Space-Time Discontinuous Petrov-Galerkin Finite Elements for Fluid Problems

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Minimum Residual and Least Squares Finite Element Methods Workshop

THE UNIVERSITY OF TEXAS AT AUSTIN

INSTITUTE FOR COMPUTATIONAL ENGINEERING & SCIENCES

Overview of DPG

A Framework for Computational Mechanics

Find $u \in U$ such that

$$b(u, v) = l(v) \quad \forall v \in V$$

with operator $B : U \to V'$ defined by $b(u, v) = \langle Bu, v\rangle_{V' \times V}$.
This gives the operator equation

$$Bu = l \quad \in V' .$$

We wish to minimize the residual $Bu - l \in V'$:

$$u_h = \underset{w_h \in U_h}{\arg\min} \frac{1}{2} \left\| Bw_h - l \right\|_{V'}^2 .$$

Dual norms are not computationally tractable. Inverse Riesz map moves the residual to a more accessible space:

$$u_h = \underset{w_h \in U_h}{\arg\min} \frac{1}{2} \left\| R_V^{-1}(Bw_h - l) \right\|_{V}^2 .$$

Overview of DPG

Petrov-Galerkin with Optimal Test Functions

Taking the Gâteaux derivative to be zero in all directions $\delta u \in U_h$ gives,

$$\left(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u\right)_V = 0, \quad \forall \delta u \in U,$$

which by definition of the Riesz map is equivalent to

$$\left\langle Bu_h - l, R_V^{-1}B\delta u_h\right\rangle = 0 \quad \forall \delta u_h \in U_h\,,$$

with optimal test functions $v_{\delta u_h} := R_V^{-1}B\delta u_h$ for each trial function δu_h.

Resulting Petrov-Galerkin System

This gives a simple bilinear form

$$b(u_h, v_{\delta u_h}) = l(v_{\delta u_h}),$$

with $v_{\delta u_h} \in V$ that solves the auxiliary problem

$$(v_{\delta u_h}, \delta v)_V = \langle R_V v_{\delta u_h}, \delta v\rangle = \langle B\delta u_h, \delta v\rangle = b(\delta u_h, \delta v) \quad \forall \delta v \in V.$$

Discontinuous Petrov-Galerkin

- Continuous test space produces global solve for optimal test functions
- Discontinuous test space results in an embarrassingly parallel solve

Hermitian Positive Definite Stiffness Matrix

Property of all minimum residual methods

$$b(u_h, v_{\delta u_h}) = (v_{u_h}, v_{\delta u_h})_V = \overline{(v_{\delta u_h}, v_{u_h})_V} = \overline{b(\delta u_h, v_{u_h})}$$

Error Representation Function

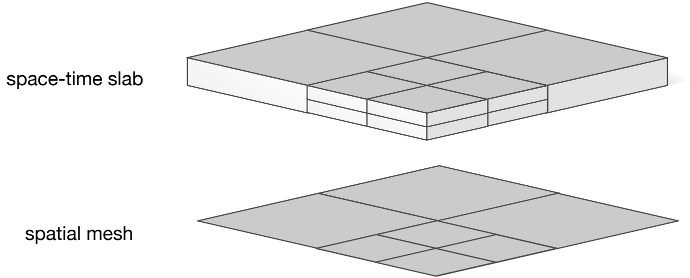
Energy norm of Galerkin error (residual) can be computed without exact solution

$$\left\|u_h - u\right\|_E = \left\|B(u_h - u)\right\|_{V'} = \left\|Bu_h - l\right\|_{V'} = \left\|R_V^{-1}(Bu_h - l)\right\|_V$$

Space-Time DPG

Extending DPG to Transient Problems

- Time stepping techniques are not ideally suited to highly adaptive grids
- Space-time FEM proposed as a solution
  - ✓ Unified treatment of space and time
  - ✓ Local space-time adaptivity (local time stepping)
  - ✓ Parallel-in-time integration (space-time multigrid)
  - ✗ Spatially stable FEM methods may not be stable in space-time
  - ✗ Need to support higher dimensional problems
- DPG provides necessary stability and adaptivity



