# A Unified Treatment of Primitive, Conservation, and Entropy Variable Formulations of Navier-Stokes with Discontinuous Petrov-Galerkin Finite Elements

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## Nonlinear Forms

### Primitive Variables

Consider the DPG Navier-Stokes derivation from previously with primitive variables:

$$\left(\frac{1}{\mu}\mathbb{D},\mathbb{S}\right)+(2\boldsymbol{u},\nabla\cdot\mathbb{S})-\left(\frac{2}{3}\boldsymbol{u},\nabla\operatorname{tr}\mathbb{S}\right)-\left\langle\frac{4}{3}\hat{\boldsymbol{u}},\mathbb{S}\boldsymbol{n}_x\right\rangle=0 \tag{1a}$$

$$\left(\frac{Pr}{C_p\mu}\boldsymbol{q},\boldsymbol{\tau}\right)-(T,\nabla\cdot\boldsymbol{\tau})+\left\langle\hat{T},\tau_n\right\rangle=0 \tag{1b}$$

$$-\left(\begin{pmatrix}\rho\boldsymbol{u}\\ \rho\end{pmatrix},\nabla_{xt}v_c\right)+\left\langle\hat{t}_c,v_c\right\rangle=(f_c,v_c) \tag{1c}$$

$$-\left(\begin{pmatrix}\rho\boldsymbol{u}\otimes\boldsymbol{u}+\rho RT\boldsymbol{I}-\mathbb{D}\\ \rho\boldsymbol{u}\end{pmatrix},\nabla_{xt}\boldsymbol{v}_m\right)+\left\langle\hat{\boldsymbol{t}}_m,\boldsymbol{v}_m\right\rangle=(\boldsymbol{f}_m,\boldsymbol{v}_m) \tag{1d}$$

$$-\left(\begin{pmatrix}\rho\boldsymbol{u}\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)+\boldsymbol{u}\rho RT+\boldsymbol{q}-\boldsymbol{u}\cdot\mathbb{D}\\ \rho\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\end{pmatrix},\nabla_{xt}v_e\right)+\left\langle\hat{t}_e,v_e\right\rangle=(f_e,v_e)\,, \tag{1e}$$

where

$$\begin{aligned}
\hat{\boldsymbol{u}}&=\operatorname{tr}(\boldsymbol{u})\\
\hat{T}&=\operatorname{tr}(T)\\
\hat{t}_c&=\operatorname{tr}(\rho\boldsymbol{u})\cdot\boldsymbol{n}_x+\operatorname{tr}(\rho)\,n_t\\
\hat{\boldsymbol{t}}_m&=\operatorname{tr}(\rho\boldsymbol{u}\otimes\boldsymbol{u}+\rho RT\boldsymbol{I}-\mathbb{D})\cdot\boldsymbol{n}_x+\operatorname{tr}(\rho\boldsymbol{u})\,n_t\\
\hat{t}_e&=\operatorname{tr}\left(\rho\boldsymbol{u}\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)+\boldsymbol{u}\rho RT+\boldsymbol{q}-\boldsymbol{u}\cdot\mathbb{D}\right)\cdot\boldsymbol{n}_x\\
&\quad+\operatorname{tr}\left(\rho\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\right)n_t\,.
\end{aligned}$$

Now define primitive fluxes for continuity, momentum, and energy equations:

$$\begin{aligned}
\boldsymbol{F}_c^p&:=\rho\boldsymbol{u}\\
\mathbb{F}_m^p&:=\rho\boldsymbol{u}\otimes\boldsymbol{u}+\rho RT\boldsymbol{I}\\
\boldsymbol{F}_e^p&:=\rho\boldsymbol{u}\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)+\boldsymbol{u}\rho RT
\end{aligned}$$

Our bilinear form is then simplified:

$$\left(\frac{1}{\mu}\mathbb{D},\mathbb{S}\right)+(2\boldsymbol{u},\nabla\cdot\mathbb{S})-\left(\frac{2}{3}\boldsymbol{u},\nabla\operatorname{tr}\mathbb{S}\right)-\left\langle\frac{4}{3}\hat{\boldsymbol{u}},\mathbb{S}\boldsymbol{n}_x\right\rangle=0 \tag{2a}$$

$$\left(\frac{Pr}{C_p\mu}\boldsymbol{q},\boldsymbol{\tau}\right)-(T,\nabla\cdot\boldsymbol{\tau})+\left\langle\hat{T},\tau_n\right\rangle=0 \tag{2b}$$

$$-\left(\begin{pmatrix}\boldsymbol{F}_c^p\\ \rho\end{pmatrix},\nabla_{xt}v_c\right)+\langle\hat{t}_c,v_c\rangle=(f_c,v_c) \tag{2c}$$

$$-\left(\begin{pmatrix}\mathbb{F}_m^p-\mathbb{D}\\ \rho\boldsymbol{u}\end{pmatrix},\nabla_{xt}\boldsymbol{v}_m\right)+\langle\hat{\boldsymbol{t}}_m,\boldsymbol{v}_m\rangle=(\boldsymbol{f}_m,\boldsymbol{v}_m) \tag{2d}$$

$$-\left(\begin{pmatrix}\boldsymbol{F}_e^p+\boldsymbol{q}-\boldsymbol{u}\cdot\mathbb{D}\\ \rho\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\end{pmatrix},\nabla_{xt}v_e\right)+\langle\hat{t}_e,v_e\rangle=(f_e,v_e)\ , \tag{2e}$$

