Pressureless Navier-Stokes Formulation

Truman E. Ellis

August 4, 2014

We can derive the compressible Navier-Stokes equations in terms of the Cauchy stress tensor. Note that

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}\,,$$

and

$$\begin{aligned}\sigma_{ii} &= 2\mu\varepsilon_{ii} + N\lambda\varepsilon_{ii}\\ &= (2\mu + N\lambda)\varepsilon_{ii}\,,\end{aligned}$$

where N is the dimension. Then

$$\begin{aligned}\varepsilon_{ij} &= \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu}\varepsilon_{kk}\delta_{ij}\\ &= \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(2\mu+N\lambda)}\sigma_{kk}\delta_{ij}\\ &= \frac{1}{2\mu}\sigma_{ij} - \frac{1}{2\mu(\frac{2\mu}{\lambda}+N)}\sigma_{kk}\delta_{ij}\,.\end{aligned}$$

Incompressible

If we assume an incompressible medium, then $\lambda \to \infty$ and

$$\begin{aligned}\varepsilon_{ij} &= \frac{1}{2\mu}\sigma_{ij} - \frac{1}{2N\mu}\sigma_{kk}\delta_{ij}\\ &= \frac{1}{2\mu}\left[\sigma_{ij} - \frac{1}{N}\sigma_{kk}\delta_{ij}\right]\,.\end{aligned}$$

This embeds the zero divergence condition. If we take the trace of both sides, we get

$$\nabla\cdot\boldsymbol{u} = \varepsilon_{ii} = \frac{1}{2\mu}\left[\sigma_{ii} - \sigma_{ii}\right] = 0\,.$$

Pressure, though not explicitly used in this formulation, is defined as

$$p = \lambda\nabla\cdot\boldsymbol{u} - \frac{1}{N}\sigma_{ii} = \frac{1}{N}\sigma_{ii}\,.$$

The space-time form of the Cauchy momentum equation is

$$\nabla_{xt} \cdot \begin{pmatrix} \rho \boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{\sigma} \\ \rho \boldsymbol{u} \end{pmatrix} = \boldsymbol{f} \,.$$

Our incompressible Navier-Stokes sytem is then

$$\begin{aligned} \frac{1}{\mu}\boldsymbol{\sigma} - \frac{1}{N\mu}\operatorname{tr}(\boldsymbol{\sigma})\boldsymbol{I} - \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right) &= 0 \\ \nabla_{xt} \cdot \begin{pmatrix} \rho \boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{\sigma} \\ \rho \boldsymbol{u} \end{pmatrix} &= \boldsymbol{f} \,. \end{aligned}$$

We multiply by test functions $\boldsymbol{\tau}$ (symmetric tensor) and $\boldsymbol{v}$ and integrate by parts over a space-time element K.

$$\begin{aligned} \left(\frac{1}{\mu}\boldsymbol{\sigma}, \boldsymbol{\tau}\right) - \left(\frac{1}{N\mu}\operatorname{tr}(\boldsymbol{\sigma})\boldsymbol{I}, \boldsymbol{\tau}\right) + (2\boldsymbol{u}, \nabla \cdot \boldsymbol{\tau}) - \langle 2\hat{\boldsymbol{u}}, \boldsymbol{\tau} \cdot \boldsymbol{n}_x \rangle &= 0 \\ -\left(\begin{pmatrix} \rho \boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{\sigma} \\ \rho \boldsymbol{u} \end{pmatrix}, \nabla_{xt} \boldsymbol{v}\right) + \left\langle \hat{\boldsymbol{t}}, \boldsymbol{v} \right\rangle &= (\boldsymbol{f}, \boldsymbol{v}) \ , \end{aligned}$$

where

$$\begin{aligned} \hat{\boldsymbol{u}} &= \operatorname{tr}(\boldsymbol{u}) \\ \hat{\boldsymbol{t}} &= \operatorname{tr}\left(\rho \boldsymbol{u} \otimes \boldsymbol{u} - \boldsymbol{\sigma}\right) \cdot \boldsymbol{n}_x + \operatorname{tr}\left(\rho \boldsymbol{u}\right) n_t \,. \end{aligned}$$

