Robustness for Transient Problems

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July 29, 2014

Consider domain $Q = \Omega \times [0, T]$ with boundary $\Gamma = \Gamma_- \cup \Gamma_+ \cup \Gamma_0 \cup \Gamma_T$ where Γ_- is the spatial inflow boundary, Γ_+ is the spatial outflow boundary, Γ_0 is the initial time boundary, and Γ_T is the final time boundary. Let Γ_h denote the entire mesh skeleton.

Assume that boundary conditions are applied on the boundary $\Gamma_0 \subset \Gamma$. Recall that, for the ultra-weak variational formulation

$$b\left(\left(u, \widehat{u}\right), v\right) = \left(u, A_h^* v\right)_{L^2(\Omega)} + \langle \widehat{u}, [\![v]\!] \rangle_{\Gamma_h \setminus \Gamma_0}$$

we can recover

$$\|u\|^2_{L^2(Q)} = b(u, v^*)$$

for conforming v^* satisfying the adjoint equation

$$\begin{aligned} A^* v^* &= u \\ v^* &= 0 \text{ on } \Gamma_h \setminus \Gamma_0. \end{aligned}$$

Together, these give necessary conditions on the test norm $\|\cdot\|_V$ such that we have L^2 robustness (this gives robustness in the variable u; for the first order formulation, conditions for σ must also be shown).

$$\|u\|^2_{L^2(Q)} = b(u, v^*) \leq \frac{b(u, v^*)}{\|v^*\|_V} \|v^*\|_V \leq \|u\|_E \|v^*\|_V$$

Thus, showing $\|v^*\|_V \lesssim \|u\|_{L^2(Q)}$ gives the result that $\|u\|_{L^2(Q)} \lesssim \|u\|_E$.

1 Convection-Diffusion

Consider convection-diffusion

$$\begin{aligned} \frac{1}{\epsilon} \boldsymbol{\sigma} - \nabla u &= 0 \\ \frac{\partial u}{\partial t} + \boldsymbol{\beta} \cdot \nabla u - \nabla \cdot \boldsymbol{\sigma} &= f \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_- \\ u &= 0 \text{ on } \Gamma_+ \\ u &= u_0 \text{ on } \Gamma_0. \end{aligned}$$

Let $\tilde{\boldsymbol{\beta}} := \begin{pmatrix} \boldsymbol{\beta} \\ 1 \end{pmatrix}$ and $\nabla_{xt} := \begin{pmatrix} \nabla \\ \frac{\partial}{\partial t} \end{pmatrix}$, then we can rewrite this as

$$\begin{aligned} \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= 0 \\ \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} u - \nabla \cdot \boldsymbol{\sigma} &= f \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_- \\ u &= 0 \text{ on } \Gamma_+ \\ u &= u_0 \text{ on } \Gamma_0 . \end{aligned}$$

We decompose the adjoint into three parts: a discontinuous part

$$\begin{aligned} \frac{1}{\epsilon}\boldsymbol{\tau}_0 + \nabla v_0 &= 0 \\ -\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_0 + \nabla \cdot \boldsymbol{\tau}_0 &= 0 \\ \boldsymbol{\tau}_0 \cdot \boldsymbol{n}_x &= \boldsymbol{\tau} \cdot \boldsymbol{n}_x \text{ on } \Gamma_- \cup \Gamma_0 \\ v_0 &= v \text{ on } \Gamma_+ \\ v_0 &= v \text{ on } \Gamma_T \\ [\![v_0]\!] &= [\![v]\!] \text{ on } \Gamma_h^0 \\ [\![\boldsymbol{\tau}_0 \cdot \boldsymbol{n}_x]\!] &= [\![\boldsymbol{\tau}_0 \cdot \boldsymbol{n}_x]\!] \text{ on } \Gamma_{hx}^0 , \end{aligned}$$

a continuous part with forcing term g

$$\begin{aligned} \frac{1}{\epsilon}\boldsymbol{\tau}_1 + \nabla v_1 &= 0 \\ -\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1 + \nabla \cdot \boldsymbol{\tau}_1 &= g \\ \boldsymbol{\tau}_1 \cdot \boldsymbol{n}_x &= 0 \text{ on } \Gamma_- \\ v_1 &= 0 \text{ on } \Gamma_+ \\ v_1 &= 0 \text{ on } \Gamma_T , \end{aligned}$$

and a continuous part with forcing f

$$\begin{aligned} \frac{1}{\epsilon}\boldsymbol{\tau}_2 + \nabla v_2 &= f \\ -\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_2 + \nabla \cdot \boldsymbol{\tau}_2 &= 0 \\ \boldsymbol{\tau}_2 \cdot \boldsymbol{n}_x &= 0 \text{ on } \Gamma_- \\ v_2 &= 0 \text{ on } \Gamma_+ \\ v_2 &= 0 \text{ on } \Gamma_T . \end{aligned}$$

(The boundary conditions can be derived by taking the ultra-weak formulation and choosing boundary conditions such that the temporal flux and spatial flux terms $\langle \widehat{u}, [\![\tau_n]\!] \rangle_{\Gamma_{out}}$ and $\left\langle \widehat{f}_n, [\![v]\!] \right\rangle_{\Gamma_{in}}$ are zero.)

