Discontinuous Petrov-Galerkin (DPG) Method With Optimal Test Functions

Progress Report

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THE UNIVERSITY OF TEXAS AT AUSTIN

INSTITUTE FOR COMPUTATIONAL ENGINEERING & SCIENCES

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Three DPG Punchlines

1. DPG Method is a Ritz method. It supports adaptivity with no preasymptotic behavior.
2. You can control the norm in which you want to converge.
3. DPG is easy to code.

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Primal DPG Method for Maxwell equations.

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Assume

$$J_S^{\text{imp}} = n \times H^{imp}$$

and look for the unknown surface current on the skeleton also in the same form.

$$\left\{ \begin{array}{l} E \in H(\text{curl}, \Omega), \; n \times E = n \times E^{\text{imp}} \text{ on } \Gamma_1 \\ \hat{h} \in \text{tr}_{\Gamma_h} H(\text{curl}, \Omega), \; n \times \hat{h} = n \times (-i\omega H^{imp}) \text{ on } \Gamma_2 \\ (\frac{1}{\mu} \boldsymbol{\nabla} \times E, \boldsymbol{\nabla}_h \times F) + ((-\omega^2 \epsilon + i\omega\sigma) E, F) + \langle n \times \hat{h}, F \rangle_{\Gamma_h} = -i\omega (J^{\text{imp}}, F) \\ \qquad \forall F \in H(\text{curl}, \Omega_h) \, . \end{array} \right.$$

FE discretization for curl-curl problem

Hexahedral meshes

$H(\text{curl})$ element for electric field E:

$$(\mathcal{P}^{p-1} \otimes \mathcal{P}^{p} \otimes \mathcal{P}^{p}) \times (\mathcal{P}^{p} \otimes \mathcal{P}^{p-1} \otimes \mathcal{P}^{p}) \times (\mathcal{P}^{p} \otimes \mathcal{P}^{p} \otimes \mathcal{P}^{p-1})$$

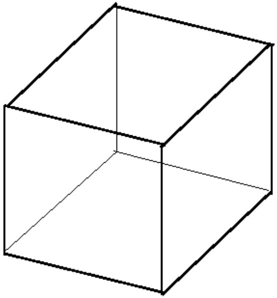
and trace of the same element for flux (surface current) $\hat{h}$.

Same element for the enriched space but with order $p + \Delta p$.

In reported experiments: $p = 2$, $\Delta p = 2$.

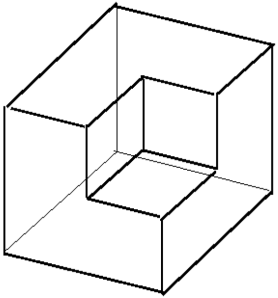
A 3D Maxwell example

Take a cube $(0, 2)^3$



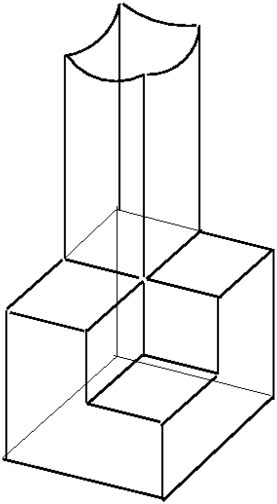
Fichera corner

Divide it into eight smaller cubes and remove one:



Fichera corner microwave

Attach a waveguide:



$\epsilon = \mu = 1, \sigma = 0$

$\omega = 5$(1.6 wavelengths in the cube)

Cut the waveguide and use the lowest propagating mode for BC along the cut.

Fichera corner microwave, $p = 2$.

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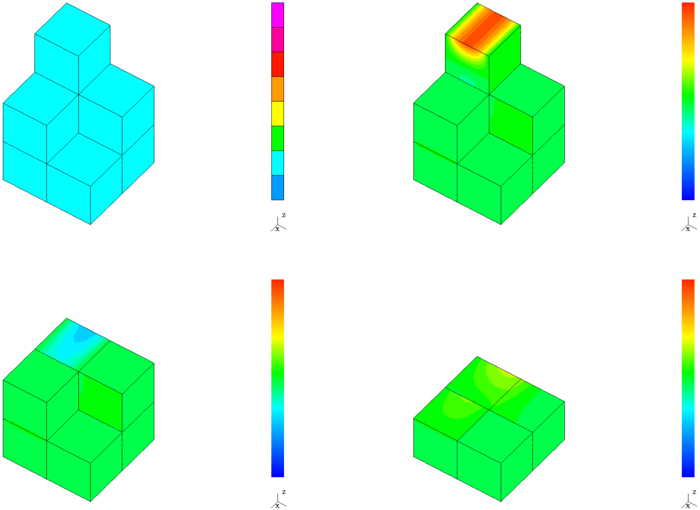
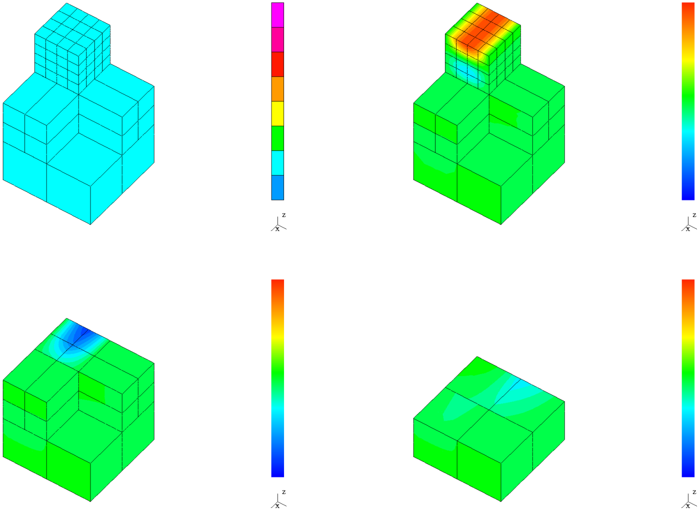


Initial mesh and real part of E_1

Fichera corner microwave, $p = 2$.



Mesh and real part of E_1 after two refinements

Fichera corner microwave, $p = 2$.