Linearization

The Jacobian is

$$\begin{aligned} &\left(\frac{1}{\mu}\boldsymbol{\sigma}, \boldsymbol{\tau}\right) - \left(\frac{1}{N\mu}\operatorname{tr}(\boldsymbol{\sigma})\boldsymbol{I}, \boldsymbol{\tau}\right) + (2\boldsymbol{u}, \nabla \cdot \boldsymbol{\tau}) - \langle 2\hat{\boldsymbol{u}}, \boldsymbol{\tau} \cdot \boldsymbol{n}_x \rangle \\ &-\left(\begin{pmatrix} \rho \Delta \boldsymbol{u} \otimes \tilde{\boldsymbol{u}} + \rho \tilde{\boldsymbol{u}} \otimes \Delta \boldsymbol{u} - \boldsymbol{\sigma} \\ \rho \Delta \boldsymbol{u} \end{pmatrix}, \nabla_{xt} \boldsymbol{v}\right) + \left\langle \hat{\boldsymbol{t}}, \boldsymbol{v} \right\rangle \ , \end{aligned}$$

with residual

$$(2\tilde{\boldsymbol{u}}, \nabla \cdot \boldsymbol{\tau}) - (\rho \tilde{\boldsymbol{u}} \otimes \tilde{\boldsymbol{u}}, \nabla \boldsymbol{v}) - (\boldsymbol{f}, \boldsymbol{v}) \ .$$

Test Norm

For the following discussion, we drop ρ (or assume $\rho = 1$). Note that $\boldsymbol{\sigma}^d = \boldsymbol{\sigma} - \frac{1}{N}\operatorname{tr}(\boldsymbol{\sigma})\boldsymbol{I}$, and $\boldsymbol{\sigma}^d \boldsymbol{\tau} = \boldsymbol{\sigma} \boldsymbol{\tau}^d$. Also note that

$$(\Delta \boldsymbol{u} \otimes \tilde{\boldsymbol{u}} + \tilde{\boldsymbol{u}} \otimes \Delta \boldsymbol{u}) \nabla \boldsymbol{v} = \tilde{\boldsymbol{u}} \cdot \left(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T\right) \boldsymbol{u} \,,$$

since

$$
\begin{aligned}
(\tilde{u}_i u_j + u_i \tilde{u}_j)\, v_{i,j} &= \tilde{u}_i u_j v_{i,j} + u_i \tilde{u}_j v_{i,j} \\
&= \tilde{u}_j u_i v_{j,i} + u_i \tilde{u}_j v_{i,j} \\
&= u_i(\tilde{u}_j(v_{i,j} + v_{j,i}))\,.
\end{aligned}
$$

Grouping terms:

$$
\begin{aligned}
&\left(\boldsymbol{\sigma}, \frac{1}{\nu}\boldsymbol{\tau}^d + \nabla \boldsymbol{v}\right) \\
\left(\boldsymbol{u}, 2\nabla\cdot\boldsymbol{\tau} - \tilde{\boldsymbol{u}}\cdot(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T) - \frac{\partial \boldsymbol{v}}{\partial t}\right)&\,.
\end{aligned}
$$

So our graph norm based on the first version is defined by

$$
\left\|\{\boldsymbol{v},\boldsymbol{\tau}\}\right\|^2 = \left\|\frac{1}{\nu}\boldsymbol{\tau} - \frac{1}{N\nu}\operatorname{tr}(\boldsymbol{\tau})\boldsymbol{I} + \nabla\boldsymbol{v}\right\|^2 + \left\|2\nabla\cdot\boldsymbol{\tau} - \tilde{\boldsymbol{u}}\cdot\left(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T\right) - \frac{\partial\boldsymbol{v}}{\partial t}\right\|^2 + \|\boldsymbol{v}\|^2 + \|\boldsymbol{\tau}\|^2\ .
$$

