

# Robustness for Transient Problems

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Assume that boundary conditions are applied on the boundary  $\Gamma_0 \subset \Gamma$ . Recall that, for the ultra-weak variational formulation

$$b((u, \hat{u}), v) = (u, A_h^* v)_{L^2(\Omega)} + \langle \hat{u}, \llbracket v \rrbracket \rangle_{\Gamma_h \setminus \Gamma_0}$$

we can recover

$$\|u\|_{L^2(\Omega)}^2 = b(u, v^*)$$

for conforming  $v^*$  satisfying the adjoint equation

$$\begin{aligned} A^* v^* &= u \\ v^* &= 0 \text{ on } \Gamma_h \setminus \Gamma_0. \end{aligned}$$

Together, these give necessary conditions on the test norm  $\|\cdot\|_V$  such that we have  $L^2$  robustness (this gives robustness in the variable  $u$ ; for the first order formulation, conditions for  $\sigma$  must also be shown).

$$\|u\|_{L^2(\Omega)}^2 = b(u, v^*) \leq \frac{b(u, v^*)}{\|v^*\|_V} \|v^*\|_V \leq \|u\|_E \|v^*\|_V$$

Thus, showing  $\|v^*\|_V \lesssim \|u\|_{L^2(\Omega)}$  gives the result that  $\|u\|_{L^2(\Omega)} \lesssim \|u\|_E$ .

## 1 Reaction-diffusion

Consider reaction diffusion

$$\begin{aligned} \frac{\partial u}{\partial t} + u - \epsilon \Delta u &= f \\ u &= 0 \text{ on } \Gamma_1 \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_2 \\ u(t=0) &= u_0. \end{aligned}$$

The adjoint equation satisfies

$$\begin{aligned} -\frac{\partial v}{\partial t} + v - \epsilon \Delta v &= u \\ v &= 0 \text{ on } \Gamma_1 \\ \frac{\partial v}{\partial n} &= 0 \text{ on } \Gamma_2 \\ v(t = T) &= 0. \end{aligned}$$

(The boundary conditions can be derived by taking the ultra-weak formulation and choosing boundary conditions such that the temporal flux and spatial flux terms  $\langle \hat{u}, \llbracket \tau_n \rrbracket \rangle_{\Gamma_1}$  and  $\langle \hat{f}_n, \llbracket v \rrbracket \rangle_{\Gamma_2}$  are zero.)

We can then derive that the test norm

$$\|v\|_V^2 = \left\| \frac{\partial v}{\partial t} \right\|^2 + \|v\|^2 + \epsilon \|\nabla v\|^2$$

provides the necessary bound  $\|v^*\|_V \lesssim \|u\|_{L^2(\Omega)}$ .

To see, this we multiply the adjoint equation by two terms as follows:

1. Multiply by  $v$  and integrate over  $\Omega \times [0, T] = Q$  to get

$$-\int_Q \frac{\partial v}{\partial t} v + \int_Q v^2 + \epsilon \int_Q |\nabla v|^2 - \epsilon \int_0^T \int_\Gamma \frac{\partial v}{\partial n} v = \int_Q uv.$$

Noting that either  $v = 0$  or  $\frac{\partial v}{\partial n} = 0$  on the boundary removes the integral over  $\Gamma$ . Next, we can factor the first term and use Young's inequality to get

$$-\int_0^T \frac{\partial}{\partial t} \int_\Omega v^2 + \|v\|_Q^2 + \epsilon \|\nabla v\|_Q^2 \leq \frac{1}{2} \|u\|_Q^2 + \frac{1}{2} \|v\|_Q^2$$

Integrating by parts the first term gives

$$-\int_\Omega v^2 \Big|_0^T + \frac{1}{2} \|v\|_Q^2 + \epsilon \|\nabla v\|_Q^2 \leq \frac{1}{2} \|u\|_Q^2$$

Using boundary condition  $v = 0$  at  $t = T$  gives

$$\frac{1}{2} \|v\|_Q^2 + \epsilon \|\nabla v\|_Q^2 \leq \int_\Omega v(t=0)^2 + \frac{1}{2} \|v\|_Q^2 + \epsilon \|\nabla v\|_Q^2 \leq \frac{1}{2} \|u\|_Q^2.$$

2. Multiply by  $-\frac{\partial v}{\partial t}$  and integrate over  $Q$ . Young's inequality changes the right hand side to

$$\int_Q \frac{\partial v^2}{\partial t} - \int_Q v \frac{\partial v}{\partial t} + \epsilon \int_Q \Delta v \frac{\partial v}{\partial t} = \int_Q -u \frac{\partial v}{\partial t} \leq \frac{1}{2} \|u\|_Q^2 + \frac{1}{2} \left\| \frac{\partial v}{\partial t} \right\|_Q^2.$$