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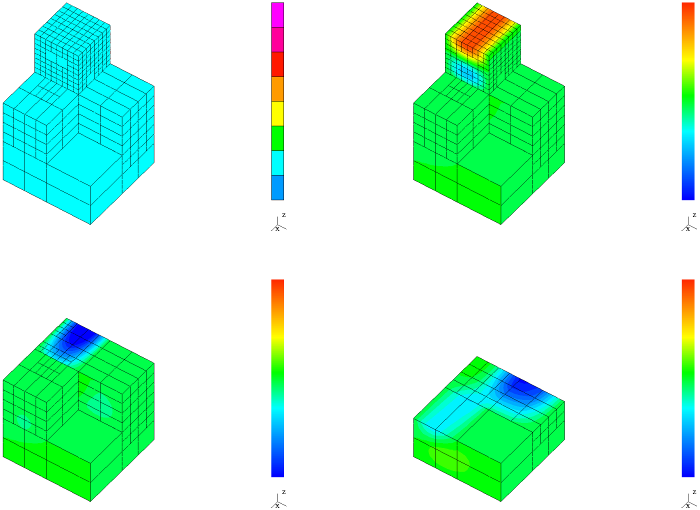
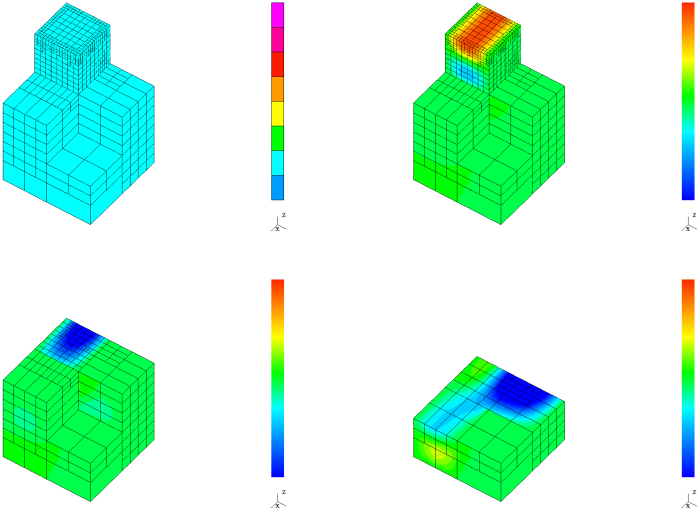


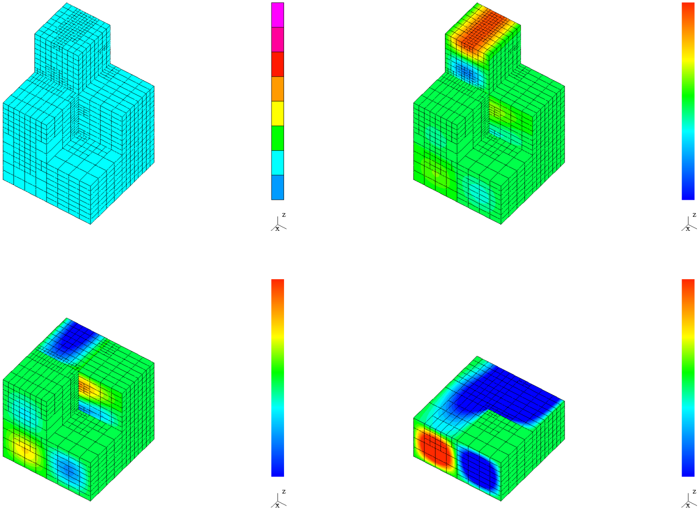
Mesh and real part of E_1 after four refinements

Fichera corner microwave, $p = 2$.



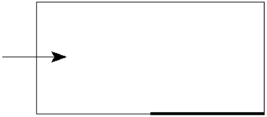
Mesh and real part of E_1 after six refinements

Fichera corner microwave, $p = 2$.



Mesh and real part of E_1 after eight refinements

From Ph.D. Dissertation of Jesse Chan: Compressible Navier-Stokes Equations: Carter's flat plate problem [1]

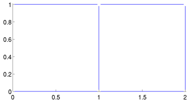


$$\mathsf{M}_\infty = 3, \mathsf{Re}_L = 1000, \mathsf{Pr} = 0.72, \gamma = 1.4, \theta_\infty = 390^o[\mathsf{R}]$$

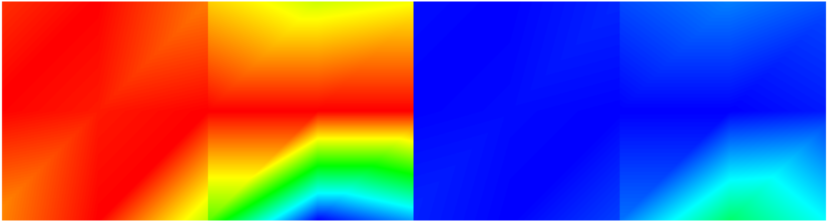
[1] L.D., J.T. Oden, W. Rachowicz, ``A New Finite Element Method for Solving Compressible Navier-Stokes Equations Based on an Operator Splitting Method and hp Adaptivity,'', *Comput. Methods Appl. Mech. Engrg.*, **84**, 275-326, 1990.

Extrapolation to Compressible NS Equations

Initial Mesh ($p = 2$):

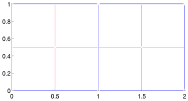


Horizontal velocity and temperature

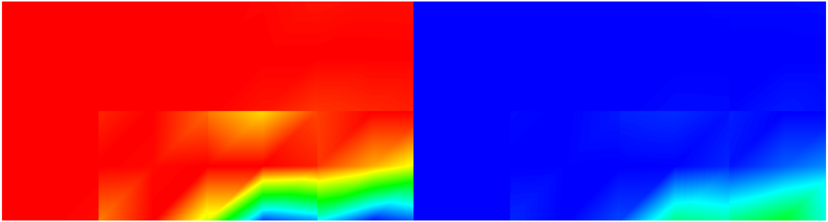


Extrapolation to Compressible NS Equations

Mesh 1:

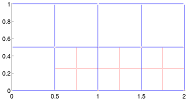


Horizontal velocity and temperature

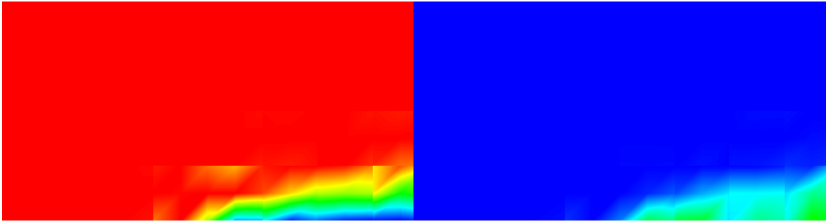


Extrapolation to Compressible NS Equations

Mesh 2:

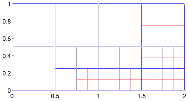


Horizontal velocity and temperature

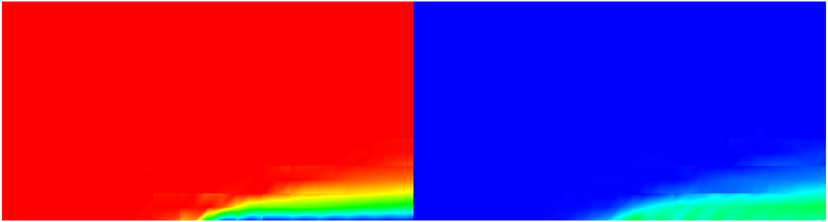


Extrapolation to Compressible NS Equations

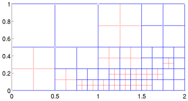
Mesh 3:



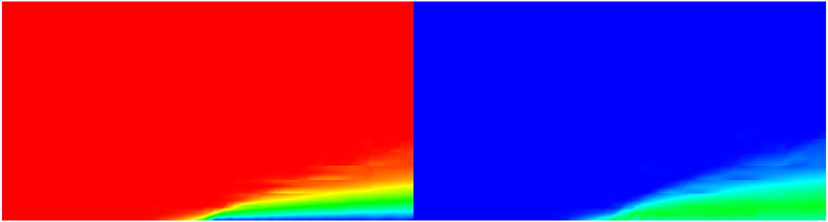
Horizontal velocity and temperature



Mesh 4:

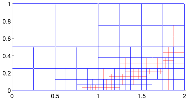


Horizontal velocity and temperature

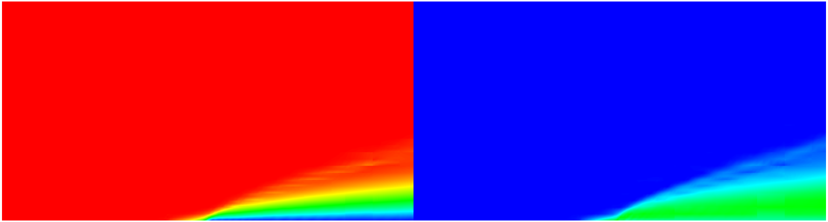


Extrapolation to Compressible NS Equations

Mesh 5:

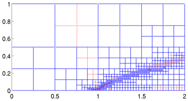


Horizontal velocity and temperature

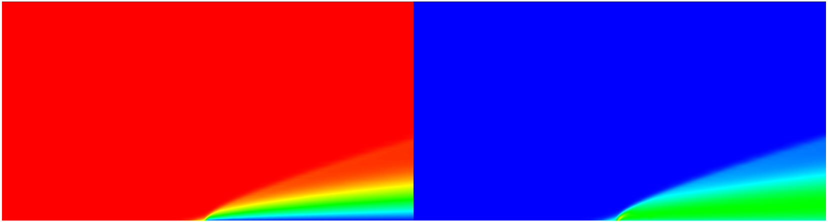


Extrapolation to Compressible NS Equations

Mesh 7:

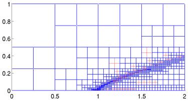


Horizontal velocity and temperature

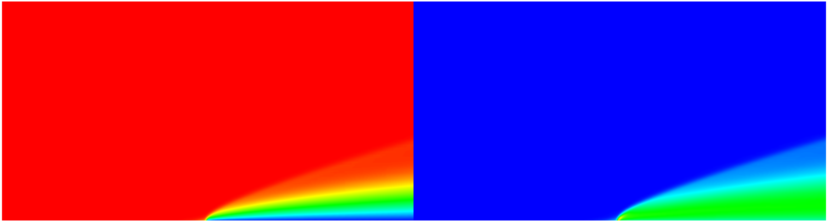


Extrapolation to Compressible NS Equations

Mesh 8:

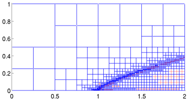


Horizontal velocity and temperature

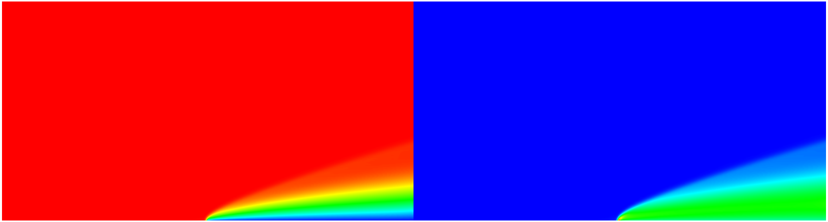


Extrapolation to Compressible NS Equations

Mesh 9:

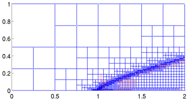


Horizontal velocity and temperature

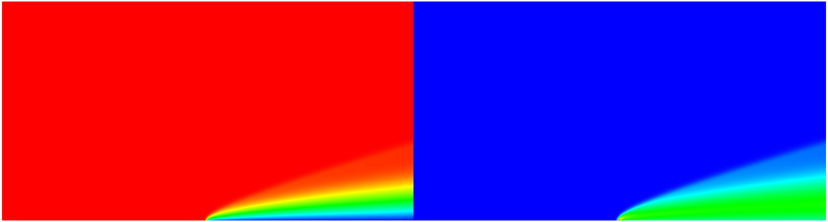


Extrapolation to Compressible NS Equations

Mesh 10:



Horizontal velocity and temperature



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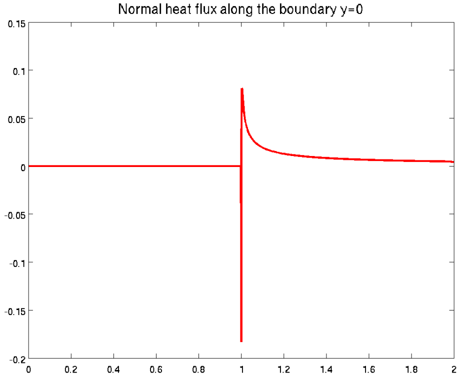


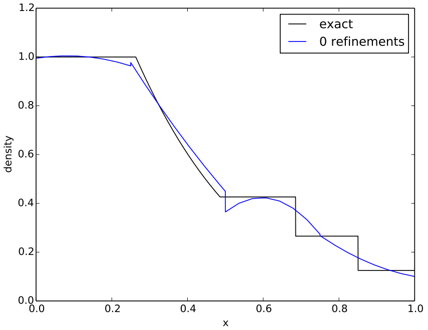
Heat flux along the plate

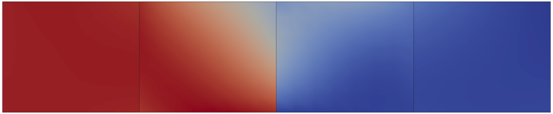
From Ph.D. Proposal of Truman Ellis: Space-Time Compressible Navier-Stokes Equations: Sod Shock Tube Problem [2]

[2] G.A. Sod, ``A Survey of Several Finite Difference Methods for Systems of Nonlinear Hyperbolic Conservation Laws'', *J. Comp. Methods Phys.*, **27**, 1-31, 1978.

