

Theorem[section] [theorem]Lemma [theorem]Corollary
Definition[section] Example[section]

Space-Time Discontinuous Petrov-Galerkin Finite Elements for Fluid Problems

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THE UNIVERSITY OF
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Overview of DPG

A Framework for Computational Mechanics

Find $u \in U$ such that

$$b(u, v) = l(v) \quad \forall v \in V$$

with operator $B : U \rightarrow V'$ defined by $b(u, v) = \langle Bu, v \rangle_{V' \times V}$.

This gives the operator equation

$$Bu = l \in V'.$$

We wish to minimize the residual $Bu - l \in V'$:

$$u_h = \arg \min_{w_h \in U_h} \frac{1}{2} \|Bw_h - l\|_{V'}^2.$$

Dual norms are not computationally tractable. Inverse Riesz map moves the residual to a more accessible space:

$$u_h = \arg \min_{w_h \in U_h} \frac{1}{2} \|R_V^{-1}(Bw_h - l)\|_V^2.$$

Petrov-Galerkin with Optimal Test Functions

Taking the Gâteaux derivative to be zero in all directions $\delta u \in U_h$ gives,

$$(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u)_V = 0, \quad \forall \delta u \in U,$$

which by definition of the Riesz map is equivalent to

$$\langle Bu_h - l, R_V^{-1}B\delta u_h \rangle = 0 \quad \forall \delta u_h \in U_h,$$

with optimal test functions $v_{\delta u_h} := R_V^{-1}B\delta u_h$ for each trial function δu_h .

Resulting Petrov-Galerkin System

This gives a simple bilinear form

$$b(u_h, v_{\delta u_h}) = l(v_{\delta u_h}),$$

with $v_{\delta u_h} \in V$ that solves the auxiliary problem

$$(v_{\delta u_h}, \delta v)_V = \langle R_V v_{\delta u_h}, \delta v \rangle = \langle B\delta u_h, \delta v \rangle = b(\delta u_h, \delta v) \quad \forall \delta v \in V.$$

Discontinuous Petrov-Galerkin

- Continuous test space produces global solve for optimal test functions
- Discontinuous test space results in an embarrassingly parallel solve

Hermitian Positive Definite Stiffness Matrix

Property of all minimum residual methods

$$b(u_h, v_{\delta u_h}) = (v_{u_h}, v_{\delta u_h})_V = \overline{(v_{\delta u_h}, v_{u_h})_V} = \overline{b(\delta u_h, v_{u_h})}$$

Error Representation Function

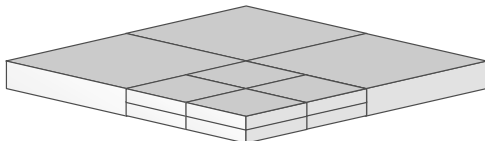
Energy norm of Galerkin error (residual) can be computed without exact solution

$$\|u_h - u\|_E = \|B(u_h - u)\|_{V'} = \|Bu_h - l\|_{V'} = \|R_V^{-1}(Bu_h - l)\|_V$$

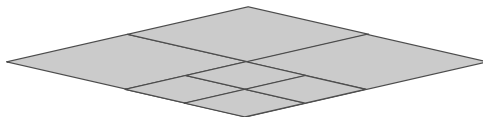
Extending DPG to Transient Problems

- Time stepping techniques are not ideally suited to highly adaptive grids
- Space-time FEM proposed as a solution
 - ✓ Unified treatment of space and time
 - ✓ Local space-time adaptivity (local time stepping)
 - ✓ Parallel-in-time integration (space-time multigrid)
 - ✗ Spatially stable FEM methods may not be stable in space-time
 - ✗ Need to support higher dimensional problems
- DPG provides necessary stability and adaptivity

space-time slab



spatial mesh



Space-Time DPG for Convection-Diffusion

Space-Time Divergence Form

Equation is parabolic in space-time.

$$\frac{\partial u}{\partial t} + \beta \cdot \nabla u - \epsilon \Delta u = f$$

This is just a composition of a constitutive law and conservation of mass.

$$\sigma - \epsilon \nabla u = 0$$

$$\frac{\partial u}{\partial t} + \nabla \cdot (\beta u - \sigma) = f$$

We can rewrite this in terms of a space-time divergence.

