

Robustness for Transient Problems

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Consider domain $Q = \Omega \times [0, T]$ with boundary $\Gamma = \Gamma_- \cup \Gamma_+ \cup \Gamma_0 \cup \Gamma_T$ where Γ_- is the spatial inflow boundary, Γ_+ is the spatial outflow boundary, Γ_0 is the initial time boundary, and Γ_T is the final time boundary. Let Γ_h denote the entire mesh skeleton.

Assume that boundary conditions are applied on the boundary $\Gamma_0 \subset \Gamma$. Recall that, for the ultra-weak variational formulation

$$b((u, \widehat{u}), v) = (u, A_h^* v)_{L^2(\Omega)} + \langle \widehat{u}, \llbracket v \rrbracket \rangle_{\Gamma_h \setminus \Gamma_0}$$

we can recover

$$\|u\|_{L^2(Q)}^2 = b(u, v^*)$$

for conforming v^* satisfying the adjoint equation

$$\begin{aligned} A^* v^* &= u \\ v^* &= 0 \text{ on } \Gamma_h \setminus \Gamma_0. \end{aligned}$$

Together, these give necessary conditions on the test norm $\|\cdot\|_V$ such that we have L^2 robustness (this gives robustness in the variable u ; for the first order formulation, conditions for σ must also be shown).

$$\|u\|_{L^2(Q)}^2 = b(u, v^*) \leq \frac{b(u, v^*)}{\|v^*\|_V} \|v^*\|_V \leq \|u\|_E \|v^*\|_V$$

Thus, showing $\|v^*\|_V \lesssim \|u\|_{L^2(Q)}$ gives the result that $\|u\|_{L^2(Q)} \lesssim \|u\|_E$.

1 Convection-Diffusion

Consider convection-diffusion

$$\begin{aligned} \frac{1}{\epsilon} \boldsymbol{\sigma} - \nabla u &= 0 \\ \frac{\partial u}{\partial t} + \boldsymbol{\beta} \cdot \nabla u - \nabla \cdot \boldsymbol{\sigma} &= f \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_- \\ u &= 0 \text{ on } \Gamma_+ \\ u &= u_0 \text{ on } \Gamma_0. \end{aligned}$$

Let $\tilde{\beta} := \begin{pmatrix} \beta \\ 1 \end{pmatrix}$ and $\nabla_{xt} := \begin{pmatrix} \nabla \\ \frac{\partial}{\partial t} \end{pmatrix}$, then we can rewrite this as

$$\begin{aligned} \frac{1}{\epsilon} \sigma - \nabla u &= 0 \\ \tilde{\beta} \cdot \nabla_{xt} u - \nabla \cdot \sigma &= f \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_- \\ u &= 0 \text{ on } \Gamma_+ \\ u &= u_0 \text{ on } \Gamma_0. \end{aligned}$$

We decompose the adjoint into three parts: a discontinuous part

$$\begin{aligned} \frac{1}{\epsilon} \tau_0 + \nabla v_0 &= 0 \\ -\tilde{\beta} \cdot \nabla_{xt} v_0 + \nabla \cdot \tau_0 &= 0 \\ \tau_0 \cdot \mathbf{n}_x &= \tau \cdot \mathbf{n}_x \text{ on } \Gamma_- \cup \Gamma_0 \\ v_0 &= v \text{ on } \Gamma_+ \\ v_0 &= v \text{ on } \Gamma_T \\ \llbracket v_0 \rrbracket &= \llbracket v \rrbracket \text{ on } \Gamma_h^0 \\ \llbracket \tau_0 \cdot \mathbf{n}_x \rrbracket &= \llbracket \tau \cdot \mathbf{n}_x \rrbracket \text{ on } \Gamma_{hx}^0, \end{aligned}$$

a continuous part with forcing term g

$$\begin{aligned} \frac{1}{\epsilon} \tau_1 + \nabla v_1 &= 0 \\ -\tilde{\beta} \cdot \nabla_{xt} v_1 + \nabla \cdot \tau_1 &= g \\ \tau_1 \cdot \mathbf{n}_x &= 0 \text{ on } \Gamma_- \\ v_1 &= 0 \text{ on } \Gamma_+ \\ v_1 &= 0 \text{ on } \Gamma_T, \end{aligned}$$

and a continuous part with forcing f

$$\begin{aligned} \frac{1}{\epsilon} \tau_2 + \nabla v_2 &= f \\ -\tilde{\beta} \cdot \nabla_{xt} v_2 + \nabla \cdot \tau_2 &= 0 \\ \tau_2 \cdot \mathbf{n}_x &= 0 \text{ on } \Gamma_- \\ v_2 &= 0 \text{ on } \Gamma_+ \\ v_2 &= 0 \text{ on } \Gamma_T. \end{aligned}$$

(The boundary conditions can be derived by taking the ultra-weak formulation and choosing boundary conditions such that the temporal flux and spatial flux terms $\langle \hat{u}, \llbracket \tau_n \rrbracket \rangle_{\Gamma_{out}}$ and $\langle \hat{t}_n, \llbracket v \rrbracket \rangle_{\Gamma_{in}}$ are zero.)

