A Unified Treatment of Primitive, Conservation, and Entropy Variable Formulations of Navier-Stokes with Discontinuous Petrov-Galerkin Finite Elements

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Abstract

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1. Motivation

1.1. Coarse Mesh Stability

Initial mesh design for computational fluid dynamics can be a time-consuming and expensive process. The stability properties and nonlinear convergence of most numerical methods rely on a minimum level of mesh resolution. This means that unless the initial computational mesh is fine enough, convergence can not be guaranteed. Any meshes below this minimum resolution level are termed to be in the "pre-asymptotic regime." This condition implies that meshes need to in some way anticipate the solution before it is known. On top of the minimum requirement that the surface meshes must adequately represent the geometry of the problem under consideration, resolution requirements on the volume mesh make the CFD practitioner's job significantly more time consuming. This is not to mention auxiliary requirements from turbulence models needed to accurately resolve boundary layers. Thus, mesh design and simulation become an iterative trial and error process. A common process is for an engineer to study the problem at hand and attempt to predict regions that require extra resolution. Then they will spend hours coaxing the mesher to produce an adequate mesh before importing the mesh into the solver. Too often, the solver will fail to converge on the given mesh due to some unforeseen mesh inadequacy on part of the solution domain. The engineer then needs to descend back into the mesher to fix the problem elements. This process is repeated until the solver is satisfied and convergence can be reached. This iterative trial and error is undesirable

while working on personal computers or modestly sized compute clusters, but becomes increasing costly as the problem is scaled up to the tens or hundreds of thousands of processors common in high performance computing environments.

In contrast to most other numerical methods, the discontinuous Petrov-Galerkin finite element method retains exceptional stability on extremely coarse meshes. DPG is also inherently very adaptive. It is possible to compute the residual error without knowledge of the exact solution, which can be used to robustly drive adaptivity. This results in a very automated technology, as the user can initialize a computation on the coarsest mesh which adequately represents the geometry then step back and let the program solve and adapt iteratively until it resolves the solution features.

1.2. DPG for High Performance Computing

Many of the features inherent in the DPG method appear promising in the context of high performance computing. Our goal is to design a method that eliminates human intervention as much as possible. The superior stability of the method promises to prevent a simulation from crashing which could eliminate expensive restarts on large systems. Preliminary studies on convection-diffusion suggest exceptional robustness of the method in terms of diminishing viscosity, promising successful application to a large class of flow problems. The adaptivity lent by the *error representation function* provides a reliable and automated way to start from a coarse mesh and only refine

toward solution features in need. This uses compute resources much more efficiently than uniform refinements, allowing larger simulations with fewer resources. These features combine to produce a high degree of automaticity. Ultimately, it is desirable that an engineer could produce a rough mesh that just captures the geometry of the problem and start a DPG simulation that automatically picks up solution features without the user needing to jump back in and fix things.

DPG is very compute intensive compared to the associated communication and memory costs. Most of the work is spent in embarrassingly parallel local solves for the optimal test functions and local stiffness matrix assembly. Additionally, the stability properties of DPG make high order stability a triviality, and in general, high order methods tend to have a more attractive compute/communication profile than low order methods. In our codes, we use QR factorization for optimal test function solves, but this factorization is recyclable as we essentially have many right hand sides. The division of degrees of freedom into internal vs skeleton unknowns produces a global system which can be statically condensed into a solve purely in terms of the skeleton degrees of freedom. In addition to significantly cutting down on the size of the global solve, this produces a embarrassingly parallel post-processing solve for the internal degrees of freedom. This property was one of the motivations behind the development of the hybridized discontinuous Galerkin [2] method. No matter what system of equations is being considered, DPG always produces a Hermitian (symmetric if real) positive definite stiffness

matrix for the global solve. This property has not really been leveraged in our simulations so far, since we have focused on direct rather than iterative solvers, but we anticipate it might be an attractive feature in the future. As compute resources scale up, many more HPC simulations are increasingly becoming coupled in multiphysics simulations. Since the only requirement for a well-defined discrete DPG method is a well-defined continuous problem, it is certainly possible that each different part of the multiphysics simulation could be discretized with DPG – no need to develop many different methods for each part of the simulation. Already DPG has been successfully applied to a wide variety of problems in computational mechanics, as noted above.

2. Overview of DPG

For a full treatment of the various ideas in DPG, please see [3]. The basic ideas are fairly straight-forward; DPG minimizes the residual in a user defined energy norm. Consider a variational problem: find $u \in U$ such that

$$b(u, v) = l(v) \quad \forall v \in V$$

with operator $B : U \to V' \quad (V'$ is the dual space to $V)$ defined by $b(u, v) = \langle Bu, v \rangle_{V' \times V}$. This gives the operator equation:

$$Bu = l \in V' .$$

We wish to minimize the residual $Bu - l$ in V':

$$u_h = \underset{w_h \in U_h}{\arg\min} \frac{1}{2} \|Bu - l\|_{V'}^2 \ .$$

This is a very natural mathematical framework based soundly in functional analysis, but it is not yet a practical method as the V' norm is not especially tractable to work with. The insight is that since we are working with Hilbert spaces, we can use the Riesz representation theorem to find a complementary object in V rather than V'. Let $R_V : V \ni v \to (v, \cdot) \in V'$ be the Riesz map. Then the inverse Riesz map (which is an isometry) lets us represent our residual in V:

$$u_h = \underset{w_h \in U_h}{\arg\min} \frac{1}{2} \left\|R_V^{-1}(Bu - l)\right\|_V^2 \ .$$

