Space-Time Discontinuous Petrov-Galerkin Finite Elements for Fluid Problems

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13th US National Congress on Computational Mechanics

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Overview of DPG

A Framework for Computational Mechanics

Find $u \in U$ such that

$$b(u, v) = l(v) \quad \forall v \in V$$

with operator $B : U \to V'$ defined by $b(u, v) = \langle Bu, v\rangle_{V' \times V}$.
This gives the operator equation

$$Bu = l \quad \in V' .$$

We wish to minimize the residual $Bu - l \in V'$:

$$u_h = \underset{w_h \in U_h}{\arg\min} \frac{1}{2} \left\| Bw_h - l \right\|_{V'}^2 .$$

Dual norms are not computationally tractable. Inverse Riesz map moves the residual to a more accessible space:

$$u_h = \underset{w_h \in U_h}{\arg\min} \frac{1}{2} \left\| R_V^{-1}(Bw_h - l) \right\|_{V}^2 .$$

Overview of DPG

Petrov-Galerkin with Optimal Test Functions

Taking the Gâteaux derivative to be zero in all directions $\delta u \in U_h$ gives,

$$\left(R_V^{-1}(Bu_h - l), R_V^{-1} B\delta u\right)_V = 0, \quad \forall \delta u \in U,$$

which by definition of the Riesz map is equivalent to

$$\left\langle Bu_h - l, R_V^{-1} B\delta u_h \right\rangle = 0 \quad \forall \delta u_h \in U_h \,,$$

with optimal test functions $v_{\delta u_h} := R_V^{-1} B\delta u_h$ for each trial function δu_h.

Resulting Petrov-Galerkin System

This gives a simple bilinear form

$$b(u_h, v_{\delta u_h}) = l(v_{\delta u_h}),$$

with $v_{\delta u_h} \in V$ that solves the auxiliary problem

$$(v_{\delta u_h}, \delta v)_V = \langle R_V v_{\delta u_h}, \delta v\rangle = \langle B\delta u_h, \delta v\rangle = b(\delta u_h, \delta v) \quad \forall \delta v \in V.$$

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Discontinuous Petrov-Galerkin

- Continuous test space produces global solve for optimal test functions
- Discontinuous test space results in an embarrassingly parallel solve

Hermitian Positive Definite Stiffness Matrix

Property of all minimum residual methods

$$b(u_h, v_{\delta u_h}) = (v_{u_h}, v_{\delta u_h})_V = \overline{(v_{\delta u_h}, v_{u_h})_V} = \overline{b(\delta u_h, v_{u_h})}$$

Error Representation Function

Energy norm of Galerkin error (residual) can be computed without exact solution

$$\|u_h - u\|_E = \|B(u_h - u)\|_{V'} = \|Bu_h - l\|_{V'} = \left\|R_V^{-1}(Bu_h - l)\right\|_V$$

Space-Time DPG

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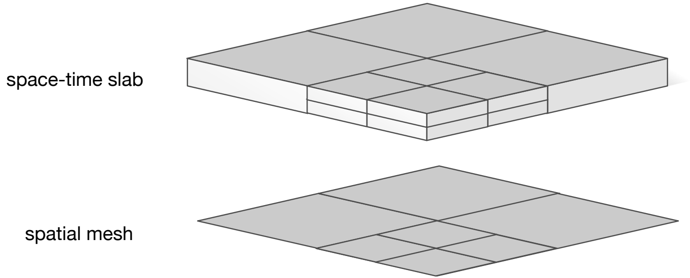
Extending DPG to Transient Problems

- Time stepping techniques are not ideally suited to highly adaptive grids
- Space-time FEM proposed as a solution
  - ✓ Unified treatment of space and time
  - ✓ Local space-time adaptivity (local time stepping)
  - ✓ Parallel-in-time integration (space-time multigrid)
  - ✗ Spatially stable FEM methods may not be stable in space-time
  - ✗ Need to support higher dimensional problems
- DPG provides necessary stability and adaptivity



Space-Time DPG for Convection-Diffusion

Space-Time Divergence Form

Equation is parabolic in space-time.

$$\frac{\partial u}{\partial t} + \boldsymbol{\beta} \cdot \nabla u - \epsilon \Delta u = f$$

This is just a composition of a constitutive law and conservation of mass.

$$\begin{aligned} \boldsymbol{\sigma} - \epsilon \nabla u &= 0 \\ \frac{\partial u}{\partial t} + \nabla \cdot (\boldsymbol{\beta} u - \boldsymbol{\sigma}) &= f \end{aligned}$$

We can rewrite this in terms of a space-time divergence.

$$\begin{aligned} \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= 0 \\ \nabla_{xt} \cdot \begin{pmatrix} \boldsymbol{\beta} u - \boldsymbol{\sigma} \\ u \end{pmatrix} &= f \end{aligned}$$

Space-Time DPG for Convection-Diffusion

Ultra-Weak Formulation with Discontinuous Test Functions

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Multiply by test function and integrate by parts over space-time element K.