## Conservation Variables

Now we wish to do a change of variables to conservation variables:

$$\begin{aligned}\rho&=\rho\\ \boldsymbol{m}&=\rho\boldsymbol{u}\\ E&=\rho\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\end{aligned}$$

We can define new fluxes in conservation variables:

$$\begin{aligned}\boldsymbol{F}_c^c&=\boldsymbol{m}\\ \mathbb{F}_m^c&=\frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\rho}+(\gamma-1)\left(E-\frac{\boldsymbol{m}\cdot\boldsymbol{m}}{2\rho}\right)\boldsymbol{I}\\ \boldsymbol{F}_e^c&=\frac{\boldsymbol{m}}{\rho}E+(\gamma-1)\left(E-\frac{\boldsymbol{m}\cdot\boldsymbol{m}}{2\rho}\right)\frac{\boldsymbol{m}}{\rho}\end{aligned}$$

and our new bilinear form is

$$\left(\frac{1}{\mu}\mathbb{D},\mathbb{S}\right)+\left(2\frac{\boldsymbol{m}}{\rho},\nabla\cdot\mathbb{S}\right)-\left(\frac{2}{3}\frac{\boldsymbol{m}}{\rho},\nabla\operatorname{tr}\mathbb{S}\right)-\left\langle\frac{4}{3}\hat{\boldsymbol{u}},\mathbb{S}\boldsymbol{n}_x\right\rangle=0 \tag{3a}$$

$$\left(\frac{Pr}{C_p\mu}\boldsymbol{q},\boldsymbol{\tau}\right)-\left(\frac{E-\frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m}}{C_v\rho},\nabla\cdot\boldsymbol{\tau}\right)+\left\langle\hat{T},\tau_n\right\rangle=0 \tag{3b}$$

$$-\left(\begin{pmatrix}\boldsymbol{F}_c^c\\ \rho\end{pmatrix},\nabla_{xt}v_c\right)+\langle\hat{t}_c,v_c\rangle=(f_c,v_c) \tag{3c}$$

$$-\left(\begin{pmatrix}\mathbb{F}_m^c-\mathbb{D}\\ \boldsymbol{m}\end{pmatrix},\nabla_{xt}\boldsymbol{v}_m\right)+\langle\hat{\boldsymbol{t}}_m,\boldsymbol{v}_m\rangle=(\boldsymbol{f}_m,\boldsymbol{v}_m) \tag{3d}$$

$$-\left(\begin{pmatrix}\boldsymbol{F}_e^c+\boldsymbol{q}-\frac{\boldsymbol{m}}{\rho}\cdot\mathbb{D}\\ E\end{pmatrix},\nabla_{xt}v_e\right)+\langle\hat{t}_e,v_e\rangle=(f_e,v_e)\ , \tag{3e}$$

## Entropy Variables

Now we wish to do a change of variables to entropy variables:

$$
\begin{aligned}
V_c &= \frac{-E + (E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m})\left(\gamma + 1 - \ln\left[\frac{(\gamma-1)(E-\frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m})}{\rho^\gamma}\right]\right)}{E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m}} \\
\boldsymbol{V}_m &= \frac{\boldsymbol{m}}{E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m}} \\
V_e &= \frac{-\rho}{E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m}}
\end{aligned}
$$

with reverse mapping:

$$
\begin{aligned}
\rho &= -\alpha V_e \\
\boldsymbol{m} &= \alpha \boldsymbol{V}_m \\
E &= \alpha\left(1 - \frac{1}{2V_e}\boldsymbol{V}_m\cdot\boldsymbol{V}_m\right)
\end{aligned}
$$

where

$$
\alpha(V_c, \boldsymbol{V}_m, V_e) = \left[\frac{\gamma-1}{(-V_e)^\gamma}\right]^{\frac{1}{\gamma-1}} \exp\left[\frac{-\gamma + V_c - \frac{1}{2V_e}\boldsymbol{V}_m\cdot\boldsymbol{V}_m}{\gamma-1}\right]
$$

We can define new fluxes in entropy variables:

$$
\begin{aligned}
\boldsymbol{F}_c^e &= \alpha\boldsymbol{V}_m \\
\mathbb{F}_m^e &= \alpha\left(-\frac{\boldsymbol{V}_m\otimes\boldsymbol{V}_m}{V_e} + (\gamma-1)\boldsymbol{I}\right) \\
\boldsymbol{F}_e^e &= \alpha\frac{\boldsymbol{V}_m}{V_e}\left(\frac{1}{2V_e}\boldsymbol{V}_m\cdot\boldsymbol{V}_m - \gamma\right)
\end{aligned}
$$

and our new bilinear form is

$$
\left(\frac{1}{\mu}\mathbb{D}, \mathbb{S}\right) - \left(2\frac{\boldsymbol{V}_m}{V_e}, \nabla\cdot\mathbb{S}\right) + \left(\frac{2}{3}\frac{\boldsymbol{V}_m}{V_e}, \nabla\operatorname{tr}\mathbb{S}\right) - \left\langle\frac{4}{3}\hat{\boldsymbol{u}}, \mathbb{S}\boldsymbol{n}_x\right\rangle = 0 \tag{4a}
$$

$$
\left(\frac{Pr}{C_p\mu}\boldsymbol{q}, \boldsymbol{\tau}\right) + \left(\frac{1}{C_v V_e}, \nabla\cdot\boldsymbol{\tau}\right) + \left\langle\hat{T}, \tau_n\right\rangle = 0 \tag{4b}
$$

