Robustness for Transient Problems

Truman E. Ellis and Jesse L. Chan

July 28, 2014

Assume that boundary conditions are applied on the boundary $\Gamma_0 \subset \Gamma$. Recall that, for the ultra-weak variational formulation

$$b\left(\left(u, \widehat{u}\right), v\right) = \left(u, A_h^* v\right)_{L^2(\Omega)} + \langle \widehat{u}, [\![v]\!] \rangle_{\Gamma_h \setminus \Gamma_0}$$

we can recover

$$\|u\|^2_{L^2(\Omega)} = b(u, v^*)$$

for conforming v^* satisfying the adjoint equation

$$\begin{aligned} A^* v^* &= u \\ v^* &= 0 \text{ on } \Gamma_h \setminus \Gamma_0. \end{aligned}$$

Together, these give necessary conditions on the test norm $\|\cdot\|_V$ such that we have L^2 robustness (this gives robustness in the variable u; for the first order formulation, conditions for σ must also be shown).

$$\|u\|^2_{L^2(\Omega)} = b(u, v^*) \leq \frac{b(u, v^*)}{\|v^*\|_V} \|v^*\|_V \leq \|u\|_E \|v^*\|_V$$

Thus, showing $\|v^*\|_V \lesssim \|u\|_{L^2(\Omega)}$ gives the result that $\|u\|_{L^2(\Omega)} \lesssim \|u\|_E$.

1 Reaction-diffusion

Consider reaction diffusion

$$\begin{aligned} \frac{\partial u}{\partial t} + u - \epsilon \Delta u &= f \\ u &= 0 \text{ on } \Gamma_1 \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_2 \\ u(t = 0) &= u_0. \end{aligned}$$

The adjoint equation satisfies

$$\begin{aligned}
-\frac{\partial v}{\partial t} + v - \epsilon \Delta v &= u \\
v &= 0 \text{ on } \Gamma_1 \\
\frac{\partial v}{\partial n} &= 0 \text{ on } \Gamma_2 \\
v(t = T) &= 0.
\end{aligned}$$

(The boundary conditions can be derived by taking the ultra-weak formulation and choosing boundary conditions such that the temporal flux and spatial flux terms $\langle \widehat{u}, [\![\tau_n]\!]\rangle_{\Gamma_1}$ and $\left\langle \widehat{f}_n, [\![v]\!]\right\rangle_{\Gamma_2}$ are zero.)

We can then derive that the test norm

$$\|v\|_V^2 = \left\|\frac{\partial v}{\partial t}\right\|^2 + \|v\|^2 + \epsilon \|\nabla v\|^2$$

provides the necessary bound $\|v^*\|_V \lesssim \|u\|_{L^2(\Omega)}$.

To see, this we multiply the adjoint equation by two terms as follows:

1. Multiply by v and integrate over $\Omega \times [0, T] = Q$ to get

$$-\int_Q \frac{\partial v}{\partial t} v + \int_Q v^2 + \epsilon \int_Q |\nabla v|^2 - \epsilon \int_0^T \int_\Gamma \frac{\partial v}{\partial n} v = \int_Q uv.$$

Noting that either $v = 0$ or $\frac{\partial v}{\partial n} = 0$ on the boundary removes the integral over Γ. Next, we can factor the first term and use Young's inequality to get

$$-\int_0^T \frac{\partial}{\partial t} \int_\Omega v^2 + \|v\|_Q^2 + \epsilon \|\nabla v\|_Q^2 \leq \frac{1}{2}\|u\|_Q^2 + \frac{1}{2}\|v\|_Q^2$$

Integrating by parts the first term gives

$$-\int_\Omega v^2 \Big|_0^T + \frac{1}{2}\|v\|_Q^2 + \epsilon \|\nabla v\|_Q^2 \leq \frac{1}{2}\|u\|_Q^2$$

Using boundary condition $v = 0$ at $t = T$ gives

$$\frac{1}{2}\|v\|_Q^2 + \epsilon \|\nabla v\|_Q^2 \leq \int_\Omega v(t=0)^2 + \frac{1}{2}\|v\|_Q^2 + \epsilon \|\nabla v\|_Q^2 \leq \frac{1}{2}\|u\|_Q^2.$$

2. Multiply by $-\frac{\partial v}{\partial t}$ and integrate over Q. Young's inequality changes the right hand side to

$$\int_Q \frac{\partial v}{\partial t}^2 - \int_Q v\frac{\partial v}{\partial t} + \epsilon \int_Q \Delta v \frac{\partial v}{\partial t} = \int_Q -u\frac{\partial v}{\partial t} \leq \frac{1}{2}\|u\|_Q^2 + \frac{1}{2}\left\|\frac{\partial v}{\partial t}\right\|_Q^2.$$

