Pressureless Navier-Stokes Formulation

Truman E. Ellis

January 22, 2014

We can derive the compressible Navier-Stokes equations in terms of the Cauchy stress tensor. Note that

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}\,,$$

and

$$\begin{aligned}\sigma_{ii} &= 2\mu\varepsilon_{ii} + N\lambda\varepsilon_{ii}\\ &= (2\mu + N\lambda)\varepsilon_{ii}\,,\end{aligned}$$

where N is the dimension. Then

$$\begin{aligned}\varepsilon_{ij} &= \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu}\varepsilon_{kk}\delta_{ij}\\ &= \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(2\mu + N\lambda)}\sigma_{kk}\delta_{ij}\\ &= \frac{1}{2\mu}\sigma_{ij} - \frac{1}{2\mu(\frac{2\mu}{\lambda} + N)}\sigma_{kk}\delta_{ij}\,.\end{aligned}$$

Incompressible

If we assume an incompressible medium, then $\lambda \to \infty$ and

$$\begin{aligned}\varepsilon_{ij} &= \frac{1}{2\mu}\sigma_{ij} - \frac{1}{2N\mu}\sigma_{kk}\delta_{ij}\\ &= \frac{1}{2\mu}\left[\sigma_{ij} - \frac{1}{N}\sigma_{kk}\delta_{ij}\right]\,.\end{aligned}$$

This embeds the zero divergence condition. If we take the trace of both sides, we get

$$\nabla \cdot \boldsymbol{u} = \varepsilon_{ii} = \frac{1}{2\mu}\left[\sigma_{ii} - \sigma_{ii}\right] = 0\,.$$

The space-time form of the Cauchy momentum equation is

$$\nabla_{xt} \cdot \begin{pmatrix} \rho\boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{\sigma} \\ \rho\boldsymbol{u} \end{pmatrix} = \boldsymbol{f}\,.$$

Our incompressible Navier-Stokes sytem is then

$$\begin{aligned}\boldsymbol{\sigma} - \frac{1}{N}\operatorname{tr}(\boldsymbol{\sigma})\boldsymbol{I} - \mu\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right) &= 0\\ \nabla_{xt}\cdot\begin{pmatrix}\rho\boldsymbol{u}\otimes\boldsymbol{u}-\boldsymbol{\sigma}\\ \rho\boldsymbol{u}\end{pmatrix} &= \boldsymbol{f}\,.\end{aligned}$$

We multiply by test functions $\boldsymbol{\tau}$ (symmetric tensor) and $\boldsymbol{v}$ and integrate by parts over a space-time element K.

$$\begin{aligned}(\boldsymbol{\sigma},\boldsymbol{\tau}) - \left(\frac{1}{N}\operatorname{tr}(\boldsymbol{\sigma})\boldsymbol{I},\boldsymbol{\tau}\right) + (2\mu\boldsymbol{u},\nabla\cdot\boldsymbol{\tau}) - \langle 2\mu\hat{\boldsymbol{u}},\boldsymbol{\tau}\cdot\boldsymbol{n}_x\rangle &= 0\\ -\left(\begin{pmatrix}\rho\boldsymbol{u}\otimes\boldsymbol{u}-\boldsymbol{\sigma}\\ \rho\boldsymbol{u}\end{pmatrix},\nabla_{xt}\boldsymbol{v}\right) + \left\langle\hat{\boldsymbol{t}},\boldsymbol{v}\right\rangle &= (\boldsymbol{f},\boldsymbol{v})\ ,\end{aligned}$$

where

$$\begin{aligned}\hat{\boldsymbol{u}} &= \operatorname{tr}(\boldsymbol{u})\\ \hat{\boldsymbol{t}}_m &= \operatorname{tr}\left(\rho\boldsymbol{u}\otimes\boldsymbol{u}-\boldsymbol{\sigma}\right)\cdot\boldsymbol{n}_x + \operatorname{tr}\left(\rho\boldsymbol{u}\right)n_t\,.\end{aligned}$$

