

Robustness for Transient Problems

Truman E. Ellis and Jesse L. Chan

July 23, 2014

Assume that boundary conditions are applied on the boundary $\Gamma_0 \subset \Gamma$. Recall that, for the ultra-weak variational formulation

$$b((u, \hat{u}), v) = (u, A_h^* v)_{L^2(\Omega)} + \langle \hat{u}, \llbracket v \rrbracket \rangle_{\Gamma_h \setminus \Gamma_0}$$

we can recover

$$\|u\|_{L^2(\Omega)}^2 = b(u, v^*)$$

for conforming v^* satisfying the adjoint equation

$$\begin{aligned} A^* v^* &= u \\ v^* &= 0 \text{ on } \Gamma_h \setminus \Gamma_0. \end{aligned}$$

Together, these give necessary conditions on the test norm $\|\cdot\|_V$ such that we have L^2 robustness (this gives robustness in the variable u ; for the first order formulation, conditions for σ must also be shown).

$$\|u\|_{L^2(\Omega)}^2 = b(u, v^*) \leq \frac{b(u, v^*)}{\|v^*\|_V} \|v^*\|_V \leq \|u\|_E \|v^*\|_V$$

Thus, showing $\|v^*\|_V \lesssim \|u\|_{L^2(\Omega)}$ gives the result that $\|u\|_{L^2(\Omega)} \lesssim \|u\|_E$.

1 Reaction-diffusion

Consider reaction diffusion

$$\begin{aligned} \frac{\partial u}{\partial t} + u - \epsilon \Delta u &= f \\ u &= 0 \text{ on } \Gamma_1 \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_2 \\ u(t=0) &= u_0. \end{aligned}$$

The adjoint equation satisfies

$$\begin{aligned} -\frac{\partial v}{\partial t} + v - \epsilon \Delta v &= u \\ v &= 0 \text{ on } \Gamma_1 \\ \frac{\partial v}{\partial n} &= 0 \text{ on } \Gamma_2 \\ v(t = T) &= 0. \end{aligned}$$

(The boundary conditions can be derived by taking the ultra-weak formulation and choosing boundary conditions such that the temporal flux and spatial flux terms $\langle \hat{u}, \llbracket \tau_n \rrbracket \rangle_{\Gamma_1}$ and $\langle \hat{f}_n, \llbracket v \rrbracket \rangle_{\Gamma_2}$ are zero.)

We can then derive that the test norm

$$\|v\|_V^2 = \left\| \frac{\partial v}{\partial t} \right\|^2 + \|v\|^2 + \epsilon \|\nabla v\|^2$$

provides the necessary bound $\|v^*\|_V \lesssim \|u\|_{L^2(\Omega)}$.

To see, this we multiply the adjoint equation by two terms as follows:

1. Multiply by v and integrate over $\Omega \times [0, T] = Q$ to get

$$-\int_Q \frac{\partial v}{\partial t} v + \int_Q v^2 + \epsilon \int_Q |\nabla v|^2 - \epsilon \int_0^T \int_\Gamma \frac{\partial v}{\partial n} v = \int_Q uv.$$

Noting that either $v = 0$ or $\frac{\partial v}{\partial n} = 0$ on the boundary removes the integral over Γ . Next, we can factor the first term and use Young's inequality to get

$$-\int_0^T \frac{\partial}{\partial t} \int_\Omega v^2 + \|v\|_Q^2 + \epsilon \|\nabla v\|_Q^2 \leq \frac{1}{2} \|u\|_Q^2 + \frac{1}{2} \|v\|_Q^2$$

Integrating by parts the first term gives

$$-\int_\Omega v^2 \Big|_0^T + \frac{1}{2} \|v\|_Q^2 + \epsilon \|\nabla v\|_Q^2 \leq \frac{1}{2} \|u\|_Q^2$$

Using boundary condition $v = 0$ at $t = T$ gives

$$\frac{1}{2} \|v\|_Q^2 + \epsilon \|\nabla v\|_Q^2 \leq \int_\Omega v(t=0)^2 + \frac{1}{2} \|v\|_Q^2 + \epsilon \|\nabla v\|_Q^2 \leq \frac{1}{2} \|u\|_Q^2.$$

