Notes on Space-Time for the Heat Equation

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For the following discussion, let (master) element K be a tensor product of a spatial component, X, and a time component, T. Let ∇ denote the spatial gradient, and $\frac{\partial}{\partial t}$ denote the time derivative.

The heat equation is

$$\frac{\partial u}{\partial t} - \epsilon \Delta u = f$$

As a first order system, this is

$$\begin{aligned} \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= 0 \\ \frac{\partial u}{\partial t} - \nabla \cdot \boldsymbol{\sigma} &= f \end{aligned}$$

From this formulation, we can deduce that

$$u, \boldsymbol{\sigma}, \nabla u, \frac{\partial u}{\partial t} - \nabla \cdot \boldsymbol{\sigma} \in L^2$$

If we view $\{-\boldsymbol{\sigma}, u\} := \mathbf{U}$ as a group variable, then the last condition tells us that $\mathbf{U} \in H(\mathrm{div}, K)$. In particular, this means that across a constant time interface (horizontal lines in Figure 1 and horizontal planes in Figure 3), the normal component, u should be continuous.

If we take the typical volume-centered, face-centered, edge-centered, node-centered approach, then we can see that u should have a volume-centered component, all face-centered components, and edge-centered components on non-horizontal edges. Of course, when we move to the DPG formulation, u takes only the volume-centered basis, while the trace $\hat{u}$ takes everything else.

Let $\mathbf{n} = (\mathbf{n_x}, n_t)^T$ be the full space-time normal vector where $\mathbf{n_x}$ is the spatial component and n_t is the temporal component.

Multiplying by test functions $\boldsymbol{\tau}$ and v, and integrating by parts

$$\begin{aligned} -\left(u, \frac{\partial v}{\partial t}\right) + \langle \hat{u}, v \cdot n_t \rangle + (\boldsymbol{\sigma}, \nabla v) - \left\langle \widehat{\boldsymbol{\sigma} \cdot \mathbf{n_x}}, v \right\rangle \\ + \left(\frac{1}{\epsilon}\boldsymbol{\sigma}, \boldsymbol{\tau}\right) + (u, \nabla \cdot \boldsymbol{\tau}) - \langle \hat{u}, \boldsymbol{\tau} \cdot \mathbf{n_x} \rangle = (f, v) \end{aligned}$$

Let's examine these terms one at a time. Volume terms 1, 3, 5, and 6 are just as they are for pure spatial DPG with the exception that vector quantities are of the dimension of the spatial dimension only. Term 2 will degenerate on constant space (as a function of time, vertical) boundaries. Terms 4 and 7 will degenerate on constant time (as a function of space, horizontal) boundaries.

Now we analyze each variable in turn. It is easy to see that u is a simple scalar valued, volume-centered L^2 variable on each space-time element. $\boldsymbol{\sigma}$ is a vector valued, volume-centered $\mathbf{L}^2$ variable with dimension equal to the spatial dimension.

Now for the trace and flux variables. $\hat{u}$ is probably the most complicated variable in the system. It will obviously have face-centered components on all faces, but it also needs to maintain full connectivity in spatial directions, necessitating the need for edge-centered components on non-horizontal edges. The face-centered bases have support only on a shared face between two elements. The edge-centered bases on the other hand have support on all faces connected to that edge. $\hat{f}$ is much simpler by comparison. It only has scalar face-centered components on non-horizontal faces.

Finally, we need to discuss the test functions. v will need to be full $H^1(K)$, while $\boldsymbol{\tau}$ will need to live in a tensor product space $H(\text{div}, X) \times L^2(T)$.

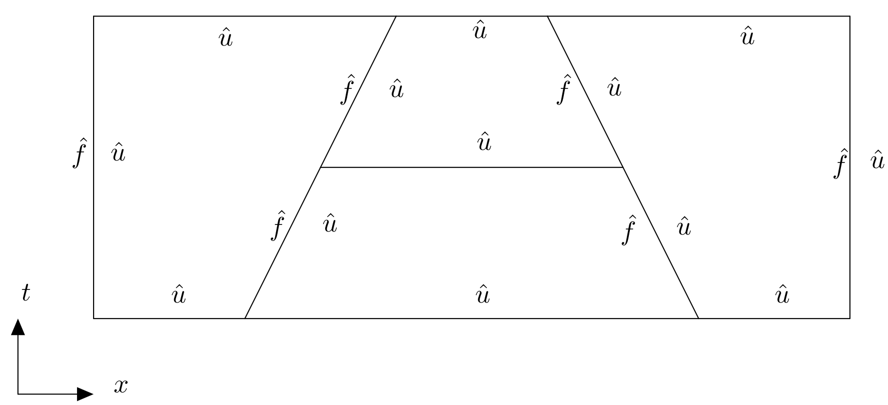

Figure 1: Allowable refinement pattern illustrating support of trace and flux variables

