PECOS

Predictive Engineering and Computational Sciences

Locally Conservative Discontinuous Petrov-Galerkin for Fluid Problems

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A Summary of DPG

Overview of Features

- Robust for singularly perturbed problems
- Stable in the preasymptotic regime
- Designed for adaptive mesh refinement

DPG is a minimum residual method:

$$u_h = \underset{w_h \in U_h}{\arg\min} \frac{1}{2} \|Bw_h - l\|^2_{V'}$$

$$\Updownarrow$$

$$b(u_h, R_V^{-1} B\delta u_h) = l(R_V^{-1} B\delta u_h) \quad \forall \delta u_h \in U_h$$

where $v_{\delta u_h} := R_V^{-1} B\delta u_h$ are the optimal test functions.

DPG for Convection-Diffusion

Start with the strong-form PDE.

$$\nabla \cdot (\beta u) - \epsilon \Delta u = g$$

Rewrite as a system of first-order equations.

$$\begin{aligned} \nabla \cdot (\beta u - \boldsymbol{\sigma}) &= g \\ \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= \mathbf{0} \end{aligned}$$

Multiply by test functions and integrate by parts over each element, K.

$$\begin{aligned} -(\beta u - \boldsymbol{\sigma}, \nabla v)_K + ((\beta u - \boldsymbol{\sigma}) \cdot \mathbf{n}, v)_{\partial K} &= (g, v)_K \\ \frac{1}{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\tau})_K + (u, \nabla \cdot \boldsymbol{\tau})_K - (u, \tau_n)_{\partial K} &= 0 \end{aligned}$$

Use the ultraweak (DPG) formulation to obtain bilinear form $b(u, v) = l(v)$.

$$\begin{aligned} -(\beta u - \boldsymbol{\sigma}, \nabla v)_K + (\hat{f}, v)_{\partial K} + \frac{1}{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\tau})_K & \\ + (u, \nabla \cdot \boldsymbol{\tau})_K - (\hat{u}, \tau_n)_{\partial K} &= (g, v)_K \end{aligned}$$

Local Conservation

The local conservation law in convection diffusion is

$$\int_{\partial K} \hat{f} = \int_K g \,,$$

which is equivalent to having $\mathbf{v}_K := \{v, \boldsymbol{\tau}\} = \{1_K, \mathbf{0}\}$ in the test space. In general, this is not satisfied by the optimal test functions. Following Moro et al[4] (also Chang and Nelson[2]), we can enforce this condition with Lagrange multipliers:

$$L(u_h, \boldsymbol{\lambda}) = \frac{1}{2} \left\| R_V^{-1}(Bu_h - l) \right\|_V^2 - \sum_K \lambda_K \underbrace{\langle Bu_h - l, \mathbf{v}_K \rangle}_{\langle \hat{f}, 1_K \rangle_{\partial K} - \langle g, 1_K \rangle_K} \,,$$

where $\boldsymbol{\lambda} = \{\lambda_1, \cdots, \lambda_N\}$.

Local Conservation

Finding the critical points of $L(u, \boldsymbol{\lambda})$, we get the following equations.

$$\frac{\partial L(u_h, \boldsymbol{\lambda})}{\partial u_h} = b(u_h, R_V^{-1} B\delta u_h) - l(R_V^{-1} B\delta u_h) - \sum_K \lambda_K b(\delta u_h, \mathbf{v}_K) = 0 \quad \forall \delta u_h \in U_h$$

$$\frac{\partial L(u_h, \boldsymbol{\lambda})}{\partial \lambda_K} = -b(u_h, \mathbf{v}_K) + l(\mathbf{v}_K) = 0 \quad \forall K$$

A few consequences:

- We've turned our minimization problem into a saddlepoint problem.
- Only need to find the optimal test function in the orthogonal complement of constants.

