Gradient of a Vector

Takes a rank 1 object and expands it to rank 2 where the second dimension comes from the gradient.

$$\nabla \boldsymbol{u} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

Dot Product of a Vector and Tensor

Takes a rank 2 object and collapses the second dimension via dot product.

$$\begin{aligned} \boldsymbol{u} \cdot \mathbb{T} &= \begin{bmatrix} \boldsymbol{u} \cdot (\mathbb{T}_{11} & \mathbb{T}_{12}) \\ \boldsymbol{u} \cdot (\mathbb{T}_{21} & \mathbb{T}_{22}) \end{bmatrix} \\ &= \begin{bmatrix} u\mathbb{T}_{11} + v\mathbb{T}_{12} \\ u\mathbb{T}_{21} + v\mathbb{T}_{22} \end{bmatrix} \end{aligned}$$

Divergence of a Tensor

Takes a rank 2 object and collapses the second dimension via divergence.

$$\begin{aligned} \nabla \cdot \mathbb{T} &= \begin{bmatrix} \nabla \cdot (\mathbb{T}_{11} & \mathbb{T}_{12}) \\ \nabla \cdot (\mathbb{T}_{21} & \mathbb{T}_{22}) \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial \mathbb{T}_{11}}{\partial x} + \frac{\partial \mathbb{T}_{12}}{\partial y} \\ \frac{\partial \mathbb{T}_{21}}{\partial x} + \frac{\partial \mathbb{T}_{22}}{\partial y} \end{bmatrix} \end{aligned}$$

Laplacian of a Vector

$$\begin{aligned} \nabla \cdot \nabla \boldsymbol{u} &= \nabla \cdot \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} \nabla \cdot (\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}) \\ \nabla \cdot (\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}) \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \end{bmatrix} \end{aligned}$$

Divergence of a Symmetric Gradient

Let $\boldsymbol{\sigma} = \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right)$, then

$$
\begin{aligned}
\nabla \cdot \boldsymbol{\sigma} &= \nabla \cdot \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \\
&= \nabla \cdot \begin{bmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} & 2\frac{\partial v}{\partial y} \end{bmatrix} \\
&= \begin{bmatrix} 2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y \partial x} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 v}{\partial y^2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \end{bmatrix}
\end{aligned}
$$