Reply to the Reviewers and the Editor

We are very grateful to the reviewer for reading our paper and the critiqe. The reviewers questions along with our answers are listed below. All changes in the revised version of the paper are in red.

I read the paper by T. Ellis, J. Chan, and L. Demkowicz. I think the paper addresses an interesting issue, and the results are interesting. On the other hand, I believe that the paper has a few rought spots that need clarification/correction, and also that it should have a written in a wider sense than the one it is currently written, especially considering that it is aimed at appearing in a survey volume like the one you are editing. Then, I detail now what are my main suggestions for this paper:

1. *Make, possibly in the concluding remarks, some comments on the relationship this work has with HdG, and also with multiscale FEM. I definitely think there is a connection between the approaches, and then I would encourage the authors to make this connection explicit (for example, the propeties of broken test functions in S 2.2.4 are very similar to the properties of Multiscale basis functions).*

   A discussion of how DPG relates to multiscale methods and HDG has been added to the introduction.

2. *In the problem presentation, is the convective field β a time-dependent function? If so, there is clearly an overlap between the distinct components of the boundary of Q. This might have an impact in the boundary conditions chosen.*

   We are not sure we understand the question. The boundary of the space-time domain $Q := \Omega \times (0,T)$ consists of three parts: $\Omega \times 0$, $\Omega \times T$ and the lateral boundary $\partial\Omega \times (0,T)$. The lateral boundary is partitioned into two parts dependent upon the advection vector that may vary in time,

   $$\begin{aligned} \Gamma_{in} &:= \ \{(x,t) \in \partial\Omega \times (0,T) \ : \ \beta(x,t)\cdot n(x) < 0\} \\ \Gamma_{out} &:= \ \{(x,t) \in \partial\Omega \times (0,T) \ : \ \beta(x,t)\cdot n(x) \geq 0\} \end{aligned}$$

   Clearly, one and only one of the defining conditions is satisfied which means that the lateral boundary is *partitioned* into the two sets.

3. *In equation (3), I believe there is a boundary term missing (this term is then present in (4), containing a numerical flux).*

   The are three variational formulations. In the first one (equation (3)), test functions satisfy boundary conditions present in the definition of the L^2-adjoint and, as a consequence, no boundary terms are present. Removing boundary conditions from test functions in the second formulation (equation (4)) results in the introduction of additional unknowns - the traces and fluxes that live on the domain boundary. Finally, the ultimate variational formulation with broken test spaces (equation (5)) uses traces and fluxes on *the whole mesh skeleton* including the domain boundary. The intermediate formulation (4) is supposed to help reader to transition to the final formulation (5).

4. *I think the first paragraph in page 6 should be in the introduction. It states the main idea of the paper, so....*

   We decided to maintain this paragraph in its current place, but added the following extra comment to the introduction.

   In the sense, the main challenge is to come up with a correct test norm. The residual is measured in the dual test norm, and the DPG method minimizes the residual. The residual can be interpreted as a special *energy norm*. In other words, the DPG method delivers an orthogonal projection in the energy norm. The task is especially challenging for singular perturbation problems. Given a trial norm, we strive to determine a quasi-optimal test norm such that the corresponding energy norm is robustly equivalent to the trial norm of choice. An additional difficulty comes from the fact that the optimal test functions should be easily approximated with a simple enrichment strategy. For convection dominated diffusion, this means that the test functions should not develop boundary layers. The task of determining the quasi optimal test norm (we call it a *robust test norm* leads then to a stability analysis for the adjoint equation which is the subject of this paper. For a more general discussion on the subject, see [2].

5. *Also in page 6, could the authors give some extra motivation on the inclusion of the L^2 norm of u in the norm? It doesn't seem to come directly from the formulation. Also, am I correct in supposing that, unless this term is included in the norm, the local problems appearing by using the broken norms given in (7) and (8) would not be well-posed?*

   The following remark has been added to the text.

   Remark. In the DPG technology, the test norm must be *localizable*, i.e.,

$$\|v\|_V^2 = \sum_K \|v\|_{V(K)}^2$$

   where $\|v\|_{V(K)}$ denotes a test norm (and not just a seminorm) for the element test space. In practice this means the addition of properly scaled L^2-terms. Without those terms, we could not invert the Riesz operator on the element level. Addition of the L^2 terms does not necessarily contradict the robustness of the norm, see the discussion in [2] on bounded below operators. An alternate strategy has been explored in [4] where we enforce element conservation property by securing the presence of a constant function in the element test space. The residual is then minimized only over the orthogonal complement to the constants which eliminates the need for adding the L^2-term to the test norm.

6. *Could the authors include some more comments in their proofs? They seem to be correct, and they follow somehow familiar arguments, but still I believe some more details should be given on how we go from one point to the next one. Also, some comments on how to link one result to the following one would be welcome, just to make the paper easier to read.*

   We have added a number of additional comments in the derivations to make the logical flow easier (all in red).

7. *At the end of Section 3 the authors could give some comments on what is the significance of the previous results. In particular, do they lead, naturally, to an error estimate of the method?*

   A couple lines have been added at the end of Section 3 to summarize.

   In conclusion, with either robust test norm, we can claim the following stability result,

$$\begin{aligned}\|u-u_h\| &\lesssim \|(u,\boldsymbol{\sigma},\hat{u},\hat{t})-(u_h,\boldsymbol{\sigma}_h,\hat{u}_h,\hat{t}_h)\|_E \\ &= \inf_{(u_h,\boldsymbol{\sigma}_h,\hat{u}_h,\hat{t}_h)} \|(u,\boldsymbol{\sigma},\hat{u},\hat{t})-(u_h,\boldsymbol{\sigma}_h,\hat{u}_h,\hat{t}_h)\|_E \,.\end{aligned}$$

   Notice that, contrary to the steady-state case, we have not been able to secure a robust L^2 bound for the stress. The best approximation error in the energy norm can be estimated locally, i.e. element-wise, see [3, 1]. This leads to an ultimate, final h estimate but not necessarily with robust constants. The loss of robustness in the best approximation error estimate is the consequence of rescaling the L^2-terms to avoid boundary layers in the optimal test functions. However, similarly to the steady-state case, with refinements, the mesh-dependent L^2-terms converge to the optimal ones so we hope to regain robustness in the limit. We do not attempt to analyze the best approximation error in this contribution and restrict ourselves to numerical experiments only.

8. *The legends in the plots are barely readable. Can the authors please make them clearer? Also, it could be interesting to see an elevation of a numerical solution containing a boundary, or inner, layer.*

   Fixed.

9. *Finally, there are several typos and inconsistencies in the references. Please unify the format, and order them alphabetically.*

   Fixed.

References

[1] J. Chan, N. Heuer, T. Bui-Thanh, and L. Demkowicz. A robust DPG method for convection-dominated diffusion problems II: Adjoint boundary conditions and mesh-dependent test norms. *Comp. Math. Appl.*, 67(4):771 – 795, 2014. High-order Finite Element Approximation for Partial Differential Equations.

[2] L.F. Demkowicz and J. Gopalakrishnan. Discontinuous Petrov-Galerkin (DPG) method. Technical Report 15-20, ICES, December 2015.

[3] L.F. Demkowicz and N. Heuer. Robust DPG method for convection-dominated diffusion problems. *SIAM J. Numer. Anal.*, 51(5):1514–2537, 2013.

[4] T.E. Ellis, L.F. Demkowicz, and J.L. Chan. Locally conservative discontinuous Petrov-Galerkin finite elements for fluid problems. *Comp. Math. Appl.*, 68(11):1530 – 1549, 2014.