The term  $\int_Q v \frac{\partial v}{\partial t}$  can be reduced to the positive contribution  $\int_\Omega v(t=0)^2$  as above. We can then take the Laplacian term, integrate by parts in space to get

$$\int_Q \Delta v \frac{\partial v}{\partial t} = \int_0^T \int_\Omega \Delta v \frac{\partial v}{\partial t} = \int_0^T \int_\Gamma \frac{\partial v}{\partial t} \frac{\partial v}{\partial n} - \int_0^T \int_\Omega \nabla \left( \frac{\partial v}{\partial t} \right) \nabla v.$$

Since either  $v = 0$  or  $\frac{\partial v}{\partial n} = 0$  on  $\Gamma$ , the first term disappears. The second term can be bounded by noting

$$- \int_0^T \int_\Omega \nabla \left( \frac{\partial v}{\partial t} \right) \nabla v = - \int_0^T \frac{\partial}{\partial t} \int_\Omega |\nabla v|^2 = - \int_\Omega |\nabla v|^2 \Big|_0^T.$$

Since  $v = 0$  at  $t = T$ ,  $\nabla v = 0$  at  $t = T$  as well, and we are left with the positive contribution  $\int_\Omega |\nabla v(t=0)|^2$ . Then,

$$\frac{1}{2} \left\| \frac{\partial v}{\partial t} \right\|_Q^2 \leq \frac{1}{2} \|u\|_Q.$$

Together, these two show that, under test norm

$$\|v\|_V^2 = \left\| \frac{\partial v}{\partial t} \right\|^2 + \|v\|^2 + \epsilon \|\nabla v\|^2,$$

the adjoint equation  $v^*$  satisfies

$$\|v^*\|_V \lesssim \|u\|_{L^2(\Omega)}$$

and thus the DPG energy norm robustly bounds the  $L^2$  norm from above

$$\|u\|_{L^2(\Omega)} \lesssim \|u\|_E.$$

## 2 Convection-diffusion

Consider convection-diffusion

$$\begin{aligned} \frac{\partial u}{\partial t} + \beta \cdot \nabla u - \epsilon \Delta u &= f \\ u &= 0 \text{ on } \Gamma_{out} \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_{in} \\ u(t=0) &= u_0. \end{aligned}$$

Let  $\tilde{\beta} := \begin{pmatrix} \beta \\ 1 \end{pmatrix}$  and  $\nabla_{xt} := \begin{pmatrix} \nabla \\ \frac{\partial}{\partial t} \end{pmatrix}$ , then we can rewrite this as

$$\begin{aligned} \tilde{\beta} \cdot \nabla_{xt} u - \epsilon \Delta u &= f \\ u &= 0 \text{ on } \Gamma_{out} \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_{in} \\ u(t=0) &= u_0. \end{aligned}$$

The adjoint equation satisfies

$$\begin{aligned} -\tilde{\beta} \cdot \nabla_{xt} v - \epsilon \Delta v &= u \\ v &= 0 \text{ on } \Gamma_{in} \\ \frac{\partial v}{\partial n} &= 0 \text{ on } \Gamma_{out} \\ v(t=T) &= 0. \end{aligned}$$

(The boundary conditions can be derived by taking the ultra-weak formulation and choosing boundary conditions such that the temporal flux and spatial flux terms  $\langle \hat{u}, \llbracket \tau_n \rrbracket \rangle_{\Gamma_{out}}$  and  $\langle \hat{f}_n, \llbracket v \rrbracket \rangle_{\Gamma_{in}}$  are zero.) The  $t=0$  and  $t=T$  boundaries can be considered as an inflow and outflow boundary respectively in space-time and we denote  $\partial Q_{in} := \Gamma_{in} \cup t=0$  while  $\partial Q_{out} := \Gamma_{out} \cup t=T$ .

We can then derive that the test norm

$$\|v\|_V^2 = \left\| \tilde{\beta} \cdot \nabla_{xt} v \right\|^2 + \epsilon \|\nabla v\|^2$$

provides the necessary bound  $\|v^*\|_V \lesssim \|u\|_{L^2(Q)}$ .

