PECOS

Predictive Engineering and Computational Sciences

Locally Conservative Discontinuous Petrov-Galerkin for Fluid Problems

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A Summary of DPG

Overview of Features

- Robust for singularly perturbed problems
- Stable in the preasymptotic regime
- Designed for adaptive mesh refinement

DPG is a minimum residual method:

$$u_h = \underset{w_h \in U_h}{\arg\min} \frac{1}{2} \|Bw_h - l\|_{V'}^2$$

$$\Updownarrow$$

$$b(u_h, R_V^{-1} B\delta u_h) = l(R_V^{-1} B\delta u_h) \quad \forall \delta u_h \in U_h$$

where $v_{\delta u_h} := R_V^{-1} B\delta u_h$ are the optimal test functions.

DPG for Convection-Diffusion

Start with the strong-form PDE.

$$\nabla \cdot (\boldsymbol{\beta} u) - \epsilon \Delta u = g$$

Rewrite as a system of first-order equations.

$$\begin{aligned} \nabla \cdot (\boldsymbol{\beta} u - \boldsymbol{\sigma}) &= g \\ \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= \mathbf{0} \end{aligned}$$

Multiply by test functions and integrate by parts over each element, K.

$$\begin{aligned} -(\boldsymbol{\beta} u - \boldsymbol{\sigma}, \nabla v)_K + ((\boldsymbol{\beta} u - \boldsymbol{\sigma}) \cdot \mathbf{n}, v)_{\partial K} &= (g, v)_K \\ \frac{1}{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\tau})_K + (u, \nabla \cdot \boldsymbol{\tau})_K - (u, \tau_n)_{\partial K} &= 0 \end{aligned}$$

Use the ultraweak (DPG) formulation to obtain bilinear form $b(u, v) = l(v)$.

$$\begin{aligned} -(\boldsymbol{\beta} u - \boldsymbol{\sigma}, \nabla v)_K + (\hat{f}, v)_{\partial K} + \frac{1}{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\tau})_K \\ + (u, \nabla \cdot \boldsymbol{\tau})_K - (\hat{u}, \tau_n)_{\partial K} = (g, v)_K \end{aligned}$$

Local Conservation

The local conservation law in convection diffusion is

$$\int_{\partial K} \hat{f} = \int_K g \,,$$

which is equivalent to having $\mathbf{v}_K := \{v, \boldsymbol{\tau}\} = \{1_K, \mathbf{0}\}$ in the test space. In general, this is not satisfied by the optimal test functions. Following Moro et al[5] (also Chang and Nelson[2]), we can enforce this condition with Lagrange multipliers:

$$L(u_h, \boldsymbol{\lambda}) = \frac{1}{2} \left\| R_V^{-1}(Bu_h - l) \right\|_V^2 - \sum_K \lambda_K \underbrace{\langle Bu_h - l, \mathbf{v}_K \rangle}_{\langle \hat{f}, 1_K \rangle_{\partial K} - \langle g, 1_K \rangle_K} \,,$$

where $\boldsymbol{\lambda} = \{\lambda_1, \cdots, \lambda_N\}$.

Local Conservation

Finding the critical points of $L(u, \boldsymbol{\lambda})$, we get the following equations[4].

$$\frac{\partial L(u_h, \boldsymbol{\lambda})}{\partial u_h} = b(u_h, R_V^{-1} B\delta u_h) - l(R_V^{-1} B\delta u_h) - \sum_K \lambda_K b(\delta u_h, \mathbf{v}_K) = 0 \quad \forall \delta u_h \in U_h$$

$$\frac{\partial L(u_h, \boldsymbol{\lambda})}{\partial \lambda_K} = -b(u_h, \mathbf{v}_K) + l(\mathbf{v}_K) = 0 \quad \forall K$$

A few consequences:

- We've turned our minimization problem into a saddlepoint problem.
- Only need to find the optimal test function in the orthogonal complement of constants.