2. Multiply by $-\frac{\partial v}{\partial t}$ and integrate over Q . Young's inequality changes the right hand side to

$$\int_Q \frac{\partial v^2}{\partial t} - \int_Q v \frac{\partial v}{\partial t} + \epsilon \int_Q \Delta v \frac{\partial v}{\partial t} = \int_Q -u \frac{\partial v}{\partial t} \leq \frac{1}{2} \|u\|_Q^2 + \frac{1}{2} \left\| \frac{\partial v}{\partial t} \right\|_Q^2.$$

The term $\int_Q v \frac{\partial v}{\partial t}$ can be reduced to the positive contribution $\int_\Omega v(t=0)^2$ as above. We can then take the Laplacian term, integrate by parts in space to get

$$\int_Q \Delta v \frac{\partial v}{\partial t} = \int_0^T \int_\Omega \Delta v \frac{\partial v}{\partial t} = \int_0^T \int_\Gamma \frac{\partial v}{\partial t} \frac{\partial v}{\partial n} - \int_0^T \int_\Omega \nabla \left(\frac{\partial v}{\partial t} \right) \nabla v.$$

Since either $v = 0$ or $\frac{\partial v}{\partial n} = 0$ on Γ , the first term disappears. The second term can be bounded by noting

$$- \int_0^T \int_\Omega \nabla \left(\frac{\partial v}{\partial t} \right) \nabla v = - \int_0^T \frac{\partial}{\partial t} \int_\Omega |\nabla v|^2 = - \int_\Omega |\nabla v|^2 \Big|_0^T.$$

Since $v = 0$ at $t = T$, $\nabla v = 0$ at $t = T$ as well, and we are left with the positive contribution $\int_\Omega |\nabla v(t=0)|^2$. Then,

$$\frac{1}{2} \left\| \frac{\partial v}{\partial t} \right\|_Q^2 \leq \frac{1}{2} \|u\|_Q.$$

Together, these two show that, under test norm

$$\|v\|_V^2 = \left\| \frac{\partial v}{\partial t} \right\|_Q^2 + \|v\|^2 + \epsilon \|\nabla v\|^2,$$

the adjoint equation v^* satisfies

$$\|v^*\|_V \lesssim \|u\|_{L^2(\Omega)}$$

and thus the DPG energy norm robustly bounds the L^2 norm from above

$$\|u\|_{L^2(\Omega)} \lesssim \|u\|_E.$$

2 Convection-diffusion

Consider convection-diffusion

$$\begin{aligned} \frac{\partial u}{\partial t} + \beta \cdot \nabla u - \epsilon \Delta u &= f \\ u &= 0 \text{ on } \Gamma_{out} \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_{in} \\ u(t=0) &= u_0. \end{aligned}$$

Let $\tilde{\beta} := \begin{pmatrix} \beta \\ 1 \end{pmatrix}$ and $\nabla_{xt} := \begin{pmatrix} \nabla \\ \frac{\partial}{\partial t} \end{pmatrix}$, then we can rewrite this as

$$\begin{aligned} \tilde{\beta} \cdot \nabla_{xt} u - \epsilon \Delta u &= f \\ u &= 0 \text{ on } \Gamma_{out} \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_{in} \\ u(t=0) &= u_0. \end{aligned}$$

The adjoint equation satisfies

$$\begin{aligned} -\tilde{\beta} \cdot \nabla_{xt} v - \epsilon \Delta v &= u \\ v &= 0 \text{ on } \Gamma_{in} \\ \frac{\partial v}{\partial n} &= 0 \text{ on } \Gamma_{out} \\ v(t=T) &= 0. \end{aligned}$$

(The boundary conditions can be derived by taking the ultra-weak formulation and choosing boundary conditions such that the temporal flux and spatial flux terms $\langle \hat{u}, \llbracket \tau_n \rrbracket \rangle_{\Gamma_{out}}$ and $\langle \hat{f}_n, \llbracket v \rrbracket \rangle_{\Gamma_{in}}$ are zero.) The $t=0$ and $t=T$ boundaries can be considered as an inflow and outflow boundary respectively in space-time and we denote $\partial Q_{in} := \Gamma_{in} \cup t=0$ while $\partial Q_{out} := \Gamma_{out} \cup t=T$.

