Discontinuous Petrov-Galerkin (DPG) Method With Optimal Test Functions

Progress Report

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THE UNIVERSITY OF TEXAS AT AUSTIN

INSTITUTE FOR COMPUTATIONAL ENGINEERING & SCIENCES

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Sandia: P.N. Bochev, K.J. Peterson, D. Ridzal and Ch. M. Siefert

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U. Tel-Aviv I. Harari

Three DPG Punchlines

1: DPG Method is a Ritz method. It supports adaptivity with no preasymptotic behavior.

2: You can control the norm in which you want to converge.

3: DPG is easy to code.

Punchline 1

DPG Method is a Ritz method. It supports adaptivity with no preasymptotic behavior.

Primal DPG Method for Maxwell equations.

Assume

$$J_S^{\text{imp}} = n \times H^{imp}$$

and look for the unknown surface current on the skeleton also in the same form.

$$\begin{cases} E \in H(\text{curl}, \Omega), \; n \times E = n \times E^{\text{imp}} \text{ on } \Gamma_1 \\ \hat{h} \in \text{tr}_{\Gamma_h} H(\text{curl}, \Omega), \; n \times \hat{h} = n \times (-i\omega H^{imp}) \text{ on } \Gamma_2 \\ (\frac{1}{\mu} \nabla \times E, \nabla_h \times F) + ((-\omega^2 \epsilon + i\omega\sigma)E, F) + \langle n \times \hat{h}, F \rangle_{\Gamma_h} = -i\omega (J^{\text{imp}}, F) \\ \qquad \forall F \in H(\text{curl}, \Omega_h) \, . \end{cases}$$

FE discretization for curl-curl problem

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Hexahedral meshes

$H(\text{curl})$ element for electric field E:

$$(\mathcal{P}^{p-1} \otimes \mathcal{P}^p \otimes \mathcal{P}^p) \times (\mathcal{P}^p \otimes \mathcal{P}^{p-1} \otimes \mathcal{P}^p) \times (\mathcal{P}^p \otimes \mathcal{P}^p \otimes \mathcal{P}^{p-1})$$

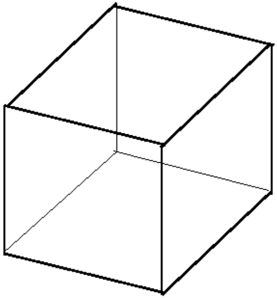
and trace of the same element for flux (surface current) $\hat{h}$.

Same element for the enriched space but with order $p + \Delta p$.

In reported experiments: $p = 2$, $\Delta p = 2$.

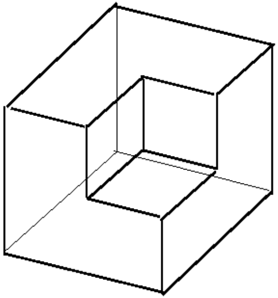
A 3D Maxwell example

Take a cube $(0,2)^3$



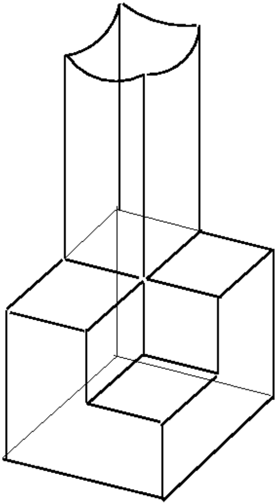
Fichera corner

Divide it into eight smaller cubes and remove one:



Fichera corner microwave

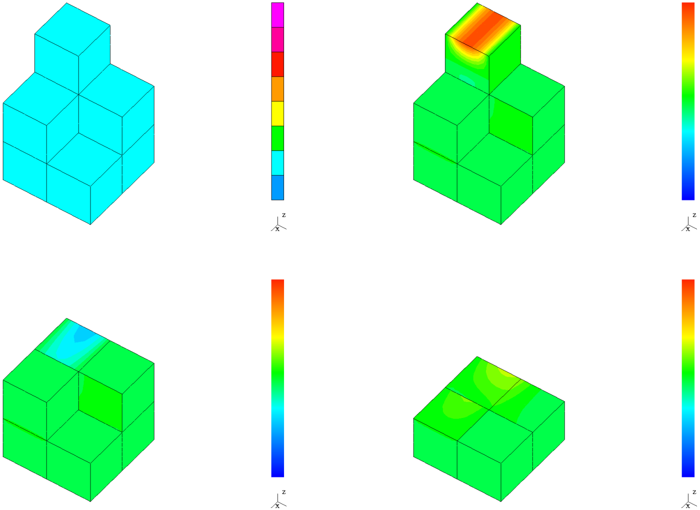
Attach a waveguide:



$\epsilon = \mu = 1, \sigma = 0$
$\omega = 5$(1.6 wavelengths in the cube)

Cut the waveguide and use the lowest propagating mode for BC along the cut.

Fichera corner microwave, $p = 2$.



Initial mesh and real part of E_1

Fichera corner microwave, $p = 2$.

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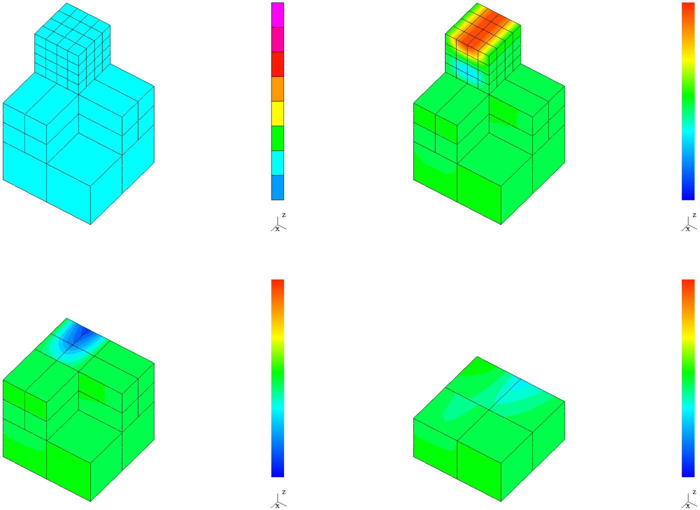

Mesh and real part of E_1 after two refinements

Fichera corner microwave, $p = 2$.

