

# Discontinuous Petrov-Galerkin (DPG) Method With Optimal Test Functions

Progress Report

Leszek Demkowicz

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# Collaboration:

Portland State: J. Gopalakrishnan

ICES: T. Bui-Thanh, O. Ghattas, B. Moser, T. Ellis, J. Zitelli

Argonne: N. Roberts

Boeing: D. Young

Basque U: D. Pardo

C.U. of Hong-Kong: W. Qiu

KAUST: V. Calo

Livermoore: J. Bramwell

Los Alamos: M. Shaskov

Rice: J. Chan

Sandia: P.N. Bochev, K.J. Peterson, D. Ridzal and Ch. M. Siefert

C.U. Chile: I. Muga, N. Heuer

U. Helsinki: A. Niemi

U. Nevada: J. Li

U. Tel-Aviv I. Harari

- 1 DPG Method is a Ritz method. It supports adaptivity with no preasymptotic behavior.
- 2 You can control the norm in which you want to converge.
- 3 DPG is easy to code.

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Assume

$$J_S^{\text{imp}} = n \times H^{\text{imp}}$$

and look for the unknown surface current on the skeleton also in the same form.

$$\left\{ \begin{array}{l} E \in H(\text{curl}, \Omega), \quad n \times E = n \times E^{\text{imp}} \text{ on } \Gamma_1 \\ \hat{h} \in \text{tr}_{\Gamma_h} H(\text{curl}, \Omega), \quad n \times \hat{h} = n \times (-i\omega H^{\text{imp}}) \text{ on } \Gamma_2 \\ (\frac{1}{\mu} \nabla \times E, \nabla_h \times F) + ((-\omega^2 \epsilon + i\omega \sigma) E, F) + \langle n \times \hat{h}, F \rangle_{\Gamma_h} = -i\omega (J^{\text{imp}}, F) \\ \forall F \in H(\text{curl}, \Omega_h). \end{array} \right.$$

Hexahedral meshes

$H(\text{curl})$  element for electric field  $E$ :

$$(\mathcal{P}^{p-1} \otimes \mathcal{P}^p \otimes \mathcal{P}^p) \times (\mathcal{P}^p \otimes \mathcal{P}^{p-1} \otimes \mathcal{P}^p) \times (\mathcal{P}^p \otimes \mathcal{P}^p \otimes \mathcal{P}^{p-1})$$

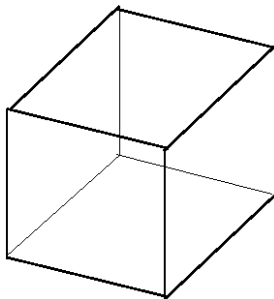
and trace of the same element for flux (surface current)  $\hat{h}$ .

Same element for the enriched space but with order  $p + \Delta p$ .

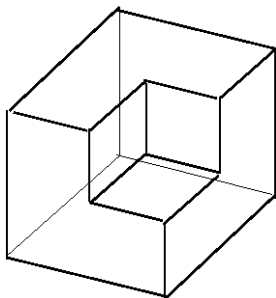
In reported experiments:  $p = 2$ ,  $\Delta p = 2$ .

# A 3D Maxwell example

Take a cube  $(0, 2)^3$

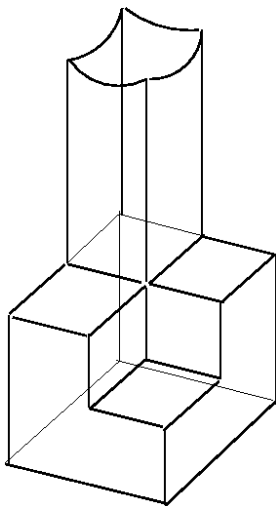


Divide it into eight smaller cubes and remove one:





Attach a waveguide:

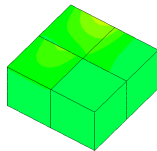
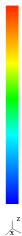
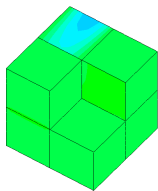
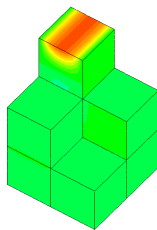
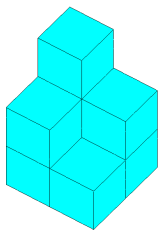


$$\epsilon = \mu = 1, \sigma = 0$$

$$\omega = 5(1.6 \text{ wavelengths in the cube})$$

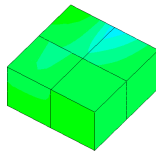
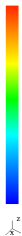
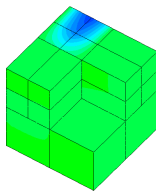
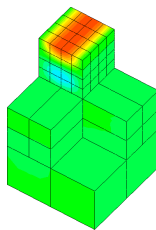
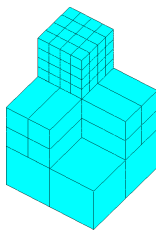
Cut the waveguide and use the lowest propagating mode for BC along the cut.

