

# Computing Longest Common Substring/Subsequence of Non-linear Texts

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**Abstract.** A non-linear text is a directed graph where each vertex is labeled with a string. In this paper, we introduce the longest common substring/subsequence problems on non-linear texts. Firstly, we present an algorithm to compute the longest common substring of non-linear texts  $G_1$  and  $G_2$  in  $O(|E_1||E_2|)$  time and  $O(|V_1||V_2|)$  space, when at least one of  $G_1$  and  $G_2$  is acyclic. Here,  $V_i$  and  $E_i$  are the sets of vertices and arcs of input non-linear text  $G_i$ , respectively, for  $1 \leq i \leq 2$ . Secondly, we present algorithms to compute the longest common subsequence of  $G_1$  and  $G_2$  in  $O(|E_1||E_2|)$  time and  $O(|V_1||V_2|)$  space, when both  $G_1$  and  $G_2$  are acyclic, and in  $O(|E_1||E_2| + |V_1||V_2| \log |\Sigma|)$  time and  $O(|V_1||V_2|)$  space if  $G_1$  and/or  $G_2$  are cyclic, where,  $\Sigma$  denotes the alphabet.

## 1 Introduction

We consider *non-linear texts*, which are directed graphs where vertices are labeled by strings. Pattern matching on non-linear texts was first considered in [3], where an  $O(N + m|E| + R \log \log m)$  time algorithm for directed acyclic graphs. Here,  $m$  is the pattern length,  $N$  is the number of vertices,  $|E|$  is the number of arcs, and  $R$  is the output size. The algorithm was improved in [6], where an  $O(n + m|E|)$  time algorithm was shown. Here,  $n$  represents the total length of the string labels in the graph. Furthermore, in [1], an  $O(n)$  time algorithm was shown for trees. The problem was solved for general directed graphs in [2], where an  $O(n + |E|)$  time algorithm was developed. The approximate matching problem for non-linear texts was also considered in [2], where they showed that the problem can be solved in  $O(m(n \log m + e))$  time when edit operations are only allowed in the pattern. Here,  $e$  denotes the number of arcs in the graph when the graph is converted so that each node is labeled by a single character. They also showed that the problem is NP-complete when edit operations are allowed on the non-linear text. Furthermore, in [5], the algorithm was improved to  $O(m(n + e))$ .

Note that previous work on pattern matching on non-linear texts assumed a linear pattern. In this paper, we study a more generalized version of the problem, and consider the *longest common substring* and *longest common subsequence* problems between two non-linear texts. Firstly, we present an algorithm to compute the longest common substring of non-linear texts  $G_1$  and  $G_2$  in  $O(|E_1||E_2|)$  time and  $O(|V_1||V_2|)$  space, where  $V_i$  and  $E_i$  are the sets of vertices and arcs of input non-linear text  $G_i$ , respectively, for  $1 \leq i \leq 2$ . The algorithm works if one of  $G_1$  and  $G_2$  is acyclic. Secondly, we present algorithms to compute the longest common subsequence in  $O(|E_1||E_2|)$  time and  $O(|V_1||V_2|)$  space if both  $G_1$  and  $G_2$  are acyclic, and in  $O(|E_1||E_2| + |V_1||V_2| \log |\Sigma|)$  time and  $O(|V_1||V_2|)$  space if  $G_1$  and/or  $G_2$  are cyclic. Cyclic non-linear texts represent infinitely many and long strings, but our algorithms solve the above problems quite efficiently. Our algorithms are natural

extension of classical dynamic programming methods to compute longest common substring/subsequence of linear strings, and hence are easy to understand.

Very recently, an algorithm for determining the longest common subsequence between two *finite* languages was shown in [7]. The algorithm is a modification of the method based on weighted transducers [4], and requires  $O(|\Sigma|^2|E_1||E_2|)$  time and space. Compared to this work, our algorithms are faster and also apply to infinite languages.

problem	text	pattern	time complexity
Substring Matching	acyclic graph	linear	$O(n + m E )$ [6]
	tree	linear	$O(n)$ [1]
	graph	linear	$O(n +  E )$ [2]
Approximate Matching	graph w/edit operations	linear	NP-complete [2]
	graph	linear w/edit operations	$O(m(n + e))$ [5]
	text1	text2	
Longest Common Substring	acyclic graph	acyclic graph	$O( E_1  E_2 )$ (this work)
	graph	acyclic graph	$O( E_1  E_2 )$ (this work)
Longest Common Subsequence	acyclic graph	acyclic graph	$O( \Sigma ^2 E_1  E_2 )$ [7]
	acyclic graph	acyclic graph	$O( E_1  E_2 )$ (this work)
	graph	graph	$O( E_1  E_2  +  V_1  V_2  \log  \Sigma )$ (this work)

**Table 1.** Algorithms on non-linear text.

## 2 Preliminaries

### 2.1 Notation

Let  $\Sigma$  be a finite alphabet, and the elements of  $\Sigma^*$  are called *strings*. The *length* of a string  $w$  is denoted by  $|w|$ . The *empty string*, denoted by  $\varepsilon$ , is a string of length 0, and thus  $|\varepsilon| = 0$ . Let  $\Sigma^+ = \Sigma^* - \{\varepsilon\}$ . Strings  $x$ ,  $y$ , and  $z$  are called a *prefix*, *substring*, and *suffix* of string  $w = xyz$ , respectively. For any string  $w$ , let *suffix*( $w$ ) denote the set of suffixes of  $w$ . The  $i$ -th symbol of a string  $w$  is denoted by  $w[i]$  for  $1 \leq i \leq |w|$ , and the substring of  $w$  that begins at position  $i$  and ends at position  $j$  is denoted by  $w[i..j]$  for  $1 \leq i \leq j \leq |w|$ . For convenience, let  $w[i..j] = \varepsilon$  for  $i > j$ . The set of substrings of a string  $w$  is denoted by *substr*( $w$ ). A string  $u$  is a *subsequence* of another string  $w$  if there exists a sequence of integers  $i_1, \dots, i_k$  with  $k \geq 0$  such that  $1 \leq i_1 < \dots < i_k \leq |w|$  and  $u = w[i_1] \dots w[i_k]$ .

