CMPS 102: Homework #5

Due on Tuesday, May 14th, 2015

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Problem 1

A rabbit wants to go through a distance of n feet. It can either do a short hop of one foot or a long hop of three feet. Denote f(n) as the number of ways that the rabbit can go through a distance of n feet.
Give an O(n) algorithm to find f(n).
a.) For a given $n \geq 0$, $f(n)$ depends directly on $f(n-1)$ and $f(n-3)$. This is because the only ways to get to n under the rules of our game are to either jump from $n-1$ or $n-3$.

b.) The subproblems are related additively, in that the solution to a problem of size n is the sum of the solutions to problems of size $n-1$ and $n-3$. This is because, as stated above, these two subproblems are the only ones that can lead to a problem of size n. Combine that with the fact that $n-1$ and $n-3$ are always different numbers, than their solution values must be different (they are different locations so the paths taken to go them must be different from one another), so we can simply add the value of $f(n-1)$ and $f(n-3)$ to get $f(n)$.

c.) $\forall n \geq 0 : f(n) = f(n-1) + f(n-3) : f(1) = f(2) = 1, f(3) = 2$
Because we are starting at an integer and subtracting either 3 or 1 each iteration, we will always bottom out at one of these base cases. This recursion leads to an unbalanced tree, and to get solution to $f(n)$ you can simply count the leaves. Intuitively this corresponds to a unique path being traced back from n to 0, the starting point for the rabbit. At each subproblem, we have only two ways to go, trace ourselves back a single foot or a 3-foot step. Since these two values are unique, any paths that lead to them are also unique, and thus they should be added to get the value for the current subproblem.

d.) Since this problem is one dimensionsal (over n), we only need an array of length n to perform memoization. Everytime $f(n)$ is called, we see is the array index at position n is empty, if so we perform our calculation by continuing the recursion and enter it into the table once we find the value, if it is not empty we simply extract the value from the array in constant time, bottoming out the recursion early.

e.) Two arrows would be pointing out from n backwards along a single axis, one landing on $n-1$ and the other landing on $n-3$.

f.) The algorithm runs in time $O(n)$, with n being the number of feet that we wish to see how many ways a rabbit can traverse. This is obvious from the algorithm, it consists of a single loop that runs $n-4$ times, doing constant work each iteration, and finally returning M[n] at the end, which is the value we are looking for.

The algorithm is given below

```
# M = memoization array of length n, starts empty
# n > 0 is the f(n) value we are querying
findpaths(n,M) :
  M[1] = 1
  M[2] = 1
  M[3] = 2
  for i in [4 to n] # inclusive of ends
    if i < 3  # make sure we do not index negatives
      M[i] = M[i-1]
    else
      M[i] = M[i-1]+M[i-3]

  return M[n]
```

Problem 2

Let $G = (V, E)$ be an undirected graph with n nodes. Recall that a subset of the nodes is called an independent set if no two of them are joined by an edge. Finding large independent sets is difficult in general; but here we'll see that it can be done efficiently if the graph is "simple" enough.
Call a graph $G = (V, E)$ a path if its nodes can be written as $v_1, v_2, ..., v_n$, with an edge between vi and v_j if and only if the numbers i and j differ by exactly 1. With each node v_i, we associate a positive integer weight w_i.
Consider, for example, the five-node path drawn in Figure 6.28. The weights are the numbers drawn inside the nodes.
The goal in this question is to solve the following problem:

Find an independent set in a path G whose total weight is as large as possible.

(a) Give an example to show that the following algorithm does not always find an independent set of maximum total weight.

```
The "heaviest-first" greedy algorithm
  Start with S equal to the empty set
  While some node remains in G
    Pick a node vi of maximum weight
    Add vi to S
    Delete vi and its neighbors from G
  Endwhile
  Return S
```

Answer :

An example of a graph which would prove the above algorithm incorrect would be :

$$1 - -9 - -10 - -8 - -5$$

The correct nodes to choose are 9 and 8 for a score of 17, but the above algorithm chooses 10 first, deleted 8 and 9 (it's neighbors), then selects 5, then 1, for a score of 16.

(b) Give an example to show that the following algorithm also does not always find an independent set of maximum total weight.

```
Let S1 be the set of all vi where i is an odd number
Let S2 be the set of all vi where i is an even number
(Note that S1 and S2 are both independent sets) Determine which of S1 or S2 has greater total weight,
```

Answer :

An example of a graph which would prove the above algorithm incorrect would be :

$$20 - -5 - -1 - -5 - -1 - -5 - -1 - -5 - -1 - -5$$

The optimal solution would choose $20, 5, 5, 5, 5 = 40$. The algorithm would partition the graph into two sets, $20, 1, 1, 1, 1 = 24$ and $5, 5, 5, 5, 5 = 25$, thus it would choose the second set getting a score of 25. This is less than the optimal solution so this algorithm is not optimal in all cases.

