Robust DPG Test Norms for Space-Time Convection-Diffusion

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Overview of DPG

A Framework for Computational Mechanics

Find $u \in U$ such that

$$b(u, v) = l(v) \quad \forall v \in V$$

with operator $B : U \to V'$ defined by $b(u, v) = \langle Bu, v \rangle_{V' \times V}$.
This gives the operator equation

$$Bu = l \quad \in V'.$$

We wish to minimize the residual $Bu - l \in V'$:

$$u_h = \arg\min_{w_h \in U_h} \frac{1}{2} \|Bw_h - l\|_{V'}^2.$$

Dual norms are not computationally tractable. Inverse Riesz map moves the residual to a more accessible space:

$$u_h = \arg\min_{w_h \in U_h} \frac{1}{2} \left\| R_V^{-1}(Bw_h - l) \right\|_V^2.$$

Overview of DPG

Petrov-Galerkin with Optimal Test Functions

Taking the Gâteaux derivative to be zero in all directions $\delta u \in U_h$ gives,

$$\left(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u \right)_V = 0, \quad \forall \delta u \in U,$$

which by definition of the Riesz map is equivalent to

$$\left\langle Bu_h - l, R_V^{-1}B\delta u_h \right\rangle = 0 \quad \forall \delta u_h \in U_h,$$

with optimal test functions $v_{\delta u_h} := R_V^{-1}B\delta u_h$ for each trial function δu_h.

Resulting Petrov-Galerkin System

This gives a simple bilinear form

$$b(u_h, v_{\delta u_h}) = l(v_{\delta u_h}),$$

with $v_{\delta u_h} \in V$ that solves the auxiliary problem

$$(v_{\delta u_h}, \delta v)_V = \langle R_V v_{\delta u_h}, \delta v \rangle = \langle B\delta u_h, \delta v \rangle = b(\delta u_h, \delta v) \quad \forall \delta v \in V.$$

Space-Time DPG

Extending DPG to Transient Problems

- Time stepping techniques are not ideally suited to highly adaptive grids
- Space-time FEM proposed as a solution
  - ✓ Unified treatment of space and time
  - ✓ Local space-time adaptivity (local time stepping)
  - ✓ Parallel-in-time integration (space-time multigrid)
  - ✗ Spatially stable FEM methods may not be stable in space-time
  - ✗ Need to support higher dimensional problems
- DPG provides necessary stability and adaptivity

Initial Space-Time Guess ➡ ↻ **Iterate** ➡ **Converge**

$$u_t = -u_x$$
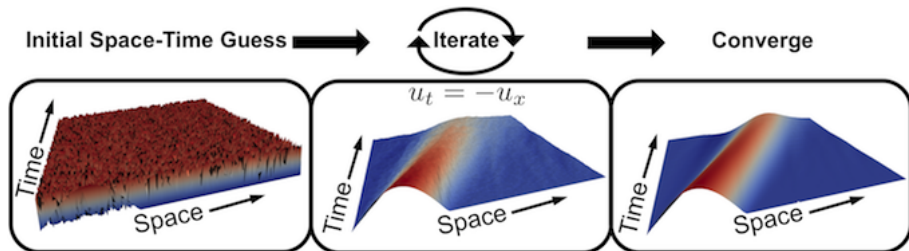
Courtesy XBraid by LLNL

Space-Time DPG for Convection-Diffusion

Space-Time Divergence Form

Equation is parabolic in space-time.

$$\frac{\partial u}{\partial t} + \boldsymbol{\beta} \cdot \nabla u - \epsilon \Delta u = f$$

This is just a composition of a constitutive law and conservation of mass.

$$\boldsymbol{\sigma} - \epsilon \nabla u = 0$$

$$\frac{\partial u}{\partial t} + \nabla \cdot (\boldsymbol{\beta} u - \boldsymbol{\sigma}) = f$$

We can rewrite this in terms of a space-time divergence.

