

Space-Time Navier-Stokes with Entropy Variables

Truman E. Ellis

October 3, 2014

Primitive Variables

Consider the DPG Navier-Stokes derivation from previously with primitive variables:

$$\left(\frac{1}{\mu}\mathbb{D}, \mathbb{S}\right) + (2\mathbf{u}, \nabla \cdot \mathbb{S}) - \left(\frac{2}{3}\mathbf{u}, \nabla \operatorname{tr} \mathbb{S}\right) - \left\langle \frac{4}{3}\hat{\mathbf{u}}, \mathbb{S}\mathbf{n}_x \right\rangle = 0 \quad (1a)$$

$$\left(\frac{Pr}{C_p\mu}\mathbf{q}, \boldsymbol{\tau}\right) - (T, \nabla \cdot \boldsymbol{\tau}) + \langle \hat{T}, \tau_n \rangle = 0 \quad (1b)$$

$$- \left(\left(\begin{array}{c} \rho\mathbf{u} \\ \rho \end{array} \right), \nabla_{xt}v_c \right) + \langle \hat{t}_c, v_c \rangle = (f_c, v_c) \quad (1c)$$

$$- \left(\left(\begin{array}{c} \rho\mathbf{u} \otimes \mathbf{u} + \rho RT \mathbf{I} - \mathbb{D} \\ \rho\mathbf{u} \end{array} \right), \nabla_{xt}\mathbf{v}_m \right) + \langle \hat{\mathbf{t}}_m, \mathbf{v}_m \rangle = (\mathbf{f}_m, \mathbf{v}_m) \quad (1d)$$

$$- \left(\left(\begin{array}{c} \rho\mathbf{u} (C_v T + \frac{1}{2}\mathbf{u} \cdot \mathbf{u}) + \mathbf{u}\rho RT + \mathbf{q} - \mathbf{u} \cdot \mathbb{D} \\ \rho (C_v T + \frac{1}{2}\mathbf{u} \cdot \mathbf{u}) \end{array} \right), \nabla_{xt}v_e \right) + \langle \hat{t}_e, v_e \rangle = (f_e, v_e), \quad (1e)$$

where

$$\begin{aligned} \hat{\mathbf{u}} &= \operatorname{tr}(\mathbf{u}) \\ \hat{T} &= \operatorname{tr}(T) \\ \hat{t}_c &= \operatorname{tr}(\rho\mathbf{u}) \cdot \mathbf{n}_x + \operatorname{tr}(\rho) n_t \\ \hat{\mathbf{t}}_m &= \operatorname{tr}(\rho\mathbf{u} \otimes \mathbf{u} + \rho RT \mathbf{I} - \mathbb{D}) \cdot \mathbf{n}_x + \operatorname{tr}(\rho\mathbf{u}) n_t \\ \hat{t}_e &= \operatorname{tr} \left(\rho\mathbf{u} \left(C_v T + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} \right) + \mathbf{u}\rho RT + \mathbf{q} - \mathbf{u} \cdot \mathbb{D} \right) \cdot \mathbf{n}_x \\ &\quad + \operatorname{tr} \left(\rho \left(C_v T + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} \right) \right) n_t. \end{aligned}$$

Now define primitive fluxes for continuity, momentum, and energy equations:

$$\begin{aligned} \mathbf{F}_c^p &:= \rho\mathbf{u} \\ \mathbb{F}_m^p &:= \rho\mathbf{u} \otimes \mathbf{u} + \rho RT \mathbf{I} \\ \mathbf{F}_e^p &:= \rho\mathbf{u} \left(C_v T + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} \right) + \mathbf{u}\rho RT \end{aligned}$$