$$
-\left(\begin{pmatrix}\boldsymbol{F}_c^e \\ -\alpha V_e\end{pmatrix}, \nabla_{xt} v_c\right) + \langle\hat{t}_c, v_c\rangle = (f_c, v_c) \tag{4c}
$$

$$
-\left(\begin{pmatrix}\mathbb{F}_m^e - \mathbb{D} \\ \alpha\boldsymbol{V}_m\end{pmatrix}, \nabla_{xt}\boldsymbol{v}_m\right) + \langle\hat{\boldsymbol{t}}_m, \boldsymbol{v}_m\rangle = (\boldsymbol{f}_m, \boldsymbol{v}_m) \tag{4d}
$$

$$
-\left(\begin{pmatrix}\boldsymbol{F}_e^e + \boldsymbol{q} + \frac{\boldsymbol{V}_m}{V_e}\cdot\mathbb{D} \\ \alpha\left(1 - \frac{1}{2V_e}\boldsymbol{V}_m\cdot\boldsymbol{V}_m\right)\end{pmatrix}, \nabla_{xt} v_e\right) + \langle\hat{t}_e, v_e\rangle = (f_e, v_e)\ , \tag{4e}
$$

## Linearization

For each change of variables, we maintain the same linear variables: $L := \left\{\boldsymbol{q}, \hat{\boldsymbol{u}}, \hat{e}, \hat{t}_c, \hat{\boldsymbol{t}}_m, \hat{t}_e\right\}$. Let $U$ be the set of variables involved in nonlinear interactions. We apply a linearization $U \approx \tilde{U} + \Delta U$ and solve

$$R_{,U}(\tilde{U})\Delta U + R(L) = -R(\tilde{U})\,,$$

where

$$\begin{aligned} R(L) = &\left(\frac{Pr}{C_p\mu}\boldsymbol{q}, \boldsymbol{\tau}\right) - (\boldsymbol{q}, \nabla v_e) - \left\langle \frac{4}{3}\hat{\boldsymbol{u}}, \mathbb{S}\boldsymbol{n}_x\right\rangle + \left\langle \hat{T}, \tau_n\right\rangle + \left\langle \hat{t}_c, v_c\right\rangle + \left\langle \hat{\boldsymbol{t}}_m, v_m\right\rangle + \left\langle \hat{t}_e, v_e\right\rangle \\ &- (f_c, v_c) - (\boldsymbol{f}_m, \boldsymbol{v}_m) - (f_e, v_e) \end{aligned}$$

### Primitive Variables

The set of nonlinear variables is $U^p := \{\rho, \boldsymbol{u}, T, \mathbb{D}\}$. Then $R_{,U^p}(\tilde{U}^p)\Delta U^p$ is

$$\begin{aligned} &\left(\frac{1}{\mu}\Delta\mathbb{D}, \mathbb{S}\right) + (2\Delta\boldsymbol{u}, \nabla\cdot\mathbb{S}) - \left(\frac{2}{3}\Delta\boldsymbol{u}, \nabla\operatorname{tr}\mathbb{S}\right) \\ &- (\Delta T, \nabla\cdot\boldsymbol{\tau}) \\ &- \left(\begin{pmatrix} \boldsymbol{F}^p_{c,U^p}\Delta U^p \\ \Delta\rho \end{pmatrix}, \nabla_{xt} v_c\right) \\ &- \left(\begin{pmatrix} \mathbb{F}^p_{m,U^p}\Delta U^p - \Delta\mathbb{D} \\ \Delta\rho\tilde{\boldsymbol{u}} + \tilde{\rho}\Delta\boldsymbol{u} \end{pmatrix}, \nabla_{xt}\boldsymbol{v}_m\right) \\ &- \left(\begin{pmatrix} \boldsymbol{F}^p_{e,U^p}\Delta U^p - \Delta\boldsymbol{u}\cdot\tilde{\mathbb{D}} - \tilde{\boldsymbol{u}}\cdot\Delta\mathbb{D} \\ C_v\Delta\rho\tilde{T} + C_v\tilde{\rho}\Delta T + \frac{1}{2}\left(\Delta\rho\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}} + \tilde{\rho}\Delta\boldsymbol{u}\cdot\tilde{\boldsymbol{u}} + \tilde{\rho}\tilde{\boldsymbol{u}}\cdot\Delta\boldsymbol{u}\right) \end{pmatrix}, \nabla_{xt} v_e\right) \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{F}^p_{c,U^p}\Delta U^p &:= \Delta\rho\tilde{\boldsymbol{u}} + \tilde{\rho}\Delta\boldsymbol{u} \\ \mathbb{F}^p_{m,U^p} &:= \Delta\rho\tilde{\boldsymbol{u}}\otimes\tilde{\boldsymbol{u}} + \tilde{\rho}\Delta\boldsymbol{u}\otimes\tilde{\boldsymbol{u}} + \tilde{\rho}\tilde{\boldsymbol{u}}\otimes\Delta\boldsymbol{u} + R\left(\Delta\rho\tilde{T} + \tilde{\rho}\Delta T\right)\boldsymbol{I} \\ \boldsymbol{F}^p_{e,U^p} &:= C_v\Delta\rho\tilde{\boldsymbol{u}}\tilde{T} + C_v\tilde{\rho}\Delta\boldsymbol{u}\tilde{T} + C_v\tilde{\rho}\tilde{\boldsymbol{u}}\Delta T \\ &+ \frac{1}{2}\Delta\rho\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}} + \frac{1}{2}\tilde{\rho}\Delta\boldsymbol{u}\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}} + \frac{1}{2}\tilde{\rho}\tilde{\boldsymbol{u}}\Delta\boldsymbol{u}\cdot\tilde{\boldsymbol{u}} + \frac{1}{2}\tilde{\rho}\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}}\cdot\Delta\boldsymbol{u} \\ &+ R\Delta\boldsymbol{u}\tilde{\rho}\tilde{T} + R\tilde{\boldsymbol{u}}\Delta\rho\tilde{T} + R\tilde{\boldsymbol{u}}\tilde{\rho}\Delta T \end{aligned}$$