Taking the Gâteaux derivative to be zero in all directions $\delta u \in U_h$ gives,

$$\left(R_V^{-1}(Bu_h - l), R_V^{-1} B\delta u\right)_V = 0, \quad \forall \delta u \in U,$$

which by definition of the Riesz map is equivalent to

$$\left\langle Bu_h - l, R_V^{-1} B\delta u_h \right\rangle = 0 \quad \forall \delta u_h \in U_h \,,$$

with optimal test functions $v_{\delta u_h} \coloneqq R_V^{-1} B\delta u_h$ for each trial function δu_h. This gives a simple bilinear form

$$b(u_h, v_{\delta u_h}) = l(v_{\delta u_h}),$$

with $v_{\delta u_h} \in V$ that solves the auxiliary problem

$$(v_{\delta u_h}, \delta v)_V = \langle R_V v_{\delta u_h}, \delta v \rangle = \langle B\delta u_h, \delta v \rangle = b(\delta u_h, \delta v) \quad \forall \delta v \in V.$$

We might call this an *optimal Petrov-Galerkin* method. We arrive at the same method by realizing the supremum in the inf-sup condition, motivating the *optimal* nomenclature. These optimal Petrov-Galerkin methods produce Hermitian, positive-definite stiffness matrices since

$$b(u_h, v_{\delta \boldsymbol{u}_h}) = (v_{u_h}, v_{\delta \boldsymbol{u}_h})_V = \overline{(v_{\delta \boldsymbol{u}_h}, v_{u_h})} = \overline{b(\delta u_h, v_{u_h})}\,.$$

We can calculate the energy norm (defined by $\|u\|_E := \|Bu\|_{V'}$) of the Galerkin error without knowing the exact solution by using the residual:

$$\|u_h - u\|_E = \|B(u_h - u)\|_{V'} = \|Bu_h - l\|_{V'} = \left\|R_V^{-1}(Bu_h - l)\right\|_V\,,$$

where we designate $R_V^{-1}(Bu_h - l)$ the *error representation function*. This has proven to be a very reliable *a-posteriori* error estimator for driving adaptivity.

Babuška's theorem [4] says that discrete stability and approximability imply convergence. That is, if M is the continuity constant for $b(u, v)$ which satisfies the discrete inf-sup condition with constant γ_h,

$$\sup_{v_h \in V_h} \frac{|b(u, v)|}{\|v_h\|_V} \geq \gamma_h \|u_h\|_U\,,$$

then the Galerkin error satisfies the bound

$$\|u_h - u\|_U \leq \frac{M}{\gamma_h} \inf_{w_h \in U_h} \|w_h - u\|_U \ .$$

Optimal test functions realize the supremum in the discrete discrete inf-sup condition such that $\gamma_h \geq \gamma$, the infinite-dimensional inf-sup constant. If we then use the energy norm for $\|\cdot\|_U$, then $M = \gamma = 1$ and Babuška's estimate implies that the optimal Petrov-Galerkin method is the most stable Petrov-Galerkin method possible.

There are still many features of the method that are left to be decided, for example the choice of U and V spaces. If V is taken to be a continuous space, then the auxiliary problem becomes global in scope, something that we would like to avoid. In order to ensure the auxiliary problem can be solved element-by-element, we take V to be a space of functions that are discontinuous between elements. (Technically, V should also be infinite dimensional, but we have found it to be sufficient to use an "enriched" space of higher polynomial dimension than the trial space [5].) The downside to using discontinuous test functions is that it introduces new interface unknowns. When the equations are integrated by parts over each element, the jump in test functions introduces new unknowns on the mesh skeleton that would have disappeared with continuous test functions. Moro *et al.* [6] handle the flux unknowns with a numerical flux in the hybridized DPG (HDPG) method, but the standard DPG method treats these as new unknowns to be solved

for. We still haven't specified our trial space U, but the rule is that for every integration by parts, a new skeleton unknown is introduced. Most DPG considerations have used the ultra-weak variational formulation and break a second order PDE into a system of first order PDEs. This introduces a trace unknown (through the constitutive law) and a flux unknown (through the conservation law) with field variables that live in L^2. But Demkowicz and Gopalakrishnan also formulated a *primal DPG* method [7] for second order equations that does not introduce a trace unknown. The overall number of interface unknowns in the primal DPG method is the same, however, since the solution is required to be H^1 conforming and the trace unknowns are essentially hidden there.

The final unresolved choice is what norm to apply to the V space. This is one of the most important factors in designing a robust DPG method as the corresponding Riesz operator needs to be inverted to solve for the optimal test functions. If the norm produces unresolved boundary layers in the auxiliary problem, then many of the attractive features of DPG may fall apart. For a full discussion of robust test norms for convection-diffusion problems, see [8, 9]. But elimination of boundary layers in the auxiliary solve is not the only requirement at play. This choice also controls what norm the residual is minimized in. Often we want this norm to be equivalent to the L^2 norm. Fortunately, we have found that it is possible to design test norms such that the implied energy norm is provably robust and equivalent to L^2 for convection-diffusion which serves as the most relevant model problem

for our research. Norms for Navier-Stokes are derived by analogy to the convection-diffusion norm.

2.1. DPG for Nonlinear Problems

In the context of linear problems, DPG has been successfully applied to a wide range of physical applications. Theoretical groundwork for DPG applied to the Poisson equation was laid in [10]. The time-harmonic Helmholtz equation was the focus of [11, 12] and [13]. Linear elasticity and plate problems were addressed in [14], [15], and [16], and Maxwell was addressed in [17, 18]. Linear fluid dynamics problems include convection-diffusion[19, 8, 9, 20, 21] and stationary Stokes flow [22, 21].