The term $\int_Q v\frac{\partial v}{\partial t}$ can be reduced to the positive contribution $\int_\Omega v(t=0)^2$ as above. We can then take the Laplacian term, integrate by parts in space to get

$$\int_Q \Delta v\frac{\partial v}{\partial t} = \int_0^T\int_\Omega \Delta v\frac{\partial v}{\partial t} = \int_0^T\int_\Gamma \frac{\partial v}{\partial t}\frac{\partial v}{\partial n} - \int_0^T\int_\Omega \nabla\left(\frac{\partial v}{\partial t}\right)\nabla v.$$

Since either $v=0$ or $\frac{\partial v}{\partial n}=0$ on Γ, the first term disappears. The second term can be bounded by noting

$$-\int_0^T\int_\Omega \nabla\left(\frac{\partial v}{\partial t}\right)\nabla v = -\int_0^T\frac{\partial}{\partial t}\int_\Omega |\nabla v|^2 = -\int_\Omega |\nabla v|^2\Big|_0^T.$$

Since $v=0$ at $t=T$, $\nabla v=0$ at $t=T$ as well, and we are left with the positive contribution $\int_\Omega |\nabla v(t=0)|^2$. Then,

$$\frac{1}{2}\left\|\frac{\partial v}{\partial t}\right\|_Q^2 \leq \frac{1}{2}\left\|u\right\|_Q.$$

Together, these two show that, under test norm

$$\left\|v\right\|_V^2 = \left\|\frac{\partial v}{\partial t}\right\|^2 + \left\|v\right\|^2 + \epsilon\left\|\nabla v\right\|^2,$$

the adjoint equation v^* satisfies

$$\left\|v^*\right\|_V \lesssim \left\|u\right\|_{L^2(\Omega)}$$

and thus the DPG energy norm robustly bounds the L^2 norm from above

$$\left\|u\right\|_{L^2(\Omega)} \lesssim \left\|u\right\|_E.$$

2 Convection-diffusion

Consider convection-diffusion

$$\begin{aligned}
\frac{\partial u}{\partial t} + \boldsymbol{\beta}\cdot\nabla u - \epsilon\Delta u &= f\\
u &= 0 \text{ on } \Gamma_{out}\\
\frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_{in}\\
u(t=0) &= u_0.
\end{aligned}$$

Let $\tilde{\boldsymbol{\beta}} := \begin{pmatrix} \boldsymbol{\beta} \\ 1 \end{pmatrix}$ and $\nabla_{xt} := \begin{pmatrix} \nabla \\ \frac{\partial}{\partial t} \end{pmatrix}$, then we can rewrite this as

$$\begin{aligned} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} u - \epsilon \Delta u &= f \\ u &= 0 \text{ on } \Gamma_{out} \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_{in} \\ u(t=0) &= u_0. \end{aligned}$$

The adjoint equation satisfies

$$\begin{aligned} -\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v - \epsilon \Delta v &= u \\ v &= 0 \text{ on } \Gamma_{in} \\ \frac{\partial v}{\partial n} &= 0 \text{ on } \Gamma_{out} \\ v(t=T) &= 0. \end{aligned}$$

(The boundary conditions can be derived by taking the ultra-weak formulation and choosing boundary conditions such that the temporal flux and spatial flux terms $\langle \widehat{u}, [\![\tau_n]\!] \rangle_{\Gamma_{out}}$ and $\left\langle \widehat{f}_n, [\![v]\!] \right\rangle_{\Gamma_{in}}$ are zero.) The $t = 0$ and $t = T$ boundaries can be considered as an inflow and outflow boundary respectively in space-time and we denote $\partial Q_{in} := \Gamma_{in} \cup t = 0$ while $\partial Q_{out} := \Gamma_{out} \cup t = T$.

We can then derive that the test norm

$$\|v\|_V^2 = \left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v\right\|^2 + \epsilon \left\|\nabla v\right\|^2$$

provides the necessary bound $\|v^*\|_V \lesssim \|u\|_{L^2(Q)}$.

To see this, we multiply the adjoint equation by two terms as follows:

1. Multiply by $-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v$ and integrate over Q to get

$$\left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v\right\| = -\int_Q u \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v - \epsilon \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \Delta v \,. \tag{1}$$