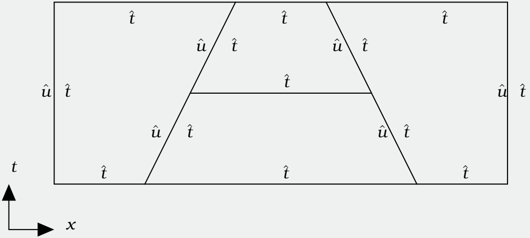
$$
\begin{aligned}
\left(\frac{1}{\epsilon}\boldsymbol{\sigma},\boldsymbol{\tau}\right)_K + (u,\nabla\cdot\boldsymbol{\tau})_K - \langle\hat{u},\boldsymbol{\tau}\cdot\boldsymbol{n}_x\rangle_{\partial K} &= 0\\
-\left(\begin{pmatrix}\boldsymbol{\beta}u-\boldsymbol{\sigma}\\ u\end{pmatrix},\nabla_{xt}v\right)_K + \langle\hat{t},v\rangle_{\partial K} &= f
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{u} &:= \operatorname{tr}(u)\\
\hat{t} &:= \operatorname{tr}(\boldsymbol{\beta}u-\boldsymbol{\sigma})\cdot\boldsymbol{n}_x\\
&\quad + \operatorname{tr}(u)\cdot n_t
\end{aligned}
$$

- Trace $\hat{u}$ defined on spatial boundaries
- Flux $\hat{t}$ defined on all boundaries

Support of Trace Variables



Bilinear form with group variables:

$$b\left(\left(u,\hat{u}\right),v\right)=\left(u,A_h^*v\right)_{L^2(\Omega_h)}+\langle\widehat{u},[\![v]\!]\rangle_{\Gamma_h}$$

For conforming v^* satisfying $A^*v^*=u$

$$\|u\|^2_{L^2(\Omega_h)}=b(u,v^*)=\frac{b(u,v^*)}{\|v^*\|_V}\|v^*\|_V$$

$$\leq\sup_{v^*\neq0}\frac{|b(u,v^*)|}{\|v^*\|}\|v^*\|=\|u\|_E\|v^*\|_V$$

Necessary robustness condition:

$$\|v^*\|_V\lesssim\|u\|_{L^2(\Omega_h)}$$

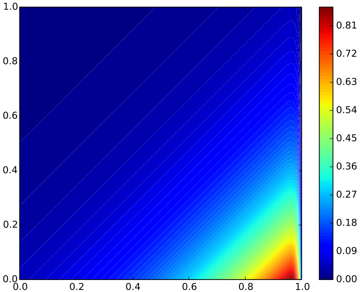
$$\Rightarrow\|u\|_{L^2(\Omega_h)}\lesssim\|u\|_E$$

Analytical Solution

$$u=e^{-lt}(e^{\lambda_1(x-1)}-e^{\lambda_2(x-1)})$$

$$\lambda_{1,2}=\frac{-1\pm\sqrt{1-4l\epsilon}}{-2\epsilon}$$

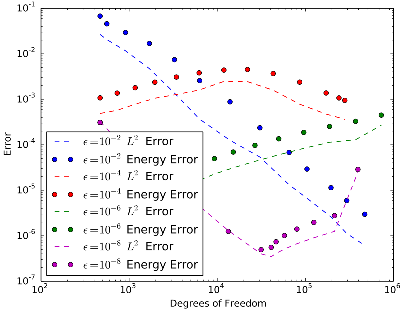
where $l=3,\quad\epsilon=10^{-2}$



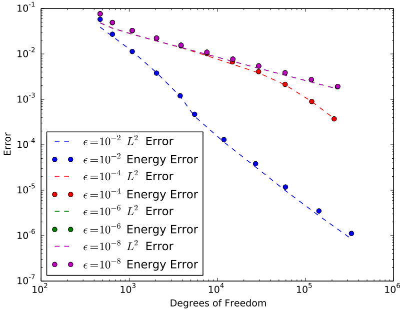
Space-Time Convection-Diffusion

Robust Norms

A norm should be: bounded by $\|u\|_{L^2(\Omega_h)}$, have good conditioning, not produce boundary layers in the optimal test function.



$$\|(v,\boldsymbol{\tau})\|^2 = \left\|\nabla\cdot\boldsymbol{\tau} - \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\right\|^2 + \left\|\frac{1}{\epsilon}\boldsymbol{\tau} + \nabla v\right\|^2 + \|v\|^2 + \|\boldsymbol{\tau}\|^2$$