Transient Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$





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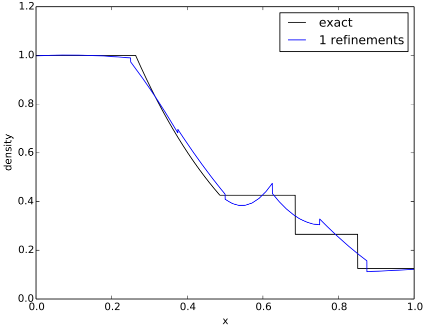

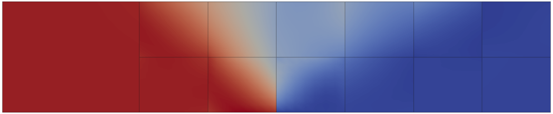

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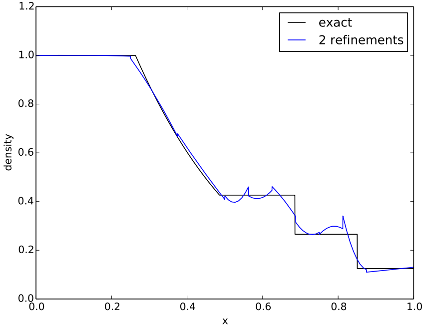

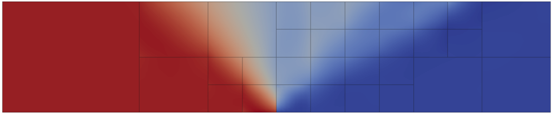

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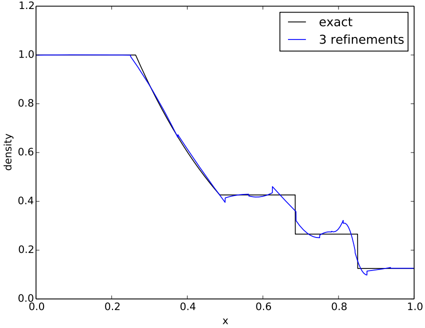

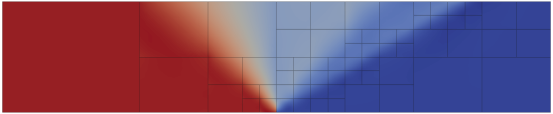

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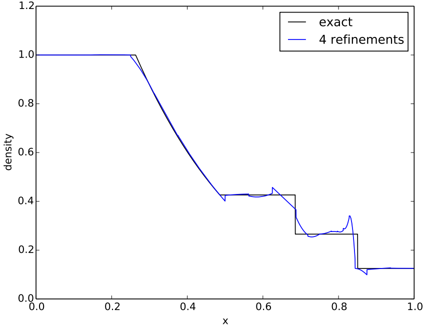

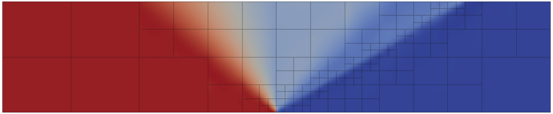

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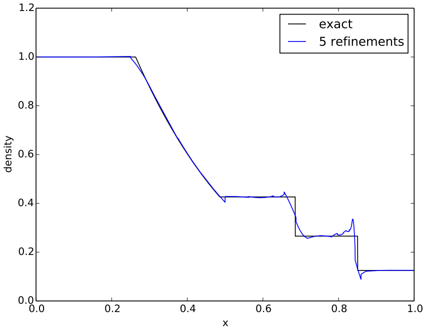

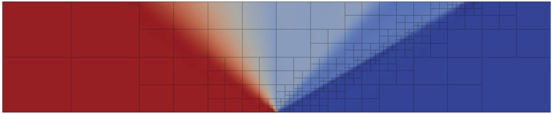

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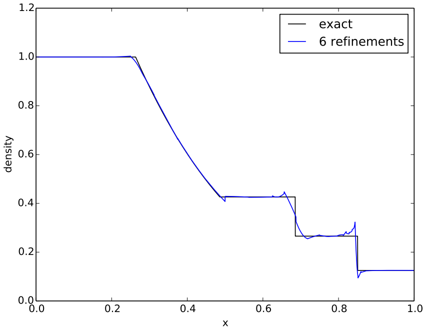

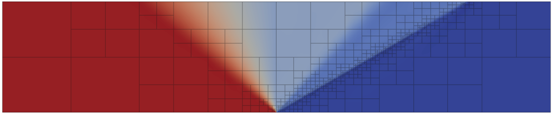

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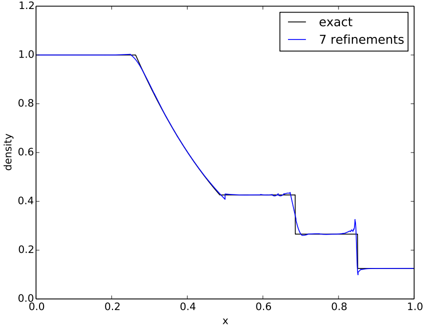

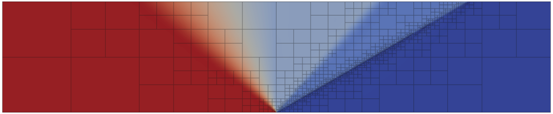

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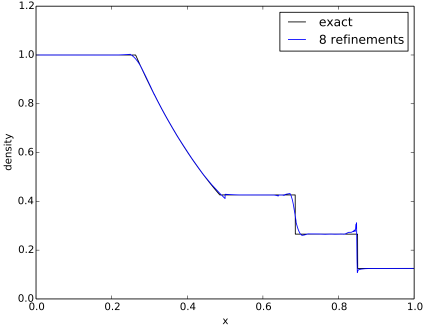