Our robust test norm for transient convection-diffusion is

$$
\begin{aligned}
\|(v,\boldsymbol{\tau})\|_V^2 \coloneqq \frac{1}{\epsilon}\|\boldsymbol{\tau}\|^2 &+ \left\|\nabla\cdot\boldsymbol{\tau} - \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt} v\right\|^2 \\
&+ \left\|\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt} v\right\|^2 + \epsilon\|\nabla v\|^2 + \|v\|^2\ ,
\end{aligned}
\tag{1}
$$

where $\tilde{\boldsymbol{\beta}} = (\boldsymbol{\beta}, 1)^T$. The analogous norm for incompressible Navier-Stokes is

$$
\begin{aligned}
\left\|\{\boldsymbol{v},\boldsymbol{\tau}\}\right\|^2 = \frac{1}{\nu}\left\|\boldsymbol{\tau} - \frac{1}{N}\operatorname{tr}(\boldsymbol{\tau})\boldsymbol{I}\right\|^2 &+ \left\|2\nabla\cdot\boldsymbol{\tau} - \tilde{\boldsymbol{u}}\cdot\left(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T\right) - \frac{\partial\boldsymbol{v}}{\partial t}\right\|^2 \\
&+ \left\|\tilde{\boldsymbol{u}}\cdot\left(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T\right) + \frac{\partial\boldsymbol{v}}{\partial t}\right\|^2 + \nu\|\nabla\boldsymbol{v}\|^2 + \|\boldsymbol{v}\|^2\ .
\end{aligned}
$$

Boundary Conditions

On spatial boundaries, we have access to the following terms for setting boundary conditions:

$$
\begin{aligned}
\hat{\boldsymbol{u}} &= \operatorname{tr}(\boldsymbol{u}) \\
\hat{\boldsymbol{t}} &= \operatorname{tr}(\rho\boldsymbol{u}\otimes\boldsymbol{u} - \boldsymbol{\sigma})\cdot\boldsymbol{n}_x\,.
\end{aligned}
$$

For 2D flow, we can expand the flux:

$$
\hat{\boldsymbol{t}} = \begin{bmatrix} (\rho u_1 u_1 - \sigma_{11})n_1 + (\rho u_1 u_2 - \sigma_{12})n_2 \\ (\rho u_1 u_2 - \sigma_{12})n_1 + (\rho u_2 u_2 - \sigma_{22})n_2 \end{bmatrix}
$$

The constitutive law gives the following relations:

$$
\begin{bmatrix} \frac{1}{2}(\sigma_{11} - \sigma_{22}) & \sigma_{12} \\ \sigma_{12} & \frac{1}{2}(\sigma_{22} - \sigma_{11}) \end{bmatrix} = \mu \begin{bmatrix} 2\frac{\partial u_1}{\partial x} & \frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y} & 2\frac{\partial u_2}{\partial y} \end{bmatrix}
$$

Alternatively we can write this in terms of the implicitly present pressure,

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = - \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} + \mu \begin{bmatrix} 2\frac{\partial u_1}{\partial x} & \frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y} & 2\frac{\partial u_2}{\partial y} \end{bmatrix}$$

This gives the following implications:

$$\begin{aligned} \sigma_{11} = 0 &\Rightarrow \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0 \\ \sigma_{22} = 0 &\Rightarrow \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0 \\ \sigma_{12} = 0 &\Rightarrow \frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y} = 0 \\ \sigma_{11} = \sigma_{22} &\Rightarrow \frac{\partial u_1}{\partial x} = 0 \\ &\Rightarrow \frac{\partial u_2}{\partial y} = 0 \end{aligned}$$

But the first two are already satisfied by continuity. In the other direction, we get the following implications:

$$\begin{aligned} \frac{\partial u_1}{\partial x} &= 0 \Rightarrow \sigma_{11} = \sigma_{22} \\ \frac{\partial u_2}{\partial y} &= 0 \Rightarrow \sigma_{11} = \sigma_{22} \\ \frac{\partial u_1}{\partial y} = \frac{\partial u_2}{\partial x} &= 0 \Rightarrow \sigma_{12} = 0 \end{aligned}$$