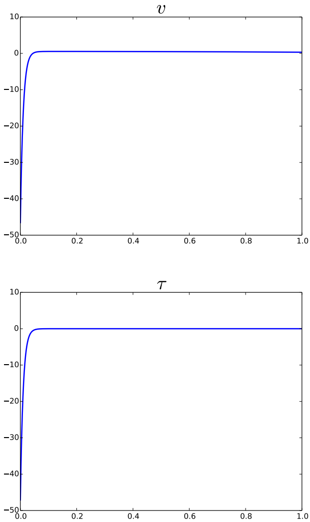


$$\|(v,\boldsymbol{\tau})\|^2 = \left\|\nabla\cdot\boldsymbol{\tau} - \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\right\|^2 + \min\left(\frac{1}{h^2},\frac{1}{\epsilon}\right)\|\boldsymbol{\tau}\|^2 + \epsilon\|\nabla v\|^2 + \|\boldsymbol{\beta}\cdot\nabla v\|^2 + \|v\|^2$$

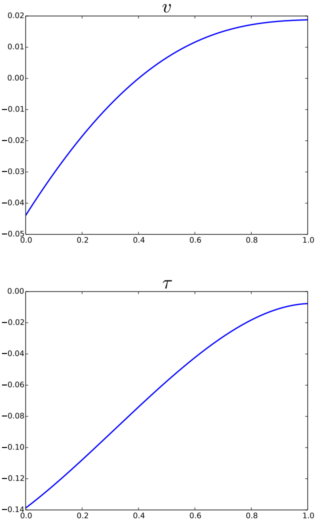
Space-Time Convection-Diffusion

Ideal Optimal Shape Functions

Graph Norm



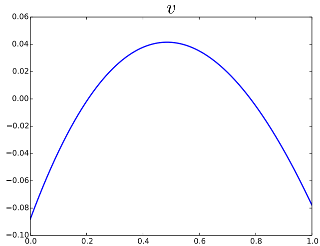
Robust Norm

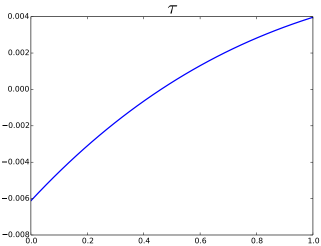


Space-Time Convection-Diffusion

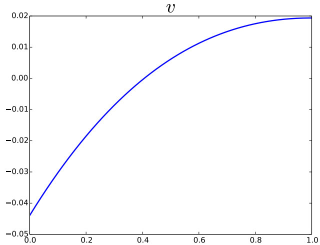
Approximated ($p = 3$) Optimal Shape Functions

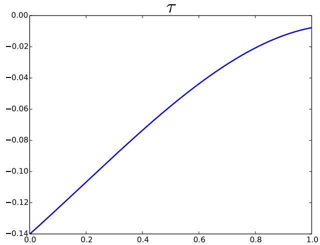
Graph Norm





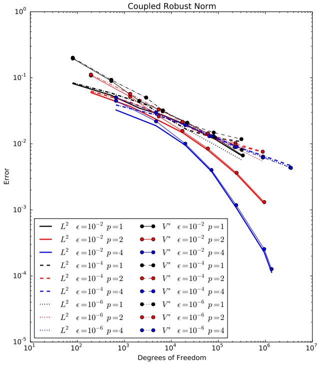
Robust Norm

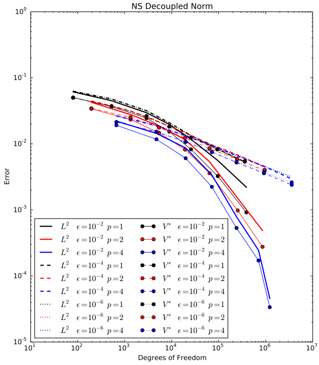




Space-Time Convection-Diffusion

Robust Norms for 2D Space-Time





$$
\begin{aligned}
\left\|(v,\boldsymbol{\tau})\right\|^2 &= \left\|\nabla\cdot\boldsymbol{\tau} - \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt} v\right\|^2 \\
&+ \min\left(\frac{1}{h^2},\frac{1}{\epsilon}\right)\left\|\boldsymbol{\tau}\right\|^2 \\
&+ \epsilon\left\|\nabla v\right\|^2 + \left\|\boldsymbol{\beta}\cdot\nabla v\right\|^2 + \left\|v\right\|^2
\end{aligned}
$$

$$
\begin{aligned}
\left\|(v,\boldsymbol{\tau})\right\|^2 &= \left\|\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt} v\right\|^2 + \left\|\nabla\cdot\boldsymbol{\tau}\right\|^2 \\
&+ \frac{1}{h^2}\left\|\boldsymbol{\tau}\right\|^2 + \left\|\nabla v\right\|^2 + \left\|v\right\|^2
\end{aligned}
$$

