

# **CMPS 102: Homework #5**

Due on Tuesday, May 12st, 2015

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## Problem 1

A rabbit wants to go through a distance of  $n$  feet. It can either do a short hop of one foot or a long hop of three feet. Denote  $f(n)$  as the number of ways that the rabbit can go through a distance of  $n$  feet.

Give an  $O(n)$  algorithm to find  $f(n)$ .

1. a.) For a given  $n \geq 0$ ,  $f(n)$  depends directly on  $f(n-1)$  and  $f(n-3)$ . This is because the only ways to get to  $n$  under the rules of our game are to either jump from  $n-1$  or  $n-3$ .
2. b.) The subproblems are related additively, in that the solution to a problem of size  $n$  is the sum of the solutions to problems of size  $n-1$  and  $n-3$ . This is because, as stated above, these two subproblems are the only ones that can lead to a problem of size  $n$ . Combine that with the fact that  $n-1$  and  $n-3$  are always different numbers, than their solution values must be different (they are different locations so the paths taken to go them must be different from one another), so we can simply add the value of  $f(n-1)$  and  $f(n-3)$  to get  $f(n)$ .
3. c.)  $\forall n \geq 0 : f(n) = f(n-1) + f(n-3) : f(1) = f(2) = 1, f(3) = 2$   
Because we are starting at an integer and subtracting either 3 or 1 each iteration, we will always bottom out at one of these base cases. This recursion leads to an unbalanced tree, and to get solution to  $f(n)$  you can simply count the leaves. Intuitively this corresponds to a unique path being traced back from  $n$  to 0, the starting point for the rabbit. At each subproblem, we have only two ways to go, trace ourselves back a single foot or a 3-foot step. Since these two values are unique, any paths that lead to them are also unique, and thus they should be added to get the value for the current subproblem.
4. d.) Since this problem is one dimensional (over  $n$ ), we only need an array of length  $n$  to perform memoization. Everytime  $f(n)$  is called, we see if the array index at position  $n$  is empty, if so we perform our calculation by continuing the recursion and enter it into the table once we find the value, if it is not empty we simply extract the value from the array in constant time, bottoming out the recursion early.
5. e.) Two arrows would be pointing out from  $n$  backwards along a single axis, one landing on  $n-1$  and the other landing on  $n-3$ .
6. f.) The algorithm runs in time  $O(n)$ , with  $n$  being the number of feet that we wish to see how many ways a rabbit can traverse. This is obvious from the algorithm, it consists of a single loop that runs  $n-4$  times, doing constant work each iteration, and finally returning  $M[n]$  at the end, which is the value we are looking for.

The algorithm is given below

```

// M = memoization array of length n, starts empty
// n > 0 is the f(n) value we are querying
findpaths(n,M) :
    M[1] = 1
5    M[2] = 1
    M[3] = 2
    for i in [4 ... n] // inclusive of ends
        if i < 3 // make sure we do not index negatives
            M[i] = M[i-1]
10    else
        M[i] = M[i-1]+M[i-3]

    return M[n]

```

## Problem 2

Let  $G = (V, E)$  be an undirected graph with  $n$  nodes. Recall that a subset of the nodes is called an independent set if no two of them are joined by an edge. Finding large independent sets is difficult in general; but here we'll see that it can be done efficiently if the graph is "simple" enough.

Call a graph  $G = (V, E)$  a path if its nodes can be written as  $v_1, v_2, \dots, v_n$ , with an edge between  $v_i$  and  $v_j$  if and only if the numbers  $i$  and  $j$  differ by exactly 1. With each node  $v_i$ , we associate a positive integer weight  $w_i$ .

Consider, for example, the five-node path drawn in Figure 6.28. The weights are the numbers drawn inside the nodes.

The goal in this question is to solve the following problem:

*Find an independent set in a path  $G$  whose total weight is as large as possible.*

(a) Give an example to show that the following algorithm does not always find an independent set of maximum total weight.

```

The "heaviest-first" greedy algorithm
  Start with S equal to the empty set
  While some node remains in G
    Pick a node  $v_i$  of maximum weight
    Add  $v_i$  to S
    Delete  $v_i$  and its neighbors from G
  Endwhile
  Return S

```

**Answer :**

An example of a graph which would prove the above algorithm incorrect would be :

$$1 - 9 - 10 - 8 - 5$$

The correct nodes to choose are 9 and 8 for a score of 17, but the above algorithm chooses 10 first, deleted 8 and 9 (it's neighbors), then selects 5, then 1, for a score of 16.

(b) Give an example to show that the following algorithm also does not always find an independent set of maximum total weight.

```

Let S1 be the set of all  $v_i$  where  $i$  is an odd number
Let S2 be the set of all  $v_i$  where  $i$  is an even number
(Note that S1 and S2 are both independent sets) Determine which of S1 or S2 has greater total weight,

```

**Answer :**

An example of a graph which would prove the above algorithm incorrect would be :

$$20 - 5 - 1 - 5 - 1 - 5 - 1 - 5 - 1 - 5$$

The optimal solution would choose  $20, 5, 5, 5, 5 = 40$ . The algorithm would partition the graph into two sets,  $20, 1, 1, 1, 1 = 24$  and  $5, 5, 5, 5, 5 = 25$ , thus it would choose the second set getting a score of 25. This is less than the optimal solution so this algorithm is not optimal in all cases.