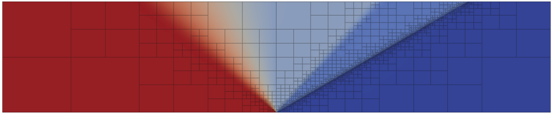

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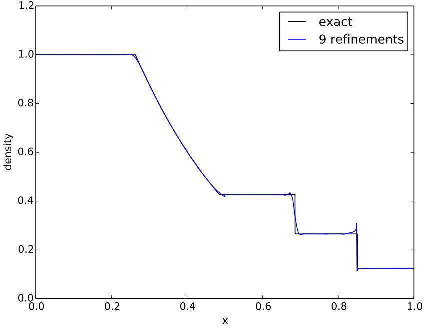

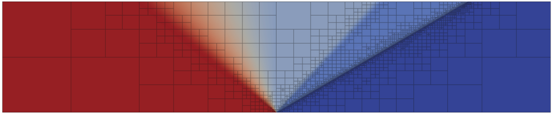

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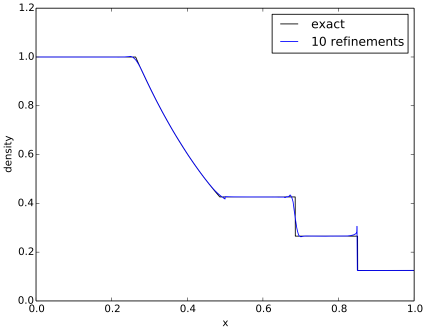

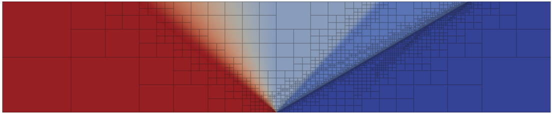

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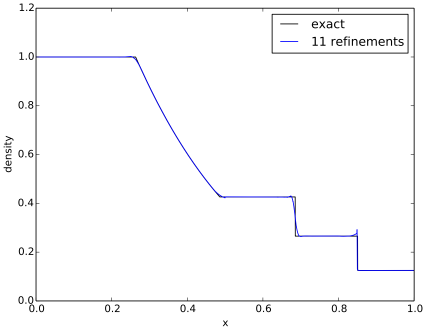

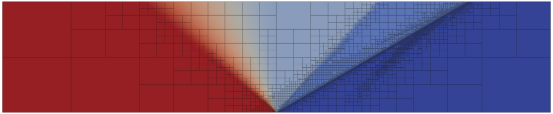

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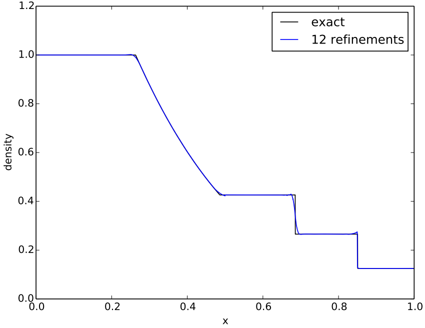

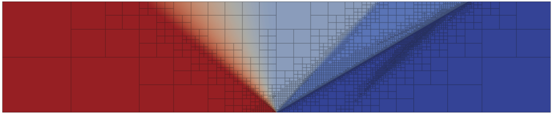

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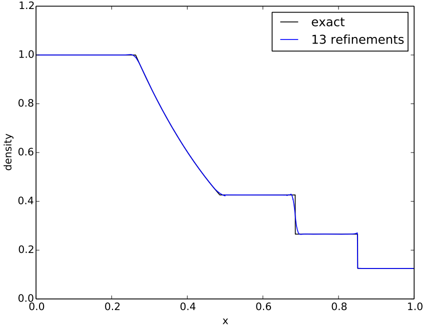
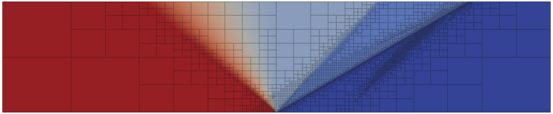
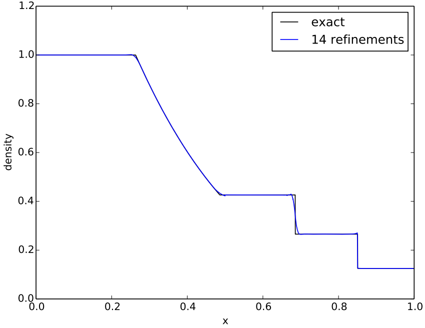
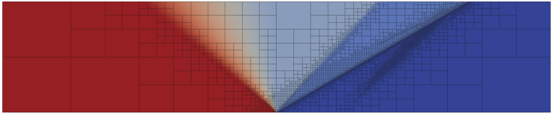




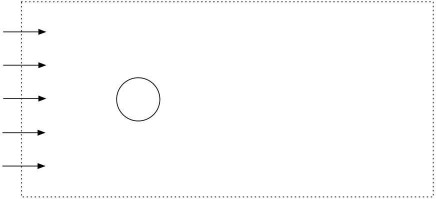
Transient Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$





From Ph.D. Dissertation of Nathan V. Roberts: Incompressible Navier-Stokes: Flow Past a Cylinder[3]

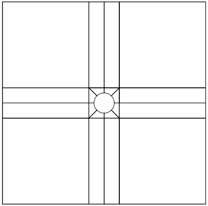


$[\mathrm{Re}_D = 40]$

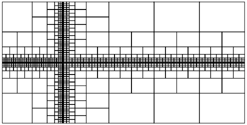
[3] L.S.G. Kovasznay. ``Hot-wire investigation of the wake behind cylinders at low Reynolds numbers.'' *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, **198**(1053):174-190, 1949.

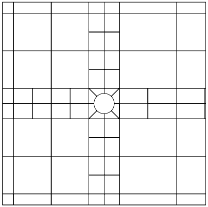
Incompressible Navier-Stokes

Domain: 240×120 cylinder diameters. Begin with a $p = 3$ preliminary mesh that simply captures geometry (detail):



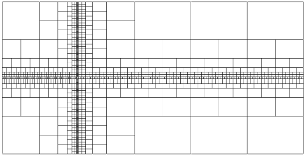
Perform initial anisotropic refinements to get aspect ratios below 2:

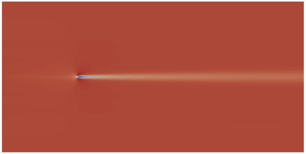




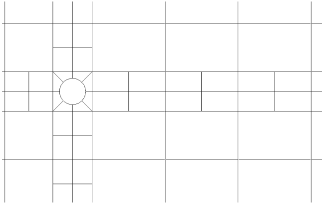
Incompressible Navier-Stokes

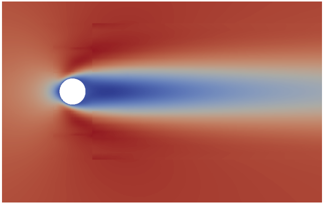
Initial Mesh ($p = 3$), and horizontal velocity solution:





Mesh detail, and horizontal velocity solution

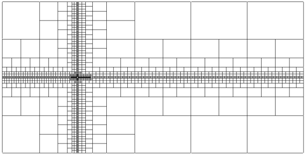


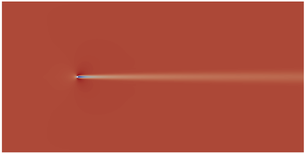


Relative energy error of solution: 9.76×10^{-2}.

