

# Robustness for Transient Problems

Truman E. Ellis and Jesse L. Chan

July 30, 2014

Consider domain  $Q = \Omega \times [0, T]$  with boundary  $\Gamma = \Gamma_- \cup \Gamma_+ \cup \Gamma_0 \cup \Gamma_T$  where  $\Gamma_-$  is the spatial inflow boundary,  $\Gamma_+$  is the spatial outflow boundary,  $\Gamma_0$  is the initial time boundary, and  $\Gamma_T$  is the final time boundary. Let  $\Gamma_h$  denote the entire mesh skeleton.

Assume that boundary conditions are applied on the boundary  $\Gamma_0 \subset \Gamma$ . Recall that, for the ultra-weak variational formulation

$$b((u, \widehat{u}), v) = (u, A_h^* v)_{L^2(\Omega)} + \langle \widehat{u}, \llbracket v \rrbracket \rangle_{\Gamma_h \setminus \Gamma_0}$$

we can recover

$$\|u\|_{L^2(Q)}^2 = b(u, v^*)$$

for conforming  $v^*$  satisfying the adjoint equation

$$\begin{aligned} A^* v^* &= u \\ v^* &= 0 \text{ on } \Gamma_h \setminus \Gamma_0. \end{aligned}$$

Together, these give necessary conditions on the test norm  $\|\cdot\|_V$  such that we have  $L^2$  robustness (this gives robustness in the variable  $u$ ; for the first order formulation, conditions for  $\sigma$  must also be shown).

$$\|u\|_{L^2(Q)}^2 = b(u, v^*) \leq \frac{b(u, v^*)}{\|v^*\|_V} \|v^*\|_V \leq \|u\|_E \|v^*\|_V$$

Thus, showing  $\|v^*\|_V \lesssim \|u\|_{L^2(Q)}$  gives the result that  $\|u\|_{L^2(Q)} \lesssim \|u\|_E$ .

## 1 Convection-Diffusion

Consider convection-diffusion

$$\begin{aligned} \frac{1}{\epsilon} \boldsymbol{\sigma} - \nabla u &= 0 \\ \frac{\partial u}{\partial t} + \boldsymbol{\beta} \cdot \nabla u - \nabla \cdot \boldsymbol{\sigma} &= f \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_- \\ u &= 0 \text{ on } \Gamma_+ \\ u &= u_0 \text{ on } \Gamma_0. \end{aligned}$$

Let  $\tilde{\beta} := \begin{pmatrix} \beta \\ 1 \end{pmatrix}$  and  $\nabla_{xt} := \begin{pmatrix} \nabla \\ \frac{\partial}{\partial t} \end{pmatrix}$ , then we can rewrite this as

$$\begin{aligned} \frac{1}{\epsilon} \sigma - \nabla u &= 0 \\ \tilde{\beta} \cdot \nabla_{xt} u - \nabla \cdot \sigma &= f \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_- \\ u &= 0 \text{ on } \Gamma_+ \\ u &= u_0 \text{ on } \Gamma_0. \end{aligned}$$

We decompose the adjoint into three parts: a discontinuous part

$$\begin{aligned} \frac{1}{\epsilon} \tau_0 + \nabla v_0 &= 0 \\ -\tilde{\beta} \cdot \nabla_{xt} v_0 + \nabla \cdot \tau_0 &= 0 \\ \tau_0 \cdot \mathbf{n}_x &= \tau \cdot \mathbf{n}_x \text{ on } \Gamma_- \cup \Gamma_0 \\ v_0 &= v \text{ on } \Gamma_+ \\ v_0 &= v \text{ on } \Gamma_T \\ \llbracket v_0 \rrbracket &= \llbracket v \rrbracket \text{ on } \Gamma_h^0 \\ \llbracket \tau_0 \cdot \mathbf{n}_x \rrbracket &= \llbracket \tau \cdot \mathbf{n}_x \rrbracket \text{ on } \Gamma_{hx}^0, \end{aligned}$$

a continuous part with forcing term  $g$

$$\begin{aligned} \frac{1}{\epsilon} \tau_1 + \nabla v_1 &= 0 \\ -\tilde{\beta} \cdot \nabla_{xt} v_1 + \nabla \cdot \tau_1 &= g \\ \tau_1 \cdot \mathbf{n}_x &= 0 \text{ on } \Gamma_- \\ v_1 &= 0 \text{ on } \Gamma_+ \\ v_1 &= 0 \text{ on } \Gamma_T, \end{aligned}$$

and a continuous part with forcing  $f$

$$\begin{aligned} \frac{1}{\epsilon} \tau_2 + \nabla v_2 &= f \\ -\tilde{\beta} \cdot \nabla_{xt} v_2 + \nabla \cdot \tau_2 &= 0 \\ \tau_2 \cdot \mathbf{n}_x &= 0 \text{ on } \Gamma_- \\ v_2 &= 0 \text{ on } \Gamma_+ \\ v_2 &= 0 \text{ on } \Gamma_T. \end{aligned}$$