and $R(\tilde{U}^p)$ is

$$\begin{aligned} &\left(\frac{1}{\mu}\tilde{\mathbb{D}}, \mathbb{S}\right) + (2\tilde{\boldsymbol{u}}, \nabla\cdot\mathbb{S}) - \left(\frac{2}{3}\tilde{\boldsymbol{u}}, \nabla\operatorname{tr}\mathbb{S}\right) \\ &- \left(\tilde{T}, \nabla\cdot\boldsymbol{\tau}\right) \\ &- \left(\begin{pmatrix} \boldsymbol{F}^p_c(\tilde{U}^p) \\ \tilde{\rho} \end{pmatrix}, \nabla_{xt} v_c\right) \\ &- \left(\begin{pmatrix} \mathbb{F}^p_m(\tilde{U}^p) - \tilde{\mathbb{D}} \\ \tilde{\rho}\tilde{\boldsymbol{u}} \end{pmatrix}, \nabla_{xt}\boldsymbol{v}_m\right) \\ &- \left(\begin{pmatrix} \boldsymbol{F}^p_e(\tilde{U}^p) - \tilde{\boldsymbol{u}}\cdot\tilde{\mathbb{D}} \\ \tilde{\rho}\left(C_v\tilde{T} + \frac{1}{2}\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}}\right) \end{pmatrix}, \nabla_{xt} v_e\right) \end{aligned}$$

## Conservation Variables

The set of nonlinear variables is $U^c := \{\rho, \boldsymbol{m}, E, \mathbb{D}\}$. Then $R_{,U^c}(\tilde{U}^c)\Delta U^c$ is

$$
\begin{aligned}
&\left(\frac{1}{\mu}\Delta\mathbb{D}, \mathbb{S}\right) + \left(2\left(\frac{\Delta\boldsymbol{m}}{\tilde{\rho}} - \frac{\tilde{\boldsymbol{m}}}{\tilde{\rho}^2}\Delta\rho\right), \nabla\cdot\mathbb{S}\right) - \left(\frac{2}{3}\left(\frac{\Delta\boldsymbol{m}}{\tilde{\rho}} - \frac{\tilde{\boldsymbol{m}}}{\tilde{\rho}^2}\Delta\rho\right), \nabla\operatorname{tr}\mathbb{S}\right) \\
&- \left(\frac{\Delta E - \frac{1}{2\tilde{\rho}}\Delta\boldsymbol{m}\cdot\tilde{\boldsymbol{m}} - \frac{1}{2\tilde{\rho}}\tilde{\boldsymbol{m}}\cdot\Delta\boldsymbol{m} + \frac{1}{2\tilde{\rho}^2}\tilde{\boldsymbol{m}}\cdot\tilde{\boldsymbol{m}}\Delta\rho}{C_v\tilde{\rho}} - \frac{\tilde{E} - \frac{1}{2\tilde{\rho}}\tilde{\boldsymbol{m}}\cdot\tilde{\boldsymbol{m}}}{C_v\tilde{\rho}^2}\Delta\rho, \nabla\cdot\boldsymbol{\tau}\right) \\
&- \left(\begin{pmatrix} \boldsymbol{F}^c_{c,U^c}\Delta U^c \\ \Delta\rho \end{pmatrix}, \nabla_{xt} v_c\right) \\
&- \left(\begin{pmatrix} \mathbb{F}^c_{m,U^c}\Delta U^c - \Delta\mathbb{D} \\ \Delta\boldsymbol{m} \end{pmatrix}, \nabla_{xt}\boldsymbol{v}_m\right) \\
&- \left(\begin{pmatrix} \boldsymbol{F}^c_{e,U^c}\Delta U^c - \frac{\Delta\boldsymbol{m}}{\tilde{\rho}}\cdot\tilde{\mathbb{D}} + \frac{\tilde{\boldsymbol{m}}}{\tilde{\rho}^2}\Delta\rho\cdot\tilde{\mathbb{D}} - \frac{\tilde{\boldsymbol{m}}}{\tilde{\rho}}\cdot\Delta\mathbb{D} \\ \Delta E \end{pmatrix}, \nabla_{xt} v_e\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\boldsymbol{F}^c_{c,U^c}\Delta U^c &= \Delta\boldsymbol{m} \\
\mathbb{F}^c_{m,U^c}\Delta U^c &= \frac{\Delta\boldsymbol{m}\otimes\tilde{\boldsymbol{m}}}{\tilde{\rho}} + \frac{\tilde{\boldsymbol{m}}\otimes\Delta\boldsymbol{m}}{\tilde{\rho}} - \frac{\tilde{\boldsymbol{m}}\otimes\tilde{\boldsymbol{m}}}{\tilde{\rho}^2}\Delta\rho \\
&\quad + (\gamma - 1)\left(\Delta E - \frac{\Delta\boldsymbol{m}\cdot\tilde{\boldsymbol{m}}}{2\tilde{\rho}} - \frac{\tilde{\boldsymbol{m}}\cdot\Delta\boldsymbol{m}}{2\tilde{\rho}} + \frac{\tilde{\boldsymbol{m}}\cdot\tilde{\boldsymbol{m}}}{2\tilde{\rho}^2}\Delta\rho\right)\boldsymbol{I} \\
\boldsymbol{F}^c_{e,U^c}\Delta U^c &= \frac{\Delta\boldsymbol{m}}{\tilde{\rho}}\tilde{E} + \frac{\tilde{\boldsymbol{m}}}{\tilde{\rho}}\Delta E - \frac{\tilde{\boldsymbol{m}}}{\tilde{\rho}^2}\tilde{E}\Delta\rho \\
&\quad + (\gamma - 1)\left(\Delta E\frac{\tilde{\boldsymbol{m}}}{\tilde{\rho}} + \tilde{E}\frac{\Delta\boldsymbol{m}}{\tilde{\rho}} - \tilde{E}\frac{\tilde{\boldsymbol{m}}}{\tilde{\rho}^2}\Delta\rho\right) \\
&\quad + (\gamma - 1)\left(-\frac{\Delta\boldsymbol{m}\tilde{\boldsymbol{m}}\cdot\tilde{\boldsymbol{m}}}{2\tilde{\rho}^2} - \frac{\tilde{\boldsymbol{m}}\Delta\boldsymbol{m}\cdot\tilde{\boldsymbol{m}}}{2\tilde{\rho}^2} - \frac{\tilde{\boldsymbol{m}}\tilde{\boldsymbol{m}}\cdot\Delta\boldsymbol{m}}{2\tilde{\rho}^2} + \frac{\tilde{\boldsymbol{m}}\tilde{\boldsymbol{m}}\cdot\tilde{\boldsymbol{m}}}{\tilde{\rho}^3}\Delta\rho\right)
\end{aligned}
$$