$$\begin{aligned} \frac{1}{\epsilon} \sigma - \nabla u &= 0 \\ \nabla_{xt} \cdot \begin{pmatrix} \beta u - \sigma \\ u \end{pmatrix} &= f \end{aligned}$$

Multiply by test function and integrate by parts over space-time element K .

$$\begin{aligned} \left(\frac{1}{\epsilon} \boldsymbol{\sigma}, \boldsymbol{\tau} \right)_K + (u, \nabla \cdot \boldsymbol{\tau})_K - \langle \hat{u}, \boldsymbol{\tau} \cdot \mathbf{n}_x \rangle_{\partial K} &= 0 \\ - \left(\begin{pmatrix} \beta u - \boldsymbol{\sigma} \\ u \end{pmatrix}, \nabla_{xt} v \right)_K + \langle \hat{t}, v \rangle_{\partial K} &= f \end{aligned}$$

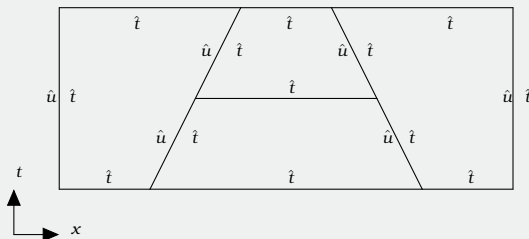
where

$$\hat{u} := \text{tr}(u)$$

$$\begin{aligned} \hat{t} &:= \text{tr}(\beta u - \boldsymbol{\sigma}) \cdot \mathbf{n}_x \\ &\quad + \text{tr}(u) \cdot n_t \end{aligned}$$

- Trace \hat{u} defined on spatial boundaries
- Flux \hat{t} defined on all boundaries

Support of Trace Variables



Robust Norms

Bilinear form with group variables:

$$b((u, \hat{u}), v) = (u, A_h^* v)_{L^2(\Omega_h)} + \langle \hat{u}, \llbracket v \rrbracket \rangle_{\Gamma_h}$$

For conforming v^* satisfying $A^* v^* = u$

$$\begin{aligned} \|u\|_{L^2(\Omega_h)}^2 &= b(u, v^*) = \frac{b(u, v^*)}{\|v^*\|_V} \|v^*\|_V \\ &\leq \sup_{v^* \neq 0} \frac{|b(u, v^*)|}{\|v^*\|} \|v^*\| = \|u\|_E \|v^*\|_V \end{aligned}$$

Necessary robustness condition:

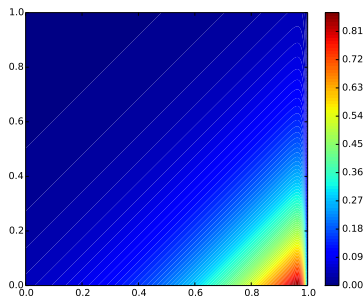
$$\begin{aligned} \|v^*\|_V &\lesssim \|u\|_{L^2(\Omega_h)} \\ \Rightarrow \|u\|_{L^2(\Omega_h)} &\lesssim \|u\|_E \end{aligned}$$

Analytical Solution

$$u = e^{-lt} (e^{\lambda_1(x-1)} - e^{\lambda_2(x-1)})$$

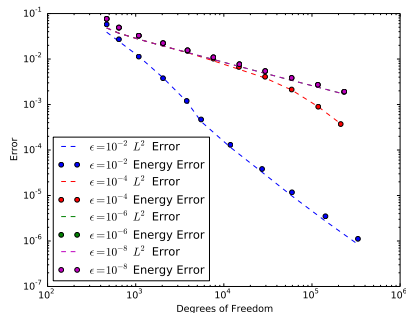
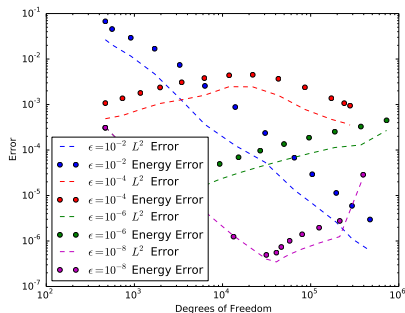
$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1 - 4l\epsilon}}{-2\epsilon}$$

where $l = 3$, $\epsilon = 10^{-2}$



Robust Norms

A norm should be: bounded by $\|u\|_{L^2(\Omega_h)}$, have good conditioning, not produce boundary layers in the optimal test function.