Problem 2 continued on next page...

Figure 6.28 A paths with weights on the nodes. The maximum weight of an independent set is 14.

(c) Give an algorithm that takes an n-node path G with weights and returns an independent set of maximum total weight. The running time should be polynomial in n, independent of the values of the weights.

```
# G is an array of integers of length n, representing
# the ordered nodes of the path. G[i] is the weight of that node.
# returns an array containing the nodes in the largest
# independent set
find_set(G)
  S = empty array # will contain the independent set
  M = array[0 to n] # n = length of G
  M[0] = 0
  for i in [1 to n] # inclusive of ends
    if G[i]+M[i-2] > M[i-1]
      S.push_back(i) # push this (ith) node into our set
      M[i] = G[i]+M[i-2]
    else
      M[i] = M[i-1] # else we just use the last node
  return S
```

The subproblems that I'm solving are the ones involving the two previous nodes. At each node i, we can either take that node and not take its neighbor (by the rules of being independent set), or we can skip the current node and take its immediate neighbor. Thus we need to know which of these two subproblems results in a larger independent set in order to make the right choice. The subproblems are related in that if a node i is in the largest independent set on a graph G, then neither of its neighbors can be in that set, so we don't have to consider those nodes.
The recurrance is $f(n) = max(v_i + f(n-2), f(n-1)) : f(0) = 0$.
We have an array to fill in as apposed to a table, this is because we are only working in a single variable, the iteration variable. We fill in the table from low to high index. The table is initialized with the 0th entry being set to zero, if there are no nodes to look at then we get no value from them.
An arrow diagram would point from node i to the two nodes preceding it.
This algorithm is also linear on n, with it's running time being $O(n)$. This is just like the previous problem, even though our recurrance relation spaws many children, there are a large number of redundent calls being made. With memoization, we simply have to fill in the n array values, starting from the first index and working our way up. On the way, we keep track of which items go into the final set by seeing which is the two options (M[n-1] or G[i]+M[i-2]) are larger, if it is the latter than we add node i into the set. Because the algorithm iterates n times and performs constant work each time, the running time is $O(n)$.

Problem 3

Let $G = (V, E)$ be a directed graph with nodes $v_1, ..., v_n$. We say that G is an ordered graph if it has the following properties.

(i) Each edge goes from a node with a lower index to a node with a higher index. That is, every directed

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edge has the form (v_i, v_j) with $i < j$.

(ii) Each node except vn has at least one edge leaving it. That is, for every node v_i, $i = 1, 2, ..., n1$, there is at least one edge of the form (v_i, v_j).

The length of a path is the number of edges in it. The goal in this question is to solve the following problem (see Figure 6.29 for an example).

Given an ordered graph G, find the length of the longest path that begins at v_1 and ends at v_n.

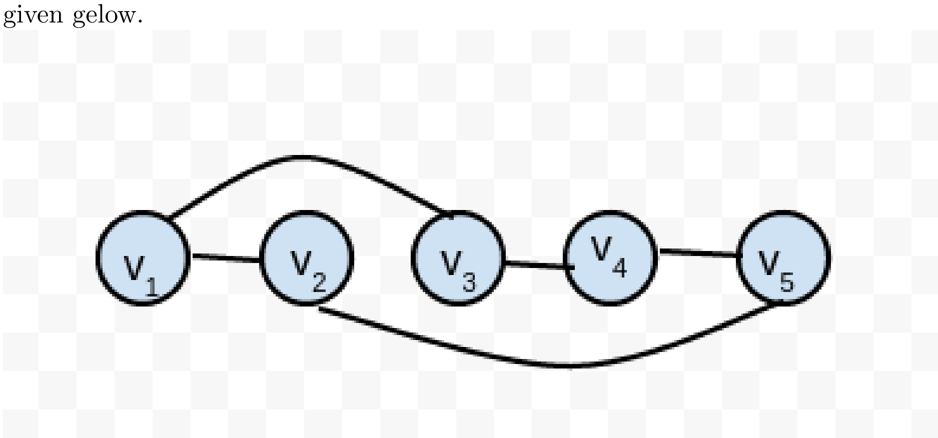
(a) Show that the following algorithm does not correctly solve this problem, by giving an example of an ordered graph on which it does not return the correct answer.

```
Set w = v_1
Set L = 0

While there is an edge out of the node w
  Choose the edge (w, v_j)
    for which j is as small as possible
  Set w = v_j
  Increase L by 1
end while
Return L as the length of the longest path
```

In your example, say what the correct answer is and also what the algorithm above finds.

