Space-Time DPG for Fluid Problems

Truman Ellis and Leszek Demkowicz

Institute for Computational Engineering and Sciences, University of Texas

Introduction

The discontinuous Petrov-Galerkin method is a novel finite element framework with exceptional stability and adaptivity properties. The DPG framework can be used to derive stable discretizations of any well-posed variational formulation. In contrast to many other numerical methods, DPG does not suffer from a pre-asymptotic regime (unstable behavior on coarse meshes). This means that a simulation can be initiated at the coarsest scale possible while automatic adaptivity resolves solution features based on a robust, built-in measure of residual error. We present preliminary work on a space-time DPG formulation that enables automatic local time stepping and a kind of parallel-in-time integration.

A Minimum Residual Method

Let U and V be trial and test Hilbert spaces for a well-posed variational problem $b(u,v) = l(v)$. In operator form this is $Bu = l$, where $B : U \to V'$. We seek to minimize the residual for the discrete space $U_h \subset U$:

$$u_h = \underset{w_h \in U_h}{\arg\min}\ \frac{1}{2}\|Bw_h - l\|^2_{V'}$$

Use the Riesz inverse to minimize in the V-norm rather than its dual:

$$\frac{1}{2}\|Bu_h - l\|^2_{V'} = \frac{1}{2}\left\|R_V^{-1}(Bu_h - l)\right\|^2_V = \frac{1}{2}\left(R_V^{-1}(Bu_h - l), R_V^{-1}(Bu_h - l)\right)_V .$$

First order optimality requires the Gâteaux derivative to be zero in all directions $\delta u \in U_h$, i.e.,

$$\left(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u\right)_V = 0, \quad \forall \delta u \in U_h.$$

By definition of the Riesz operator, this is equivalent to

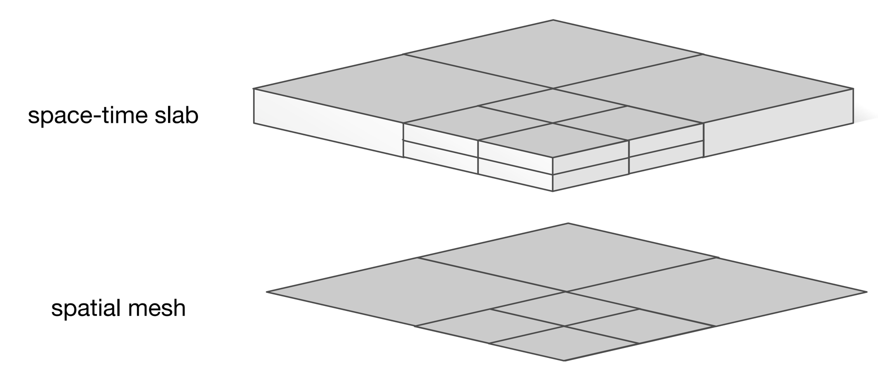
$$\left\langle Bu_h - l, R_V^{-1}B\delta u_h\right\rangle = 0 \quad \forall \delta u_h \in U_h .$$

Identify $v_{\delta u_h} := R_V^{-1}B\delta u_h$ as the optimal test function for trial function δu_h. This gives us

$$b(u_h, v_{\delta u_h}) = l(v_{\delta u_h}).$$

Space-Time DPG

- Time stepping techniques are not ideally suited to highly adaptive grids
- Space-time FEM proposed as a solution
  - ✓ Unified treatment of space and time
  - ✓ Local space-time adaptivity (local time stepping)
  - ✓ Parallel-in-time integration
  - ✗ Spatially stable FEM methods may not be stable in space-time
  - ✗ Need to support higher dimensional problems
- DPG provides necessary stability and adaptivity



Ultra-Weak Transient Convection-Diffusion

Start with the strong-form PDE.

$$\frac{\partial u}{\partial t} + \nabla \cdot (\boldsymbol{\beta} u) - \epsilon \Delta u = g$$

