Robustness for Transient Problems

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Consider domain $Q = \Omega \times [0, T]$ with boundary $\Gamma = \Gamma_- \cup \Gamma_+ \cup \Gamma_0 \cup \Gamma_T$ where Γ_- is the spatial inflow boundary, Γ_+ is the spatial outflow boundary, Γ_0 is the initial time boundary, and Γ_T is the final time boundary. Let Γ_h denote the entire mesh skeleton.

Assume that boundary conditions are applied on the boundary $\Gamma_0 \subset \Gamma$. Recall that, for the ultra-weak variational formulation

$$b\left(\left(u, \widehat{u}\right), v\right) = \left(u, A_h^* v\right)_{L^2(\Omega)} + \langle \widehat{u}, [\![v]\!] \rangle_{\Gamma_h \setminus \Gamma_0}$$

we can recover

$$\|u\|_{L^2(Q)}^2 = b(u, v^*)$$

for conforming v^* satisfying the adjoint equation

$$\begin{aligned} A^* v^* &= u \\ v^* &= 0 \text{ on } \Gamma_h \setminus \Gamma_0. \end{aligned}$$

Together, these give necessary conditions on the test norm $\|\cdot\|_V$ such that we have L^2 robustness (this gives robustness in the variable u; for the first order formulation, conditions for σ must also be shown).

$$\|u\|_{L^2(Q)}^2 = b(u, v^*) \leq \frac{b(u, v^*)}{\|v^*\|_V} \|v^*\|_V \leq \|u\|_E \|v^*\|_V$$

Thus, showing $\|v^*\|_V \lesssim \|u\|_{L^2(Q)}$ gives the result that $\|u\|_{L^2(Q)} \lesssim \|u\|_E$.

1 Reaction-Diffusion

Consider reaction diffusion

$$\begin{aligned} \frac{\partial u}{\partial t} + u - \epsilon \Delta u &= f \\ u &= 0 \text{ on } \Gamma_1 \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_2 \\ u &= u_0 \text{ on } \Gamma_0. \end{aligned}$$

The adjoint equation satisfies

$$\begin{aligned}-\frac{\partial v}{\partial t}+v-\epsilon\Delta v&=u\\ v&=0 \text{ on } \Gamma_1\\ \frac{\partial v}{\partial n}&=0 \text{ on } \Gamma_2\\ v&=0 \text{ on } \Gamma_T.\end{aligned}$$

(The boundary conditions can be derived by taking the ultra-weak formulation and choosing boundary conditions such that the temporal flux and spatial flux terms $\langle \widehat{u}, [\![\tau_n]\!]\rangle_{\Gamma_1}$ and $\left\langle \widehat{f}_n, [\![v]\!]\right\rangle_{\Gamma_2}$ are zero.)

We can then derive that the test norm

$$\|v\|_V^2=\left\|\frac{\partial v}{\partial t}\right\|^2+\|v\|^2+\epsilon\|\nabla v\|^2$$

provides the necessary bound $\|v^*\|_V \lesssim \|u\|_{L^2(\Omega)}$.

To see, this we multiply the adjoint equation by two terms as follows:

1. Multiply by v and integrate over Q to get

$$-\int_Q \frac{\partial v}{\partial t}v+\int_Q v^2+\epsilon\int_Q |\nabla v|^2-\epsilon\int_0^T\int_\Gamma \frac{\partial v}{\partial n}v=\int_Q uv.$$

   Noting that either $v=0$ or $\frac{\partial v}{\partial n}=0$ on the boundary removes the integral over Γ. Next, we can factor the first term and use Young's inequality to get

$$-\int_0^T \frac{\partial}{\partial t}\int_\Omega v^2+\|v\|_Q^2+\epsilon\|\nabla v\|_Q^2 \le \frac{1}{2}\|u\|_Q^2+\frac{1}{2}\|v\|_Q^2$$

   Integrating by parts the first term gives

$$-\int_\Omega v^2\bigg|_0^T+\frac{1}{2}\|v\|_Q^2+\epsilon\|\nabla v\|_Q^2 \le \frac{1}{2}\|u\|_Q^2$$