In this section, following [23, 24], we extend the idea of DPG to nonlinear problems. In an abstract sense, we consider the discrete nonlinear variational problem

$$\langle B(u_h) - \ell, v\rangle_{V^*\times V} \coloneqq \langle R(u_h), v\rangle_{V^*\times V} = 0, \quad \forall v \in V.$$

The linearization of this problem leads to the Newton iteration, defined by the step

$$\langle B'(u_h)\Delta u, v\rangle_{V^*\times V} = \langle R(u_h), v\rangle_{V^*\times V}\,,$$

where $B'(u_h)$ is the Gateux derivative of B given u_h. It can be shown that DPG applied to this linearized problem leads to a Gauss-Newton iteration,

where

$$\min_{\Delta u} \|B'(u_h)\Delta u - r(u_h)\|^2_{V'}$$

is solved at every iteration [25].

The robust stability properties of DPG appear to carry over into the nonlinear regime as well; Moro, Peraire and Nguyen observed [6, 26] that on a single element, the HDPG method (referring to DPG applied to a hybridized Discontinuous Galerkin (HDG) method) converges, whereas HDG on its own does not. Moreover, HDPG required an order of magnitude less artificial diffusion to converge to a nonlinear solution of the compressible Navier-Stokes. Chan *et al.* [25] showed that for both the viscous Burgers' equation and the compressible Navier Stokes equations at high Mach/Reynolds number, DPG converges on an extremely coarse mesh, allowing the use of an adaptive scheme to capture shock and boundary layer phenomena starting from an initial mesh of $O(1)$ elements. Roberts reports similar results boasting similar stability for high Reynolds numbers for the incompressible Navier-Stokes equations [23].

3. Space-Time DPG

Previous explorations of the DPG method focused exclusively on steady state problems. The easiest extension of steady DPG to transient problems would be to do an implicit time stepping technique in time and use DPG for only the spatial solve at each time step. We did indeed explore this approach, but it didn't seem to be a natural fit with the adaptive features of

DPG. Clearly the CourantFriedrichsLewy (CFL) condition was not binding since we were interested in implicit time integration schemes, but it can be a guiding principle for temporal accuracy in this case. So if we are interested in temporally accurate solutions, we are limited by the fact that our smallest mesh elements (which may be orders of magnitude smaller than the largest elements) are constrained to proceed at a much smaller time step than the mesh as a whole. We can either restrict the whole mesh to the smallest time step, or we can attempt some sort of local time stepping. A space-time DPG formulation presents an attractive choice as we will be able to preserve our natural adaptivity from the steady problems while extending it in time. Thus we achieve an adaptive solution technique for transient problems in a unified framework. The obvious downside to such an approach is that for 2D spatial problems, we now have to compute on a three dimensional mesh while a spatially 3D problem becomes four dimensional.

3.1. Heat Equation

The simplest parabolic space-time problem we can consider is the heat equation. We start with a general d-dimensional spatial derivation and later simplify to spatially 1D with a few numerical experiments.

Let $\Omega(t) \subset \mathbb{R}^d$ be the spatial domain with boundary $\partial\Omega$. The heat equation is

$$\frac{\partial u}{\partial t} - \epsilon \Delta u = f\,, \quad \boldsymbol{x} \in \Omega\,,\ t \in (t_0, T) \tag{1}$$

where u is unknown temperature, ϵ is the diffusion scale, f is the source

term, t_0 is the start time, and T is the final time. Let $Q \subset \mathbb{R}^{d+1}$ denote the full space-time domain which is then tessellated into space-time elements K.

The second order formulation of the heat equation is really just a composition of Fourier's law and conservation of energy:

$$\begin{aligned} \boldsymbol{\sigma} - \epsilon \nabla u &= 0 \\ \frac{\partial u}{\partial t} - \nabla \cdot \boldsymbol{\sigma} &= f \,, \end{aligned} \tag{2}$$

where $\boldsymbol{\sigma}$ is the heat flux. The key insight that we will use over and over in the following problems is that we can rewrite our conservation equation in terms of a space-time divergence operator: $\nabla_{xt} \cdot () := \nabla \cdot () + \frac{\partial ()}{\partial t}$. Our new system is then

$$\begin{aligned} \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= 0 \\ \nabla_{xt} \cdot \begin{pmatrix} -\boldsymbol{\sigma} \\ u \end{pmatrix} &= f \,. \end{aligned} \tag{3}$$

We now proceed with the standard DPG practice and multiply by test functions $\boldsymbol{\tau}$ and v and integrate by parts over each space-time element K:

$$\begin{aligned} \left(\frac{1}{\epsilon}\boldsymbol{\sigma}, \boldsymbol{\tau}\right) + (u, \nabla \cdot \boldsymbol{\tau}) - \langle \hat{u}, \boldsymbol{\tau} \cdot \boldsymbol{n}_x \rangle &= 0 \\ - \left(\begin{pmatrix} -\boldsymbol{\sigma} \\ u \end{pmatrix}, \nabla_{xt} v \right) + \left\langle \hat{t}, v \right\rangle &= f \,, \end{aligned} \tag{4}$$

where $\boldsymbol{\tau}$ is in a broken tensor product space of $\boldsymbol{H}(\mathrm{div})$ spatially multiplied by

L^2 functions temporally, while v is in a broken H^1 space in full space-time. Also

$$\hat{u} := \mathrm{tr}(u)$$
$$\hat{t} := \mathrm{tr}(-\boldsymbol{\sigma}) \cdot \boldsymbol{n}_x + \mathrm{tr}(u) \cdot n_t$$

are new unknowns that live on the mesh skeleton introduced by the integration by parts (parenthesis indicate volume integrals while angle brackets indicate surface integrals). Note that the constitutive law was only integrated by parts over spatial dimensions, which means that *spatial trace* $\hat{u}$ only exists on mesh boundaries with a nonzero spatial normal component. On the other hand, *flux* $\hat{t}$ exists on all mesh boundaries, but changes nature between pure spatial and temporal edges while taking on a mixed nature on slanted boundaries. We illustrate the support of these skeleton variables in Figure 1.