We can then derive that the test norm

$$\begin{aligned} \|(v, \boldsymbol{\tau})\|_{V,K}^2 &:= \frac{1}{\epsilon} \|\boldsymbol{\tau}\|_K^2 + \left\| \nabla \cdot \boldsymbol{\tau} - \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \right\|_K^2 \\ &\quad + \|\boldsymbol{\beta} \cdot \nabla v\|_K^2 + \epsilon \|\nabla v\|_K^2 + \|v\|_K^2, \end{aligned} \quad (1)$$

provides the necessary bound $\|v^*\|_V \lesssim \|u\|_{L^2(Q)}$.

In the following lemmas we establish the following bounds:

- Bound on $\|(v_0, \boldsymbol{\tau}_0)\|_V$.
- Bound on $\|(v_1, \boldsymbol{\tau}_1)\|_V$. Lemma 1.1 gives $\left\| \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1 \right\| \leq \|g\|$. Since $\nabla \cdot \boldsymbol{\tau}_1 = g + \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1$,

$$\|\nabla \cdot \boldsymbol{\tau}_1\| \leq \|g\| + \left\| \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1 \right\| \leq 2\|g\|.$$

Or, the fact that $\nabla \cdot \boldsymbol{\tau} - \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1 = g$ clearly gives

$$\left\| \nabla \cdot \boldsymbol{\tau} - \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1 \right\| = \|g\|.$$

Also, clearly

$$\|\boldsymbol{\beta} \cdot \nabla v_1\| \leq \left\| \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1 \right\| \leq \|g\|.$$

Lemma 1.2 gives $\|v_1\|^2 + \epsilon \|\nabla v_1\|^2 \leq \|g\|^2$. Since $\epsilon^{1/2} \nabla v_1 = -\epsilon^{-1/2} \boldsymbol{\tau}_1$,

$$\frac{1}{\epsilon} \|\boldsymbol{\tau}_1\|^2 \leq \|g\|^2.$$

Thus, all $(v_1, \boldsymbol{\tau}_1)$ terms in (1) are accounted for.

- Bound on $\|(v_2, \boldsymbol{\tau}_2)\|_V$. The fact that $\nabla \cdot \boldsymbol{\tau} - \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v = 0$ clearly gives

$$\left\| \nabla \cdot \boldsymbol{\tau} - \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_2 \right\| = 0 \leq \|\boldsymbol{f}\|.$$

Lemma 1.2 gives $\|v_2\|^2 + \epsilon \|\nabla v_2\|^2 \leq \epsilon \|\boldsymbol{f}\|^2$. Since $\epsilon^{1/2} \nabla v_2 = \boldsymbol{f} - \epsilon^{-1/2} \boldsymbol{\tau}_2$,

$$\frac{1}{\epsilon} \|\boldsymbol{\tau}_2\|^2 \leq (1 + \epsilon) \|\boldsymbol{f}\|^2.$$

Finally,

$$\|\boldsymbol{\beta} \cdot \nabla v_2\| \leq \|\boldsymbol{\beta}\|_\infty \|\nabla v_2\| \leq \|\boldsymbol{\beta}\|_\infty \|\boldsymbol{f}\|.$$

Thus, all $(v_2, \boldsymbol{\tau}_2)$ terms in (1) are accounted for.

Our goal is to analyze the stability properties of the adjoint equations by deriving bounds of the form $\|(v_1, \boldsymbol{\tau}_1)\|_V \leq \|g\|_L^2(Q)$ and $\|(v_2, \boldsymbol{\tau}_2)\|_V \leq \|f\|_L^2(Q)$.