A *directed graph* is an ordered pair  $(V, E)$  of set  $V$  of *vertices* and set  $E \subseteq V \times V$  of *arcs*. A *path* in a directed graph  $G = (V, E)$  is a sequence  $v_0, \dots, v_k$  of vertices such that  $(v_{i-1}, v_i) \in E$  for every  $i = 1, \dots, k$ . For any vertex  $v \in V$ , let  $P(v)$  denote the set of paths that end at vertex  $v$ . The set of all paths in  $G$  is denoted by  $P(G)$ , namely,  $P(G) = \{P(v) \mid v \in V\}$ .

### 2.2 Longest common substring problem

The *longest common substring* problem is, given two strings  $x$  and  $y$ , to compute the length of longest common substrings of them. Although this problem can be solved in  $O(|x| + |y|)$  time using the generalized suffix tree of  $x$  and  $y$ , we here mention a

dynamic programming based solution. Letting  $C_{i,j}$  denote the maximum length of common suffixes of  $x[1..i]$  and  $y[1..j]$ , it suffices to compute the maximum of  $C_{i,j}$  over all the pairs  $(i, j)$ . Since we have

$$C_{i,j} = \begin{cases} 1 + C_{i-1,j-1} & \text{if } i, j > 0 \text{ and } x[i] = y[j]; \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

the problem can be solved in  $O(|x||y|)$  time.

### 2.3 Longest common subsequence problem

The *longest common subsequence* problem is, given two strings  $x$  and  $y$ , to compute the length of longest common subsequences of them. It is well-known that, this problem can be solved in  $O(|x||y|)$  time by using the following recurrence:

$$C_{i,j} = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0; \\ 1 + C_{i-1,j-1} & \text{if } i, j > 0 \text{ and } x[i] = y[j]; \\ \max(C_{i-1,j}, C_{i,j-1}) & \text{if } i, j > 0 \text{ and } x[i] \neq y[j], \end{cases} \quad (2)$$

where  $C_{i,j}$  is the length of longest common subsequence of  $x[1..i]$  and  $y[1..j]$ .

### 2.4 Non-linear texts

A *non-linear text* is a directed graph with vertices labeled by strings, namely, it is a directed graph  $G = (V, E, L)$  where  $V$  is the set of vertices,  $E$  is the set of arcs, and  $L : V \rightarrow \Sigma^+$  is a labeling function that maps nodes  $v \in V$  to non-empty strings  $L(v) \in \Sigma^+$ . For a path  $p = v_0, \dots, v_k \in P(G)$ , let  $L(p)$  denote the string spelled out by  $p$ , namely  $L(p) = L(v_0) \cdots L(v_k)$ . The size  $|G|$  of a non-linear text  $G = (V, E, L)$  is  $|V| + |E| + \sum_{v \in V} |L(v)|$ . Let  $\text{substr}(G)$ ,  $\text{suffix}(G)$ , and  $\text{subseq}(G)$  be the sets of substrings, suffixes and subsequences of a non-linear text  $G = (V, E, L)$ , namely,

$$\begin{aligned} \text{substr}(G) &= \{\text{substr}(L(p)) \mid p \in P(G)\}, \\ \text{suffix}(G) &= \{\text{suffix}(L(p)) \mid p \in P(G)\}, \\ \text{subseq}(G) &= \{\text{subseq}(L(p)) \mid p \in P(G)\}. \end{aligned}$$

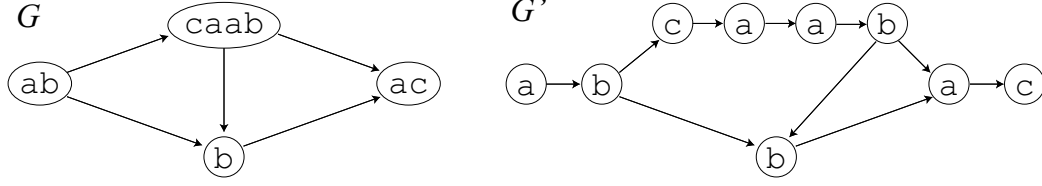
For a non-linear text  $G = (V, E, L)$ , consider a non-linear text  $G' = (V', E', L')$  such that  $L' : V' \rightarrow \Sigma$ ,

$$\begin{aligned} V' &= \{v_{i,j} \mid L'(v_{i,j}) = L(v_i)[j], v_i \in V, 1 \leq j \leq |L(v_i)|\}, \text{ and} \\ E' &= \{(v_{i,|L(v_i)|}, v_{k,1}) \mid (v_i, v_k) \in E\} \cup \{(v_{i,j}, v_{i,j+1}) \mid v_i \in V, 1 \leq j < |L(v_i)|\}. \end{aligned}$$

Namely,  $G'$  is a non-linear text in which each vertex is labeled with a single character and  $\text{substr}(G') = \text{substr}(G)$ . An example is shown in Figure 1. Since  $|V'| = \sum_{v \in V} |L(v)|$ ,  $|E'| = |E| + \sum_{v \in V} (|L(v)| - 1)$ , and  $\sum_{v' \in V'} |L(v')| = \sum_{v \in V} |L(v)|$ , we have  $|G'| = O(|G|)$ . We remark that given  $G$ , we can easily construct  $G'$  in  $O(|G|)$  time. Observe that  $\text{subseq}(G) = \text{subseq}(G')$  also holds.

In the sequel we only consider non-linear texts where each vertex is labeled with a single character. For any non-linear text  $G = (V, E, L)$  such that  $L(v) \in \Sigma$  for any  $v \in V$ , it trivially holds that  $\text{substr}(G) = \{L(p) \mid p \in P(G)\}$ .

We sometimes call strings in  $\Sigma^*$  linear strings or linear texts, in order to clearly distinguish them from non-linear texts.



**Figure 1.** A non-linear text  $G = (V, E, L)$  with  $L : V \rightarrow \Sigma^+$  and its corresponding non-linear text  $G' = (V', E', L')$  with  $L' : V' \rightarrow \Sigma$ .

### 3 Computing Longest Common Substring of Non-linear Texts

In this section, we tackle the problem of computing the length of longest common substrings of two input non-linear texts. The problem is formalized as follows.

*Problem 1 (Longest common substring problem for non-linear texts).*

**Input:** Non-linear texts  $G_1 = (V_1, E_1, L_1)$  and  $G_2 = (V_2, E_2, L_2)$ .

**Output:** The length of a longest string in  $\text{substr}(G_1) \cap \text{substr}(G_2)$ .

For example, see the non-linear texts  $G_1$  and  $G_2$  of Figure 2. The solution to the above problem is 5, since there is a longest common substring **abbaa** of  $G_1$  and  $G_2$ .