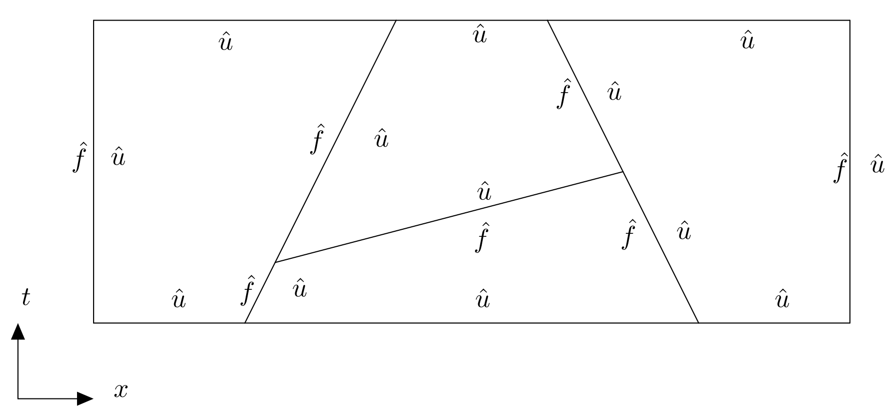

Figure 2: Questionable refinement pattern - as the division approaches horizontal, $\hat{f}$ degenerates

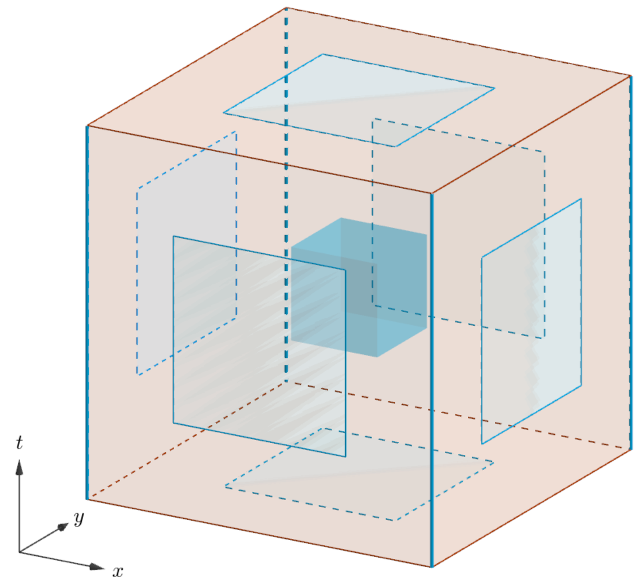

Figure 3: Support for u and $\hat{u}$ basis functions

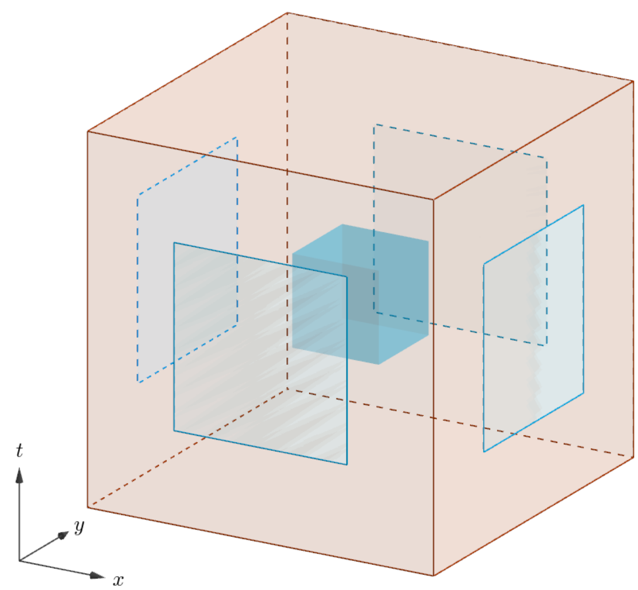

Figure 4: Support for $\boldsymbol{\sigma}$ and $\hat{f}$ basis functions