$$\begin{aligned} \|(v, \tau)\|^2 &= \left\| \nabla \cdot \tau - \tilde{\beta} \cdot \nabla_{xt} v \right\|^2 \\ &+ \left\| \frac{1}{\epsilon} \tau + \nabla v \right\|^2 + \|v\|^2 + \|\tau\|^2 \end{aligned}$$

$$\begin{aligned} \|(v, \tau)\|^2 &= \left\| \nabla \cdot \tau - \tilde{\beta} \cdot \nabla_{xt} v \right\|^2 \\ &+ \min \left(\frac{1}{h^2}, \frac{1}{\epsilon} \right) \|\tau\|^2 \\ &+ \epsilon \|\nabla v\|^2 + \|\beta \cdot \nabla v\|^2 + \|v\|^2 \end{aligned}$$

Space-Time Convection-Diffusion

Ideal Optimal Shape Functions

Graph Norm

Robust Norm

Space-Time Convection-Diffusion

Approximated ($p = 3$) Optimal Shape Functions

Graph Norm

Robust Norm

Robust Norms for 2D Space-Time

Confusion/SpaceTimeRobustness/Coupled/SpaceTimeRobustness

$$\begin{aligned} \|(v, \tau)\|^2 = & \left\| \nabla \cdot \tau - \tilde{\beta} \cdot \nabla_{xt} v \right\|^2 \\ & + \min \left(\frac{1}{h^2}, \frac{1}{\epsilon} \right) \|\tau\|^2 \\ & + \epsilon \|\nabla v\|^2 + \|\beta \cdot \nabla v\|^2 + \|v\|^2 \end{aligned}$$

$$\begin{aligned} \|(v, \tau)\|^2 = & \left\| \tilde{\beta} \cdot \nabla_{xt} v \right\|^2 + \|\nabla \cdot \tau\|^2 \\ & + \frac{1}{h^2} \|\tau\|^2 + \|\nabla v\|^2 + \|v\|^2 \end{aligned}$$

Space-Time Incompressible Navier-Stokes

Space-Time Divergence Form

First order space-time divergence form:

$$\begin{aligned}\frac{1}{\nu} \boldsymbol{\sigma}_1 - \nabla u_1 &= 0 \\ \frac{1}{\nu} \boldsymbol{\sigma}_2 - \nabla u_2 &= 0 \\ \nabla_{xt} \cdot \left(\left(\begin{array}{c} \mathbf{u} \otimes \mathbf{u} - \left(\begin{array}{c} \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2 \end{array} \right) + p \mathbf{I} \\ \mathbf{u} \end{array} \right) \right) &= \mathbf{f} \\ \nabla \cdot \mathbf{u} &= 0 \\ \int_{\Omega} p &= 0\end{aligned}$$

Space-Time Incompressible Navier-Stokes

Ultra-Weak Formulation with Discontinuous Test Functions

Multiplying by τ_1 , τ_2 , \mathbf{u} , and q , and integrating by parts

$$\begin{aligned} \left(\frac{1}{\nu} \boldsymbol{\sigma}_1, \boldsymbol{\tau}_1 \right) + (u_1, \nabla \cdot \boldsymbol{\tau}_1) - \langle \hat{u}_1, \tau_{1n} \rangle &= 0 \\ \left(\frac{1}{\nu} \boldsymbol{\sigma}_2, \boldsymbol{\tau}_2 \right) + (u_2, \nabla \cdot \boldsymbol{\tau}_2) - \langle \hat{u}_2, \tau_{2n} \rangle &= 0 \\ - \left(\left(\begin{array}{c} \mathbf{u} \otimes \mathbf{u} - \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} + p\mathbf{I} \\ \mathbf{u} \end{array} \right), \nabla_{xt} \mathbf{v} \right) + \langle \hat{\mathbf{t}}, \mathbf{v} \rangle &= (\mathbf{f}, \mathbf{v}) \\ - (\mathbf{u}, \nabla q) + \langle \widehat{\mathbf{u} \cdot \mathbf{n}}, q \rangle &= 0 \\ \int_{\Omega} p &= 0 \end{aligned}$$