Space-Time Incompressible Navier-Stokes

Space-Time Divergence Form

First order space-time divergence form:

$$\frac{1}{\nu}\boldsymbol{\sigma}_1 - \nabla u_1 = 0$$

$$\frac{1}{\nu}\boldsymbol{\sigma}_2 - \nabla u_2 = 0$$

$$\nabla_{xt} \cdot \left(\left(\begin{array}{c} \boldsymbol{u} \otimes \boldsymbol{u} - \begin{pmatrix} \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2 \end{pmatrix} + p\boldsymbol{I} \\ \boldsymbol{u} \end{array} \right) \right) = \boldsymbol{f}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

$$\int_\Omega p = 0$$

Space-Time Incompressible Navier-Stokes

Ultra-Weak Formulation with Discontinuous Test Functions

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Multiplying by $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, $\boldsymbol{u}$, and q, and integrating by parts

$$\left(\frac{1}{\nu}\boldsymbol{\sigma}_1, \boldsymbol{\tau}_1\right) + (u_1, \nabla \cdot \boldsymbol{\tau}_1) - \langle \hat{u}_1, \tau_{1n} \rangle = 0$$

$$\left(\frac{1}{\nu}\boldsymbol{\sigma}_2, \boldsymbol{\tau}_2\right) + (u_2, \nabla \cdot \boldsymbol{\tau}_2) - \langle \hat{u}_2, \tau_{2n} \rangle = 0$$

$$-\left(\begin{pmatrix} \boldsymbol{u} \otimes \boldsymbol{u} - \begin{pmatrix} \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2 \end{pmatrix} + p\boldsymbol{I} \\ \boldsymbol{u} \end{pmatrix}, \nabla_{xt}\boldsymbol{v}\right) + \langle \hat{\boldsymbol{t}}, \boldsymbol{v} \rangle = (\boldsymbol{f}, \boldsymbol{v})$$

$$-(\boldsymbol{u}, \nabla q) + \langle \widehat{\boldsymbol{u} \cdot \boldsymbol{n}}, q \rangle = 0$$

$$\int_\Omega p = 0$$

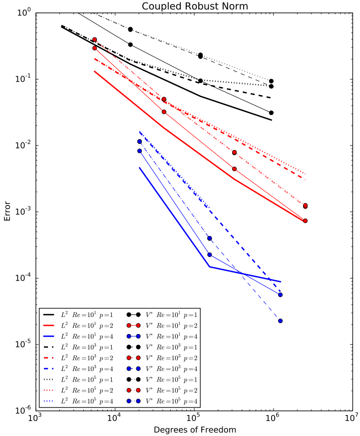
where

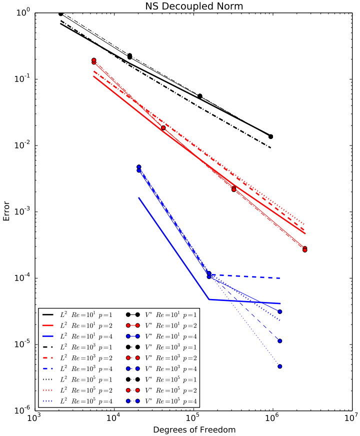
$$\hat{\boldsymbol{t}} = \operatorname{tr}\left(\begin{pmatrix} \boldsymbol{u} \otimes \boldsymbol{u} - \begin{pmatrix} \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_2 \end{pmatrix} + p\boldsymbol{I} \\ \boldsymbol{u} \end{pmatrix} \cdot \boldsymbol{n}_{xt}\right)$$

Space-Time Incompressible Navier-Stokes

Taylor-Green Vortex Problem

$$\boldsymbol{u} = \begin{pmatrix} e^{-2\nu t}\sin x \cos y \\ -e^{-2\nu t}\cos x \sin y \end{pmatrix}$$





Space-Time Navier-Stokes

First Order System with Primitive Variables

Assuming Stokes hypothesis, ideal gas law, and constant viscosity:

$$\frac{1}{\mu}\mathbb{D} - \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right) + \frac{2}{3}\nabla \cdot \boldsymbol{u}\mathbb{I} = 0$$

$$\frac{Pr}{C_p\mu}\boldsymbol{q} + \nabla T = 0$$

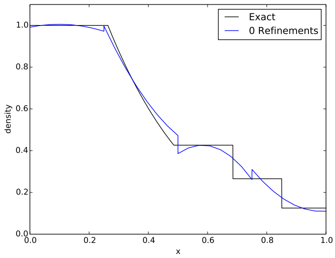
$$\nabla_{xt} \cdot \begin{pmatrix} \rho\boldsymbol{u} \\ \rho \end{pmatrix} = f_c$$

