

Space-Time Navier-Stokes with Entropy Variables

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Nonlinear Forms

Primitive Variables

Consider the DPG Navier-Stokes derivation from previously with primitive variables:

$$\left(\frac{1}{\mu}\mathbb{D}, \mathbb{S}\right) + (2\mathbf{u}, \nabla \cdot \mathbb{S}) - \left(\frac{2}{3}\mathbf{u}, \nabla \operatorname{tr} \mathbb{S}\right) - \left\langle \frac{4}{3}\hat{\mathbf{u}}, \mathbb{S}\mathbf{n}_x \right\rangle = 0 \quad (1a)$$

$$\left(\frac{Pr}{C_p\mu}\mathbf{q}, \boldsymbol{\tau}\right) - (T, \nabla \cdot \boldsymbol{\tau}) + \left\langle \hat{T}, \tau_n \right\rangle = 0 \quad (1b)$$

$$- \left(\left(\begin{array}{c} \rho\mathbf{u} \\ \rho \end{array} \right), \nabla_{xt} v_c \right) + \langle \hat{t}_c, v_c \rangle = (f_c, v_c) \quad (1c)$$

$$- \left(\left(\begin{array}{c} \rho\mathbf{u} \otimes \mathbf{u} + \rho RT \mathbf{I} - \mathbb{D} \\ \rho\mathbf{u} \end{array} \right), \nabla_{xt} \mathbf{v}_m \right) + \langle \hat{\mathbf{t}}_m, \mathbf{v}_m \rangle = (\mathbf{f}_m, \mathbf{v}_m) \quad (1d)$$

$$- \left(\left(\begin{array}{c} \rho\mathbf{u} (C_v T + \frac{1}{2}\mathbf{u} \cdot \mathbf{u}) + \mathbf{u} \rho RT + \mathbf{q} - \mathbf{u} \cdot \mathbb{D} \\ \rho (C_v T + \frac{1}{2}\mathbf{u} \cdot \mathbf{u}) \end{array} \right), \nabla_{xt} v_e \right) + \langle \hat{t}_e, v_e \rangle = (f_e, v_e) , \quad (1e)$$

where

$$\begin{aligned} \hat{\mathbf{u}} &= \operatorname{tr}(\mathbf{u}) \\ \hat{T} &= \operatorname{tr}(T) \\ \hat{t}_c &= \operatorname{tr}(\rho\mathbf{u}) \cdot \mathbf{n}_x + \operatorname{tr}(\rho) n_t \\ \hat{\mathbf{t}}_m &= \operatorname{tr}(\rho\mathbf{u} \otimes \mathbf{u} + \rho RT \mathbf{I} - \mathbb{D}) \cdot \mathbf{n}_x + \operatorname{tr}(\rho\mathbf{u}) n_t \\ \hat{t}_e &= \operatorname{tr} \left(\rho\mathbf{u} \left(C_v T + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} \right) + \mathbf{u} \rho RT + \mathbf{q} - \mathbf{u} \cdot \mathbb{D} \right) \cdot \mathbf{n}_x \\ &\quad + \operatorname{tr} \left(\rho \left(C_v T + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} \right) \right) n_t . \end{aligned}$$

Now define primitive fluxes for continuity, momentum, and energy equations:

$$\begin{aligned} \mathbf{F}_c^p &:= \rho\mathbf{u} \\ \mathbb{F}_m^p &:= \rho\mathbf{u} \otimes \mathbf{u} + \rho RT \mathbf{I} \\ \mathbf{F}_e^p &:= \rho\mathbf{u} \left(C_v T + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} \right) + \mathbf{u} \rho RT \end{aligned}$$

