DPG for Engineers

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Abstract

We discuss and explain the discontinuous Petrov-Galerkin finite element method with an engineering audience in mind. As such, we try not to delve into too much mathematics.

1 Motivation and Introductory Concepts

The discontinuous Petrov-Galerkin finite element method is a promising new framework for automated computing of a broad class of partial differential equations.

1.1 A Framework for Automated Computing

1.2 Mathematical Fundamentals

1.2.1 Banach, Hilbert, and Sobolev Spaces

Consider a bounded domain, $\Omega \in \mathbb{R}^d$ and a vector space of functions defined on Ω. A Banach space is a complete normed vector space. An example of which is $L_1(\Omega)$ which is the set of all functions defined on Ω such that the L_1 norm $\|u\|_{L_1(\Omega)} := \int_\Omega |u|$ is finite. A Hilbert space is a Banach space where the norm is defined through an inner product. An example of which is $L_2(\Omega)$ for which the norm is $\|u\|_{L_2(\Omega)} := (u,u)^{\frac{1}{2}}_{L_2(\Omega)} := \left(\int_\Omega u^2\right)^{\frac{1}{2}}$. Finally, a Sobolev space is a Hilbert space where the inner product contains derivatives of the function. A common Sobolev space is $H^1(\Omega)$ where $\|u\|^2_{H^1(\Omega)} := \|u\|^2_{L_2(\Omega)} + \|\nabla u\|^2_{L_2(\Omega)}$.

1.2.2 Variational Formulations, Dual Spaces, and Riesz Maps

Consider a Banach space U. The dual space, U' is the set of all linear functionals on U such that the pairing $\langle u, w\rangle_{U\times U'} \in \mathbb{R}$ for all $u \in U$ and $w \in U'$. Any PDE problem can abstractly be written as: find solution $u \in U$ such that

$$Bu = l$$

where $B : U \to V'$ maps functions to some dual space and $l \in U'$. By the calculus of variations, this is equivalent to the problem: find solution $u \in U$ such that

$$\langle Bu, v\rangle = \langle l, v\rangle$$

for all functions $v \in V$ and v is called a test function. This defines a bilinear form $b(u, v) = f(v)$ where u is often called the trial function. Now let U and V be Hilbert rather than merely Banach. The Riesz representation theorem defines a bijective, isometric mapping from every member in V to a corresponding member in the dual space V'. We denote this mapping $R_V : V \to V'$. Thus, we can take our operator equation representing the strong form of our PDE

$$Bu - l = 0 \quad \in V'$$

and map it back to V via the inverse Riesz map:

$$R_V^{-1}(Bu - l) = 0 \quad \in V.$$

1.3 Ritz Methods and Least Squares

1.4 SUPG – Or Stability Through the Appropriate Test Space

The finite element method provides an attractive mathematically sound framework for computational mechanics, yet for many years it was thought to be inadequate for solution of anything but symmetric problems such Poisson and elasticity. A breakthrough came through with the 1990 invention of the streamline-upwind/Petrov-Galerkin (SUPG) method by Tom Hughes [1] and his collaborators. Effectively, SUPG adds an upwind bias to test functions to imitate upwinded finite difference techniques and was really the first finite element method to successfully solve fluid flow problems. One of the key insights was that by choosing the correct test space, you could improve the stability of the finite element method. This is an important concept in the development of the DPG framework.

2 Derivation of the DPG Framework

We present two concrete examples, Poisson and convection-diffusion, before generalizing in the next section.

2.1 DPG for Poisson

Consider the Poisson problem with a mix of Dirichlet and Neumann boundary conditions: find $u \in U$ such that

$$\begin{aligned} -\Delta u &= f \quad && \text{on } \Omega \\ u &= g \quad && \text{on } \Gamma_g \\ \nabla u \cdot \boldsymbol{n} &= h \quad && \text{on } \Gamma_h \,. \end{aligned}$$

We begin by selecting a solution trial space. For comparison purposes to other finite element methods, we choose $U = H^1(\Omega)$. From our functional analysis discussion, we know that we can write our residual

$$-\Delta u - f \quad \in V' \,.$$

Since V' is a Hilbert space, we have some norm that we can measure the residual in.

3 DPG for Well Posed PDEs

4 Lessons from Several Application Domains

4.1 Helmholtz

4.2 Convection-Diffusion

4.3 Space-Time

References

[1] A. Brooks, T. Hughes, Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations, Comput. Methods Appl. Mech. Eng. (1990) 199–259.