It is worth noting that one technique used to control the cost of the extra dimensional solve is to divide the space-time domain into a number of space-time slabs. Since information does not travel backwards in time, we can march forward solving and adapting on each times slab before moving to the next one. The flux on the final time boundary of one slab acts as the initial condition on the next. Under mild mesh regularity assumptions, one can show that marching with slabs delivers the same solution as the single slab DPG method, see [27] for a related discussion for the convection problem.

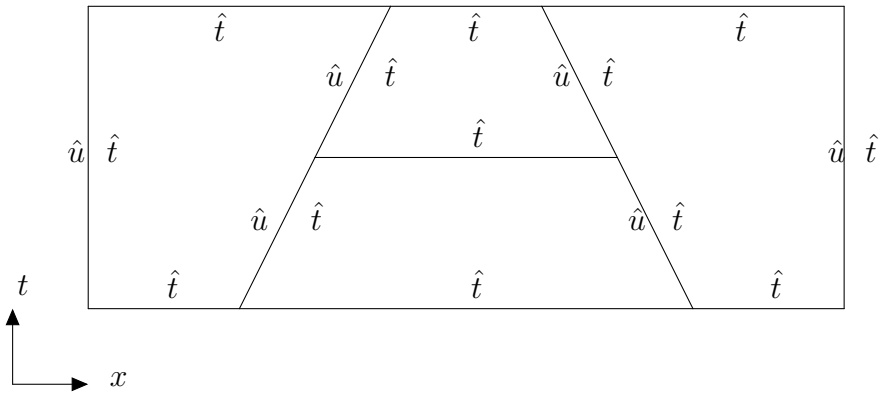

Figure 1: Support of flux and spatial trace variables

3.1.1. Numerical Experiments

If we consider a domain $\Omega = [0,1]^2$ with an initial condition of $u = \cos(2\pi x)$ with zero flux conditions at the boundaries, the exact solution is

$$u = \cos(2\pi x)e^{-4\pi^2 \epsilon t}\,.$$

We ran this with $\epsilon = 10^{-2}$ on a sequence of uniform meshes and $p = 2$ for the field representation of u. We were able to achieve the expected third order convergence as demonstrated in Figure 2.

4. Transient Compressible Navier-Stokes

We make a large jump from convection-diffusion to the compressible Navier-Stokes equations. The following discussion holds in any dimension, but the provided results are only for spatially 1D flows (at the time of this

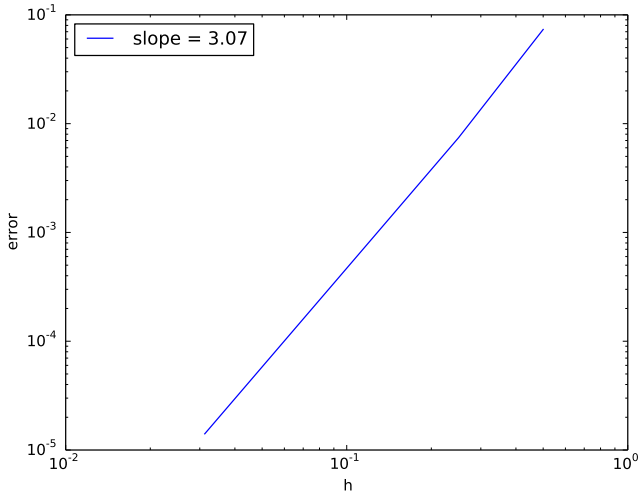

Figure 2: L^2 convergence of u for the space-time heat equation

writing, our code only supported a maximum of two dimensional meshes). The compressible Navier-Stokes equations are

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\boldsymbol{u}\\ \rho e_0\end{bmatrix} + \nabla\cdot\begin{bmatrix}\rho\boldsymbol{u}\\ \rho\boldsymbol{u}\otimes\boldsymbol{u}+p\mathbb{I}-\mathbb{D}\\ \rho\boldsymbol{u}e_0+\boldsymbol{u}p+\boldsymbol{q}-\boldsymbol{u}\cdot\mathbb{D}\end{bmatrix} = \begin{bmatrix}f_c\\ \boldsymbol{f}_m\\ f_e\end{bmatrix}, \tag{5}$$

where ρ is the density, $\boldsymbol{u}$ is the velocity, p is the pressure, $\mathbb{I}$ is the identity matrix, $\mathbb{D}$ is the deviatoric stress tensor or viscous stress, e_0 is the total energy, $\boldsymbol{q}$ is the heat flux, and f_c, $\boldsymbol{f}_m$, and f_e are the source terms for the continuity, momentum, and energy equations, respectively. We might assume

Stokes hypothesis that $\lambda = -\frac{2}{3}\mu$, and

$$\mathbb{D} = 2\mu \boldsymbol{S}^* = 2\mu \left[\frac{1}{2} \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T \right) - \frac{1}{3} \nabla \cdot \boldsymbol{u} \mathbb{I} \right] ,$$

where $\boldsymbol{S}^*$ is the trace-less viscous strain rate tensor and μ might follow a Sutherland viscosity law, but the one dimensional flows considered in this paper are a special case for which we simply set $\mathbb{D} = 2\mu \frac{\partial u}{\partial x}$ with constant μ. For the sake of the derivations, we use the more complicated case derived from Stokes hypothesis, but with a constant viscosity. It would be a fairly straightforward process to introduce a varying viscosity law if that were desired. Our heat flux is given by Fourier's law:

$$\boldsymbol{q} = -C_p \frac{\mu}{Pr} \nabla T ,$$

where C_p is the specific heat at constant pressure and Pr is the Prandtl number. We need to close these equations with an equation of state. The ideal gas assumption gives the following relations (but other equations of state would be fairly straightforward to implement)

$$\gamma := \frac{C_p}{C_v} , \quad p = \rho R T , \quad e = C_v T , \quad C_p - C_v = R ,$$

where γ is the ratio of specific heats, C_v is the specific heat at constant volume, R is the gas constant, e is the internal energy, T is the temperature, and γ, C_p, C_v, and R are constant properties of the fluid. The total energy

is defined by

$$e_0 = e + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\,.$$

We can write our first order system of equations in space-time as follows:

$$\frac{1}{\mu}\mathbb{D} - \left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right) + \frac{2}{3}\nabla\cdot\boldsymbol{u}\mathbb{I} = 0 \tag{6a}$$

$$\frac{Pr}{C_p\mu}\boldsymbol{q} + \nabla T = 0 \tag{6b}$$

$$\nabla_{xt}\cdot\begin{pmatrix}\rho\boldsymbol{u}\\ \\ \rho\end{pmatrix} = f_c \tag{6c}$$

$$\nabla_{xt}\cdot\begin{pmatrix}\rho\boldsymbol{u}\otimes\boldsymbol{u} + \rho RT\mathbb{I} - \mathbb{D}\\ \\ \rho\boldsymbol{u}\end{pmatrix} = \boldsymbol{f}_m \tag{6d}$$

$$\nabla_{xt}\cdot\begin{pmatrix}\rho\boldsymbol{u}\left(C_vT + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) + \rho RT\boldsymbol{u} + \boldsymbol{q} - \boldsymbol{u}\cdot\mathbb{D}\\ \\ \rho\left(C_vT + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\end{pmatrix} = f_e\,, \tag{6e}$$

where our solution variables are ρ, $\boldsymbol{u}$, T, $\mathbb{D}$, and $\boldsymbol{q}$.

4.1. Derivation of Space-Time DPG Formulation

We start with (6) and multiply by test functions $\mathbb{S}$ (symmetric trace-free), $\boldsymbol{\tau}$, v_c, $\boldsymbol{v}_m$, v_e, then integrate by parts over each space-time element K:

$$\left(\frac{1}{\mu}\mathbb{D}, \mathbb{S}\right) + (2\boldsymbol{u}, \nabla \cdot \mathbb{S}) - \langle 2\hat{\boldsymbol{u}}, \mathbb{S}\boldsymbol{n}_x \rangle = 0 \tag{7a}$$

$$\left(\frac{Pr}{C_p\mu}\boldsymbol{q}, \boldsymbol{\tau}\right) - (T, \nabla \cdot \boldsymbol{\tau}) + \left\langle \hat{T}, \tau_n \right\rangle = 0 \tag{7b}$$

$$-\left(\begin{pmatrix} \rho\boldsymbol{u} \\ \rho \end{pmatrix}, \nabla_{xt} v_c\right) + \langle \hat{t}_c, v_c \rangle = (f_c, v_c) \tag{7c}$$

$$-\left(\begin{pmatrix} \rho\boldsymbol{u} \otimes \boldsymbol{u} + \rho RT\boldsymbol{I} - \mathbb{D} \\ \rho\boldsymbol{u} \end{pmatrix}, \nabla_{xt} \boldsymbol{v}_m\right) + \langle \hat{\boldsymbol{t}}_m, \boldsymbol{v}_m \rangle = (\boldsymbol{f}_m, \boldsymbol{v}_m) \tag{7d}$$

$$-\left(\begin{pmatrix} \rho\boldsymbol{u}\left(C_vT + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) + \boldsymbol{u}\rho RT + \boldsymbol{q} - \boldsymbol{u}\cdot\mathbb{D} \\ \rho\left(C_vT + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) \end{pmatrix}, \nabla_{xt} v_e\right) + \langle \hat{t}_e, v_e \rangle = (f_e, v_e)\ , \tag{7e}$$

where

$$
\begin{aligned}
\hat{\boldsymbol{u}} &= \mathrm{tr}(\boldsymbol{u}) \\
\hat{T} &= \mathrm{tr}(T) \\
\hat{t}_c &= \mathrm{tr}\left(\rho \boldsymbol{u}\right) \cdot \boldsymbol{n}_x + \mathrm{tr}\left(\rho\right) n_t \\
\hat{\boldsymbol{t}}_m &= \mathrm{tr}\left(\rho \boldsymbol{u} \otimes \boldsymbol{u} + \rho R T \boldsymbol{I} - \mathbb{D}\right) \cdot \boldsymbol{n}_x + \mathrm{tr}\left(\rho \boldsymbol{u}\right) n_t \\
\hat{t}_e &= \mathrm{tr}\left(\rho \boldsymbol{u} \left(C_v T + \frac{1}{2} \boldsymbol{u} \cdot \boldsymbol{u}\right) + \boldsymbol{u} \rho R T + \boldsymbol{q} - \boldsymbol{u} \cdot \mathbb{D}\right) \cdot \boldsymbol{n}_x \\
&\quad + \mathrm{tr}\left(\rho \left(C_v T + \frac{1}{2} \boldsymbol{u} \cdot \boldsymbol{u}\right)\right) n_t \,.
\end{aligned}
$$

Note that integrating $\mathbb{S}$ against the symmetric gradient only picks up the symmetric part. This is a much more complicated system of equations than we had for the space-time heat equation, but the situation has many similarities. Test function $\boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div}, K)$ where the divergence is taken only over spatial dimensions, $v_c, v_e \in H^1(K)$, and $\boldsymbol{v}_m \in \boldsymbol{H}^1(K)$. These are all familiar spaces from our work with the heat equation. Test functions are usually taken to duplicate the structure of the trial functions. Since $\mathbb{D}$ is symmetric and trace-free, $\mathbb{S}$ is taken to be the same, each row of which needs to be in $\boldsymbol{H}(\mathrm{div}, K)$. In 2D the symmetric trace-free condition produces a tensor with the following structure:

$$
\begin{bmatrix}
S_{11} & S_{12} \\
S_{12} & -S_{11}
\end{bmatrix}
$$

(two independent variables) and in 3D, it would look like

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & -S_{11} - S_{22} \end{bmatrix}$$

(five independent variables).