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Equation is parabolic in space-time.

$$\frac{\partial u}{\partial t} + \boldsymbol{\beta} \cdot \nabla u - \epsilon \Delta u = f$$

This is just a composition of a constitutive law and conservation of mass.

$$\begin{aligned} \boldsymbol{\sigma} - \epsilon \nabla u &= 0 \\ \frac{\partial u}{\partial t} + \nabla \cdot (\boldsymbol{\beta} u - \boldsymbol{\sigma}) &= f \end{aligned}$$

We can rewrite this in terms of a space-time divergence.

$$\begin{aligned} \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= 0 \\ \nabla_{xt} \cdot \begin{pmatrix} \boldsymbol{\beta} u - \boldsymbol{\sigma} \\ u \end{pmatrix} &= f \end{aligned}$$

Space-Time DPG for Convection-Diffusion

Ultra-Weak Formulation with Discontinuous Test Functions

Multiply by test function and integrate by parts over space-time element K.

$$\left(\frac{1}{\epsilon}\boldsymbol{\sigma},\boldsymbol{\tau}\right)_K+(u,\nabla\cdot\boldsymbol{\tau})_K-\langle\hat{u},\boldsymbol{\tau}\cdot\boldsymbol{n}_x\rangle_{\partial K}=0$$

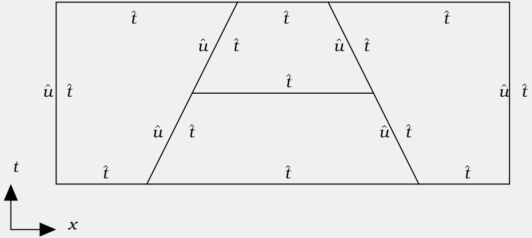
$$-\left(\begin{pmatrix}\boldsymbol{\beta}u-\boldsymbol{\sigma}\\ u\end{pmatrix},\nabla_{xt}v\right)_K+\langle\hat{t},v\rangle_{\partial K}=f$$

where

$$\hat{u}:=\operatorname{tr}(u)$$
$$\hat{t}:=\operatorname{tr}(\boldsymbol{\beta}u-\boldsymbol{\sigma})\cdot\boldsymbol{n}_x+\operatorname{tr}(u)\cdot n_t$$

- Trace $\hat{u}$ defined on spatial boundaries
- Flux $\hat{t}$ defined on all boundaries

Support of Trace Variables



Space-Time Convection-Diffusion

L^2 Equivalent Norms

Bilinear form with group variables:

$$b\left((u,\hat{u}),v\right) = (u, A_h^* v)_{L^2(\Omega_h)} + \langle \widehat{u}, [\![v]\!]\rangle_{\Gamma_h}$$

For conforming v^* satisfying $A^* v^* = u$

$$\|u\|^2_{L^2(\Omega_h)} = b(u,v^*) = \frac{b(u,v^*)}{\|v^*\|_V}\|v^*\|_V$$

$$\leq \sup_{v^*\neq 0}\frac{|b(u,v^*)|}{\|v^*\|}\|v^*\| = \|u\|_E\|v^*\|_V$$

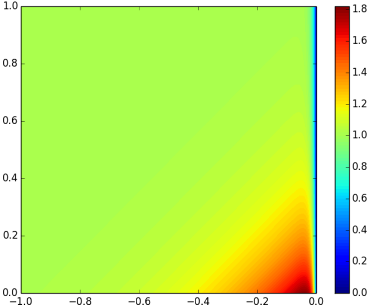
Necessary robustness condition:

$$\|v^*\|_V \lesssim \|u\|_{L^2(\Omega_h)}$$

$$\Rightarrow \|u\|_{L^2(\Omega_h)} \lesssim \|u\|_E$$

Analytical Solution

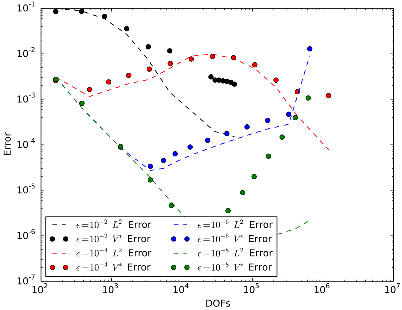
$$e^{-lt}\left(e^{\lambda_1(x-1)} - e^{\lambda_2(x-1)}\right) + \left(1 - e^{\frac{1}{\epsilon}x}\right)$$