For simplicity, let us first consider the case where the two input non-linear texts are both acyclic.

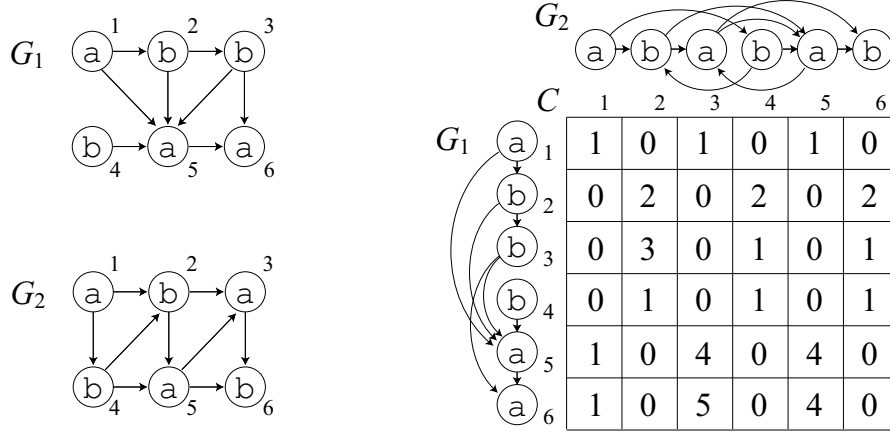
**Theorem 2.** *If  $G_1$  and  $G_2$  are acyclic, then Problem 1 can be solved in  $O(|E_1||E_2|)$  time and  $O(|V_1||V_2|)$  space.*

*Proof.* Let  $v_{1,i}$  and  $v_{2,j}$  denote the  $i$ -th and  $j$ -th vertex in topological ordering in  $G_1$  and in  $G_2$ , for  $1 \leq i \leq |V_1|$  and  $1 \leq j \leq |V_2|$ , respectively. Let  $C_{i,j}$  denote the length of a longest string in  $\text{suffix}(L_1(P(v_{1,i}))) \cap \text{suffix}(L_2(P(v_{2,j})))$ .  $C_{i,j}$  can be calculated as follows.

1. If  $L_1(v_{1,i}) = L_2(v_{2,j})$ , there are two cases to consider:
  - (a) If there are no arcs to  $v_{1,i}$  or to  $v_{2,j}$ , i.e.,  $P(v_{1,i}) = \{v_{1,i}\}$  or  $P(v_{2,j}) = \{v_{2,j}\}$ , then clearly  $C_{i,j} = 1$ .
  - (b) Otherwise, let  $v_{1,k}$  and  $v_{2,\ell}$  be any nodes s.t.  $(v_{1,k}, v_{1,i}) \in E_1$  and  $(v_{2,\ell}, v_{2,j}) \in E_2$ , respectively. Let  $z$  be a longest string in  $\text{suffix}(L_1(P(v_{1,i}))) \cap \text{suffix}(L_2(P(v_{2,j})))$ . Assume on the contrary that there exists a string  $y \in \text{suffix}(L_1(P(v_{1,k}))) \cap \text{suffix}(L_2(P(v_{2,\ell})))$  such that  $|y| > |z| - 1$ . This contradicts that  $z$  is a longest common suffix of  $L_1(P(v_{1,i}))$  and  $L_2(P(v_{2,j}))$ , since  $L_1(v_{1,i}) = L_2(v_{2,j})$ . Hence  $y \leq |z| - 1$ . If  $v_{1,k}$  and  $v_{2,\ell}$  are vertices satisfying  $C_{k,\ell} = |z| - 1$ , then  $C_{i,j} = C_{k,\ell} + 1$ . Note that such  $v_{1,k}$  and  $v_{2,\ell}$  always exist.
2. If  $L_1(v_{1,i}) \neq L_2(v_{2,j})$ , then trivially  $\text{suffix}(L_1(P(v_{1,i}))) \cap \text{suffix}(L_2(P(v_{2,j}))) = \{\varepsilon\}$ . Hence  $C_{i,j} = 0$ .

Consequently we obtain the following recurrence:

$$C_{i,j} = \begin{cases} 1 + \max(\{C_{k,\ell} \mid (v_{1,k}, v_{1,i}) \in E_1, (v_{2,\ell}, v_{2,j}) \in E_2\} \cup \{0\}) & \text{if } L_1(v_{1,i}) = L_2(v_{2,j}); \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$



**Figure 2.** Example of dynamic programming for computing the length of a longest common substring of non-linear texts  $G_1$  and  $G_2$ . Each vertex is annotated with its topological order. In this example,  $\max C_{i,j} = 5$  and the longest common substring is **abbaa**.

We use dynamic programming to compute  $C_{i,j}$  for all  $1 \leq i \leq |V_1|$  and  $1 \leq j \leq |V_2|$ . Consider to compute  $\max\{C_{k,\ell} \mid (v_{1,k}, v_{1,i}) \in E_1, (v_{2,\ell}, v_{2,j}) \in E_2\}$ . For each fixed  $(v_{1,k}, v_{1,i}) \in E_1$ , we refer the value of  $C_{k,\ell}$  for all  $1 \leq \ell < j$  such that  $(v_{2,\ell}, v_{2,j}) \in E_2$ , in  $O(|E_2|)$  time. Therefore, the total time complexity for computing  $\max\{C_{k,\ell} \mid (v_{1,k}, v_{1,i}) \in E_1, (v_{2,\ell}, v_{2,j}) \in E_2\}$  is  $O(|E_1||E_2|)$ . Since we can sort vertices of  $G_1$  and  $G_2$  in topological ordering in linear time, the total time complexity is  $O(|E_1||E_2|)$ . The space complexity is clearly  $O(|V_1||V_2|)$ .  $\square$

An example of computing  $C_{i,j}$  using dynamic programming is shown in Figure 2.

We remark that the recurrence of (3) is a natural generalization of that of (1) for computing the longest common substring of linear texts.