and $R(\tilde{U}^p)$ is

$$
\begin{aligned}
&\left(\frac{1}{\mu}\tilde{\mathbb{D}}, \mathbb{S}\right) + \left(2\frac{\tilde{\boldsymbol{m}}}{\tilde{\rho}}, \nabla\cdot\mathbb{S}\right) - \left(\frac{2}{3}\frac{\tilde{\boldsymbol{m}}}{\tilde{\rho}}, \nabla\operatorname{tr}\mathbb{S}\right) \\
&- \left(\frac{\tilde{E} - \frac{1}{2\tilde{\rho}}\tilde{\boldsymbol{m}}\cdot\tilde{\boldsymbol{m}}}{C_v\tilde{\rho}}, \nabla\cdot\boldsymbol{\tau}\right) \\
&- \left(\begin{pmatrix} \boldsymbol{F}^c_c \\ \tilde{\rho} \end{pmatrix}, \nabla_{xt} v_c\right) \\
&- \left(\begin{pmatrix} \mathbb{F}^c_m - \tilde{\mathbb{D}} \\ \tilde{\boldsymbol{m}} \end{pmatrix}, \nabla_{xt}\boldsymbol{v}_m\right) \\
&- \left(\begin{pmatrix} \boldsymbol{F}^c_e - \frac{\tilde{\boldsymbol{m}}}{\tilde{\rho}}\cdot\tilde{\mathbb{D}} \\ \tilde{E} \end{pmatrix}, \nabla_{xt} v_e\right)
\end{aligned}
$$

### Entropy Variables

The set of nonlinear variables is $U^e := \{V_c, \boldsymbol{V}_m, V_e, \mathbb{D}\}$. Then $R_{,U^e}(\tilde{U}^e)\Delta U^e$ is

$$\left(\frac{1}{\mu}\Delta\mathbb{D}, \mathbb{S}\right) - \left(2\left(\frac{\Delta\boldsymbol{V}_m}{\tilde{V}_e} - \frac{\tilde{\boldsymbol{V}}_m}{\tilde{V}_e^2}\Delta V_e\right), \nabla\cdot\mathbb{S}\right) + \left(\frac{2}{3}\left(\frac{\Delta\boldsymbol{V}_m}{\tilde{V}_e} - \frac{\tilde{\boldsymbol{V}}_m}{\tilde{V}_e^2}\Delta V_e\right), \nabla\operatorname{tr}\mathbb{S}\right)$$
$$- \left(\frac{1}{C_v V_e^2}\Delta V_e, \nabla\cdot\boldsymbol{\tau}\right)$$
$$- \left(\begin{pmatrix} \boldsymbol{F}^e_{c,U^e}\Delta U^e \\ -\alpha_{,U^e}\Delta U^e\tilde{V}_e - \alpha\Delta V_e \end{pmatrix}, \nabla_{xt} v_c\right)$$
$$- \left(\begin{pmatrix} \mathbb{F}^e_{m,U^e}\Delta U^e - \Delta\mathbb{D} \\ \alpha_{,U^e}\Delta U^e\tilde{\boldsymbol{V}}_m + \alpha\Delta\boldsymbol{V}_m \end{pmatrix}, \nabla_{xt}\boldsymbol{v}_m\right)$$
$$- \left(\begin{pmatrix} \boldsymbol{F}^e_{e,U^e}\Delta U^e + \frac{\Delta\boldsymbol{V}_m}{\tilde{V}_e}\cdot\tilde{\mathbb{D}} + \frac{\tilde{\boldsymbol{V}}_m}{\tilde{V}_e}\cdot\Delta\mathbb{D} - \frac{\tilde{\boldsymbol{V}}_m}{\tilde{V}_e^2}\cdot\tilde{\mathbb{D}}\Delta V_e \\ \alpha_{,U^e}\Delta U^e\left(1 - \frac{1}{2\tilde{V}_e}\tilde{\boldsymbol{V}}_m\cdot\tilde{\boldsymbol{V}}_m\right) - \alpha\frac{1}{\tilde{V}_e}\tilde{\boldsymbol{V}}_m\cdot\Delta\boldsymbol{V}_m + \alpha\frac{1}{2\tilde{V}_e^2}\tilde{\boldsymbol{V}}_m\cdot\tilde{\boldsymbol{V}}_m\Delta V_e \end{pmatrix}, \nabla_{xt} v_e\right)$$