# Fichera corner microwave, $p = 2$ .



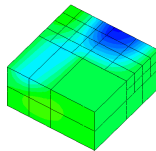
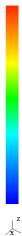
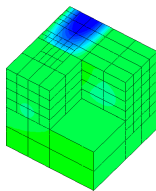
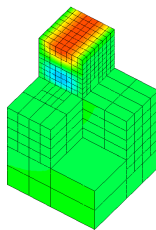
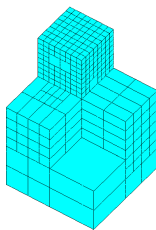
Initial mesh and real part of  $E_1$

# Fichera corner microwave, $p = 2$ .



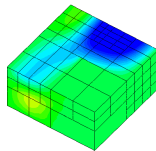
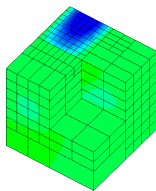
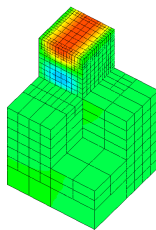
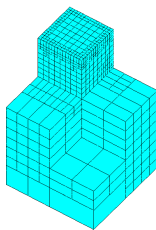
Mesh and real part of  $E_1$  after two refinements

# Fichera corner microwave, $p = 2$ .



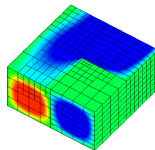
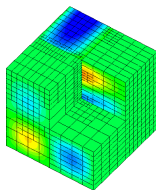
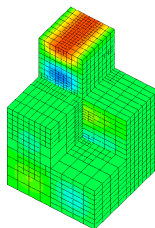
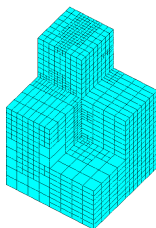
Mesh and real part of  $E_1$  after four refinements

# Fichera corner microwave, $p = 2$ .



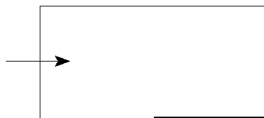
Mesh and real part of  $E_1$  after six refinements

# Fichera corner microwave, $p = 2$ .



Mesh and real part of  $E_1$  after eight refinements

# From Ph.D. Dissertation of Jesse Chan: Compressible Navier-Stokes Equations: Carter's flat plate problem <sup>1</sup>



$$M_{\infty} = 3, \text{Re}_L = 1000, \text{Pr} = 0.72, \gamma = 1.4, \theta_{\infty} = 390^{\circ}[\text{R}]$$

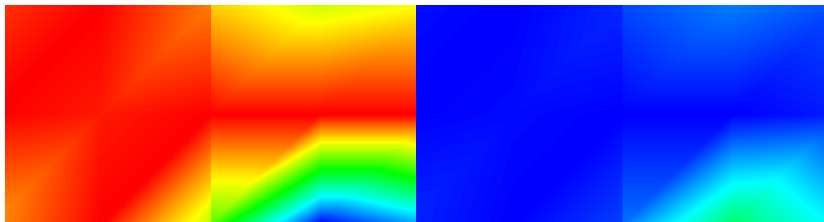
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<sup>1</sup> L.D., J.T. Oden, W. Rachowicz, "A New Finite Element Method for Solving Compressible Navier-Stokes Equations Based on an Operator Splitting Method and  $h_p$  Adaptivity," *Comput. Methods Appl. Mech. Engrg.*, **84**, 275-326, 1990.

Initial Mesh ( $p = 2$ ):

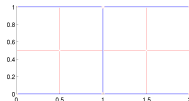


Horizontal velocity and temperature

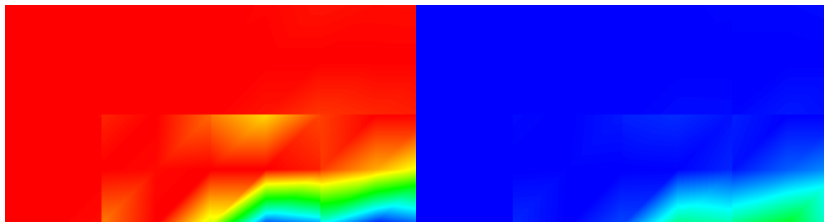




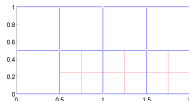
Mesh 1:



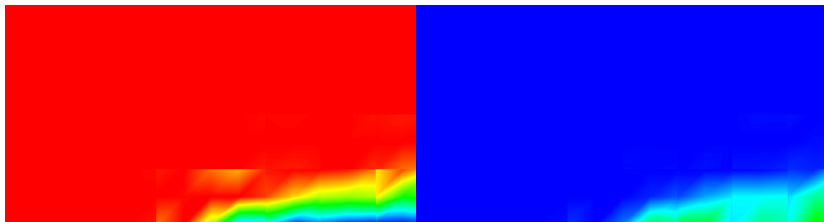
Horizontal velocity and temperature



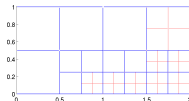
Mesh 2:



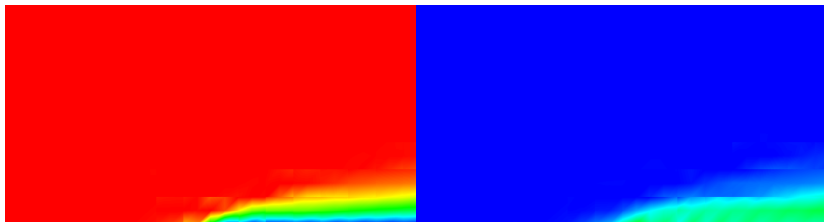
Horizontal velocity and temperature



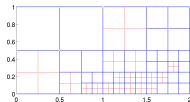
Mesh 3:



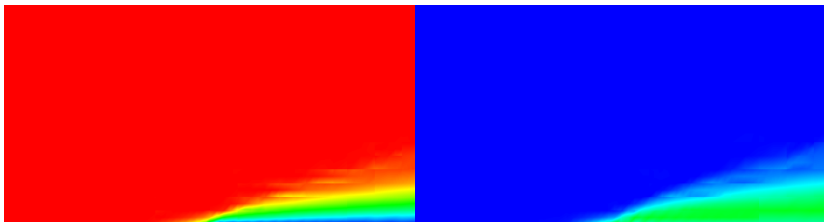
Horizontal velocity and temperature



Mesh 4:

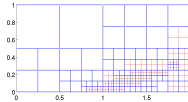


Horizontal velocity and temperature

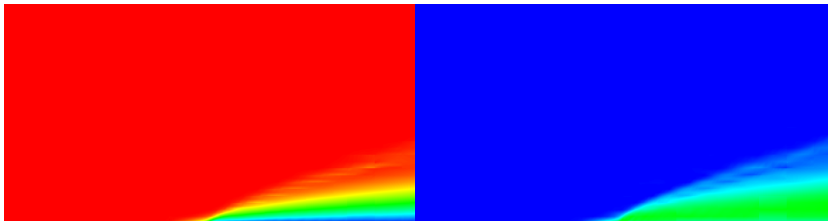


# Extrapolation to Compressible NS Equations

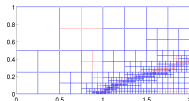
Mesh 5:



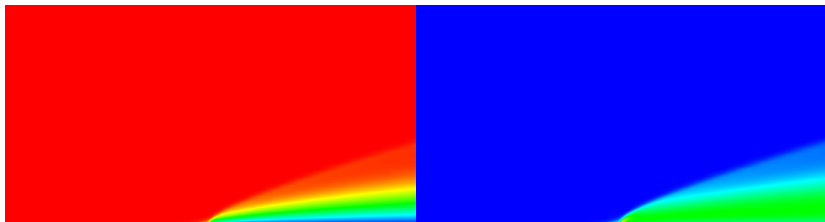
Horizontal velocity and temperature



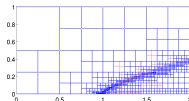
Mesh 7:



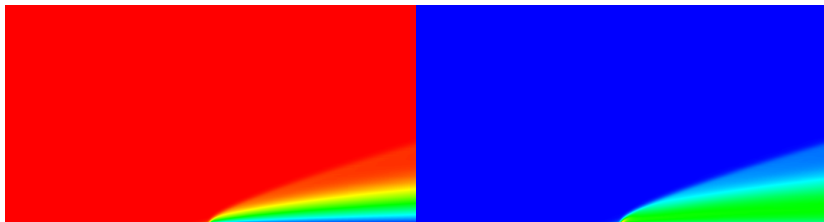
Horizontal velocity and temperature



Mesh 8:

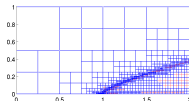


Horizontal velocity and temperature

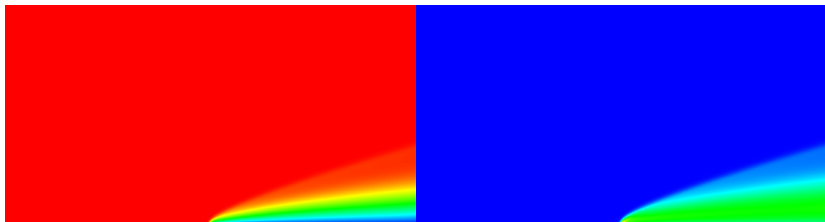


# Extrapolation to Compressible NS Equations

Mesh 9:

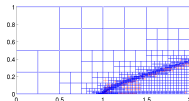


Horizontal velocity and temperature

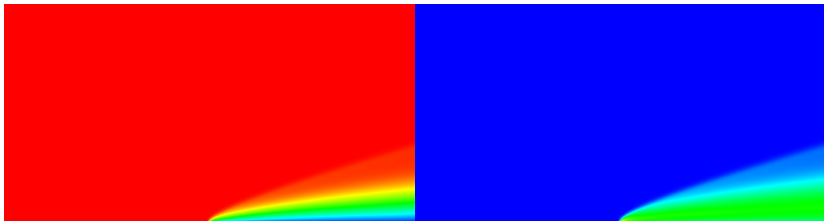




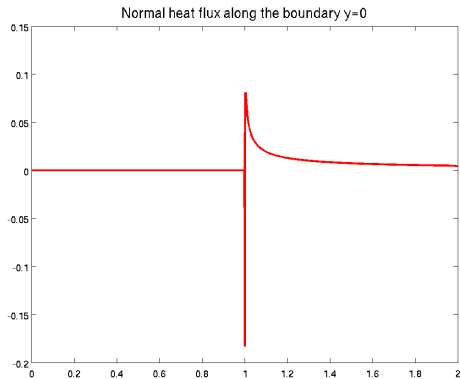
Mesh 10:



Horizontal velocity and temperature



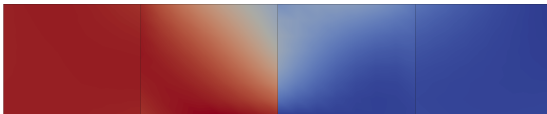
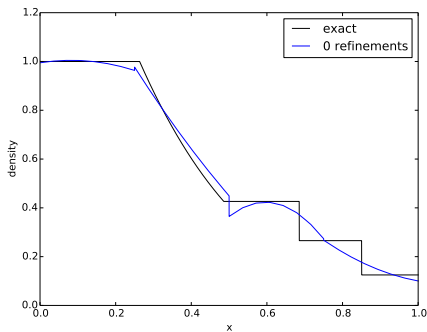
# Extrapolation to Compressible NS Equations



Heat flux along the plate

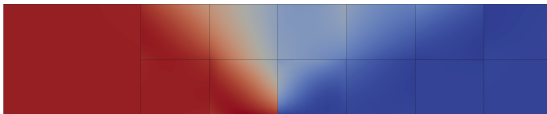
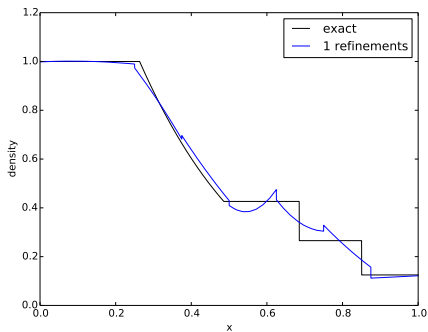
# Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