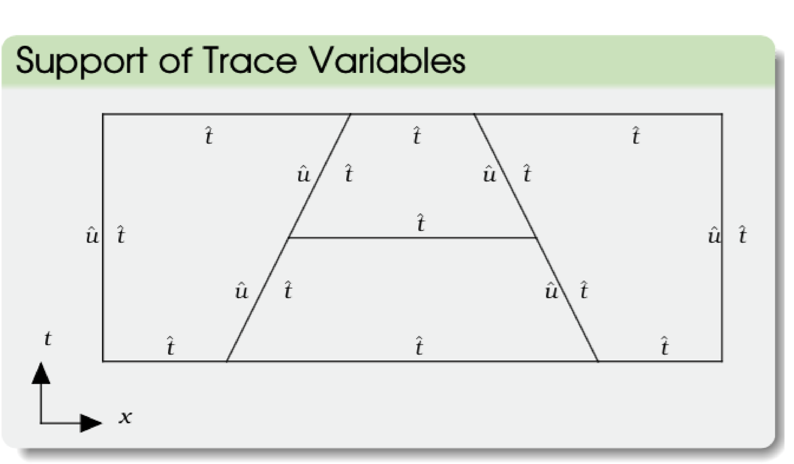
Rewrite as a system of first-order equations.

$$\frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u = \mathbf{0}$$

$$\nabla_{xt} \cdot \begin{pmatrix} \boldsymbol{\beta} u - \boldsymbol{\sigma} \\ u \end{pmatrix} = g$$

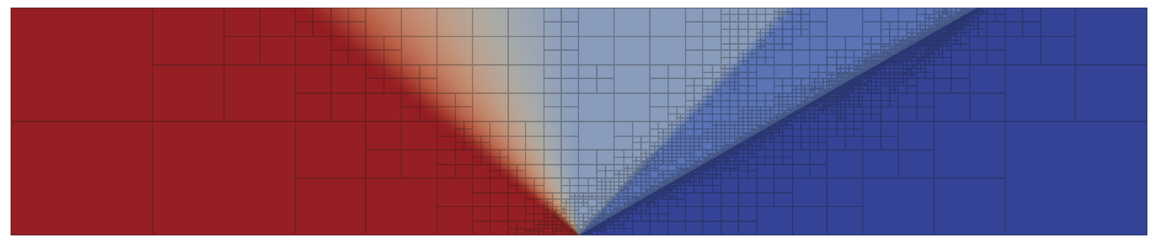
Multiply by test functions and integrate by parts over each element, K. Declare traces and fluxes to be independent unknowns and incorporate boundary conditions to obtain the final variational formulation.

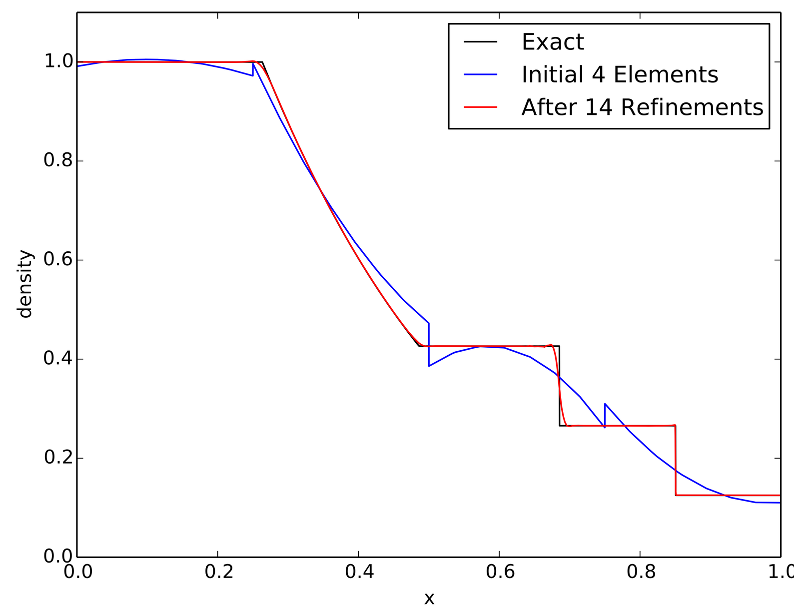
$$\frac{1}{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\tau})_K + (u, \nabla \cdot \boldsymbol{\tau})_K - \langle \hat{u}, \tau_n \rangle_{\partial K}$$
$$- \left(\begin{pmatrix} \boldsymbol{\beta} u - \boldsymbol{\sigma} \\ u \end{pmatrix}, \nabla_{xt} v \right)_K + \left\langle \hat{f}, v \right\rangle_{\partial K} = (g, v)_K$$