where

$$\begin{aligned}
\boldsymbol{F}^e_{c,U^e}\Delta U^e &= \alpha_{,U^e}\Delta U^e\tilde{\boldsymbol{V}}_m + \alpha\Delta\boldsymbol{V}_m \\
\mathbb{F}^e_{m,U^e}\Delta U^e &= \alpha_{,U^e}\Delta U^e\left(-\frac{\tilde{\boldsymbol{V}}_m\otimes\tilde{\boldsymbol{V}}_m}{\tilde{V}_e} + (\gamma-1)\boldsymbol{I}\right) \\
&\quad + \alpha\left(-\frac{\Delta\boldsymbol{V}_m\otimes\tilde{\boldsymbol{V}}_m}{\tilde{V}_e} - \frac{\tilde{\boldsymbol{V}}_m\otimes\Delta\boldsymbol{V}_m}{\tilde{V}_e} + \frac{\tilde{\boldsymbol{V}}_m\otimes\tilde{\boldsymbol{V}}_m}{\tilde{V}_e^2}\Delta V_e\right) \\
\boldsymbol{F}^e_{e,U^e}\Delta U^e &= \alpha_{,U^e}\Delta U^e\frac{\tilde{\boldsymbol{V}}_m}{\tilde{V}_e}\left(\frac{1}{2\tilde{V}_e}\tilde{\boldsymbol{V}}_m\cdot\tilde{\boldsymbol{V}}_m - \gamma\right) \\
&\quad + \alpha\left(\frac{\Delta\boldsymbol{V}_m}{\tilde{V}_e}\left(\frac{1}{2\tilde{V}_e}\tilde{\boldsymbol{V}}_m\cdot\tilde{\boldsymbol{V}}_m - \gamma\right) - \frac{\tilde{\boldsymbol{V}}_m}{V_e^2}\left(\frac{1}{2\tilde{V}_e}\tilde{\boldsymbol{V}}_m\cdot\tilde{\boldsymbol{V}}_m - \gamma\right)\Delta V_e\right. \\
&\quad \left. + \frac{\tilde{\boldsymbol{V}}_m}{\tilde{V}_e}\left(\frac{1}{\tilde{V}_e}\tilde{\boldsymbol{V}}_m\cdot\Delta\boldsymbol{V}_m - \frac{1}{2\tilde{V}_e^2}\tilde{\boldsymbol{V}}_m\cdot\tilde{\boldsymbol{V}}_m\Delta V_e\right)\right)
\end{aligned}$$

$$\begin{aligned}
\alpha_{,U^e}\Delta U^e &= \left[\frac{\gamma-1}{(-\tilde{V}_e)^\gamma}\right]^{\frac{2-\gamma}{\gamma-1}}\gamma(-\tilde{V}_e)^{-(\gamma+1)}\exp\left[\frac{-\gamma+\tilde{V}_c-\frac{1}{2\tilde{V}_e}\tilde{\boldsymbol{V}}_m\cdot\tilde{\boldsymbol{V}}_m}{\gamma-1}\right]\Delta V_e \\
&\quad + \left[\frac{\gamma-1}{(-\tilde{V}_e)^\gamma}\right]^{\frac{1}{\gamma-1}}\exp\left[\frac{-\gamma+\tilde{V}_c-\frac{1}{2\tilde{V}_e}\tilde{\boldsymbol{V}}_m\cdot\tilde{\boldsymbol{V}}_m}{\gamma-1}\right]\frac{1}{\gamma-1} \\
&\quad \left(\Delta V_c - \frac{1}{\tilde{V}_e}\tilde{\boldsymbol{V}}_m\cdot\Delta\boldsymbol{V}_m + \frac{1}{2\tilde{V}_e^2}\tilde{\boldsymbol{V}}_m\cdot\tilde{\boldsymbol{V}}_m\Delta V_e\right)
\end{aligned}$$

and $R(\tilde{U}^p)$ is

$$
\begin{aligned}
\left(\frac{1}{\mu}\tilde{\mathbb{D}}, \mathbb{S}\right) - \left(2\frac{\tilde{\boldsymbol{V}}_m}{\tilde{V}_e}, \nabla\cdot\mathbb{S}\right) + \left(\frac{2}{3}\frac{\tilde{\boldsymbol{V}}_m}{V_e}, \nabla\,\mathrm{tr}\,\mathbb{S}\right)& \\
+ \left(\frac{1}{C_v\tilde{V}_e}, \nabla\cdot\boldsymbol{\tau}\right)& \\
- \left(\begin{pmatrix} \boldsymbol{F}^e_c \\ -\alpha\tilde{V}_e \end{pmatrix}, \nabla_{xt} v_c\right)& \\
- \left(\begin{pmatrix} \mathbb{F}^e_m - \tilde{\mathbb{D}} \\ \alpha\tilde{\boldsymbol{V}}_m \end{pmatrix}, \nabla_{xt} \boldsymbol{v}_m\right)& \\
- \left(\begin{pmatrix} \boldsymbol{F}^e_e + \frac{\tilde{\boldsymbol{V}}_m}{\tilde{V}_e}\cdot\tilde{\mathbb{D}} \\ \alpha\left(1 - \frac{1}{2\tilde{V}_e}\tilde{\boldsymbol{V}}_m\cdot\tilde{\boldsymbol{V}}_m\right) \end{pmatrix}, \nabla_{xt} v_e\right)&
\end{aligned}
$$