Linearization

The Jacobian is

$$\begin{aligned}&(\boldsymbol{\sigma},\boldsymbol{\tau}) - \left(\frac{1}{N}\operatorname{tr}(\boldsymbol{\sigma})\boldsymbol{I},\boldsymbol{\tau}\right) + (2\mu\boldsymbol{u},\nabla\cdot\boldsymbol{\tau}) - \langle 2\mu\hat{\boldsymbol{u}},\boldsymbol{\tau}\cdot\boldsymbol{n}_x\rangle\\ &-\left(\begin{pmatrix}\rho\Delta\boldsymbol{u}\otimes\tilde{\boldsymbol{u}}+\rho\tilde{\boldsymbol{u}}\otimes\Delta\boldsymbol{u}-\boldsymbol{\sigma}\\ \rho\Delta\boldsymbol{u}\end{pmatrix},\nabla_{xt}\boldsymbol{v}\right) + \left\langle\hat{\boldsymbol{t}},\boldsymbol{v}\right\rangle\ ,\end{aligned}$$

with residual

$$-\left(\rho\tilde{\boldsymbol{u}}\otimes\tilde{\boldsymbol{u}},\nabla\boldsymbol{v}\right) - (\boldsymbol{f},\boldsymbol{v})\ .$$

Test Norm

For the following discussion, we drop ρ (or assume $\rho = 1$). Note that $\boldsymbol{\sigma}^d = \boldsymbol{\sigma} - \frac{1}{N}\operatorname{tr}(\boldsymbol{\sigma})\boldsymbol{I}$, and $\boldsymbol{\sigma}^d\boldsymbol{\tau} = \boldsymbol{\sigma}\boldsymbol{\tau}^d$. Also note that

$$(\Delta\boldsymbol{u}\otimes\tilde{\boldsymbol{u}}+\tilde{\boldsymbol{u}}\otimes\Delta\boldsymbol{u})\nabla\boldsymbol{v} = \tilde{\boldsymbol{u}}\cdot\left(\nabla\boldsymbol{v}+(\nabla\boldsymbol{v})^T\right)\boldsymbol{u}\,,$$

since

$$\begin{aligned}\left(\tilde{u}_iu_j + u_i\tilde{u}_j\right)v_{i,j} &= \tilde{u}_iu_jv_{i,j} + u_i\tilde{u}_jv_{i,j}\\ &= \tilde{u}_ju_iv_{j,i} + u_i\tilde{u}_jv_{i,j}\\ &= u_i(\tilde{u}_j(v_{i,j}+v_{j,i}))\,.\end{aligned}$$

Grouping terms:

$$\left(\boldsymbol{\sigma}, \boldsymbol{\tau}^d + \nabla \boldsymbol{v}\right)$$
$$\left(\boldsymbol{u}, 2\mu\nabla \cdot \boldsymbol{\tau} - \tilde{\boldsymbol{u}} \cdot (\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T) - \frac{\partial v}{\partial t}\right) .$$

Alternatively, if we divided the first equation by 2μ, we would have gotten:

$$\left(\boldsymbol{\sigma}, \frac{1}{2\mu}\boldsymbol{\tau}^d + \nabla \boldsymbol{v}\right)$$
$$\left(\boldsymbol{u}, \nabla \cdot \boldsymbol{\tau} - \tilde{\boldsymbol{u}} \cdot (\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T) - \frac{\partial v}{\partial t}\right) .$$

So our graph norm based on the first version is defined by

$$\|\{\boldsymbol{v}, \boldsymbol{\tau}\}\|^2 = \left\|\boldsymbol{\tau} - \frac{1}{N}\operatorname{tr}(\boldsymbol{\tau})\boldsymbol{I} + \nabla \boldsymbol{v}\right\|^2 + \left\|2\mu\nabla \cdot \boldsymbol{\tau} - \tilde{\boldsymbol{u}} \cdot \left(\nabla \boldsymbol{v} + (\boldsymbol{v})^T\right) - \frac{\partial v}{\partial t}\right\|^2 + \|v\|^2 .$$

Compressible

These ideas probably won't work for compressible flow, but this is retained for reference. Alternatively, if we assume the Stokes hypothesis that $\lambda = -\frac{2}{3}\mu$, we instead get

$$\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{1}{2\mu(N-3)}\sigma_{kk}\delta_{ij} .$$