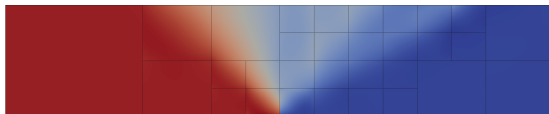
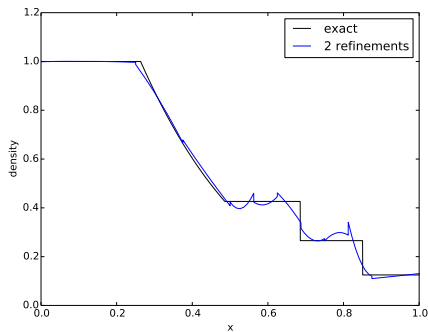
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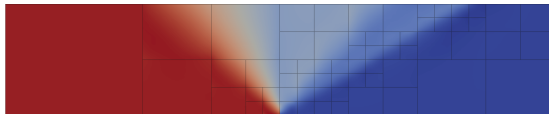
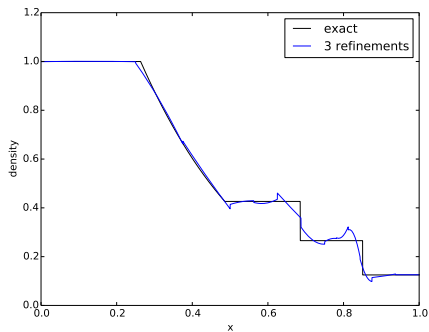
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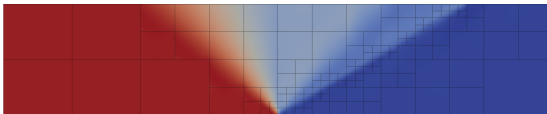
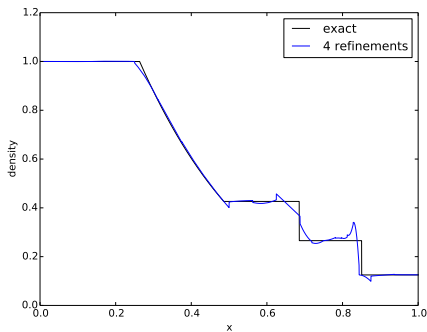
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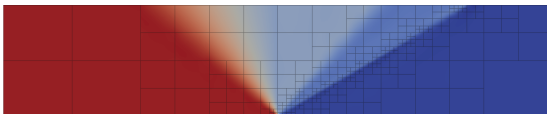
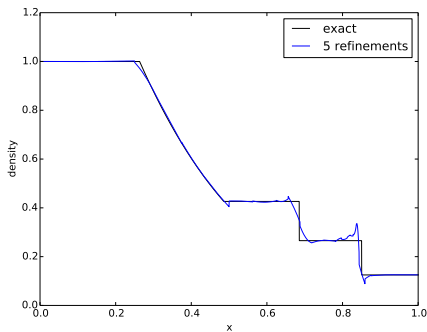
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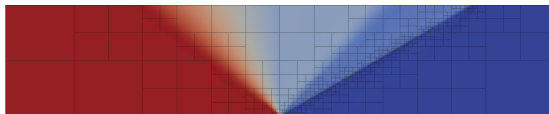
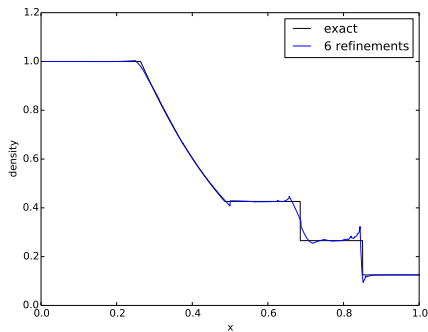
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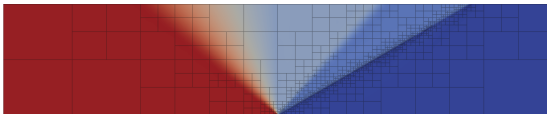
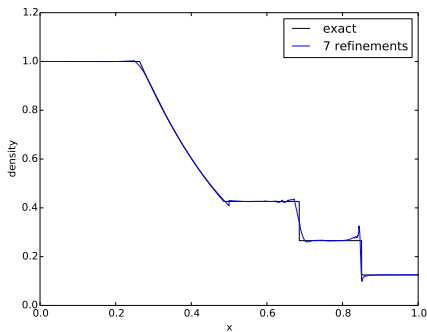
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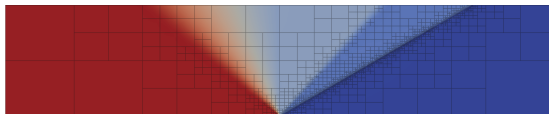
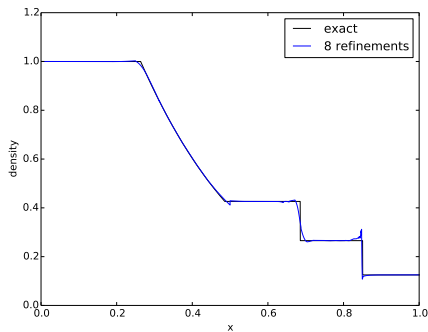
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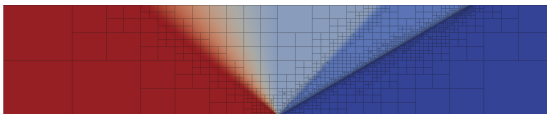
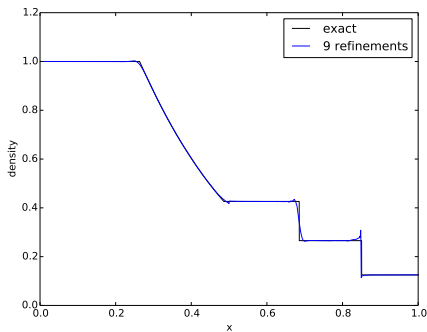
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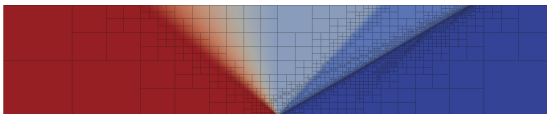
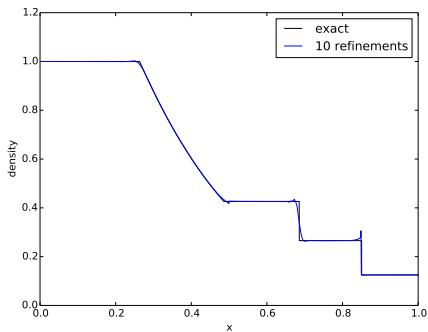
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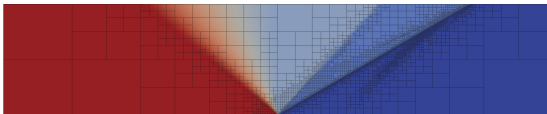
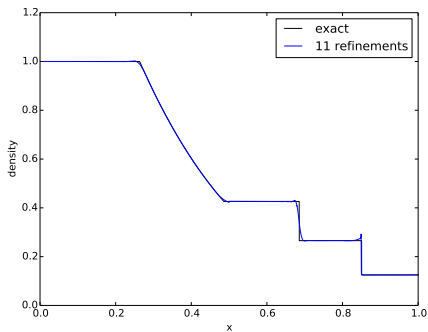
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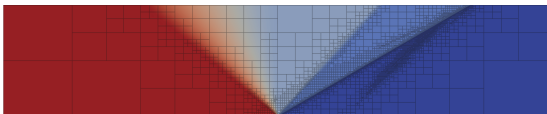
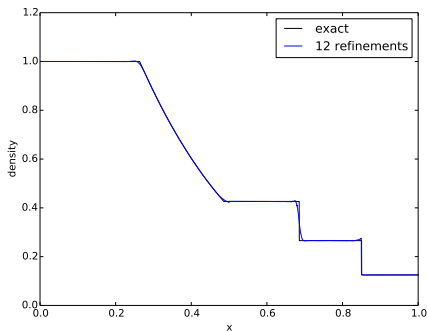
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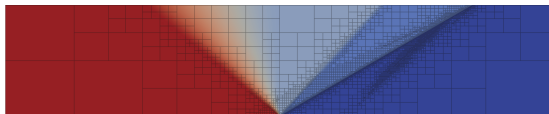
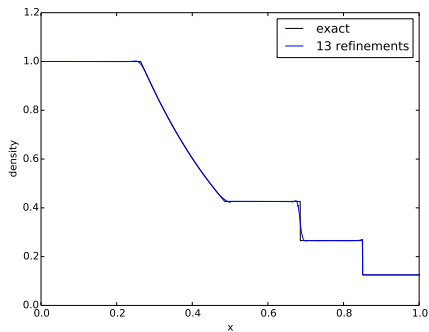
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# Compressible Navier-Stokes