# Entropy Norms

Denote primitive, conservation, and entropy variables as $W$, $U$, and $V$ respectively.

## Entropy Metrics and Symmetrizers

### Conservation Variables

Consider entropy function $H(U)$. The entropy metric we want to control is

$$(\delta U, H_{,UU}\delta U) = (\delta U, V_{,U}\delta U)$$

where

$$
V_{,U}(U) = \begin{bmatrix}
\frac{4\gamma\rho^2E^2 - 4\gamma\rho E\boldsymbol{m}\cdot\boldsymbol{m} + (1+\gamma)(\boldsymbol{m}\cdot\boldsymbol{m})^2}{\rho(\boldsymbol{m}\cdot\boldsymbol{m}-2\rho E)^2} & -\frac{2\boldsymbol{m}\boldsymbol{m}\cdot\boldsymbol{m}}{(\boldsymbol{m}\cdot\boldsymbol{m}-2\rho E)^2} & -\frac{4\rho(\rho E - \boldsymbol{m}\cdot\boldsymbol{m})}{(\boldsymbol{m}\cdot\boldsymbol{m}-2\rho E)^2} \\
 & \frac{2\rho(2\rho E + \boldsymbol{m}\cdot\boldsymbol{m})}{(\boldsymbol{m}\cdot\boldsymbol{m}-2\rho E)^2} & -\frac{4\rho^2\boldsymbol{m}}{(\boldsymbol{m}\cdot\boldsymbol{m}-2\rho E)^2} \\
Symm. & & \frac{4\rho^3}{(\boldsymbol{m}\cdot\boldsymbol{m}-2\rho E)^2}
\end{bmatrix}
$$

Let $A_0^c = V_{,U}(U)$ denote the symmetrizer for conservation variables.

### Primitive Variables

Consider a change of variables to primitive variables: $\delta W = U_{,W}\delta W$. Our entropy metric is then

$$(U_{,W}\delta W, V_{,U}U_{,W}\delta W) = \left(\delta W, U_{,W}^T V_{,U} U_{,W}\delta W\right)$$

Then

$$
U_{,W} = \begin{bmatrix}
1 & 0 & 0 \\
\boldsymbol{u} & \rho & 0 \\
C_vT + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} & \rho\boldsymbol{u} & C_v\rho
\end{bmatrix}
$$

where $V_{,U}$ in primitive variables is

$$
V_{,U}(W) = \begin{bmatrix}
\frac{\gamma}{\rho} + \frac{(\boldsymbol{u}\cdot\boldsymbol{u})^2}{4\rho C_v^2T^2} & -\frac{\boldsymbol{u}\boldsymbol{u}\cdot\boldsymbol{u}}{2\rho C_v^2T^2} & -\frac{(C_vT - \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u})}{\rho C_v^2T^2} \\
 & \frac{C_vT + 2\boldsymbol{u}\cdot\boldsymbol{u}}{2\rho C_v^2T^2} & -\frac{\boldsymbol{u}}{\rho C_v^2T^2} \\
Symm. & & \frac{1}{\rho C_v^2T^2}
\end{bmatrix}
$$

and

$$U_{,W}^T V_{,U} U_{,W} = \begin{bmatrix} \frac{\gamma}{\rho} + \frac{(\boldsymbol{u}\cdot\boldsymbol{u})^2}{4\rho C_v^2 T^2} & 0 & 0 \\ 0 & \frac{\rho(CvT+2\boldsymbol{u}\cdot\boldsymbol{u})}{2C_v^2T^2} & 0 \\ 0 & 0 & \frac{\rho}{T^2} \end{bmatrix}$$

Let $A_0^p = U_{,W}^T V_{,U} U_{,W}$ denote the symmetrizer for primitive variables.

**Entropy Variables**

Consider a change of variables to entropy variables: $\delta V = U_{,V}\delta V$. Our entropy metric is then

$$(U_{,V}\delta V, V_{,U}U_{,V}\delta V) = \left(\delta V, U_{,V}^T\delta V\right) = \left(\delta V, A_0^{-1}\delta V\right)$$

with

$$U_{,V} = \frac{\alpha}{\gamma-1}\begin{bmatrix} -V_e & \boldsymbol{V}_m & 1-\frac{\boldsymbol{V}_m\cdot\boldsymbol{V}_m}{2V_e} \\ & \gamma-1-\frac{\boldsymbol{V}_m\cdot\boldsymbol{V}_m}{2V_e} & \left(\gamma-\frac{\boldsymbol{V}_m\cdot\boldsymbol{V}_m}{2V_e}\right)\frac{\boldsymbol{V}_m}{V_e} \\ Symm. & & \frac{4\gamma V_e^2-4\gamma V_e\boldsymbol{V}_m\cdot\boldsymbol{V}_m+(\boldsymbol{V}_m\cdot\boldsymbol{V}_m)^2}{4V_e^3} \end{bmatrix}$$

Let $A_0^e = U_{,V}(V)$ denote the symmetrizer for entropy variables.

