Inviscid Burgers Equation Notes

Strong form:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0$$

Space-time divergence form:

$$\nabla_{xt} \cdot \begin{pmatrix} \frac{1}{2}u^2 \\ u \end{pmatrix} = 0$$

Integrate by Parts, then Linearize

Ultra-weak form:

$$-\left(\begin{pmatrix} \frac{1}{2}u^2 \\ u \end{pmatrix}, \nabla_{xt} v\right) + \langle \hat{t}, v \rangle = 0$$

where $\hat{t} = \begin{pmatrix} \frac{1}{2}u^2 \\ u \end{pmatrix} \cdot \boldsymbol{n}_{xt}$.

Linearized form:

$$-\left(\begin{pmatrix} \tilde{u}\Delta u \\ \Delta u \end{pmatrix}, \nabla_{xt} v\right) + \langle \hat{t}, v \rangle = \left(\begin{pmatrix} \frac{1}{2}\tilde{u}^2 \\ \tilde{u} \end{pmatrix}, \nabla_{xt} v\right)$$

I don't see a way of relating the definition of $\hat{t}$ as a linear term of Δu. As the background flow, $\tilde{u}$ converges to the exact solution, Δu converges to zero. Ideally, for a converged solution, $\hat{t} = \begin{pmatrix} \frac{1}{2}\tilde{u}^2 \\ \tilde{u} \end{pmatrix} \cdot \boldsymbol{n}_{xt}$, but the current LinearTerm code does not allow this possibility.

Linearize, then Integrate by Parts

Linearized form:

$$\left(\nabla_{xt} \cdot \begin{pmatrix} \tilde{u}\Delta u \\ \Delta u \end{pmatrix}, v\right) = -\left(\nabla_{xt} \cdot \begin{pmatrix} \frac{1}{2}\tilde{u}^2 \\ \tilde{u} \end{pmatrix}, v\right)$$

Ultra-weak form:

$$-\left(\left(\begin{array}{c}\tilde{u}\Delta u\\ \Delta u\end{array}\right),\nabla_{xt}v\right)+\left\langle\Delta\hat{t},v\right\rangle=\left(\left(\begin{array}{c}\frac{1}{2}\tilde{u}^2\\ \tilde{u}\end{array}\right),\nabla_{xt}v\right)-\left\langle\tilde{\hat{t}},v\right\rangle$$

where $\Delta\hat{t}=\operatorname{tr}\left(\begin{array}{c}\tilde{u}\Delta u\\ \Delta u\end{array}\right)\cdot\boldsymbol{n}_{xt}$ and $\tilde{\hat{t}}=\operatorname{tr}\left(\begin{array}{c}\frac{1}{2}\tilde{u}^2\\ \tilde{u}\end{array}\right)\cdot\boldsymbol{n}_{xt}$.