Notes on Steady Incompressible Navier-Stokes

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Non-Conservative Form

The steady incompressible Navier-Stokes equations are:

$$\begin{aligned} \nabla \boldsymbol{u} \cdot \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p &= \mathbf{f} \\ \nabla \cdot \boldsymbol{u} &= 0 \\ \int_\Omega p &= 0 \end{aligned}$$

In 2D we get a system

$$\begin{aligned} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \nabla u_1 - \nu \Delta u_1 + \frac{\partial p}{\partial x} &= f_1 \\ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \nabla u_2 - \nu \Delta u_2 + \frac{\partial p}{\partial y} &= f_2 \\ \nabla \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= 0 \\ \int_\Omega p &= 0 \end{aligned}$$

As a first order system, this is

$$\begin{aligned}
\boldsymbol{\sigma}_1 - \nabla u_1 &= 0\\
\boldsymbol{\sigma}_2 - \nabla u_2 &= 0\\
\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \boldsymbol{\sigma}_1 - \nu \nabla \cdot \boldsymbol{\sigma}_1 + \frac{\partial p}{\partial x} &= f_1\\
\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \boldsymbol{\sigma}_2 - \nu \nabla \cdot \boldsymbol{\sigma}_2 + \frac{\partial p}{\partial y} &= f_2\\
\nabla \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= 0\\
\int_\Omega p &= 0
\end{aligned}$$

Multiplying by $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, v_1, v_2, v_d, and integrating by parts:

$$\begin{aligned}
(\boldsymbol{\sigma}_1, \boldsymbol{\tau}_1) + (u_1, \nabla \cdot \boldsymbol{\tau}_1) - \langle \hat{u}_1, \tau_{1n} \rangle &= 0\\
(\boldsymbol{\sigma}_2, \boldsymbol{\tau}_2) + (u_2, \nabla \cdot \boldsymbol{\tau}_2) - \langle \hat{u}_2, \tau_{2n} \rangle &= 0\\
\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \boldsymbol{\sigma}_1, v_1 \right) + (\nu \boldsymbol{\sigma}_1, \nabla v_1) - \left(p, \frac{\partial v_1}{\partial x} \right) + \left\langle \hat{t}, v_1 \right\rangle &= (f_1, v_1)\\
\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \boldsymbol{\sigma}_2, v_2 \right) + (\nu \boldsymbol{\sigma}_2, \nabla v_2) - \left(p, \frac{\partial v_2}{\partial y} \right) + \left\langle \hat{t}, v_2 \right\rangle &= (f_2, v_2)\\
-\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \nabla v_d \right) + \left\langle \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix}, v_d \cdot \mathbf{n} \right\rangle &= 0\\
\int_\Omega p &= 0
\end{aligned}$$

where $\hat{t} := (\widehat{-\boldsymbol{\sigma} + p}\mathbf{I})\mathbf{n}$.

Note that if $\boldsymbol{v} := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, then

$$-\left(p, \frac{\partial v_1}{\partial x} \right) + \langle p, v_1 \cdot n_x \rangle - \left(p, \frac{\partial v_2}{\partial y} \right) + \langle p, v_2 \cdot n_y \rangle = -(p, \nabla \cdot \boldsymbol{v}) + \langle p, v_n \rangle$$

Linearizing:

$$
\begin{aligned}
(\Delta\boldsymbol{\sigma}_1,\boldsymbol{\tau}_1)+(\Delta u_1,\nabla\cdot\boldsymbol{\tau}_1)-\langle\hat{u}_1,\tau_{1n}\rangle &= -(\boldsymbol{\sigma}_1,\boldsymbol{\tau}_1)-(u_1,\nabla\cdot\boldsymbol{\tau}_1)\\
(\Delta\boldsymbol{\sigma}_2,\boldsymbol{\tau}_2)+(u_2,\nabla\cdot\boldsymbol{\tau}_2)-\langle\hat{u}_2,\tau_{2n}\rangle &= -(\boldsymbol{\sigma}_2,\boldsymbol{\tau}_2)-(u_2,\nabla\cdot\boldsymbol{\tau}_2)\\
\left(\begin{pmatrix}\Delta u_1\\ \Delta u_2\end{pmatrix}\cdot\boldsymbol{\sigma}_1,v_1\right)+\left(\begin{pmatrix}u_1\\ u_2\end{pmatrix}\cdot\Delta\boldsymbol{\sigma}_1,v_1\right)+(\nu\Delta\boldsymbol{\sigma}_1,\nabla v_1)-\left(p,\frac{\partial v_1}{\partial x}\right)+\langle\hat{t},v_1\rangle&\\
&= (f_1,v_1)-\left(\begin{pmatrix}u_1\\ u_2\end{pmatrix}\cdot\boldsymbol{\sigma}_1,v_1\right)-(\nu\boldsymbol{\sigma}_1,\nabla v_1)\\
\left(\begin{pmatrix}\Delta u_1\\ \Delta u_2\end{pmatrix}\cdot\boldsymbol{\sigma}_2,v_2\right)+\left(\begin{pmatrix}u_1\\ u_2\end{pmatrix}\cdot\Delta\boldsymbol{\sigma}_2,v_2\right)+(\nu\Delta\boldsymbol{\sigma}_2,\nabla v_2)-\left(p,\frac{\partial v_2}{\partial y}\right)+\langle\hat{t},v_2\rangle&\\
&= (f_2,v_2)-\left(\begin{pmatrix}u_1\\ u_2\end{pmatrix}\cdot\boldsymbol{\sigma}_2,v_2\right)-(\nu\boldsymbol{\sigma}_2,\nabla v_2)\\
-\left(\begin{pmatrix}\Delta u_1\\ \Delta u_2\end{pmatrix},\nabla v_d\right)+\left\langle\begin{pmatrix}\hat{u}_1\\ \hat{u}_2\end{pmatrix},v_d\cdot\mathbf{n}\right\rangle &= \left(\begin{pmatrix}u_1\\ u_2\end{pmatrix},\nabla v_d\right)\\
\int_\Omega p &= 0
\end{aligned}
$$