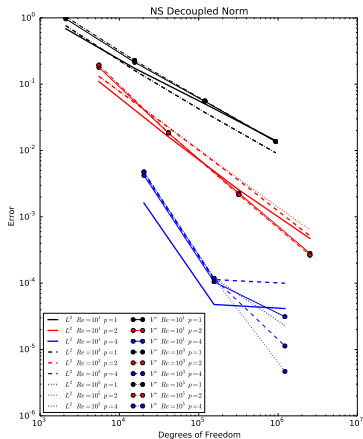
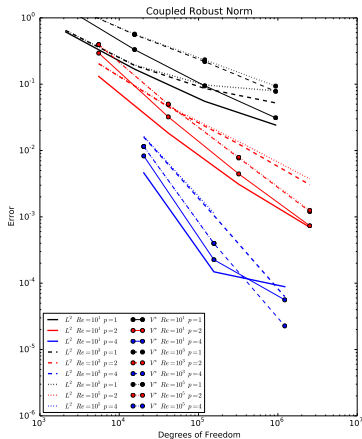
where

$$\hat{\mathbf{t}} = \text{tr} \left(\left(\begin{array}{c} \mathbf{u} \otimes \mathbf{u} - \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} + p\mathbf{I} \\ \mathbf{u} \end{array} \right) \cdot \mathbf{n}_{xt} \right)$$

Space-Time Incompressible Navier-Stokes

Taylor-Green Vortex Problem

$$\mathbf{u} = \begin{pmatrix} e^{-2\nu t} \sin x \cos y \\ -e^{-2\nu t} \cos x \sin y \end{pmatrix}$$



Assuming Stokes hypothesis, ideal gas law, and constant viscosity:

$$\frac{1}{\mu} \mathbb{D} - (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + \frac{2}{3} \nabla \cdot \mathbf{u} \mathbb{I} = 0$$

$$\frac{Pr}{C_p \mu} \mathbf{q} + \nabla T = 0$$

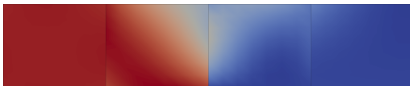
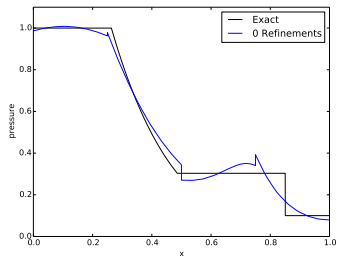
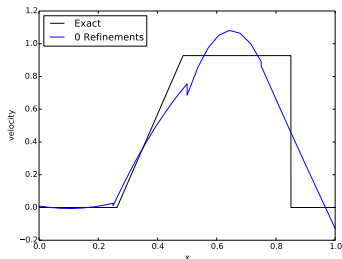
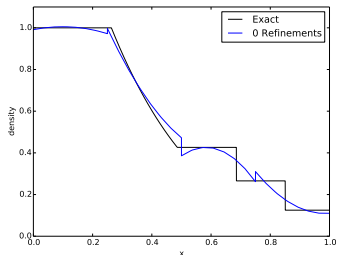
$$\nabla_{xt} \cdot \begin{pmatrix} \rho \mathbf{u} \\ \rho \end{pmatrix} = f_c$$

$$\nabla_{xt} \cdot \begin{pmatrix} \rho \mathbf{u} \otimes \mathbf{u} + \rho RT \mathbb{I} - \mathbb{D} \\ \rho \mathbf{u} \end{pmatrix} = \mathbf{f}_m$$

$$\nabla_{xt} \cdot \begin{pmatrix} \rho \mathbf{u} (C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u}) + \rho RT \mathbf{u} + \mathbf{q} - \mathbf{u} \cdot \mathbb{D} \\ \rho (C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u}) \end{pmatrix} = f_e,$$

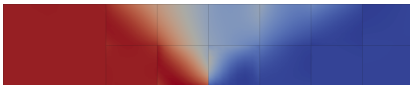
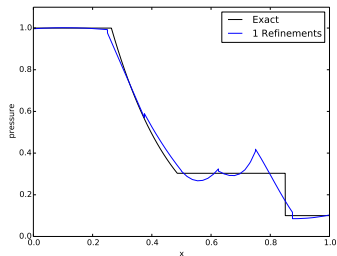
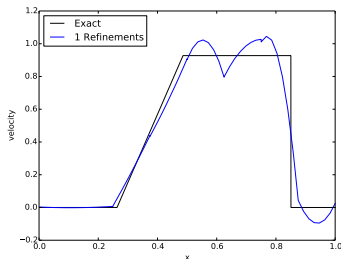
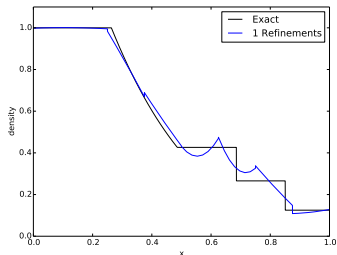
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