Note that

$$
\begin{aligned}
-\int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \Delta v = & -\int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla \cdot \nabla v \\
= & -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \int_Q \nabla(\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v) \cdot \nabla v \\
= & -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \int_Q (\nabla \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v) \cdot \nabla v \\
& + \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla \nabla_{xt} v \cdot \nabla v \\
= & -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \int_Q (\nabla \boldsymbol{\beta} \cdot \nabla v) \cdot \nabla v \\
& + \frac{1}{2} \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} (\nabla v \cdot \nabla v) \\
= & -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \int_Q (\nabla \boldsymbol{\beta} \cdot \nabla v) \cdot \nabla v \\
& + \frac{1}{2} \int_\Gamma \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n} (\nabla v \cdot \nabla v) - \frac{1}{2} \int_Q \nabla_{xt} \cdot \tilde{\boldsymbol{\beta}} (\nabla v \cdot \nabla v) \\
= & -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \int_Q (\nabla \boldsymbol{\beta} \cdot \nabla v) \cdot \nabla v \\
& + \frac{1}{2} \int_\Gamma \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n} (\nabla v \cdot \nabla v) - \frac{1}{2} \int_Q \nabla \cdot \boldsymbol{\beta} (\nabla v \cdot \nabla v) \\
= & -\int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \frac{1}{2} \int_\Gamma \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n} (\nabla v \cdot \nabla v) \\
& + \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \boldsymbol{I}) \nabla v
\end{aligned}
$$

Plugging this into (1), we get

$$
\begin{aligned}
\left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v\right\| &= -\int_Q u\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \boldsymbol{I}) \nabla v \\
&\quad - \epsilon \int_{\Gamma_x} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \nabla v \cdot \boldsymbol{n}_x + \epsilon \frac{1}{2} \int_{\Gamma} \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n} (\nabla v \cdot \nabla v) \\
&= -\int_Q u\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \boldsymbol{I}) \nabla v \\
&\quad - \int_{\Gamma_-} \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v \underbrace{\nabla v \cdot \boldsymbol{n}_x}_{=0} - \int_{\Gamma_+} \left(\underbrace{\frac{\partial v}{\partial t}}_{=0} + \boldsymbol{\beta} \cdot \nabla v \right) \nabla v \cdot \boldsymbol{n}_x \\
&\quad + \frac{1}{2} \int_{\Gamma_-} \underbrace{\boldsymbol{\beta} \cdot \boldsymbol{n}_x}_{<0} (\nabla v \cdot \nabla v) + \frac{1}{2} \int_{\Gamma_+} \boldsymbol{\beta} \cdot \boldsymbol{n}_x (\nabla v \cdot \nabla v) \\
&\quad + \frac{1}{2} \int_{\Gamma_0} \underbrace{n_t}_{<0} (\nabla v \cdot \nabla v) + \frac{1}{2} \int_{\Gamma_T} n_t \underbrace{(\nabla v \cdot \nabla v)}_{=0} \\
&\leq -\int_Q u\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \boldsymbol{I}) \nabla v \\
&\quad + \int_{\Gamma_+} \left(-\frac{\partial v}{\partial \boldsymbol{n}_x} \boldsymbol{\beta} + \frac{1}{2} \boldsymbol{\beta} \cdot \boldsymbol{n}_x \nabla v \right) \cdot \nabla v \\
&= -\int_Q u\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \boldsymbol{I}) \nabla v \\
&\quad + \int_{\Gamma_+} \left(-\frac{\partial v}{\partial \boldsymbol{n}_x} \boldsymbol{\beta} + \frac{1}{2} \boldsymbol{\beta} \cdot \boldsymbol{n}_x \frac{\partial v}{\partial \boldsymbol{n}_x} \boldsymbol{n}_x \right) \cdot \frac{\partial v}{\partial \boldsymbol{n}_x} \boldsymbol{n}_x \\
&= -\int_Q u\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \boldsymbol{I}) \nabla v \\
&\quad \underbrace{- \frac{1}{2} \int_{\Gamma_+} \left(\frac{\partial v}{\partial \boldsymbol{n}_x} \right)^2 \boldsymbol{\beta} \cdot \boldsymbol{n}_x}_{<0} \\
&\leq -\int_Q u\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \epsilon \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \boldsymbol{I}) \nabla v \\
&\leq -\frac{\|u\|}{2} + \frac{\left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v\right\|}{2} + \epsilon \int_Q \nabla v (\nabla \boldsymbol{\beta} - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \boldsymbol{I}) \nabla v \\
&\leq -\frac{\|u\|}{2} + \frac{\left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v\right\|}{2} + \epsilon C \left\|\nabla v\right\|^2
\end{aligned}
$$

2. Define $w = e^{T-t} v$ and note that $\frac{\partial w}{\partial t} = \left(\frac{\partial v}{\partial t} - v\right) e^{T-t}$ while $\nabla w = \cancelto{0}{\nabla e^{T-t}} v + e^{T-t} \nabla v$ and $\nabla \cdot (\boldsymbol{\beta} w) = \nabla \cdot (\boldsymbol{\beta}) e^{T-t} v + \boldsymbol{\beta} \cdot e^{T-t} \nabla v$ and $\Delta w = e^{T-t} \Delta v$.

