Notes on Space-Time for the Heat Equation

Truman E. Ellis

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Introduction

For the following discussion, let (master) element K be a tensor product of a spatial component, X, and a time component, T. Let ∇ denote the spatial gradient, and $\frac{\partial}{\partial t}$ denote the time derivative. Let $\mathbf{n} = (\mathbf{n_x}, n_t)^T$ be the full space-time normal vector where $\mathbf{n_x}$ is the spatial component and n_t is the temporal component.

We make a few assumptions about the space-time mesh. Figures 1 and 2 represent two possible temporal refinements of the center element. We will see later in the discussion that flux variables $\hat{f}$ are not well defined on constant-time interfaces, while trace variables are defined on all element interfaces. Mathematically, the mesh in Figure 2 is perfectly valid, but for practical reasons we think it would be best to avoid these kinds of temporal refinements. For one, we would have to add logic that adds new flux variables if the new element division does not follow a constant-time contour. Also, the new flux variable becomes ill-defined as the interface approaches "horizontal". Of course, this assumption of only making constant-time temporal refinements doesn't guarantee that the won't run into this "near-horizontal" condition, but if "top" and "bottom" faces are guaranteed to be "horizontal" then mesh quality guidelines should make such situations undesirable for other reasons. We can do some experiments with the current code to approximate how catastrophic this "near-horizontal" refinement would be. This is analogous to pure convection when the convection vector aligns (or nearly aligns) with the mesh. In which case, the cross-stream flux degenerates. This is something I'll be looking into soon.

Function Spaces

The heat equation is

$$\frac{\partial u}{\partial t} - \epsilon \Delta u = f$$

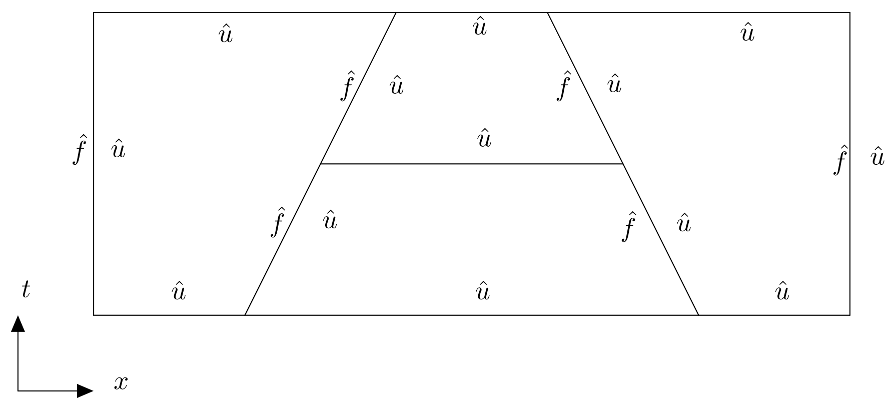

Figure 1: Allowable refinement pattern illustrating support of trace and flux variables

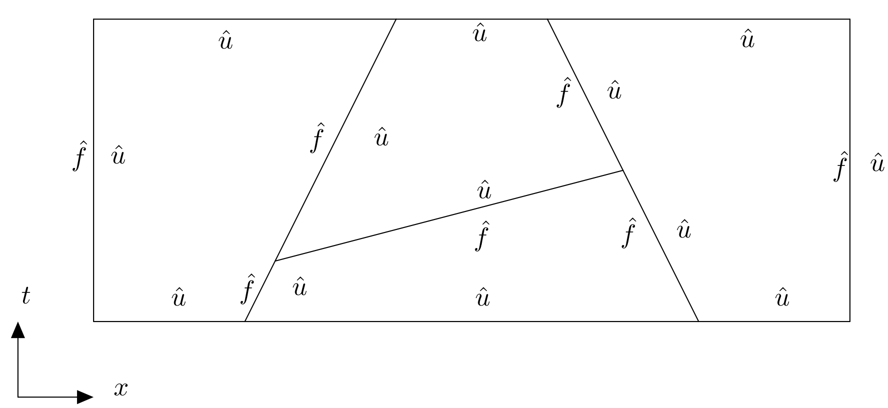

Figure 2: Questionable refinement pattern - as the division approaches horizontal, $\hat{f}$ degenerates

As a first order system, this is

$$\frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} - \nabla \cdot \boldsymbol{\sigma} = f \tag{2}$$

Before we transition to thinking of field, trace, and flux variables, we want to examine what kind of spaces u and $\boldsymbol{\sigma}$ should live in. From the above formulation we can deduce that

$$u, \boldsymbol{\sigma}, \nabla u, \frac{\partial u}{\partial t} - \nabla \cdot \boldsymbol{\sigma} \in L^2$$

If we view $\{-\boldsymbol{\sigma}, u\} := \mathbf{U}$ as a group variable, then the last condition tells us that $\mathbf{U} \in H(\mathrm{div}, K)$. In particular, this means that across a constant-time interface, $u \cdot n_t$ should be continuous, and across a non-constant-time interface, $-\boldsymbol{\sigma} \cdot \mathbf{n_x} + u \cdot n_t$ should be continuous.