   Using boundary condition $v=0$ at $t=T$ gives

$$\frac{1}{2}\|v\|_Q^2+\epsilon\|\nabla v\|_Q^2 \le \int_\Omega v(t=0)^2+\frac{1}{2}\|v\|_Q^2+\epsilon\|\nabla v\|_Q^2 \le \frac{1}{2}\|u\|_Q^2.$$

2. Multiply by $-\frac{\partial v}{\partial t}$ and integrate over Q. Young's inequality changes the right hand side to

$$\int_Q \frac{\partial v}{\partial t}^2-\int_Q v\frac{\partial v}{\partial t}+\epsilon\int_Q \Delta v\frac{\partial v}{\partial t}=\int_Q -u\frac{\partial v}{\partial t}\le \frac{1}{2}\|u\|_Q^2+\frac{1}{2}\left\|\frac{\partial v}{\partial t}\right\|_Q^2.$$

The term $\int_Q v \frac{\partial v}{\partial t}$ can be reduced to the positive contribution $\int_\Omega v(t=0)^2$ as above. We can then take the Laplacian term, integrate by parts in space to get

$$\int_Q \Delta v \frac{\partial v}{\partial t} = \int_0^T \int_\Omega \Delta v \frac{\partial v}{\partial t} = \int_0^T \int_\Gamma \frac{\partial v}{\partial t} \frac{\partial v}{\partial n} - \int_0^T \int_\Omega \nabla \left(\frac{\partial v}{\partial t} \right) \nabla v.$$

Since either $v = 0$ or $\frac{\partial v}{\partial n} = 0$ on Γ, the first term disappears. The second term can be bounded by noting

$$-\int_0^T \int_\Omega \nabla \left(\frac{\partial v}{\partial t} \right) \nabla v = -\int_0^T \frac{\partial}{\partial t} \int_\Omega |\nabla v|^2 = -\int_\Omega |\nabla v|^2 \Big|_0^T.$$

Since $v = 0$ at $t = T$, $\nabla v = 0$ at $t = T$ as well, and we are left with the positive contribution $\int_\Omega |\nabla v(t=0)|^2$. Then,

$$\frac{1}{2} \left\| \frac{\partial v}{\partial t} \right\|_Q^2 \leq \frac{1}{2} \|u\|_Q .$$

Together, these two show that, under test norm

$$\|v\|_V^2 = \left\| \frac{\partial v}{\partial t} \right\|^2 + \|v\|^2 + \epsilon \|\nabla v\|^2 ,$$

the adjoint equation v^* satisfies

$$\|v^*\|_V \lesssim \|u\|_{L^2(\Omega)}$$

and thus the DPG energy norm robustly bounds the L^2 norm from above

$$\|u\|_{L^2(\Omega)} \lesssim \|u\|_E .$$

2 Convection-Diffusion

Consider convection-diffusion

$$\begin{aligned} \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= 0 \\ \frac{\partial u}{\partial t} + \boldsymbol{\beta} \cdot \nabla u - \nabla \cdot \boldsymbol{\sigma} &= f \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_- \\ u &= 0 \text{ on } \Gamma_+ \\ u &= u_0 \text{ on } \Gamma_0. \end{aligned}$$

Let $\tilde{\boldsymbol{\beta}} := \begin{pmatrix} \boldsymbol{\beta} \\ 1 \end{pmatrix}$ and $\nabla_{xt} := \begin{pmatrix} \nabla \\ \frac{\partial}{\partial t} \end{pmatrix}$, then we can rewrite this as

$$\begin{aligned}
\frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= 0 \\
\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} u - \nabla \cdot \boldsymbol{\sigma} &= f \\
\frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_- \\
u &= 0 \text{ on } \Gamma_+ \\
u &= u_0 \text{ on } \Gamma_0 .
\end{aligned}$$

The adjoint equation satisfies

$$\begin{aligned}
-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v - \epsilon \Delta v &= u \\
\frac{\partial v}{\partial n} &= 0 \text{ on } \Gamma_- \\
v &= 0 \text{ on } \Gamma_+ \\
v &= 0 \text{ on } \Gamma_T .
\end{aligned}$$

(The boundary conditions can be derived by taking the ultra-weak formulation and choosing boundary conditions such that the temporal flux and spatial flux terms $\langle \widehat{u}, [\![\tau_n]\!] \rangle_{\Gamma_{out}}$ and $\left\langle \widehat{f}_n, [\![v]\!] \right\rangle_{\Gamma_{in}}$ are zero.)