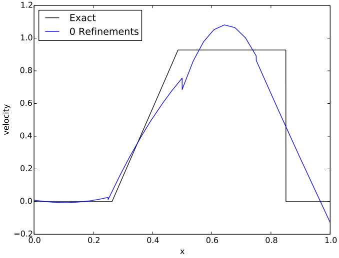
$$\nabla_{xt} \cdot \begin{pmatrix} \rho\boldsymbol{u} \otimes \boldsymbol{u} + \rho RT\mathbb{I} - \mathbb{D} \\ \rho\boldsymbol{u} \end{pmatrix} = \boldsymbol{f}_m$$

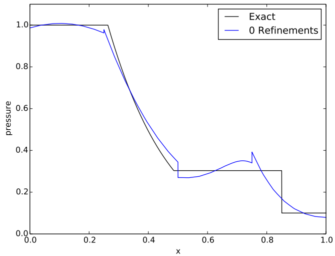
$$\nabla_{xt} \cdot \begin{pmatrix} \rho\boldsymbol{u}\left(C_v T + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) + \rho RT\boldsymbol{u} + \boldsymbol{q} - \boldsymbol{u}\cdot\mathbb{D} \\ \rho\left(C_v T + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) \end{pmatrix} = f_e\,,$$

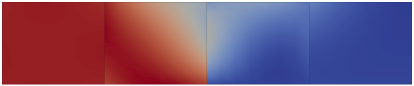
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$









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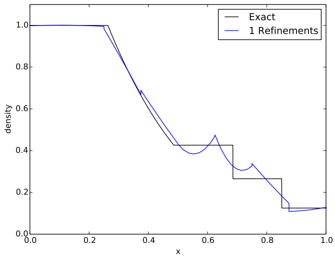

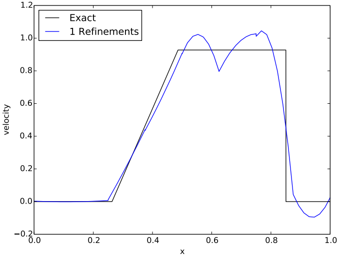

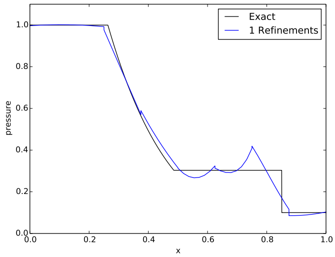

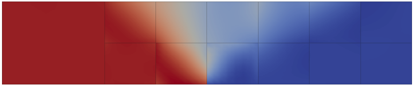

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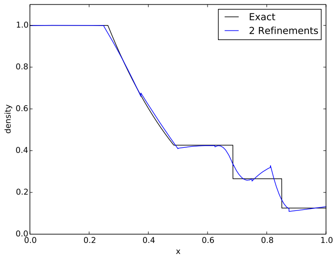

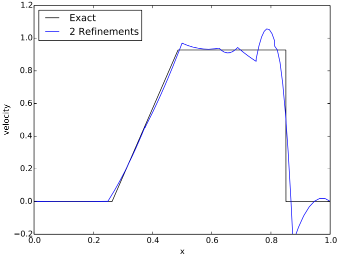

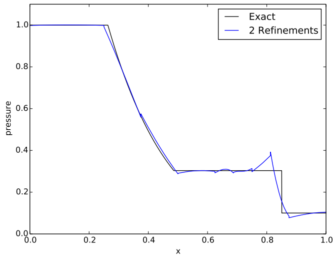

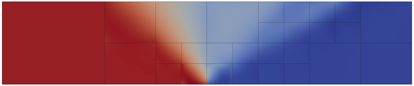

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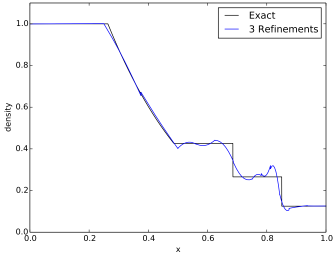

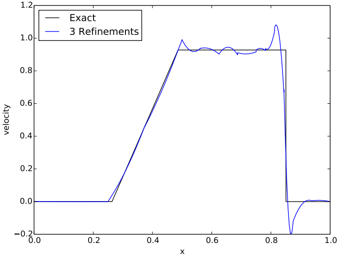

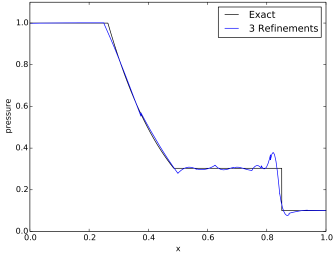

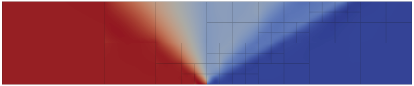

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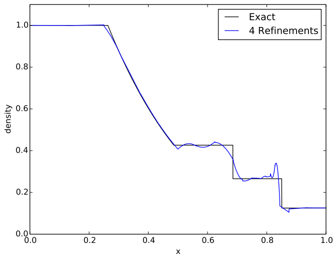