Furthermore, we can solve Problem 1 in case where *only one of the input non-linear texts is acyclic*:

**Theorem 3.** *If at least one of  $G_1$  and  $G_2$  is acyclic, then Problem 1 can be solved in  $O(|E_1||E_2|)$  time and  $O(|V_1||V_2|)$  space.*

*Proof.* Assume w.l.o.g. that  $G_1$  is acyclic. Recall the proof of Theorem 2. A key observation is that it indeed suffices to sort one of the input non-linear texts in topological ordering.

For any vertex  $v_{2,j} \in V_2$  and positive integer  $h$ , let  $P_h(v_{2,j})$  denote the set of paths of length not greater than  $h$ , which end at vertex  $v_{2,j}$ . Assume we have sorted vertices of  $G_1$ . Let  $C_{i,j}$  denote the length of a longest string in  $\text{suffix}(L_1(P(v_{1,i}))) \cap \text{suffix}(L_2(P_r(v_{2,j})))$ , where  $r$  is the length of a longest path in  $P(v_{1,i})$ . We compute  $C_{1,j}$  for each vertex  $v_{2,j} \in V_2$  by:  $C_{1,j} = 1$  if  $L_1(v_{1,1}) = L_2(v_{2,j})$  and  $C_{1,j} = 0$  otherwise. Then we compute  $C_{i,j}$  for all  $i > 1$  using the same recurrence as (3). Since the length of any element in  $\text{substr}(G_1) \cap \text{substr}(G_2)$  is not greater than that of the longest path in  $G_1$ ,  $\max\{C_{i,j} \mid 1 \leq i \leq |V_1|, 1 \leq j \leq |V_2|\}$  equals to the length of a longest string in  $\text{substr}(G_1) \cap \text{substr}(G_2)$ . Consequently,  $G_2$  does not have to be acyclic.  $\square$

A pseudo-code of our algorithm to solve the longest common substring problem for non-linear texts is shown in Algorithm 1.

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**Algorithm 1:** Computing the length of longest common substring of non-linear texts.

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**Input:** Acyclic non-linear text  $G_1 = (V_1, E_1, L_1)$  and non-linear text  $G_2 = (V_2, E_2, L_2)$ .  
**Output:** Length of a longest string in  $\text{substr}(G_1) \cap \text{substr}(G_2)$ .

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1 topological sort  $G_1$ ;
2  $n \leftarrow |V_1|$ ;  $m \leftarrow |V_2|$ ;
3 Let  $C$  be an  $n \times m$  integer array;
4 for  $i \leftarrow 1$  to  $n$  do
5   for  $j \leftarrow 1$  to  $m$  do
6     if  $f(v_{1,i}) = f(v_{2,j})$  then
7        $C_{i,j} \leftarrow 1$ ;
8       forall  $v_{1,k}$  s.t.  $(v_{1,k}, v_{1,i}) \in E_1$  do
9         forall  $v_{2,\ell}$  s.t.  $(v_{2,\ell}, v_{2,j}) \in E_2$  do
10          if  $C_{i,j} < 1 + C_{k,\ell}$  then
11             $C_{i,j} \leftarrow 1 + C_{k,\ell}$ ;
12     else
13        $C_{i,j} \leftarrow 0$ ;
14 return  $\max\{C_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ ;
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## 4 Computing Longest Common Subsequence Problem of Non-linear Texts

In this section, we tackle the problem of computing the length of longest common subsequence of two input non-linear texts. The problem is formalized as follows.

*Problem 4 (Longest common subsequence problem for non-linear texts).*

**Input:** Non-linear texts  $G_1 = (V_1, E_1, L_1)$  and  $G_2 = (V_2, E_2, L_2)$ .

**Output:** The length of a longest string in  $\text{subseq}(G_1) \cap \text{subseq}(G_2)$ .

For example, see the non-linear texts  $G_1$  and  $G_2$  of Figure 3. The solution to the above problem is 4, since there is a longest common subsequence **acdb** of  $G_1$  and  $G_2$ .

In the sequel we present our algorithm to solve the above problem in case where both  $G_1$  and  $G_2$  are acyclic.

**Theorem 5.** *If  $G_1$  and  $G_2$  are acyclic, then Problem 4 can be solved in  $O(|E_1||E_2|)$  time and  $O(|V_1||V_2|)$  space.*

*Proof.* Let  $v_{1,i}$  and  $v_{2,j}$  denote the  $i$ -th and  $j$ -th vertex in topological ordering in  $G_1$  and in  $G_2$ , for  $1 \leq i \leq |V_1|$  and  $1 \leq j \leq |V_2|$ , respectively. Let  $C_{i,j}$  denote the length of a longest string in  $\text{subseq}(L_1(P(v_{1,i}))) \cap \text{subseq}(L_2(P(v_{2,j})))$ .  $C_{i,j}$  can be calculated as follows.

1. If  $L_1(v_{1,i}) = L_2(v_{2,j})$ , there are two cases to consider:
  - (a) If there are no arcs to  $v_{1,i}$  or to  $v_{2,j}$ , i.e.,  $P(v_{1,i}) = \{v_{1,i}\}$  or  $P(v_{2,j}) = \{v_{2,j}\}$ , then clearly  $C_{i,j} = 1$ .
  - (b) Otherwise, let  $v_{1,k}$  and  $v_{2,\ell}$  be any nodes s.t.  $(v_{1,k}, v_{1,i}) \in E_1$  and  $(v_{2,\ell}, v_{2,j}) \in E_2$ , respectively. Let  $z$  be a longest string in  $\text{subseq}(L_1(P(v_{1,i}))) \cap \text{subseq}(L_2(P(v_{2,j})))$ . Assume on the contrary that there exists a string  $y \in \text{subseq}(L_1(P(v_{1,k}))) \cap \text{subseq}(L_2(P(v_{2,\ell})))$  such that  $|y| > |z| - 1$ . This contradicts that  $z$  is a longest common subsequence of  $L_1(P(v_{1,i}))$  and  $L_2(P(v_{2,j}))$ , since  $L_1(v_{1,i}) = L_2(v_{2,j})$ . Hence  $|y| \leq |z| - 1$ . If  $v_{1,k}$  and  $v_{2,\ell}$  are vertices satisfying  $C_{k,\ell} = |z| - 1$ , then  $C_{i,j} = C_{k,\ell} + 1$ . Note that such  $v_{1,k}$  and  $v_{2,\ell}$  always exist.