Our bilinear form is then simplified:

$$\left(\frac{1}{\mu}\mathbb{D}, \mathbb{S}\right) + (2\mathbf{u}, \nabla \cdot \mathbb{S}) - \left(\frac{2}{3}\mathbf{u}, \nabla \operatorname{tr} \mathbb{S}\right) - \left\langle \frac{4}{3}\hat{\mathbf{u}}, \mathbb{S}\mathbf{n}_x \right\rangle = 0 \quad (2a)$$

$$\left(\frac{Pr}{C_p\mu}\mathbf{q}, \boldsymbol{\tau}\right) - (T, \nabla \cdot \boldsymbol{\tau}) + \langle \hat{T}, \tau_n \rangle = 0 \quad (2b)$$

$$- \left(\left(\frac{\mathbf{F}_c^p}{\rho} \right), \nabla_{xt} v_c \right) + \langle \hat{t}_c, v_c \rangle = (f_c, v_c) \quad (2c)$$

$$- \left(\left(\frac{\mathbb{F}_m^p - \mathbb{D}}{\rho\mathbf{u}} \right), \nabla_{xt} \mathbf{v}_m \right) + \langle \hat{\mathbf{t}}_m, \mathbf{v}_m \rangle = (\mathbf{f}_m, \mathbf{v}_m) \quad (2d)$$

$$- \left(\left(\frac{\mathbf{F}_e^p + \mathbf{q} - \mathbf{u} \cdot \mathbb{D}}{\rho(C_v T + \frac{1}{2}\mathbf{u} \cdot \mathbf{u})} \right), \nabla_{xt} v_e \right) + \langle \hat{t}_e, v_e \rangle = (f_e, v_e) , \quad (2e)$$

Conservation Variables

Now we wish to do a change of variables to conservation variables:

$$\begin{aligned} \rho &= \rho \\ \mathbf{m} &= \rho\mathbf{u} \\ E &= \rho \left(C_v T + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} \right) \end{aligned}$$

We can define new fluxes in conservation variables:

$$\begin{aligned} \mathbf{F}_c^c &= \mathbf{m} \\ \mathbb{F}_m^c &= \frac{\mathbf{m} \otimes \mathbf{m}}{\rho} + (\gamma - 1) \left(E - \frac{\mathbf{m} \cdot \mathbf{m}}{2\rho} \right) \mathbf{I} \\ \mathbf{F}_e^c &= \frac{\mathbf{m}}{\rho} E + (\gamma - 1) \left(E - \frac{\mathbf{m} \cdot \mathbf{m}}{2\rho} \right) \frac{\mathbf{m}}{\rho} \end{aligned}$$

and our new bilinear form is

$$\left(\frac{1}{\mu}\mathbb{D}, \mathbb{S}\right) + \left(2\frac{\mathbf{m}}{\rho}, \nabla \cdot \mathbb{S}\right) - \left(\frac{2}{3}\frac{\mathbf{m}}{\rho}, \nabla \operatorname{tr} \mathbb{S}\right) - \left\langle \frac{4}{3}\hat{\mathbf{u}}, \mathbb{S}\mathbf{n}_x \right\rangle = 0 \quad (3a)$$

$$\left(\frac{Pr}{C_p\mu}\mathbf{q}, \boldsymbol{\tau}\right) - \left(\frac{E - \frac{1}{2\rho}\mathbf{m} \cdot \mathbf{m}}{C_v\rho}, \nabla \cdot \boldsymbol{\tau}\right) + \langle \hat{T}, \tau_n \rangle = 0 \quad (3b)$$

$$- \left(\left(\frac{\mathbf{F}_c^c}{\rho} \right), \nabla_{xt} v_c \right) + \langle \hat{t}_c, v_c \rangle = (f_c, v_c) \quad (3c)$$

$$- \left(\left(\frac{\mathbb{F}_m^c - \mathbb{D}}{\rho\mathbf{m}} \right), \nabla_{xt} \mathbf{v}_m \right) + \langle \hat{\mathbf{t}}_m, \mathbf{v}_m \rangle = (\mathbf{f}_m, \mathbf{v}_m) \quad (3d)$$

$$- \left(\left(\frac{\mathbf{F}_e^c + \mathbf{q} - \frac{\mathbf{m}}{\rho} \cdot \mathbb{D}}{E} \right), \nabla_{xt} v_e \right) + \langle \hat{t}_e, v_e \rangle = (f_e, v_e) , \quad (3e)$$

