Robustness for Transient Problems

Truman E. Ellis and Jesse L. Chan

August 7, 2014

Consider domain $Q = \Omega \times [0, T]$ with boundary $\Gamma = \Gamma_- \cup \Gamma_+ \cup \Gamma_0 \cup \Gamma_T$ where Γ_- is the spatial inflow boundary, Γ_+ is the spatial outflow boundary, Γ_0 is the initial time boundary, and Γ_T is the final time boundary. Let Γ_h denote the entire mesh skeleton.

Assume that boundary conditions are applied on the boundary $\Gamma_0 \subset \Gamma$. Recall that, for the ultra-weak variational formulation

$$b\left((u, \widehat{u}), v\right) = (u, A_h^* v)_{L^2(\Omega)} + \langle \widehat{u}, [\![v]\!] \rangle_{\Gamma_h \setminus \Gamma_0}$$

we can recover

$$\|u\|_{L^2(Q)}^2 = b(u, v^*)$$

for conforming v^* satisfying the adjoint equation

$$A^* v^* = u$$
$$v^* = 0 \text{ on } \Gamma_h \setminus \Gamma_0.$$

Together, these give necessary conditions on the test norm $\|\cdot\|_V$ such that we have L^2 robustness (this gives robustness in the variable u; for the first order formulation, conditions for σ must also be shown).

$$\|u\|_{L^2(Q)}^2 = b(u, v^*) \leq \frac{b(u, v^*)}{\|v^*\|_V} \|v^*\|_V \leq \|u\|_E \|v^*\|_V$$

Thus, showing $\|v^*\|_V \lesssim \|u\|_{L^2(Q)}$ gives the result that $\|u\|_{L^2(Q)} \lesssim \|u\|_E$.

1   Convection-Diffusion

Consider convection-diffusion

$$\frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u = 0$$

$$\frac{\partial u}{\partial t} + \boldsymbol{\beta} \cdot \nabla u - \nabla \cdot \boldsymbol{\sigma} = f$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_-$$

$$u = 0 \text{ on } \Gamma_+$$

$$u = u_0 \text{ on } \Gamma_0.$$

Let $\tilde{\boldsymbol{\beta}} := \begin{pmatrix} \boldsymbol{\beta} \\ 1 \end{pmatrix}$ and $\nabla_{xt} := \begin{pmatrix} \nabla \\ \frac{\partial}{\partial t} \end{pmatrix}$, then we can rewrite this as

$$\frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u = 0$$

$$\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} u - \nabla \cdot \boldsymbol{\sigma} = f$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_-$$

$$u = 0 \text{ on } \Gamma_+$$

$$u = u_0 \text{ on } \Gamma_0.$$

We decompose the adjoint into three parts: a discontinuous part

$$\frac{1}{\epsilon}\boldsymbol{\tau}_0 + \nabla v_0 = 0$$

$$-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_0 + \nabla \cdot \boldsymbol{\tau}_0 = 0$$

$$\boldsymbol{\tau}_0 \cdot \boldsymbol{n}_x = \boldsymbol{\tau} \cdot \boldsymbol{n}_x \text{ on } \Gamma_- \cup \Gamma_0$$

$$v_0 = v \text{ on } \Gamma_+$$

$$v_0 = v \text{ on } \Gamma_T$$

$$[\![v_0]\!] = [\![v]\!] \text{ on } \Gamma_h^0$$

$$[\![\boldsymbol{\tau}_0 \cdot \boldsymbol{n}_x]\!] = [\![\boldsymbol{\tau}_0 \cdot \boldsymbol{n}_x]\!] \text{ on } \Gamma_{hx}^0,$$

a continuous part with forcing term g

$$\frac{1}{\epsilon}\boldsymbol{\tau}_1 + \nabla v_1 = 0$$

$$-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1 + \nabla \cdot \boldsymbol{\tau}_1 = g$$