2. If  $L_1(v_{1,i}) \neq L_2(v_{2,j})$ , there are two cases to consider:

- (a) If there are no arcs to  $v_{1,i}$  and to  $v_{2,j}$ , i.e.,  $P(v_{1,i}) = \{v_{1,i}\}$  and  $P(v_{2,j}) = \{v_{2,j}\}$ , then clearly  $C_{i,j} = 0$ .
- (b) Otherwise, let  $v_{1,k}$  and  $v_{2,\ell}$  be any nodes s.t.  $(v_{1,k}, v_{1,i}) \in E_1$  and  $(v_{2,\ell}, v_{2,j}) \in E_2$ , respectively. Let  $z$  be a longest string in  $\text{subseq}(L_1(P(v_{1,i}))) \cap \text{subseq}(L_2(P(v_{2,j})))$ . Assume on the contrary that there exists a string  $y \in \text{subseq}(L_1(P(v_{1,k}))) \cap \text{subseq}(L_2(P(v_{2,j})))$  such that  $|y| > |z|$ . This contradicts that  $z$  is a longest common subsequence of  $L_1(P(v_{1,i}))$  and  $L_2(P(v_{2,j}))$ , since  $\text{subseq}(L_1(P(v_{1,k}))) \cap \text{subseq}(L_2(P(v_{2,j}))) \subseteq \text{subseq}(L_1(P(v_{1,i}))) \cap \text{subseq}(L_2(P(v_{2,j})))$ . Hence  $|y| \leq |z|$ . If  $v_{1,k}$  is a vertex satisfying  $C_{k,j} = |z|$ , then  $C_{i,j} = C_{k,j}$ . Similarly, if  $v_{2,\ell}$  is a vertex satisfying  $C_{i,\ell} = |z|$ , then  $C_{i,j} = C_{i,\ell}$ . Note that such  $v_{1,k}$  or  $v_{2,\ell}$  always exists.

Consequently we obtain the following recurrence:

$$C_{i,j} = \begin{cases} 1 + \max(\{C_{k,\ell} \mid (v_{1,k}, v_{1,i}) \in E_1, (v_{2,\ell}, v_{2,j}) \in E_2\} \cup \{0\}) & \text{if } L_1(v_{1,i}) = L_2(v_{2,j}); \\ \max\left(\{C_{k,j} \mid (v_{1,k}, v_{1,i}) \in E_1\} \cup \{C_{i,\ell} \mid (v_{2,\ell}, v_{2,j}) \in E_2\} \cup \{0\}\right) & \text{otherwise.} \end{cases} \quad (4)$$

We use dynamic programming to compute  $C_{i,j}$  for all  $1 \leq i \leq |V_1|$  and  $1 \leq j \leq |V_2|$ .

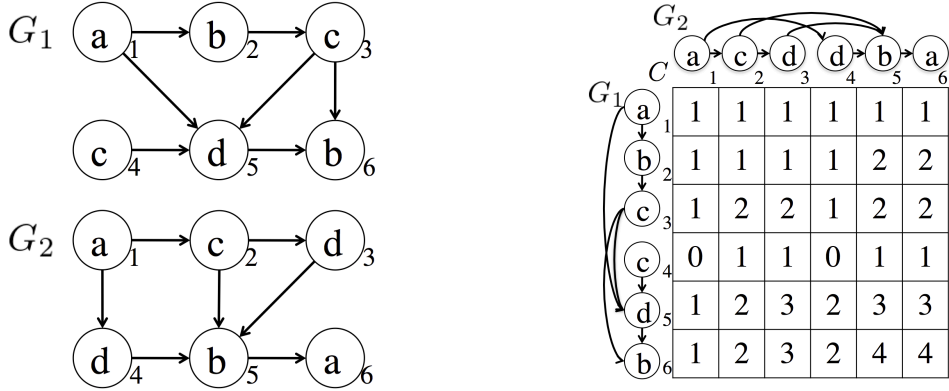
By similar arguments to the proof of Theorem 2, computing  $\max\{C_{k,\ell} \mid (v_{1,k}, v_{1,i}) \in E_1, (v_{2,\ell}, v_{2,j}) \in E_2\}$  takes  $O(|E_1||E_2|)$  time.

Consider to compute  $\max\{C_{k,j}, C_{i,\ell} \mid (v_{1,k}, v_{1,i}) \in E_1, (v_{2,k}, v_{2,j}) \in E_2\}$ . For each fixed  $(v_{1,k}, v_{1,i}) \in E_1$ , we refer the value of  $C_{k,j}$  for all  $1 \leq j \leq |V_2|$  in  $O(|V_2|)$  time. Similarly, for each fixed  $(v_{2,\ell}, v_{2,j}) \in E_2$ , we refer the value of  $C_{i,\ell}$  for all  $1 \leq i \leq |V_1|$  in  $O(|V_1|)$  time. Therefore, the total time cost for computing  $\max\{C_{k,j}, C_{i,\ell} \mid (v_{1,k}, v_{1,i}) \in E_1, (v_{2,\ell}, v_{2,j}) \in E_2\}$  is  $O(|V_2||E_1| + |V_1||E_2|)$ .

Since we can sort vertices of  $G_1$  and  $G_2$  in topological ordering in linear time, the total time complexity is  $O(|E_1||E_2|)$ . The space complexity is clearly  $O(|V_1||V_2|)$ .  $\square$

An example of computing  $C_{i,j}$  using dynamic programming is show in Figure 3. We remark that the recurrence of (4) is a natural generalization of that of (2) for computing the longest common subsequence of linear texts.

Algorithm 2 shows a pseudo-code of our algorithm to solve Problem 4 in case where both  $G_1$  and  $G_2$  are acyclic.



**Figure 3.** Example of dynamic programming for computing the length of a longest common subsequence of non-linear texts  $G_1$  and  $G_2$ . Each vertex is annotated with its topological order. In this example,  $\max C_{i,j} = 4$  and the longest common subsequence is **acdb**.