We can then derive that the test norm

$$\begin{aligned}
\|(v, \boldsymbol{\tau})\|_{V,K}^2 \coloneqq \min\left\{\frac{1}{\epsilon}, \frac{1}{\mu(K)}\right\} \|\boldsymbol{\tau}\|_K^2 + \left\|\nabla \cdot \boldsymbol{\tau} - \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v\right\|_K^2 \\
+ \left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v\right\|_K^2 + \epsilon \|\nabla v\|_K^2 + \|v\|_K^2 \ ,
\end{aligned} \tag{1}$$

provides the necessary bound $\|v^*\|_V \lesssim \|u\|_{L^2(Q)}$.

To see this, we multiply the adjoint equation by two terms as follows:

1. Multiply by $-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v$ and integrate over Q to get

$$\left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v\right\| = -\int_Q u \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v - \epsilon \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \Delta v \, . \tag{2}$$

Note that

$$
\begin{aligned}
-\int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \Delta v &= -\int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla \cdot \nabla v \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \int_Q \nabla(\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v) \cdot \nabla v \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \int_Q (\nabla \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v) \cdot \nabla v \\
&\quad + \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla \nabla_{xt} v \cdot \nabla v \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \int_Q (\nabla \boldsymbol{\beta} \cdot \nabla v) \cdot \nabla v \\
&\quad + \frac{1}{2} \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} (\nabla v \cdot \nabla v) \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \int_Q (\nabla \boldsymbol{\beta} \cdot \nabla v) \cdot \nabla v \\
&\quad + \frac{1}{2} \int_\Gamma \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n} (\nabla v \cdot \nabla v) - \frac{1}{2} \int_Q \nabla_{xt} \cdot \tilde{\boldsymbol{\beta}} (\nabla v \cdot \nabla v) \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \int_Q (\nabla \boldsymbol{\beta} \cdot \nabla v) \cdot \nabla v \\
&\quad + \frac{1}{2} \int_\Gamma \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n} (\nabla v \cdot \nabla v) - \frac{1}{2} \int_Q \nabla \cdot \boldsymbol{\beta} (\nabla v \cdot \nabla v) \\
&= -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \frac{1}{2} \int_\Gamma \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n} (\nabla v \cdot \nabla v) \\
&\quad + \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \boldsymbol{I}) \nabla v
\end{aligned}
$$