$$\boldsymbol{\tau}_1 \cdot \boldsymbol{n}_x = 0 \text{ on } \Gamma_-$$

$$v_1 = 0 \text{ on } \Gamma_+$$

$$v_1 = 0 \text{ on } \Gamma_T,$$

and a continuous part with forcing $\boldsymbol{f}$

$$\frac{1}{\epsilon}\boldsymbol{\tau}_2 + \nabla v_2 = \boldsymbol{f}$$

$$-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_2 + \nabla \cdot \boldsymbol{\tau}_2 = 0$$

$$\boldsymbol{\tau}_2 \cdot \boldsymbol{n}_x = 0 \text{ on } \Gamma_-$$

$$v_2 = 0 \text{ on } \Gamma_+$$

$$v_2 = 0 \text{ on } \Gamma_T.$$

(The boundary conditions can be derived by taking the ultra-weak formulation and choosing boundary conditions such that the temporal flux and spatial flux terms $\langle \widehat{u}, [\![\tau_n]\!] \rangle_{\Gamma_{out}}$ and $\langle \widehat{t}_n, [\![v]\!] \rangle_{\Gamma_{in}}$ are zero.)

We can then derive that the test norm

$$\|(v,\boldsymbol{\tau})\|_{V,K}^2 := \frac{1}{\epsilon}\|\boldsymbol{\tau}\|_K^2 + \left\|\nabla\cdot\boldsymbol{\tau} - \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\right\|_K^2$$
$$+ \left\|\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\right\|_K^2 + \epsilon\|\nabla v\|_K^2 + \|v\|_K^2 , \tag{1}$$

provides the necessary bound $\|v^*\|_V \lesssim \|u\|_{L^2(Q)}$.

In the following lemmas we establish the following bounds:

- Bound on $\|(v_0,\boldsymbol{\tau}_0)\|_V$.

- Bound on $\|(v_1,\boldsymbol{\tau}_1)\|_V$. Lemma 1.1 gives $\left\|\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v_1\right\| \le \|g\|$. Since $\nabla\cdot\boldsymbol{\tau}_1 = g + \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v_1$,

$$\|\nabla\cdot\boldsymbol{\tau}_1\| \le \|g\| + \left\|\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v_1\right\| \le 2\|g\| .$$

  Lemma 1.2 gives $\|v_1\|^2 + \epsilon\|\nabla v_1\|^2 \le \|g\|^2$. Since $\epsilon^{1/2}\nabla v_1 = \epsilon^{-1/2}\boldsymbol{\tau}_1$,

$$\frac{1}{\epsilon}\|\boldsymbol{\tau}_1\|^2 \le \|g\|^2 .$$

  Thus, all the terms in (1) are accounted for.

- Bound on $\|(v_2,\boldsymbol{\tau}_2)\|_V$.

Our goal is to analyze the stability properties of the adjoint equations by deriving bounds of the form $\|(v_1,\boldsymbol{\tau}_1)\|_V \le \|g\|_L^2(Q)$ and $\|(v_2,\boldsymbol{\tau}_2)\|_V \le \|f\|_L^2(Q)$.
Insert conditions on $\boldsymbol{\beta}$

Lemma 1.1. *For the above conditions on $\boldsymbol{\beta}$,*

$$\left\|\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v_1\right\| \le \|g\|$$

and since $-\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v_1 + \nabla\cdot\boldsymbol{\tau}_1 = g$,

$$\|\nabla\cdot\boldsymbol{\tau}_1\| \le \|g\| .$$

Proof. Multiply by $-\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v$ and integrate over Q to get

$$\left\|\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\right\| = -\int_Q g\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v + \int_Q \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\nabla\cdot\boldsymbol{\tau} . \tag{2}$$