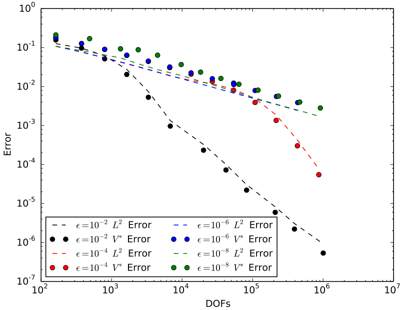


Space-Time Convection-Diffusion

L^2 Equivalent Norms

A norm should be: bounded by $\|u\|_{L^2(\Omega_h)}$, have good conditioning, not produce boundary layers in the optimal test function.





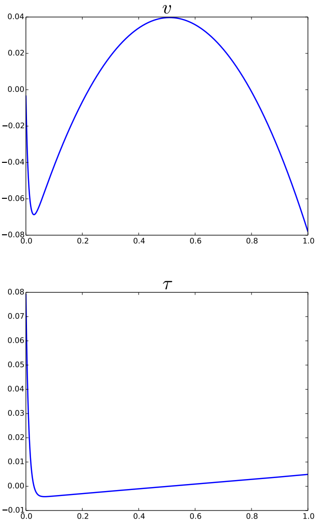
$$\|(v,\boldsymbol{\tau})\|^2 = \left\|\nabla\cdot\boldsymbol{\tau} - \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt} v\right\|^2 + \left\|\frac{1}{\epsilon}\boldsymbol{\tau} + \nabla v\right\|^2 + \|v\|^2 + \|\boldsymbol{\tau}\|^2$$

$$\|(v,\boldsymbol{\tau})\|^2 = \left\|\nabla\cdot\boldsymbol{\tau} - \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt} v\right\|^2 + \min\left(\frac{1}{h^2},\frac{1}{\epsilon}\right)\|\boldsymbol{\tau}\|^2 + \epsilon\|\nabla v\|^2 + \|\boldsymbol{\beta}\cdot\nabla v\|^2 + \|v\|^2$$