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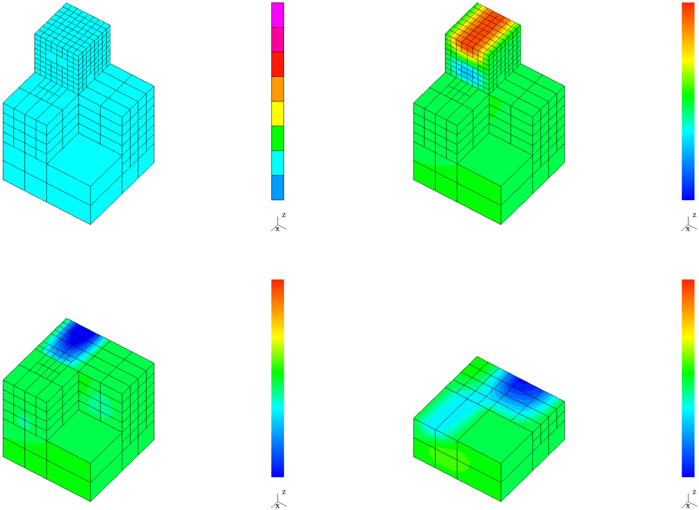
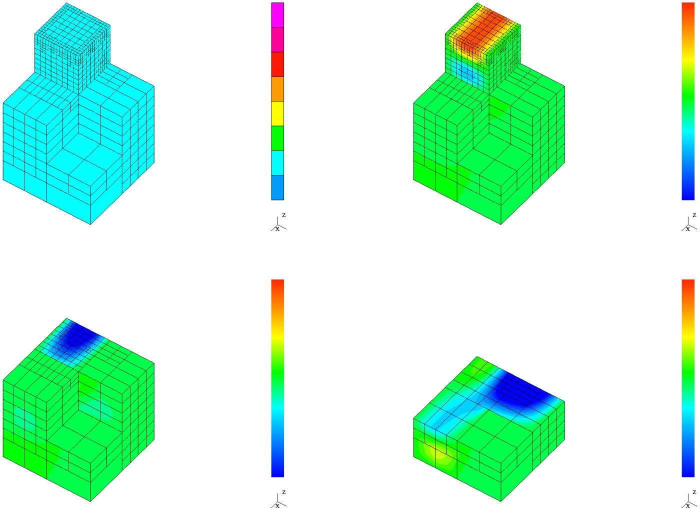


Mesh and real part of E_1 after four refinements

Fichera corner microwave, $p = 2$.



Mesh and real part of E_1 after six refinements

Fichera corner microwave, $p = 2$.

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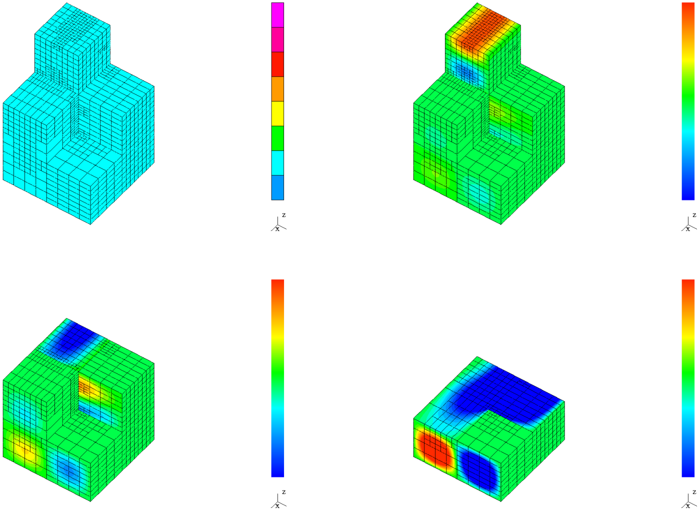
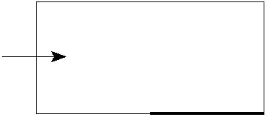


Mesh and real part of E_1 after eight refinements

From Ph.D. Dissertation of Jesse Chan: Compressible Navier-Stokes Equations: Carter's flat plate problem [1]

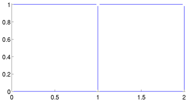


$$\mathsf{M}_\infty = 3, \mathsf{Re}_L = 1000, \mathsf{Pr} = 0.72, \gamma = 1.4, \theta_\infty = 390^o[\mathsf{R}]$$

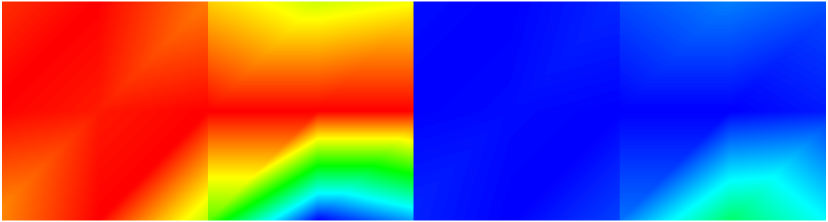
[1] L.D., J.T. Oden, W. Rachowicz, ``A New Finite Element Method for Solving Compressible Navier-Stokes Equations Based on an Operator Splitting Method and hp Adaptivity,'', *Comput. Methods Appl. Mech. Engrg.*, **84**, 275-326, 1990.

