Notes on Steady Incompressible Navier-Stokes

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Non-Conservative Form

The steady incompressible Navier-Stokes equations are:

$$
\begin{aligned}
\nabla \boldsymbol{u} \cdot \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p &= \mathbf{f} \\
\nabla \cdot \boldsymbol{u} &= 0 \\
\int_\Omega p &= 0
\end{aligned}
$$

In 2D we get a system

$$
\begin{aligned}
\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \nabla u_1 - \nu \Delta u_1 + \frac{\partial p}{\partial x} &= f_1 \\
\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \nabla u_2 - \nu \Delta u_2 + \frac{\partial p}{\partial y} &= f_2 \\
\nabla \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= 0 \\
\int_\Omega p &= 0
\end{aligned}
$$

As a first order system, this is

$$
\begin{aligned}
\boldsymbol{\sigma}_1 - \nabla u_1 &= 0\\
\boldsymbol{\sigma}_2 - \nabla u_2 &= 0\\
\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \boldsymbol{\sigma}_1 - \nu \nabla \cdot \boldsymbol{\sigma}_1 + \frac{\partial p}{\partial x} &= f_1\\
\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \boldsymbol{\sigma}_2 - \nu \nabla \cdot \boldsymbol{\sigma}_2 + \frac{\partial p}{\partial y} &= f_2\\
\nabla \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= 0\\
\int_\Omega p &= 0
\end{aligned}
$$

Multiplying by $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, v_1, v_2, v_d, and integrating by parts:

$$
\begin{aligned}
(\boldsymbol{\sigma}_1, \boldsymbol{\tau}_1) + (u_1, \nabla \cdot \boldsymbol{\tau}_1) - \langle \hat{u}_1, \tau_{1n} \rangle &= 0\\
(\boldsymbol{\sigma}_2, \boldsymbol{\tau}_2) + (u_2, \nabla \cdot \boldsymbol{\tau}_2) - \langle \hat{u}_2, \tau_{2n} \rangle &= 0\\
\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \boldsymbol{\sigma}_1, v_1 \right) + (\nu \boldsymbol{\sigma}_1, \nabla v_1) - \left(p, \frac{\partial v_1}{\partial x} \right) + \left\langle \hat{t}, v_1 \right\rangle &= (f_1, v_1)\\
\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \boldsymbol{\sigma}_2, v_2 \right) + (\nu \boldsymbol{\sigma}_2, \nabla v_2) - \left(p, \frac{\partial v_2}{\partial y} \right) + \left\langle \hat{t}, v_2 \right\rangle &= (f_2, v_2)\\
- \left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \nabla v_d \right) + \left\langle \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix}, v_d \cdot \mathbf{n} \right\rangle &= 0\\
\int_\Omega p &= 0
\end{aligned}
$$

where $\hat{t} := (\widehat{-\boldsymbol{\sigma} + p}\mathbf{I})\mathbf{n}$.

Note that if $\boldsymbol{v} := \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, then

$$
-\left(p, \frac{\partial v_1}{\partial x} \right) + \langle p, v_1 \cdot n_x \rangle - \left(p, \frac{\partial v_2}{\partial y} \right) + \langle p, v_2 \cdot n_y \rangle = -\left(p, \nabla \cdot \boldsymbol{v} \right) + \langle p, v_n \rangle
$$

Linearizing:

$$
\begin{aligned}
(\Delta\boldsymbol{\sigma}_1, \boldsymbol{\tau}_1) + (\Delta u_1, \nabla\cdot\boldsymbol{\tau}_1) - \langle \hat{u}_1, \tau_{1n}\rangle &= -(\boldsymbol{\sigma}_1, \boldsymbol{\tau}_1) - (u_1, \nabla\cdot\boldsymbol{\tau}_1)\\
(\Delta\boldsymbol{\sigma}_2, \boldsymbol{\tau}_2) + (u_2, \nabla\cdot\boldsymbol{\tau}_2) - \langle \hat{u}_2, \tau_{2n}\rangle &= -(\boldsymbol{\sigma}_2, \boldsymbol{\tau}_2) - (u_2, \nabla\cdot\boldsymbol{\tau}_2)\\
\left(\begin{pmatrix}\Delta u_1\\ \Delta u_2\end{pmatrix}\cdot\boldsymbol{\sigma}_1, v_1\right) + \left(\begin{pmatrix}u_1\\ u_2\end{pmatrix}\cdot\Delta\boldsymbol{\sigma}_1, v_1\right) + (\nu\Delta\boldsymbol{\sigma}_1, \nabla v_1) - \left(p, \frac{\partial v_1}{\partial x}\right) + \langle\hat{t}, v_1\rangle&\\
= (f_1, v_1) - \left(\begin{pmatrix}u_1\\ u_2\end{pmatrix}\cdot\boldsymbol{\sigma}_1, v_1\right) - (\nu\boldsymbol{\sigma}_1, \nabla v_1)&\\
\left(\begin{pmatrix}\Delta u_1\\ \Delta u_2\end{pmatrix}\cdot\boldsymbol{\sigma}_2, v_2\right) + \left(\begin{pmatrix}u_1\\ u_2\end{pmatrix}\cdot\Delta\boldsymbol{\sigma}_2, v_2\right) + (\nu\Delta\boldsymbol{\sigma}_2, \nabla v_2) - \left(p, \frac{\partial v_2}{\partial y}\right) + \langle\hat{t}, v_2\rangle&\\
= (f_2, v_2) - \left(\begin{pmatrix}u_1\\ u_2\end{pmatrix}\cdot\boldsymbol{\sigma}_2, v_2\right) - (\nu\boldsymbol{\sigma}_2, \nabla v_2)&\\
-\left(\begin{pmatrix}\Delta u_1\\ \Delta u_2\end{pmatrix}, \nabla v_d\right) + \left\langle\begin{pmatrix}\hat{u}_1\\ \hat{u}_2\end{pmatrix}, v_d\cdot\mathbf{n}\right\rangle &= \left(\begin{pmatrix}u_1\\ u_2\end{pmatrix}, \nabla v_d\right)\\
\int_\Omega p &= 0
\end{aligned}
$$