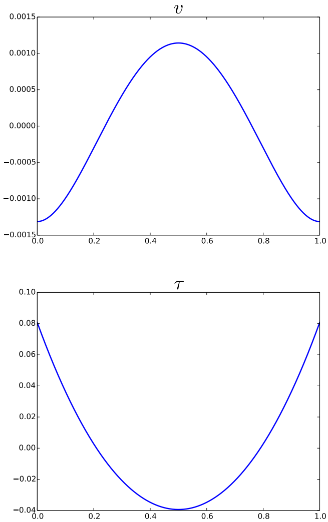
Steady Convection-Diffusion

Ideal Optimal Shape Functions

Graph Norm

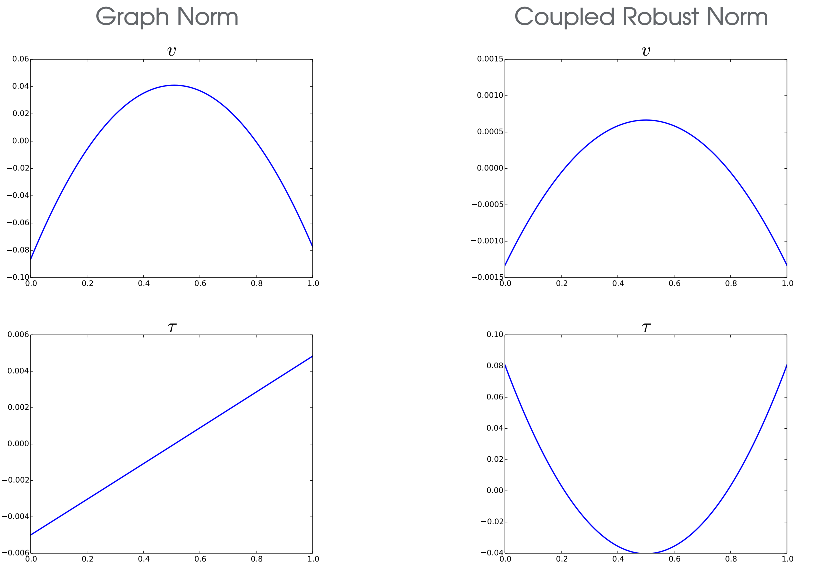


Coupled Robust Norm



Steady Convection-Diffusion

Approximated ($p = 3$) Optimal Shape Functions



Steady Convection-Diffusion

Two Robust Norms for Steady Convection-Diffusion

The following norms are robust for steady convection-diffusion.

The robust norm was derived in[1]:

$$\left\|(v,\boldsymbol{\tau})\right\|^2 = \left\|\boldsymbol{\beta}\cdot\nabla v\right\|^2 + \epsilon\left\|\nabla v\right\|^2 + \min\left(\frac{\epsilon}{h^2},1\right)\left\|v\right\|^2 + \left\|\nabla\cdot\boldsymbol{\tau}\right\|^2 + \min\left(\frac{1}{h^2},\frac{1}{\epsilon}\right)\left\|\boldsymbol{\tau}\right\|^2 .$$

The case for the coupled robust norm was made in[2]:

$$\left\|(v,\boldsymbol{\tau})\right\|^2 = \left\|\boldsymbol{\beta}\cdot\nabla v\right\|^2 + \epsilon\left\|\nabla v\right\|^2 + \min\left(\frac{\epsilon}{h^2},1\right)\left\|v\right\|^2 + \left\|\nabla\cdot\boldsymbol{\tau}-\boldsymbol{\beta}\cdot\nabla v\right\|^2 + \min\left(\frac{1}{h^2},\frac{1}{\epsilon}\right)\left\|\boldsymbol{\tau}\right\|^2 .$$

[1] J. Chan et al. ``A robust DPG method for convection-dominated diffusion problems II: Adjoint boundary conditions and mesh-dependent test norms''. In: *Comp. Math. Appl.* 67.4 (2014). High-order Finite Element Approximation for Partial Differential Equations, pp. 771 –795.

[2] J.L. Chan. ``A DPG Method for Convection-Diffusion Problems''. PhD thesis. University of Texas at Austin, 2013.

Space-Time Convection-Diffusion

Two Robust Norms for Transient Convection-Diffusion

Let $\tilde{\beta} := \begin{pmatrix} \beta \\ 1 \end{pmatrix}$ and $\nabla_{xt} v := \begin{pmatrix} \nabla v \\ \frac{\partial v}{\partial t} \end{pmatrix}$.

The following norms are robust for space-time convection-diffusion.

Robust Norm:

$$\|(v, \boldsymbol{\tau})\|^2 = \left\|\tilde{\beta} \cdot \nabla_{xt} v\right\|^2 + \epsilon \|\nabla v\|^2 + \min\left(\frac{\epsilon}{h^2}, 1\right) \|v\|^2 + \|\nabla \cdot \boldsymbol{\tau}\|^2 + \min\left(\frac{1}{h^2}, \frac{1}{\epsilon}\right) \|\boldsymbol{\tau}\|^2 .$$

Coupled Robust Norm

$$\|(v, \boldsymbol{\tau})\|^2 = \left\|\tilde{\beta} \cdot \nabla_{xt} v\right\|^2 + \epsilon \|\nabla v\|^2 + \min\left(\frac{\epsilon}{h^2}, 1\right) \|v\|^2 + \left\|\nabla \cdot \boldsymbol{\tau} - \tilde{\beta} \cdot \nabla_{xt} v\right\|^2 + \min\left(\frac{1}{h^2}, \frac{1}{\epsilon}\right) \|\boldsymbol{\tau}\|^2 .$$

