Derivation of Space-Time DPG for the Euler Equations in Conservation Form

Truman E. Ellis

September 23, 2013

The 1D Euler equations in conservation form are

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ m\\ E\end{bmatrix}+\frac{\partial}{\partial x}\begin{bmatrix}m\\ m^2/\rho+p\\ (E+p)m/\rho\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

If we define

$$\nabla := \begin{bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial t}\end{bmatrix}$$

then we can rewrite the 1D Euler equations as a system.

$$\nabla\cdot\begin{bmatrix}m\\ \rho\end{bmatrix}=0$$

$$\nabla\cdot\begin{bmatrix}m^2/\rho+p\\ m\end{bmatrix}=0$$

$$\nabla\cdot\begin{bmatrix}(E+p)m/\rho\\ E\end{bmatrix}=0$$

Multiplying each equation by v_m (for mass), v_x (for x-momentum), and v_e (for energy), and integrating by parts over spacetime Q:

$$\int_{\partial Q} v_m\begin{bmatrix}m\\ \rho\end{bmatrix}\cdot\mathbf{n}-\int_Q\begin{bmatrix}m\\ \rho\end{bmatrix}\cdot\nabla v_m=0$$

$$\int_{\partial Q} v_x\begin{bmatrix}m^2/\rho+p\\ m\end{bmatrix}\cdot\mathbf{n}-\int_Q\begin{bmatrix}m^2/\rho+p\\ m\end{bmatrix}\cdot\nabla v_x=0$$

$$\int_{\partial Q} v_e\begin{bmatrix}(E+p)m/\rho\\ E\end{bmatrix}\cdot\mathbf{n}-\int_Q\begin{bmatrix}(E+p)m/\rho\\ E\end{bmatrix}\cdot\nabla v_e=0$$

Now identify the fluxes

$$\hat{F}_m := \begin{bmatrix}m\\ \rho\end{bmatrix}\cdot\mathbf{n}$$

$$\hat{F}_x := \begin{bmatrix}m^2/\rho+p\\ m\end{bmatrix}\cdot\mathbf{n}$$

$$\hat{F}_e := \begin{bmatrix}(E+p)m/\rho\\ E\end{bmatrix}\cdot\mathbf{n}$$

Assuming an ideal gas equation of state, linearize the volume terms.

$$F_m(\mathbf{U}) := \begin{bmatrix} m \\ \rho \end{bmatrix} \approx \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\rho \\ \Delta m \\ \Delta E \end{bmatrix}$$

$$F_x(\mathbf{U}) := \begin{bmatrix} \frac{3-\gamma}{2}\frac{m^2}{\rho} + (\gamma-1)E \\ m \end{bmatrix} \approx \begin{bmatrix} \frac{\gamma-3}{2}\frac{m^2}{\rho^2} & (3-\gamma)\frac{m}{\rho} & (\gamma-1) \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta\rho \\ \Delta m \\ \Delta E \end{bmatrix}$$

$$F_e(\mathbf{U}) := \begin{bmatrix} \frac{1-\gamma}{2}\frac{m^3}{\rho^2} + \gamma\frac{m}{\rho}E \\ E \end{bmatrix} \approx \begin{bmatrix} (\gamma-1)\frac{m^3}{\rho^3} - \gamma\frac{m}{\rho^2}E & \frac{3(1-\gamma)}{2}\frac{m^2}{\rho^2} + \gamma\frac{E}{\rho} & \gamma\frac{m}{\rho} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\rho \\ \Delta m \\ \Delta E \end{bmatrix}$$

Finally, our space-time DPG formulation of the Euler equations is as follows.

Given background flow quantities ρ, u, $e \in L^2(Q)$, find $\Delta\rho$, Δu, $\Delta e \in L^2(Q)$, and $\hat{F}_m$, $\hat{F}_x$, $\hat{F}_e \in H^{-\frac{1}{2}}$ such that

$$\int_{\partial Q} \hat{F}_m v_m - \int_Q \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\rho \\ \Delta m \\ \Delta E \end{bmatrix} \cdot \nabla v_m = \int_Q \begin{bmatrix} m \\ \rho \end{bmatrix} \cdot \nabla v_m$$

$$\int_{\partial Q} \hat{F}_x v_x - \int_Q \begin{bmatrix} \frac{\gamma-3}{2}\frac{m^2}{\rho^2} & (3-\gamma)\frac{m}{\rho} & (\gamma-1) \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta\rho \\ \Delta m \\ \Delta E \end{bmatrix} \cdot \nabla v_x = \int_Q \begin{bmatrix} \frac{3-\gamma}{2}\frac{m^2}{\rho} + (\gamma-1)E \\ m \end{bmatrix} \cdot \nabla v_x$$

$$\int_{\partial Q} \hat{F}_e v_e - \int_Q \begin{bmatrix} (\gamma-1)\frac{m^3}{\rho^3} - \gamma\frac{m}{\rho^2}E & \frac{3(1-\gamma)}{2}\frac{m^2}{\rho^2} + \gamma\frac{E}{\rho} & \gamma\frac{m}{\rho} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta\rho \\ \Delta m \\ \Delta E \end{bmatrix} \cdot \nabla v_e = \int_Q \begin{bmatrix} \frac{1-\gamma}{2}\frac{m^3}{\rho^2} + \gamma\frac{m}{\rho}E \\ E \end{bmatrix} \cdot \nabla v_e$$

for all v_m, v_x, and v_e.

As a first cut, we will use the graph norm.