Optimal Test Functions

For each $\mathbf{u} = \{u, \boldsymbol{\sigma}, \hat{u}, \hat{f}\} \in \mathbf{U}_h$, find $\mathbf{v_u} = \{v_\mathbf{u}, \boldsymbol{\tau}_\mathbf{u}\} \in \mathbf{V}$ such that

$$(\mathbf{v_u}, \mathbf{w})_\mathbf{V} = b(\mathbf{u}, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}$$

where $\mathbf{V}$ becomes $\mathbf{V}_{p+\Delta p}$ in order to make this computationally tractable. We recently developed this modification to the *robust test norm* [1] which behaves better in the presence of singularities.

$$\left\|(v, \boldsymbol{\tau})\right\|^2_{\mathbf{V},\Omega_h} = \left\|\min\left\{\frac{1}{\sqrt{\epsilon}}, \frac{1}{\sqrt{|K|}}\right\}\boldsymbol{\tau}\right\|^2 + \left\|\nabla\cdot\boldsymbol{\tau} - \boldsymbol{\beta}\cdot\nabla v\right\|^2$$

$$+ \left\|\boldsymbol{\beta}\cdot\nabla v\right\|^2 + \epsilon\left\|\nabla v\right\|^2 \quad \underbrace{+ \left\|v\right\|^2}_{\text{No longer necessary}}$$

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$$+ \|\boldsymbol{\beta} \cdot \nabla v\|^2 + \epsilon \|\nabla v\|^2 \underbrace{+ \left(\frac{1}{|K|} \int_K v \right)^2}_{\text{Zero mean term}}$$

Stability Analysis

We follow Brezzi's theory for an abstract mixed problem:

$$\begin{cases} \boldsymbol{u} \in \boldsymbol{U}, p \in Q \\ a(\boldsymbol{u}, \boldsymbol{w}) + c(p, \boldsymbol{w}) & = l(\boldsymbol{w}) \quad \forall \boldsymbol{w} \in \boldsymbol{U} \\ c(q, \boldsymbol{u}) & = g(q) \quad \forall q \in Q \end{cases}$$

where a, c, l, g denote the appropriate bilinear and linear forms. Note that $a(\boldsymbol{u}, \boldsymbol{w}) = b(\boldsymbol{u}, T\boldsymbol{w}) = (T\boldsymbol{u}, T\boldsymbol{w})_V$.

Let ψ denote the $H(\text{div}, \Omega)$ extension of flux $\hat{f}$ that realizes the minimum in the definition of the quotient (minimum energy extension) norm.

The norm for the Lagrange multipliers λ_K is implied by the quotient norm used for $H^{-1/2}(\Gamma_h)$ and continuity bound for form $c(p, \boldsymbol{w})$:

$$||\boldsymbol{\lambda}|| := \left(\sum_K \mu(K) \lambda_K^2 \right)^{1/2}$$

Inf Sup Condition relating spaces $\boldsymbol{U}$ and Q

$$\sup_{\boldsymbol{w}\in\boldsymbol{U}} \frac{|c(p,\boldsymbol{w})|}{||\boldsymbol{w}||_{\boldsymbol{U}}} \geq \beta||p||_Q$$

Let

$$R\ :\ L^2(\Omega) \ni q \rightarrow \boldsymbol{\psi} \in \boldsymbol{H}(\mathrm{div},\Omega) \cap \boldsymbol{H}^1(\Omega) = \boldsymbol{H}^1(\Omega)$$

be the continuous right inverse of the divergence operator constructed by Costabel and McIntosh[3]. Let $\boldsymbol{\psi}_h$ denote the classical, lowest order Raviart-Thomas (RT) interpolant of function

$$\boldsymbol{\psi} = R(\sum_K \lambda_K 1_K)\,.$$

Note that $\mathrm{div}\boldsymbol{\psi}_h = \mathrm{div}\boldsymbol{\psi} = \lambda_K$ in element K.

Inf Sup Condition relating spaces $\boldsymbol{U}$ and Q

Classical h-interpolation interpolation error estimate for the lowest error Raviart-Thomas elements and continuity of operator R imply the stability estimate:

$$\begin{aligned} ||\boldsymbol{\psi}_h|| &\leq ||\boldsymbol{\psi}_h - \boldsymbol{\psi}|| + ||\boldsymbol{\psi}|| \\ &\leq Ch||\boldsymbol{\psi}||_{H^1} + ||\boldsymbol{\psi}|| \\ &\leq C||\mathrm{div}\boldsymbol{\psi}|| = C(\textstyle\sum_K \mu(K)\lambda_K^2)^{1/2} \end{aligned}$$