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**Algorithm 2:** Computing the length of longest common subsequence of acyclic non-linear texts

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**Input:** Two acyclic non-linear texts  $G_1 = (V_1, E_1, L_1), G_2 = (V_2, E_2, L_2)$   
**Output:** Length of a longest string in  $subseq(G_1) \cap subseq(G_2)$

```

1 topological sort  $G_1$ ;
2 topological sort  $G_2$ ;
3  $n \leftarrow |V_1|$ ;  $m \leftarrow |V_2|$ ;
4 Let  $C$  be an  $n \times m$  integer array;
5 for  $i \leftarrow 1$  to  $n$  do
6   for  $j \leftarrow 1$  to  $m$  do
7     if  $f(v_{1,i}) = f(v_{2,j})$  then
8        $C_{i,j} \leftarrow 1$ ;
9       forall  $v_{1,k}$  s.t.  $(v_{1,k}, v_{1,i}) \in E_1$  do
10        forall  $v_{2,\ell}$  s.t.  $(v_{2,\ell}, v_{2,j}) \in E_2$  do
11          if  $C_{i,j} < 1 + C_{k,\ell}$  then
12             $C_{i,j} \leftarrow 1 + C_{k,\ell}$ ;
13       else
14          $C_{i,j} \leftarrow 0$ ;
15         forall  $v_{1,k}$  s.t.  $(v_{1,k}, v_{1,i}) \in E_1$  do
16           if  $C_{i,j} < C_{k,j}$  then
17              $C_{i,j} \leftarrow C_{k,j}$ ;
18         forall  $v_{2,\ell}$  s.t.  $(v_{2,\ell}, v_{2,j}) \in E_2$  do
19           if  $C_{i,j} < C_{i,\ell}$  then
20              $C_{i,j} \leftarrow C_{i,\ell}$ ;
21 return  $\max\{C_{i,j} \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ ;

```

---



## 5 Computing Longest Common Subsequence of Cyclic Non-linear Texts

In this section, we present an algorithm to solve Problem 4 in case where the input non-linear texts are cyclic. We output  $\infty$  if  $\text{subseq}(G_1) \cap \text{subseq}(G_2)$  is infinite, and do the length of a longest string in  $\text{subseq}(G_1) \cap \text{subseq}(G_2)$  otherwise.

We transform a cyclic non-linear text  $G = (V, E, L)$  into an acyclic non-linear text  $G' = (V', E', L')$  based on the strongly connected components. For each vertex  $v \in V$ , let  $[v]$  denote the set of vertices that belong to the same strongly connected component. Formally,  $G'$  is defined as

$$\begin{aligned} V' &= \{[v] \mid v \in V\}, \\ E' &= \{([v], [u]) \mid [v] \neq [u], (v', u') \in E \text{ for some } v' \in [v], u' \in [u]\} \cup \{(v, v) \mid |[v]| \geq 2\}, \end{aligned}$$

and  $L'([v]) = \{L(v) \mid v \in [v]\} \subseteq \Sigma$ . We regard each  $[v]$  as a single vertex that is contracted from vertices in  $[v]$ . Observe that  $\text{subseq}(G') = \text{subseq}(G)$ .

An example of transformed acyclic non-linear texts is shown in Figure 4.

**Theorem 6.** *If  $G_1$  and/or  $G_2$  are cyclic, then Problem 4 can be solved in  $O(|E_1||E_2| + |V_1||V_2| \log |\Sigma|)$  time and  $O(|V_1||V_2|)$  space.*

*Proof.* We first transform cyclic non-linear texts  $G_1$  and  $G_2$  into corresponding acyclic non-linear texts  $G'_1$  and  $G'_2$ , as described previously. Let  $v'_{1,i}$  and  $v'_{2,j}$  denote the  $i$ -th and  $j$ -th vertex in topological ordering in  $G'_1$  and  $G'_2$ , for  $1 \leq i \leq |V'_1|$  and  $1 \leq j \leq |V'_2|$ , respectively. Let  $S_1$  and  $S_2$  denote the sets of vertices which has a loop, namely,  $S_1 = \{L'_1(v'_{1,i}) \mid (v'_{1,i}, v'_{1,i}) \in E'_1\}$  and  $S_2 = \{L'_2(v'_{2,j}) \mid (v'_{2,j}, v'_{2,j}) \in E'_2\}$ . If  $S_1 \cap S_2 \neq \emptyset$ , then let  $c$  be any character in  $S_1 \cap S_2$ . Clearly an infinite repetition  $c^*$  of  $c$  is a common subsequence of  $G_1$  and  $G_2$ , and hence we output  $\infty$ .

In the sequel, consider the case where  $S_1 \cap S_2 = \emptyset$ . In this case, it is clear that  $\text{subseq}(G_1) \cap \text{subseq}(G_2)$  is finite. Let  $C_{i,j}$  denote the length of a longest string in  $\text{subseq}(L'_1(P(v'_{1,i}))) \cap \text{subseq}(L'_2(P(v'_{2,j})))$ .  $C_{i,j}$  can be calculated as follows.

1. If  $L'(v'_{1,i}) \cap L'(v'_{2,j}) \neq \emptyset$ , there are two cases to consider:
  - (a) If there are no arcs to  $v'_{1,i}$  or to  $v'_{2,j}$ , i.e.,  $P(v'_{1,i}) = \{v'_{1,i}\}$  or  $P(v'_{2,j}) = \{v'_{2,j}\}$ , then clearly  $C_{i,j} = 1$ .
  - (b) Otherwise, let  $v'_{1,k}$  and  $v'_{2,\ell}$  be any nodes s.t.  $(v'_{1,k}, v'_{1,i}) \in E'_1$  and  $(v'_{2,\ell}, v'_{2,j}) \in E'_2$ , respectively. Let  $z$  be a longest string in  $\text{subseq}(L'_1(P(v'_{1,i}))) \cap \text{subseq}(L'_2(P(v'_{2,j})))$ . Assume on the contrary that there exists a string  $y \in \text{subseq}(L'_1(P(v'_{1,k}))) \cap \text{subseq}(L'_2(P(v'_{2,\ell})))$  such that  $|y| > |z| - 1$ . This contradicts that  $z$  is a longest common subsequence of  $L'_1(P(v'_{1,i}))$  and  $L'_2(P(v'_{2,j}))$ , since  $L'_1(v'_{1,i}) \cap L'_2(v'_{2,j}) \neq \emptyset$ . Hence  $|y| \leq |z| - 1$ . If  $v'_{1,k}$  and  $v'_{2,\ell}$  are vertices satisfying  $C_{k,\ell} = |z| - 1$ , then  $C_{i,j} = C_{k,\ell} + 1$ . Note that such  $v'_{1,k}$  and  $v'_{2,\ell}$  always exist.
2. If  $L'(v'_{1,i}) \cap L'(v'_{2,j}) = \emptyset$ , there are two cases to consider:
  - (a) If there are no arcs to  $v'_{1,i}$  and to  $v'_{2,j}$ , i.e.,  $P(v'_{1,i}) = \{v'_{1,i}\}$  and  $P(v'_{2,j}) = \{v'_{2,j}\}$ , then clearly  $C_{i,j} = 0$ .
  - (b) Otherwise, let  $v'_{1,k}$  and  $v'_{2,\ell}$  be any nodes s.t.  $(v'_{1,k}, v'_{1,i}) \in E'_1$  and  $(v'_{2,\ell}, v'_{2,j}) \in E'_2$ , respectively. Let  $z$  be a longest string in  $\text{subseq}(L'_1(P(v'_{1,i}))) \cap \text{subseq}(L'_2(P(v'_{2,j})))$ . Assume on the contrary that there exists a string  $y \in \text{subseq}(L'_1(P(v'_{1,k}))) \cap \text{subseq}(L'_2(P(v'_{2,\ell})))$  such that  $|y| > |z|$ . This contradicts that  $z$  is a longest common subsequence of  $L'_1(P(v'_{1,i}))$  and  $L'_2(P(v'_{2,j}))$ , since  $\text{subseq}(L'_1(P(v'_{1,k}))) \cap \text{subseq}(L'_2(P(v'_{2,j}))) \neq \emptyset$ .

