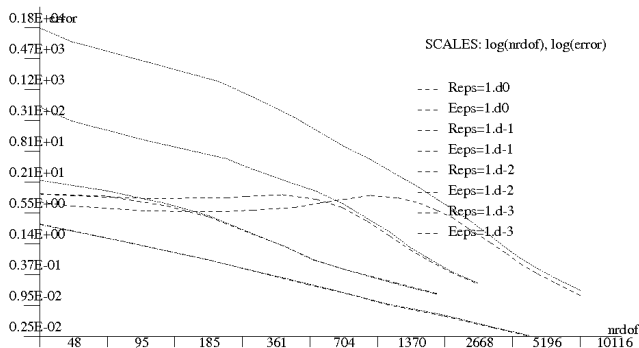
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Lin/Stynes example, $\epsilon = 10^0, 10^{-1}, 10^{-2}$.

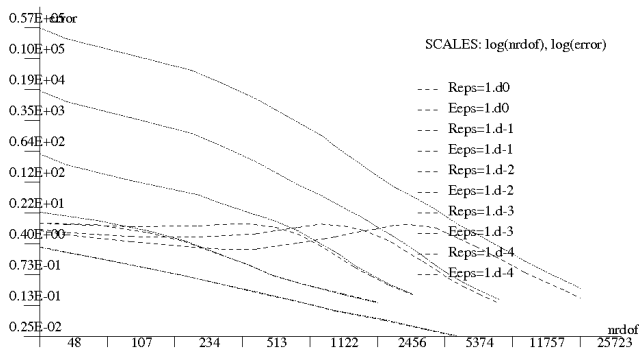


Residual and “balanced” error of u for h -adaptive solution, $p = 2$

Lin/Stynes example, $\epsilon = 10^0, 10^{-1}, 10^{-2}, 10^{-3}$.



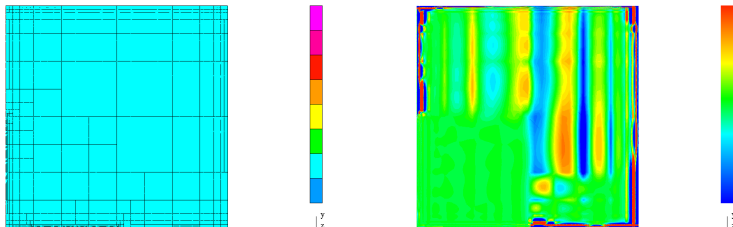
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Question: Can we select the test norm in such a way that the DPG method would deliver high accuracy in a preselected subdomain, e.g. $(0, \frac{1}{2})^2 \subset (0, 1)^2$?

Answer: Yes!



Optimal mesh and the corresponding pointwise error (range $(-0.001 - 0.001)$).

Three DPG Punchlines

- 1 DPG Method is a Ritz method. It supports adaptivity with no preasymptotic behavior.
- 2 You can control the norm in which you want to converge.
- 3 DPG is easy to code.

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