Note that

$$\frac{1}{\epsilon}\int_Q \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\nabla\cdot\boldsymbol{\tau} = -\int_Q \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\nabla\cdot\nabla v$$

$$= -\int_{\Gamma_x}\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\nabla v\cdot\boldsymbol{n}_x + \int_Q \nabla(\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v)\cdot\nabla v$$

$$= -\int_{\Gamma_x}\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\nabla v\cdot\boldsymbol{n}_x + \int_Q (\nabla\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v)\cdot\nabla v$$

$$+ \int_Q \tilde{\boldsymbol{\beta}}\cdot\nabla\nabla_{xt}v\cdot\nabla v$$

$$= -\int_{\Gamma_x}\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\nabla v\cdot\boldsymbol{n}_x + \int_Q (\nabla\boldsymbol{\beta}\cdot\nabla v)\cdot\nabla v$$

$$+ \frac{1}{2}\int_Q \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}(\nabla v\cdot\nabla v)$$

$$= -\int_{\Gamma_x}\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\nabla v\cdot\boldsymbol{n}_x + \int_Q (\nabla\boldsymbol{\beta}\cdot\nabla v)\cdot\nabla v$$

$$+ \frac{1}{2}\int_\Gamma \tilde{\boldsymbol{\beta}}\cdot\boldsymbol{n}(\nabla v\cdot\nabla v) - \frac{1}{2}\int_Q \nabla_{xt}\cdot\tilde{\boldsymbol{\beta}}(\nabla v\cdot\nabla v)$$

$$= -\int_{\Gamma_x}\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\nabla v\cdot\boldsymbol{n}_x + \int_Q (\nabla\boldsymbol{\beta}\cdot\nabla v)\cdot\nabla v$$

$$+ \frac{1}{2}\int_\Gamma \tilde{\boldsymbol{\beta}}\cdot\boldsymbol{n}(\nabla v\cdot\nabla v) - \frac{1}{2}\int_Q \nabla\cdot\boldsymbol{\beta}(\nabla v\cdot\nabla v)$$

$$= -\int_{\Gamma_x}\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\nabla v\cdot\boldsymbol{n}_x + \frac{1}{2}\int_\Gamma \tilde{\boldsymbol{\beta}}\cdot\boldsymbol{n}(\nabla v\cdot\nabla v)$$

$$+ \int_Q \nabla v(\nabla\boldsymbol{\beta} - \frac{1}{2}\nabla\cdot\boldsymbol{\beta}\boldsymbol{I})\nabla v$$

4

Plugging this into (2), we get

$$\left\| \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \right\| = -\int_Q g\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2}\nabla \cdot \boldsymbol{\beta} \boldsymbol{I})\nabla v$$

$$- \epsilon \int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \epsilon \frac{1}{2} \int_\Gamma \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n}(\nabla v \cdot \nabla v)$$

$$= -\int_Q g\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2}\nabla \cdot \boldsymbol{\beta} \boldsymbol{I})\nabla v$$

$$- \int_{\Gamma_-} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \underbrace{\nabla v \cdot \boldsymbol{n}_x}_{=0} - \int_{\Gamma_+} \left(\underbrace{\frac{\partial v}{\partial t}}_{=0} + \boldsymbol{\beta} \cdot \nabla v \right) \nabla v \cdot \boldsymbol{n}_x$$

$$+ \frac{1}{2} \int_{\Gamma_-} \underbrace{\boldsymbol{\beta} \cdot \boldsymbol{n}_x}_{<0}(\nabla v \cdot \nabla v) + \frac{1}{2} \int_{\Gamma_+} \boldsymbol{\beta} \cdot \boldsymbol{n}_x(\nabla v \cdot \nabla v)$$

$$+ \frac{1}{2} \int_{\Gamma_0} \underbrace{n_t}_{<0}(\nabla v \cdot \nabla v) + \frac{1}{2} \int_{\Gamma_T} n_t \underbrace{(\nabla v \cdot \nabla v)}_{=0}$$

$$\leq -\int_Q g\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2}\nabla \cdot \boldsymbol{\beta} \boldsymbol{I})\nabla v$$

$$+ \int_{\Gamma_+} \left(-\frac{\partial v}{\partial \boldsymbol{n}_x}\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{n}_x \nabla v \right) \cdot \nabla v$$

