Locally Conservative DPG for Fluid Problems

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Introduction

The discontinuous Petrov-Galerkin finite element method has been described as least squares finite elements with a twist. The key difference is that least squares methods seek to minimize the residual of the solution in the L^2 norm, while DPG seeks the minimization in a dual norm realized through the inverse Riesz map. Exact mass conservation has been an issue that has plagued least squares finite elements for a long time. In this work, we augment our DPG system with Lagrange multipliers in order to exactly enforce local conservation. Effectively, this turns our minimization problem into a constrained minimization problem. We note that standard DPG, while not guaranteed to be conservative, appears to be nearly conservative in practice.

DPG is a Minimum Residual Method

Let U and V be trial and test Hilbert spaces for a well-posed variational problem $b(u,v)=l(v)$. In operator form this is $Bu=l$, where $B:U\to V'$. We seek to minimize the residual for the discrete space $U_h\subset U$:

$$u_h = \underset{w_h\in U_h}{\arg\min}\ \frac{1}{2}\left\|Bw_h-l\right\|_{V'}^2$$

Use the Riesz inverse to minimize in the V-norm rather than its dual:

$$\begin{aligned}\frac{1}{2}\left\|Bu_h-l\right\|_{V'}^2 &= \frac{1}{2}\left\|R_V^{-1}(Bu_h-l)\right\|_V^2\\ &= \frac{1}{2}\left(R_V^{-1}(Bu_h-l),R_V^{-1}(Bu_h-l)\right)_V .\end{aligned}$$

First order optimality requires the Gâteaux derivative to be zero in all directions $\delta u\in U_h$, i.e.,

$$\left(R_V^{-1}(Bu_h-l),R_V^{-1}B\delta u\right)_V=0,\quad \forall\delta u\in U.$$

By definition of the Riesz operator, this is equivalent to

$$\left\langle Bu_h-l,R_V^{-1}B\delta u_h\right\rangle=0\quad\forall\delta u_h\in U_h\,.$$

Identify $v_{\delta u_h}:=R_V^{-1}B\delta u_h$ as the optimal test function for trial function δu_h. This gives us

$$b(u_h,v_{\delta u_h})=l(v_{\delta u_h}).$$

DPG for Convection-Diffusion

Start with the strong-form PDE.

$$\nabla\cdot(\boldsymbol{\beta}u)-\epsilon\Delta u=g$$

Rewrite as a system of first-order equations.

$$\begin{aligned}\nabla\cdot(\boldsymbol{\beta}u-\boldsymbol{\sigma})&=g\\ \frac{1}{\epsilon}\boldsymbol{\sigma}-\nabla u&=\mathbf{0}\end{aligned}$$

Multiply by test functions and integrate by parts over each element, K.

$$\begin{aligned}&-(\boldsymbol{\beta}u-\boldsymbol{\sigma},\nabla v)_K+((\boldsymbol{\beta}u-\boldsymbol{\sigma})\cdot\mathbf{n},v)_{\partial K}=(g,v)_K\\ &\frac{1}{\epsilon}(\boldsymbol{\sigma},\boldsymbol{\tau})_K+(u,\nabla\cdot\boldsymbol{\tau})_K-(u,\tau_n)_{\partial K}=0\end{aligned}$$

Use the ultraweak (DPG) formulation to obtain bilinear form $b(u,v)=l(v)$.

$$\begin{aligned}-(\boldsymbol{\beta}u-\boldsymbol{\sigma},\nabla v)_K+(\hat{f},v)_{\partial K}+\frac{1}{\epsilon}(\boldsymbol{\sigma},\boldsymbol{\tau})_K&\\ +(u,\nabla\cdot\boldsymbol{\tau})_K-(\hat{u},\tau_n)_{\partial K}&=(g,v)_K\end{aligned}$$

Local Conservation

The local conservation law in convection diffusion is

$$\int_{\partial K}\hat{f}=\int_K g\,,$$

which is equivalent to having $\mathbf{v}_K:=\{v,\boldsymbol{\tau}\}=\{1_K,\mathbf{0}\}$ in the test space. In general, this is not satisfied by the optimal test functions. Following Moro et al[1], we can enforce this condition with Lagrange multipliers:

$$L(u_h,\boldsymbol{\lambda})=\frac{1}{2}\left\|R_V^{-1}(Bu_h-l)\right\|_V^2-\sum_K\lambda_K\underbrace{\langle Bu_h-l,\mathbf{v}_K\rangle}_{\langle\hat{f},1_K\rangle_{\partial K}-\langle g,1_K\rangle_K},$$

where $\boldsymbol{\lambda}=\{\lambda_1,\cdots,\lambda_N\}$.