Insert conditions on $\boldsymbol{\beta}$

Lemma 1.1. *For the above conditions on β ,*

$$\left\| \tilde{\beta} \cdot \nabla_{xt} v_1 \right\| \leq \|g\|.$$

Proof. Multiply by $-\tilde{\beta} \cdot \nabla_{xt} v$ and integrate over Q to get

$$\left\| \tilde{\beta} \cdot \nabla_{xt} v \right\|^2 = - \int_Q g \tilde{\beta} \cdot \nabla_{xt} v + \int_Q \tilde{\beta} \cdot \nabla_{xt} v \nabla \cdot \tau. \quad (2)$$

Note that

$$\begin{aligned} \frac{1}{\epsilon} \int_Q \tilde{\beta} \cdot \nabla_{xt} v \nabla \cdot \tau &= - \int_Q \tilde{\beta} \cdot \nabla_{xt} v \nabla \cdot \nabla v \\ &= - \int_{\Gamma_x} \tilde{\beta} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \int_Q \nabla (\tilde{\beta} \cdot \nabla_{xt} v) \cdot \nabla v \\ &= - \int_{\Gamma_x} \tilde{\beta} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \int_Q (\nabla \tilde{\beta} \cdot \nabla_{xt} v) \cdot \nabla v \\ &\quad + \int_Q \tilde{\beta} \cdot \nabla \nabla_{xt} v \cdot \nabla v \\ &= - \int_{\Gamma_x} \tilde{\beta} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \int_Q (\nabla \beta \cdot \nabla v) \cdot \nabla v \\ &\quad + \frac{1}{2} \int_Q \tilde{\beta} \cdot \nabla_{xt} (\nabla v \cdot \nabla v) \\ &= - \int_{\Gamma_x} \tilde{\beta} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \int_Q (\nabla \beta \cdot \nabla v) \cdot \nabla v \\ &\quad + \frac{1}{2} \int_{\Gamma} \tilde{\beta} \cdot \mathbf{n} (\nabla v \cdot \nabla v) - \frac{1}{2} \int_Q \nabla_{xt} \cdot \tilde{\beta} (\nabla v \cdot \nabla v) \\ &= - \int_{\Gamma_x} \tilde{\beta} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \int_Q (\nabla \beta \cdot \nabla v) \cdot \nabla v \\ &\quad + \frac{1}{2} \int_{\Gamma} \tilde{\beta} \cdot \mathbf{n} (\nabla v \cdot \nabla v) - \frac{1}{2} \int_Q \nabla \cdot \beta (\nabla v \cdot \nabla v) \\ &= - \int_{\Gamma_x} \tilde{\beta} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \frac{1}{2} \int_{\Gamma} \tilde{\beta} \cdot \mathbf{n} (\nabla v \cdot \nabla v) \\ &\quad + \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \end{aligned}$$

Plugging this into (2), we get

$$\begin{aligned}
\left\| \tilde{\beta} \cdot \nabla_{xt} v \right\|^2 &= - \int_Q g \tilde{\beta} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \\
&\quad - \epsilon \int_{\Gamma_x} \tilde{\beta} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \epsilon \frac{1}{2} \int_{\Gamma} \tilde{\beta} \cdot \mathbf{n} (\nabla v \cdot \nabla v) \\
&= - \int_Q g \tilde{\beta} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \\
&\quad - \int_{\Gamma_-} \tilde{\beta} \cdot \nabla_{xt} v \underbrace{\nabla v \cdot \mathbf{n}_x}_{=0} - \int_{\Gamma_+} \left(\underbrace{\frac{\partial v}{\partial t}}_{=0} + \beta \cdot \nabla v \right) \nabla v \cdot \mathbf{n}_x \\
&\quad + \frac{1}{2} \int_{\Gamma_-} \underbrace{\beta \cdot \mathbf{n}_x}_{<0} (\nabla v \cdot \nabla v) + \frac{1}{2} \int_{\Gamma_+} \beta \cdot \mathbf{n}_x (\nabla v \cdot \nabla v) \\
&\quad + \frac{1}{2} \int_{\Gamma_0} \underbrace{n_t}_{<0} (\nabla v \cdot \nabla v) + \frac{1}{2} \int_{\Gamma_T} n_t \underbrace{(\nabla v \cdot \nabla v)}_{=0} \\
&\leq - \int_Q g \tilde{\beta} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \\
&\quad + \int_{\Gamma_+} \left(- \frac{\partial v}{\partial \mathbf{n}_x} \beta + \frac{1}{2} \beta \cdot \mathbf{n}_x \nabla v \right) \cdot \nabla v \\
&= - \int_Q g \tilde{\beta} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \\
&\quad + \int_{\Gamma_+} \left(- \frac{\partial v}{\partial \mathbf{n}_x} \beta + \frac{1}{2} \beta \cdot \mathbf{n}_x \frac{\partial v}{\partial \mathbf{n}_x} \mathbf{n}_x \right) \cdot \frac{\partial v}{\partial \mathbf{n}_x} \mathbf{n}_x \\
&= - \int_Q g \tilde{\beta} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \\
&\quad - \underbrace{\frac{1}{2} \int_{\Gamma_+} \left(\frac{\partial v}{\partial \mathbf{n}_x} \right)^2 \beta \cdot \mathbf{n}_x}_{<0} \\
&\leq - \int_Q g \tilde{\beta} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \\
&\leq \frac{\|g\|^2}{2} + \frac{\left\| \tilde{\beta} \cdot \nabla_{xt} v \right\|^2}{2} + \epsilon \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \\
&\leq \frac{\|g\|^2}{2} + \frac{\left\| \tilde{\beta} \cdot \nabla_{xt} v \right\|^2}{2} + \epsilon C \|\nabla v\|^2
\end{aligned}$$