Sod Shock-Tube Results

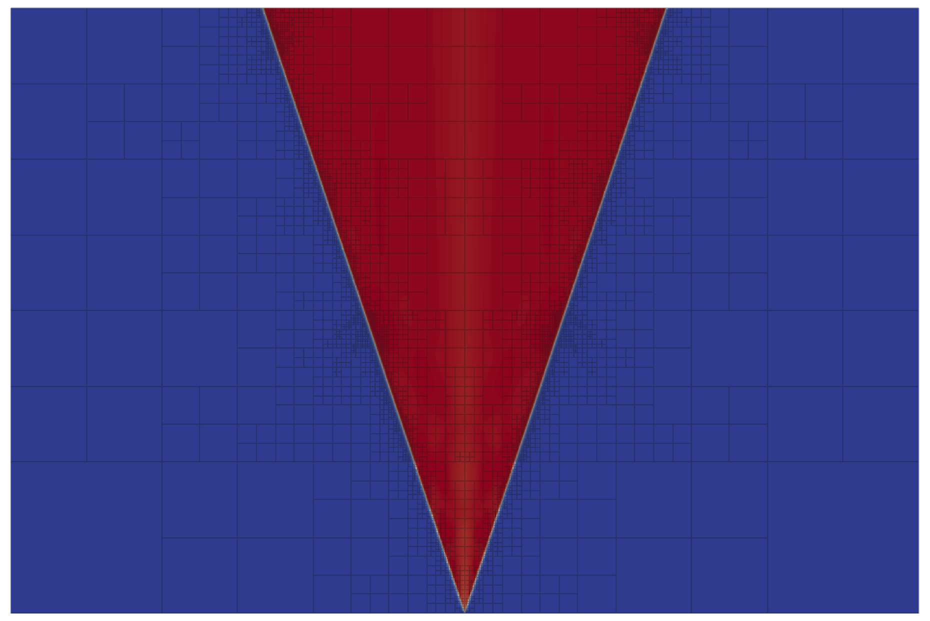
Extension to other fluid problems like compressible Navier-Stokes equation is straightforward.
Sod and Noh results computed with small physical viscosity (10^{-5} and 10^{-3} respectively), but no shock capturing or artificial viscosity.

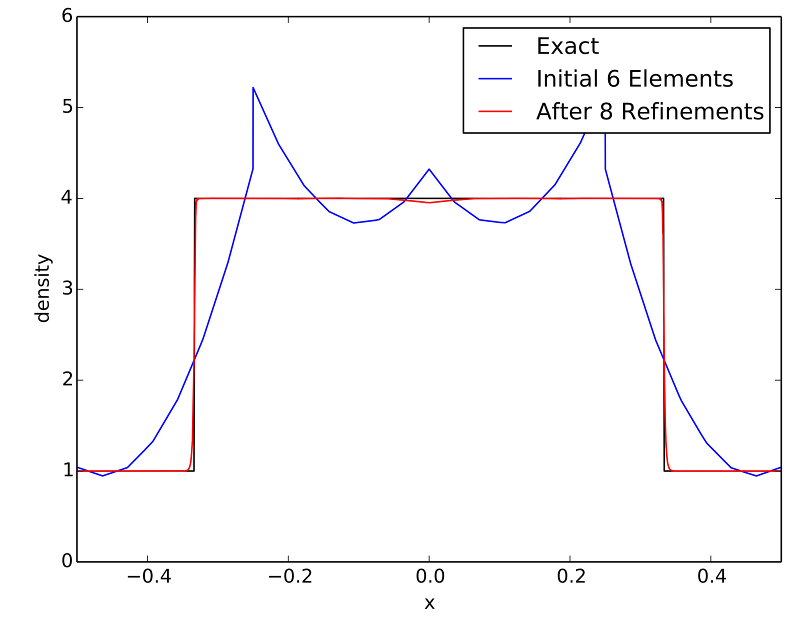




Noh Implosion Results

Computed with a sequence of 4 time slabs.





Conclusions

- DPG provides a stable and adaptive framework for space-time
- Technique is suitable to any well-posed transient problem
- Time slabs can reduce global solve cost
- Proof of robustness for convection-diffusion in preparation

Concurrent Work

- Multiphysics
  - Heat conduction (Poisson and Heat equation)
  - Wave problems (Helmholtz and Maxwell)
  - Linear elasticity and plate problems
  - Convection-Diffusion, Stokes, incompressible Navier-Stokes, compressible Navier-Stokes, Euler
- Natively nonlinear DPG
- DPG for non-Hilbert L^p spaces
- Local conservation
- Iterative solvers
- Entropy scaling for physically meaningful test norms
- General polyhedral elements

Recommended References

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[3] T.E. Ellis, L.F. Demkowicz, and J.L. Chan.
Locally conservative discontinuous Petrov-Galerkin finite elements for fluid problems.
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