$\text{subseq}(L'_2(P(v'_{2,j}))) \subseteq \text{subseq}(L'_1(P(v'_{1,i}))) \cap \text{subseq}(L'_2(P(v'_{2,j})))$ . Hence  $|y| \leq |z|$ . If  $v'_{1,k}$  is a vertex satisfying  $C_{k,j} = |z|$ , then  $C_{i,j} = C_{k,j}$ . Similarly, if  $v'_{2,\ell}$  is a vertex satisfying  $C_{i,\ell} = |z|$ , then  $C_{i,j} = C_{i,\ell}$ . Note that such  $v'_{1,k}$  ( $k \neq i$ ) or  $v'_{2,\ell}$  ( $\ell \neq j$ ) always exists.

Consequently we obtain the following recurrence:

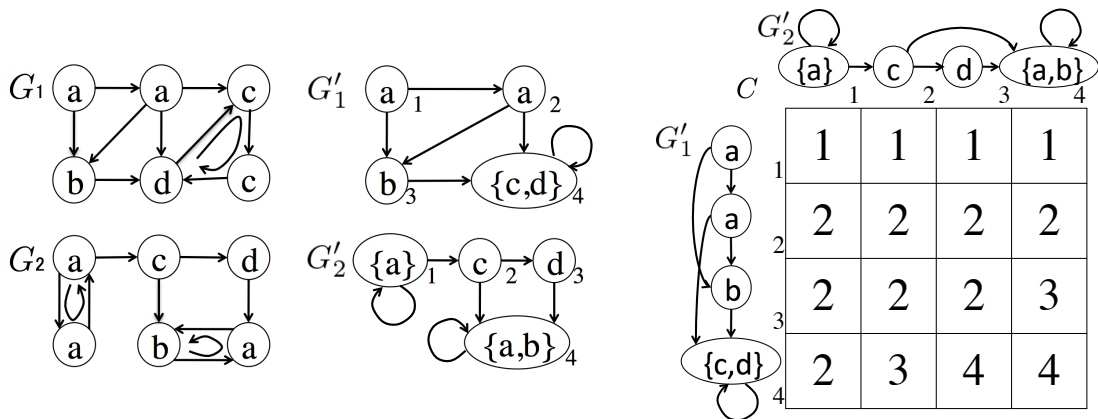
$$C_{i,j} = \begin{cases} 1 + \max(\{C_{k,\ell} \mid (v'_{1,k}, v'_{1,i}) \in E'_1, (v'_{2,\ell}, v'_{2,j}) \in E'_2\} \cup \{0\}) & \text{If } L'(v'_{1,i}) \cap L'(v'_{2,j}) \neq \emptyset; \\ \max\left(\{C_{k,j} \mid (v'_{1,k}, v'_{1,i}) \in E_1\} \cup \{C_{i,\ell} \mid (v'_{2,\ell}, v'_{2,j}) \in E_2\}\right) \cup \{0\} & \text{otherwise.} \end{cases} \quad (5)$$

It is well-known that we can transform  $G_1$  and  $G_2$  into  $G'_1$  and  $G'_2$  in linear time, based on strongly connected components.

For each self-loop such as  $(v'_{1,i}, v'_{1,i}) \in E_1$ , we refer the value of  $C_{i,j}$  for all  $1 \leq j \leq |V'_2|$  in  $O(|V'_2|)$  time. Similarly, for each self-loop such as  $(v'_{2,j}, v'_{2,j}) \in E_2$ , we refer the value of  $C_{i,j}$  for all  $1 \leq i \leq |V'_1|$  in  $O(|V'_1|)$  time. For the other arcs, we can compute  $C_{i,j}$  for all  $1 \leq i \leq |V'_1|$  and  $1 \leq j \leq |V'_2|$  using dynamic programming in  $O(|E'_1| \cdot |E'_2|)$  time, in a similar way as the previous section. Therefore the total time cost for computing  $C_{i,j}$  is  $O(|E'_1| \cdot |E'_2|)$ .