□

Lemma 1.2. *For the duration of this lemma, let $v := v_1 + v_2$. Then, for the*

above conditions on β ,

$$\|v\|^2 + \epsilon \|\nabla v\|^2 \leq \|g\|^2 + \epsilon \|\mathbf{f}\|^2.$$

Proof. Define $w = e^t v$ and note that $\frac{\partial w}{\partial t} = \left(\frac{\partial v}{\partial t} + v\right) e^t$ while all spatial derivatives go through. Multiplying the adjoint by w and integrating over Q gives

$$-\int_Q \tilde{\beta} \cdot \nabla_{xt} v w - \epsilon \Delta v w = \int_Q g w - \epsilon \int_Q \nabla \cdot \mathbf{f} w$$

or

$$-\int_Q e^t v \tilde{\beta} \cdot \nabla_{xt} v - \epsilon \int_Q e^t v \Delta v = \int_Q e^t g v - \epsilon \int_Q e^t v \nabla \cdot \mathbf{f}$$

Integrating by parts:

$$\begin{aligned} \int_Q \nabla_{xt} \cdot (e^t \tilde{\beta} v) v - \int_{\Gamma} e^t \tilde{\beta} \cdot \mathbf{n} v^2 + \epsilon \int_Q e^t \nabla v \cdot \nabla v - \epsilon \int_{\Gamma_x} e^t v \cdot \nabla v \cdot \mathbf{n}_x \\ = \int_Q e^t g v + \epsilon \int_Q e^t \nabla v \cdot \mathbf{f} - \epsilon \int_{\Gamma_x} e^t v \mathbf{f} \cdot \mathbf{n}_x \end{aligned}$$

Note that $\nabla_{xt} \cdot e^t v \tilde{\beta} = e^t (\tilde{\beta} \nabla_{xt} v + v)$ if $\nabla \cdot \beta = 0$. Moving some terms to the right hand side, we get

$$\begin{aligned} \int_Q e^t v^2 + \int_Q \epsilon e^t \nabla v \cdot \nabla v \\ = \int_Q e^t g v + \epsilon \int_Q e^t \nabla v \cdot \mathbf{f} - \epsilon \int_{\Gamma_x} e^t v \mathbf{f} \cdot \mathbf{n}_x \\ - \int_Q e^t \tilde{\beta} \cdot \nabla_{xt} v v + \int_{\Gamma} e^t \tilde{\beta} \cdot \mathbf{n} v^2 + \epsilon \int_{\Gamma_x} e^t v \cdot \nabla v \cdot \mathbf{n}_x \end{aligned}$$