Let $\hat{f}$ be the trace of $\boldsymbol{\psi}_h$, then

$$\begin{aligned} \sup_{\hat{f}\in H^{-1/2}(\Gamma_h)} \frac{|\sum_K \lambda_K \langle \hat{f}, 1_K\rangle_{\partial K}|}{||\hat{f}||_{H^{-1/2}(\Gamma_h)}} &\geq \frac{|\sum_K \lambda_K \int_K \mathrm{div}\boldsymbol{\psi}_h \, 1_K|}{||\boldsymbol{\psi}_h||_{H(\mathrm{div},\Omega)}} \\ &\geq \frac{1}{C}(\sum_K \mu(K)\lambda_K^2)^{1/2} \end{aligned}$$

Inf Sup in Kernel Condition

We characterize the "kernel" space:

$$\begin{aligned} \boldsymbol{U}_0 := & \quad \{\boldsymbol{w} \in \boldsymbol{U} \,:\, c(q, \boldsymbol{w}) = 0 \quad \forall q \in Q\} \\ = & \quad \{(u, \boldsymbol{\sigma}, \hat{u}, \hat{t}) \,:\, \langle \hat{t}, 1_K \rangle = 0 \quad \forall K\} \end{aligned}$$

With $\boldsymbol{u} \in \boldsymbol{U}_0$, we have then:

$$\begin{aligned} \sup_{\boldsymbol{w} \in \boldsymbol{U}_0} \frac{|a(\boldsymbol{u}, \boldsymbol{w})|}{||\boldsymbol{w}||_{\boldsymbol{U}}} \geq \frac{|b(\boldsymbol{u}, T\boldsymbol{u})|}{||\boldsymbol{u}||} &= \frac{|b(\boldsymbol{u}, T\boldsymbol{u})|}{||T\boldsymbol{u}||} \frac{||T\boldsymbol{u}||}{||\boldsymbol{u}||} \\ &= \sup_{(v, \boldsymbol{\tau})} \frac{|b((u, \boldsymbol{\sigma}, \hat{u}, \hat{t}), (v, \boldsymbol{\tau}))|}{||(v, \boldsymbol{\tau})||} \frac{||T\boldsymbol{u}||}{||\boldsymbol{u}||} \geq \gamma^2 ||(u, \boldsymbol{\sigma}, \hat{u}, \hat{t})|| \end{aligned}$$

where γ is the stability constant for the standard continuous DPG formulation.

The FE error is bounded by the best approximation error. Note that the exact Lagrange multipliers are zero, so the best approximation error involves only the solution $(u, \boldsymbol{\sigma}, \hat{u}, \hat{t})$.

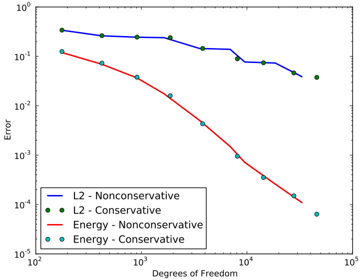
Erickson-Johnson Problem

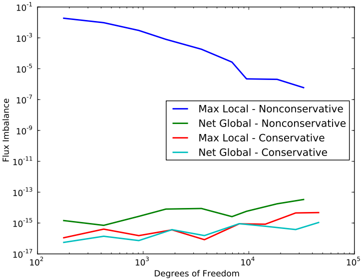
On domain $\Omega = [0,1]^2$, with $\beta = (1,0)^T$, $f = 0$ and boundary conditions

$$\hat{f} = u_0, \quad \beta_n \leq 0\,, \qquad \hat{u} = 0, \quad \beta_n > 0$$

Separation of variabes gives an analytic solution

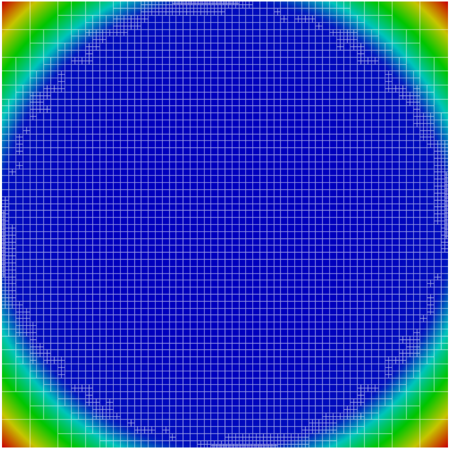
$$u(x,y) = C_0 + \sum_{n=1}^{\infty} C_n \frac{\exp(r_2(x-1)) - \exp(r_1(x-1))}{r_1 \exp(-r_2) - r_2 \exp(-r_1)} \cos(n\pi y)$$