Plugging this into (2), we get

$$
\begin{aligned}
\left\|\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\right\| &= -\int_Q u\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v + \epsilon\int_Q \nabla v(\nabla\boldsymbol{\beta} - \frac{1}{2}\nabla\cdot\boldsymbol{\beta}\boldsymbol{I})\nabla v \\
&\quad -\epsilon\int_{\Gamma_x}\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\nabla v\cdot\boldsymbol{n}_x + \epsilon\frac{1}{2}\int_\Gamma\tilde{\boldsymbol{\beta}}\cdot\boldsymbol{n}(\nabla v\cdot\nabla v) \\
&= -\int_Q u\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v + \epsilon\int_Q \nabla v(\nabla\boldsymbol{\beta} - \frac{1}{2}\nabla\cdot\boldsymbol{\beta}\boldsymbol{I})\nabla v \\
&\quad -\int_{\Gamma_-}\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\underbrace{\nabla v\cdot\boldsymbol{n}_x}_{=0} - \int_{\Gamma_+}\left(\underbrace{\frac{\partial v}{\partial t}}_{=0} + \boldsymbol{\beta}\cdot\nabla v\right)\nabla v\cdot\boldsymbol{n}_x \\
&\quad +\frac{1}{2}\int_{\Gamma_-}\underbrace{\boldsymbol{\beta}\cdot\boldsymbol{n}_x}_{<0}(\nabla v\cdot\nabla v) + \frac{1}{2}\int_{\Gamma_+}\boldsymbol{\beta}\cdot\boldsymbol{n}_x(\nabla v\cdot\nabla v) \\
&\quad +\frac{1}{2}\int_{\Gamma_0}\underbrace{n_t}_{<0}(\nabla v\cdot\nabla v) + \frac{1}{2}\int_{\Gamma_T} n_t\underbrace{(\nabla v\cdot\nabla v)}_{=0} \\
&\le -\int_Q u\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v + \epsilon\int_Q \nabla v(\nabla\boldsymbol{\beta} - \frac{1}{2}\nabla\cdot\boldsymbol{\beta}\boldsymbol{I})\nabla v \\
&\quad +\int_{\Gamma_+}\left(-\frac{\partial v}{\partial\boldsymbol{n}_x}\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{n}_x\nabla v\right)\cdot\nabla v \\
&= -\int_Q u\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v + \epsilon\int_Q \nabla v(\nabla\boldsymbol{\beta} - \frac{1}{2}\nabla\cdot\boldsymbol{\beta}\boldsymbol{I})\nabla v \\
&\quad +\int_{\Gamma_+}\left(-\frac{\partial v}{\partial\boldsymbol{n}_x}\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{n}_x\frac{\partial v}{\partial\boldsymbol{n}_x}\boldsymbol{n}_x\right)\cdot\frac{\partial v}{\partial\boldsymbol{n}_x}\boldsymbol{n}_x \\
&= -\int_Q u\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v + \epsilon\int_Q \nabla v(\nabla\boldsymbol{\beta} - \frac{1}{2}\nabla\cdot\boldsymbol{\beta}\boldsymbol{I})\nabla v \\
&\quad \underbrace{-\frac{1}{2}\int_{\Gamma_+}\left(\frac{\partial v}{\partial\boldsymbol{n}_x}\right)^2\boldsymbol{\beta}\cdot\boldsymbol{n}_x}_{<0} \\
&\le -\int_Q u\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v + \epsilon\int_Q \nabla v(\nabla\boldsymbol{\beta} - \frac{1}{2}\nabla\cdot\boldsymbol{\beta}\boldsymbol{I})\nabla v \\
&\le -\frac{\|u\|}{2} + \frac{\left\|\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\right\|}{2} + \epsilon\int_Q \nabla v(\nabla\boldsymbol{\beta} - \frac{1}{2}\nabla\cdot\boldsymbol{\beta}\boldsymbol{I})\nabla v \\
&\le -\frac{\|u\|}{2} + \frac{\left\|\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\right\|}{2} + \epsilon C\|\nabla v\|^2
\end{aligned}
$$

2. Define $w = e^{T-t}v$ and note that $\frac{\partial w}{\partial t} = \left(\frac{\partial v}{\partial t} - v\right)e^{T-t}$ while $\nabla w = \cancelto{0}{\nabla e^{T-t}}v + e^{T-t}\nabla v$ and $\nabla\cdot(\boldsymbol{\beta}w) = \nabla\cdot(\boldsymbol{\beta})e^{T-t}v + \boldsymbol{\beta}\cdot e^{T-t}\nabla v$ and $\Delta w = e^{T-t}\Delta v$.

Also, $\nabla_{xt} w = \frac{\partial e^{T-t} v}{\partial t} + \nabla e^{T-t} v = e^{T-t}(\nabla_{xt} v - v)$. Plugging this into the adjoint equation, we get

$$-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt}(w) - \epsilon \Delta w = u - \epsilon \nabla \cdot \boldsymbol{\sigma}$$

or

$$\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt}(v) - v + \epsilon \Delta v = e^{t-T}(-u + \epsilon \nabla \cdot \boldsymbol{\sigma})$$

Multiply by $-v$ and integrate to get

$$\int_Q -\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v v + v^2 - \epsilon \Delta v v = \int_Q e^{t-T} u v - \epsilon \int_Q e^{t-T} \nabla \cdot \boldsymbol{\sigma} v$$