In order to better see the structure in the following derivation, we introduce some terms to simplify things. First we identify the Euler fluxes for the continuity, momentum, and energy equations:

$$\begin{aligned} \boldsymbol{F}_c &:= \rho \boldsymbol{u} \\ \mathbb{F}_m &:= \rho \boldsymbol{u} \otimes \boldsymbol{u} + \rho R T \mathbb{I} \\ \boldsymbol{F}_e &:= \rho \boldsymbol{u} \left(C_v T + \frac{1}{2} \boldsymbol{u} \cdot \boldsymbol{u} \right) + \rho R T \boldsymbol{u} \,. \end{aligned}$$

The viscous fluxes are

$$\begin{aligned} \boldsymbol{K}_c &:= \boldsymbol{0} \\ \mathbb{K}_m &:= \mathbb{D} \\ \boldsymbol{K}_e &:= -\boldsymbol{q} + \boldsymbol{u} \cdot \mathbb{D} \,, \end{aligned}$$

the conserved terms are

$$
\begin{aligned}
C_c &:= \rho \\
\boldsymbol{C}_m &:= \rho \boldsymbol{u} \\
C_e &:= \rho(C_v T + \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}) \,,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{M}_{\mathbb{D}} &:= \frac{1}{\mu}\mathbb{D} \\
\boldsymbol{M}_{\boldsymbol{q}} &:= \frac{Pr}{C_p \mu}\boldsymbol{q} \\
\boldsymbol{G}_{\mathbb{D}} &:= 2\boldsymbol{u} \\
G_{\boldsymbol{q}} &:= -T \,.
\end{aligned}
$$

Finally, we define group terms

$$
\begin{aligned}
F &:= \{\boldsymbol{F}_c \,,\, \mathbb{F}_m \,,\, \boldsymbol{F}_e\} \\
K &:= \{\boldsymbol{K}_c \,,\, \mathbb{K}_m \,,\, \boldsymbol{K}_e\} \\
C &:= \{C_c \,,\, \boldsymbol{C}_m \,,\, C_e\} \\
M &:= \{\mathbb{M}_{\mathbb{D}} \,,\, \boldsymbol{M}_{\boldsymbol{q}}\} \\
G &:= \{\boldsymbol{G}_{\mathbb{D}} \,,\, G_{\boldsymbol{q}}\} \\
f &:= \{f_c \,,\, \boldsymbol{f}_m \,,\, f_e\}
\end{aligned}
$$

and group variables

$$
\begin{aligned}
W &:= \{\rho\,,\, \boldsymbol{u}\,,\, T\} \\
\hat{W} &:= \left\{2\hat{\boldsymbol{u}}\,,\, -\hat{T}\right\} \\
\hat{t} &:= \left\{\hat{t}_e\,,\, \hat{\boldsymbol{t}}_m,\,,\, \hat{t}_e\right\} \\
\Psi &:= \{\mathbb{S}\,,\, \boldsymbol{\tau}\} \\
V &:= \{v_c\,,\, \boldsymbol{v}_m,\,,\, v_e\}\ .
\end{aligned}
$$

Our Navier-Stokes variational form then looks like

$$
(M, \Psi) + (G, \nabla \cdot \Psi) - \left\langle \hat{W}, \Psi \cdot \boldsymbol{n}_x \right\rangle = 0 \tag{8a}
$$

$$
-\left(\begin{pmatrix} F - K \\ \\ C \end{pmatrix}, \nabla_{xt} V\right) + \left\langle \hat{t}, V \right\rangle = (f, V)\ . \tag{8b}
$$

4.2. Conservation Variables

We now wish to perform a change of variables to conservation variables (density, momentum, and total energy):

$$
\begin{aligned}
\rho &= \rho \\
\boldsymbol{m} &= \rho \boldsymbol{u} \\
E &= \rho\left(C_v T + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\ .
\end{aligned}
$$

We start be rewriting our Euler fluxes, viscous fluxes, and conserved quantities: The structure of the equations is identical, we just need to substitute new definitions for several terms (terms not enumerated are identical):

$$
\begin{aligned}
F &= \left\{ \boldsymbol{m}\,,\ \frac{\boldsymbol{m}\otimes\boldsymbol{m}}{\rho} + (\gamma-1)\left(E - \frac{\boldsymbol{m}\cdot\boldsymbol{m}}{2\rho}\right)\mathbb{I}\,,\ \gamma E\frac{\boldsymbol{m}}{\rho} - (\gamma-1)\frac{\boldsymbol{m}\cdot\boldsymbol{m}}{2\rho^2}\boldsymbol{m} \right\} \\
K &= \left\{ \mathbf{0}\,,\ \mathbb{D}\,,\ -\boldsymbol{q} + \frac{\boldsymbol{m}}{\rho}\cdot\mathbb{D} \right\} \\
C &= \{\rho\,,\ \boldsymbol{m}\ E\} \\
G &= \left\{ 2\frac{\boldsymbol{m}}{\rho}\,,\ -\frac{E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m}}{C_v\rho} \right\}\,.
\end{aligned}
$$