$$\frac{1}{\epsilon} \boldsymbol{\sigma} - \nabla u = 0$$

$$\nabla_{xt} \cdot \begin{pmatrix} \boldsymbol{\beta} u - \boldsymbol{\sigma} \\ u \end{pmatrix} = f$$

Space-Time DPG for Convection-Diffusion

Ultra-Weak Formulation with Discontinuous Test Functions

Multiply by test function and integrate by parts over space-time element K.

$$\left(\frac{1}{\epsilon}\boldsymbol{\sigma}, \boldsymbol{\tau}\right)_K + (u, \nabla \cdot \boldsymbol{\tau})_K - \langle \hat{u}, \boldsymbol{\tau} \cdot \boldsymbol{n}_x \rangle_{\partial K} = 0$$

$$-\left(\begin{pmatrix} \boldsymbol{\beta} u - \boldsymbol{\sigma} \\ u \end{pmatrix}, \nabla_{xt} v\right)_K + \langle \hat{t}, v \rangle_{\partial K} = f$$

where

$$\hat{u} := \mathrm{tr}(u)$$
$$\hat{t} := \mathrm{tr}(\boldsymbol{\beta} u - \boldsymbol{\sigma}) \cdot \boldsymbol{n}_x$$
$$\quad + \mathrm{tr}(u) \cdot n_t$$

- Trace $\hat{u}$ defined on spatial boundaries
- Flux $\hat{t}$ defined on all boundaries

Support of Trace Variables

Space-Time Convection-Diffusion
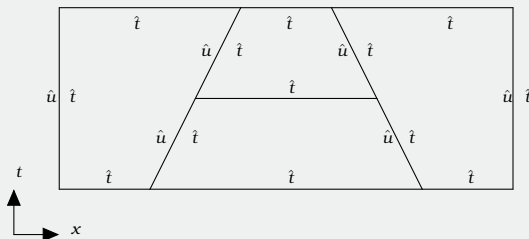
L^2 Equivalent Norms

Bilinear form with group variables:

$$b\left((u, \hat{u}), v\right) = (u, A_h^* v)_{L^2(\Omega_h)} + \langle \hat{u}, [\![v]\!] \rangle_{\Gamma_h}$$
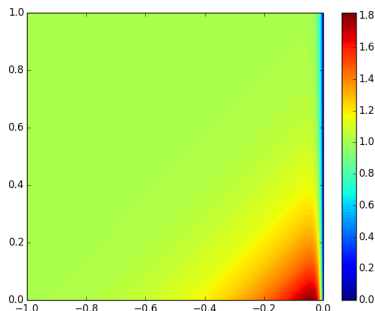
For conforming v^* satisfying $A^* v^* = u$

$$\|u\|_{L^2(\Omega_h)}^2 = b(u, v^*) = \frac{b(u, v^*)}{\|v^*\|_V} \|v^*\|_V$$

$$\leq \sup_{v^* \neq 0} \frac{|b(u, v^*)|}{\|v^*\|} \|v^*\| = \|u\|_E \|v^*\|_V$$

Necessary robustness condition:

$$\|v^*\|_V \lesssim \|u\|_{L^2(\Omega_h)}$$

$$\Rightarrow \|u\|_{L^2(\Omega_h)} \lesssim \|u\|_E$$

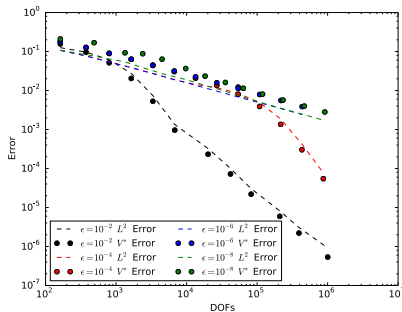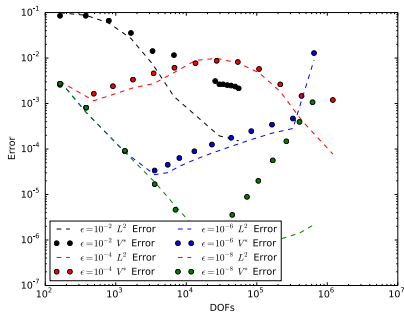**Analytical Solution**

$$e^{-lt}(e^{\lambda_1(x-1)} - e^{\lambda_2(x-1)})$$
$$+ \left(1 - e^{\frac{1}{\epsilon}x}\right)$$