Entropy Variables

Now we wish to do a change of variables to entropy variables:

$$\begin{aligned} V_c &= \frac{-E + (E - \frac{1}{2\rho} \mathbf{m} \cdot \mathbf{m}) \left(\gamma + 1 - \ln \left[\frac{(\gamma-1)(E - \frac{1}{2\rho} \mathbf{m} \cdot \mathbf{m})}{\rho^\gamma} \right] \right)}{E - \frac{1}{2\rho} \mathbf{m} \cdot \mathbf{m}} \\ \mathbf{V}_m &= \frac{\mathbf{m}}{E - \frac{1}{2\rho} \mathbf{m} \cdot \mathbf{m}} \\ V_e &= \frac{-\rho}{E - \frac{1}{2\rho} \mathbf{m} \cdot \mathbf{m}} \end{aligned}$$

with reverse mapping:

$$\begin{aligned} \rho &= -\alpha V_e \\ \mathbf{m} &= \alpha \mathbf{V}_m \\ E &= \alpha \left(1 - \frac{1}{2V_e} \mathbf{V}_m \cdot \mathbf{V}_m \right) \end{aligned}$$

where

$$\alpha(V_c, \mathbf{V}_m, V_e) = \left[\frac{\gamma - 1}{-(V_e)^\gamma} \right]^{\frac{1}{\gamma-1}} \exp \left[\frac{-\gamma + V_c - \frac{1}{2V_e} \mathbf{V}_m \cdot \mathbf{V}_m}{\gamma - 1} \right]$$

We can define new fluxes in entropy variables:

$$\begin{aligned} \mathbf{F}_c^e &= \alpha \mathbf{V}_m \\ \mathbb{F}_m^e &= -\alpha \frac{\mathbf{V}_m \otimes \mathbf{V}_m}{V_e} + \alpha(\gamma - 1) \mathbf{I} \\ \mathbf{F}_e^e &= -\alpha \frac{\mathbf{V}_m}{V_e} \left(1 - \frac{1}{2V_e} \mathbf{V}_m \cdot \mathbf{V}_m \right) - \alpha(\gamma - 1) \frac{\mathbf{V}_m}{V_e} \end{aligned}$$

and our new bilinear form is

$$\left(\frac{1}{\mu} \mathbb{D}, \mathbb{S} \right) - \left(2 \frac{\mathbf{V}_m}{V_e}, \nabla \cdot \mathbb{S} \right) + \left(\frac{2}{3} \frac{\mathbf{V}_m}{V_e}, \nabla \operatorname{tr} \mathbb{S} \right) - \left\langle \frac{4}{3} \hat{\mathbf{u}}, \mathbb{S} \mathbf{n}_x \right\rangle = 0 \quad (4a)$$

$$\left(\frac{Pr}{C_p \mu} \mathbf{q}, \boldsymbol{\tau} \right) + \left(\frac{1}{C_v V_e}, \nabla \cdot \boldsymbol{\tau} \right) + \langle \hat{T}, \tau_n \rangle = 0 \quad (4b)$$

$$- \left(\begin{pmatrix} \mathbf{F}_c^e \\ -\alpha V_e \end{pmatrix}, \nabla_{xt} v_c \right) + \langle \hat{t}_c, v_c \rangle = (f_c, v_c) \quad (4c)$$

$$- \left(\begin{pmatrix} \mathbb{F}_m^e - \mathbb{D} \\ \alpha \mathbf{V}_m \end{pmatrix}, \nabla_{xt} \mathbf{v}_m \right) + \langle \hat{\mathbf{t}}_m, \mathbf{v}_m \rangle = (\mathbf{f}_m, \mathbf{v}_m) \quad (4d)$$

$$- \left(\begin{pmatrix} \mathbf{F}_e^e + \mathbf{q} + \frac{\mathbf{V}_m}{V_e} \cdot \mathbb{D} \\ \alpha \left(1 - \frac{1}{2V_e} \mathbf{V}_m \cdot \mathbf{V}_m \right) \end{pmatrix}, \nabla_{xt} v_e \right) + \langle \hat{t}_e, v_e \rangle = (f_e, v_e), \quad (4e)$$