$$= -\int_Q g\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2}\nabla \cdot \boldsymbol{\beta} \boldsymbol{I})\nabla v$$

$$+ \int_{\Gamma_+} \left(-\frac{\partial v}{\partial \boldsymbol{n}_x}\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{n}_x \frac{\partial v}{\partial \boldsymbol{n}_x}\boldsymbol{n}_x \right) \cdot \frac{\partial v}{\partial \boldsymbol{n}_x}\boldsymbol{n}_x$$

$$= -\int_Q g\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2}\nabla \cdot \boldsymbol{\beta} \boldsymbol{I})\nabla v$$

$$\underbrace{-\frac{1}{2} \int_{\Gamma_+} \left(\frac{\partial v}{\partial \boldsymbol{n}_x} \right)^2 \boldsymbol{\beta} \cdot \boldsymbol{n}_x}_{<0}$$

$$\leq -\int_Q g\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2}\nabla \cdot \boldsymbol{\beta} \boldsymbol{I})\nabla v$$

$$\leq -\frac{\|g\|}{2} + \frac{\left\| \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \right\|}{2} + \epsilon \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2}\nabla \cdot \boldsymbol{\beta} \boldsymbol{I})\nabla v$$

$$\leq -\frac{\|g\|}{2} + \frac{\left\| \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \right\|}{2} + \epsilon C \|\nabla v\|^2$$

$$\square$$

Lemma 1.2. *For the above conditions on $\boldsymbol{\beta}$,*

$$\|v\|^2 + \epsilon \|\nabla v\|^2 \leq \|g\|^2 + \epsilon \|f\|^2 \ .$$

Note that $\boldsymbol{\tau} = \epsilon \nabla v$, so we already have control of $\|\boldsymbol{\tau}\|$.

Proof. Define $w = e^{T-t}v$ and note that $\frac{\partial w}{\partial t} = \left(\frac{\partial v}{\partial t} - v\right)e^{T-t}$ while $\nabla w = \underbrace{\nabla e^{T-t}v}_{0} + e^{T-t}\nabla v$ and $\nabla \cdot (\boldsymbol{\beta}w) = \nabla \cdot (\boldsymbol{\beta})e^{T-t}v + \boldsymbol{\beta} \cdot e^{T-t}\nabla v$ and $\Delta w = e^{T-t}\Delta v$. Also, $\nabla_{xt}w = \frac{\partial e^{T-t}v}{\partial t} + \nabla e^{T-t}v = e^{T-t}(\nabla_{xt}v - v)$. Plugging this into the adjoint equation, we get

$$-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt}(w) - \epsilon \Delta w = g - \epsilon \nabla \cdot \boldsymbol{f}$$

or

$$\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt}(v) - v + \epsilon \Delta v = e^{t-T}(-g + \epsilon \nabla \cdot \boldsymbol{f})$$

Multiply by $-v$ and integrate to get

$$\int_Q -\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt}vv + v^2 - \epsilon \Delta vv = \int_Q e^{t-T}gv - \epsilon \int_Q e^{t-T}\nabla \cdot \boldsymbol{f}v$$

Then

$$\|v\|^2 = \int_Q e^{t-T}gv - \epsilon \int_Q e^{t-T}\nabla \cdot \boldsymbol{f}v + \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt}vv + \epsilon \int_Q \Delta vv$$

$$= \int_Q e^{t-T}gv - \epsilon \int_Q e^{t-T}\nabla \cdot \boldsymbol{f}v + \frac{1}{2}\int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt}(v)^2 - \epsilon \int_Q \nabla v \nabla v + \epsilon \int_\Gamma v\nabla v \cdot \boldsymbol{n}$$

