Robustness for Transient Problems

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Consider domain $Q = \Omega \times [0, T]$ with boundary $\Gamma = \Gamma_- \cup \Gamma_+ \cup \Gamma_0 \cup \Gamma_T$ where Γ_- is the spatial inflow boundary, Γ_+ is the spatial outflow boundary, Γ_0 is the initial time boundary, and Γ_T is the final time boundary. Let Γ_h denote the entire mesh skeleton.

Assume that boundary conditions are applied on the boundary $\Gamma_0 \subset \Gamma$. Recall that, for the ultra-weak variational formulation

$$b\left((u, \widehat{u}), v\right) = (u, A_h^* v)_{L^2(\Omega)} + \langle \widehat{u}, [\![v]\!] \rangle_{\Gamma_h \setminus \Gamma_0}$$

we can recover

$$\|u\|_{L^2(Q)}^2 = b(u, v^*)$$

for conforming v^* satisfying the adjoint equation

$$\begin{aligned} A^* v^* &= u \\ v^* &= 0 \text{ on } \Gamma_h \setminus \Gamma_0. \end{aligned}$$

Together, these give necessary conditions on the test norm $\|\cdot\|_V$ such that we have L^2 robustness (this gives robustness in the variable u; for the first order formulation, conditions for σ must also be shown).

$$\|u\|_{L^2(Q)}^2 = b(u, v^*) \leq \frac{b(u, v^*)}{\|v^*\|_V} \|v^*\|_V \leq \|u\|_E \|v^*\|_V$$

Thus, showing $\|v^*\|_V \lesssim \|u\|_{L^2(Q)}$ gives the result that $\|u\|_{L^2(Q)} \lesssim \|u\|_E$.

1 Convection-Diffusion

Consider convection-diffusion

$$\begin{aligned} \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= 0 \\ \frac{\partial u}{\partial t} + \boldsymbol{\beta} \cdot \nabla u - \nabla \cdot \boldsymbol{\sigma} &= f \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_- \\ u &= 0 \text{ on } \Gamma_+ \\ u &= u_0 \text{ on } \Gamma_0. \end{aligned}$$

Let $\tilde{\boldsymbol{\beta}} := \begin{pmatrix} \boldsymbol{\beta} \\ 1 \end{pmatrix}$ and $\nabla_{xt} := \begin{pmatrix} \nabla \\ \frac{\partial}{\partial t} \end{pmatrix}$, then we can rewrite this as

$$
\begin{aligned}
\frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= 0 \\
\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} u - \nabla \cdot \boldsymbol{\sigma} &= f \\
\frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_- \\
u &= 0 \text{ on } \Gamma_+ \\
u &= u_0 \text{ on } \Gamma_0 .
\end{aligned}
$$

We decompose the adjoint into three parts: a discontinuous part

$$
\begin{aligned}
\frac{1}{\epsilon}\boldsymbol{\tau}_0 + \nabla v_0 &= 0 \\
-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_0 + \nabla \cdot \boldsymbol{\tau}_0 &= 0 \\
\boldsymbol{\tau}_0 \cdot \boldsymbol{n}_x &= \boldsymbol{\tau} \cdot \boldsymbol{n}_x \text{ on } \Gamma_- \cup \Gamma_0 \\
v_0 &= v \text{ on } \Gamma_+ \\
v_0 &= v \text{ on } \Gamma_T \\
[\![v_0]\!] &= [\![v]\!] \text{ on } \Gamma_h^0 \\
[\![\boldsymbol{\tau}_0 \cdot \boldsymbol{n}_x]\!] &= [\![\boldsymbol{\tau}_0 \cdot \boldsymbol{n}_x]\!] \text{ on } \Gamma_{hx}^0 ,
\end{aligned}
$$

a continuous part with forcing term g

$$
\begin{aligned}
\frac{1}{\epsilon}\boldsymbol{\tau}_1 + \nabla v_1 &= 0 \\
-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1 + \nabla \cdot \boldsymbol{\tau}_1 &= g \\
\boldsymbol{\tau}_1 \cdot \boldsymbol{n}_x &= 0 \text{ on } \Gamma_- \\
v_1 &= 0 \text{ on } \Gamma_+ \\
v_1 &= 0 \text{ on } \Gamma_T ,
\end{aligned}
$$

and a continuous part with forcing $\boldsymbol{f}$

$$
\begin{aligned}
\frac{1}{\epsilon}\boldsymbol{\tau}_2 + \nabla v_2 &= \boldsymbol{f} \\
-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_2 + \nabla \cdot \boldsymbol{\tau}_2 &= 0 \\
\boldsymbol{\tau}_2 \cdot \boldsymbol{n}_x &= 0 \text{ on } \Gamma_- \\
v_2 &= 0 \text{ on } \Gamma_+ \\
v_2 &= 0 \text{ on } \Gamma_T .
\end{aligned}
$$