Linearization

For each change of variables, we maintain the same linear variables: $L := \{\mathbf{q}, \hat{\mathbf{u}}, \hat{e}, \hat{t}_c, \hat{\mathbf{t}}_m, \hat{t}_e\}$. Let U be the set of variables involved in nonlinear interactions. We apply a linearization $U \approx \tilde{U} + \Delta U$ and solve

$$R_U(\tilde{U})\Delta U + R(L) = -R(\tilde{U}),$$

where

$$\begin{aligned} R(L) = & \left(\frac{Pr}{C_p \mu} \mathbf{q}, \boldsymbol{\tau} \right) - (\mathbf{q}, \nabla v_e) - \left\langle \frac{4}{3} \hat{\mathbf{u}}, \mathbb{S} \mathbf{n}_x \right\rangle + \langle \hat{T}, \tau_n \rangle + \langle \hat{t}_c, v_c \rangle + \langle \hat{\mathbf{t}}_m, v_m \rangle + \langle \hat{t}_e, v_e \rangle \\ & - (f_c, v_c) - (\mathbf{f}_m, \mathbf{v}_m) - (f_e, v_e) \end{aligned}$$

Primitive Variables

The set of nonlinear variables is $U^p := \{\rho, \mathbf{u}, T, \mathbb{D}\}$. Then $R_{U^p}(\tilde{U}^p)\Delta U^p$ is

$$\begin{aligned} & \left(\frac{1}{\mu} \Delta \mathbb{D}, \mathbb{S} \right) + (2\Delta \mathbf{u}, \nabla \cdot \mathbb{S}) - \left(\frac{2}{3} \Delta \mathbf{u}, \nabla \text{tr } \mathbb{S} \right) \\ & \quad - (\Delta T, \nabla \cdot \boldsymbol{\tau}) \\ & \quad - \left(\left(\begin{array}{c} \mathbf{F}_{c,U^p}^p \Delta U^p \\ \Delta \rho \end{array} \right), \nabla_{xt} v_c \right) \\ & \quad - \left(\left(\begin{array}{c} \mathbb{F}_{m,U^p}^p \Delta U^p - \Delta \mathbb{D} \\ \Delta \rho \tilde{\mathbf{u}} + \tilde{\rho} \Delta \mathbf{u} \end{array} \right), \nabla_{xt} \mathbf{v}_m \right) \\ & - \left(\left(\begin{array}{c} \mathbf{F}_{e,U^p}^p \Delta U^p - \Delta \mathbf{u} \cdot \tilde{\mathbb{D}} - \tilde{\mathbf{u}} \cdot \Delta \mathbb{D} \\ C_v \Delta \rho \tilde{T} + C_v \tilde{\rho} \Delta T + \frac{1}{2} (\Delta \rho \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} + \tilde{\rho} \Delta \mathbf{u} \cdot \tilde{\mathbf{u}} + \tilde{\rho} \tilde{\mathbf{u}} \cdot \Delta \mathbf{u}) \end{array} \right), \nabla_{xt} v_e \right) \end{aligned}$$

where

$$\begin{aligned} \mathbf{F}_{c,U^p}^p \Delta U^p &:= \Delta \rho \tilde{\mathbf{u}} + \tilde{\rho} \Delta \mathbf{u} \\ \mathbb{F}_{m,U^p}^p &:= \Delta \rho \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} + \tilde{\rho} \Delta \mathbf{u} \otimes \tilde{\mathbf{u}} + \tilde{\rho} \tilde{\mathbf{u}} \otimes \Delta \mathbf{u} + R(\Delta \rho \tilde{T} + \tilde{\rho} \Delta T) \mathbf{I} \\ \mathbf{F}_{e,U^p}^p &:= C_v \Delta \rho \tilde{T} + C_v \tilde{\rho} \Delta \mathbf{u} \tilde{T} + C_v \tilde{\rho} \tilde{\mathbf{u}} \Delta T \\ & \quad + \frac{1}{2} \Delta \rho \tilde{\mathbf{u}} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} + \frac{1}{2} \tilde{\rho} \Delta \mathbf{u} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} + \frac{1}{2} \tilde{\rho} \tilde{\mathbf{u}} \Delta \mathbf{u} \cdot \tilde{\mathbf{u}} + \frac{1}{2} \tilde{\rho} \tilde{\mathbf{u}} \tilde{\mathbf{u}} \cdot \Delta \mathbf{u} \\ & \quad + R \Delta \mathbf{u} \tilde{\rho} \tilde{T} + R \tilde{\mathbf{u}} \Delta \rho \tilde{T} + R \tilde{\mathbf{u}} \tilde{\rho} \Delta T \end{aligned}$$