We can then derive that the test norm

$$\|(v,\boldsymbol{\tau})\|_{V,K}^2 \coloneqq \min\left\{\frac{1}{\epsilon},\frac{1}{\mu(K)}\right\}\|\boldsymbol{\tau}\|_K^2 + \left\|\nabla\cdot\boldsymbol{\tau} - \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\right\|_K^2$$
$$+\left\|\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\right\|_K^2 + \epsilon\|\nabla v\|_K^2 + \|v\|_K^2\,, \tag{1}$$

Our goal is to analyze the stability properties of the adjoint equations by deriving bounds of the form $\|(v_1,\boldsymbol{\tau}_1)\|_V \leq \|g\|_L^2(Q)$ and $\|(v_2,\boldsymbol{\tau}_2)\|_V \leq \|f\|_L^2(Q)$. provides the necessary bound $\|v^*\|_V \lesssim \|u\|_{L^2(Q)}$.

Insert conditions on $\boldsymbol{\beta}$

Lemma 1.1. *For the above conditions on* $\boldsymbol{\beta}$*,*

$$\left\|\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v_1\right\| \leq \|g\|$$

and since $-\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v_1 + \nabla\cdot\boldsymbol{\tau}_1 = g$*,*

$$\|\nabla\cdot\boldsymbol{\tau}_1\| \leq \|g\|\,.$$

Proof. Multiply by $-\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v$ and integrate over Q to get

$$\left\|\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\right\| = -\int_Q u\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v - \epsilon\int_Q\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\Delta v\,. \tag{2}$$

Note that

$$
\begin{aligned}
-\int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \Delta v &= -\int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla \cdot \nabla v \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \int_Q \nabla(\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v) \cdot \nabla v \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \int_Q (\nabla \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v) \cdot \nabla v \\
&\quad + \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla \nabla_{xt} v \cdot \nabla v \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \int_Q (\nabla \boldsymbol{\beta} \cdot \nabla v) \cdot \nabla v \\
&\quad + \frac{1}{2} \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} (\nabla v \cdot \nabla v) \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \int_Q (\nabla \boldsymbol{\beta} \cdot \nabla v) \cdot \nabla v \\
&\quad + \frac{1}{2} \int_\Gamma \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n} (\nabla v \cdot \nabla v) - \frac{1}{2} \int_Q \nabla_{xt} \cdot \tilde{\boldsymbol{\beta}} (\nabla v \cdot \nabla v) \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \int_Q (\nabla \boldsymbol{\beta} \cdot \nabla v) \cdot \nabla v \\
&\quad + \frac{1}{2} \int_\Gamma \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n} (\nabla v \cdot \nabla v) - \frac{1}{2} \int_Q \nabla \cdot \boldsymbol{\beta} (\nabla v \cdot \nabla v) \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \frac{1}{2} \int_\Gamma \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n} (\nabla v \cdot \nabla v) \\
&\quad + \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \boldsymbol{I}) \nabla v
\end{aligned}
$$