Extrapolation to Compressible NS Equations

Initial Mesh ($p = 2$):

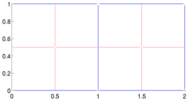


Horizontal velocity and temperature

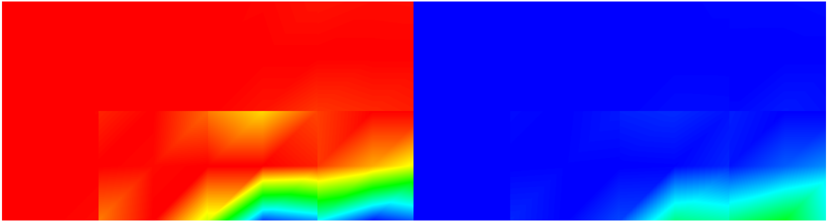


Extrapolation to Compressible NS Equations

Mesh 1:

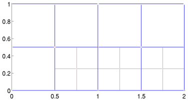


Horizontal velocity and temperature

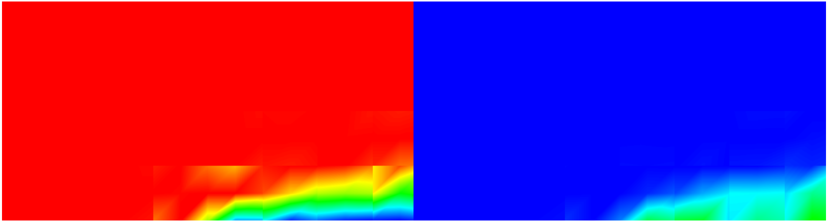


Extrapolation to Compressible NS Equations

Mesh 2:

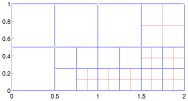


Horizontal velocity and temperature

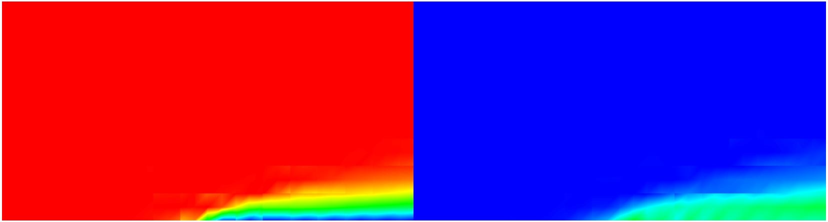


Extrapolation to Compressible NS Equations

Mesh 3:

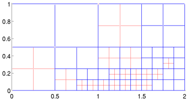


Horizontal velocity and temperature

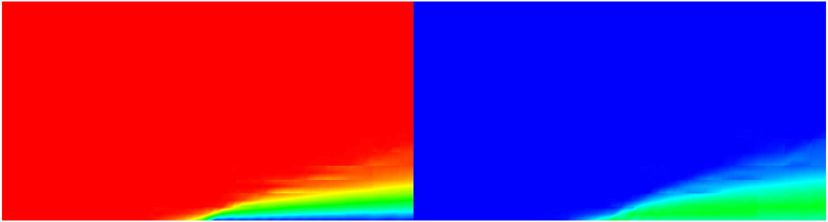


Extrapolation to Compressible NS Equations

Mesh 4:



Horizontal velocity and temperature



Extrapolation to Compressible NS Equations

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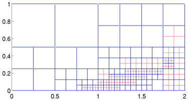
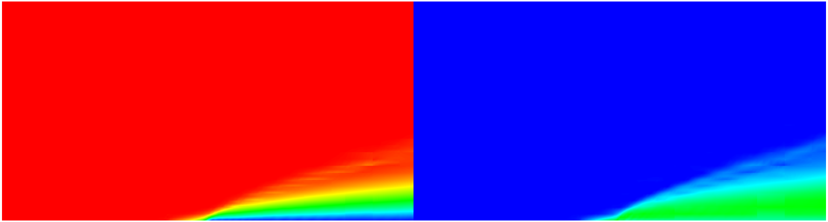
Mesh 5:

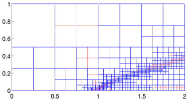


Horizontal velocity and temperature

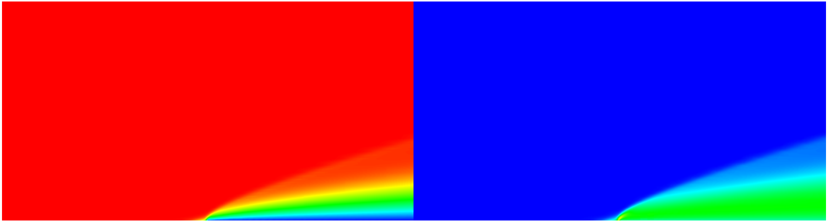


Extrapolation to Compressible NS Equations

Mesh 7:

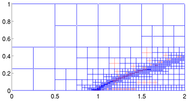


Horizontal velocity and temperature

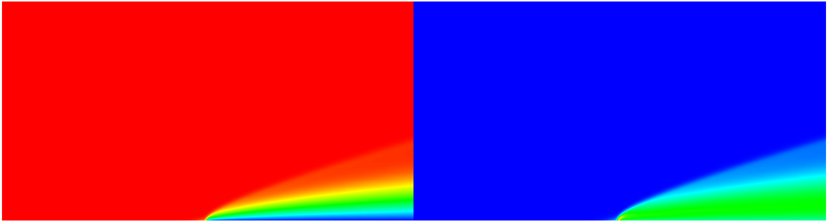


Extrapolation to Compressible NS Equations

Mesh 8:

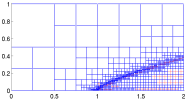


Horizontal velocity and temperature

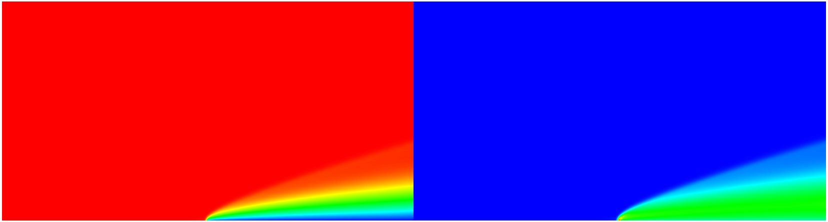


Extrapolation to Compressible NS Equations

Mesh 9:

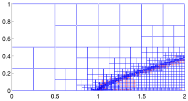


Horizontal velocity and temperature

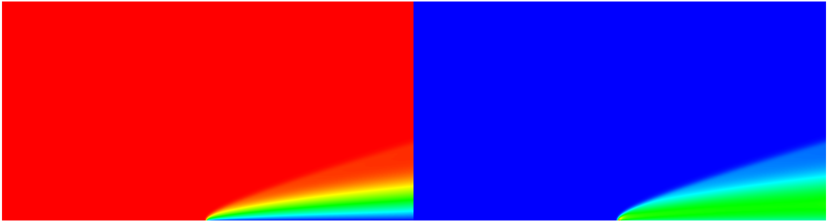


Extrapolation to Compressible NS Equations

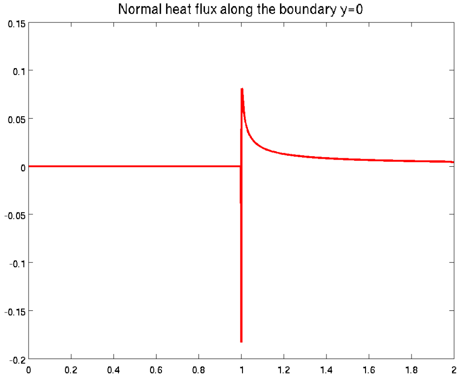
Mesh 10:



Horizontal velocity and temperature



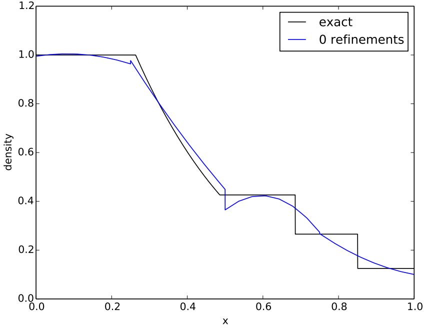
Extrapolation to Compressible NS Equations

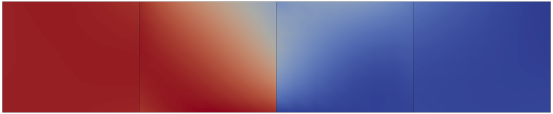


Heat flux along the plate

Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$





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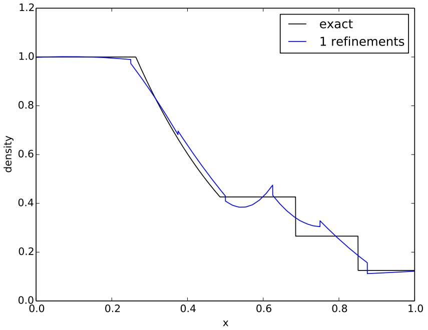

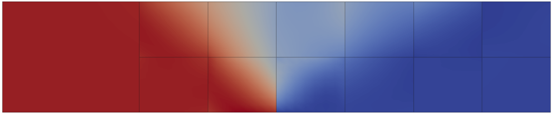

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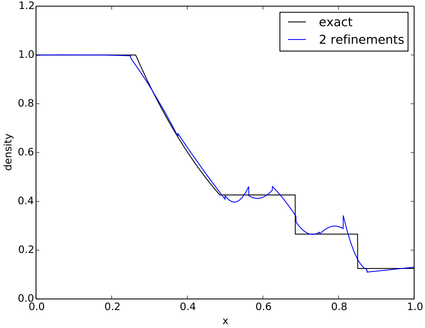

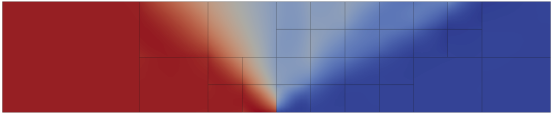

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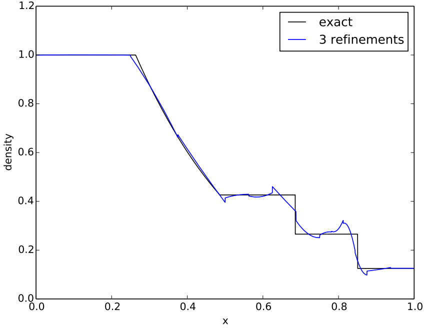

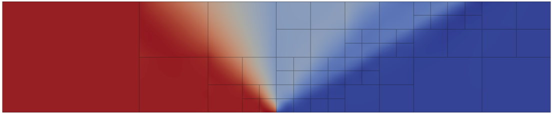

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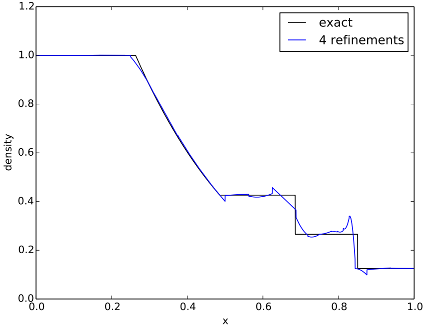