(The boundary conditions can be derived by taking the ultra-weak formulation and choosing boundary conditions such that the temporal flux and spatial flux terms  $\langle \hat{u}, \llbracket \tau_n \rrbracket \rangle_{\Gamma_{out}}$  and  $\langle \hat{t}_n, \llbracket v \rrbracket \rangle_{\Gamma_{in}}$  are zero.)

We can then derive that the test norm

$$\begin{aligned} \|(v, \boldsymbol{\tau})\|_{V,K}^2 &:= \min \left\{ \frac{1}{\epsilon}, \frac{1}{\mu(K)} \right\} \|\boldsymbol{\tau}\|_K^2 + \left\| \nabla \cdot \boldsymbol{\tau} - \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \right\|_K^2 \\ &\quad + \left\| \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \right\|_K^2 + \epsilon \|\nabla v\|_K^2 + \|v\|_K^2, \end{aligned} \quad (1)$$

provides the necessary bound  $\|v^*\|_V \lesssim \|u\|_{L^2(Q)}$ .

Our goal is to analyze the stability properties of the adjoint equations by deriving bounds of the form  $\|(v_1, \boldsymbol{\tau}_1)\|_V \leq \|g\|_L^2(Q)$  and  $\|(v_2, \boldsymbol{\tau}_2)\|_V \leq \|f\|_L^2(Q)$ .

**Insert conditions on  $\boldsymbol{\beta}$**

**Lemma 1.1.** *For the above conditions on  $\boldsymbol{\beta}$ ,*

$$\left\| \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1 \right\| \leq \|g\|$$

and since  $-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1 + \nabla \cdot \boldsymbol{\tau}_1 = g$ ,

$$\|\nabla \cdot \boldsymbol{\tau}_1\| \leq \|g\|.$$

*Proof.* Multiply by  $-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v$  and integrate over  $Q$  to get

$$\left\| \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \right\| = - \int_Q g \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla \cdot \boldsymbol{\tau}. \quad (2)$$

Note that

$$\begin{aligned}
\frac{1}{\epsilon} \int_Q \tilde{\beta} \cdot \nabla_{xt} v \nabla \cdot \boldsymbol{\tau} &= - \int_Q \tilde{\beta} \cdot \nabla_{xt} v \nabla \cdot \nabla v \\
&= - \int_{\Gamma_x} \tilde{\beta} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \int_Q \nabla (\tilde{\beta} \cdot \nabla_{xt} v) \cdot \nabla v \\
&= - \int_{\Gamma_x} \tilde{\beta} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \int_Q (\nabla \tilde{\beta} \cdot \nabla_{xt} v) \cdot \nabla v \\
&\quad + \int_Q \tilde{\beta} \cdot \nabla \nabla_{xt} v \cdot \nabla v \\
&= - \int_{\Gamma_x} \tilde{\beta} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \int_Q (\nabla \beta \cdot \nabla v) \cdot \nabla v \\
&\quad + \frac{1}{2} \int_Q \tilde{\beta} \cdot \nabla_{xt} (\nabla v \cdot \nabla v) \\
&= - \int_{\Gamma_x} \tilde{\beta} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \int_Q (\nabla \beta \cdot \nabla v) \cdot \nabla v \\
&\quad + \frac{1}{2} \int_{\Gamma} \tilde{\beta} \cdot \mathbf{n} (\nabla v \cdot \nabla v) - \frac{1}{2} \int_Q \nabla_{xt} \cdot \tilde{\beta} (\nabla v \cdot \nabla v) \\
&= - \int_{\Gamma_x} \tilde{\beta} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \int_Q (\nabla \beta \cdot \nabla v) \cdot \nabla v \\
&\quad + \frac{1}{2} \int_{\Gamma} \tilde{\beta} \cdot \mathbf{n} (\nabla v \cdot \nabla v) - \frac{1}{2} \int_Q \nabla \cdot \beta (\nabla v \cdot \nabla v) \\
&= - \int_{\Gamma_x} \tilde{\beta} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \frac{1}{2} \int_{\Gamma} \tilde{\beta} \cdot \mathbf{n} (\nabla v \cdot \nabla v) \\
&\quad + \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v
\end{aligned}$$