Plugging this into (2), we get

$$
\begin{aligned}
\left\|\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\right\| &= -\int_Q u\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v+\epsilon\int_Q\nabla v(\nabla\boldsymbol{\beta}-\frac{1}{2}\nabla\cdot\boldsymbol{\beta}\boldsymbol{I})\nabla v\\
&\quad -\epsilon\int_{\Gamma_x}\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\nabla v\cdot\boldsymbol{n}_x+\epsilon\frac{1}{2}\int_\Gamma\tilde{\boldsymbol{\beta}}\cdot\boldsymbol{n}(\nabla v\cdot\nabla v)\\
&= -\int_Q u\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v+\epsilon\int_Q\nabla v(\nabla\boldsymbol{\beta}-\frac{1}{2}\nabla\cdot\boldsymbol{\beta}\boldsymbol{I})\nabla v\\
&\quad -\int_{\Gamma_-}\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\underbrace{\nabla v\cdot\boldsymbol{n}_x}_{=0}-\int_{\Gamma_+}\left(\underbrace{\frac{\partial v}{\partial t}}_{=0}+\boldsymbol{\beta}\cdot\nabla v\right)\nabla v\cdot\boldsymbol{n}_x\\
&\quad +\frac{1}{2}\int_{\Gamma_-}\underbrace{\boldsymbol{\beta}\cdot\boldsymbol{n}_x}_{<0}(\nabla v\cdot\nabla v)+\frac{1}{2}\int_{\Gamma_+}\boldsymbol{\beta}\cdot\boldsymbol{n}_x(\nabla v\cdot\nabla v)\\
&\quad +\frac{1}{2}\int_{\Gamma_0}\underbrace{n_t}_{<0}(\nabla v\cdot\nabla v)+\frac{1}{2}\int_{\Gamma_T}n_t\underbrace{(\nabla v\cdot\nabla v)}_{=0}\\
&\le -\int_Q u\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v+\epsilon\int_Q\nabla v(\nabla\boldsymbol{\beta}-\frac{1}{2}\nabla\cdot\boldsymbol{\beta}\boldsymbol{I})\nabla v\\
&\quad +\int_{\Gamma_+}\left(-\frac{\partial v}{\partial\boldsymbol{n}_x}\boldsymbol{\beta}+\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{n}_x\nabla v\right)\cdot\nabla v\\
&= -\int_Q u\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v+\epsilon\int_Q\nabla v(\nabla\boldsymbol{\beta}-\frac{1}{2}\nabla\cdot\boldsymbol{\beta}\boldsymbol{I})\nabla v\\
&\quad +\int_{\Gamma_+}\left(-\frac{\partial v}{\partial\boldsymbol{n}_x}\boldsymbol{\beta}+\frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{n}_x\frac{\partial v}{\partial\boldsymbol{n}_x}\boldsymbol{n}_x\right)\cdot\frac{\partial v}{\partial\boldsymbol{n}_x}\boldsymbol{n}_x\\
&= -\int_Q u\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v+\epsilon\int_Q\nabla v(\nabla\boldsymbol{\beta}-\frac{1}{2}\nabla\cdot\boldsymbol{\beta}\boldsymbol{I})\nabla v\\
&\quad \underbrace{-\frac{1}{2}\int_{\Gamma_+}\left(\frac{\partial v}{\partial\boldsymbol{n}_x}\right)^2\boldsymbol{\beta}\cdot\boldsymbol{n}_x}_{<0}\\
&\le -\int_Q u\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v+\epsilon\int_Q\nabla v(\nabla\boldsymbol{\beta}-\frac{1}{2}\nabla\cdot\boldsymbol{\beta}\boldsymbol{I})\nabla v\\
&\le -\frac{\|u\|}{2}+\frac{\left\|\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\right\|}{2}+\epsilon\int_Q\nabla v(\nabla\boldsymbol{\beta}-\frac{1}{2}\nabla\cdot\boldsymbol{\beta}\boldsymbol{I})\nabla v\\
&\le -\frac{\|u\|}{2}+\frac{\left\|\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\right\|}{2}+\epsilon C\left\|\nabla v\right\|^2
\end{aligned}
$$

□

Lemma 1.2. *For the above conditions on* $\boldsymbol{\beta}$*,*

$$\|v\|^2+\epsilon\|\nabla v\|^2\le\|g\|^2+\epsilon\|f\|^2\ .$$

Note that $\boldsymbol{\tau} = \epsilon \nabla v$, so we already have control of $\|\boldsymbol{\tau}\|$.

Proof. Define $w = e^{T-t} v$ and note that $\frac{\partial w}{\partial t} = \left(\frac{\partial v}{\partial t} - v\right) e^{T-t}$ while $\nabla w = \cancelto{0}{\nabla e^{T-t}} v + e^{T-t} \nabla v$ and $\nabla \cdot (\boldsymbol{\beta} w) = \nabla \cdot (\boldsymbol{\beta}) e^{T-t} v + \boldsymbol{\beta} \cdot e^{T-t} \nabla v$ and $\Delta w = e^{T-t} \Delta v$. Also, $\nabla_{xt} w = \frac{\partial e^{T-t} v}{\partial t} + \nabla e^{T-t} v = e^{T-t}(\nabla_{xt} v - v)$. Plugging this into the adjoint equation, we get

$$-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt}(w) - \epsilon \Delta w = u - \epsilon \nabla \cdot \boldsymbol{\sigma}$$

or

$$\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt}(v) - v + \epsilon \Delta v = e^{t-T}(-u + \epsilon \nabla \cdot \boldsymbol{\sigma})$$

Multiply by $-v$ and integrate to get

$$\int_Q -\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v v + v^2 - \epsilon \Delta v v = \int_Q e^{t-T} u v - \epsilon \int_Q e^{t-T} \nabla \cdot \boldsymbol{\sigma} v$$

Then

$$\begin{aligned}
\|v\|^2 &= \int_Q e^{t-T} u v - \epsilon \int_Q e^{t-T} \nabla \cdot \boldsymbol{\sigma} v + \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v v + \epsilon \int_Q \Delta v v \\
&= \int_Q e^{t-T} u v - \epsilon \int_Q e^{t-T} \nabla \cdot \boldsymbol{\sigma} v + \frac{1}{2} \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt}(v)^2 - \epsilon \int_Q \nabla v \nabla v + \epsilon \int_\Gamma v \nabla v \cdot \boldsymbol{n}
\end{aligned}$$