Conservative Form

$$
\begin{aligned}
\nabla\cdot\left(\frac{1}{2}\boldsymbol{u}\otimes\boldsymbol{u} - \nu\nabla\boldsymbol{u} + p\boldsymbol{I}\right) &= \begin{pmatrix}f_1\\ f_2\end{pmatrix}\\
\nabla\cdot\boldsymbol{u} &= 0\\
\int_\Omega p &= 0
\end{aligned}
$$

As a first order system, this is

$$
\begin{aligned}
\boldsymbol{\sigma}_1 - \nabla u_1 &= 0\\
\boldsymbol{\sigma}_2 - \nabla u_2 &= 0\\
\nabla\cdot\left(\frac{1}{2}\boldsymbol{u}\otimes\boldsymbol{u} - \nu\begin{pmatrix}\boldsymbol{\sigma}_1\\ \boldsymbol{\sigma}_2\end{pmatrix} + p\boldsymbol{I}\right) &= \begin{pmatrix}f_1\\ f_2\end{pmatrix}\\
\nabla\cdot\boldsymbol{u} &= 0\\
\int_\Omega p &= 0
\end{aligned}
$$

Multiplying by $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, $\boldsymbol{u}$, and q, and integrating by parts

$$
\begin{aligned}
(\boldsymbol{\sigma}_1, \boldsymbol{\tau}_1) + (u_1, \nabla \cdot \boldsymbol{\tau}_1) - \langle \hat{u}_1, \tau_{1n} \rangle &= 0 \\
(\boldsymbol{\sigma}_2, \boldsymbol{\tau}_2) + (u_2, \nabla \cdot \boldsymbol{\tau}_2) - \langle \hat{u}_2, \tau_{2n} \rangle &= 0 \\
-\left(\frac{1}{2}\boldsymbol{u} \otimes \boldsymbol{u} - \nu \begin{pmatrix} \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2 \end{pmatrix} + p\boldsymbol{I}, \nabla \boldsymbol{v} \right) + \left\langle \hat{\boldsymbol{f}}, \boldsymbol{v} \right\rangle &= \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \\
-(\boldsymbol{u}, \nabla q) + \langle \widehat{\boldsymbol{u} \cdot \boldsymbol{n}}, q \rangle &= 0 \\
\int_\Omega p &= 0
\end{aligned}
$$

where

$$
\hat{\boldsymbol{f}} = \operatorname{tr}\left(\left(\frac{1}{2}\boldsymbol{u} \otimes \boldsymbol{u} - \nu \begin{pmatrix} \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2 \end{pmatrix} + p\boldsymbol{I} \right) \cdot \boldsymbol{n} \right)
$$

Linearizing:

$$
\begin{aligned}
(\boldsymbol{\sigma}_1, \boldsymbol{\tau}_1) + (\Delta u_1, \nabla \cdot \boldsymbol{\tau}_1) - \langle \hat{u}_1, \tau_{1n} \rangle &= -(u_1, \nabla \cdot \boldsymbol{\tau}_1) \\
(\boldsymbol{\sigma}_2, \boldsymbol{\tau}_2) + (\Delta u_2, \nabla \cdot \boldsymbol{\tau}_2) - \langle \hat{u}_2, \tau_{2n} \rangle &= -(u_2, \nabla \cdot \boldsymbol{\tau}_2) \\
-\left(\frac{1}{2}\Delta\boldsymbol{u} \otimes \boldsymbol{u} + \frac{1}{2}\boldsymbol{u} \otimes \Delta\boldsymbol{u} - \nu \begin{pmatrix} \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2 \end{pmatrix} + p\boldsymbol{I}, \nabla \boldsymbol{v} \right) + \left\langle \hat{\boldsymbol{f}}, \boldsymbol{v} \right\rangle &= \frac{1}{2}(\boldsymbol{u} \otimes \boldsymbol{u}, \nabla \boldsymbol{v}) \\
-(\Delta\boldsymbol{u}, \nabla q) + \langle \widehat{\boldsymbol{u} \cdot \boldsymbol{n}}, q \rangle &= (\boldsymbol{u}, \nabla q) \\
\int_\Omega p &= 0
\end{aligned}
$$

Grouping things together, we get

$$
\begin{aligned}
&(\boldsymbol{\sigma}_1, \boldsymbol{\tau}_1 + \nu \nabla v_1) \\
&(\boldsymbol{\sigma}_2, \boldsymbol{\tau}_2 + \nu \nabla v_2) \\
\left(\Delta u_1, \nabla \cdot \boldsymbol{\tau}_1 - \frac{1}{2}\left(\boldsymbol{u} \cdot \nabla v_1 + \boldsymbol{u} \cdot \frac{\partial \boldsymbol{v}}{\partial x} \right) - \frac{\partial q}{\partial x} \right)& \\
\left(\Delta u_2, \nabla \cdot \boldsymbol{\tau}_2 - \frac{1}{2}\left(\boldsymbol{u} \cdot \nabla v_2 + \boldsymbol{u} \cdot \frac{\partial \boldsymbol{v}}{\partial y} \right) - \frac{\partial q}{\partial y} \right)& \\
&(p, -\nabla \cdot \boldsymbol{v})
\end{aligned}
$$