and $R(\tilde{U}^p)$ is

$$\begin{aligned} & \left(\frac{1}{\mu} \tilde{\mathbb{D}}, \mathbb{S} \right) + (2\tilde{\mathbf{u}}, \nabla \cdot \mathbb{S}) - \left(\frac{2}{3} \tilde{\mathbf{u}}, \nabla \text{tr } \mathbb{S} \right) \\ & \quad - (\tilde{T}, \nabla \cdot \boldsymbol{\tau}) \\ & \quad - \left(\left(\begin{array}{c} \mathbf{F}_c^p(\tilde{U}^p) \\ \tilde{\rho} \end{array} \right), \nabla_{xt} v_c \right) \\ & \quad - \left(\left(\begin{array}{c} \mathbb{F}_m^p(\tilde{U}^p) - \tilde{\mathbb{D}} \\ \tilde{\rho} \tilde{\mathbf{u}} \end{array} \right), \nabla_{xt} \mathbf{v}_m \right) \\ & \quad - \left(\left(\begin{array}{c} \mathbf{F}_e^p(\tilde{U}^p) - \tilde{\mathbf{u}} \cdot \tilde{\mathbb{D}} \\ \tilde{\rho} (C_v \tilde{T} + \frac{1}{2} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}}) \end{array} \right), \nabla_{xt} v_e \right) \end{aligned}$$

Conservation Variables

The set of nonlinear variables is $U^c := \{\rho, \mathbf{m}, E, \mathbb{D}\}$. Then $R_{U^c}(\tilde{U}^c)\Delta U^c$ is

$$\begin{aligned} & \left(\frac{1}{\mu} \Delta \mathbb{D}, \mathbb{S} \right) + \left(2 \left(\frac{\Delta \mathbf{m}}{\tilde{\rho}} - \frac{\tilde{\mathbf{m}}}{\tilde{\rho}^2} \Delta \rho \right), \nabla \cdot \mathbb{S} \right) - \left(\frac{2}{3} \left(\frac{\Delta \mathbf{m}}{\tilde{\rho}} - \frac{\tilde{\mathbf{m}}}{\tilde{\rho}^2} \Delta \rho \right), \nabla \operatorname{tr} \mathbb{S} \right) \\ & - \left(\frac{\Delta E - \frac{1}{2\tilde{\rho}} \Delta \mathbf{m} \cdot \tilde{\mathbf{m}} - \frac{1}{2\tilde{\rho}} \tilde{\mathbf{m}} \cdot \Delta \mathbf{m} + \frac{1}{2\tilde{\rho}^2} \tilde{\mathbf{m}} \cdot \tilde{\mathbf{m}} \Delta \rho}{C_v \tilde{\rho}} - \frac{\tilde{E} - \frac{1}{2\tilde{\rho}} \tilde{\mathbf{m}} \cdot \tilde{\mathbf{m}}}{C_v \tilde{\rho}^2} \Delta \rho, \nabla \cdot \boldsymbol{\tau} \right) \\ & - \left(\left(\begin{array}{c} \mathbf{F}_{c,U^c}^c \Delta U^c \\ \Delta \rho \end{array} \right), \nabla_{xt} v_c \right) \\ & - \left(\left(\begin{array}{c} \mathbb{F}_{m,U^c}^c \Delta U^c - \Delta \mathbb{D} \\ \Delta \mathbf{m} \end{array} \right), \nabla_{xt} \mathbf{v}_m \right) \\ & - \left(\left(\begin{array}{c} \mathbf{F}_{e,U^c}^c \Delta U^c - \frac{\Delta \mathbf{m}}{\tilde{\rho}} \cdot \tilde{\mathbb{D}} + \frac{\tilde{\mathbf{m}}}{\tilde{\rho}^2} \Delta \rho \cdot \tilde{\mathbb{D}} - \frac{\tilde{\mathbf{m}}}{\tilde{\rho}} \cdot \Delta \mathbb{D} \\ \Delta E \end{array} \right), \nabla_{xt} v_e \right) \end{aligned}$$