Adjoint Operator

Consider the problem with homogeneous boundary conditions

$$
\begin{aligned}
\frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= 0 \\
\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} u - \nabla \cdot \boldsymbol{\sigma} &= f \\
\beta_n u - \epsilon \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_- \\
u &= 0 \text{ on } \Gamma_+ \\
u &= u_0 \text{ on } \Gamma_0 .
\end{aligned}
$$

The adjoint operator A^* is given by

$$
A^*(v, \boldsymbol{\tau}) = \left(\frac{1}{\epsilon}\boldsymbol{\tau} + \nabla v, -\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \nabla \cdot \boldsymbol{\tau} \right) .
$$

Robust Norms for Transient Convection-Diffusion

Controlling Different Field Variables

We decompose the continuous adjoint problem $A^*(\boldsymbol{\tau}, v) = (\boldsymbol{\sigma}, u)$ into

Continuous part with forcing u

$$
\begin{aligned}
\frac{1}{\epsilon}\boldsymbol{\tau}_1 + \nabla v_1 &= 0 \\
-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1 + \nabla \cdot \boldsymbol{\tau}_1 &= u
\end{aligned}
\qquad
\begin{aligned}
\boldsymbol{\tau}_1 \cdot \boldsymbol{n}_x &= 0 \text{ on } \Gamma_- \\
v_1 &= 0 \text{ on } \Gamma_+ \\
v_1 &= 0 \text{ on } \Gamma_T
\end{aligned}
$$

Continuous part with forcing $\boldsymbol{\sigma}$

$$
\begin{aligned}
\frac{1}{\epsilon}\boldsymbol{\tau}_2 + \nabla v_2 &= \boldsymbol{\sigma} \\
-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_2 + \nabla \cdot \boldsymbol{\tau}_2 &= 0
\end{aligned}
\qquad
\begin{aligned}
\boldsymbol{\tau}_2 \cdot \boldsymbol{n}_x &= 0 \text{ on } \Gamma_- \\
v_2 &= 0 \text{ on } \Gamma_+ \\
v_2 &= 0 \text{ on } \Gamma_T
\end{aligned}
$$

Robust Norms for Transient Convection-Diffusion

Proved Bounds at Our Disposal

Proofs of these lemmas can be found in[3].

Lemma (1)

If $\nabla \cdot \boldsymbol{\beta} = 0$*, we can bound*

$$\|v\|^2 + \epsilon \|\nabla v\|^2 \leq \|u\|^2 + \epsilon \|\boldsymbol{\sigma}\|^2$$

where $v = v_1 + v_2$.

Lemma (2)

If $\left\| \nabla \boldsymbol{\beta} - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \boldsymbol{I} \right\|_{L^\infty} \leq C_{\boldsymbol{\beta}}$*, we can bound*

$$\left\| \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1 \right\| \lesssim \|u\| \,.$$

[3] T.E. Ellis, J.L. Chan, and L.F. Demkowicz. *Robust DPG Methods for Transient Convection-Diffusion.* Tech. rep. 15-21. ICES, Oct. 2015.

Bound on $\left\|(v_1, \boldsymbol{\tau}_1)\right\|$

Lemma (2) $\Rightarrow$ $$\left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1\right\| \lesssim \|u\|$$

Lemma (2) $\Rightarrow$ $$\|\nabla \cdot \boldsymbol{\tau}_1\| \leq \|u\| + \left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1\right\| \lesssim 2\|u\|$$

Lemma (2) $\Rightarrow$ $$\left\|\nabla \cdot \boldsymbol{\tau}_1 - \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1\right\| = \|u\|$$

Lemma (1) $\Rightarrow$ $$\|v_1\|^2 + \epsilon \|\nabla v_1\|^2 \leq \|u\|^2$$

Lemma (1) $\Rightarrow$ $$\frac{1}{\epsilon}\|\boldsymbol{\tau}_1\| = \epsilon \|\nabla v_1\| \leq \|u\|$$