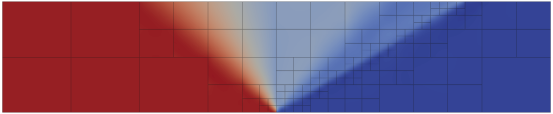

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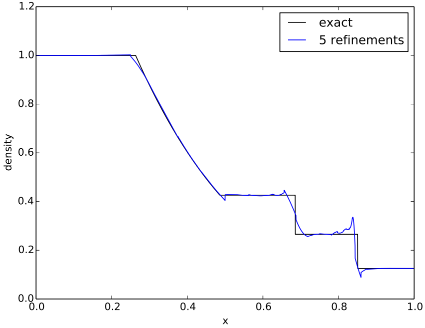

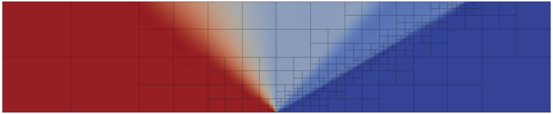

Compressible Navier-Stokes

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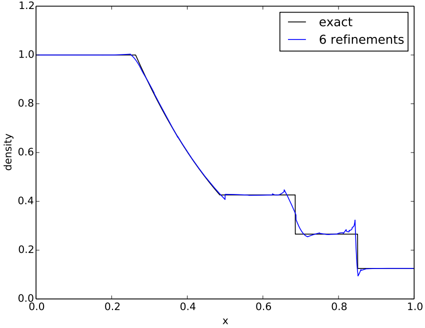

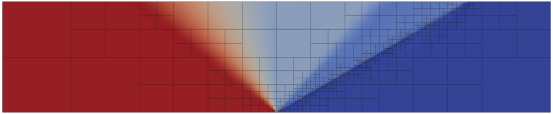

Compressible Navier-Stokes

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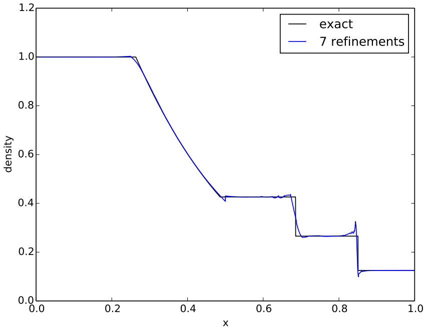

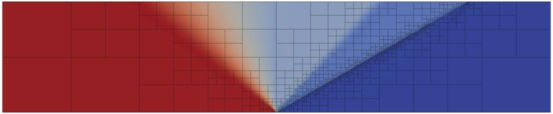

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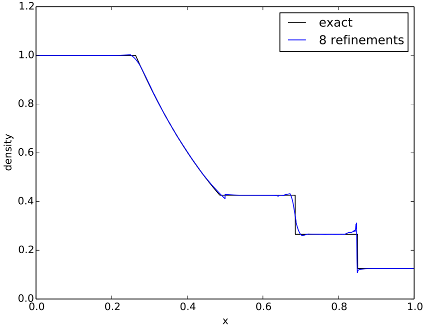

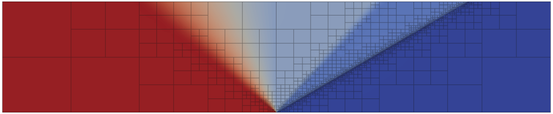

Compressible Navier-Stokes

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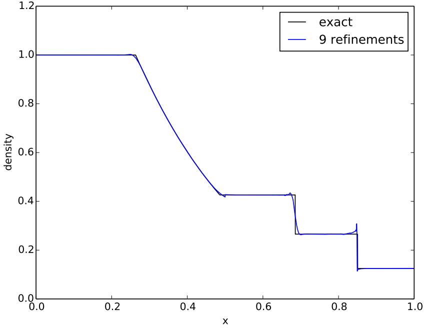

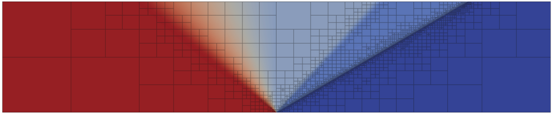

Compressible Navier-Stokes

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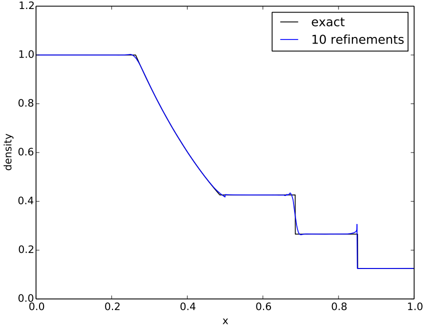
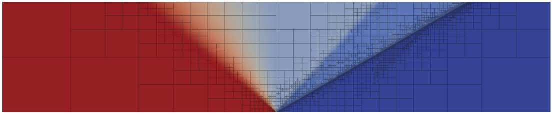
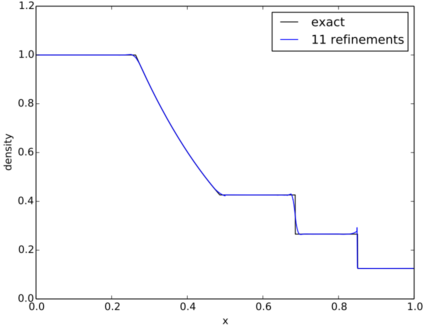
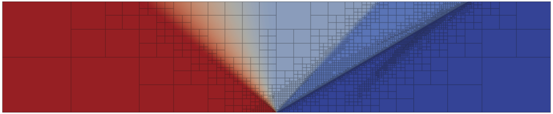




