Time Stepping Notes

Truman E. Ellis

October 17, 2013

We wish to solve the system

$$\frac{\partial U}{\partial t} + f(U) = 0$$

For a 6 stage ESDIRK algorithm, we have the Butcher Tableau:

$$\begin{array}{c|cccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c_1 & a_{10} & a_{11} & 0 & 0 & 0 & 0 \\
c_2 & a_{20} & a_{21} & a_{22} & 0 & 0 & 0 \\
c_3 & a_{30} & a_{31} & a_{32} & a_{33} & 0 & 0 \\
c_4 & a_{40} & a_{41} & a_{42} & a_{43} & a_{44} & 0 \\
c_5 & a_{50} & a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \\
\hline
 & b_0 & b_1 & b_2 & b_3 & b_4 & b_5
\end{array}$$

For ESDIRK schemes, we enforce a "stiffly accurate" condition by requiring that $a_{sj} = b_j$, where s is the number of stages minus one. We also have the condition that the final stage equals the value at the next time step. So there is no reason to reassemble from the various stages.

We solve a series of equations for $k = 0, \cdots, 5$

$$\frac{U^k}{a_{kk}\Delta t} + f(U^k) = \frac{U_n}{a_{kk}\Delta t} - \sum_{j=0}^{k-1} \frac{a_{kj}}{a_{kk}} f(U^j)$$

From the first equation we see that $U^0 = U_n$. And we have that $U_{n+1} = U^s$.

For a nonlinear system, define residual

$$R(U^k) = \frac{U^k}{a_{kk}\Delta t} + f(U^k) - \frac{U_n}{a_{kk}\Delta t} + \sum_{j=0}^{k-1} \frac{a_{kj}}{a_{kk}} f(U^j)$$

Given an approximate solution $\tilde{U}^k$, we wish to solve for an increment ΔU such that $U^k = \tilde{U}^k + \Delta U$ is a better approximation of the exact solution. Approximating $R(U^k)$ by $R(\tilde{U}^k) + R'(\tilde{U}^k)\Delta U = 0$, we obtain a linearized equation

$$\frac{\Delta U}{a_{kk}\Delta t} + f'(\tilde{U}^k)\Delta U = \frac{U_n}{a_{kk}\Delta t} - \frac{\tilde{U}^k}{a_{kk}\Delta t} - f(\tilde{U}^k) - \sum_{j=0}^{k-1} \frac{a_{kj}}{a_{kk}} f(U^j)$$

For implicit Euler, this is just

$$R(U) = \frac{U}{\Delta t} + f(U) - \frac{U_n}{\Delta t}$$

$$\frac{\Delta U}{\Delta t} + f'(\tilde{U})\Delta U = \frac{U_n}{\Delta t} - \frac{\tilde{U}}{\Delta t} - f(\tilde{U})$$

Case Studies

Convection-Diffusion

$$\frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u = 0$$

$$\frac{\partial u}{\partial t} + \nabla \cdot (\boldsymbol{\beta} u - \boldsymbol{\sigma}) - f = 0$$

$$\left(\frac{1}{\epsilon}\boldsymbol{\sigma}, \boldsymbol{\tau}\right) + (u, \nabla \cdot \boldsymbol{\tau}) - \langle \hat{u}, \tau_n \rangle = 0$$

$$\left(\frac{\partial u}{\partial t}, v\right) - (\boldsymbol{\beta} u - \boldsymbol{\sigma}, \nabla v) + \left\langle \hat{f}, v \right\rangle - (f, v) = 0$$

Here

$$f(U) = \left(\frac{1}{\epsilon}\boldsymbol{\sigma}, \boldsymbol{\tau}\right) + (u, \nabla \cdot \boldsymbol{\tau}) - \langle \hat{u}, \tau_n \rangle - (\boldsymbol{\beta} u - \boldsymbol{\sigma}, \nabla v) + \left\langle \hat{f}, v \right\rangle - (f, v)$$

But we choose not to split the traces and fluxes, so instead we have

$$f(U) = \left(\frac{1}{\epsilon}\boldsymbol{\sigma}, \boldsymbol{\tau}\right) + (u, \nabla \cdot \boldsymbol{\tau}) - (\boldsymbol{\beta} u - \boldsymbol{\sigma}, \nabla v) - (f, v)$$

while

$$f'(U) = \left(\frac{1}{\epsilon}\boldsymbol{\sigma}, \boldsymbol{\tau}\right) + (u, \nabla \cdot \boldsymbol{\tau}) - \langle \hat{u}, \tau_n \rangle - (\boldsymbol{\beta} u - \boldsymbol{\sigma}, \nabla v) + \left\langle \hat{f}, v \right\rangle$$

So, given initial solution u_n and approximate solution $\tilde{U} = \{\tilde{u}, \tilde{\boldsymbol{\sigma}}\}$, we arrive at a system

$$\begin{aligned}\frac{\delta u}{\Delta t} + \left(\frac{1}{\epsilon}\delta\boldsymbol{\sigma}, \boldsymbol{\tau}\right) + (\delta u, \nabla \cdot \boldsymbol{\tau}) - \langle \hat{u}, \tau_n \rangle - (\boldsymbol{\beta}\delta u - \delta\boldsymbol{\sigma}, \nabla v) + \left\langle \hat{f}, v \right\rangle \\ = \frac{u_n}{\Delta t} + \frac{\tilde{u}}{\Delta t} - \left(\frac{1}{\epsilon}\tilde{\boldsymbol{\sigma}}, \boldsymbol{\tau}\right) - (\tilde{u}, \nabla \cdot \boldsymbol{\tau}) + (\boldsymbol{\beta}\tilde{u} - \boldsymbol{\sigma}, \nabla v) + (f, v)\end{aligned}$$

If we let $\tilde{U} = 0$, then $U_{n+1} = \tilde{U} + \delta U = \delta U$ and since this is a linear equation, we arrive at

$$\frac{u}{\Delta t} + \left(\frac{1}{\epsilon}\boldsymbol{\sigma}, \boldsymbol{\tau}\right) + (u, \nabla \cdot \boldsymbol{\tau}) - \langle \hat{u}, \tau_n \rangle - (\boldsymbol{\beta} u - \boldsymbol{\sigma}, \nabla v) + \left\langle \hat{f}, v \right\rangle = \frac{u_n}{\Delta t} + (f, v)$$

which is exactly our linear system.