Robustness for transient problems

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Assume that boundary conditions are applied on the boundary $\Gamma_0 \subset \Gamma$. Recall that, for the ultra-weak variational formulation

$$b\left(\left(u,\widehat{u}\right),v\right) = \left(u, A_h^* v\right)_{L^2(\Omega)} + \langle \widehat{u}, [\![v]\!]\rangle_{\Gamma_h \setminus \Gamma_0}$$

we can recover

$$\|u\|^2_{L^2(\Omega)} = b(u, v^*)$$

for conforming v^* satisfying the adjoint equation

$$\begin{aligned} A^* v^* &= u \\ v^* &= 0 \text{ on } \Gamma_h \setminus \Gamma_0. \end{aligned}$$

Together, these give necessary conditions on the test norm $\|\cdot\|_V$ such that we have L^2 robustness (this gives robustness in the variable u; for the first order formulation, conditions for σ must also be shown).

$$\|u\|^2_{L^2(\Omega)} = b(u, v^*) \leq \frac{b(u, v^*)}{\|v^*\|_V} \|v^*\|_V \leq \|u\|_E \|v^*\|_V$$

Thus, showing $\|v^*\|_V \lesssim \|u\|_{L^2(\Omega)}$ gives the result that $\|u\|_{L^2(\Omega)} \lesssim \|u\|_E$.

1 Reaction-diffusion

Consider reaction diffusion

$$\begin{aligned} \frac{\partial u}{\partial t} + u - \epsilon \Delta u &= f \\ u &= 0 \text{ on } \Gamma_1 \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_2 \\ u(t=0) &= u_0. \end{aligned}$$

The adjoint equation satisfies

$$\begin{aligned}-\frac{\partial v}{\partial t}+v-\epsilon\Delta v&=u\\ v&=0\text{ on }\Gamma_1\\ \frac{\partial v}{\partial n}&=0\text{ on }\Gamma_2\\ v(t=T)&=0.\end{aligned}$$

(The boundary conditions can be derived by taking the ultra-weak formulation and choosing boundary conditions such that the temporal flux and spatial flux terms $\langle\widehat{u},[\![\tau_n]\!]\rangle_{\Gamma_1}$ and $\left\langle\widehat{f}_n,[\![v]\!]\right\rangle_{\Gamma_2}$ are zero.)

We can then derive that the test norm

$$\|v\|_V^2=\left\|\frac{\partial v}{\partial t}\right\|^2+\|v\|^2+\epsilon\|\nabla v\|^2$$

provides the necessary bound $\|v^*\|_V\lesssim\|u\|_{L^2(\Omega)}$.

To see, this we multiply the adjoint equation by two terms as follows:

1. Multiply by v and integrate over $\Omega\times[0,T]=Q$ to get

$$-\int_Q\frac{\partial v}{\partial t}v+\int_Q v^2+\epsilon\int_Q|\nabla v|^2-\epsilon\int_0^T\int_\Gamma\frac{\partial v}{\partial n}v=\int_Q uv.$$

Noting that either $v=0$ or $\frac{\partial v}{\partial n}=0$ on the boundary removes the integral over Γ. Next, we can factor the first term and use Young's inequality to get

$$-\int_0^T\frac{\partial}{\partial t}\int_\Omega v^2+\|v\|_Q^2+\epsilon\|\nabla v\|_Q^2\le\frac{1}{2}\|u\|_Q^2+\frac{1}{2}\|v\|_Q^2$$

Integrating by parts the first term gives

$$-\int_\Omega v^2\Big|_0^T+\frac{1}{2}\|v\|_Q^2+\epsilon\|\nabla v\|_Q^2\le\frac{1}{2}\|u\|_Q^2$$

Using boundary condition $v=0$ at $t=T$ gives

$$\frac{1}{2}\|v\|_Q^2+\epsilon\|\nabla v\|_Q^2\le\int_\Omega v(t=0)^2+\frac{1}{2}\|v\|_Q^2+\epsilon\|\nabla v\|_Q^2\le\frac{1}{2}\|u\|_Q^2.$$

2. Multiply by $-\frac{\partial v}{\partial t}$ and integrate over Q. Young's inequality changes the right hand side to

$$\int_Q\frac{\partial v}{\partial t}^2-\int_Q v\frac{\partial v}{\partial t}+\epsilon\int_Q\Delta v\frac{\partial v}{\partial t}=\int_Q-u\frac{\partial v}{\partial t}\le\frac{1}{2}\|u\|_Q^2+\frac{1}{2}\left\|\frac{\partial v}{\partial t}\right\|_Q^2.$$

The term $\int_Q v\frac{\partial v}{\partial t}$ can be reduced to the positive contribution $\int_\Omega v(t=0)^2$ as above. We can then take the Laplacian term, integrate by parts in space to get

$$\int_Q \Delta v \frac{\partial v}{\partial t} = \int_0^T \int_\Omega \Delta v \frac{\partial v}{\partial t} = \int_0^T \int_\Gamma \frac{\partial v}{\partial t}\frac{\partial v}{\partial n} - \int_0^T \int_\Omega \nabla\left(\frac{\partial v}{\partial t}\right)\nabla v.$$

Since either $v=0$ or $\frac{\partial v}{\partial n}=0$ on Γ, the first term disappears. The second term can be bounded by noting

$$-\int_0^T \int_\Omega \nabla\left(\frac{\partial v}{\partial t}\right)\nabla v = -\int_0^T \frac{\partial}{\partial t}\int_\Omega |\nabla v|^2 = -\int_\Omega |\nabla v|^2\Big|_0^T.$$

Since $v=0$ at $t=T$, $\nabla v=0$ at $t=T$ as well, and we are left with the positive contribution $\int_\Omega |\nabla v(t=0)|^2$. Then,

$$\frac{1}{2}\left\|\frac{\partial v}{\partial t}\right\|_Q^2 \leq \frac{1}{2}\|u\|_Q.$$

Together, these two show that, under test norm

$$\|v\|_V^2 = \left\|\frac{\partial v}{\partial t}\right\|^2 + \|v\|^2 + \epsilon\|\nabla v\|^2,$$

the adjoint equation v^* satisfies

$$\|v^*\|_V \lesssim \|u\|_{L^2(\Omega)}$$

and thus the DPG energy norm robustly bounds the L^2 norm from above

$$\|u\|_{L^2(\Omega)} \lesssim \|u\|_E.$$

2 Convection-diffusion

Truman, your turn :).

3 Robustness for transient problems given spatial robustness

Suppose we have the transient problem

$$\frac{\partial u}{\partial t} + Au = f$$

with initial condition $u(x,0)=u_0$. Suppose that DPG is robust under the ultra-weak variational formulation for the steady problem

$$(u, A_h^* v)_{L^2(\Omega)} + \langle \widehat{u}, [\![v]\!]\rangle_{\Gamma_h\setminus\Gamma_0} = (f,v)$$

with test norm $\|v\|_V$. Then, can we show that

$$\|v\|_{V,t} \coloneqq \|v\|_V + \left\|\frac{\partial v}{\partial t}\right\|_{L^2(\Omega)}$$

also leads to a robust upper bound of the L^2 norm by the DPG energy norm? I believe this may be possible. The adjoint equation for robustness for the transient problem gives

$$-\frac{\partial v}{\partial t} + A^* v = u$$

with $v = 0$ at $t = T$...