(The boundary conditions can be derived by taking the ultra-weak formulation and choosing boundary conditions such that the temporal flux and spatial flux terms $\langle \widehat{u}, [\![\tau_n]\!] \rangle_{\Gamma_{out}}$ and $\langle \hat{t}_n, [\![v]\!] \rangle_{\Gamma_{in}}$ are zero.)

We can then derive that the test norm

$$\|(v,\boldsymbol{\tau})\|_{V,K}^2 \coloneqq \frac{1}{\epsilon}\|\boldsymbol{\tau}\|_K^2 + \left\|\nabla\cdot\boldsymbol{\tau} - \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\right\|_K^2 + \|\boldsymbol{\beta}\cdot\nabla v\|_K^2 + \epsilon\|\nabla v\|_K^2 + \|v\|_K^2\,, \tag{1}$$

provides the necessary bound $\|v^*\|_V \lesssim \|u\|_{L^2(Q)}$.

In the following lemmas we establish the following bounds:

- Bound on $\|(v_0,\boldsymbol{\tau}_0)\|_V$.
- Bound on $\|(v_1,\boldsymbol{\tau}_1)\|_V$. Lemma 1.1 gives $\left\|\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v_1\right\| \leq \|g\|$. Since $\nabla\cdot\boldsymbol{\tau}_1 = g + \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v_1$,

  $$\|\nabla\cdot\boldsymbol{\tau}_1\| \leq \|g\| + \left\|\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v_1\right\| \leq 2\|g\|\,.$$

  Or, the fact that $\nabla\cdot\boldsymbol{\tau} - \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v_1 = g$ clearly gives

  $$\left\|\nabla\cdot\boldsymbol{\tau} - \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v_1\right\| = \|g\|\,.$$

  Also, clearly

  $$\|\boldsymbol{\beta}\cdot\nabla v_1\| \leq \left\|\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v_1\right\| \leq \|g\|\,.$$

  Lemma 1.2 gives $\|v_1\|^2 + \epsilon\|\nabla v_1\|^2 \leq \|g\|^2$. Since $\epsilon^{1/2}\nabla v_1 = -\epsilon^{-1/2}\boldsymbol{\tau}_1$,

  $$\frac{1}{\epsilon}\|\boldsymbol{\tau}_1\|^2 \leq \|g\|^2\,.$$

  Thus, all $(v_1,\boldsymbol{\tau}_1)$ terms in (1) are accounted for.
- Bound on $\|(v_2,\boldsymbol{\tau}_2)\|_V$. The fact that $\nabla\cdot\boldsymbol{\tau} - \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v = 0$ clearly gives

  $$\left\|\nabla\cdot\boldsymbol{\tau} - \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v_2\right\| = 0 \leq \|\boldsymbol{f}\|\,.$$

  Lemma 1.2 gives $\|v_2\|^2 + \epsilon\|\nabla v_2\|^2 \leq \epsilon\|\boldsymbol{f}\|^2$. Since $\epsilon^{1/2}\nabla v_2 = \boldsymbol{f} - \epsilon^{-1/2}\boldsymbol{\tau}_2$,

  $$\frac{1}{\epsilon}\|\boldsymbol{\tau}_2\|^2 \leq (1+\epsilon)\|\boldsymbol{f}\|^2\,.$$

  Finally,

  $$\|\boldsymbol{\beta}\cdot\nabla v_2\| \leq \|\boldsymbol{\beta}\|_\infty\|\nabla v_2\| \leq \|\boldsymbol{\beta}\|_\infty\|\boldsymbol{f}\|\,.$$

  Thus, all $(v_2,\boldsymbol{\tau}_2)$ terms in (1) are accounted for.