Optimal Test Functions

For each $\mathbf{u} = \{u, \boldsymbol{\sigma}, \hat{u}, \hat{f}\} \in \mathbf{U}_h$, find $\mathbf{v_u} = \{v_\mathbf{u}, \boldsymbol{\tau}_\mathbf{u}\} \in \mathbf{V}$ such that

$$(\mathbf{v_u}, \mathbf{w})_\mathbf{V} = b(\mathbf{u}, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}$$

where $\mathbf{V}$ becomes $\mathbf{V}_{p+\Delta p}$ in order to make this computationally tractable. We recently developed this modification to the *robust test norm* [1] which behaves better in the presence of singularities.

$$\|(v, \boldsymbol{\tau})\|^2_{\mathbf{V},\Omega_h} = \left\| \min\left\{\frac{1}{\sqrt{\epsilon}}, \frac{1}{\sqrt{|K|}}\right\} \boldsymbol{\tau} \right\|^2 + \|\nabla \cdot \boldsymbol{\tau} - \boldsymbol{\beta} \cdot \nabla v\|^2$$
$$+ \|\boldsymbol{\beta} \cdot \nabla v\|^2 + \epsilon \|\nabla v\|^2 \quad \underbrace{+ \|v\|^2}_{\text{No longer necessary}}$$

Optimal Test Functions

For each $\mathbf{u} = \{u, \boldsymbol{\sigma}, \hat{u}, \hat{f}\} \in \mathbf{U}_h$, find $\mathbf{v_u} = \{v_\mathbf{u}, \boldsymbol{\tau}_\mathbf{u}\} \in \mathbf{V}$ such that

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$$\begin{aligned}\left\|(v, \boldsymbol{\tau})\right\|^2_{\mathbf{V},\Omega_h} = & \left\|\min\left\{\frac{1}{\sqrt{\epsilon}}, \frac{1}{\sqrt{|K|}}\right\}\boldsymbol{\tau}\right\|^2 + \left\|\nabla\cdot\boldsymbol{\tau} - \boldsymbol{\beta}\cdot\nabla v\right\|^2 \\ & + \left\|\boldsymbol{\beta}\cdot\nabla v\right\|^2 + \epsilon\left\|\nabla v\right\|^2 \underbrace{+ \left(\frac{1}{|K|}\int_K v\right)^2}_{\text{Zero mean term}}\end{aligned}$$

Stability Analysis

We follow Brezzi's theory for an abstract mixed problem:

$$\left\{\begin{array}{lll} \boldsymbol{u} \in \boldsymbol{U}, p \in Q & & \\ a(\boldsymbol{u}, \boldsymbol{w}) + c(p, \boldsymbol{w}) & = l(\boldsymbol{w}) & \forall \boldsymbol{w} \in \boldsymbol{U} \\ c(q, \boldsymbol{u}) & = g(q) & \forall q \in Q \end{array}\right.$$

where a, c, l, g denote the appropriate bilinear and linear forms. Note that $a(\boldsymbol{u}, \boldsymbol{w}) = b(\boldsymbol{u}, R_V^{-1} B\boldsymbol{w}) = (R_V^{-1} B\boldsymbol{u}, R_V^{-1} B\boldsymbol{w})_V$.
Let ψ denote the $\boldsymbol{H}(\mathrm{div}, \Omega)$ extension of flux $\hat{f}$ that realizes the minimum in the definition of the quotient (minimum energy extension) norm.
The norm for the Lagrange multipliers λ_K is implied by the quotient norm used for $H^{-1/2}(\Gamma_h)$ and continuity bound for form $c(p, \boldsymbol{w})$:

$$||\boldsymbol{\lambda}|| := \left(\sum_K \mu(K) \lambda_K^2 \right)^{1/2}$$

Inf Sup Condition

The inf-sup condition relating spaces $\boldsymbol{U}$ and Q is

$$\sup_{\boldsymbol{w}\in\boldsymbol{U}} \frac{|c(p,\boldsymbol{w})|}{||\boldsymbol{w}||_{\boldsymbol{U}}} \geq \beta ||p||_Q$$

Let

$$R\ :\ L^2(\Omega) \ni q \to \boldsymbol{\psi} \in \boldsymbol{H}(\mathrm{div},\Omega) \cap \boldsymbol{H}^1(\Omega) = \boldsymbol{H}^1(\Omega)$$

be the continuous right inverse of the divergence operator constructed by Costabel and McIntosh[3]. Let $\boldsymbol{\psi}_h$ denote the classical, lowest order Raviart-Thomas (RT) interpolant of function

$$\boldsymbol{\psi} = R(\sum_K \lambda_K 1_K)$$

Note that $\mathrm{div}\boldsymbol{\psi}_h = \mathrm{div}\boldsymbol{\psi} = \lambda_K$ in element K.