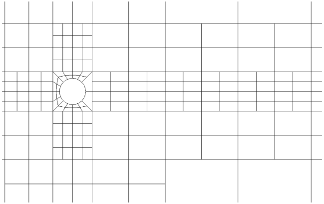
Incompressible Navier-Stokes

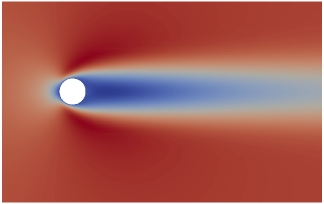
Mesh after 1 h-refinement, and horizontal velocity solution:





Mesh detail, and horizontal velocity solution

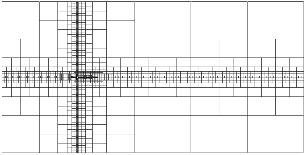


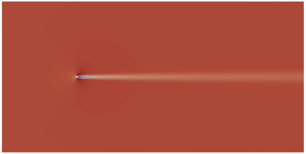


Relative energy error of solution: 1.81×10^{-2}.

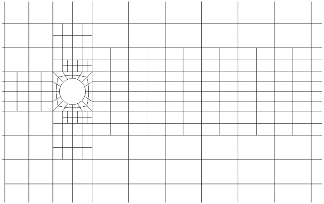
Incompressible Navier-Stokes

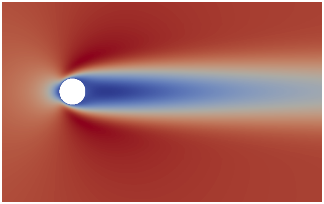
Mesh after 2 h-refinements, and horizontal velocity solution:





Mesh detail, and horizontal velocity solution

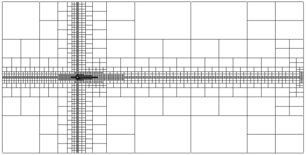


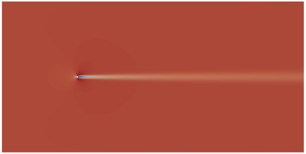


Relative energy error of solution: 3.26×10^{-3}.

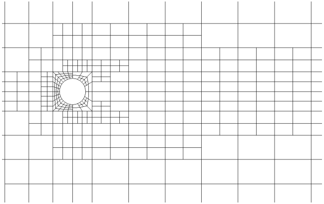
Incompressible Navier-Stokes

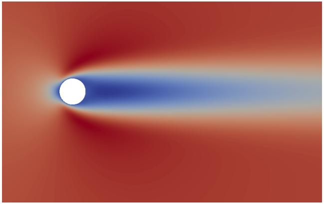
Mesh after 3 h-refinements, and horizontal velocity solution:





Mesh detail, and horizontal velocity solution

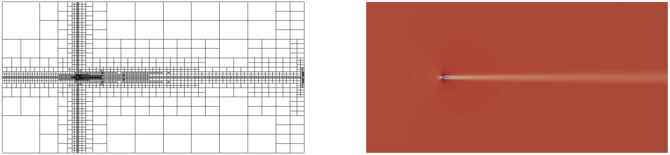




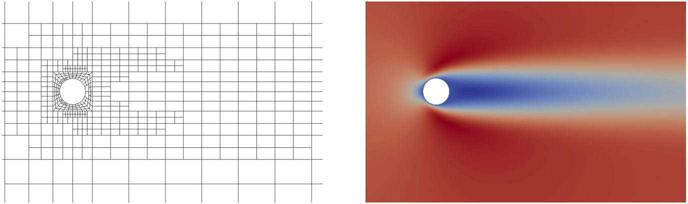
Relative energy error of solution: 1.29×10^{-3}.

Incompressible Navier-Stokes

Mesh after 4 h-refinements, and horizontal velocity solution:



Mesh detail, and horizontal velocity solution



Relative energy error of solution: 5.74×10^{-4}.

Chan, Ellis, and Roberts results were computed using

Camellia: A Software Framework for DPG[4]

Camellia:

- is easy to use (Stokes cavity flow h-adaptive driver takes $\approx$ 130 lines of code)
- supports MPI, h- and p- adaptivity
- built atop Trilinos
- supports 1-3D spatial meshes (1D and 3D just added)
- space-time meshes coming soon
- open source release planned for 2015

[4] Nathan V. Roberts. ``Camellia: A software framework for discontinuous Petrov-Galerkin methods.'' *Computers & Mathematics with Applications*, 2014 (in review).

Three DPG Punchlines

1. DPG Method is a Ritz method. It supports adaptivity with no preasymptotic behavior.
2. You can control the norm in which you want to converge.
3. DPG is easy to code.

The simplest singular perturbation problem: reaction-dominated diffusion [5]

[5] L.D. and I. Harari, ``Primal DPG Method for Reaction dominated Diffusion'', in preparation.

The simplest singular perturbation problem:

Reaction-dominated diffusion

$$\left\{\begin{array}{rll} u & = 0 & \text{on } \Gamma \\ -\epsilon^2 \Delta u + c(x) u & = f & \text{in } \Omega \end{array}\right.$$

where $0 < c_0 \leq c(x) \leq c_1$.