Our goal is to analyze the stability properties of the adjoint equations by deriving bounds of the form $\|(v_1,\boldsymbol{\tau}_1)\|_V \leq \|g\|_L^2(Q)$ and $\|(v_2,\boldsymbol{\tau}_2)\|_V \leq \|f\|_L^2(Q)$.

Insert conditions on $\boldsymbol{\beta}$

Lemma 1.1. *For the above conditions on $\boldsymbol{\beta}$,*

$$\left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1\right\| \leq \|g\| \,.$$

Proof. Multiply by $-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v$ and integrate over Q to get

$$\left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v\right\| = -\int_Q g\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla \cdot \boldsymbol{\tau} \,. \tag{2}$$

Note that

$$\begin{aligned}
\frac{1}{\epsilon}\int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla \cdot \boldsymbol{\tau} &= -\int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla \cdot \nabla v \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \int_Q \nabla(\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v) \cdot \nabla v \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \int_Q (\nabla\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v) \cdot \nabla v \\
&\quad + \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla\nabla_{xt} v \cdot \nabla v \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \int_Q (\nabla\boldsymbol{\beta} \cdot \nabla v) \cdot \nabla v \\
&\quad + \frac{1}{2}\int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt}(\nabla v \cdot \nabla v) \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \int_Q (\nabla\boldsymbol{\beta} \cdot \nabla v) \cdot \nabla v \\
&\quad + \frac{1}{2}\int_\Gamma \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n}(\nabla v \cdot \nabla v) - \frac{1}{2}\int_Q \nabla_{xt} \cdot \tilde{\boldsymbol{\beta}}(\nabla v \cdot \nabla v) \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \int_Q (\nabla\boldsymbol{\beta} \cdot \nabla v) \cdot \nabla v \\
&\quad + \frac{1}{2}\int_\Gamma \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n}(\nabla v \cdot \nabla v) - \frac{1}{2}\int_Q \nabla \cdot \boldsymbol{\beta}(\nabla v \cdot \nabla v) \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \frac{1}{2}\int_\Gamma \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n}(\nabla v \cdot \nabla v) \\
&\quad + \int_Q \nabla v(\nabla\boldsymbol{\beta} - \frac{1}{2}\nabla \cdot \boldsymbol{\beta}\boldsymbol{I})\nabla v
\end{aligned}$$

Plugging this into (2), we get

$$
\begin{aligned}
\left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v\right\| = & -\int_Q g\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \boldsymbol{I}) \nabla v \\
& - \epsilon \int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \epsilon \frac{1}{2} \int_\Gamma \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n} (\nabla v \cdot \nabla v) \\
= & -\int_Q g\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \boldsymbol{I}) \nabla v \\
& - \int_{\Gamma_-} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \underbrace{\nabla v \cdot \boldsymbol{n}_x}_{=0} - \int_{\Gamma_+} \left(\underbrace{\frac{\partial v}{\partial t}}_{=0} + \boldsymbol{\beta} \cdot \nabla v \right) \nabla v \cdot \boldsymbol{n}_x \\
& + \frac{1}{2} \int_{\Gamma_-} \underbrace{\boldsymbol{\beta} \cdot \boldsymbol{n}_x}_{<0} (\nabla v \cdot \nabla v) + \frac{1}{2} \int_{\Gamma_+} \boldsymbol{\beta} \cdot \boldsymbol{n}_x (\nabla v \cdot \nabla v) \\
& + \frac{1}{2} \int_{\Gamma_0} \underbrace{n_t}_{<0} (\nabla v \cdot \nabla v) + \frac{1}{2} \int_{\Gamma_T} n_t \underbrace{(\nabla v \cdot \nabla v)}_{=0} \\
\leq & -\int_Q g\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \boldsymbol{I}) \nabla v \\
& + \int_{\Gamma_+} \left(-\frac{\partial v}{\partial \boldsymbol{n}_x} \boldsymbol{\beta} + \frac{1}{2} \boldsymbol{\beta} \cdot \boldsymbol{n}_x \nabla v \right) \cdot \nabla v \\
= & -\int_Q g\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \boldsymbol{I}) \nabla v \\
& + \int_{\Gamma_+} \left(-\frac{\partial v}{\partial \boldsymbol{n}_x} \boldsymbol{\beta} + \frac{1}{2} \boldsymbol{\beta} \cdot \boldsymbol{n}_x \frac{\partial v}{\partial \boldsymbol{n}_x} \boldsymbol{n}_x \right) \cdot \frac{\partial v}{\partial \boldsymbol{n}_x} \boldsymbol{n}_x \\
= & -\int_Q g\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \boldsymbol{I}) \nabla v \\
& \underbrace{- \frac{1}{2} \int_{\Gamma_+} \left(\frac{\partial v}{\partial \boldsymbol{n}_x} \right)^2 \boldsymbol{\beta} \cdot \boldsymbol{n}_x}_{<0} \\
\leq & -\int_Q g\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \boldsymbol{I}) \nabla v \\
\leq & -\frac{\|g\|}{2} + \frac{\left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v\right\|}{2} + \epsilon \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \boldsymbol{I}) \nabla v \\
\leq & -\frac{\|g\|}{2} + \frac{\left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v\right\|}{2} + \epsilon C \left\|\nabla v\right\|^2
\end{aligned}
$$