Let  $\Sigma_1$  and  $\Sigma_2$  be the sets of characters that appear in  $G_1$  and  $G_2$ , respectively. The time cost to compute  $S_1 \cap S_2$  is  $O(|\Sigma_1| \log |\Sigma_2| + |\Sigma_2| \log |\Sigma_1|)$  using a balanced tree. Assume  $S_1 \cap S_2 = \emptyset$ , and consider to compute  $L'(v'_{1,i}) \cap L'(v'_{2,j})$ . If  $|L'(v'_{1,i})| > 1$  and  $|L'(v'_{2,j})| > 1$ , then we know  $L'(v'_{1,i}) \cap L'(v'_{2,j}) = \emptyset$  since  $S_1 \cap S_2 = \emptyset$ . If  $|L'(v'_{1,i})| = 1$  and/or  $|L'(v'_{2,j})| = 1$ , then  $L'(v'_{1,i}) \cap L'(v'_{2,j})$  can be computed in  $O(\log |\Sigma|)$  time using a balanced tree, where  $|\Sigma| = \max\{|\Sigma_1|, |\Sigma_2|\}$ . Therefore the total time cost to compare  $L'(v'_{1,i})$  and  $L'(v'_{2,j})$  for all  $1 \leq i \leq |V'_1|$  and  $1 \leq j \leq |V'_2|$  is  $O(|V'_1||V'_2| \log |\Sigma|)$ . The total time complexity becomes  $O(|E_1| + |E_2| + |E'_1||E'_2| + |V'_1||V'_2| \log |\Sigma| + |\Sigma_1| \log |\Sigma_2| + |\Sigma_2| \log |\Sigma_1|) = O(|E_1||E_2| + |V_1||V_2| \log |\Sigma|)$ , since  $|\Sigma_1| \leq |V_1|$  and  $|\Sigma_2| \leq |V_2|$ . The total space complexity is  $O(|V'_1||V'_2| + |\Sigma_1| \log |\Sigma_2| + |\Sigma_2| \log |\Sigma_1|) = O(|V_1||V_2|)$ .  $\square$

An example of computing  $C_{i,j}$  using dynamic programming is shown in Figure 4. A pseudo-code of our algorithm is shown in Algorithm 3.



**Figure 4.** Example of dynamic programming for computing the length of a longest common subsequence of non-linear texts  $G_1$  and  $G_2$ .  $G'_1$  and  $G'_2$  are non-linear texts which are transformed from  $G_1$  and  $G_2$  by grouping vertices into strongly connected components. Each vertex is annotated with its topological order. In this example,  $\max C_{i,j} = 4$  and the longest common subsequence is **aacd**.

---

**Algorithm 3:** Computing the length of longest common subsequence of cyclic non-linear texts

---

**Input:** Two non-linear texts  $G_1 = (V_1, E_1, L_1), G_2 = (V_2, E_2, L_2)$   
**Output:** Length of a longest string in  $\text{subseq}(G_1) \cap \text{subseq}(G_2)$

```

1  $G'_1 \leftarrow$  Strongly Connected Components  $G_1$ ;
2  $G'_2 \leftarrow$  Strongly Connected Components  $G_2$ ;
3 Let  $S_1$  be a set of vertices which belong to cycles in  $G_1$ ;
4 Let  $S_2$  be a set of vertices which belong to cycles in  $G_2$ ;
5 if  $S_1 \cap S_2 \neq \emptyset$  then
6   return  $\infty$ ;
7 else
8   topological sort  $G'_1$ ;
9   topological sort  $G'_2$ ;
10  Let  $C$  be an  $|V'_1| \times |V'_2|$  integer array;
11  for  $i \leftarrow 1$  to  $|V'_1|$  do
12    for  $j \leftarrow 1$  to  $|V'_2|$  do
13      if  $(v'_{1,i}, v'_{1,i}) \in E'_1$  then
14        if  $(v'_{2,j}, v'_{2,j}) \in E'_2$  then
15           $C_{i,j} \leftarrow$  Vertex-mismatch  $(v'_{1,i}, v'_{2,j})$ ;
16        else if  $L(v'_{1,i}) \supseteq L(v'_{2,j})$  then
17           $C_{i,j} \leftarrow$  Vertex-match  $(v'_{1,i}, v'_{2,j})$ ;
18        else
19           $C_{i,j} \leftarrow$  Vertex-mismatch  $(v'_{1,i}, v'_{2,j})$ ;
20      else if  $(v'_{2,j}, v'_{2,j}) \in E'_2$  then
21        if  $L(v'_{1,i}) \subseteq L(v'_{2,j})$  then
22           $C_{i,j} \leftarrow$  Vertex-match  $(v'_{1,i}, v'_{2,j})$ ;
23        else
24           $C_{i,j} \leftarrow$  Vertex-mismatch  $(v'_{1,i}, v'_{2,j})$ ;
25      else if  $L(v'_{1,i}) = L(v'_{2,j})$  then
26         $C_{i,j} \leftarrow$  Vertex-match  $(v'_{1,i}, v'_{2,j})$ ;
27      else
28         $C_{i,j} \leftarrow$  Vertex-mismatch  $(v'_{1,i}, v'_{2,j})$ ;
29 return  $\max\{C_{i,j} \mid 1 \leq i \leq |V'_1|, 1 \leq j \leq |V'_2|\}$ ;
```

---



---

**Algorithm 4:** Vertex-match( $v_{1,i}, v_{2,j}$ )

---

```

1  $C_{i,j} \leftarrow 1$ 
2 forall  $v_{1,k}$  s.t.  $(v_{1,k}, v_{1,i}) \in E_1$  do
3   forall  $v_{2,\ell}$  s.t.  $(v_{2,\ell}, v_{2,j}) \in E_2$  do
4     if  $C_{i,j} < 1 + C_{k,\ell}$  then
5        $C_{i,j} \leftarrow 1 + C_{k,\ell}$ 
6 return  $C_{i,j}$ 
```

---

**Algorithm 5:** Vertex-mismatch( $v_{1,i}, v_{2,j}$ )

---

```

1  $C_{i,j} \leftarrow 0$ 
2 forall  $v_{1,k}$  s.t.  $(v_{1,k}, v_{1,i}) \in E_1$  do
3   if  $C_{i,j} < C_{k,j}$  then
4      $C_{i,j} \leftarrow C_{k,j}$ 
5 forall  $v_{2,\ell}$  s.t.  $(v_{2,\ell}, v_{2,j}) \in E_2$  do
6   if  $C_{i,j} < C_{i,\ell}$  then
7      $C_{i,j} \leftarrow C_{i,\ell}$ 
8 return  $C_{i,j}$ 

```

---

## 6 Conclusions

We considered the longest common substring and subsequence problems between two non-linear texts. We showed that when the texts are acyclic, the problem can be solved in  $O(|E_1||E_2|)$  time and  $O(|V_1||V_2|)$  space by a dynamic programming approach. Furthermore, we extend our algorithm and consider the case where the texts can contain cycles, and presented an  $O(|E_1||E_2| + |V_1||V_2| \log |\Sigma|)$  time and  $O(|V_1||V_2|)$  space algorithm for the longest common subsequence problem. The longest common substring between general graphs is an open problem.

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