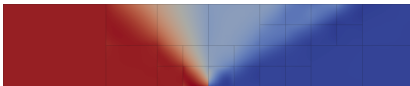
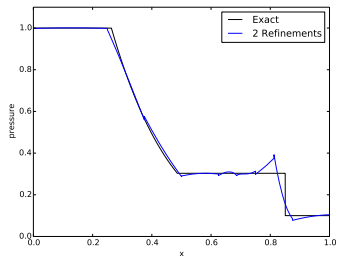
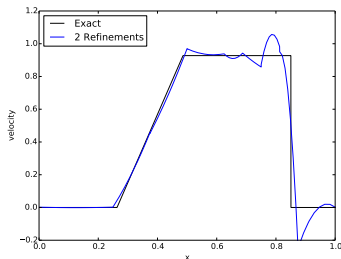
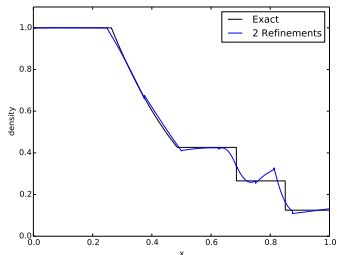
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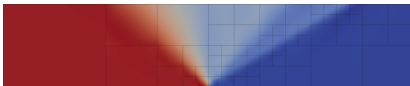
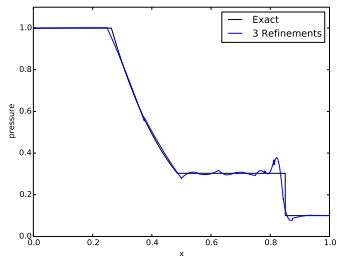
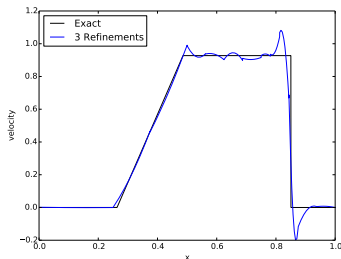
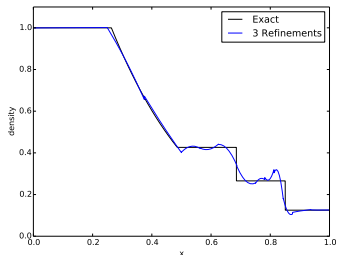
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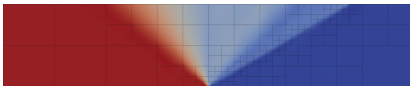
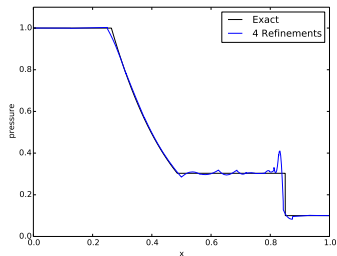
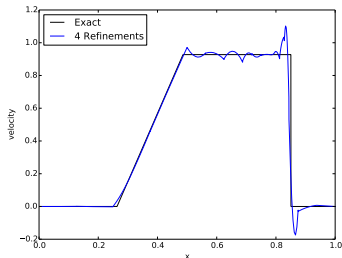
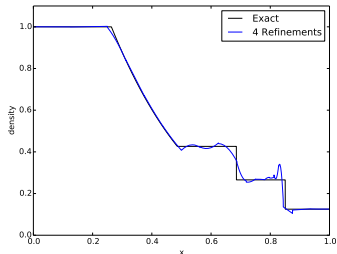
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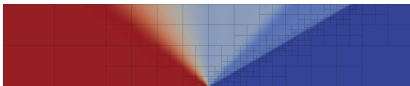
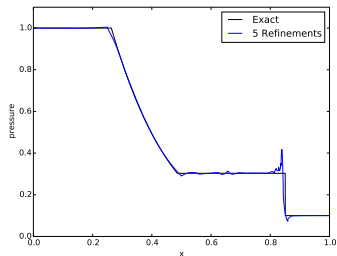