Inf Sup Condition

Classical h-interpolation error estimate for the lowest error Raviart-Thomas elements and continuity of operator R imply the stability estimate:

$$
\begin{aligned}
||\psi_h|| &\leq ||\psi_h - \psi|| + ||\psi|| \\
&\leq Ch||\psi||_{H^1} + ||\psi|| \\
&\leq C||\mathrm{div}\psi|| = C(\textstyle\sum_K \mu(K)\lambda_K^2)^{1/2}
\end{aligned}
$$

Let $\hat{f}$ be the trace of ψ_h, then

$$
\begin{aligned}
\sup_{\hat{f}\in H^{-1/2}(\Gamma_h)} \frac{|\sum_K \lambda_K \langle \hat{f}, 1_K\rangle_{\partial K}|}{||\hat{f}||_{H^{-1/2}(\Gamma_h)}} &\geq \frac{|\sum_K \lambda_K \int_K \mathrm{div}\psi_h \, 1_K|}{||\psi_h||_{H(\mathrm{div},\Omega)}} \\
&\geq \frac{1}{C}(\sum_K \mu(K)\lambda_K^2)^{1/2}
\end{aligned}
$$

Inf Sup in Kernel Condition

We characterize the "kernel" space:

$$\begin{aligned} \boldsymbol{U}_0 := & \quad \{\boldsymbol{w} \in \boldsymbol{U} \,:\, c(q, \boldsymbol{w}) = 0 \quad \forall q \in Q\} \\ = & \quad \{(u, \boldsymbol{\sigma}, \hat{u}, \hat{t}) \,:\, \langle \hat{t}, 1_K \rangle = 0 \quad \forall K\} \end{aligned}$$

With $\boldsymbol{u} \in \boldsymbol{U}_0$, we have then:

$$\begin{aligned} \sup_{\boldsymbol{w} \in \boldsymbol{U}_0} \frac{|a(\boldsymbol{u}, \boldsymbol{w})|}{||\boldsymbol{w}||_{\boldsymbol{U}}} \geq \frac{|b(\boldsymbol{u}, T\boldsymbol{u})|}{||\boldsymbol{u}||} &= \frac{|b(\boldsymbol{u}, T\boldsymbol{u})|}{||T\boldsymbol{u}||} \frac{||T\boldsymbol{u}||}{||\boldsymbol{u}||} \\ &= \sup_{(v, \boldsymbol{\tau})} \frac{|b((u, \boldsymbol{\sigma}, \hat{u}, \hat{t}), (v, \boldsymbol{\tau}))|}{||(v, \boldsymbol{\tau})||} \frac{||T\boldsymbol{u}||}{||\boldsymbol{u}||} \geq \gamma^2 ||(u, \boldsymbol{\sigma}, \hat{u}, \hat{t})|| \end{aligned}$$

where γ is the stability constant for the standard DPG formulation. The FE error is bounded by the best approximation error. Note that the exact Lagrange multipliers are zero, so the best approximation error involves only the solution $(u, \boldsymbol{\sigma}, \hat{u}, \hat{t})$.

Robustness Analysis

- We prove robustness of the restricted DPG method by switching to the energy norm in Brezzi's stability analysis.
- The inf-sup in kernel condition is simple. Upon replacing the original norm of solution $\boldsymbol{u}$ with the energy norm, γ and the continuity constant become one.
- In order to investigate the robustness of inf-sup constant β, we need to understand what the energy norm of flux variable $\hat{f}$ is.
- For an element, K, we solve for the optimal test functions, $v_K \in H^1(K)$, and $\boldsymbol{\tau}_K \in \boldsymbol{H}(\mathrm{div}, K)$ corresponding to flux $\hat{f}$:

$$((v_K, \boldsymbol{\tau}_K), (\delta v, \delta \boldsymbol{\tau}))_V = \left\langle \hat{f}, \delta v \right\rangle \quad \forall \delta v \in H^1(K),\, \delta \boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div}, K)$$

- The energy norm for $\hat{f}$ is then

$$||\hat{f}||_E^2 = \sum_K ||(v_K, \boldsymbol{\tau}_K)||_V^2$$

Robustness Analysis

- Need to establish conditions under which the inf-sup constant is independent of viscosity.

