

# Pressureless Navier-Stokes Formulation

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January 29, 2014

We can derive the compressible Navier-Stokes equations in terms of the Cauchy stress tensor. Note that

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij},$$

and

$$\begin{aligned}\sigma_{ii} &= 2\mu\varepsilon_{ii} + N\lambda\varepsilon_{ii} \\ &= (2\mu + N\lambda)\varepsilon_{ii},\end{aligned}$$

where  $N$  is the dimension. Then

$$\begin{aligned}\varepsilon_{ij} &= \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu}\varepsilon_{kk}\delta_{ij} \\ &= \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(2\mu + N\lambda)}\sigma_{kk}\delta_{ij} \\ &= \frac{1}{2\mu}\sigma_{ij} - \frac{1}{2\mu(\frac{2\mu}{\lambda} + N)}\sigma_{kk}\delta_{ij}.\end{aligned}$$

## Incompressible

If we assume an incompressible medium, then  $\lambda \rightarrow \infty$  and

$$\begin{aligned}\varepsilon_{ij} &= \frac{1}{2\mu}\sigma_{ij} - \frac{1}{2N\mu}\sigma_{kk}\delta_{ij} \\ &= \frac{1}{2\mu}\left[\sigma_{ij} - \frac{1}{N}\sigma_{kk}\delta_{ij}\right].\end{aligned}$$

This embeds the zero divergence condition. If we take the trace of both sides, we get

$$\nabla \cdot \mathbf{u} = \varepsilon_{ii} = \frac{1}{2\mu}[\sigma_{ii} - \sigma_{ii}] = 0.$$

The space-time form of the Cauchy momentum equation is

$$\nabla_{xt} \cdot \begin{pmatrix} \rho \mathbf{u} \otimes \mathbf{u} - \boldsymbol{\sigma} \\ \rho \mathbf{u} \end{pmatrix} = \mathbf{f}.$$

Our incompressible Navier-Stokes system is then

$$\begin{aligned} \frac{1}{\mu} \boldsymbol{\sigma} - \frac{1}{N\mu} \text{tr}(\boldsymbol{\sigma}) \mathbf{I} - \left( \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) &= 0 \\ \nabla_{xt} \cdot \begin{pmatrix} \rho \mathbf{u} \otimes \mathbf{u} - \boldsymbol{\sigma} \\ \rho \mathbf{u} \end{pmatrix} &= \mathbf{f}. \end{aligned}$$

We multiply by test functions  $\boldsymbol{\tau}$  (symmetric tensor) and  $\mathbf{v}$  and integrate by parts over a space-time element  $K$ .

$$\begin{aligned} \left( \frac{1}{\mu} \boldsymbol{\sigma}, \boldsymbol{\tau} \right) - \left( \frac{1}{N\mu} \text{tr}(\boldsymbol{\sigma}) \mathbf{I}, \boldsymbol{\tau} \right) + (2\mathbf{u}, \nabla \cdot \boldsymbol{\tau}) - \langle 2\hat{\mathbf{u}}, \boldsymbol{\tau} \cdot \mathbf{n}_x \rangle &= 0 \\ - \left( \begin{pmatrix} \rho \mathbf{u} \otimes \mathbf{u} - \boldsymbol{\sigma} \\ \rho \mathbf{u} \end{pmatrix}, \nabla_{xt} \mathbf{v} \right) + \langle \hat{\mathbf{t}}, \mathbf{v} \rangle &= (\mathbf{f}, \mathbf{v}), \end{aligned}$$

where

$$\begin{aligned} \hat{\mathbf{u}} &= \text{tr}(\mathbf{u}) \\ \hat{\mathbf{t}}_m &= \text{tr}(\rho \mathbf{u} \otimes \mathbf{u} - \boldsymbol{\sigma}) \cdot \mathbf{n}_x + \text{tr}(\rho \mathbf{u}) n_t. \end{aligned}$$