where

$$\begin{aligned} \mathbf{F}_{c,U^c}^c \Delta U^c &= \Delta \mathbf{m} \\ \mathbb{F}_{m,U^c}^c \Delta U^c &= \frac{\Delta \mathbf{m} \otimes \tilde{\mathbf{m}}}{\tilde{\rho}} + \frac{\tilde{\mathbf{m}} \otimes \Delta \mathbf{m}}{\tilde{\rho}} - \frac{\tilde{\mathbf{m}} \otimes \tilde{\mathbf{m}}}{\tilde{\rho}^2} \Delta \rho \\ &+ (\gamma - 1) \left(\Delta E - \frac{\Delta \mathbf{m} \cdot \tilde{\mathbf{m}}}{2\tilde{\rho}} - \frac{\tilde{\mathbf{m}} \cdot \Delta \mathbf{m}}{2\tilde{\rho}} + \frac{\tilde{\mathbf{m}} \cdot \tilde{\mathbf{m}}}{2\tilde{\rho}^2} \Delta \rho \right) \mathbf{I} \\ \mathbf{F}_{e,U^c}^c \Delta U^c &= \frac{\Delta \mathbf{m}}{\tilde{\rho}} \tilde{E} + \frac{\tilde{\mathbf{m}}}{\tilde{\rho}} \Delta E - \frac{\tilde{\mathbf{m}}}{\tilde{\rho}^2} \tilde{E} \Delta \rho \\ &+ (\gamma - 1) \left(\Delta E \frac{\tilde{\mathbf{m}}}{\tilde{\rho}} + \tilde{E} \frac{\Delta \mathbf{m}}{\tilde{\rho}} - \tilde{E} \frac{\tilde{\mathbf{m}}}{\tilde{\rho}^2} \Delta \rho \right) \\ &+ (\gamma - 1) \left(-\frac{\Delta \mathbf{m} \tilde{\mathbf{m}} \cdot \tilde{\mathbf{m}}}{2\tilde{\rho}^2} - \frac{\tilde{\mathbf{m}} \Delta \mathbf{m} \cdot \tilde{\mathbf{m}}}{2\tilde{\rho}^2} - \frac{\tilde{\mathbf{m}} \tilde{\mathbf{m}} \cdot \Delta \mathbf{m}}{2\tilde{\rho}^2} + \frac{\tilde{\mathbf{m}} \tilde{\mathbf{m}} \cdot \tilde{\mathbf{m}}}{\tilde{\rho}^3} \Delta \rho \right) \end{aligned}$$

and $R(\tilde{U}^p)$ is

$$\begin{aligned} & \left(\frac{1}{\mu} \Delta \mathbb{D}, \mathbb{S} \right) + \left(2 \frac{\tilde{\mathbf{m}}}{\tilde{\rho}}, \nabla \cdot \mathbb{S} \right) - \left(\frac{2}{3} \frac{\tilde{\mathbf{m}}}{\tilde{\rho}}, \nabla \operatorname{tr} \mathbb{S} \right) \\ & - \left(\frac{\tilde{E} - \frac{1}{2\tilde{\rho}} \tilde{\mathbf{m}} \cdot \tilde{\mathbf{m}}}{C_v \tilde{\rho}}, \nabla \cdot \boldsymbol{\tau} \right) \\ & - \left(\left(\begin{array}{c} \mathbf{F}_c^c \\ \tilde{\rho} \end{array} \right), \nabla_{xt} v_c \right) \\ & - \left(\left(\begin{array}{c} \mathbb{F}_m^c - \tilde{\mathbb{D}} \\ \tilde{\mathbf{m}} \end{array} \right), \nabla_{xt} \mathbf{v}_m \right) \\ & - \left(\left(\begin{array}{c} \mathbf{F}_e^c - \frac{\tilde{\mathbf{m}}}{\tilde{\rho}} \cdot \tilde{\mathbb{D}} \\ \tilde{E} \end{array} \right), \nabla_{xt} v_e \right) \end{aligned}$$