Then

$$\begin{aligned}
\|v\|^2 &= \int_Q e^{t-T} u v - \epsilon \int_Q e^{t-T} \nabla \cdot \boldsymbol{\sigma} v + \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v v + \epsilon \int_Q \Delta v v \\
&= \int_Q e^{t-T} u v - \epsilon \int_Q e^{t-T} \nabla \cdot \boldsymbol{\sigma} v + \frac{1}{2} \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} (v)^2 - \epsilon \int_Q \nabla v \nabla v + \epsilon \int_\Gamma v \nabla v \cdot \boldsymbol{n}
\end{aligned}$$

Or

$$\begin{aligned}
\|v\|^2 + \epsilon \|\nabla v\|^2 &= \int_Q e^{t-T} u v - \epsilon \int_Q e^{t-T} \nabla \cdot \boldsymbol{\sigma} v \\
&\quad - \frac{1}{2} \int_Q \underbrace{\nabla_{xt} \cdot \tilde{\boldsymbol{\beta}}}_{=0} (v)^2 + \frac{1}{2} \int_\Gamma v^2 \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n} + \epsilon \int_{\Gamma_x} v \nabla v \cdot \boldsymbol{n}_x \\
&= \int_Q e^{t-T} u v + \epsilon \int_Q e^{t-T} \boldsymbol{\sigma} \cdot \nabla v - \epsilon \int_{\Gamma_-} v \underbrace{\boldsymbol{\sigma} \cdot \boldsymbol{n}_x}_{=\epsilon \frac{\partial v}{\partial n} = 0} - \epsilon \int_{\Gamma_+} \underbrace{v}_{=0} \boldsymbol{\sigma} \cdot \boldsymbol{n}_x \\
&\quad + \frac{1}{2} \int_{\Gamma_-} v^2 \underbrace{\boldsymbol{\beta} \cdot \boldsymbol{n}_x}_{<0} + \frac{1}{2} \int_{\Gamma_+} \underbrace{v^2}_{=0} \boldsymbol{\beta} \cdot \boldsymbol{n}_x \\
&\quad + \frac{1}{2} \int_{\Gamma_0} \underbrace{v^2(-n_t)}_{<0} + \frac{1}{2} \int_{\Gamma_T} \underbrace{v^2}_{=0} n_t \\
&\quad + \epsilon \int_{\Gamma_-} v \underbrace{\nabla v \cdot \boldsymbol{n}_x}_{=0} + \epsilon \int_{\Gamma_+} \underbrace{v}_{=0} \nabla v \cdot \boldsymbol{n}_x \\
&\leq \left\| e^{t-T} \right\|_{L_\infty(Q)} \left(\int_Q u v + \epsilon \int_Q \boldsymbol{\sigma} \cdot \nabla v \right) \\
&\leq \left(\frac{\|u\|^2}{2} + \frac{\epsilon \|\boldsymbol{\sigma}\|^2}{2} + \frac{\|v\|^2}{2} + \frac{\epsilon \|\nabla v\|^2}{2} \right)
\end{aligned}$$

3. Note that $\boldsymbol{\tau} = \epsilon \nabla v$, so we already have control of $\|\boldsymbol{\tau}\|$.
4. We also have that

3 Robustness for transient problems given spatial robustness

Suppose we have the transient problem

$$\frac{\partial u}{\partial t} + Au = f$$

with initial condition $u(x, 0) = u_0$. Suppose that DPG is robust under the ultra-weak variational formulation for the steady problem

$$(u, A_h^* v)_{L^2(\Omega)} + \langle \widehat{u}, [\![v]\!] \rangle_{\Gamma_h \setminus \Gamma_0} = (f, v)$$

with test norm $\|v\|_V$. Then, can we show that

$$\|v\|_{V,t} \coloneqq \|v\|_V + \left\| \frac{\partial v}{\partial t} \right\|_{L^2(\Omega)}$$

also leads to a robust upper bound of the L^2 norm by the DPG energy norm? I believe this may be possible. The adjoint equation for robustness for the transient problem gives

$$-\frac{\partial v}{\partial t} + A^* v = u$$

with $v = 0$ at $t = T$...