## Linearization

The Jacobian is

$$\begin{aligned} \left( \frac{1}{\mu} \boldsymbol{\sigma}, \boldsymbol{\tau} \right) - \left( \frac{1}{N\mu} \text{tr}(\boldsymbol{\sigma}) \mathbf{I}, \boldsymbol{\tau} \right) + (2\mathbf{u}, \nabla \cdot \boldsymbol{\tau}) - \langle 2\hat{\mathbf{u}}, \boldsymbol{\tau} \cdot \mathbf{n}_x \rangle \\ - \left( \begin{pmatrix} \rho \Delta \mathbf{u} \otimes \tilde{\mathbf{u}} + \rho \tilde{\mathbf{u}} \otimes \Delta \mathbf{u} - \boldsymbol{\sigma} \\ \rho \Delta \mathbf{u} \end{pmatrix}, \nabla_{xt} \mathbf{v} \right) + \langle \hat{\mathbf{t}}, \mathbf{v} \rangle, \end{aligned}$$

with residual

$$(2\tilde{\mathbf{u}}, \nabla \cdot \boldsymbol{\tau}) - (\rho \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}}, \nabla \mathbf{v}) - (\mathbf{f}, \mathbf{v}).$$

## Test Norm

For the following discussion, we drop  $\rho$  (or assume  $\rho = 1$ ). Note that  $\boldsymbol{\sigma}^d = \boldsymbol{\sigma} - \frac{1}{N} \text{tr}(\boldsymbol{\sigma}) \mathbf{I}$ , and  $\boldsymbol{\sigma}^d \boldsymbol{\tau} = \boldsymbol{\sigma} \boldsymbol{\tau}^d$ . Also note that

$$(\Delta \mathbf{u} \otimes \tilde{\mathbf{u}} + \tilde{\mathbf{u}} \otimes \Delta \mathbf{u}) \nabla \mathbf{v} = \tilde{\mathbf{u}} \cdot \left( \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right) \mathbf{u},$$

since

$$\begin{aligned} (\tilde{u}_i u_j + u_i \tilde{u}_j) v_{i,j} &= \tilde{u}_i u_j v_{i,j} + u_i \tilde{u}_j v_{i,j} \\ &= \tilde{u}_j u_i v_{j,i} + u_i \tilde{u}_j v_{i,j} \\ &= u_i (\tilde{u}_j (v_{i,j} + v_{j,i})). \end{aligned}$$

Grouping terms:

$$\begin{aligned} & \left( \boldsymbol{\sigma}, \frac{1}{\nu} \boldsymbol{\tau}^d + \nabla \mathbf{v} \right) \\ & \left( \mathbf{u}, 2\nabla \cdot \boldsymbol{\tau} - \tilde{\mathbf{u}} \cdot (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) - \frac{\partial \mathbf{v}}{\partial t} \right) . \\ \|\{\mathbf{v}, \boldsymbol{\tau}\}\|^2 &= \left\| \frac{1}{\nu} \boldsymbol{\tau} - \frac{1}{N\nu} \text{tr}(\boldsymbol{\tau}) \mathbf{I} + \nabla \mathbf{v} \right\|^2 + \left\| 2\nabla \cdot \boldsymbol{\tau} - \tilde{\mathbf{u}} \cdot (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) - \frac{\partial \mathbf{v}}{\partial t} \right\|^2 + \|\mathbf{v}\|^2 + \|\boldsymbol{\tau}\|^2 . \end{aligned}$$

## Compressible

These ideas probably won't work for compressible flow, but this is retained for reference. Alternatively, if we assume the Stokes hypothesis that  $\lambda = -\frac{2}{3}\mu$ , we instead get

$$\varepsilon_{ij} = \frac{1}{2\mu} \sigma_{ij} - \frac{1}{2\mu(N-3)} \sigma_{kk} \delta_{ij} .$$