To see this, we multiply the adjoint equation by two terms as follows:

1. Multiply by  $-\tilde{\beta} \cdot \nabla_{xt} v$  and integrate over  $Q$  to get

$$\left\| \tilde{\beta} \cdot \nabla_{xt} v \right\| = - \int_Q u \tilde{\beta} \cdot \nabla_{xt} v - \epsilon \int_Q \tilde{\beta} \cdot \nabla_{xt} v \Delta v. \quad (1)$$

Note that

$$\begin{aligned}
-\int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \Delta v &= -\int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla \cdot \nabla v \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \int_Q \nabla(\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v) \cdot \nabla v \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \int_Q (\nabla \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v) \cdot \nabla v \\
&\quad + \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla \nabla_{xt} v \cdot \nabla v \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \int_Q (\nabla \boldsymbol{\beta} \cdot \nabla v) \cdot \nabla v \\
&\quad + \frac{1}{2} \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} (\nabla v \cdot \nabla v) \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \int_Q (\nabla \boldsymbol{\beta} \cdot \nabla v) \cdot \nabla v \\
&\quad + \frac{1}{2} \int_{\Gamma} \tilde{\boldsymbol{\beta}} \cdot \mathbf{n} (\nabla v \cdot \nabla v) - \frac{1}{2} \int_Q \nabla_{xt} \cdot \tilde{\boldsymbol{\beta}} (\nabla v \cdot \nabla v) \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \int_Q (\nabla \boldsymbol{\beta} \cdot \nabla v) \cdot \nabla v \\
&\quad + \frac{1}{2} \int_{\Gamma} \tilde{\boldsymbol{\beta}} \cdot \mathbf{n} (\nabla v \cdot \nabla v) - \frac{1}{2} \int_Q \nabla \cdot \boldsymbol{\beta} (\nabla v \cdot \nabla v) \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \frac{1}{2} \int_{\Gamma} \tilde{\boldsymbol{\beta}} \cdot \mathbf{n} (\nabla v \cdot \nabla v) \\
&\quad + \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \mathbf{I}) \nabla v
\end{aligned}$$

Plugging this into (1), we get

$$\begin{aligned}
\|\tilde{\beta} \cdot \nabla_{xt} v\| &= - \int_Q u \tilde{\beta} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \\
&\quad - \epsilon \int_{\Gamma_x} \tilde{\beta} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \epsilon \frac{1}{2} \int_{\Gamma} \tilde{\beta} \cdot \mathbf{n} (\nabla v \cdot \nabla v) \\
&= - \int_Q u \tilde{\beta} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \\
&\quad - \int_{\Gamma_-} \tilde{\beta} \cdot \nabla_{xt} v \underbrace{\nabla v \cdot \mathbf{n}_x}_{=0} - \int_{\Gamma_+} \left( \underbrace{\frac{\partial v}{\partial t}}_{=0} + \beta \cdot \nabla v \right) \nabla v \cdot \mathbf{n}_x \\
&\quad + \frac{1}{2} \int_{\Gamma_-} \underbrace{\beta \cdot \mathbf{n}_x}_{<0} (\nabla v \cdot \nabla v) + \frac{1}{2} \int_{\Gamma_+} \beta \cdot \mathbf{n}_x (\nabla v \cdot \nabla v) \\
&\quad + \frac{1}{2} \int_{\Gamma_0} \underbrace{n_t}_{<0} (\nabla v \cdot \nabla v) + \frac{1}{2} \int_{\Gamma_T} n_t \underbrace{(\nabla v \cdot \nabla v)}_{=0} \\
&\leq - \int_Q u \tilde{\beta} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \\
&\quad + \int_{\Gamma_+} \left( - \frac{\partial v}{\partial \mathbf{n}_x} \beta + \frac{1}{2} \beta \cdot \mathbf{n}_x \nabla v \right) \cdot \nabla v \\
&= - \int_Q u \tilde{\beta} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \\
&\quad + \int_{\Gamma_+} \left( - \frac{\partial v}{\partial \mathbf{n}_x} \beta + \frac{1}{2} \beta \cdot \mathbf{n}_x \frac{\partial v}{\partial \mathbf{n}_x} \mathbf{n}_x \right) \cdot \frac{\partial v}{\partial \mathbf{n}_x} \mathbf{n}_x \\
&= - \int_Q u \tilde{\beta} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \\
&\quad - \underbrace{\frac{1}{2} \int_{\Gamma_+} \left( \frac{\partial v}{\partial \mathbf{n}_x} \right)^2 \beta \cdot \mathbf{n}_x}_{<0} \\
&\leq - \int_Q u \tilde{\beta} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \\
&\leq - \frac{\|u\|}{2} + \frac{\|\tilde{\beta} \cdot \nabla_{xt} v\|}{2} + \epsilon \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \\
&\leq - \frac{\|u\|}{2} + \frac{\|\tilde{\beta} \cdot \nabla_{xt} v\|}{2} + \epsilon C \|\nabla v\|^2
\end{aligned}$$

2. Define  $w = e^{T-t} v$  and note that  $\frac{\partial w}{\partial t} = (\frac{\partial v}{\partial t} - v) e^{T-t}$  while  $\nabla w = \nabla e^{T-t} v + e^{T-t} \nabla v$  and  $\nabla \cdot (\beta w) = \nabla \cdot (\beta) e^{T-t} v + \beta \cdot e^{T-t} \nabla v$  and  $\Delta w = e^{T-t} \Delta v$ .