We can guarantee robust control

$$\|(u, 0)\|_{L^2(\Omega_h)} \lesssim \|(u, \boldsymbol{\sigma})\|_E .$$

Robust Norms for Transient Convection-Diffusion

Control of $\boldsymbol{\sigma}$

Bound on $\left\|(v_2, \boldsymbol{\tau}_2)\right\|$

$$\text{Definition} \Rightarrow \quad \left\|\nabla \cdot \boldsymbol{\tau}_2 - \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_2\right\| = 0 \leq \|\boldsymbol{\sigma}\|$$

$$\text{Lemma (1)} \Rightarrow \quad \|v_2\|^2 + \epsilon \|\nabla v_2\|^2 \leq \epsilon \|\boldsymbol{\sigma}\|^2$$

$$\text{Lemma (1)} \Rightarrow \quad \frac{1}{\epsilon} \|\boldsymbol{\tau}_2\| = \|\boldsymbol{\sigma}\| + \epsilon \|\nabla v_2\| = (1+\epsilon) \|\boldsymbol{\sigma}\|$$

We have not been able to prove bounds on $\left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_2\right\|$ or $\|\nabla \cdot \boldsymbol{\tau}_2\|$.

We can **not** guarantee robust control

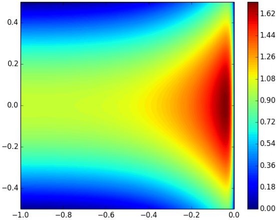
$$\|(0, \boldsymbol{\sigma})\|_{L^2(\Omega_h)} \not\lesssim \|(u, \boldsymbol{\sigma})\|_E \ .$$

Robust Norms for Transient Convection-Diffusion

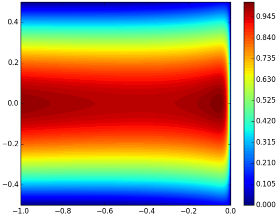
Transient Analytical Solution

Transient impulse decays to Eriksson-Johnson steady state solution.

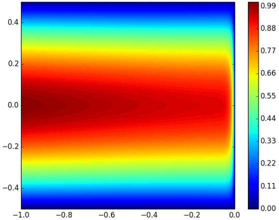
$$u = \exp(-lt)\left[\exp(\lambda_1 x) - \exp(\lambda_2 x)\right] + \cos(\pi y)\frac{\exp(s_1 x) - \exp(r_1 x)}{\exp(-s_1) - \exp(-r_1)}$$



$t = 0.0$



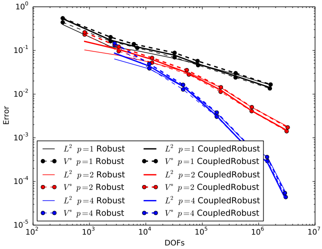
$t = 0.5$



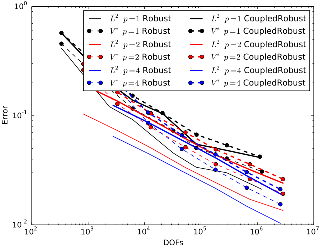
$t = 1.0$

Robust Norms for Transient Convection-Diffusion

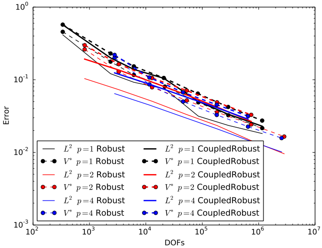
Robust Convergence to Analytical Solution



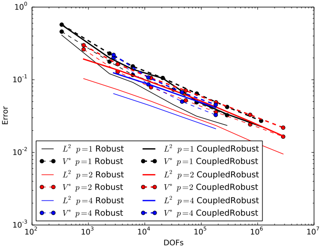
$\epsilon = 10^{-2}$



$\epsilon = 10^{-4}$



$\epsilon = 10^{-6}$



$\epsilon = 10^{-8}$

Thank You!

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