Space-Time Navier-Stokes with Entropy Variables

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October 2, 2014

The Navier-Stokes equations are

$$\nabla_{xt} \cdot \begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{U} \end{pmatrix} = \boldsymbol{\mathcal{F}} \tag{1}$$

where

$$\boldsymbol{U} = \begin{pmatrix} U_c \\ \boldsymbol{U}_m \\ U_e \end{pmatrix} = \rho \begin{pmatrix} 1 \\ \boldsymbol{u} \\ e_0 \end{pmatrix}$$

and

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{F}_1 - \sum_{j=1}^d K_{1j} \boldsymbol{U}_{,j} & \boldsymbol{F}_2 - \sum_{j=1}^d K_{2j} \boldsymbol{U}_{,j} & \boldsymbol{F}_3 - \sum_{j=1}^d K_{3j} \boldsymbol{U}_{,j} \end{bmatrix}$$

Multiplying by test function $\boldsymbol{W}$ and integrating by parts, we get

$$-\left(\begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{U} \end{pmatrix}, \nabla_{xt} \boldsymbol{W} \right) + \left\langle \hat{\boldsymbol{T}}, \boldsymbol{W} \right\rangle = (\boldsymbol{\mathcal{F}}, \boldsymbol{W}) \tag{2}$$

Assuming summation on i, we can also write this

$$-(\boldsymbol{U}, \boldsymbol{W}_{,t}) - \left(\boldsymbol{F}_i - \sum_{j=1}^d K_{ij} \boldsymbol{U}_{,j}, \boldsymbol{W}_{,i} \right) + \left\langle \hat{\boldsymbol{T}}, \boldsymbol{W} \right\rangle = (\boldsymbol{\mathcal{F}}, \boldsymbol{W}) \tag{3}$$

Consider a change of variables: $\boldsymbol{U} = \boldsymbol{U}(\boldsymbol{V})$ and linearization $\boldsymbol{V} \approx \tilde{\boldsymbol{V}} + \Delta \boldsymbol{V}$. Then

$$\begin{aligned}
&-\left(\boldsymbol{U}(\tilde{\boldsymbol{V}}) + \boldsymbol{U}_{,\boldsymbol{V}}(\tilde{\boldsymbol{V}}) \Delta \boldsymbol{V}, \boldsymbol{W}_{,t} \right) \\
&-\left(\boldsymbol{F}_i(\boldsymbol{U}(\tilde{\boldsymbol{V}})) + \boldsymbol{F}_{i,\boldsymbol{U}}(\boldsymbol{U}(\tilde{\boldsymbol{V}})) \boldsymbol{U}_{,\boldsymbol{V}}(\tilde{\boldsymbol{V}}) \Delta \boldsymbol{V}, \boldsymbol{W}_{,i} \right) \\
&-\sum_{j=1}^d \left(K_{ij}(\boldsymbol{U}(\tilde{\boldsymbol{V}})) \boldsymbol{U}_{,V} \tilde{\boldsymbol{V}}_{,j} + K_{ij,\boldsymbol{U}} \boldsymbol{U}_{,\boldsymbol{V}}(\tilde{\boldsymbol{V}}) \boldsymbol{U}_{,\boldsymbol{V}}(\tilde{\boldsymbol{V}}) \tilde{\boldsymbol{V}}_{,j} \Delta \boldsymbol{V} + K_{ij} \boldsymbol{U}_{,\boldsymbol{V}}(\tilde{\boldsymbol{V}}) \Delta \boldsymbol{V}_{,j}, \boldsymbol{W}_{,i} \right) \\
&+\left\langle \hat{\boldsymbol{T}}, \boldsymbol{W}_{,i} \right\rangle = (\boldsymbol{\mathcal{F}}, \boldsymbol{W})
\end{aligned}$$