Answer : This algorithm is too greedy, so all we need to do is trick it into picking a very close neighbor originally, and set it up so this neighbor only leads to a very short path between it and v_n. An example is given gelow.



In the above graph, all edges are going left to right, my software wouldn't draw curved arrows so I just used lines. The algorithm given would start at v_1, and select the edge to v_2, since that is the closest vertex which is reachable directly. It will then be forced to take the single edge from v_2 to v_5, which finishes the algorithm with a score of 2. The optimal solution however would be to go from v_1 to v_3, v_3 to v_4, then v_4 to v_5, for a score of 3.

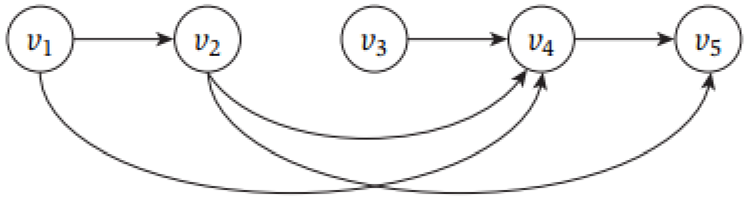


Figure 6.29 The correct answer for this ordered graph is 3: The longest path from v_1 to v_n uses the three edges (v_1, v_2),(v_2, v_4), and (v_4, v_5).

(b) Give an efficient algorithm that takes an ordered graph G and returns the length of the longest path that begins at v_1 and ends at v_n.(Again, the length of a path is the number of edges in the path.)

Answer : This one took me a while to think about, because I was confused on what to concentrate on, the vertices or the edges. I ended up coming up with following notation :

$$E_i = \{e_{i1}e_{i2} \dots e_{ip}\}$$

the set of all edges that end at vertex i (i is the target). Each edge $e_{ik} \in E_i$ has a source labeled $e_{ik}.s$.

The sub-problems that we are worried about solving is finding the maximum length path from vertex 1 to any vertex that comes before the current vertex and has an edge connecting it to the current vertex. Thus for a given vertex n with edge set E_n, we need to find the lengths of the longest path from the start vertex to all vertices that are the source of an edge in E_n.
The subproblems are related in that if we know the length of the longest path from the start to a vertex that is before the current one and connected to the current one, then the length of the longest path could be 1 + the length of the longest path to that vertex. To find the true longest path, we take the max of the longest paths to every previous vertex and add one.
The recurrance is $f(i) = 1 + max(f(k) : k = e.s \ \forall e \in E_i)$ and $f(1) = 0$. This says that we look at the longest path for all vertices before and adjacent to the current vertex, and add one to the max of these values. The base case is on the first vertex, there is no where to go so the longest path $f(1) = 0$.
We once again only have an array to fill in as opposed to a table, because this problem only needs to memoize what is the best path for each of the n vertices. The recursion states that the nth problem depends on all problems i such that there is an edge (i, n), so there would be arrows pointing backwards towards all adjacent vertices. The table is filled in algorithmically left to right, starting by filling in the first index with 0 and moving up, using the recurrance and the edge lists to fill in the individual sub-problems.

My algorithm goes through all vertices, starting at the beginning, and fills in the memoization array using the recurrance defined above. One issue is that since this is a directed graph, natural graph representations don't know which edges point in to which vertices, only what points out. This means that for each iteration of the loop i, we have to go through each vertex $k < i$ and check if there is an edge (k, i), one that points to the current vertex. If so we add the value of the max length for that vertex k to a list, which we end up taking the max value from. This guarentees that we find the longest path to the current vertex, which must be 1 plus the maximum of the longest paths to all precending, adjacent vertices.

Note that this problem didn't ask to find the longest path, just the length, which simplifies the algorithm slighly by lessening the book-keeping.

```
# G = adj. matrix representation of graph on n vertices [1 to n].
```

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```
# G[i][k] = 1 if an edge from vertex i to vertex k, 0 otherwise
# returns an integer that gives the longest path length
find_longest_path(G)
  M = array[1 to n] # will hold max length vals for all sub-problems
  V = empty max heap # will hold temp. values for max function
  M[1] = 0
  for i in [1 to n] # inclusive of ends
    for j in [1 to i-1] # only go to current vertex, not passed
      if(G[j][i] == 1) #if edges from j to i
        V.push_back(M[j]) # add the length value to the list, note that because
                          # j < i, M[j] is defined.
    M[i] = 1 + max(V) # set length 1 + max length of all adjacent, preceding vertices
    V.empty() # clear the list for the next iteration

  return M[n]
```

Problem 4

In a word processor, the goal of “pretty-printing” is to take text with a ragged right margin, like this,

```
Call me Ishmael.
Some years ago,
never mind how long precisely,
having little or no money in my purse,
and nothing particular to interest me on shore,
I thought I would sail about a little
and see the watery part of the world.
```

and turn it into text whose right margin is as “even” as possible, like this.