Or

$$\|v\|^2 + \epsilon \|\nabla v\|^2 = \int_Q e^{t-T}gv - \epsilon \int_Q e^{t-T}\nabla \cdot \boldsymbol{f}v$$

$$- \frac{1}{2}\int_Q \underbrace{\nabla_{xt} \cdot \tilde{\boldsymbol{\beta}}}_{=0}(v)^2 + \frac{1}{2}\int_\Gamma v^2 \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n} + \epsilon \int_{\Gamma_x} v\nabla v \cdot \boldsymbol{n}_x$$

$$= \int_Q e^{t-T}gv + \epsilon \int_Q e^{t-T}\boldsymbol{f} \cdot \nabla v - \epsilon \int_{\Gamma_-} v \underbrace{\boldsymbol{f} \cdot \boldsymbol{n}_x}_{=\epsilon \frac{\partial v}{\partial n}=0} - \epsilon \int_{\Gamma_+} \underbrace{v}_{=0} \boldsymbol{f} \cdot \boldsymbol{n}_x$$

$$+ \frac{1}{2}\int_{\Gamma_-} v^2 \underbrace{\boldsymbol{\beta} \cdot \boldsymbol{n}_x}_{<0} + \frac{1}{2}\int_{\Gamma_+} \underbrace{v^2}_{=0}\boldsymbol{\beta} \cdot \boldsymbol{n}_x$$

$$+ \frac{1}{2}\int_{\Gamma_0} \underbrace{v^2(-n_t)}_{<0} + \frac{1}{2}\int_{\Gamma_T} \underbrace{v^2}_{=0} n_t$$

$$+ \epsilon \int_{\Gamma_-} v \underbrace{\nabla v \cdot \boldsymbol{n}_x}_{=0} + \epsilon \int_{\Gamma_+} \underbrace{v}_{=0}\nabla v \cdot \boldsymbol{n}_x$$

$$\leq \|e^{t-T}\|_{L_\infty(Q)}\left(\int_Q gv + \epsilon \int_Q \boldsymbol{f} \cdot \nabla v\right)$$

$$\leq \left(\frac{\|g\|^2}{2} + \frac{\epsilon \|\boldsymbol{f}\|^2}{2} + \frac{\|v\|^2}{2} + \frac{\epsilon \|\nabla v\|^2}{2}\right)$$

$\square$

2 Robustness for transient problems given spatial robustness

Suppose we have the transient problem

$$\frac{\partial u}{\partial t} + Au = f$$

with initial condition $u(x,0) = u_0$. Suppose that DPG is robust under the ultra-weak variational formulation for the steady problem

$$(u, A_h^* v)_{L^2(\Omega)} + \langle \widehat{u}, [\![v]\!] \rangle_{\Gamma_h \backslash \Gamma_0} = (f, v)$$

with test norm $\|v\|_V$. Then, can we show that

$$\|v\|_{V,t} := \|v\|_V + \left\| \frac{\partial v}{\partial t} \right\|_{L^2(\Omega)}$$

also leads to a robust upper bound of the L^2 norm by the DPG energy norm? I believe this may be possible. The adjoint equation for robustness for the transient problem gives

$$-\frac{\partial v}{\partial t} + A^* v = u$$

with $v = 0$ at $t = T$...

3 Transient Eriksson-Johnson

We can derive a transient Eriksson-Johnson solution using separation of variables. Consider

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \epsilon \Delta u = 0$$

with boundary conditions

$$u = 0 \text{ on } \Gamma_+,$$

$$u - \epsilon \frac{\partial u}{\partial n} = u_0 - \epsilon \frac{\partial u_0}{\partial n} \text{ on } \Gamma_-,$$

$$\epsilon \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_0,$$

and initial condition $u(x,y,0) = u_0(x,y)$ that satisfies the given boundary data. Assuming that $u(x,y,t) = X(x,y)T(t)$ and $Lu = \frac{\partial u}{\partial x} - \epsilon \Delta u$, we can plug this into the equation

$$\frac{\partial u}{\partial t} + Lu = 0$$

and rearrange to get

$$-\frac{\frac{\partial T}{\partial t}}{T} = \frac{LX}{X} = C.$$

This assumes then that $\frac{\partial T}{\partial t} = -CT$, or that $T(t) = e^{-Ct}$, and that $LX = CX$, or that X is made up of the eigenfunctions of L.