Our bilinear form is then simplified:

$$\left(\frac{1}{\mu}\mathbb{D}, \mathbb{S}\right) + (2\mathbf{u}, \nabla \cdot \mathbb{S}) - \left(\frac{2}{3}\mathbf{u}, \nabla \operatorname{tr} \mathbb{S}\right) - \left\langle \frac{4}{3}\hat{\mathbf{u}}, \mathbb{S}\mathbf{n}_x \right\rangle = 0 \quad (2a)$$

$$\left(\frac{Pr}{C_p\mu}\mathbf{q}, \boldsymbol{\tau}\right) - (T, \nabla \cdot \boldsymbol{\tau}) + \langle \hat{T}, \tau_n \rangle = 0 \quad (2b)$$

$$- \left(\left(\frac{\mathbf{F}_c^p}{\rho} \right), \nabla_{xt} v_c \right) + \langle \hat{t}_c, v_c \rangle = (f_c, v_c) \quad (2c)$$

$$- \left(\left(\frac{\mathbb{F}_m^p - \mathbb{D}}{\rho\mathbf{u}} \right), \nabla_{xt} \mathbf{v}_m \right) + \langle \hat{\mathbf{t}}_m, \mathbf{v}_m \rangle = (\mathbf{f}_m, \mathbf{v}_m) \quad (2d)$$

$$- \left(\left(\frac{\mathbf{F}_e^p + \mathbf{q} - \mathbf{u} \cdot \mathbb{D}}{\rho(C_v T + \frac{1}{2}\mathbf{u} \cdot \mathbf{u})} \right), \nabla_{xt} v_e \right) + \langle \hat{t}_e, v_e \rangle = (f_e, v_e) , \quad (2e)$$

Conservation Variables

Now we wish to do a change of variables to conservation variables:

$$\begin{aligned} \rho &= \rho \\ \mathbf{m} &= \rho\mathbf{u} \\ E &= \rho \left(C_v T + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} \right) \end{aligned}$$

We can define new fluxes in conservation variables:

$$\begin{aligned} \mathbf{F}_c^c &= \mathbf{m} \\ \mathbb{F}_m^c &= \frac{\mathbf{m} \otimes \mathbf{m}}{\rho} + (\gamma - 1) \left(E - \frac{\mathbf{m} \cdot \mathbf{m}}{2\rho} \right) \mathbf{I} \\ \mathbf{F}_e^c &= \frac{\mathbf{m}}{\rho} E + (\gamma - 1) \left(E - \frac{\mathbf{m} \cdot \mathbf{m}}{2\rho} \right) \frac{\mathbf{m}}{\rho} \end{aligned}$$

and our new bilinear form is

$$\left(\frac{1}{\mu}\mathbb{D}, \mathbb{S}\right) + \left(2\frac{\mathbf{m}}{\rho}, \nabla \cdot \mathbb{S}\right) - \left(\frac{2}{3}\frac{\mathbf{m}}{\rho}, \nabla \operatorname{tr} \mathbb{S}\right) - \left\langle \frac{4}{3}\hat{\mathbf{u}}, \mathbb{S}\mathbf{n}_x \right\rangle = 0 \quad (3a)$$

$$\left(\frac{Pr}{C_p\mu}\mathbf{q}, \boldsymbol{\tau}\right) - \left(\frac{E - \frac{1}{2\rho}\mathbf{m} \cdot \mathbf{m}}{C_v\rho}, \nabla \cdot \boldsymbol{\tau}\right) + \langle \hat{T}, \tau_n \rangle = 0 \quad (3b)$$

$$- \left(\left(\frac{\mathbf{F}_c^c}{\rho} \right), \nabla_{xt} v_c \right) + \langle \hat{t}_c, v_c \rangle = (f_c, v_c) \quad (3c)$$

$$- \left(\left(\frac{\mathbb{F}_m^c - \mathbb{D}}{\mathbf{m}} \right), \nabla_{xt} \mathbf{v}_m \right) + \langle \hat{\mathbf{t}}_m, \mathbf{v}_m \rangle = (\mathbf{f}_m, \mathbf{v}_m) \quad (3d)$$

$$- \left(\left(\frac{\mathbf{F}_e^c + \mathbf{q} - \frac{\mathbf{m}}{\rho} \cdot \mathbb{D}}{E} \right), \nabla_{xt} v_e \right) + \langle \hat{t}_e, v_e \rangle = (f_e, v_e) , \quad (3e)$$