Compressible Navier-Stokes

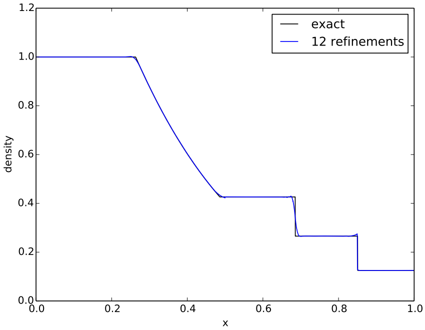
Sod Shock Tube with $\mu = 10^{-5}$

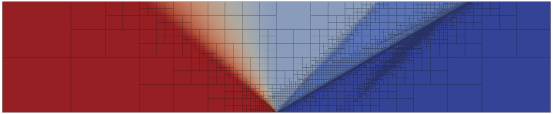




Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$





Compressible Navier-Stokes

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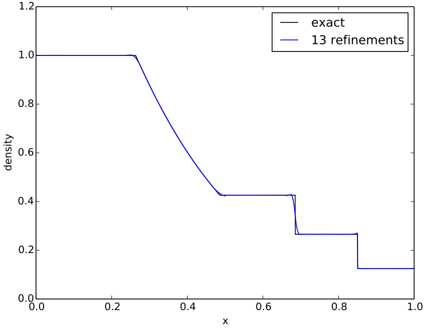
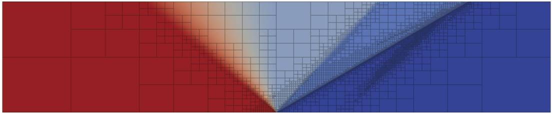
Sod Shock Tube with $\mu = 10^{-5}$





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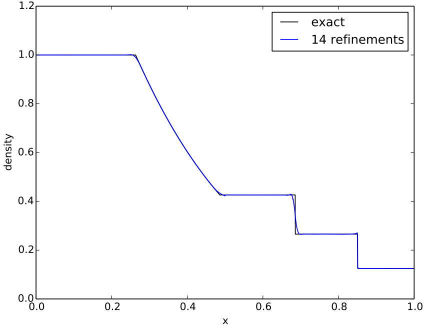
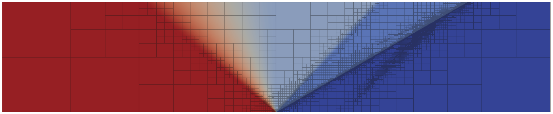
Sod Shock Tube with $\mu = 10^{-5}$





Punchline 2

You can control the norm in which you want to converge.

The simplest singular perturbation problem: reaction-dominated diffusion [2]

[2] L.D. and I. Harari, ``Primal DPG Method for Reaction dominated Diffusion'', in preparation.

The simplest singular perturbation problem: Reaction-dominated diffusion

$$\begin{cases} u &= 0 \quad \text{on } \Gamma \\ -\epsilon^2 \Delta u + c(x)u &= f \quad \text{in } \Omega \end{cases}$$

where $0 < c_0 \le c(x) \le c_1$.
Standard variational formulation:

$$\begin{cases} u \in H^1(\Omega) \\ \epsilon^2(\nabla u, \nabla v) + (cu, v) = (f, v) \quad v \in H^1(\Omega) \end{cases}$$

Standard Galerkin method delivers the best approximation error in the energy norm:

$$\|u\|^2_{\epsilon^k} := \epsilon^k \|\nabla u\|^2 + \|c^{1/2} u\|^2, \quad k = 2$$

Convergence in ``balanced'' norm

Fact: Under favorable regularity conditions, the solution is *uniformly* bounded in data f in a ``balanced'' norm [3] :

$$\|u\|_\epsilon^2 := \epsilon\|\nabla u\|^2 + \|c^{1/2}u\|^2$$

i.e.

$$\|u\|_\epsilon \lesssim \|f\|_{\text{appropriate}}$$

Question: Can we select the test norm in such a way that the DPG method will be *robust* in the balanced norm ?

$$\|u - u_h\|_\epsilon + \|\hat{t} - \hat{t}_h\|_? \lesssim \inf_{w_h} \|u - w_h\|_\epsilon + \inf_{\hat{r}_h} \|\hat{t} - \hat{r}_h\|_?$$

[3] R. Lin and M. Stynes, ``A balanced finite element method for singularly perturbed reaction-diffusion problems'', *SIAM J. Numer. Anal.*, **50(5):** 2729--2743, 2012.

A bit of history: Optimal test functions of Barret and M[4]

For each $w \in U_h$, determine the corresponding v_w that solves the auxiliary variational problem:

$$\left\{ \begin{array}{l} v_w \in H_0^1(\Omega) \\ \underbrace{\epsilon^2(\nabla \delta u, \nabla v_w) + (c\,\delta u, v_w)}_{\text{the bilinear form we have}} = \underbrace{\epsilon(\nabla \delta u, w) + (c\,\delta u, w)}_{\text{the bilinear form we want}} \quad \forall \delta u \in H_0^1(\Omega) \end{array} \right.$$

With the optimal test functions, the Galerkin orthogonality for the original form changes into Galerkin orthogonality in the desired, ``balanced'' norm:

$$\epsilon^2(\nabla(u-u_h), \nabla v_w) + (c\,(u-u_h), v_w) = 0 \quad \Longrightarrow \quad \epsilon(\nabla(u-u_h), \nabla v_u) + (c\,(u-u_h), w) = 0$$

Consequently, the PG solution delivers the best approximation error in the desired norm.