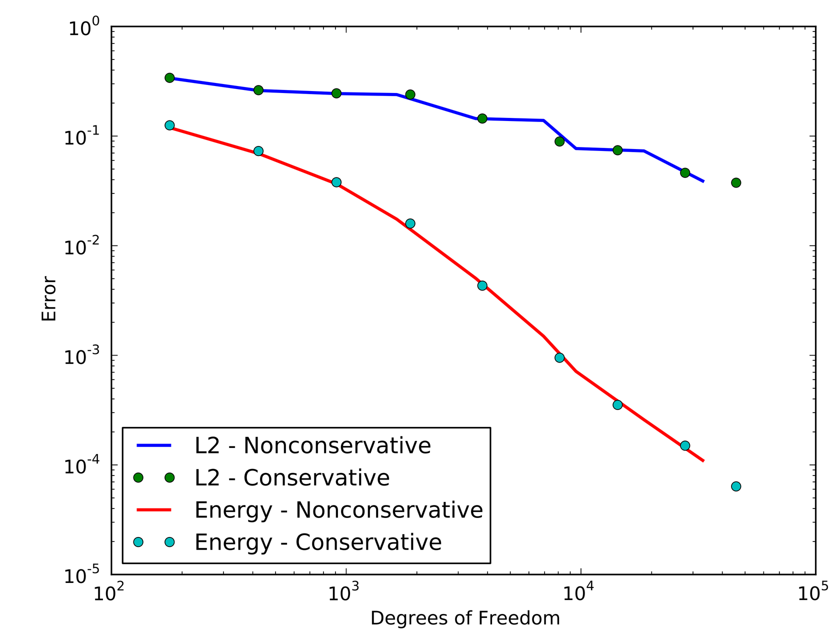
Finding the critical points of $L(u,\boldsymbol{\lambda})$, we get the following equations.

$$\begin{aligned}\frac{\partial L(u_h,\boldsymbol{\lambda})}{\partial u_h}&=b(u_h,R_V^{-1}B\delta u_h)-l(R_V^{-1}B\delta u_h)\\ &\quad-\sum_K\lambda_K b(\delta u_h,\mathbf{v}_K)=0\quad\forall\delta u_h\in U_h\\ \frac{\partial L(u_h,\boldsymbol{\lambda})}{\partial\lambda_K}&=-b(u_h,\mathbf{v}_K)+l(\mathbf{v}_K)=0\quad\forall K\end{aligned}$$

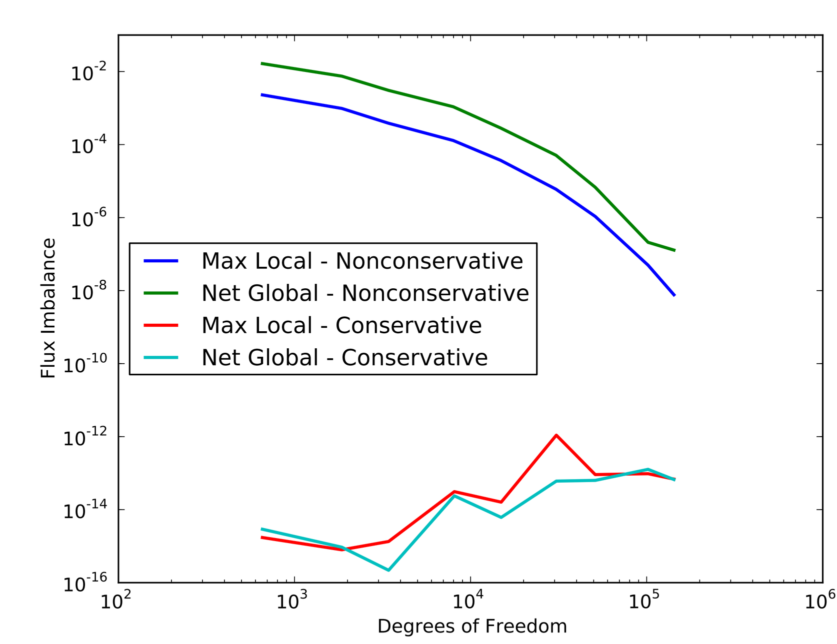
One of that consequences of enforcing local conservation via Lagrange multipliers is that we've replaced our symmetric positive-definite system with a saddlepoint problem.