## Robust Norm

Consider domain $Q = \Omega\times[0,T]$ with boundary $\Gamma = \Gamma_-\cup\Gamma_+\cup\Gamma_0\cup\Gamma_T$ where $\Gamma_-$ is the spatial inflow boundary, $\Gamma_+$ is the spatial outflow boundary, $\Gamma_0$ is the initial time boundary, and $\Gamma_T$ is the final time boundary. Let $\Gamma_h$ denote the entire mesh skeleton.

Assume that boundary conditions are applied on the boundary $\Gamma_0\subset\Gamma$. Recall that, for the ultra-weak variational formulation

$$b\left((u,\widehat{u}),v\right) = (u, A_h^*v)_{L^2(\Omega)} + \langle\widehat{u}, [\![v]\!]\rangle_{\Gamma_h\setminus\Gamma_0}$$

If we have conforming $v^*$ such that

$$\begin{aligned} A^*v^* &= A_0u \\ v^* &= 0 \text{ on } \Gamma_h\setminus\Gamma_0. \end{aligned}$$

then

$$\left\|A_0^{\frac{1}{2}}u\right\|^2 = (u, A_0u) = (u, A^*v^*) = b(u,v^*) = \frac{b(u,v^*)}{\|v^*\|_V}\|v^*\|_V \le \|u\|_E\|v^*\|_V\ .$$

Thus, we need to develop an adjoint norm such that $A^*v^* = A_0u$ and $\left\|A_0^{\frac{1}{2}}u\right\|_V \le \|v^*\|_V$.

We start by rewriting our linearized bilinear form as

$$\begin{aligned} (M\Sigma,\Psi) + (GU,\nabla\Psi) + \left\langle H\hat{U},\Psi\right\rangle &= -R_\Sigma((\tilde{U},\tilde{\Sigma}),\Psi) \\ -\left(\begin{pmatrix} \mathcal{F}U - K\Sigma \\ CU \end{pmatrix}, \nabla_{xt}V\right) + \left\langle\hat{T},V\right\rangle &= (f,V) - R_U((\tilde{U},\tilde{\Sigma}),V) \end{aligned}$$

where $U$ denotes the primary variables, $\hat{U}$ represents the trace variables, $\hat{T}$ represents the flux variables, $\Sigma$ represents the viscous (and heat) stress variables, $\Psi$ represents the test functions applied to the constitutive laws, $V$ represents the test functions applied to the conservation laws, $\mathcal{F}U$ are the Euler fluxes, $K\Sigma$ is the viscous (and heat) contribution to the conservation laws, $C(U)$ represents the conserved quantity for each conservation law, $(M\Sigma,\Psi)$, $(GU,\nabla\Psi)$, and $(H\hat{U},\Psi)$ are bilinear forms representing the $\Sigma$, $U$, and $\hat{U}$ contributions to the constitutive laws, $R_\Sigma$ is the constitutive residual, $R_U$ is the conservative residual, and $f$ represents any source terms. The exact form of each of these depends on whether we are considering primitive variables, conservation variables, or entropy variables.

We define our adjoint equations by grouping terms by $\Sigma$ and $U$ and weighting the second equation by $A_0 U$:

$$M^*\Psi + K^*\nabla V = 0$$
$$-\begin{pmatrix} \mathcal{F}^* \\ C^* \end{pmatrix} \cdot \nabla_{xt} V + G^* \cdot \nabla\Psi = A_0 U$$

we use the similarity in form between this and our convection-diffusion adjoint equation to define a (hopefully) robust norm for Navier-Stokes. For convection-diffusion, we derived a bound $\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \leq \|u\|$ by multiplying both sides by $-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v$ and integrating over $Q$ to get something like

$$\left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v\right\|^2 \leq \frac{\|u\|^2}{2} + \frac{\left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v\right\|^2}{2} + \epsilon C \left\|\nabla v\right\|^2 .$$

Without proof, we postulate that we would get a similar bound for Navier-Stokes if we multiply both sides of the second equation by $-A_0^{-1} \begin{pmatrix} \mathcal{F}^* \\ C^* \end{pmatrix} \cdot \nabla_{xt} V$ and integrate over $Q$. By analogy, we would hope to get a bound like

$$\begin{aligned} \left\|A_0^{-\frac{1}{2}} \begin{pmatrix} \mathcal{F}^* \\ C^* \end{pmatrix} \cdot \nabla_{xt} V\right\|^2 &\leq \int_Q A_0 U A_0^{-1} \begin{pmatrix} \mathcal{F}^* \\ C^* \end{pmatrix} \cdot \nabla_{xt} V + \epsilon C \left\|\nabla V\right\|^2 \\ &\leq \frac{\left\|A_0^{\frac{1}{2}} U\right\|^2}{2} + \frac{\left\|A_0^{-\frac{1}{2}} \begin{pmatrix} \mathcal{F}^* \\ C^* \end{pmatrix} \cdot \nabla_{xt} V\right\|^2}{2} + \epsilon C \left\|\nabla V\right\|^2 \end{aligned}$$

indicating that including a term such as $\left\|A_0^{-\frac{1}{2}} \begin{pmatrix} \mathcal{F}^* \\ C^* \end{pmatrix} \cdot \nabla_{xt} V\right\|$ in our norm will produce a robust bound.