4

- J.W. Barret and K. W. Morton, ``Approximate Symmetrization and Petrov-Galerkin Methods for Diffusion-Convection Problems'', *Comp. Meth. Appl. Mech and Engng.*, **46**, 97 (1984).
- L. D. and J. T. Oden, ``An Adaptive Characteristic Petrov-Galerkin Finite Element Method for Convection-Dominated Linear and Nonlinear Parabolic Problems in One Space Variable'', *Journal of Computational Physics*, **68(1):** 188--273, 1986.

Constructing optimal test norm [5]

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Theorem

Let v_u be the Barret-Morton optimal test function corresponding to u. Let $\|v_u\|_V$ be a test norm such that

$$\|v_u\|_V \lesssim \|u\|_\epsilon$$

Then

$$\|u - u_h\|_\epsilon \lesssim \|u - u_h\|_E = \inf_{w_h \in U_h} \|u - w_h\|_E \leq \textit{BAE estimate}$$

Proof:

$$\begin{aligned}
\|u\|_\epsilon^2 &= \epsilon(\nabla u, \nabla u) + (cu, u) = \epsilon^2(\nabla u, \nabla v_u) + (cu, v_u) \\
&= b((u, \hat{t}), v_u) \leq \frac{b((u,\hat{t}), v_u)}{\|v_u\|_V} \|v_u\|_V \\
&\leq \sup_v \frac{b((u,\hat{t}), v_u)}{\|v\|_V} \|v_u\|_V = \|(u, \hat{t})\|_E \|v_u\|_V \\
&\lesssim \|(u, \hat{t})\|_E \|u\|_\epsilon
\end{aligned}$$

[5] L. D., M. Heuer, ``Robust DPG Method for Convection-Dominated Diffusion Problems,'' *SIAM J. Num. Anal*, **51:** 2514--2537, 2013.

Constructing optimal test norm

The point: Construction of the optimal test norm is reduced to the stability (robustness) analysis for the Barret-Morton test functions.

Lemma

Let

$$\|v\|_V^2 := \epsilon^3 \|\nabla v\|^3 + \|c^{1/2} v\|^2$$

Then

$$\|v_u\| \lesssim \|u\|_\epsilon$$

In order to avoid boundary layers in the optimal test functions (that we cannot resolve using simple enriched space) we scale the reaction term with a mesh-dependent factor:

$$\|v\|_{V,mod}^2 := \epsilon^3 \|\nabla v\|^3 + \min\{1, \frac{\epsilon^3}{h^2}\} \|c^{1/2} v\|^2$$

Manufactured solution of Lin and Stynes, $\epsilon = 10^{-1}$

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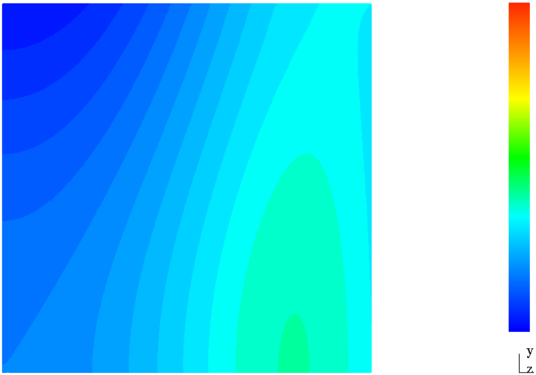
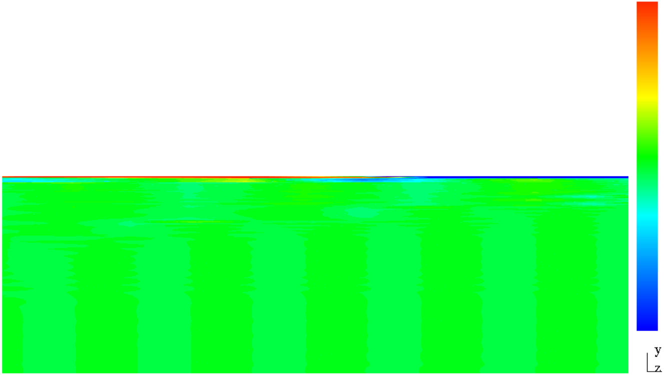


The functions exhibits strong boundary layers invisible in this scale.

Range: (-0.6,0.6)

Manufactured solution of Lin and Stynes, $\epsilon = 10^{-1}$



Zoom on the north boundary layer.

Optimal mesh for $\epsilon = 10^{-1}$

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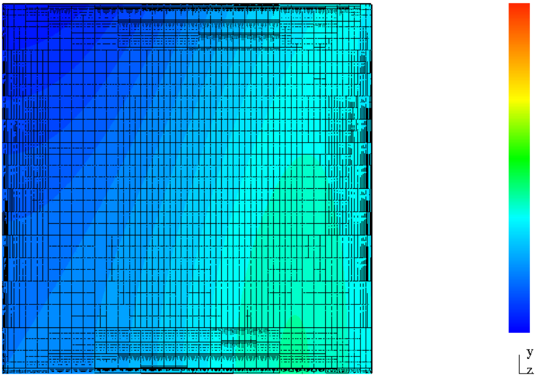
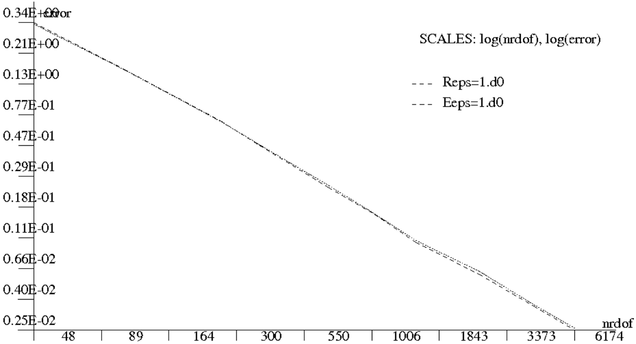