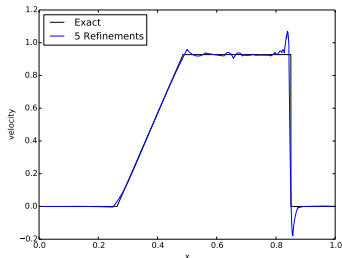
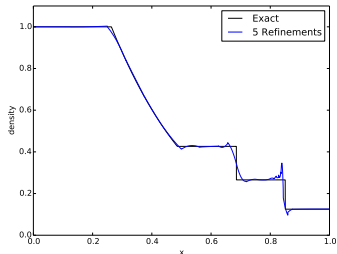
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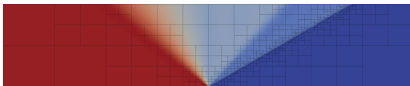
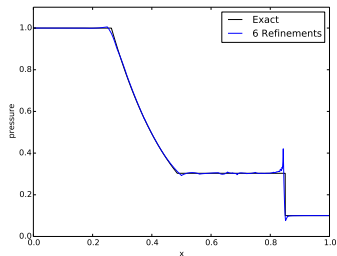
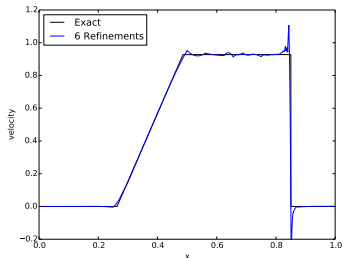
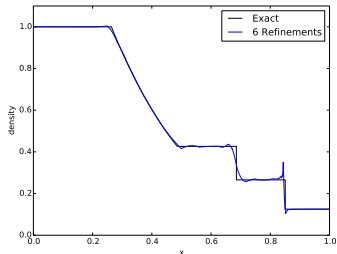
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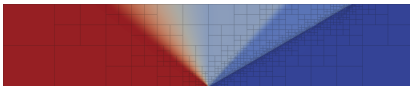
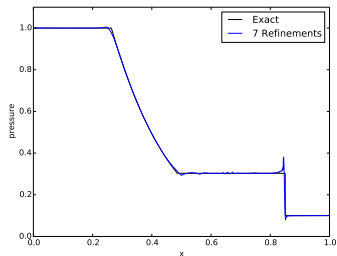
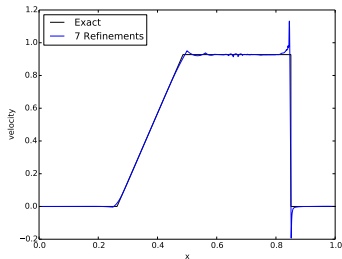
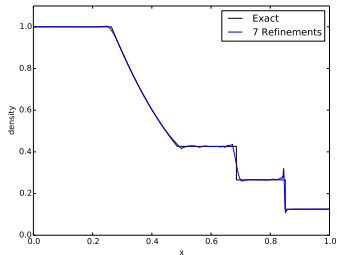
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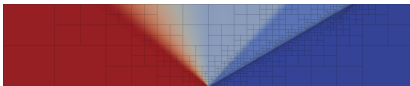
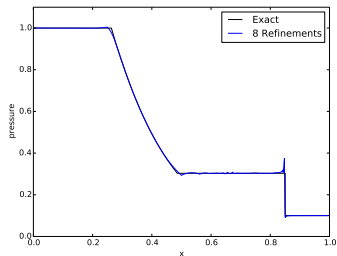
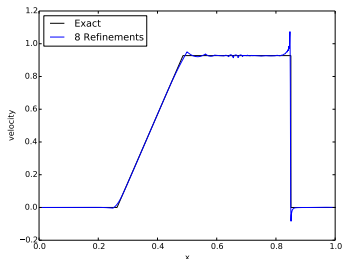
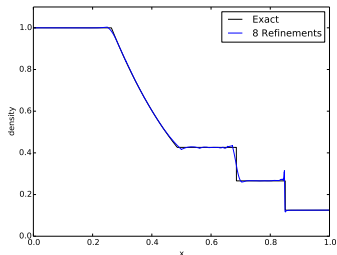
Compressible Navier-Stokes