Entropy Variables

Now we wish to do a change of variables to entropy variables:

$$\begin{aligned} V_c &= \frac{-E + (E - \frac{1}{2\rho} \mathbf{m} \cdot \mathbf{m}) \left(\gamma + 1 - \ln \left[\frac{(\gamma-1)(E - \frac{1}{2\rho} \mathbf{m} \cdot \mathbf{m})}{\rho^\gamma} \right] \right)}{E - \frac{1}{2\rho} \mathbf{m} \cdot \mathbf{m}} \\ \mathbf{V}_m &= \frac{\mathbf{m}}{E - \frac{1}{2\rho} \mathbf{m} \cdot \mathbf{m}} \\ V_e &= \frac{-\rho}{E - \frac{1}{2\rho} \mathbf{m} \cdot \mathbf{m}} \end{aligned}$$

with reverse mapping:

$$\begin{aligned} \rho &= -\alpha V_e \\ \mathbf{m} &= \alpha \mathbf{V}_m \\ E &= \alpha \left(1 - \frac{1}{2V_e} \mathbf{V}_m \cdot \mathbf{V}_m \right) \end{aligned}$$

where

$$\alpha(V_c, \mathbf{V}_m, V_e) = \left[\frac{\gamma - 1}{-(V_e)^\gamma} \right]^{\frac{1}{\gamma-1}} \exp \left[\frac{-\gamma + V_c - \frac{1}{2V_e} \mathbf{V}_m \cdot \mathbf{V}_m}{\gamma - 1} \right]$$

We can define new fluxes in entropy variables:

$$\begin{aligned} \mathbf{F}_c^e &= \alpha \mathbf{V}_m \\ \mathbb{F}_m^e &= -\alpha \frac{\mathbf{V}_m \otimes \mathbf{V}_m}{V_e} + \alpha(\gamma - 1) \mathbf{I} \\ \mathbf{F}_e^e &= -\alpha \frac{\mathbf{V}_m}{V_e} \left(1 - \frac{1}{2V_e} \mathbf{V}_m \cdot \mathbf{V}_m \right) - \alpha(\gamma - 1) \frac{\mathbf{V}_m}{V_e} \end{aligned}$$

and our new bilinear form is

$$\left(\frac{1}{\mu} \mathbb{D}, \mathbb{S} \right) - \left(2 \frac{\mathbf{V}_m}{V_e}, \nabla \cdot \mathbb{S} \right) + \left(\frac{2}{3} \frac{\mathbf{V}_m}{V_e}, \nabla \operatorname{tr} \mathbb{S} \right) - \left\langle \frac{4}{3} \hat{\mathbf{u}}, \mathbb{S} \mathbf{n}_x \right\rangle = 0 \quad (4a)$$

$$\left(\frac{Pr}{C_p \mu} \mathbf{q}, \boldsymbol{\tau} \right) + \left(\frac{1}{C_v V_e}, \nabla \cdot \boldsymbol{\tau} \right) + \langle \hat{T}, \tau_n \rangle = 0 \quad (4b)$$

$$- \left(\left(\begin{array}{c} \mathbf{F}_c^e \\ \rho \end{array} \right), \nabla_{xt} v_c \right) + \langle \hat{t}_c, v_c \rangle = (f_c, v_c) \quad (4c)$$

$$- \left(\left(\begin{array}{c} \mathbb{F}_m^e - \mathbb{D} \\ \mathbf{m} \end{array} \right), \nabla_{xt} \mathbf{v}_m \right) + \langle \hat{\mathbf{t}}_m, \mathbf{v}_m \rangle = (\mathbf{f}_m, \mathbf{v}_m) \quad (4d)$$

$$- \left(\left(\begin{array}{c} \mathbf{F}_e^e + \mathbf{q} + \frac{\mathbf{V}_m}{V_e} \cdot \mathbb{D} \\ E \end{array} \right), \nabla_{xt} v_e \right) + \langle \hat{t}_e, v_e \rangle = (f_e, v_e) , \quad (4e)$$