We can then derive that the test norm

$$\|v\|_V^2 = \left\| \tilde{\beta} \cdot \nabla_{xt} v \right\|^2 + \epsilon \|\nabla v\|^2$$

provides the necessary bound $\|v^*\|_V \lesssim \|u\|_{L^2(Q)}$.

To see this, we multiply the adjoint equation by two terms as follows:

1. Multiply by $-\tilde{\beta} \cdot \nabla_{xt} v$ and integrate over Q to get

$$\left\| \tilde{\beta} \cdot \nabla_{xt} v \right\| = - \int_Q u \tilde{\beta} \cdot \nabla_{xt} v - \epsilon \int_Q \tilde{\beta} \cdot \nabla_{xt} v \Delta v. \quad (1)$$

Note that

$$\begin{aligned}
-\int_Q \tilde{\beta} \cdot \nabla_{xt} v \Delta v &= -\int_Q \tilde{\beta} \cdot \nabla_{xt} v \nabla \cdot \nabla v \\
&= -\int_{\Gamma_x} \tilde{\beta} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \int_Q \nabla(\tilde{\beta} \cdot \nabla_{xt} v) \cdot \nabla v \\
&= -\int_{\Gamma_x} \tilde{\beta} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \int_Q (\nabla \tilde{\beta} \cdot \nabla_{xt} v) \cdot \nabla v \\
&\quad + \int_Q \tilde{\beta} \cdot \nabla \nabla_{xt} v \cdot \nabla v \\
&= -\int_{\Gamma_x} \tilde{\beta} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \int_Q (\nabla \beta \cdot \nabla v) \cdot \nabla v \\
&\quad + \frac{1}{2} \int_Q \tilde{\beta} \cdot \nabla_{xt} (\nabla v \cdot \nabla v) \\
&= -\int_{\Gamma_x} \tilde{\beta} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \int_Q (\nabla \beta \cdot \nabla v) \cdot \nabla v \\
&\quad + \frac{1}{2} \int_{\Gamma} \tilde{\beta} \cdot \mathbf{n} (\nabla v \cdot \nabla v) - \frac{1}{2} \int_Q \nabla_{xt} \cdot \tilde{\beta} (\nabla v \cdot \nabla v) \\
&= -\int_{\Gamma_x} \tilde{\beta} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \int_Q (\nabla \beta \cdot \nabla v) \cdot \nabla v \\
&\quad + \frac{1}{2} \int_{\Gamma} \tilde{\beta} \cdot \mathbf{n} (\nabla v \cdot \nabla v) - \frac{1}{2} \int_Q \nabla \cdot \beta (\nabla v \cdot \nabla v) \\
&= -\int_{\Gamma_x} \tilde{\beta} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \frac{1}{2} \int_{\Gamma} \tilde{\beta} \cdot \mathbf{n} (\nabla v \cdot \nabla v) \\
&\quad + \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v
\end{aligned}$$

Plugging this into (1), we get

$$\begin{aligned}
\|\tilde{\beta} \cdot \nabla_{xt} v\| &= -\int_Q u \tilde{\beta} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \\
&\quad - \epsilon \int_{\Gamma_x} \tilde{\beta} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \epsilon \frac{1}{2} \int_{\Gamma} \tilde{\beta} \cdot \mathbf{n} (\nabla v \cdot \nabla v)
\end{aligned}$$

3 Robustness for transient problems given spatial robustness

Suppose we have the transient problem

$$\frac{\partial u}{\partial t} + Au = f$$

with initial condition $u(x, 0) = u_0$. Suppose that DPG is robust under the ultra-weak variational formulation for the steady problem

$$(u, A_h^* v)_{L^2(\Omega)} + \langle \hat{u}, \llbracket v \rrbracket \rangle_{\Gamma_h \setminus \Gamma_0} = (f, v)$$

with test norm $\|v\|_V$. Then, can we show that

$$\|v\|_{V,t} := \|v\|_V + \left\| \frac{\partial v}{\partial t} \right\|_{L^2(\Omega)}$$

also leads to a robust upper bound of the L^2 norm by the DPG energy norm? I believe this may be possible. The adjoint equation for robustness for the transient problem gives

$$-\frac{\partial v}{\partial t} + A^* v = u$$

with $v = 0$ at $t = T$...