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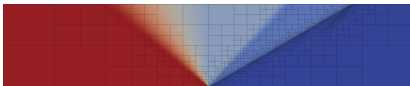
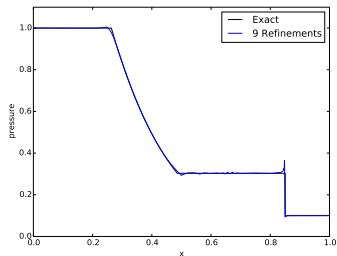
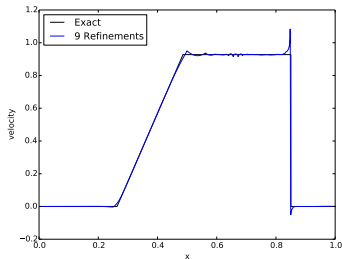
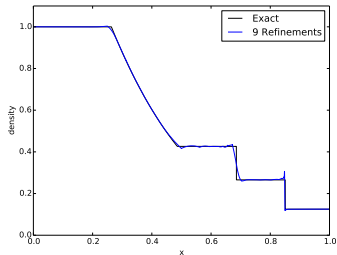
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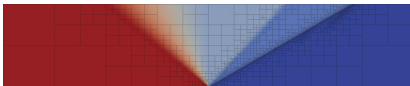
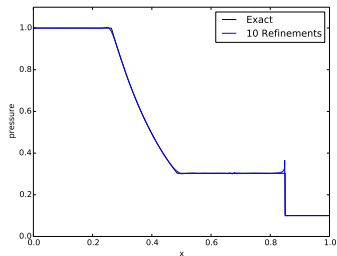
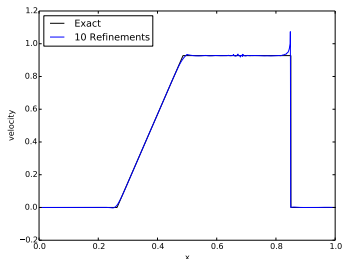
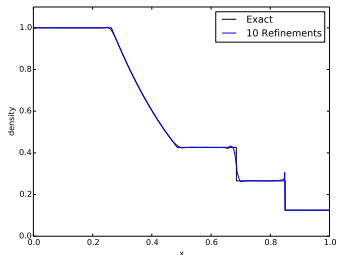
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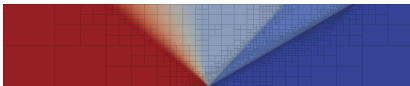
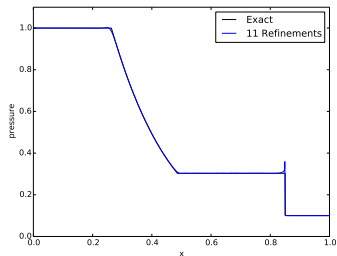
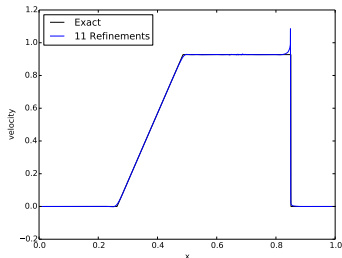
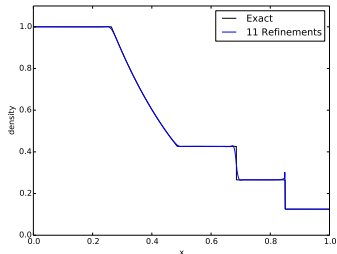
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