Plugging this into (2), we get

$$\begin{aligned}
\left\| \tilde{\beta} \cdot \nabla_{xt} v \right\| &= - \int_Q g \tilde{\beta} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \\
&\quad - \epsilon \int_{\Gamma_x} \tilde{\beta} \cdot \nabla_{xt} v \nabla v \cdot \mathbf{n}_x + \epsilon \frac{1}{2} \int_{\Gamma} \tilde{\beta} \cdot \mathbf{n} (\nabla v \cdot \nabla v) \\
&= - \int_Q g \tilde{\beta} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \\
&\quad - \int_{\Gamma_-} \tilde{\beta} \cdot \nabla_{xt} v \underbrace{\nabla v \cdot \mathbf{n}_x}_{=0} - \int_{\Gamma_+} \left( \underbrace{\frac{\partial v}{\partial t}}_{=0} + \beta \cdot \nabla v \right) \nabla v \cdot \mathbf{n}_x \\
&\quad + \frac{1}{2} \int_{\Gamma_-} \underbrace{\beta \cdot \mathbf{n}_x}_{<0} (\nabla v \cdot \nabla v) + \frac{1}{2} \int_{\Gamma_+} \beta \cdot \mathbf{n}_x (\nabla v \cdot \nabla v) \\
&\quad + \frac{1}{2} \int_{\Gamma_0} \underbrace{n_t}_{<0} (\nabla v \cdot \nabla v) + \frac{1}{2} \int_{\Gamma_T} \underbrace{n_t}_{=0} (\nabla v \cdot \nabla v) \\
&\leq - \int_Q g \tilde{\beta} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \\
&\quad + \int_{\Gamma_+} \left( - \frac{\partial v}{\partial \mathbf{n}_x} \beta + \frac{1}{2} \beta \cdot \mathbf{n}_x \nabla v \right) \cdot \nabla v \\
&= - \int_Q g \tilde{\beta} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \\
&\quad + \int_{\Gamma_+} \left( - \frac{\partial v}{\partial \mathbf{n}_x} \beta + \frac{1}{2} \beta \cdot \mathbf{n}_x \frac{\partial v}{\partial \mathbf{n}_x} \mathbf{n}_x \right) \cdot \frac{\partial v}{\partial \mathbf{n}_x} \mathbf{n}_x \\
&= - \int_Q g \tilde{\beta} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \\
&\quad - \underbrace{\frac{1}{2} \int_{\Gamma_+} \left( \frac{\partial v}{\partial \mathbf{n}_x} \right)^2 \beta \cdot \mathbf{n}_x}_{<0} \\
&\leq - \int_Q g \tilde{\beta} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \\
&\leq - \frac{\|g\|}{2} + \frac{\left\| \tilde{\beta} \cdot \nabla_{xt} v \right\|}{2} + \epsilon \int_Q \nabla v (\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}) \nabla v \\
&\leq - \frac{\|g\|}{2} + \frac{\left\| \tilde{\beta} \cdot \nabla_{xt} v \right\|}{2} + \epsilon C \|\nabla v\|^2
\end{aligned}$$

□

**Lemma 1.2.** *For the above conditions on  $\beta$ ,*

$$\|v\|^2 + \epsilon \|\nabla v\|^2 \leq \|g\|^2 + \epsilon \|f\|^2.$$

Note that  $\boldsymbol{\tau} = \epsilon \nabla v$ , so we already have control of  $\|\boldsymbol{\tau}\|$ .

*Proof.* Define  $w = e^{T-t}v$  and note that  $\frac{\partial w}{\partial t} = (\frac{\partial v}{\partial t} - v)e^{T-t}$  while  $\nabla w = \nabla e^{T-t}v + e^{T-t}\nabla v$  and  $\nabla \cdot (\boldsymbol{\beta}w) = \nabla \cdot (\boldsymbol{\beta})e^{T-t}v + \boldsymbol{\beta} \cdot e^{T-t}\nabla v$  and  $\Delta w = e^{T-t}\Delta v$ . Also,  $\nabla_{xt}w = \frac{\partial e^{T-t}v}{\partial t} + \nabla e^{T-t}v = e^{T-t}(\nabla_{xt}v - v)$ . Plugging this into the adjoint equation, we get

$$-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt}(w) - \epsilon \Delta w = g - \epsilon \nabla \cdot \mathbf{f}$$

or

$$\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt}(v) - v + \epsilon \Delta v = e^{t-T}(-g + \epsilon \nabla \cdot \mathbf{f})$$