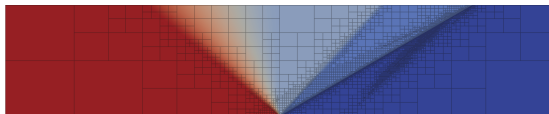
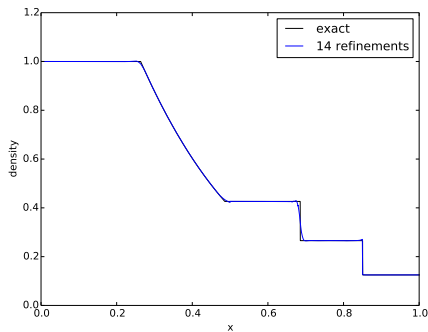
Sod Shock Tube with  $\mu = 10^{-5}$





# Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



# Three DPG Punchlines

- 1 DPG Method is a Ritz method. It supports adaptivity with no preasymptotic behavior.
- 2 You can control the norm in which you want to converge.
- 3 DPG is easy to code.

# The simplest singular perturbation problem: reaction-dominated diffusion<sup>2</sup>

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<sup>2</sup>L.D. and I. Harari, "Primal DPG Method for Reaction dominated Diffusion", in preparation.

# The simplest singular perturbation problem: Reaction-dominated diffusion

$$\begin{cases} u = 0 & \text{on } \Gamma \\ -\epsilon^2 \Delta u + c(x)u = f & \text{in } \Omega \end{cases}$$

where  $0 < c_0 \leq c(x) \leq c_1$ .

Standard variational formulation:

$$\begin{cases} u \in H^1(\Omega) \\ \epsilon^2(\nabla u, \nabla v) + (cu, v) = (f, v) \quad v \in H^1(\Omega) \end{cases}$$

Standard Galerkin method delivers the best approximation error in the energy norm:

$$\|u\|_{\epsilon^k}^2 := \epsilon^k \|\nabla u\|^2 + \|c^{1/2}u\|^2, \quad k = 2$$

**Fact:** Under favorable regularity conditions, the solution is *uniformly* bounded in data  $f$  in a “balanced” norm<sup>3</sup>:

$$\|u\|_{\epsilon}^2 := \epsilon \|\nabla u\|^2 + \|c^{1/2}u\|^2$$

i.e.

$$\|u\|_{\epsilon} \lesssim \|f\|_{\text{appropriate}}$$

**Question:** Can we select the test norm in such a way that the DPG method will be *robust* in the balanced norm?

$$\|u - u_h\|_{\epsilon} + \|\hat{t} - \hat{t}_h\|_? \lesssim \inf_{w_h} \|u - w_h\|_{\epsilon} + \inf_{\hat{r}_h} \|\hat{t} - \hat{r}_h\|_?$$

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<sup>3</sup>R. Lin and M. Stynes, “A balanced finite element method for singularly perturbed reaction-diffusion problems”, *SIAM J. Numer. Anal.*, **50**(5): 2729–2743, 2012.

# A bit of history:

Optimal test functions of Barret and Morton<sup>4 5</sup>

For each  $w \in U_h$ , determine the corresponding  $v_w$  that solves the auxiliary variational problem:

$$\begin{cases} v_w \in H_0^1(\Omega) \\ \underbrace{\epsilon^2(\nabla \delta u, \nabla v_w) + (c \delta u, v_w)}_{\text{the bilinear form we have}} = \underbrace{\epsilon(\nabla \delta u, w) + (c \delta u, w)}_{\text{the bilinear form we want}} \quad \forall \delta u \in H_0^1(\Omega) \end{cases}$$

With the optimal test functions, the Galerkin orthogonality for the original form changes into Galerkin orthogonality in the desired, “balanced” norm:

$$\epsilon^2(\nabla(u - u_h), \nabla v_w) + (c(u - u_h), v_w) = 0 \quad \implies \quad \epsilon(\nabla(u - u_h), \nabla v_u) + (c(u - u_h), w) = 0$$

Consequently, the PG solution delivers the best approximation error in the desired norm.

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<sup>4</sup>J.W. Barret and K. W. Morton, “Approximate Symmetrization and Petrov-Galerkin Methods for Diffusion-Convection Problems”, *Comp. Meth. Appl. Mech and Engng.*, **46**, 97 (1984).