Space-Time Convection-Diffusion

L^2 Equivalent Norms

A norm should be: bounded by $\|u\|_{L^2(\Omega_h)}$, have good conditioning, not produce boundary layers in the optimal test function.
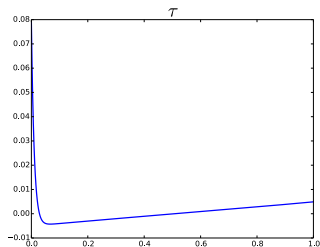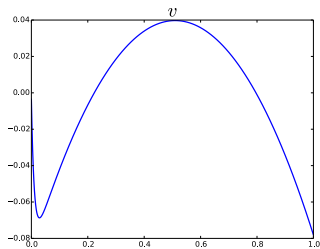

$$\|(v, \boldsymbol{\tau})\|^2 = \left\| \nabla \cdot \boldsymbol{\tau} - \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \right\|^2$$
$$+ \left\| \frac{1}{\epsilon} \boldsymbol{\tau} + \nabla v \right\|^2 + \|v\|^2 + \|\boldsymbol{\tau}\|^2$$

$$\|(v, \boldsymbol{\tau})\|^2 = \left\| \nabla \cdot \boldsymbol{\tau} - \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \right\|^2$$
$$+ \min \left(\frac{1}{h^2}, \frac{1}{\epsilon} \right) \|\boldsymbol{\tau}\|^2$$
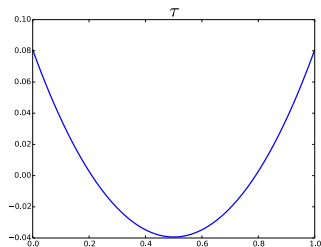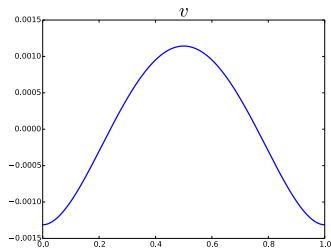$$+ \epsilon \|\nabla v\|^2 + \|\boldsymbol{\beta} \cdot \nabla v\|^2 + \|v\|^2$$

Steady Convection-Diffusion

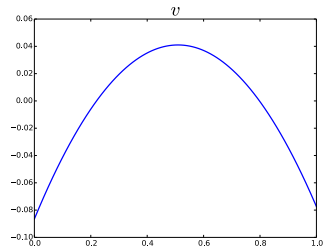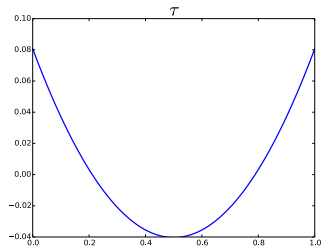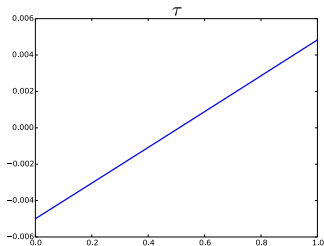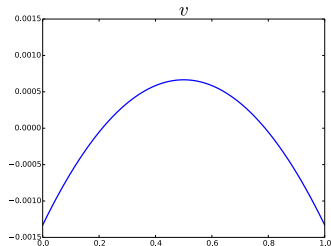Ideal Optimal Shape Functions