Or

$$\begin{aligned}
\|v\|^2 + \epsilon \|\nabla v\|^2 &= \int_Q e^{t-T} u v - \epsilon \int_Q e^{t-T} \nabla \cdot \boldsymbol{\sigma} v \\
&\quad - \frac{1}{2} \int_Q \underbrace{\nabla_{xt} \cdot \tilde{\boldsymbol{\beta}}}_{=0} (v)^2 + \frac{1}{2} \int_\Gamma v^2 \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n} + \epsilon \int_{\Gamma_x} v \nabla v \cdot \boldsymbol{n}_x \\
&= \int_Q e^{t-T} u v + \epsilon \int_Q e^{t-T} \boldsymbol{\sigma} \cdot \nabla v - \epsilon \int_{\Gamma_-} v \underbrace{\boldsymbol{\sigma} \cdot \boldsymbol{n}_x}_{=\epsilon \frac{\partial v}{\partial n} = 0} - \epsilon \int_{\Gamma_+} \underbrace{v}_{=0} \boldsymbol{\sigma} \cdot \boldsymbol{n}_x \\
&\quad + \frac{1}{2} \int_{\Gamma_-} v^2 \underbrace{\boldsymbol{\beta} \cdot \boldsymbol{n}_x}_{<0} + \frac{1}{2} \int_{\Gamma_+} \underbrace{v^2}_{=0} \boldsymbol{\beta} \cdot \boldsymbol{n}_x \\
&\quad + \frac{1}{2} \int_{\Gamma_0} \underbrace{v^2(-n_t)}_{<0} + \frac{1}{2} \int_{\Gamma_T} \underbrace{v^2}_{=0} n_t \\
&\quad + \epsilon \int_{\Gamma_-} v \underbrace{\nabla v \cdot \boldsymbol{n}_x}_{=0} + \epsilon \int_{\Gamma_+} \underbrace{v}_{=0} \nabla v \cdot \boldsymbol{n}_x \\
&\leq \left\| e^{t-T} \right\|_{L_\infty(Q)} \left(\int_Q u v + \epsilon \int_Q \boldsymbol{\sigma} \cdot \nabla v \right) \\
&\leq \left(\frac{\|u\|^2}{2} + \frac{\epsilon \|\boldsymbol{\sigma}\|^2}{2} + \frac{\|v\|^2}{2} + \frac{\epsilon \|\nabla v\|^2}{2} \right)
\end{aligned}$$

$\square$

2 Robustness for transient problems given spatial robustness

Suppose we have the transient problem

$$\frac{\partial u}{\partial t} + Au = f$$

with initial condition $u(x,0) = u_0$. Suppose that DPG is robust under the ultra-weak variational formulation for the steady problem

$$(u, A_h^* v)_{L^2(\Omega)} + \langle \widehat{u}, [\![v]\!] \rangle_{\Gamma_h \setminus \Gamma_0} = (f, v)$$

with test norm $\|v\|_V$. Then, can we show that

$$\|v\|_{V,t} \coloneqq \|v\|_V + \left\| \frac{\partial v}{\partial t} \right\|_{L^2(\Omega)}$$

also leads to a robust upper bound of the L^2 norm by the DPG energy norm? I believe this may be possible. The adjoint equation for robustness for the transient problem gives

$$-\frac{\partial v}{\partial t} + A^* v = u$$

with $v = 0$ at $t = T$...

3 Transient Eriksson-Johnson

We can derive a transient Eriksson-Johnson solution using separation of variables. Consider

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \epsilon \Delta u = 0$$

with boundary conditions

$$\begin{aligned} u &= 0 \text{ on } \Gamma_+, \\ u - \epsilon \frac{\partial u}{\partial n} &= u_0 - \epsilon \frac{\partial u_0}{\partial n} \text{ on } \Gamma_-, \\ \epsilon \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_0, \end{aligned}$$

and initial condition $u(x,y,0) = u_0(x,y)$ that satisfies the given boundary data. Assuming that $u(x,y,t) = X(x,y)T(t)$ and $Lu = \frac{\partial u}{\partial x} - \epsilon \Delta u$, we can plug this into the equation

$$\frac{\partial u}{\partial t} + Lu = 0$$

and rearrange to get

$$-\frac{\frac{\partial T}{\partial t}}{T} = \frac{LX}{X} = C.$$

This assumes then that $\frac{\partial T}{\partial t} = -CT$, or that $T(t) = e^{-Ct}$, and that $LX = CX$, or that X is made up of the eigenfunctions of L.