Special Cases

Now let's consider application of boundary conditions on specific parts of the domain. First we will consider the right boundary where $\boldsymbol{n} = [1, 0]$. Here,

$$\hat{\boldsymbol{t}} = \begin{bmatrix} \rho u_1 u_1 - \sigma_{11} \\ \rho u_1 u_2 - \sigma_{12} \end{bmatrix}$$

- If $u_1 = 0$, then we have control over $-\sigma_{11}$ and $-\sigma_{12}$.
- If $u_2 = 0$, then we have control over $\rho u_1 u_1 - \sigma_{11}$ and $-\sigma_{12}$.
- If $\sigma_{11} = 0$, then we have control over $\rho u_1 u_1$ and $\rho u_1 u_2 - \sigma_{12}$.
- If $\sigma_{12} = 0$, then we have control over $\rho u_1 u_1 - \sigma_{11}$ and $\rho u_1 u_2$.

On the other hand, if $\boldsymbol{n} = [0, 1]$ on the top boundary,

$$\hat{\boldsymbol{t}} = \begin{bmatrix} \rho u_1 u_2 - \sigma_{12} \\ \rho u_2 u_2 - \sigma_{22} \end{bmatrix}$$

- If $u_1 = 0$, then we have control over $-\sigma_{12}$ and $\rho u_2 u_2 - \sigma_{22}$.
- If $u_2 = 0$, then we have control over $-\sigma_{12}$ and $\rho u_2 u_2 - \sigma_{22}$.
- If $\sigma_{12} = 0$, then we have control over $\rho u_1 u_2$ and $\rho u_2 u_2 - \sigma_{22}$.
- If $\sigma_{22} = 0$, then we have control over $\rho u_1 u_2 - \sigma_{12}$ and $\rho u_2 u_2$.

Boundary Layer Problem

Consider the common setup for a flat plate boundary layer problem: an *inflow* boundary, a *top* boundary, an *upstream* boundary just upstream of the *plate* boundary and an *outflow* boundary. Flow is oriented left to right with the *inflow* to the left and the *outflow* to the right.

Inflow We have four options for setting *inflow* boundary conditions: $\hat{u}_1$, $\hat{u}_2$, $\hat{t}_1 = \operatorname{tr}(-\rho u_1 u_1 + \sigma_{11})$, and $\hat{t}_2 = \operatorname{tr}(-\rho u_1 u_2 + \sigma_{12})$. Past experience suggests assuming that σ_{11} and σ_{12} are negligible and setting $\hat{t}_1 = -1$ and $\hat{t}_2 = 0$.

Top and Upstream A symmetry condition seems appropriate here. We have access to $\hat{u}_1$, $\hat{u}_2$, $\hat{t}_1 = \operatorname{tr}(\pm\rho u_1 u_2 \mp \sigma_{12})$, and $\hat{t}_2 = \operatorname{tr}(\pm\rho u_2 u_2 \mp \sigma_{22})$. Symmetry implies that we should set $u_2 = 0$ which simplifies $\hat{t}_1 = \operatorname{tr}(\mp\sigma_{12})$ and $\hat{t}_2 = \operatorname{tr}(\mp\sigma_{22})$. The second symmetry condition should then probably be $\hat{t}_1 = 0$.

Plate We have access to the same variables as the *upstream* boundary, but previous experience suggests setting $\hat{u}_1 = \hat{u}_2 = 0$ to be correct.

Outflow We have four options for setting *inflow* boundary conditions: $\hat{u}_1$, $\hat{u}_2$, $\hat{t}_1 = \operatorname{tr}(\rho u_1 u_1 - \sigma_{11})$, and $\hat{t}_2 = \operatorname{tr}(\rho u_1 u_2 - \sigma_{12})$. Previously we have just set no outflow boundary condition and achieved decent results.

Problem Unfortunately, the above boundary conditions seem to allow the trivial solution for all trace a field variables. The fluxes will alternate between -1, 0, and 1 depending on cell normals in a way consistent with local conservation. So clearly there is something deficient with the above boundary conditions, but the solution is not immediately obvious. The first guess would be that we need to set an outflow condition, but the correct one is not obvious.