If we take the typical volume-centered, face-centered, edge-centered, node-centered approach, then we can see that u should have a volume-centered component, all face-centered components, and edge-centered components on non-constant-time edges. On the other hand, $\boldsymbol{\sigma}$ should have a volume-centered component and face-centered components on all non-constant time interfaces.

Now we split u and $\boldsymbol{\sigma}$ into field, trace, and flux parts. It is easy to see that our new field variable u is a simple scalar valued, volume-centered L^2 variable on each space-time element. And the field variable $\boldsymbol{\sigma}$ is a vector valued, volume-centered $\mathbf{L}^2$ variable with dimension equal to the spatial dimension. We illustrate the support in Figure 3 as a blue volume centered box on a 3D space-time master element.

Now for the trace and flux variables. $\hat{u}$ is probably the most complicated trial variable in the system. It will obviously have face-centered components on all faces, but it also needs to maintain full continuity in spatial directions, necessitating the need for edge-centered components on non-horizontal edges. The face-centered bases have support only on a shared face between two elements. The edge-centered bases on the other hand have support on all faces connected to that edge. We illustrate the support in Figure 4. $\hat{f}$ is much simpler by comparison. It only has scalar face-centered components on non-horizontal faces. Again, its support is shown in Figure 5.

Finally, we need to discuss the test functions. v will basically just be a combination of u and $\hat{u}$ as shown in Figure 6 just as $\boldsymbol{\tau}$ will be the combination of $\boldsymbol{\sigma}$ and $\hat{f}$ shown in Figure 7.

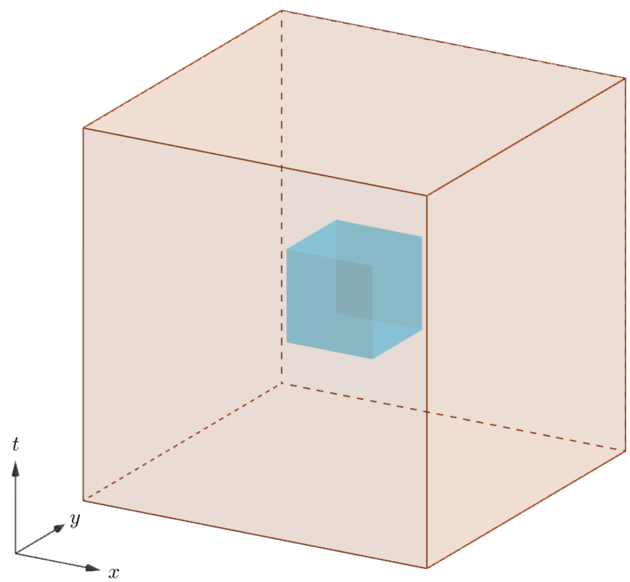

Figure 3: Support for u and $\boldsymbol{\sigma}$

Practical Aspects

Now we return to the first order system from Equation 2. Multiplying by test functions $\boldsymbol{\tau}$ and v, and integrating by parts we get the bilinear form

$$\begin{aligned}
-\left(u, \frac{\partial v}{\partial t}\right) + \langle \hat{u}, v \cdot n_t \rangle + (\boldsymbol{\sigma}, \nabla v) - \left\langle \widehat{\boldsymbol{\sigma} \cdot \mathbf{n}_{\mathbf{x}}}, v \right\rangle \\
+ \left(\frac{1}{\epsilon}\boldsymbol{\sigma}, \boldsymbol{\tau}\right) + (u, \nabla \cdot \boldsymbol{\tau}) - \langle \hat{u}, \boldsymbol{\tau} \cdot \mathbf{n}_{\mathbf{x}} \rangle = (f, v)
\end{aligned}$$

where $\widehat{\boldsymbol{\sigma} \cdot \mathbf{n}_{\mathbf{x}}} \equiv \hat{f}$.

Let's examine these terms one at a time. Clearly, volume terms 1, 3, 5, and 6 are just as they are for pure spatial DPG with the exception that vector quantities are of the dimension of the spatial dimension only. Term 2 will degenerate on constant space (as a function of time) boundaries, but we probably don't need any special logic to handle this, because at times non-constant-time interfaces will have nonzero n_t components (though this will probably only happen with moving interfaces). Terms 4 and 7 will degenerate on constant-time (as a function of space) boundaries. We will probably want to treat constant-time interfaces uniquely and not even attempt to integrate these terms because they are not well defined there.