<sup>5</sup>L. D. and J. T. Oden, “An Adaptive Characteristic Petrov-Galerkin Finite Element Method for Convection-Dominated Linear and Nonlinear Parabolic Problems in One Space Variable”, *Journal of Computational Physics*, **68(1)**: 188–273, 1986.

## Theorem

Let  $v_u$  be the Barret-Morton optimal test function corresponding to  $u$ . Let  $\|v_u\|_V$  be a test norm such that

$$\|v_u\|_V \lesssim \|u\|_\epsilon$$

Then

$$\|u - u_h\|_\epsilon \lesssim \|u - u_h\|_E = \inf_{w_h \in U_h} \|u - w_h\|_E \leq \text{BAE estimate}$$

**Proof:**

$$\begin{aligned} \|u\|_\epsilon^2 &= \epsilon(\nabla u, \nabla u) + (cu, u) = \epsilon^2(\nabla u, \nabla v_u) + (cu, v_u) \\ &= b((u, \hat{t}), v_u) \leq \frac{b((u, \hat{t}), v_u)}{\|v_u\|_V} \|v_u\|_V \\ &\leq \sup_v \frac{b((u, \hat{t}), v_u)}{\|v\|_V} \|v_u\|_V = \|(u, \hat{t})\|_E \|v_u\|_V \\ &\lesssim \|(u, \hat{t})\|_E \|u\|_\epsilon \end{aligned}$$

<sup>6</sup>L. D., M. Heuer, "Robust DPG Method for Convection-Dominated Diffusion Problems," *SIAM J. Num. Anal.*, **51**: 2514–2537, 2013.

**The point:** Construction of the optimal test norm is reduced to the stability (robustness) analysis for the Barret-Morton test functions.

## Lemma

Let

$$\|v\|_V^2 := \epsilon^3 \|\nabla v\|^3 + \|c^{1/2}v\|^2$$

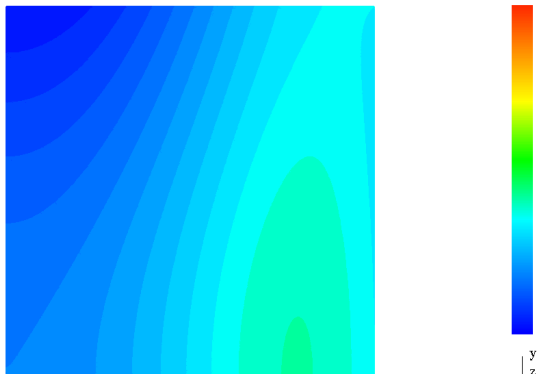
Then

$$\|v_u\| \lesssim \|u\|_\epsilon$$

In order to avoid boundary layers in the optimal test functions (that we cannot resolve using simple enriched space) we scale the reaction term with a mesh-dependent factor:

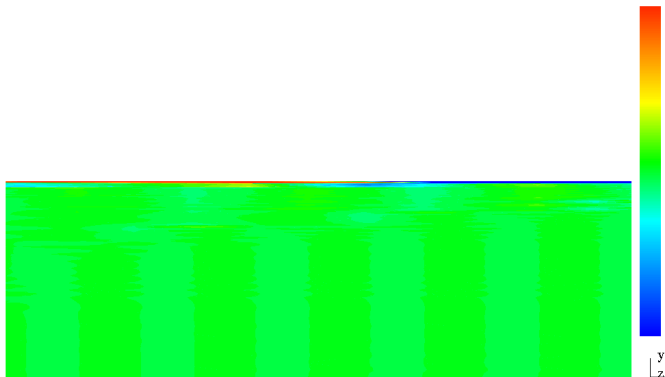
$$\|v\|_{V,mod}^2 := \epsilon^3 \|\nabla v\|^3 + \min\left\{1, \frac{\epsilon^3}{h^2}\right\} \|c^{1/2}v\|^2$$





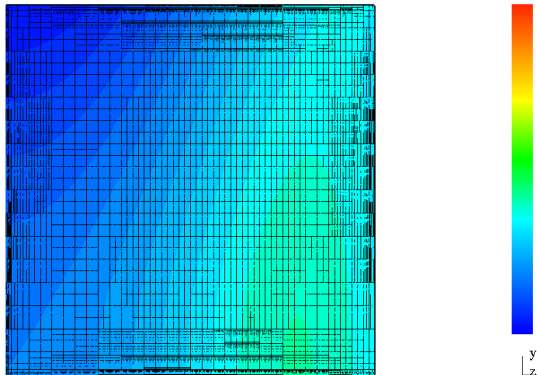
The functions exhibits strong boundary layers invisible in this scale.

Range:  $(-0.6, 0.6)$

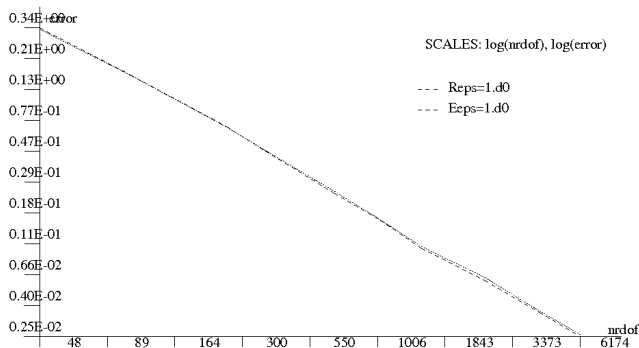


Zoom on the north boundary layer.