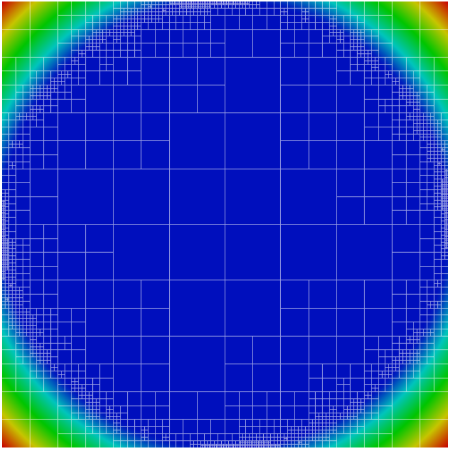


Vortex Problem

After 6 refinements, $\epsilon = 10^{-4}$, $\beta = (-y, x)^T$



Nonconservative



Conservative

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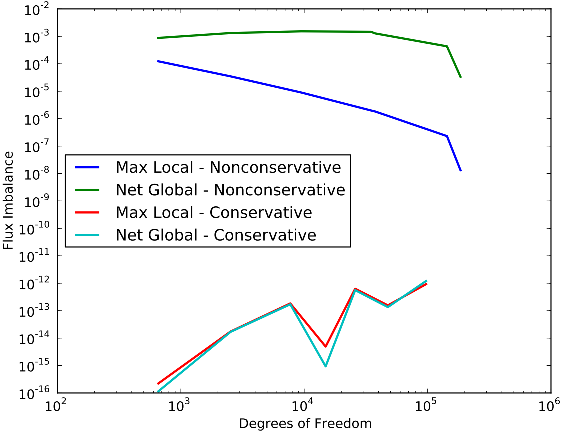
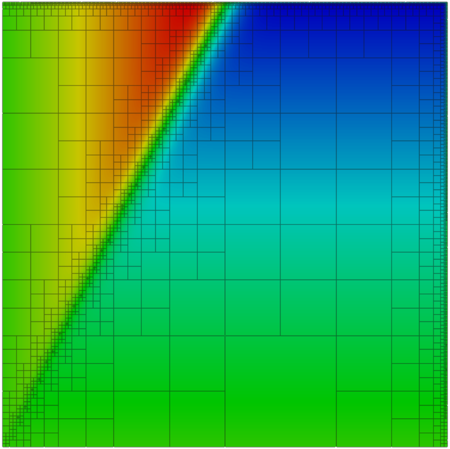


Discontinuous Source Problem

After 8 refinements, $\epsilon = 10^{-3}$, $\boldsymbol{\beta} = (0.5, 1)^T/\sqrt{1.25}$, $\hat{g} = \begin{cases} 1, & y >= 2x \\ 0, & y < 2x \end{cases}$



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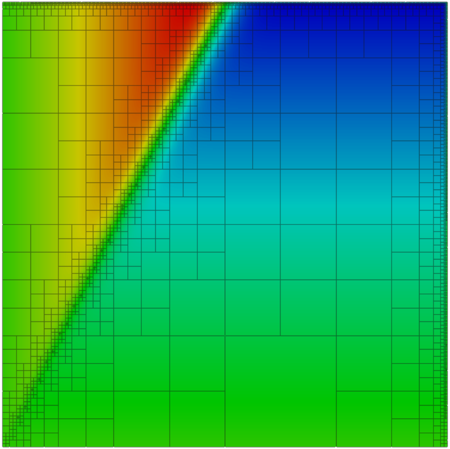