$$\sup_{\hat{f}} \frac{|\sum_K \lambda_K \langle \hat{f}, 1_K \rangle|}{||\hat{f}||_E} \geq \beta \left(\sum_K \mu(K) \lambda_K^2 \right)^{1/2}$$

- Select $\hat{f}$ as the trace of the Raviart-Thomas interpolant $\boldsymbol{\psi}_h$ of $\boldsymbol{\psi} = R(\sum_K \lambda_K 1_K)$.
- Proceed as in the previous analysis, but evaluation of the norm of $\hat{f}$ requires a local solve.

$$((v, \boldsymbol{\tau}),(\delta v, \delta\boldsymbol{\tau}))_V = \left\langle \hat{f}, \delta v \right\rangle_{\partial K} = \int_K \mathrm{div} \boldsymbol{\psi}_h \delta v = \int_K \mathrm{div} \boldsymbol{\psi} \delta v$$
$$= \int_K \lambda_K \delta v = \lambda_K (1_K, \delta v)_K \quad \forall \delta v \in H^1(K) \, \forall \delta\boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div}, K)$$

Robustness Analysis

- We need an upper bound for the energy norm of $(v_h, \boldsymbol{\tau}_h)$:

$$\left(\sum_K ||(v, \boldsymbol{\tau})||_V^2\right)^{1/2}$$

- Substituting $(v, \boldsymbol{\tau})$ for $(\delta v, \delta\boldsymbol{\tau})$ in the previous slide:

$$||(v, \boldsymbol{\tau})||_V^2 = \lambda_K (1_K, v_k)$$

- With a robust stability estimate $(1_K, v_K) \leq C\mu(K)^{1/2}||(v, \boldsymbol{\tau})||_K$,

$$||(v, \boldsymbol{\tau})||_V \leq C\mu(K)^{1/2}|\lambda_K|$$

$$\sum_K ||(v, \boldsymbol{\tau})||_V^2 \leq C^2 \sum_K \mu(K)\lambda_K^2$$

which leads to the robust estimate of inf-sup constant β.

Robustness Analysis

- Finally, we need a continuity estimate on

$$\sum_K \lambda_K \langle \hat{f}, 1_K \rangle$$

- Testing with $(1_K, \mathbf{0})$ in the local problem, we obtain

$$((v, \boldsymbol{\tau}), (1_K, \mathbf{0}))_V = \langle \hat{f}, 1_K \rangle_{\partial K}$$

- With a robust estimate $|((v, \boldsymbol{\tau}), (1_K, \mathbf{0}))_V| \leq C\mu(K)^{1/2} ||(v, \boldsymbol{\tau})||_V$,

$$\begin{aligned} |\sum_K \lambda_K \langle \hat{f}, 1_K \rangle| &\leq C \left(\sum_K \mu(K) \lambda_K^2 \right)^{1/2} \left(\sum_K ||(v, \boldsymbol{\tau})||_V^2 \right)^{1/2} \\ &= C \left(\sum_K \mu(K) \lambda_K^2 \right)^{1/2} ||\hat{f}||_E \end{aligned}$$

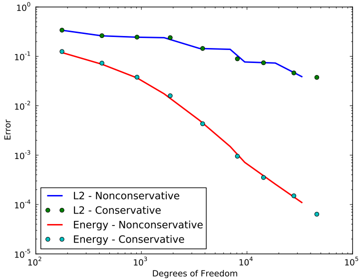
Erickson-Johnson Problem

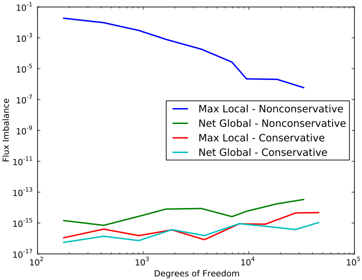
On domain $\Omega = [0,1]^2$, with $\beta = (1,0)^T$, $f = 0$ and boundary conditions