```


Call me Ishmael. Some years ago, never
mind how long precisely, having little
or no money in my purse, and nothing
particular to interest me on shore, I
thought I would sail about a little
and see the watery part of the world.
```

To make this precise enough for us to start thinking about how to write a pretty-printer for text, we need to figure out what it means for the right margins to be “even.” So suppose our text consists of a sequence of words, $W = w_1, w_2, ..., w_n$, where w_i consists of c_i characters. We have a maximum line length of L. We will assume we have a fixed-width font and ignore issues of punctuation or hyphenation.

A formatting of W consists of a partition of the words in W into lines. In the words assigned to a single line, there should be a space after each word except the last; and so if $w_j, w_{j+1}, ..., w_k$ are assigned to one line, then we should have

$$[\sum_{i=j}^{k-1}(c_i + 1)] + c_k \leq L.$$

We will call an assignment of words to a line valid if it satisfies this inequality. The difference between the left-hand side and the right-hand side will be called the slack of the linethat is, the number of spaces left at the right margin.

Give an efficient algorithm to find a partition of a set of words W into valid lines, so that the sum of the squares of the slacks of all lines (including the last line) is minimized.

Problem 5

Gerrymandering is the practice of carving up electoral districts in very careful ways so as to lead to outcomes that favor a particular political party. Recent court challenges to the practice have argued that through this calculated redistricting, large numbers of voters are being effectively (and intentionally) disenfranchised.
Computers, it turns out, have been implicated as the source of some of the "villainy" in the news coverage on this topic: Thanks to powerful software, gerrymandering has changed from an activity carried out by a bunch of people with maps,pencil,and paper into the industrial-strength process that it is today. Why is gerrymandering a computational problem? There are database issues involved in tracking voter demographics down to the level of individual streets and houses; and there are algorithmic issues involved in grouping voters into districts. Let's think a bit about what these latter issues look like.
Suppose we have a set of n precincts $P1, P2, ..., Pn$, each containing m registered voters. We're supposed to divide these precincts into two districts, each consisting of $n/2$ of the precincts. Now, for each precinct, we have information on how many voters are registered to each of two political parties. (Suppose, for simplicity, that every voter is registered to one of these two.) We'll say that the set of precincts is susceptible to gerrymandering if it is possible to perform the division into two districts in such a way that the same party holds a majority in both districts.
Give an algorithm to determine whether a given set of precincts is susceptible to gerrymandering; the running time of your algorithm should be polynomial in n and m.

Example. Suppose we have $n = 4$ precincts, and the following information on registered voters.

Precinct	1	2	3	4
Number registered for party A	55	43	60	47
Number registered for party B	45	57	40	53

This set of precincts is susceptible since, if we grouped precincts 1 and 4 into one district, and precincts 2 and 3 into the other, then party A would have a majority in both districts. (Presumably, the "we" who are doing the grouping here are members of party A.) This example is a quick illustration of the basic unfairness in gerrymandering: Although party A holds only a slim majority in the overall population (205 to 195), it ends up with a majority in not one but both districts.

Problem 6

Recall the scheduling problem from Section 4.2 in which we sought to minimize the maximum lateness. There are n jobs, each with a deadline d_i and a required processing time t_i, and all jobs are available to be scheduled starting at time s. For a job i to be done, it needs to be assigned a period from $s_i \geq s$ to $f_i = s_i + t_i$, and different jobs should be assigned nonoverlapping intervals. As usual, such an assignment of times will be called a schedule.

Problem 6 continued on next page...

In this problem, we consider the same setup, but want to optimize a different objective. In particular, we consider the case in which each job must either be done by its deadline or not at all. We'll say that a subset J of the jobs is schedulable if there is a schedule for the jobs in J so that each of them finishes by its deadline. Your problem is to select a schedulable subset of maximum possible size and give a schedule for this subset that allows each job to finish by its deadline.

(a) Prove that there is an optimal solution J (i.e., a schedulable set of maximum size) in which the jobs in J are scheduled in increasing order of their deadlines.

(b) Assume that all deadlines d_i and required times t_i are integers. Give an algorithm to find an optimal solution. Your algorithm should run in time polynomial in the number of jobs n, and the maximum deadline $D = max_i d_i$.