4.3. Entropy Variables

We perform one more change of variables to entropy variables:

$$
\begin{aligned}
V_c &= \frac{-E + (E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m})\left(\gamma + 1 - \ln\left[\frac{(\gamma-1)(E-\frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m})}{\rho^\gamma}\right]\right)}{E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m}} \\
\boldsymbol{V}_m &= \frac{\boldsymbol{m}}{E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m}} \\
V_e &= \frac{-\rho}{E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m}}
\end{aligned}
$$

with reverse mapping:

$$\begin{aligned}\rho &= -\alpha V_e \\ \boldsymbol{m} &= \alpha \boldsymbol{V}_m \\ E &= \alpha \left(1 - \frac{1}{2V_e}\boldsymbol{V}_m \cdot \boldsymbol{V}_m\right)\end{aligned}$$

where

$$\alpha(V_c, \boldsymbol{V}_m, V_e) = \left[\frac{\gamma - 1}{(-V_e)^\gamma}\right]^{\frac{1}{\gamma-1}} \exp\left[\frac{-\gamma + V_c - \frac{1}{2V_e}\boldsymbol{V}_m \cdot \boldsymbol{V}_m}{\gamma - 1}\right]$$

The new definitions are

$$\begin{aligned}F &= \left\{\alpha \boldsymbol{V}_m \, , \alpha\left(-\frac{\boldsymbol{V}_m \otimes \boldsymbol{V}_m}{V_e} + (\gamma - 1)\mathbb{I}\right) \, , \alpha\frac{\boldsymbol{V}_m}{V_e}\left(\frac{1}{2V_e}\boldsymbol{V}_m \cdot \boldsymbol{V}_m - \gamma\right)\right\} \\ K &= \left\{\boldsymbol{0} \, , \mathbb{D} \, , -\boldsymbol{q} - \frac{\boldsymbol{V}_m}{V_e} \cdot \mathbb{D}\right\} \\ C &= \left\{-\alpha V_e \, , \alpha \boldsymbol{V}_m \, , \alpha\left(1 - \frac{1}{2V_e}\boldsymbol{V}_m \cdot \boldsymbol{V}_m\right)\right\} \\ G &= \left\{-2\frac{\boldsymbol{V}_m}{V_e} \, , \frac{1}{C_v V_e}\right\} .\end{aligned}$$

We denote the set of primary conservation variables as $U = \{\rho, \boldsymbol{m}, E\}$ and entropy variables as $V = \{V_c, \boldsymbol{V}_m, V_e\}$.

4.4. Linearization

Our nonlinear residual is then

$$
\begin{aligned}
&(M, \Psi) + (G, \nabla \cdot \Psi) - \left\langle \hat{W}, \Psi \cdot \boldsymbol{n}_x \right\rangle \\
&- \left(\begin{pmatrix} F - K \\ C \end{pmatrix}, \nabla_{xt} V \right) + \left\langle \hat{t}, V \right\rangle - (f, V) = 0 \,.
\end{aligned}
$$

The terms in angle brackets will always be linear, but in general the M, G, F, K, and C terms may have nonlinear interactions between variables. Therefore we split these terms up into a value for the previous guess (denoted with a tilde) and an update (denoted with a Δ):

$$
\begin{aligned}
&\left(\tilde{M}, \Psi\right) + (\Delta M, \Psi) + \left(\tilde{G}, \nabla \cdot \Psi\right) + (\Delta G, \nabla \cdot \Psi) - \left\langle \hat{W}, \Psi \cdot \boldsymbol{n}_x \right\rangle \\
&- \left(\begin{pmatrix} \tilde{F} - \tilde{K} \\ \tilde{C} \end{pmatrix}, \nabla_{xt} V \right) - \left(\begin{pmatrix} \Delta F - \Delta K \\ \Delta C \end{pmatrix}, \nabla_{xt} V \right) \\
&+ \left\langle \hat{t}, V \right\rangle - (f, V) = 0 \,.
\end{aligned}
$$

Moving known quantities to the right hand side, we get our bilinear form

$$
\begin{aligned}
&(\Delta M, \Psi) + (\Delta G, \nabla \cdot \Psi) - \left\langle \hat{W}, \Psi \cdot \boldsymbol{n}_x \right\rangle \\
&- \left(\begin{pmatrix} \Delta F - \Delta K \\ \Delta C \end{pmatrix}, \nabla_{xt} V \right) + \langle \hat{t}, V \rangle \\
&= (f, V) - \left(\tilde{M}, \Psi \right) - \left(\tilde{G}, \nabla \cdot \Psi \right) + \left(\begin{pmatrix} \tilde{F} - \tilde{K} \\ \tilde{C} \end{pmatrix}, \nabla_{xt} V \right) .
\end{aligned}
$$

All that remains is to define the linearized terms for each set of variables. We start with primitive variables, for which

$$
\begin{aligned}
\Delta \boldsymbol{F}_c &= \Delta\rho\tilde{\boldsymbol{u}} + \tilde{\rho}\Delta\boldsymbol{u} \\
\Delta \mathbb{F}_m &= \Delta\rho\tilde{\boldsymbol{u}} \otimes \tilde{\boldsymbol{u}} + \tilde{\rho}\Delta\boldsymbol{u} \otimes \tilde{\boldsymbol{u}} + \tilde{\rho}\tilde{\boldsymbol{u}} \otimes \Delta\boldsymbol{u} + R\left(\Delta\rho\tilde{T} + \tilde{\rho}\Delta T\right)\mathbb{I} \\
\Delta \boldsymbol{F}_e &= C_v\Delta\rho\tilde{\boldsymbol{u}}\tilde{T} + C_v\tilde{\rho}\Delta\boldsymbol{u}\tilde{T} + C_v\tilde{\rho}\tilde{\boldsymbol{u}}\Delta T \\
&\quad + \frac{1}{2}\Delta\rho\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{u}} + \frac{1}{2}\tilde{\rho}\Delta\boldsymbol{u}\tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{u}} + \tilde{\rho}\tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}} \cdot \Delta\boldsymbol{u} \\
&\quad + R\Delta\boldsymbol{u}\tilde{\rho}\tilde{T} + R\tilde{\boldsymbol{u}}\Delta\rho\tilde{T} + R\tilde{\boldsymbol{u}}\tilde{\rho}\Delta T
\end{aligned}
$$