Multiply by  $-v$  and integrate to get

$$\int_Q -\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt}vv + v^2 - \epsilon \Delta vv = \int_Q e^{t-T}gv - \epsilon \int_Q e^{t-T}\nabla \cdot \mathbf{f}v$$

Then

$$\begin{aligned} \|v\|^2 &= \int_Q e^{t-T}gv - \epsilon \int_Q e^{t-T}\nabla \cdot \mathbf{f}v + \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt}vv + \epsilon \int_Q \Delta vv \\ &= \int_Q e^{t-T}gv - \epsilon \int_Q e^{t-T}\nabla \cdot \mathbf{f}v + \frac{1}{2} \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt}(v)^2 - \epsilon \int_Q \nabla v \nabla v + \epsilon \int_{\Gamma} v \nabla v \cdot \mathbf{n} \end{aligned}$$

Or

$$\begin{aligned} \|v\|^2 + \epsilon \|\nabla v\|^2 &= \int_Q e^{t-T}gv - \epsilon \int_Q e^{t-T}\nabla \cdot \mathbf{f}v \\ &\quad - \frac{1}{2} \int_Q \underbrace{\nabla_{xt} \cdot \tilde{\boldsymbol{\beta}}(v)}_{=0} + \frac{1}{2} \int_{\Gamma} v^2 \tilde{\boldsymbol{\beta}} \cdot \mathbf{n} + \epsilon \int_{\Gamma_x} v \nabla v \cdot \mathbf{n}_x \\ &= \int_Q e^{t-T}gv + \epsilon \int_Q e^{t-T} \mathbf{f} \cdot \nabla v - \epsilon \int_{\Gamma_-} v \underbrace{\mathbf{f} \cdot \mathbf{n}_x}_{=\epsilon \frac{\partial v}{\partial n} = 0} - \epsilon \int_{\Gamma_+} \underbrace{v}_{=0} \mathbf{f} \cdot \mathbf{n}_x \\ &\quad + \frac{1}{2} \int_{\Gamma_-} v^2 \underbrace{\boldsymbol{\beta} \cdot \mathbf{n}_x}_{<0} + \frac{1}{2} \int_{\Gamma_+} \underbrace{v^2}_{=0} \boldsymbol{\beta} \cdot \mathbf{n}_x \\ &\quad + \frac{1}{2} \int_{\Gamma_0} \underbrace{v^2(-n_t)}_{<0} + \frac{1}{2} \int_{\Gamma_T} \underbrace{v^2}_{=0} n_t \\ &\quad + \epsilon \int_{\Gamma_-} v \underbrace{\nabla v \cdot \mathbf{n}_x}_{=0} + \epsilon \int_{\Gamma_+} \underbrace{v}_{=0} \nabla v \cdot \mathbf{n}_x \\ &\leq \|e^{t-T}\|_{L^\infty(Q)} \left( \int_Q gv + \epsilon \int_Q \mathbf{f} \cdot \nabla v \right) \\ &\leq \left( \frac{\|g\|^2}{2} + \frac{\epsilon \|\mathbf{f}\|^2}{2} + \frac{\|v\|^2}{2} + \frac{\epsilon \|\nabla v\|^2}{2} \right) \end{aligned}$$

□

## 2 Robustness for transient problems given spatial robustness

Suppose we have the transient problem

$$\frac{\partial u}{\partial t} + Au = f$$

with initial condition  $u(x, 0) = u_0$ . Suppose that DPG is robust under the ultra-weak variational formulation for the steady problem

$$(u, A_h^* v)_{L^2(\Omega)} + \langle \hat{u}, \llbracket v \rrbracket \rangle_{\Gamma_h \setminus \Gamma_0} = (f, v)$$

with test norm  $\|v\|_V$ . Then, can we show that

$$\|v\|_{V,t} := \|v\|_V + \left\| \frac{\partial v}{\partial t} \right\|_{L^2(\Omega)}$$

also leads to a robust upper bound of the  $L^2$  norm by the DPG energy norm? I believe this may be possible. The adjoint equation for robustness for the transient problem gives

$$-\frac{\partial v}{\partial t} + A^* v = u$$

with  $v = 0$  at  $t = T$ ...