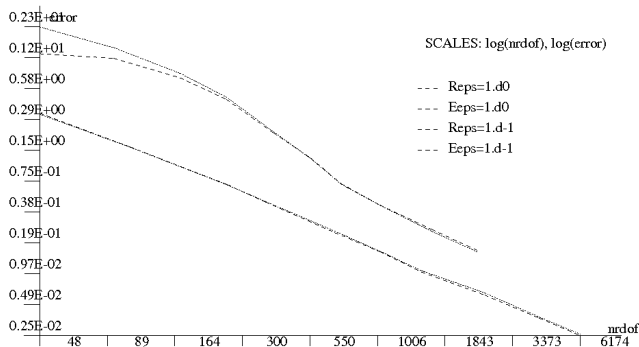
# Optimal mesh for $\epsilon = 10^{-1}$



Optimal  $h$ -adaptive mesh and numerical solution for  $\epsilon = 10^{-1}$

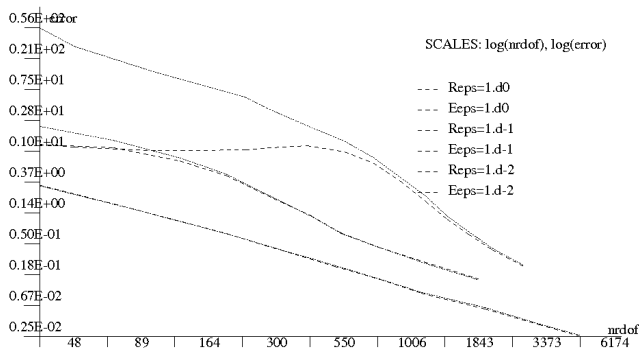


Residual and “balanced” error of  $u$  for  $h$ -adaptive solution,  $p = 2$



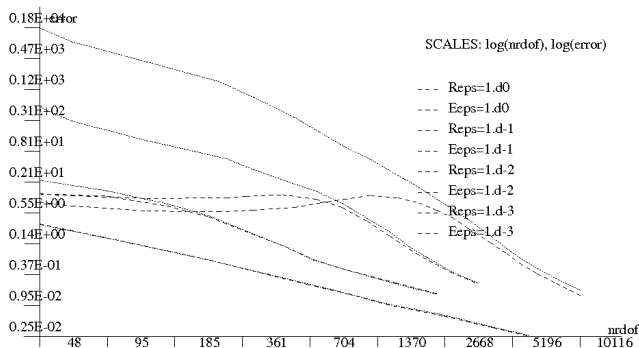
Residual and “balanced” error of  $u$  for  $h$ -adaptive solution,  $p = 2$

# Lin/Stynes example, $\epsilon = 10^0, 10^{-1}, 10^{-2}$ .

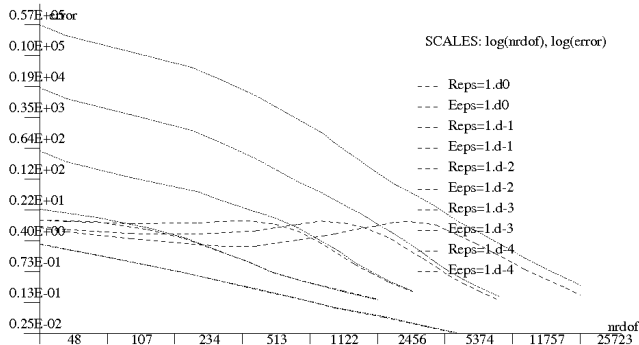


Residual and “balanced” error of  $u$  for  $h$ -adaptive solution,  $p = 2$

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Residual and “balanced” error of  $u$  for  $h$ -adaptive solution,  $p = 2$

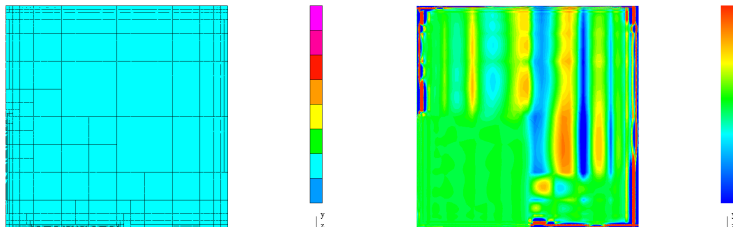


Residual and "balanced" error of  $u$  for  $h$ -adaptive solution,  $p = 2$



**Question:** Can we select the test norm in such a way that the DPG method would deliver high accuracy in a preselected subdomain, e.g.  $(0, \frac{1}{2})^2 \subset (0, 1)^2$  ?

**Answer:** Yes!



Optimal mesh and the corresponding pointwise error (range  $(-0.001 - 0.001)$ ).

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- Elasticity, shells (volumetric, shear, membrane locking)
- Metamaterials