Standard variational formulation:

$$\left\{\begin{array}{l} u \in H^1(\Omega) \\ \\ \epsilon^2 (\nabla u, \nabla v) + (cu, v) = (f, v) \quad v \in H^1(\Omega) \end{array}\right.$$

Standard Galerkin method delivers the best approximation error in the energy norm:

$$\|u\|^2_{\epsilon^k} := \epsilon^k \|\nabla u\|^2 + \|c^{1/2} u\|^2, \quad k = 2$$

Convergence in ``balanced'' norm

Fact: Under favorable regularity conditions, the solution is *uniformly* bounded in data f in a ``balanced'' norm [6] :

$$\|u\|_\epsilon^2 := \epsilon \|\nabla u\|^2 + \|c^{1/2}u\|^2$$

i.e.

$$\|u\|_\epsilon \lesssim \|f\|_{\text{appropriate}}$$

Question: Can we select the test norm in such a way that the DPG method will be *robust* in the balanced norm ?

$$\|u - u_h\|_\epsilon + \|\hat{t} - \hat{t}_h\|_? \lesssim \inf_{w_h} \|u - w_h\|_\epsilon + \inf_{\hat{r}_h} \|\hat{t} - \hat{r}_h\|_?$$

[6] R. Lin and M. Stynes, ``A balanced finite element method for singularly perturbed reaction-diffusion problems'', *SIAM J. Numer. Anal.*, **50(5):** 2729--2743, 2012.

A bit of history:

Optimal test functions of Barret and Morton [7] [8]

For each $w \in U_h$, determine the corresponding v_w that solves the auxiliary variational problem:

$$\left\{ \begin{array}{l} v_w \in H_0^1(\Omega) \\ \underbrace{\epsilon^2(\boldsymbol{\nabla}\delta u, \boldsymbol{\nabla} v_w) + (c\,\delta u, v_w)}_{\text{the bilinear form we have}} = \underbrace{\epsilon(\boldsymbol{\nabla}\delta u, w) + (c\,\delta u, w)}_{\text{the bilinear form we want}} \quad \forall \delta u \in H_0^1(\Omega) \end{array} \right.$$

With the optimal test functions, the Galerkin orthogonality for the original form changes into Galerkin orthogonality in the desired, ``balanced'' norm:

$$\epsilon^2(\boldsymbol{\nabla}(u-u_h), \boldsymbol{\nabla} v_w) + (c\,(u-u_h), v_w) = 0 \quad \Longrightarrow \quad \epsilon(\boldsymbol{\nabla}(u-u_h), \boldsymbol{\nabla} v_u) + (c\,(u-u_h), w) = 0$$

Consequently, the PG solution delivers the best approximation error in the desired norm.

[7] J.W. Barret and K. W. Morton, ``Approximate Symmetrization and Petrov-Galerkin Methods for Diffusion-Convection Problems'', *Comp. Meth. Appl. Mech and Engng.*, **46**, 97 (1984).

[8] L. D. and J. T. Oden, ``An Adaptive Characteristic Petrov-Galerkin Finite Element Method for Convection-Dominated Linear and Nonlinear Parabolic Problems in One Space Variable'', *Journal of Computational Physics*, **68(1):** 188--273, 1986.

Constructing optimal test norm [9]

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Theorem

Let v_u be the Barret-Morton optimal test function corresponding to u. Let $\|v_u\|_V$ be a test norm such that

$$\|v_u\|_V \lesssim \|u\|_\epsilon$$

Then

$$\|u - u_h\|_\epsilon \lesssim \|u - u_h\|_E = \inf_{w_h \in U_h} \|u - w_h\|_E \leq \text{BAE estimate}$$

Proof:

$$\begin{aligned}
\|u\|_\epsilon^2 &= \epsilon(\nabla u, \nabla u) + (cu, u) = \epsilon^2(\nabla u, \nabla v_u) + (cu, v_u) \\
&= b((u, \hat{t}), v_u) \leq \frac{b((u,\hat{t}), v_u)}{\|v_u\|_V} \|v_u\|_V \\
&\leq \sup_v \frac{b((u,\hat{t}), v_u)}{\|v\|_V} \|v_u\|_V = \|(u, \hat{t})\|_E \|v_u\|_V \\
&\lesssim \|(u, \hat{t})\|_E \|u\|_\epsilon
\end{aligned}$$

[9] L. D., M. Heuer, ``Robust DPG Method for Convection-Dominated Diffusion Problems,'' *SIAM J. Num. Anal*, **51:** 2514--2537, 2013.

Constructing optimal test norm

The point: Construction of the optimal test norm is reduced to the stability (robustness) analysis for the Barret-Morton test functions.

Lemma

Let

$$\|v\|_V^2 := \epsilon^3 \|\nabla v\|^3 + \|c^{1/2} v\|^2$$

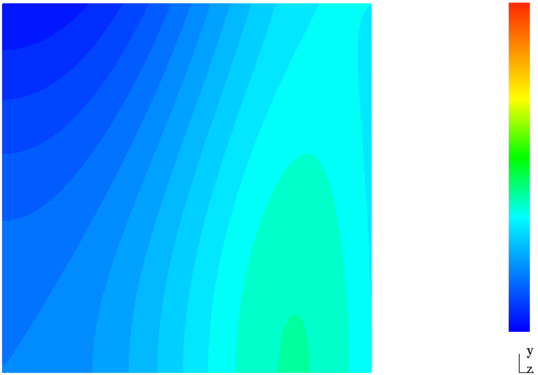
Then

$$\|v_u\| \lesssim \|u\|_\epsilon$$

In order to avoid boundary layers in the optimal test functions (that we cannot resolve using simple enriched space) we scale the reaction term with a mesh-dependent factor:

$$\|v\|_{V,mod}^2 := \epsilon^3 \|\nabla v\|^3 + \min\{1, \frac{\epsilon^3}{h^2}\} \|c^{1/2} v\|^2$$

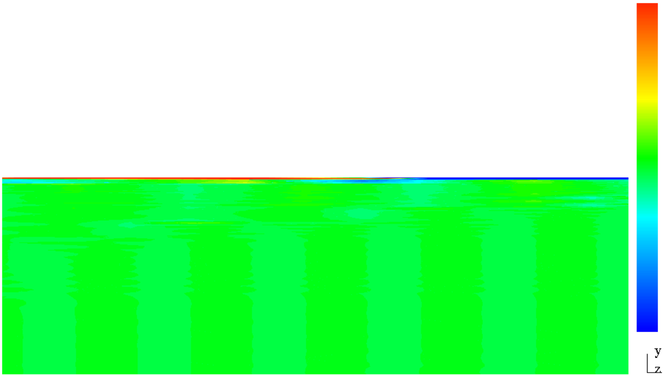
Manufactured solution of Lin and Stynes, $\epsilon = 10^{-1}$



The functions exhibits strong boundary layers invisible in this scale.