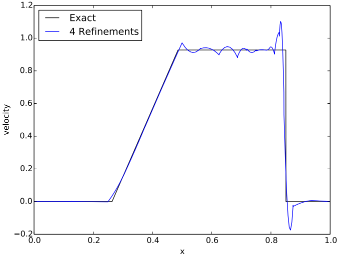

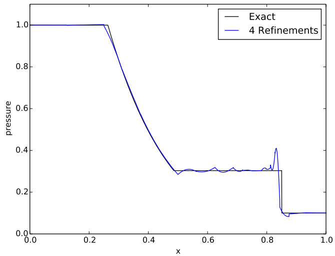

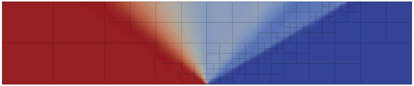

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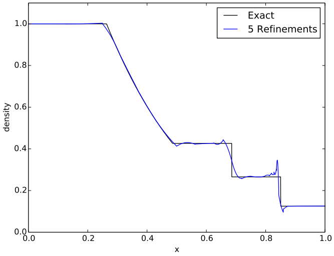

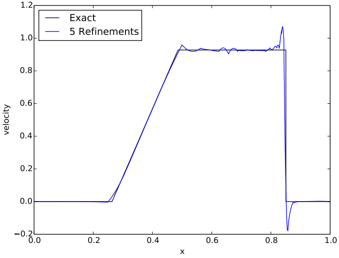

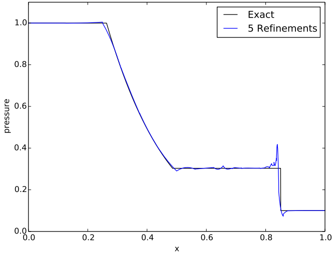

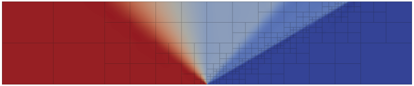

Compressible Navier-Stokes

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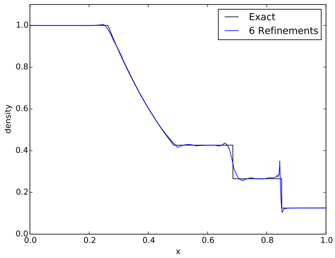

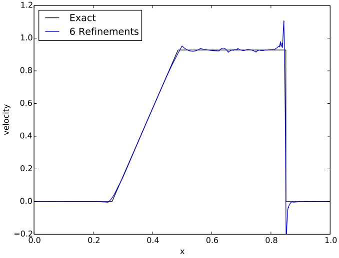

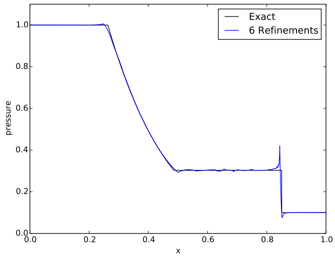

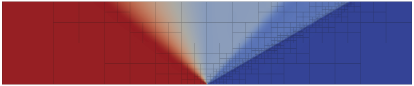

Compressible Navier-Stokes

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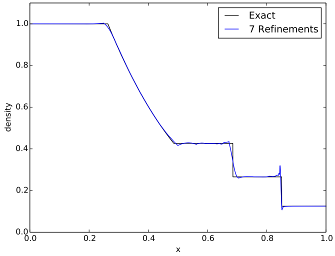
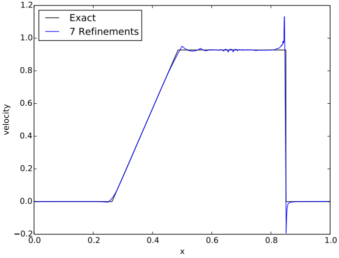
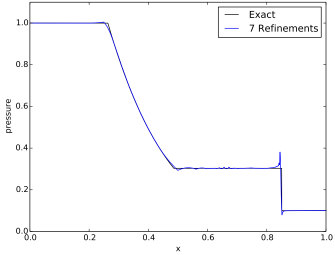
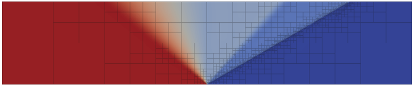
Sod Shock Tube with $\mu = 10^{-5}$



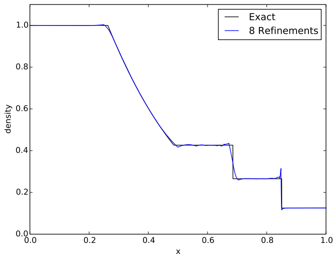


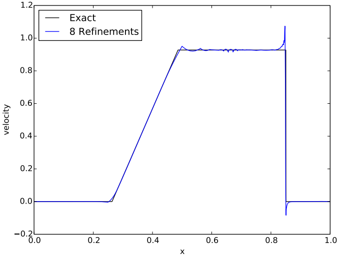


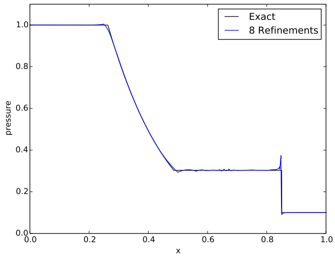


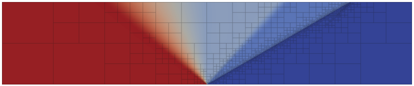
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