Convection-Diffusion Results

The locally conservative DPG formulation maintains nearly identical error convergence behavior as standard DPG for the Erickson-Johnson problem.

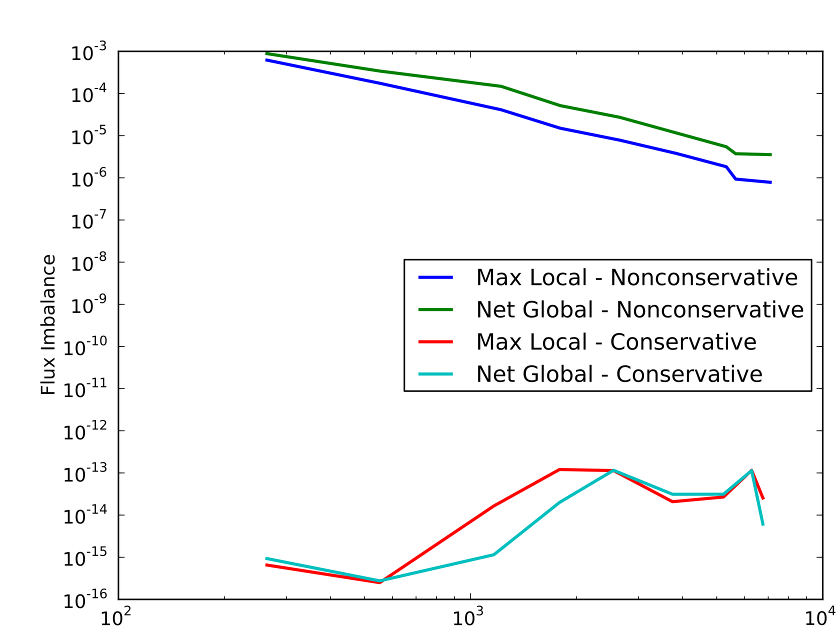


The locally conservative DPG formulation maintains flux imbalances close to machine precision, even for a discontinuous source term. Standard DPG becomes more conservative under refinement.

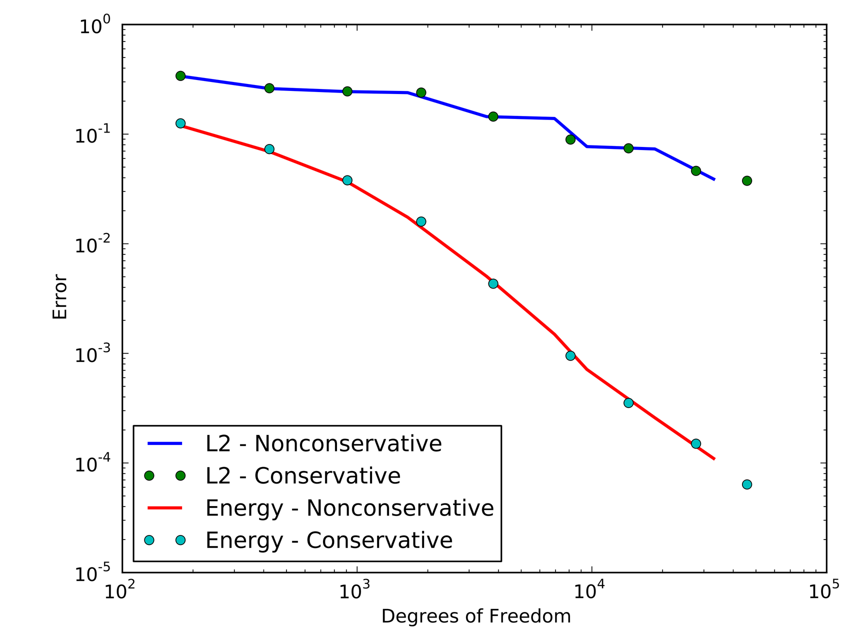


Inviscid Burgers' Equation Results

Extension of the theory to other fluid problems like the inviscid Burgers' equation is fairly trivial.



Stokes Equation Results



Conclusions

- We've turned our minimization problem into a saddlepoint problem.
- The computational cost is one extra degree of freedom per element.
- Enforcement occasionally changes the refinement strategy.
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- Standard DPG is nearly conservative in practice.
- Usually we get nearly identical results with better conservation.

References

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