□

Lemma 1.2. *For the duration of this lemma, let $v := v_1 + v_2$. Then, for the* *above conditions on $\boldsymbol{\beta}$,*

$$\|v\|^2 + \epsilon \|\nabla v\|^2 \le \|g\|^2 + \epsilon \|\boldsymbol{f}\|^2 \ .$$

Proof. Define $w = e^t v$ and note that $\frac{\partial w}{\partial t} = \left(\frac{\partial v}{\partial t} + v\right) e^t$ while all spatial derivatives go through. Multiplying the adjoint by w and integrating over Q gives

$$-\int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v w - \epsilon \Delta v w = \int_Q g w - \epsilon \int_Q \nabla \cdot \boldsymbol{f} w$$

or

$$-\int_Q e^t v \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v - \epsilon \int_Q e^t v \Delta v = \int_Q e^t g v - \epsilon \int_Q e^t v \nabla \cdot \boldsymbol{f}$$

Integrating by parts:

$$\begin{aligned}\int_Q \nabla_{xt} \cdot \left(e^t \tilde{\boldsymbol{\beta}} v\right) v - \int_\Gamma e^t \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n} v^2 + \epsilon \int_Q e^t \nabla v \cdot \nabla v - \epsilon \int_{\Gamma_x} e^t v \cdot \nabla v \cdot \boldsymbol{n}_x \\ = \int_Q e^t g v + \epsilon \int_Q e^t \nabla v \cdot \boldsymbol{f} - \epsilon \int_{\Gamma_x} e^t v \boldsymbol{f} \cdot \boldsymbol{n}_x\end{aligned}$$

Note that $\nabla_{xt} \cdot e^t v \tilde{\boldsymbol{\beta}} = e^t(\tilde{\boldsymbol{\beta}} \nabla_{xt} v + v)$ if $\nabla \cdot \boldsymbol{\beta} = 0$. Dividing both sides by e^t and moving some terms to the right hand side, we get

$$\begin{aligned}&\int_Q v^2 + \int_Q \epsilon \nabla v \cdot \nabla v \\ &\quad = \int_Q g v + \epsilon \int_Q \nabla v \cdot \boldsymbol{f} - \epsilon \int_{\Gamma_x} v \boldsymbol{f} \cdot \boldsymbol{n}_x \\ &\quad\quad - \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v v \quad + \int_\Gamma \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n} v^2 \quad + \epsilon \int_{\Gamma_x} v \cdot \nabla v \cdot \boldsymbol{n}_x\end{aligned}$$

