# Space-Time DPG:
# Designing a Method for Parallel CFD

Truman Ellis

Leszek Demkowicz, Nathan Roberts, Jesse Chan, Robert Moser

THE UNIVERSITY OF TEXAS AT AUSTIN

INSTITUTE FOR COMPUTATIONAL ENGINEERING & SCIENCES

# Motivation

# Heat Equation

## The simplest nontrivial space-time problem

Equation is elliptic in space, but hyperbolic in time.

$$\frac{\partial u}{\partial t} - \epsilon \Delta u = f$$

This is really just a composite of Fourier's law and conservation of energy.

$$\begin{aligned} \boldsymbol{\sigma} - \epsilon \nabla u &= 0 \\ \frac{\partial u}{\partial t} - \nabla \cdot \boldsymbol{\sigma} &= f \end{aligned}$$

We can rewrite this in terms of a space-time divergence.

$$\begin{aligned} \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= 0 \\ \nabla_{xt} \cdot \begin{pmatrix} -\boldsymbol{\sigma} \\ u \end{pmatrix} &= f \end{aligned}$$

# Heat Equation



Multiply by test function and integrate by parts over space-time element K.

$$\left(\frac{1}{\epsilon}\boldsymbol{\sigma},\boldsymbol{\tau}\right)+\left(u,\nabla\cdot\boldsymbol{\tau}\right)-\langle\hat{u},\boldsymbol{\tau}\cdot\boldsymbol{n}_x\rangle=0$$

$$-\left(\begin{pmatrix}-\boldsymbol{\sigma}\\ u\end{pmatrix},\nabla_{xt}v\right)+\langle\hat{t},v\rangle=f$$

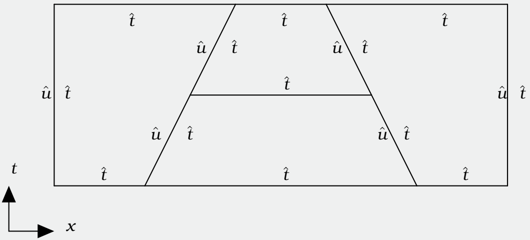
where

$$\hat{u}:=\operatorname{tr}(u)$$

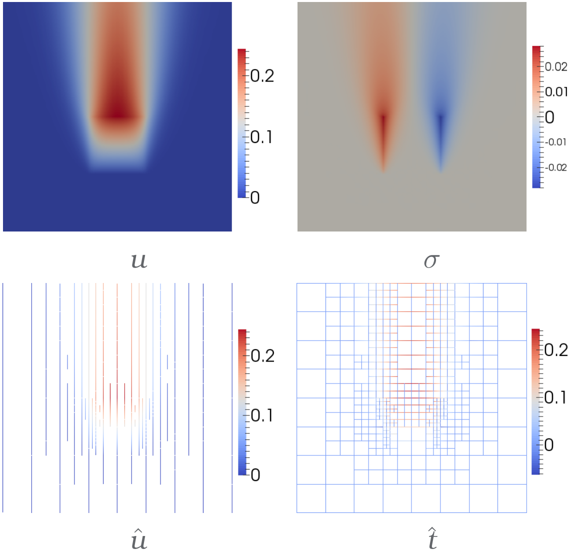
$$\hat{t}:=\operatorname{tr}(-\boldsymbol{\sigma})\cdot\boldsymbol{n}_x+\operatorname{tr}(u)\cdot n_t$$

- Trace $\hat{u}$ defined on spatial boundaries
- Flux $\hat{t}$ defined on all boundaries