Also,  $\nabla_{xt}w = \frac{\partial e^{T-t}v}{\partial t} + \nabla e^{T-t}v = e^{T-t}(\nabla_{xt}v - v)$ . Plugging this into the adjoint equation, we get

$$-\tilde{\beta} \cdot \nabla_{xt}(w) - \epsilon \Delta w = u - \epsilon \nabla \cdot \sigma$$

or

$$\tilde{\beta} \cdot \nabla_{xt}(v) - v + \epsilon \Delta v = e^{t-T}(-u + \epsilon \nabla \cdot \sigma)$$

Multiply by  $-v$  and integrate to get

$$\int_Q -\tilde{\beta} \cdot \nabla_{xt}vv + v^2 - \epsilon \Delta vv = \int_Q e^{t-T}uv - \epsilon \int_Q e^{t-T} \nabla \cdot \sigma v$$

Then

$$\begin{aligned} \|v\|^2 &= \int_Q e^{t-T}uv - \epsilon \int_Q e^{t-T} \nabla \cdot \sigma v + \int_Q \tilde{\beta} \cdot \nabla_{xt}vv + \epsilon \int_Q \Delta vv \\ &= \int_Q e^{t-T}uv - \epsilon \int_Q e^{t-T} \nabla \cdot \sigma v + \frac{1}{2} \int_Q \tilde{\beta} \cdot \nabla_{xt}(v)^2 - \epsilon \int_Q \nabla v \nabla v + \epsilon \int_{\Gamma} v \nabla v \cdot \mathbf{n} \end{aligned}$$

Or

$$\begin{aligned} \|v\|^2 + \epsilon \|\nabla v\|^2 &= \int_Q e^{t-T}uv - \epsilon \int_Q e^{t-T} \nabla \cdot \sigma v \\ &\quad - \frac{1}{2} \int_Q \underbrace{\nabla_{xt} \cdot \tilde{\beta}(v)}_{=0}^2 + \frac{1}{2} \int_{\Gamma} v^2 \tilde{\beta} \cdot \mathbf{n} + \epsilon \int_{\Gamma_x} v \nabla v \cdot \mathbf{n}_x \\ &= \int_Q e^{t-T}uv + \epsilon \int_Q e^{t-T} \sigma \cdot \nabla v - \epsilon \int_{\Gamma_-} v \underbrace{\sigma \cdot \mathbf{n}_x}_{=\epsilon \frac{\partial v}{\partial n}=0} - \epsilon \int_{\Gamma_+} \underbrace{v}_{=0} \sigma \cdot \mathbf{n}_x \\ &\quad + \frac{1}{2} \int_{\Gamma_-} v^2 \underbrace{\beta \cdot \mathbf{n}_x}_{<0} + \frac{1}{2} \int_{\Gamma_+} \underbrace{v^2}_{=0} \beta \cdot \mathbf{n}_x \\ &\quad + \frac{1}{2} \int_{\Gamma_0} \underbrace{v^2(-n_t)}_{<0} + \frac{1}{2} \int_{\Gamma_T} \underbrace{v^2}_{=0} n_t \\ &\quad + \epsilon \int_{\Gamma_-} v \underbrace{\nabla v \cdot \mathbf{n}_x}_{=0} + \epsilon \int_{\Gamma_+} \underbrace{v}_{=0} \nabla v \cdot \mathbf{n}_x \\ &\leq \|e^{t-T}\|_{L^\infty(Q)} \left( \int_Q uv + \epsilon \int_Q \sigma \cdot \nabla v \right) \\ &\leq \left( \frac{\|u\|^2}{2} + \frac{\epsilon \|\sigma\|^2}{2} + \frac{\|v\|^2}{2} + \frac{\epsilon \|\nabla v\|^2}{2} \right) \end{aligned}$$

### 3 Robustness for transient problems given spatial robustness

Suppose we have the transient problem

$$\frac{\partial u}{\partial t} + Au = f$$

with initial condition  $u(x, 0) = u_0$ . Suppose that DPG is robust under the ultra-weak variational formulation for the steady problem

$$(u, A_h^* v)_{L^2(\Omega)} + \langle \hat{u}, \llbracket v \rrbracket \rangle_{\Gamma_h \setminus \Gamma_0} = (f, v)$$

with test norm  $\|v\|_V$ . Then, can we show that

$$\|v\|_{V,t} := \|v\|_V + \left\| \frac{\partial v}{\partial t} \right\|_{L^2(\Omega)}$$

also leads to a robust upper bound of the  $L^2$  norm by the DPG energy norm? I believe this may be possible. The adjoint equation for robustness for the transient problem gives

$$-\frac{\partial v}{\partial t} + A^* v = u$$

with  $v = 0$  at  $t = T$ ...