Graph Norm

Coupled Robust Norm

Steady Convection-Diffusion

Approximated ($p = 3$) Optimal Shape Functions

Steady Convection-Diffusion

Two Robust Norms for Steady Convection-Diffusion

The following norms are robust for steady convection-diffusion.
The robust norm was derived in[1]:

$$\|(v, \boldsymbol{\tau})\|^2 = \|\boldsymbol{\beta} \cdot \nabla v\|^2 + \epsilon \|\nabla v\|^2 + \min\left(\frac{\epsilon}{h^2}, 1\right) \|v\|^2$$

$$+ \|\nabla \cdot \boldsymbol{\tau}\|^2 + \min\left(\frac{1}{h^2}, \frac{1}{\epsilon}\right) \|\boldsymbol{\tau}\|^2 .$$

The case for the coupled robust norm was made in[2]:

$$\|(v, \boldsymbol{\tau})\|^2 = \|\boldsymbol{\beta} \cdot \nabla v\|^2 + \epsilon \|\nabla v\|^2 + \min\left(\frac{\epsilon}{h^2}, 1\right) \|v\|^2$$

$$+ \|\nabla \cdot \boldsymbol{\tau} - \boldsymbol{\beta} \cdot \nabla v\|^2 + \min\left(\frac{1}{h^2}, \frac{1}{\epsilon}\right) \|\boldsymbol{\tau}\|^2 .$$

[1] J. Chan et al. ``A robust DPG method for convection-dominated diffusion problems II: Adjoint boundary conditions and mesh-dependent test norms''. In: *Comp. Math. Appl.* 67.4 (2014). High-order Finite Element Approximation for Partial Differential Equations, pp. 771 –795.
[2] J.L. Chan. ``A DPG Method for Convection-Diffusion Problems''. PhD thesis. University of Texas at Austin, 2013.

Space-Time Convection-Diffusion

Two Robust Norms for Transient Convection-Diffusion

Let $\tilde{\boldsymbol{\beta}} := \begin{pmatrix} \boldsymbol{\beta} \\ 1 \end{pmatrix}$ and $\nabla_{xt} v := \begin{pmatrix} \nabla v \\ \frac{\partial v}{\partial t} \end{pmatrix}$.

The following norms are robust for space-time convection-diffusion.

Robust Norm:

$$\|(v, \boldsymbol{\tau})\|^2 = \left\| \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \right\|^2 + \epsilon \|\nabla v\|^2 + \min\left(\frac{\epsilon}{h^2}, 1 \right) \|v\|^2$$
$$+ \|\nabla \cdot \boldsymbol{\tau}\|^2 + \min\left(\frac{1}{h^2}, \frac{1}{\epsilon} \right) \|\boldsymbol{\tau}\|^2 \, .$$

Coupled Robust Norm

$$\|(v, \boldsymbol{\tau})\|^2 = \left\| \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \right\|^2 + \epsilon \|\nabla v\|^2 + \min\left(\frac{\epsilon}{h^2}, 1 \right) \|v\|^2$$
$$+ \left\| \nabla \cdot \boldsymbol{\tau} - \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \right\|^2 + \min\left(\frac{1}{h^2}, \frac{1}{\epsilon} \right) \|\boldsymbol{\tau}\|^2 \, .$$

Adjoint Operator

Consider the problem with homogeneous boundary conditions

$$\frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u = 0$$

$$\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} u - \nabla \cdot \boldsymbol{\sigma} = f$$

$$\beta_n u - \epsilon \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_-$$

$$u = 0 \text{ on } \Gamma_+$$

$$u = u_0 \text{ on } \Gamma_0.$$

The adjoint operator A^* is given by

$$A^*(v, \boldsymbol{\tau}) = \left(\frac{1}{\epsilon}\boldsymbol{\tau} + \nabla v, -\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \nabla \cdot \boldsymbol{\tau} \right).$$

Robust Norms for Transient Convection-Diffusion

Controlling Different Field Variables

We decompose the continuous adjoint problem $A^*(\boldsymbol{\tau}, v) = (\boldsymbol{\sigma}, u)$ into