## Support of Trace Variables



# Pulsed Source Problem



# Compressible Navier-Stokes

THE UNIVERSITY OF TEXAS AT AUSTIN

The compressible Navier-Stokes equations are

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\boldsymbol{u}\\ \rho e_0\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho\boldsymbol{u}\\ \rho\boldsymbol{u}\otimes\boldsymbol{u}+p\boldsymbol{I}-\mathbb{D}\\ \rho\boldsymbol{u}e_0+\boldsymbol{u}p+\boldsymbol{q}-\boldsymbol{u}\cdot\mathbb{D}\end{bmatrix}=\begin{bmatrix}f_c\\ \boldsymbol{f}_m\\ f_e\end{bmatrix},$$

where

$$\mathbb{D}=2\mu\boldsymbol{S}^*=2\mu\left[\frac{1}{2}\left(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T\right)-\frac{1}{3}\nabla\cdot\boldsymbol{u}\boldsymbol{I}\right],$$

$$\boldsymbol{q}=-C_p\frac{\mu}{Pr}\nabla T\,,$$

and (assuming an ideal gas EOS)

$$p=\rho RT\,.$$

# Compressible Navier-Stokes

THE UNIVERSITY OF TEXAS AT AUSTIN

Writing this in space-time in terms of $\rho$, $\boldsymbol{u}$, $T$, $\mathbb{D}$, and $\boldsymbol{q}$:

$$\begin{aligned}
\mathbb{D} - \mu\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right) + \frac{2\mu}{3}\nabla\cdot\boldsymbol{u}\boldsymbol{I} &= 0 \\
\boldsymbol{q} + C_p\frac{\mu}{Pr}\nabla T &= 0 \\
\nabla_{xt}\cdot\begin{pmatrix}\rho\boldsymbol{u}\\ \rho\end{pmatrix} &= f_c \\
\nabla_{xt}\cdot\begin{pmatrix}\rho\boldsymbol{u}\otimes\boldsymbol{u} + \rho RT\boldsymbol{I} - \mathbb{D}\\ \rho\boldsymbol{u}\end{pmatrix} &= \boldsymbol{f}_m \\
\nabla_{xt}\cdot\begin{pmatrix}\rho\boldsymbol{u}\left(C_vT + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) + \boldsymbol{u}\rho RT + \boldsymbol{q} - \boldsymbol{u}\cdot\mathbb{D}\\ \rho\left(C_vT + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\end{pmatrix} &= f_e\,.
\end{aligned}$$

# Compressible Navier-Stokes

Multiplying by test functions and integrating by parts:

$$
\begin{aligned}
(\mathbb{D},\mathbb{S}) + (2\mu\boldsymbol{u},\nabla\cdot\mathbb{S}) - \left(\frac{2\mu}{3}\boldsymbol{u},\nabla\operatorname{tr}\mathbb{S}\right) - \langle 2\mu\hat{\boldsymbol{u}},\mathbb{S}\boldsymbol{n}_x\rangle + \left\langle\frac{2\mu}{3}\hat{\boldsymbol{u}},\mathbb{S}\boldsymbol{n}_x\right\rangle &= 0\\
(\boldsymbol{q},\boldsymbol{\tau}) - \left(C_p\frac{\mu}{Pr}T,\nabla\cdot\boldsymbol{\tau}\right) + \left\langle C_p\frac{\mu}{Pr}\hat{T},\tau_n\right\rangle &= 0\\
-\left(\begin{pmatrix}\rho\boldsymbol{u}\\ \rho\end{pmatrix},\nabla_{xt}v_c\right) + \langle\hat{t}_c,v_c\rangle &= (f_c,v_c)\\
-\left(\begin{pmatrix}\rho\boldsymbol{u}\otimes\boldsymbol{u}+\rho RT\boldsymbol{I}-\mathbb{D}\\ \rho\boldsymbol{u}\end{pmatrix},\nabla_{xt}\boldsymbol{v}_m\right) + \langle\hat{\boldsymbol{t}}_m,\boldsymbol{v}_m\rangle &= (\boldsymbol{f}_m,\boldsymbol{v}_m)\\
-\left(\begin{pmatrix}\rho\boldsymbol{u}\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)+\boldsymbol{u}\rho RT+\boldsymbol{q}-\boldsymbol{u}\cdot\mathbb{D}\\ \rho\left(C_vT+\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\end{pmatrix},\nabla_{xt}v_e\right) + \langle\hat{t}_e,v_e\rangle &= (f_e,v_e)\ ,
\end{aligned}
$$

where $\hat{u}$ and $\hat{T}$ are spatial traces and $\hat{t}_c$, $\hat{\boldsymbol{t}}_m$, and $\hat{t}_e$ are fluxes.