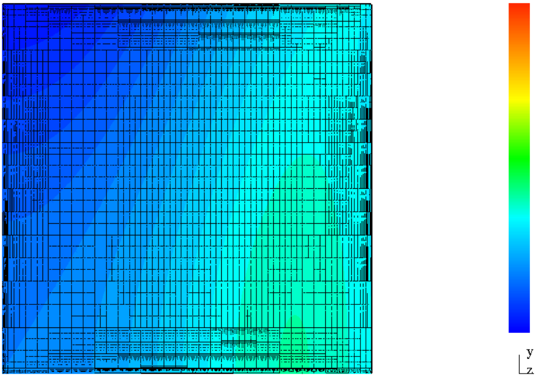
Range: (-0.6,0.6)

Manufactured solution of Lin and Stynes, $\epsilon = 10^{-1}$



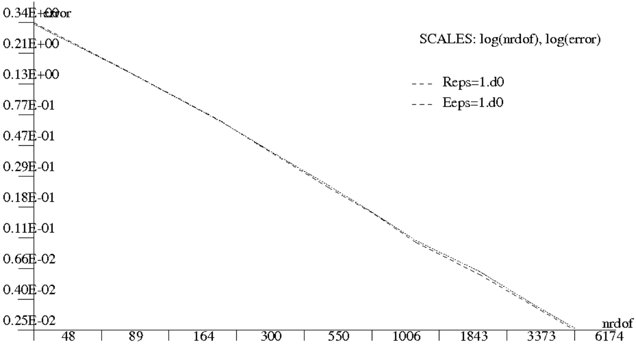
Zoom on the north boundary layer.

Optimal mesh for $\epsilon = 10^{-1}$



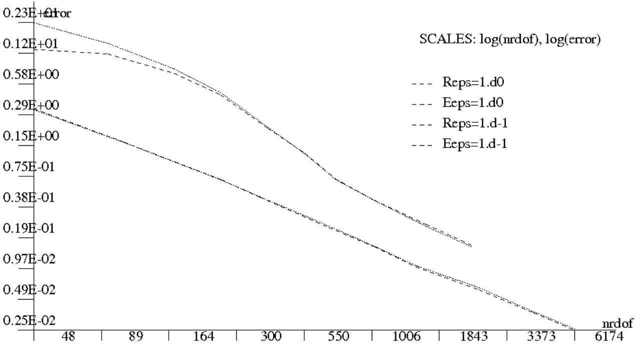
Optimal h-adaptive mesh and numerical solution for $\epsilon = 10^{-1}$

Lin/Stynes example, $\epsilon = 1$



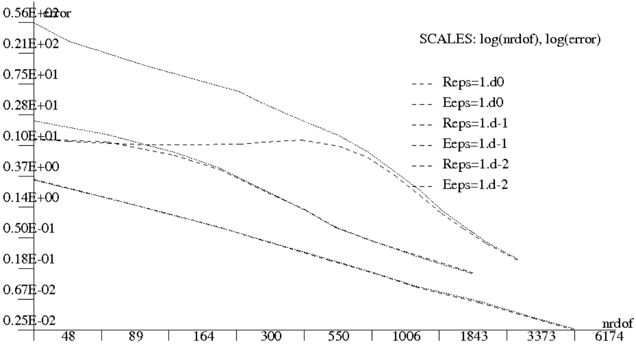
Residual and ``balanced'' error of u for h-adaptive solution, $p = 2$

Lin/Stynes example, $\epsilon = 10^0, 10^{-1}$



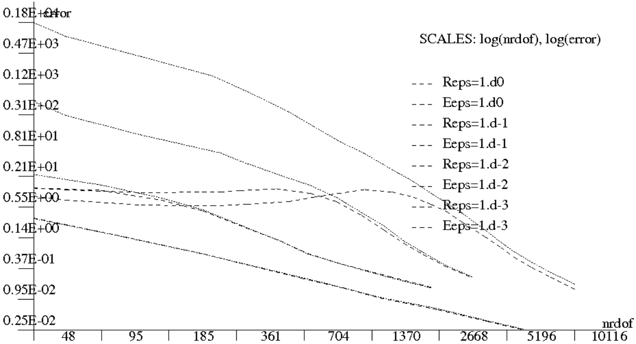
Residual and ``balanced'' error of u for h-adaptive solution, $p = 2$

Lin/Stynes example, $\epsilon = 10^0, 10^{-1}, 10^{-2}$.



Residual and "balanced" error of u for h-adaptive solution, $p = 2$

Lin/Stynes example, $\epsilon = 10^0, 10^{-1}, 10^{-2}, 10^{-3}$.



Residual and ``balanced'' error of u for h-adaptive solution, $p = 2$

Lin/Stynes example, $\epsilon = 10^0, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$

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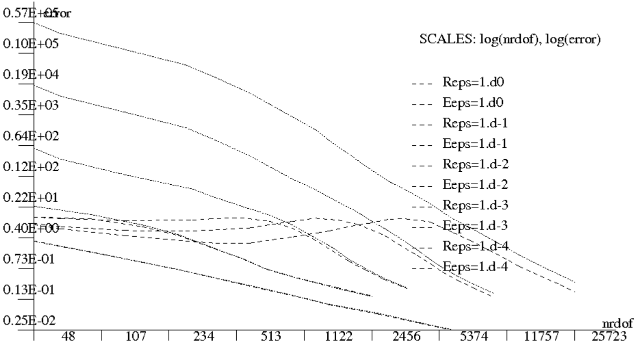
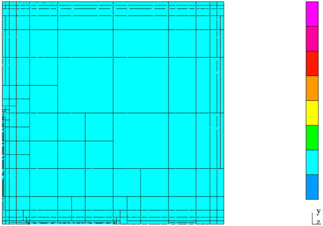
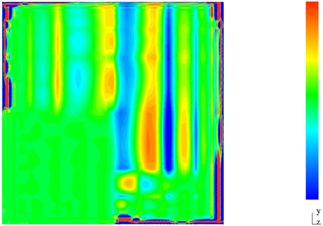


Residual and ``balanced'' error of u for h-adaptive solution, $p = 2$

Other tricks we can play: zooming on the solution

Question: Can we select the test norm in such a way that the DPG method would deliver high accuracy in a preselected subdomain, e.g. $(0, \frac{1}{2})^2 \subset (0, 1)^2$?
Answer: Yes!





Optimal mesh and the corresponding pointwise error (range $(-0.001 - 0.001)$.

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Other Applications

- Wave propagation problems (sonars, full wave form inversion in geomechanics, cloaking)

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- Wave propagation problems (sonars, full wave form inversion in geomechanics, cloaking)
- Elasticity, shells (volumetric, shear, membrane locking)
- Metamaterials