Continuous part with forcing u

$$\frac{1}{\epsilon}\boldsymbol{\tau}_1 + \nabla v_1 = 0$$

$$-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1 + \nabla \cdot \boldsymbol{\tau}_1 = u$$

$$\boldsymbol{\tau}_1 \cdot \boldsymbol{n}_x = 0 \text{ on } \Gamma_-$$
$$v_1 = 0 \text{ on } \Gamma_+$$
$$v_1 = 0 \text{ on } \Gamma_T$$

Continuous part with forcing $\boldsymbol{\sigma}$

$$\frac{1}{\epsilon}\boldsymbol{\tau}_2 + \nabla v_2 = \boldsymbol{\sigma}$$

$$-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_2 + \nabla \cdot \boldsymbol{\tau}_2 = 0$$

$$\boldsymbol{\tau}_2 \cdot \boldsymbol{n}_x = 0 \text{ on } \Gamma_-$$
$$v_2 = 0 \text{ on } \Gamma_+$$
$$v_2 = 0 \text{ on } \Gamma_T$$

Robust Norms for Transient Convection-Diffusion

Proved Bounds at Our Disposal

Proofs of these lemmas can be found in[3].

Lemma (1)

If $\nabla \cdot \boldsymbol{\beta} = 0$, we can bound

$$\|v\|^2 + \epsilon \|\nabla v\|^2 \leq \|u\|^2 + \epsilon \|\boldsymbol{\sigma}\|^2$$

where $v = v_1 + v_2$.

Lemma (2)

If $\left\|\nabla \boldsymbol{\beta} - \frac{1}{2}\nabla \cdot \boldsymbol{\beta}\boldsymbol{I}\right\|_{L^\infty} \leq C_{\boldsymbol{\beta}}$, we can bound

$$\left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1\right\| \lesssim \|u\|.$$

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[3]T.E. Ellis, J.L. Chan, and L.F. Demkowicz. *Robust DPG Methods for Transient Convection-Diffusion.* Tech. rep. 15-21. ICES, Oct. 2015.

Robust Norms for Transient Convection-Diffusion

Control of u

Bound on $\|(v_1, \boldsymbol{\tau}_1)\|$

Lemma (2) $\Rightarrow$ $\qquad \left\| \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1 \right\| \lesssim \|u\|$

Lemma (2) $\Rightarrow$ $\qquad \|\nabla \cdot \boldsymbol{\tau}_1\| \leq \|u\| + \left\| \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1 \right\| \lesssim 2\|u\|$

Lemma (2) $\Rightarrow$ $\qquad \left\| \nabla \cdot \boldsymbol{\tau}_1 - \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1 \right\| = \|u\|$

Lemma (1) $\Rightarrow$ $\qquad \|v_1\|^2 + \epsilon \|\nabla v_1\|^2 \leq \|u\|^2$

Lemma (1) $\Rightarrow$ $\qquad \dfrac{1}{\epsilon} \|\boldsymbol{\tau}_1\| = \epsilon \|\nabla v_1\| \leq \|u\|$

We can guarantee robust control

$$\|(u, 0)\|_{L^2(\Omega_h)} \lesssim \|(u, \boldsymbol{\sigma})\|_E \ .$$

Robust Norms for Transient Convection-Diffusion

Control of $\boldsymbol{\sigma}$

Bound on $\|(v_2, \boldsymbol{\tau}_2)\|$

Definition $\Rightarrow$ $\left\| \nabla \cdot \boldsymbol{\tau}_2 - \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_2 \right\| = 0 \leq \|\boldsymbol{\sigma}\|$

Lemma (1) $\Rightarrow$ $\|v_2\|^2 + \epsilon \|\nabla v_2\|^2 \leq \epsilon \|\boldsymbol{\sigma}\|^2$

Lemma (1) $\Rightarrow$ $\dfrac{1}{\epsilon} \|\boldsymbol{\tau}_2\| = \|\boldsymbol{\sigma}\| + \epsilon \|\nabla v_2\| = (1 + \epsilon) \|\boldsymbol{\sigma}\|$

We have not been able to prove bounds on $\left\| \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_2 \right\|$ or $\|\nabla \cdot \boldsymbol{\tau}_2\|$.