$$\hat{f} = u_0, \quad \beta_n \leq 0\,, \qquad \hat{u} = 0, \quad \beta_n > 0$$

Separation of variabes gives an analytic solution

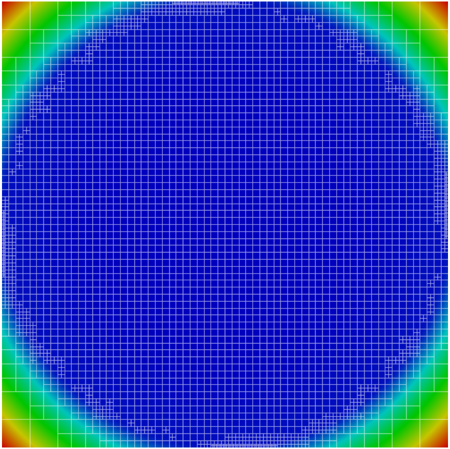
$$u(x,y) = C_0 + \sum_{n=1}^{\infty} C_n \frac{\exp(r_2(x-1)) - \exp(r_1(x-1))}{r_1 \exp(-r_2) - r_2 \exp(-r_1)} \cos(n\pi y)$$





Vortex Problem

After 6 refinements, $\epsilon = 10^{-4}$, $\beta = (-y, x)^T$



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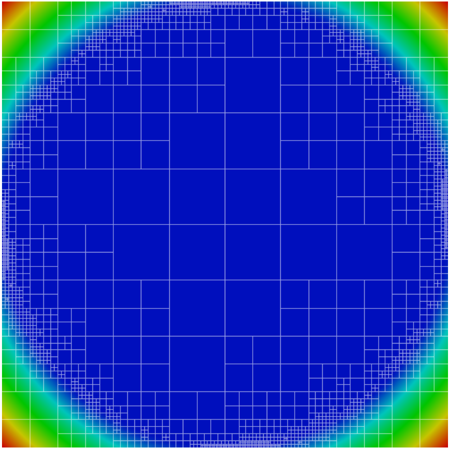

Conservative

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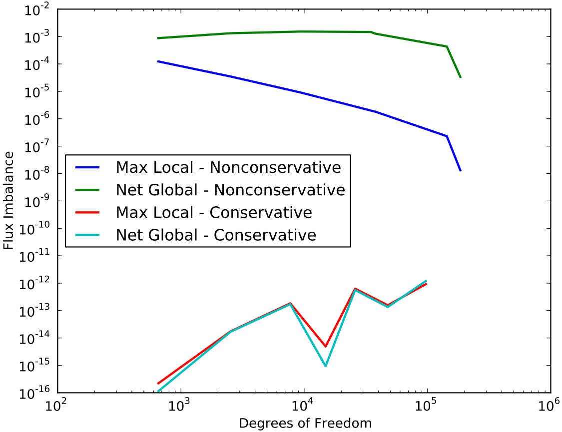
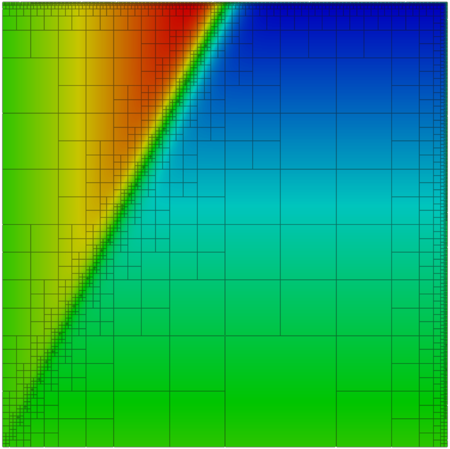


Discontinuous Source Problem

After 8 refinements, $\epsilon = 10^{-3}$, $\beta = (0.5, 1)^T/\sqrt{1.25}$, $\hat{g} = \begin{cases} 1, & y >= 2x \\ 0, & y < 2x \end{cases}$



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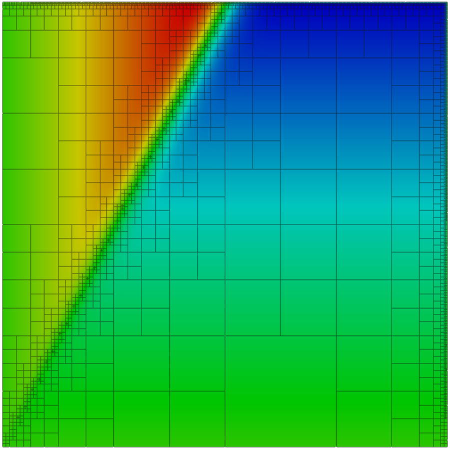