Note that $1 \leq \|e^t\|_\infty = e^T$, then

$$\begin{aligned}
& \|v\|^2 + \epsilon \|\nabla v\|^2 \\
& \leq e^T \left(\int_Q gv + \epsilon \int_Q \nabla v \cdot \mathbf{f} - \epsilon \int_{\Gamma_-} v \underbrace{\mathbf{f} \cdot \mathbf{n}_x}_{=\tau_n^0 + \frac{\partial v}{\partial \mathbf{n}_x}} - \epsilon \int_{\Gamma_+} \underbrace{v}_{=0} \mathbf{f} \cdot \mathbf{n}_x \right. \\
& \quad \left. - \int_Q \tilde{\beta} \cdot \nabla_{xt} v v + \int_\Gamma \tilde{\beta} \cdot \mathbf{n} v^2 + \epsilon \int_{\Gamma_-} v \cdot \nabla v \cdot \mathbf{n}_x + \epsilon \int_{\Gamma_+} \underbrace{v}_{=0} \frac{\partial v}{\partial \mathbf{n}_x} \right) \\
& = e^T \left(\int_Q gv + \epsilon \int_Q \nabla v \cdot \mathbf{f} - \epsilon \int_{\Gamma_-} v \frac{\partial v}{\partial \mathbf{n}_x} + \epsilon \int_{\Gamma_+} v \frac{\partial v}{\partial \mathbf{n}_x} \right. \\
& \quad \left. - \frac{1}{2} \int_Q \tilde{\beta} \cdot \nabla_{xt} v^2 + \int_\Gamma \tilde{\beta} \cdot \mathbf{n} v^2 \right) \\
& = e^T \left(\int_Q gv + \epsilon \int_Q \nabla v \cdot \mathbf{f} \right. \\
& \quad \left. + \frac{1}{2} \int_Q \nabla_{xt} \cdot \tilde{\beta} v^2 - \frac{1}{2} \int_\Gamma \tilde{\beta} \cdot \mathbf{n} v^2 + \int_\Gamma \tilde{\beta} \cdot \mathbf{n} v^2 \right) \\
& = e^T \left(\int_Q gv + \epsilon \int_Q \nabla v \cdot \mathbf{f} \right. \\
& \quad \left. + \frac{1}{2} \left(\int_{\Gamma_0} \underbrace{-v^2}_{\leq 0} + \int_{\Gamma_T} \cancel{v^2}^0 + \int_{\Gamma_-} \underbrace{\beta \cdot \mathbf{n}_x v^2}_{\leq 0} + \int_{\Gamma_+} \beta \cdot \mathbf{n}_x \cancel{v^2}^0 \right) \right) \\
& \leq e^T \left(\int_Q gv + \epsilon \int_Q \nabla v \cdot \mathbf{f} \right) \\
& \leq e^T \left(\frac{\|g\|^2}{2} + \epsilon \frac{\|\mathbf{f}\|^2}{2} + \frac{\|v\|^2}{2} + \epsilon \frac{\|\nabla v\|^2}{2} \right)
\end{aligned}$$

□

2 Entropy Norm and Navier-Stokes

The adjoint for Navier-Stokes looks like

$$\begin{aligned}
-\frac{\partial v}{\partial t} - A \cdot \nabla v - (K_{ij} v_{,j})_{,i} &= A_0 U \\
\boldsymbol{\tau} \cdot \mathbf{n}_x &= 0 \text{ on } \Gamma_- \\
v &= 0 \text{ on } \Gamma_+ \\
v &= 0 \text{ on } \Gamma_T,
\end{aligned}$$

We want to create a norm that is bounded by $(U, A_0 U) = \left\| A_0^{\frac{1}{2}} U \right\|^2$.

Lemma 2.1. *For the above conditions on β ,*

$$\left\| \tilde{\beta} \cdot \nabla_{xt} v_1 \right\| \leq \|g\|.$$

Proof.

□

3 Robustness for transient problems given spatial robustness

Suppose we have the transient problem

$$\frac{\partial u}{\partial t} + Au = f$$

with initial condition $u(x, 0) = u_0$. Suppose that DPG is robust under the ultra-weak variational formulation for the steady problem

$$(u, A_h^* v)_{L^2(\Omega)} + \langle \hat{u}, \llbracket v \rrbracket \rangle_{\Gamma_h \setminus \Gamma_0} = (f, v)$$

with test norm $\|v\|_V$. Then, can we show that

$$\|v\|_{V,t} := \|v\|_V + \left\| \frac{\partial v}{\partial t} \right\|_{L^2(\Omega)}$$

also leads to a robust upper bound of the L^2 norm by the DPG energy norm? I believe this may be possible. The adjoint equation for robustness for the transient problem gives

$$-\frac{\partial v}{\partial t} + A^* v = u$$

with $v = 0$ at $t = T$...

4 Transient Eriksson-Johnson

We can derive a transient Eriksson-Johnson solution using separation of variables. Consider

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \epsilon \Delta u = 0$$

with boundary conditions

$$\begin{aligned} u &= 0 \text{ on } \Gamma_+, \\ u - \epsilon \frac{\partial u}{\partial n} &= u_0 - \epsilon \frac{\partial u_0}{\partial n} \text{ on } \Gamma_-, \\ \epsilon \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_0, \end{aligned}$$

and initial condition $u(x, y, 0) = u_0(x, y)$ that satisfies the given boundary data. Assuming that $u(x, y, t) = X(x, y)T(t)$ and $Lu = \frac{\partial u}{\partial x} - \epsilon \Delta u$, we can plug this into the equation

$$\frac{\partial u}{\partial t} + Lu = 0$$

and rearrange to get

$$-\frac{\frac{\partial T}{\partial t}}{T} = \frac{LX}{X} = C.$$

This assumes then that $\frac{\partial T}{\partial t} = -CT$, or that $T(t) = e^{-Ct}$, and that $LX = CX$, or that X is made up of the eigenfunctions of L .