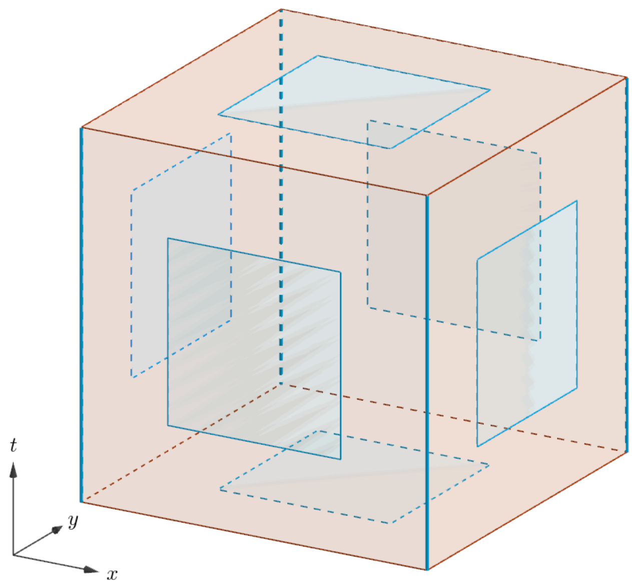

Figure 4: Support for $\hat{u}$

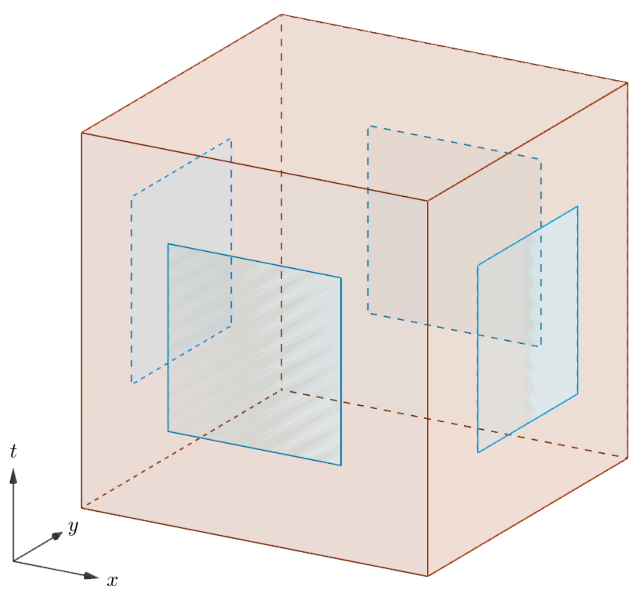

Figure 5: Support for $\hat{f}$

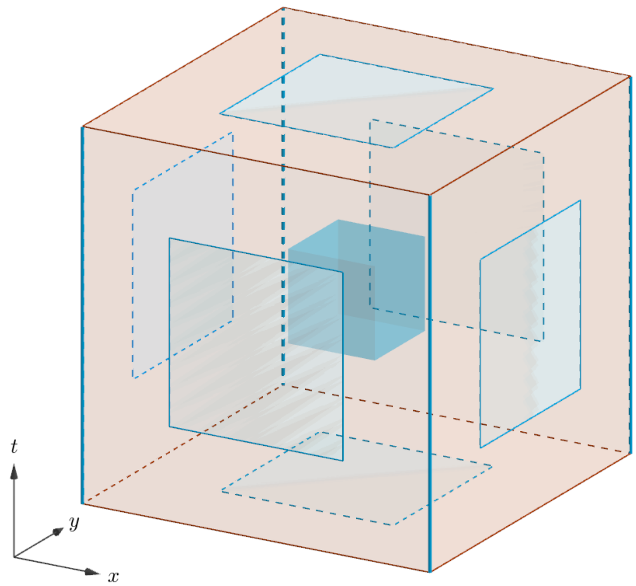

Figure 6: Support for v

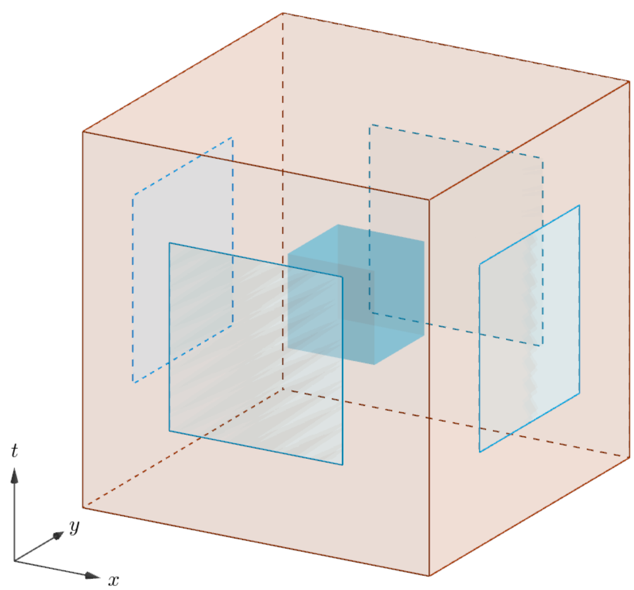

Figure 7: Support for $\boldsymbol{\tau}$