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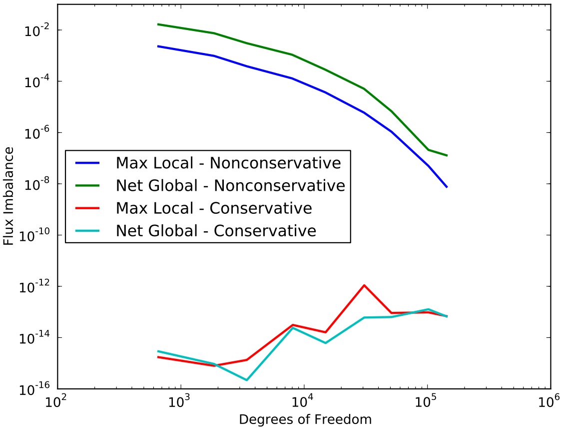
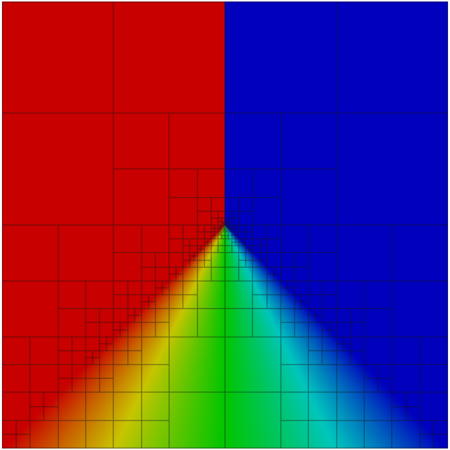


Inviscid Burgers' Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \quad \Leftrightarrow \quad \nabla_{x,t} \cdot \begin{pmatrix} \frac{u^2}{2} \\ u \end{pmatrix} = 0$$



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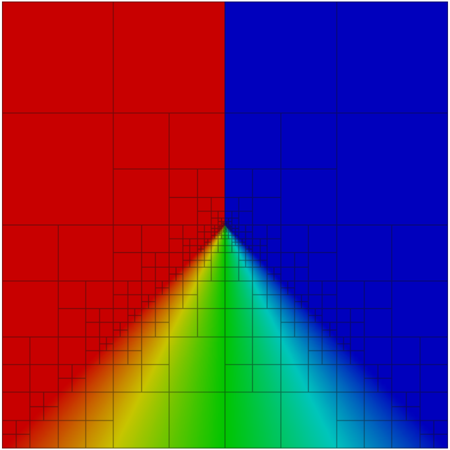

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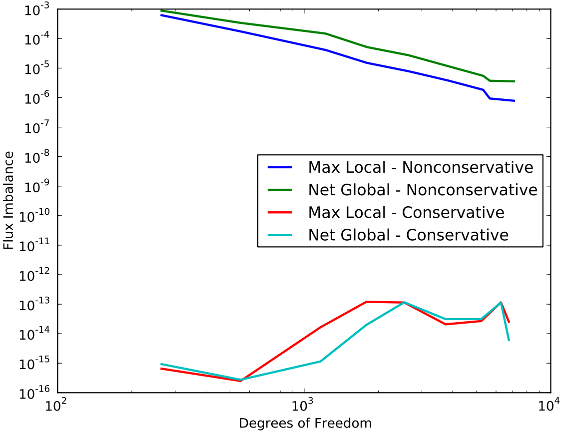
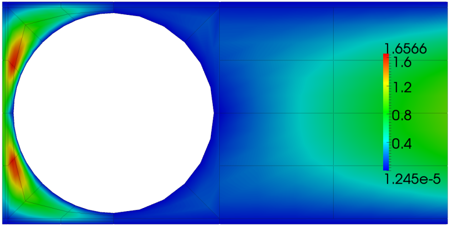
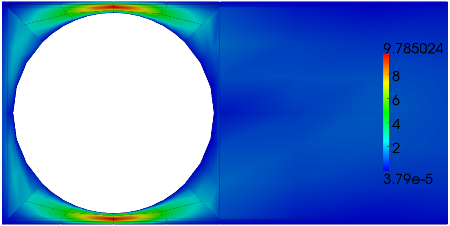
Inviscid Burgers' Equation