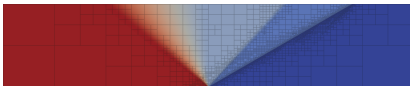
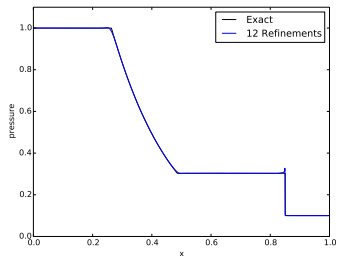
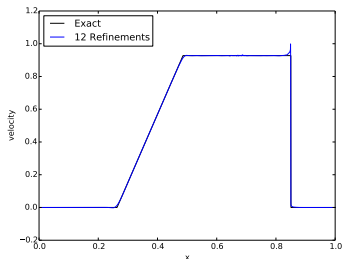
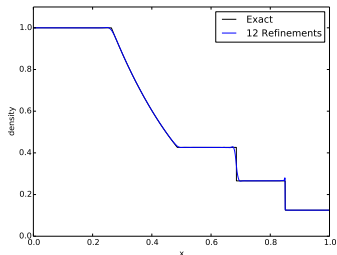
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



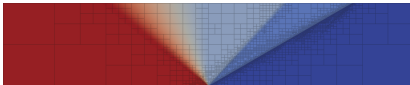
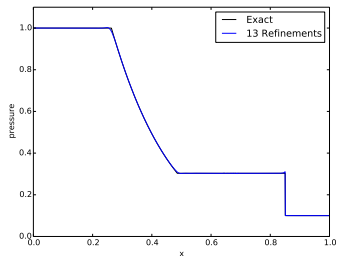
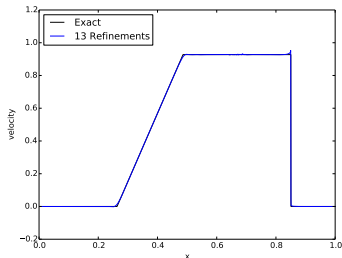
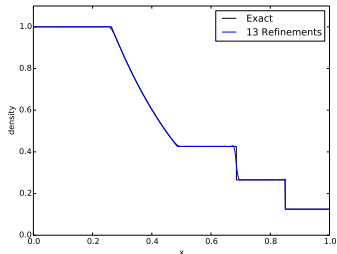
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



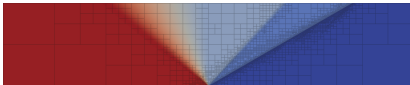
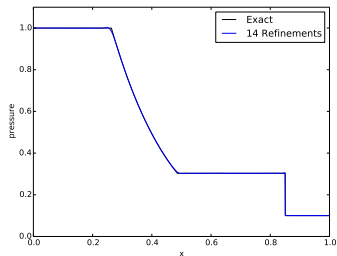
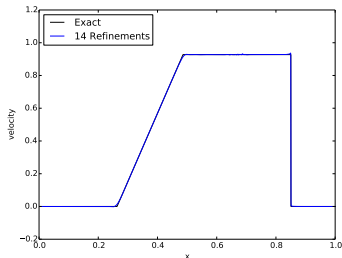
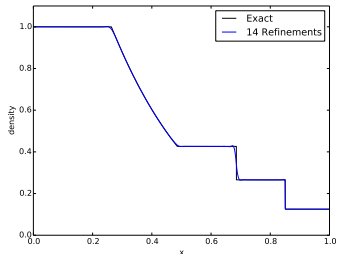
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



Compressible Navier-Stokes

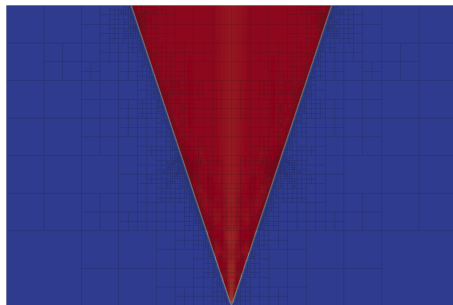
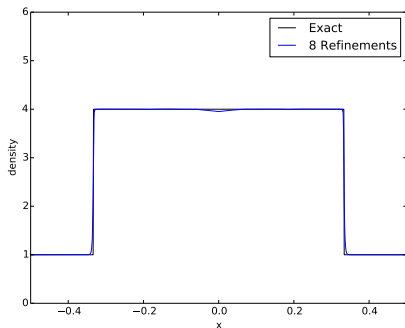
Sod Shock Tube with $\mu = 10^{-5}$



Compressible Navier-Stokes

Noh Implosion with $\mu = 10^{-3}$

Infinitely strong shock propagation.



Sequence of 4 time slabs

Past and Present Topics in DPG Research

- Multiphysics
 - Heat conduction (Poisson and Heat equation)
 - Wave problems (Helmholtz and Maxwell)
 - Linear elasticity and plate problems
 - Convection-Diffusion, Stokes, incompressible Navier-Stokes, compressible Navier-Stokes, Euler
- Natively nonlinear DPG
- DPG for non-Hilbert L^p spaces
- Local conservation
- Iterative solvers
- Entropy scaling for physically meaningful test norms
- General polyhedral elements

Thank You!

Recommended References

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- ▶ L.F. Demkowicz and N. Heuer. "Robust DPG Method for Convection-Dominated Diffusion Problems". In: *SIAM J. Numer. Anal.* 51.5 (2013), pp. 1514–2537.
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- ▶ N. Roberts, T. Bui-Thanh, and L. Demkowicz. "The DPG method for the Stokes problem". In: *Comp. Math. Appl.* 67.4 (2014). High-order Finite Element Approximation for Partial Differential Equations, pp. 966 –995.