or

$$
\begin{aligned}
&\|v\|^2 + \epsilon \|\nabla v\|^2 \\
&\quad = \int_Q gv + \epsilon \int_Q \nabla v \cdot \boldsymbol{f} - \epsilon \int_{\Gamma_-} v \underbrace{\boldsymbol{f} \cdot \boldsymbol{n}_x}_{=\cancelto{0}{\tau_h} + \frac{\partial v}{\partial \boldsymbol{n}_x}} - \epsilon \int_{\Gamma_+} \underbrace{v}_{=0} \boldsymbol{f} \cdot \boldsymbol{n}_x \\
&\qquad - \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v v \quad + \int_\Gamma \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n} v^2 \quad + \epsilon \int_{\Gamma_-} v \cdot \nabla v \cdot \boldsymbol{n}_x \quad + \epsilon \int_{\Gamma_+} \underbrace{v}_{=0} \frac{\partial v}{\partial \boldsymbol{n}_x} \\
&\quad = \int_Q gv + \epsilon \int_Q \nabla v \cdot \boldsymbol{f} \cancel{- \epsilon \int_{\Gamma_-} v \frac{\partial v}{\partial \boldsymbol{n}_x} + \epsilon \int_{\Gamma_x} v \frac{\partial v}{\partial \boldsymbol{n}_x}} \\
&\qquad - \frac{1}{2} \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v^2 \quad + \int_\Gamma \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n} v^2 \\
&\quad = \int_Q gv + \epsilon \int_Q \nabla v \cdot \boldsymbol{f} \\
&\qquad + \frac{1}{2} \int_Q \cancelto{0}{\nabla_{xt} \cdot \tilde{\boldsymbol{\beta}} v^2} \quad - \frac{1}{2} \int_\Gamma \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n} v^2 \quad + \int_\Gamma \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n} v^2 \\
&\quad = \int_Q gv + \epsilon \int_Q \nabla v \cdot \boldsymbol{f} \\
&\qquad + \frac{1}{2} \left(\int_{\Gamma_0} \underbrace{-v^2}_{\leq 0} \quad + \int_{\Gamma_T} \cancelto{0}{v^2} + \int_{\Gamma_-} \underbrace{\boldsymbol{\beta} \cdot \boldsymbol{n}_x v^2}_{\leq 0} \quad + \int_{\Gamma_+} \boldsymbol{\beta} \cdot \boldsymbol{n}_x \cancelto{0}{v^2} \right) \\
&\quad \leq \int_Q gv + \epsilon \int_Q \nabla v \cdot \boldsymbol{f} \\
&\quad \leq \frac{\|g\|^2}{2} + \epsilon \frac{\|\boldsymbol{f}\|^2}{2} + \frac{\|v\|^2}{2} + \epsilon \frac{\|\nabla v\|^2}{2}
\end{aligned}
$$

$\square$

2 Robustness for transient problems given spatial robustness

Suppose we have the transient problem

$$\frac{\partial u}{\partial t} + Au = f$$

with initial condition $u(x, 0) = u_0$. Suppose that DPG is robust under the ultra-weak variational formulation for the steady problem

$$(u, A_h^* v)_{L^2(\Omega)} + \langle \widehat{u}, [\![v]\!] \rangle_{\Gamma_h \setminus \Gamma_0} = (f, v)$$

with test norm $\|v\|_V$. Then, can we show that

$$\|v\|_{V,t} := \|v\|_V + \left\| \frac{\partial v}{\partial t} \right\|_{L^2(\Omega)}$$

also leads to a robust upper bound of the L^2 norm by the DPG energy norm? I believe this may be possible. The adjoint equation for robustness for the transient problem gives

$$-\frac{\partial v}{\partial t} + A^* v = u$$

with $v = 0$ at $t = T$...

3 Transient Eriksson-Johnson

We can derive a transient Eriksson-Johnson solution using separation of variables. Consider

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \epsilon \Delta u = 0$$

with boundary conditions

$$\begin{aligned} u &= 0 \text{ on } \Gamma_+, \\ u - \epsilon \frac{\partial u}{\partial n} &= u_0 - \epsilon \frac{\partial u_0}{\partial n} \text{ on } \Gamma_-, \\ \epsilon \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_0, \end{aligned}$$

and initial condition $u(x, y, 0) = u_0(x, y)$ that satisfies the given boundary data. Assuming that $u(x, y, t) = X(x, y)T(t)$ and $Lu = \frac{\partial u}{\partial x} - \epsilon \Delta u$, we can plug this into the equation

$$\frac{\partial u}{\partial t} + Lu = 0$$

and rearrange to get

$$-\frac{\frac{\partial T}{\partial t}}{T} = \frac{LX}{X} = C.$$

This assumes then that $\frac{\partial T}{\partial t} = -CT$, or that $T(t) = e^{-Ct}$, and that $LX = CX$, or that X is made up of the eigenfunctions of L.