Stokes Flow Around a Cylinder



1 Refinement



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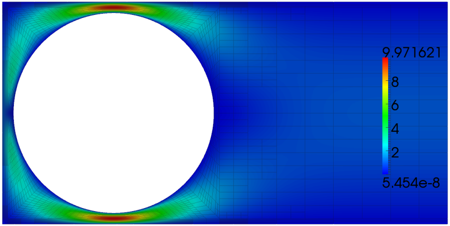
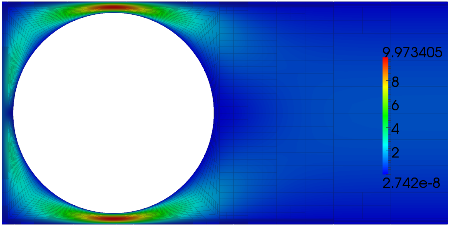


6 Refinements

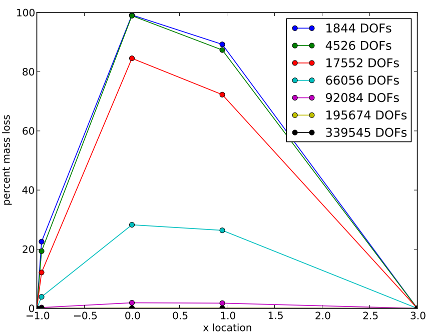


6 Refinements

Nonconservative

Conservative

Stokes Flow Around a Cylinder



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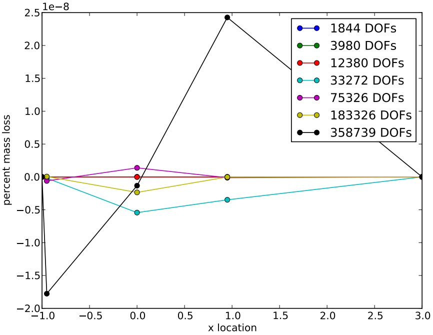

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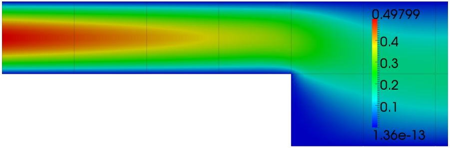
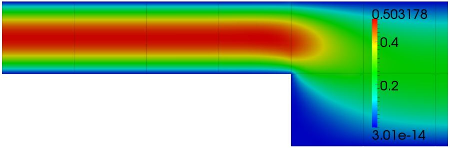
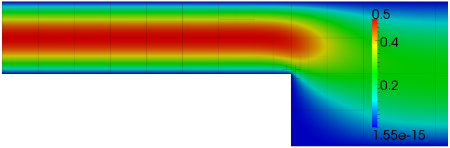
Stokes Flow Backward Facing Step



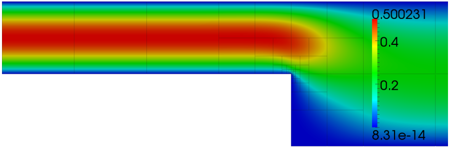
Initial Mesh



Initial Mesh



8 Refinements

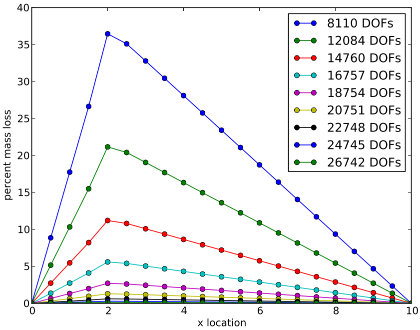


8 Refinements

Nonconservative

Conservative

Stokes Flow Backward Facing Step



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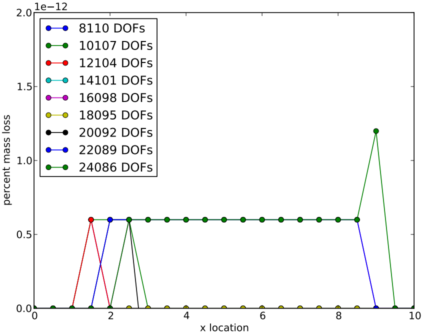

Conservative

Summary

What have we done?

- We've turned our minimization problem into a saddlepoint problem.
- The change is computationally feasible.
- Mathematically, it gets rid of troublesome term.

Does it make a difference?

- Enforcement changes refinement strategy.
- Some improvement on condition number for local solves.
- Standard DPG is nearly conservative for many of the problems considered, but seems to suffer from mass loss similar to other LSFEM methods.
- For some problems, local conservation allows us to converge to a reasonable solution on a much coarser mesh.

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