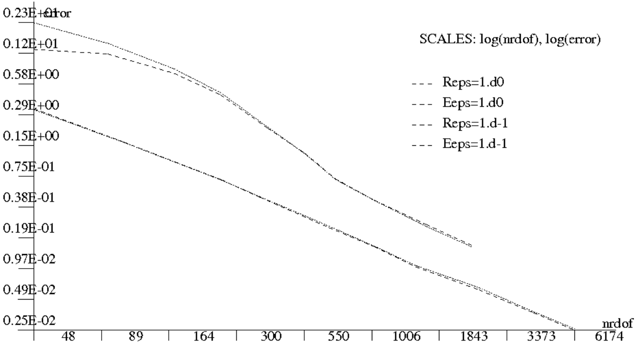
Optimal h-adaptive mesh and numerical solution for $\epsilon = 10^{-1}$

Lin/Stynes example, $\epsilon = 1$



Residual and ``balanced'' error of u for h-adaptive solution, $p = 2$

Lin/Stynes example, $\epsilon = 10^0, 10^{-1}$



Residual and ``balanced'' error of u for h-adaptive solution, $p = 2$

Lin/Stynes example, $\epsilon = 10^0, 10^{-1}, 10^{-2}$.

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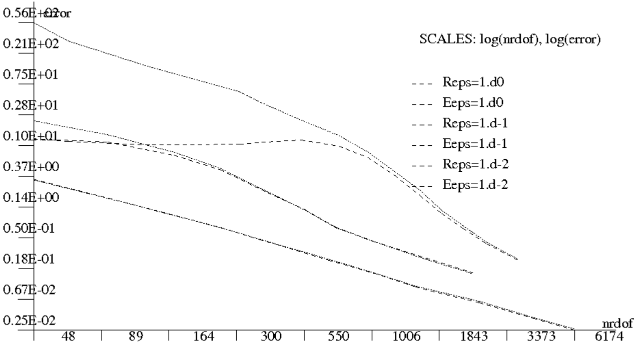

Residual and ``balanced'' error of u for h-adaptive solution, $p = 2$

Lin/Stynes example, $\epsilon = 10^0, 10^{-1}, 10^{-2}, 10^{-3}$.

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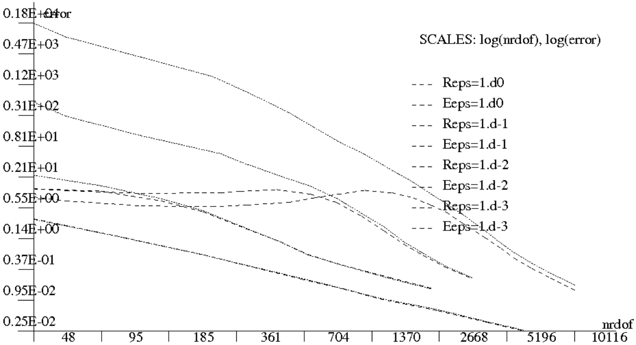
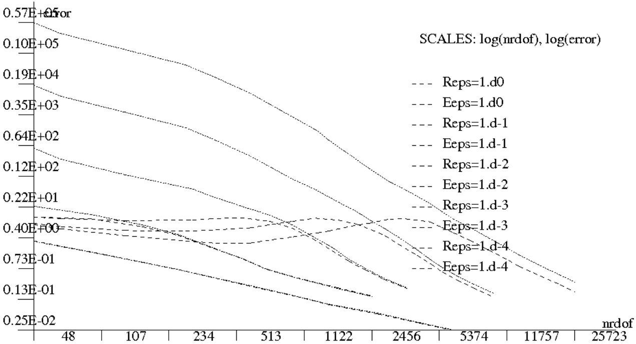


Residual and ``balanced'' error of u for h-adaptive solution, $p = 2$

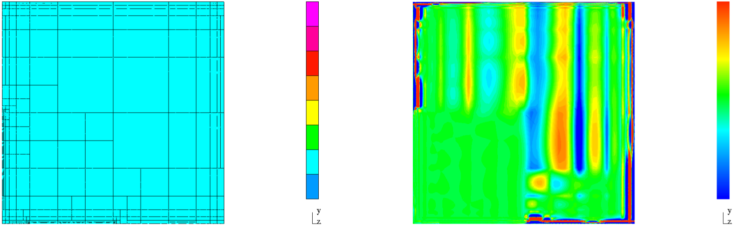
Lin/Stynes example, $\epsilon = 10^0, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-}$



Residual and ``balanced'' error of u for h-adaptive solution, $p = 2$

Other tricks we can play: zooming on the solution

Question: Can we select the test norm in such a way that the DPG method would deliver high accuracy in a preselected subdomain, e.g. $(0, \frac{1}{2})^2 \subset (0,1)^2$?
Answer: Yes!



Optimal mesh and the corresponding pointwise error (range $(-0.001 - 0.001)$.

Punchline 3

DPG is easy to code.

Other Applications

- Wave propagation problems (sonars, full wave form inversion in geomechanics, cloaking)

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- Wave propagation problems (sonars, full wave form inversion in geomechanics, cloaking)
- Elasticity, shells (volumetric, shear, membrane locking)
- Metamaterials