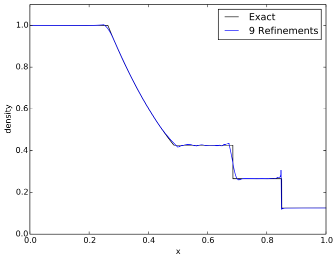


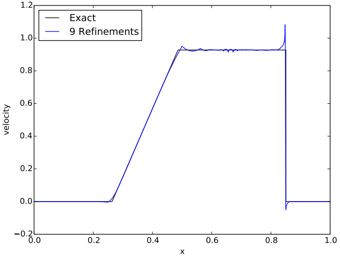


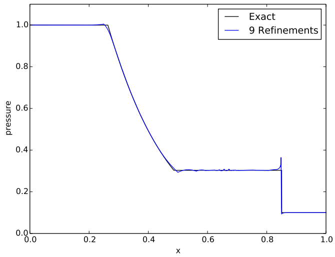


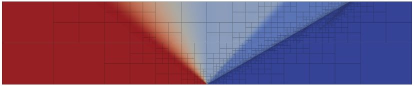
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



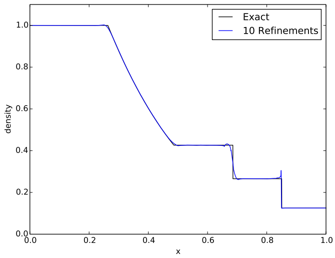


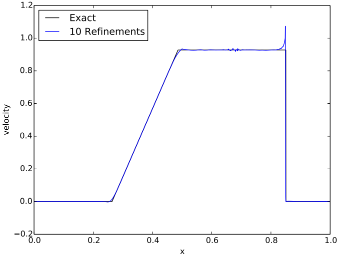


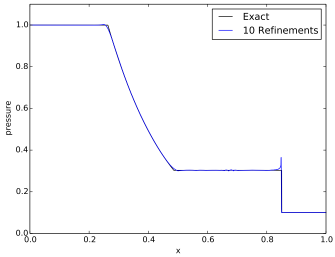


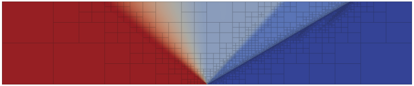
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



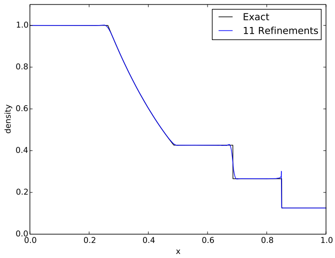


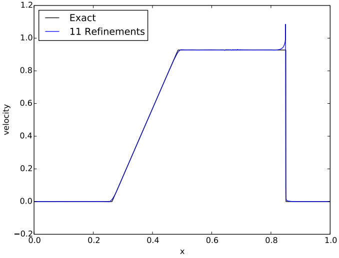


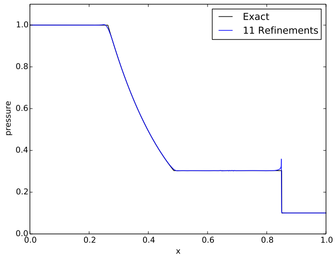


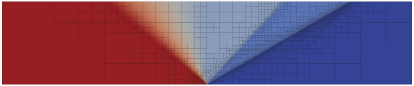
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



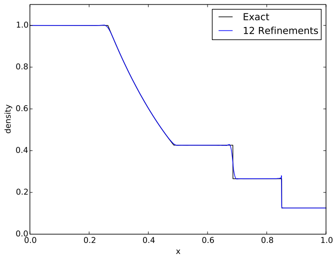


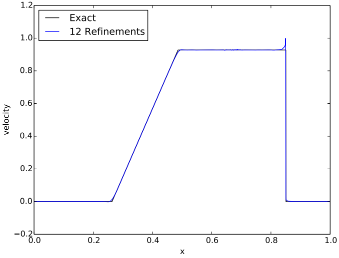


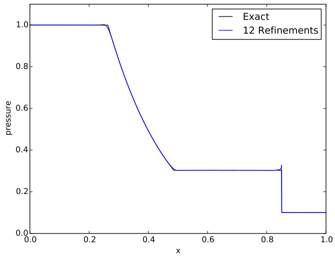


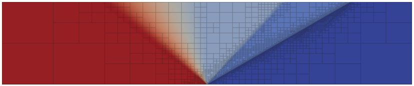
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