Also, $\nabla_{xt} w = \frac{\partial e^{T-t} v}{\partial t} + \nabla e^{T-t} v = e^{T-t}(\nabla_{xt} v - v)$. Plugging this into the adjoint equation, we get

$$-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt}(w) - \epsilon \Delta w = u - \epsilon \nabla \cdot \boldsymbol{\sigma}$$

or

$$\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt}(v) - v + \epsilon \Delta v = e^{t-T}(-u + \epsilon \nabla \cdot \boldsymbol{\sigma})$$

Multiply by $-v$ and integrate to get

$$\int_Q -\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v v + v^2 - \epsilon \Delta v v = \int_Q e^{t-T} u v - \epsilon \int_Q e^{t-T} \nabla \cdot \boldsymbol{\sigma} v$$

Then

$$\begin{aligned}
\|v\|^2 &= \int_Q e^{t-T} u v - \epsilon \int_Q e^{t-T} \nabla \cdot \boldsymbol{\sigma} v + \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v v + \epsilon \int_Q \Delta v v \\
&= \int_Q e^{t-T} u v - \epsilon \int_Q e^{t-T} \nabla \cdot \boldsymbol{\sigma} v + \frac{1}{2} \int_Q \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt}(v)^2 - \epsilon \int_Q \nabla v \nabla v + \epsilon \int_\Gamma v \nabla v \cdot \boldsymbol{n}
\end{aligned}$$

Or

$$\begin{aligned}
\|v\|^2 + \epsilon \|\nabla v\|^2 &= \int_Q e^{t-T} u v - \epsilon \int_Q e^{t-T} \nabla \cdot \boldsymbol{\sigma} v \\
&\quad - \frac{1}{2} \int_Q \underbrace{\nabla_{xt} \cdot \tilde{\boldsymbol{\beta}}}_{=0} (v)^2 + \frac{1}{2} \int_\Gamma v^2 \tilde{\boldsymbol{\beta}} \cdot \boldsymbol{n} + \epsilon \int_{\Gamma_x} v \nabla v \cdot \boldsymbol{n}_x \\
&= \int_Q e^{t-T} u v + \epsilon \int_Q e^{t-T} \boldsymbol{\sigma} \nabla v - \int_{\Gamma_-} v \underbrace{\epsilon \boldsymbol{\sigma} \cdot \boldsymbol{n}_x}_{=\frac{\partial v}{\partial n}=0} - \epsilon \int_{\Gamma_+} \underbrace{v}_{=0} \boldsymbol{\sigma} \cdot \boldsymbol{n}_x \\
&\quad + \frac{1}{2} \int_{\Gamma_-} v^2 \underbrace{\boldsymbol{\beta} \cdot \boldsymbol{n}_x}_{<0} + \frac{1}{2} \int_{\Gamma_+} \underbrace{v^2}_{=0} \boldsymbol{\beta} \cdot \boldsymbol{n}_x \\
&\quad + \frac{1}{2} \int_{\Gamma_0} \underbrace{v^2(-n_t)}_{<0} + \frac{1}{2} \int_{\Gamma_T} \underbrace{v^2}_{=0} n_t \\
&\quad + \epsilon \int_{\Gamma_-} v \underbrace{\nabla v \cdot \boldsymbol{n}_x}_{=0} + \epsilon \int_{\Gamma_+} \underbrace{v}_{=0} \nabla v \cdot \boldsymbol{n}_x \\
&\leq \left\| e^{t-T} \right\|_{L_\infty(Q)} \left(\int_Q u v + \epsilon \int_Q u \nabla v \right) \\
&\leq \left((1+\epsilon) \frac{\|u\|^2}{2} + \frac{\|v\|^2}{2} + \frac{\epsilon \|\nabla v\|^2}{2} \right)
\end{aligned}$$

3 Robustness for transient problems given spatial robustness

Suppose we have the transient problem

$$\frac{\partial u}{\partial t} + Au = f$$

with initial condition $u(x,0) = u_0$. Suppose that DPG is robust under the ultra-weak variational formulation for the steady problem

$$(u, A_h^* v)_{L^2(\Omega)} + \langle \widehat{u}, [\![v]\!] \rangle_{\Gamma_h \setminus \Gamma_0} = (f, v)$$

with test norm $\|v\|_V$. Then, can we show that

$$\|v\|_{V,t} \coloneqq \|v\|_V + \left\| \frac{\partial v}{\partial t} \right\|_{L^2(\Omega)}$$

also leads to a robust upper bound of the L^2 norm by the DPG energy norm? I believe this may be possible. The adjoint equation for robustness for the transient problem gives

$$-\frac{\partial v}{\partial t} + A^* v = u$$

with $v = 0$ at $t = T$...