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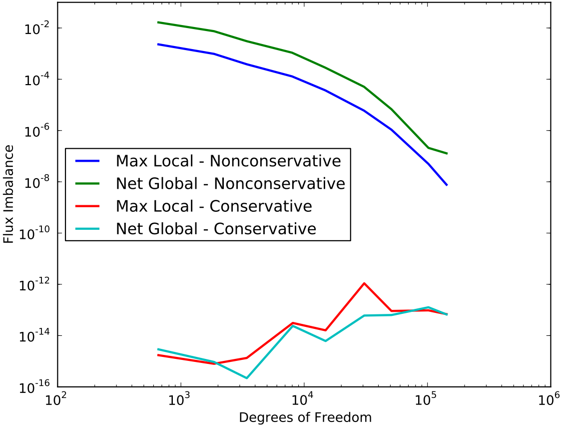
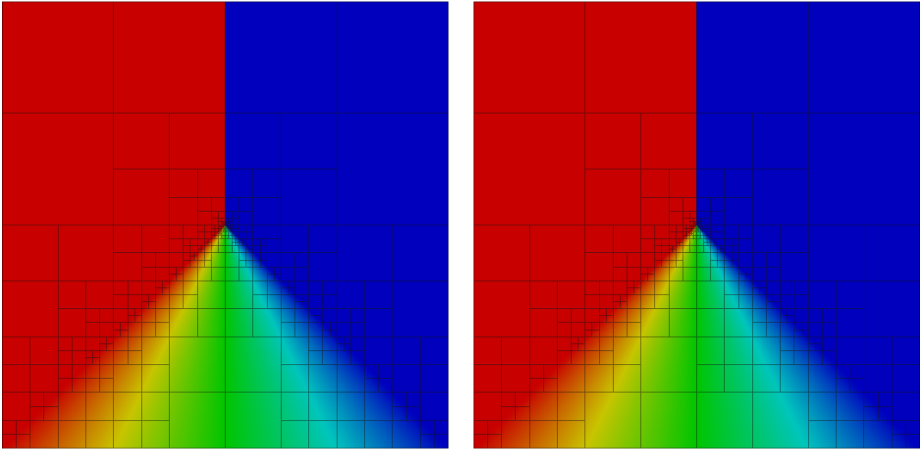


Inviscid Burgers' Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \quad \Leftrightarrow \quad \nabla_{x,t} \cdot \begin{pmatrix} \frac{u^2}{2} \\ u \end{pmatrix} = 0$$



Nonconservative Conservative

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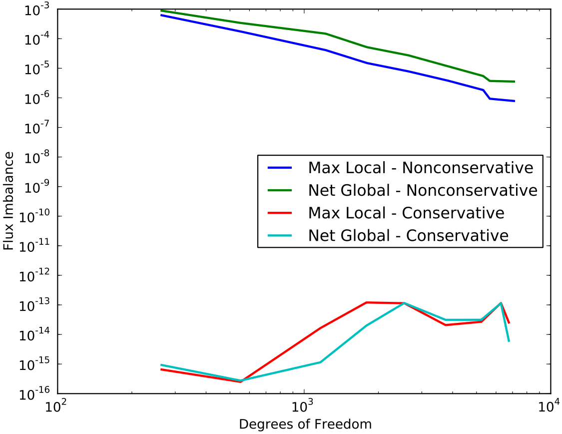
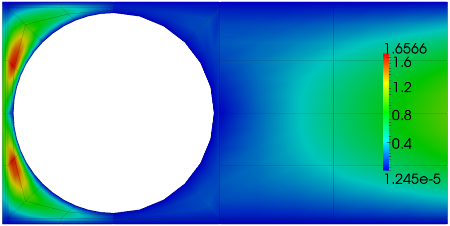
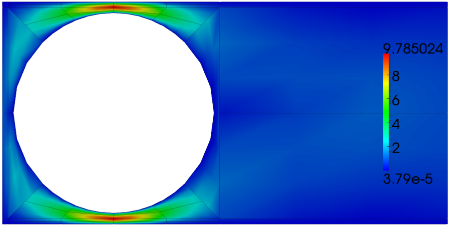