$$
\begin{aligned}
\Delta K &= \left\{0\,,\ \Delta\mathbb{D}\,,\ \Delta\boldsymbol{u}\cdot\tilde{\mathbb{D}}+\tilde{\boldsymbol{u}}\cdot\Delta\mathbb{D}\right\}\\
\Delta C_c &= \Delta\rho\\
\Delta\boldsymbol{C}_m &= \Delta\rho\tilde{\boldsymbol{u}}+\tilde{\rho}\Delta\boldsymbol{u}\\
\Delta C_e &= C_v\Delta\rho\tilde{T}+C_v\tilde{\rho}\Delta T+\frac{1}{2}\Delta\rho\tilde{\boldsymbol{u}}\cdot\tilde{\boldsymbol{u}}+\tilde{\rho}\tilde{\boldsymbol{u}}\cdot\Delta\boldsymbol{u}\\
\Delta M &= \left\{\frac{1}{\mu}\Delta\mathbb{D}\,,\ \frac{Pr}{C_p\mu}\Delta\boldsymbol{q}\right\}\\
\Delta G &= \left\{2\Delta\boldsymbol{u}\,,\ -\Delta T\right\}\,.
\end{aligned}
$$

With conservation variables, we have

$$
\begin{aligned}
\Delta\boldsymbol{F}_c &= \Delta\boldsymbol{m}\\
\Delta\mathbb{F}_m &= \frac{\Delta\boldsymbol{m}\otimes\tilde{\boldsymbol{m}}}{\tilde{\rho}}+\frac{\tilde{\boldsymbol{m}}\otimes\Delta\boldsymbol{m}}{\tilde{\rho}}-\frac{\tilde{\boldsymbol{m}}\otimes\tilde{\boldsymbol{m}}}{\tilde{\rho}^2}\Delta\rho\\
&\quad+(\gamma-1)\left(\Delta E-\frac{\tilde{\boldsymbol{m}}\cdot\Delta\boldsymbol{m}}{\tilde{\rho}}+\frac{\tilde{\boldsymbol{m}}\cdot\tilde{\boldsymbol{m}}}{2\tilde{\rho}^2}\Delta\rho\right)\mathbb{I}\\
\Delta\boldsymbol{F}_e &= \gamma\left(\Delta E\frac{\tilde{\boldsymbol{m}}}{\tilde{\rho}}+\tilde{E}\frac{\Delta\boldsymbol{m}}{\tilde{\rho}}-\tilde{E}\frac{\tilde{\boldsymbol{m}}}{\tilde{\rho}^2}\Delta\rho\right)\\
&\quad+(\gamma-1)\left(-\frac{\Delta\boldsymbol{m}\tilde{\boldsymbol{m}}\cdot\tilde{\boldsymbol{m}}}{2\tilde{\rho}^2}-\frac{\tilde{\boldsymbol{m}}\tilde{\boldsymbol{m}}\cdot\Delta\boldsymbol{m}}{\tilde{\rho}^2}+\frac{\tilde{\boldsymbol{m}}\tilde{\boldsymbol{m}}\cdot\tilde{\boldsymbol{m}}}{\tilde{\rho}^3}\Delta\rho\right)
\end{aligned}
$$

$$\Delta K = \left\{ 0\,,\, \Delta\mathbb{D}\,,\, \frac{\Delta \boldsymbol{m}}{\tilde{\rho}} \cdot \tilde{\mathbb{D}} - \frac{\tilde{\boldsymbol{u}}}{\tilde{\rho}^2} \cdot \tilde{\mathbb{D}} \Delta\rho + \frac{\tilde{\boldsymbol{m}}}{\tilde{\rho}} \cdot \Delta\mathbb{D} \right\}$$

$$\Delta C = \{ \Delta\rho\,,\, \Delta \boldsymbol{m}\,,\, \Delta E \}$$

$$\Delta M = \left\{ \frac{1}{\mu}\Delta\mathbb{D}\,,\, \frac{Pr}{C_p \mu}\Delta \boldsymbol{q} \right\}$$

$$\Delta \boldsymbol{G}_{\mathbb{D}} = 2\left(\frac{\Delta \boldsymbol{m}}{\tilde{\rho}} - \frac{\tilde{\boldsymbol{m}}}{\tilde{\rho}^2}\Delta\rho \right)$$

$$\Delta G_{\boldsymbol{q}} = -\frac{\Delta E - \frac{1}{\tilde{\rho}}\tilde{\boldsymbol{m}} \cdot \Delta \boldsymbol{m} + \frac{1}{2\tilde{\rho}^2}\tilde{\boldsymbol{m}} \cdot \tilde{\boldsymbol{m}} \Delta\rho}{C_v \tilde{\rho}} + \frac{\tilde{E} - \frac{1}{2\tilde{\rho}}\tilde{\boldsymbol{m}} \cdot \tilde{\boldsymbol{m}}}{C_v \tilde{\rho}^2}\Delta\rho\,.$$

4.5. Test Norm

4.6. Numerical Experiments

5. Conclusions and Future Work

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