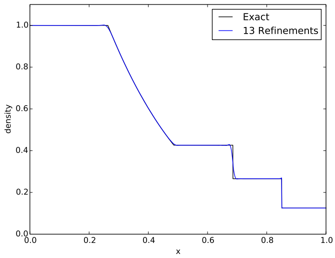


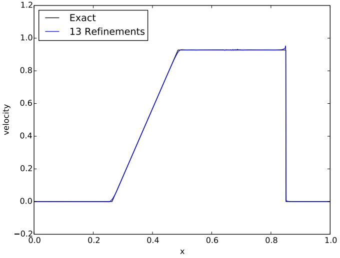


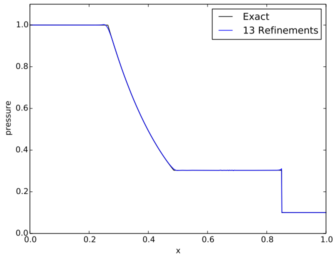


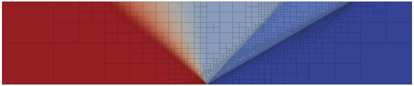
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



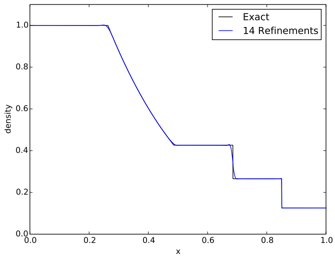


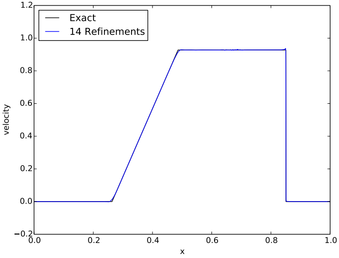


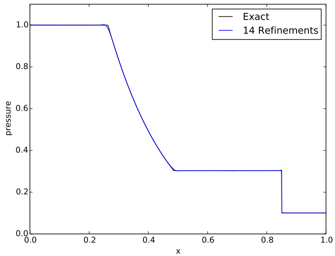


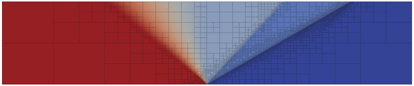
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$





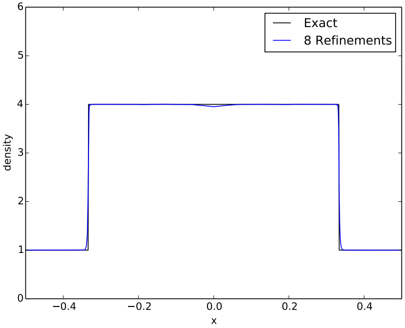


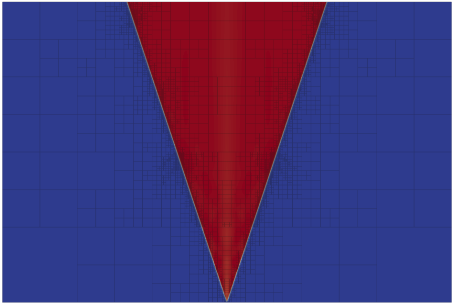


Compressible Navier-Stokes

Noh Implosion with $\mu = 10^{-3}$

Infinitely strong shock propagation.





Sequence of 4 time slabs

Related Research

Past and Present Topics in DPG Research

- Multiphysics
  - Heat conduction (Poisson and Heat equation)
  - Wave problems (Helmholtz and Maxwell)
  - Linear elasticity and plate problems
  - Convection-Diffusion, Stokes, incompressible Navier-Stokes, compressible Navier-Stokes, Euler
- Natively nonlinear DPG
- DPG for non-Hilbert L^p spaces
- Local conservation
- Iterative solvers
- Entropy scaling for physically meaningful test norms
- General polyhedral elements

Thank You!

Recommended References

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- T.E. Ellis, L.F. Demkowicz, and J.L. Chan. ``Locally Conservative Discontinuous Petrov-Galerkin Finite Elements For Fluid Problems''. In: *Comp. Math. Appl.* 68.11 (2014), pp. 1530 –1549.
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- L.F. Demkowicz and N. Heuer. ``Robust DPG Method for Convection-Dominated Diffusion Problems''. In: *SIAM J. Numer. Anal.* 51.5 (2013), pp. 1514–2537.
- J. Chan et al. ``A robust DPG method for convection-dominated diffusion problems II: Adjoint boundary conditions and mesh-dependent test norms''. In: *Comp. Math. Appl.* 67.4 (2014). High-order Finite Element Approximation for Partial Differential Equations, pp. 771 –795.
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