Stokes Flow Around a Cylinder



1 Refinement



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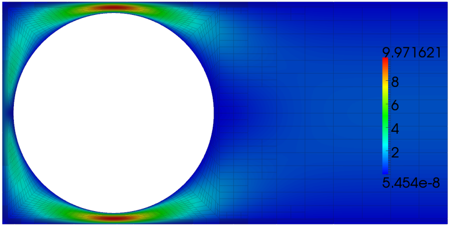
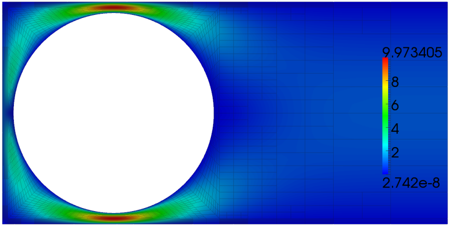


6 Refinements



6 Refinements

Nonconservative

Conservative

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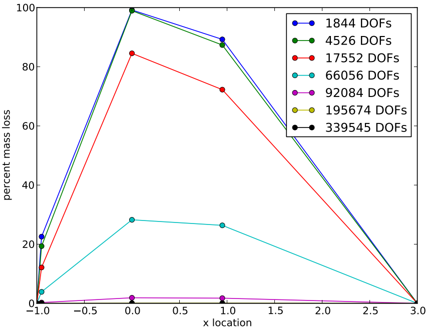

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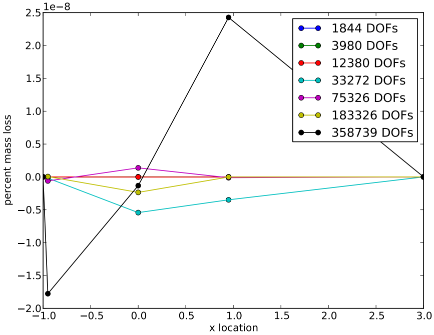
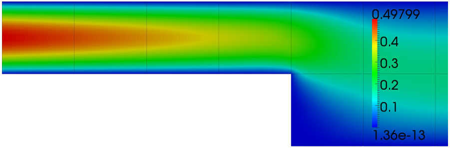
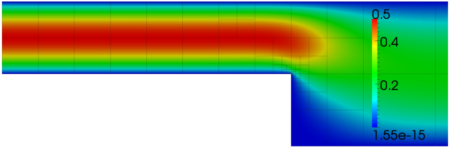


Conservative

Stokes Flow Backward Facing Step

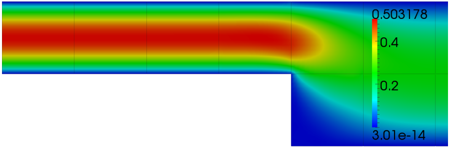


Initial Mesh



8 Refinements

Nonconservative



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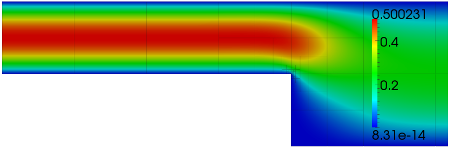

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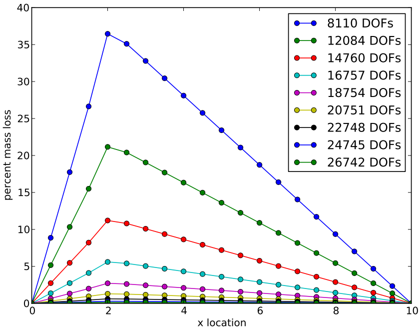

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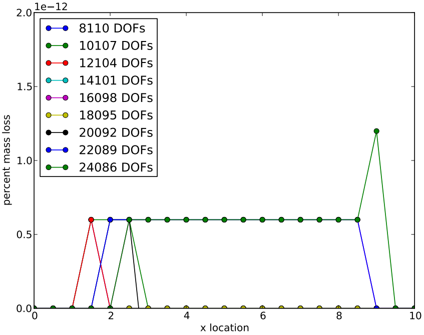

Conservative

Summary

What have we done?

- We've turned our minimization problem into a saddlepoint problem.
- The change is computationally feasible.
- Mathematically, it gets rid of troublesome term.

Does it make a difference?

- Enforcement changes refinement strategy.
- Standard DPG is nearly conservative in practice.
- Usually we get the same results with better conservation.
- Some improvement on condition number for local solves.

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We need to study the effect on real fluid dynamics.

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