We can **not** guarantee robust control

$$\|(0, \boldsymbol{\sigma})\|_{L^2(\Omega_h)} \lesssim\!\!\!\!/\;\; \|(u, \boldsymbol{\sigma})\|_E \;.$$
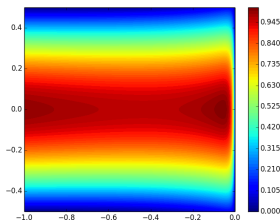
Robust Norms for Transient Convection-Diffusion

Transient Analytical Solution

Transient impulse decays to Eriksson-Johnson steady state solution.
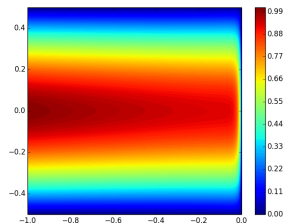
$$u = \exp(-lt)\left[\exp(\lambda_1 x) - \exp(\lambda_2 x)\right] + \cos(\pi y)\frac{\exp(s_1 x) - \exp(r_1 x)}{\exp(-s_1) - \exp(-r_1)}$$
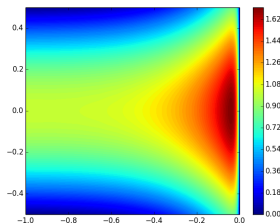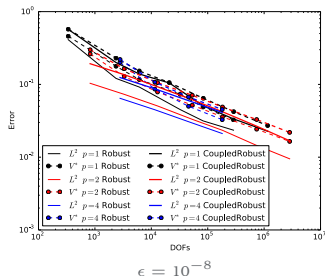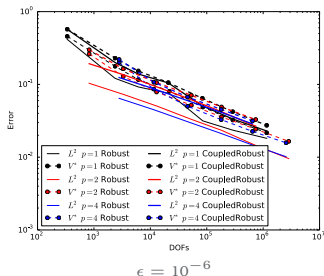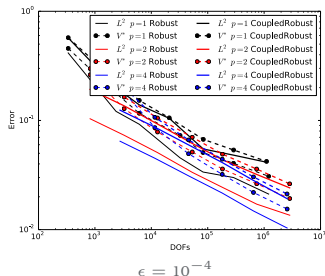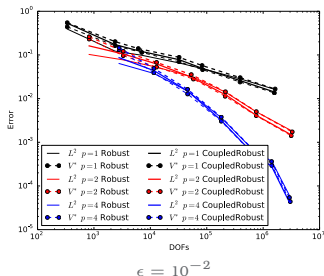

$t = 0.0$ $t = 0.5$ $t = 1.0$

Robust Norms for Transient Convection-Diffusion

Robust Convergence to Analytical Solution

$\epsilon = 10^{-2}$

$\epsilon = 10^{-4}$

$\epsilon = 10^{-6}$

$\epsilon = 10^{-8}$

Thank You!

Recommended References

- J. Chan et al. ``A robust DPG method for convection-dominated diffusion problems II: Adjoint boundary conditions and mesh-dependent test norms''. In: *Comp. Math. Appl.* 67.4 (2014). High-order Finite Element Approximation for Partial Differential Equations, pp. 771 –795.

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- L.F. Demkowicz and J. Gopalakrishnan. ``Recent Developments in Discontinuous Galerkin Finite Element Methods for Partial Differential Equations (eds. X. Feng, O. Karakashian, Y. Xing)''. In: vol. 157. IMA Volumes in Mathematics and its Applications, 2014. Chap. An Overview of the DPG Method, pp. 149–180.

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- N.V. Roberts, L.F. Demkowicz, and R.D. Moser. ``A discontinuous Petrov-Galerkin methodology for adaptive solutions to the incompressible Navier-Stokes equations''. In: *J. Comput. Phys.* 301 (2015), pp. 456 –483. ISSN: 0021-9991.