Conservative Form

$$
\begin{aligned}
\nabla\cdot(\boldsymbol{u}\otimes\boldsymbol{u}-\nu\nabla\boldsymbol{u}+p\boldsymbol{I}) &= \begin{pmatrix}f_1\\ f_2\end{pmatrix}\\
\nabla\cdot\boldsymbol{u} &= 0\\
\int_\Omega p &= 0
\end{aligned}
$$

As a first order system, this is

$$
\begin{aligned}
\boldsymbol{\sigma}_1-\nabla u_1 &= 0\\
\boldsymbol{\sigma}_2-\nabla u_2 &= 0\\
\nabla\cdot\left(\boldsymbol{u}\otimes\boldsymbol{u}-\nu\begin{pmatrix}\boldsymbol{\sigma}_1\\ \boldsymbol{\sigma}_2\end{pmatrix}+p\boldsymbol{I}\right) &= \begin{pmatrix}f_1\\ f_2\end{pmatrix}\\
\nabla\cdot\boldsymbol{u} &= 0\\
\int_\Omega p &= 0
\end{aligned}
$$

Multiplying by $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, $\boldsymbol{u}$, and q, and integrating by parts

$$\begin{aligned}
(\boldsymbol{\sigma}_1, \boldsymbol{\tau}_1) + (u_1, \nabla\cdot\boldsymbol{\tau}_1) - \langle \hat{u}_1, \tau_{1n}\rangle &= 0\\
(\boldsymbol{\sigma}_2, \boldsymbol{\tau}_2) + (u_2, \nabla\cdot\boldsymbol{\tau}_2) - \langle \hat{u}_2, \tau_{2n}\rangle &= 0\\
-\left(\boldsymbol{u}\otimes\boldsymbol{u} - \nu\begin{pmatrix}\boldsymbol{\sigma}_1\\ \boldsymbol{\sigma}_2\end{pmatrix} + p\boldsymbol{I}, \nabla\boldsymbol{v}\right) + \left\langle \hat{\boldsymbol{f}}, \boldsymbol{v}\right\rangle &= \begin{pmatrix} f_1\\ f_2\end{pmatrix}\\
-(\boldsymbol{u}, \nabla q) + \langle \widehat{\boldsymbol{u}\cdot\boldsymbol{n}}, q\rangle &= 0\\
\int_\Omega p &= 0
\end{aligned}$$

where

$$\hat{\boldsymbol{f}} = \operatorname{tr}\left(\left(\boldsymbol{u}\otimes\boldsymbol{u} - \nu\begin{pmatrix}\boldsymbol{\sigma}_1\\ \boldsymbol{\sigma}_2\end{pmatrix} + p\boldsymbol{I}\right)\cdot\boldsymbol{n}\right)$$

Linearizing:

$$\begin{aligned}
(\boldsymbol{\sigma}_1, \boldsymbol{\tau}_1) + (\Delta u_1, \nabla\cdot\boldsymbol{\tau}_1) - \langle \hat{u}_1, \tau_{1n}\rangle &= -(u_1, \nabla\cdot\boldsymbol{\tau}_1)\\
(\boldsymbol{\sigma}_2, \boldsymbol{\tau}_2) + (\Delta u_2, \nabla\cdot\boldsymbol{\tau}_2) - \langle \hat{u}_2, \tau_{2n}\rangle &= -(u_2, \nabla\cdot\boldsymbol{\tau}_2)\\
-\left(\Delta\boldsymbol{u}\otimes\boldsymbol{u} + \boldsymbol{u}\otimes\Delta\boldsymbol{u} - \nu\begin{pmatrix}\boldsymbol{\sigma}_1\\ \boldsymbol{\sigma}_2\end{pmatrix} + p\boldsymbol{I}, \nabla\boldsymbol{v}\right) + \left\langle \hat{\boldsymbol{f}}, \boldsymbol{v}\right\rangle &= (\boldsymbol{u}\otimes\boldsymbol{u}, \nabla\boldsymbol{v})\\
-(\Delta\boldsymbol{u}, \nabla q) + \langle \widehat{\boldsymbol{u}\cdot\boldsymbol{n}}, q\rangle &= (\boldsymbol{u}, \nabla q)\\
\int_\Omega p &= 0
\end{aligned}$$

Grouping things together, we get

$$\begin{aligned}
(\boldsymbol{\sigma}_1, \boldsymbol{\tau}_1 + \nu\nabla v_1)\\
(\boldsymbol{\sigma}_2, \boldsymbol{\tau}_2 + \nu\nabla v_2)\\
\left(\Delta u_1, \nabla\cdot\boldsymbol{\tau}_1 - \left(\boldsymbol{u}\cdot\nabla v_1 + \boldsymbol{u}\cdot\frac{\partial\boldsymbol{v}}{\partial x}\right) - \frac{\partial q}{\partial x}\right)\\
\left(\Delta u_2, \nabla\cdot\boldsymbol{\tau}_2 - \left(\boldsymbol{u}\cdot\nabla v_2 + \boldsymbol{u}\cdot\frac{\partial\boldsymbol{v}}{\partial y}\right) - \frac{\partial q}{\partial y}\right)\\
(p, -\nabla\cdot\boldsymbol{v})
\end{aligned}$$