**Figure 6.28** A paths with weights on the nodes. The maximum weight of an independent set is 14.

(c) Give an algorithm that takes an  $n$ -node path  $G$  with weights and returns an independent set of maximum total weight. The running time should be polynomial in  $n$ , independent of the values of the weights.

```

// G is an array of integers of length  $n$ , representing
// the ordered nodes of the path.  $G[i]$  is the weight of that node.
// returns an array containing the nodes in the largest
// independent set
5 find_set(G)
    S = empty array // will contain the independent set
    M = array[1 ... n] //  $n$  = length of G
    M[0] = 0
    for  $i$  in [1 ... n] // inclusive of ends
10     if  $G[i] + M[i-2] > M[i-1]$ 
        S.push_back(i) // push this ( $i$ th) node into our set
        M[i] =  $G[i] + M[i-2]$ 
    else
        M[i] = M[i-1] // else we just use the last node
15 return S
  
```

The subproblems that I'm solving are the ones involving the two previous nodes. At each node  $i$ , we can either take that node and not take its neighbor (by the rules of being independent set), or we can skip the current node and take its immediate neighbor. Thus we need to know which of these two subproblems results in a larger independent set in order to make the right choice. The subproblems are related in that if a node  $i$  is in the largest independent set on a graph  $G$ , then neither of its neighbors can be in that set, so we don't have to consider those nodes.

The recurrence is  $f(n) = \max(v_i + f(n-2), f(n-1)) : f(0) = 0$ .

We have an array to fill in as apposed to a table, this is because we are only working in a single variable, the iteration variable. We fill in the table from low to high index. The table is initialized with the 0th entry being set to zero, if there are no nodes to look at then we get no value from them.

An arrow diagram would point from node  $i$  to the two nodes preceding it.

This algorithm is also linear on  $n$ , with it's running time being  $O(n)$ . This is just like the previous problem, even though our recurrence relation spawns many children, there are a large number of redundant calls being made. With memoization, we simply have to fill in the  $n$  array values, starting from the first index and working our way up. On the way, we keep track of which items go into the final set by seeing which is the two options ( $M[n-1]$  or  $G[i] + M[i-2]$ ) are larger, if it is the latter than we add node  $i$  into the set. Because the algorithm iterates  $n$  times and performs constant work each time, the running time is  $O(n)$ .

### Problem 3

Let  $G = (V, E)$  be a directed graph with nodes  $v_1, \dots, v_n$ . We say that  $G$  is an ordered graph if it has the following properties.

(i) Each edge goes from a node with a lower index to a node with a higher index. That is, every directed

edge has the form  $(v_i, v_j)$  with  $i < j$ .

(ii) Each node except  $v_n$  has at least one edge leaving it. That is, for every node  $v_i$ ,  $i = 1, 2, \dots, n-1$ , there is at least one edge of the form  $(v_i, v_j)$ .

The length of a path is the number of edges in it. The goal in this question is to solve the following problem (see Figure 6.29 for an example).

*Given an ordered graph  $G$ , find the length of the longest path that begins at  $v_1$  and ends at  $v_n$ .*

(a) Show that the following algorithm does not correctly solve this problem, by giving an example of an ordered graph on which it does not return the correct answer.

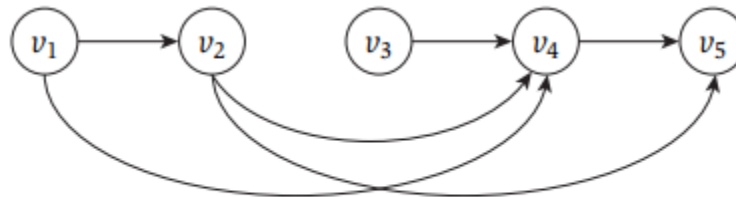
```

Set w = v_1
Set L = 0

While there is an edge out of the node w
  Choose the edge (w, v_j)
  for which j is as small as possible
  Set w = v_j
  Increase L by 1
end while
10 Return L as the length of the longest path

```

In your example, say what the correct answer is and also what the algorithm above finds.



**Figure 6.29** The correct answer for this ordered graph is 3: The longest path from  $v_1$  to  $v_n$  uses the three edges  $(v_1, v_2)$ ,  $(v_2, v_4)$ , and  $(v_4, v_5)$ .

(b) Give an efficient algorithm that takes an ordered graph  $G$  and returns the length of the longest path that begins at  $v_1$  and ends at  $v_n$ . (Again, the length of a path is the number of edges in the path.)

## Problem 4

In a word processor, the goal of “pretty-printing” is to take text with a ragged right margin, like this,

```

Call me Ishmael.
Some years ago,
never mind how long precisely,
having little or no money in my purse,
5 and nothing particular to interest me on shore,
I thought I would sail about a little
and see the watery part of the world.

```

and turn it into text whose right margin is as “even” as possible, like this.

```

Call me Ishmael. Some years ago, never
mind how long precisely, having little
or no money in my purse, and nothing
5 particular to interest me on shore, I
thought I would sail about a little
and see the watery part of the world.

```

To make this precise enough for us to start thinking about how to write a pretty-printer for text, we need to figure out what it means for the right margins to be “even.” So suppose our text consists of a sequence of words,  $W = w_1, w_2, \dots, w_n$ , where  $w_i$  consists of  $c_i$  characters. We have a maximum line length of  $L$ . We will assume we have a fixed-width font and ignore issues of punctuation or hyphenation.

A formatting of  $W$  consists of a partition of the words in  $W$  into lines. In the words assigned to a single line, there should be a space after each word except the last; and so if  $w_j, w_{j+1}, \dots, w_k$  are assigned to one line, then we should have

$$\left[ \sum_{i=j}^{k-1} (c_i + 1) \right] + c_k \leq L.$$

We will call an assignment of words to a line valid if it satisfies this inequality. The difference between the left-hand side and the right-hand side will be called the slack of the line; that is, the number of spaces left at the right margin.

Give an efficient algorithm to find a partition of a set of words  $W$  into valid lines, so that the sum of the squares of the slacks of all lines (including the last line) is minimized.

## Problem 5

*Gerrymandering* is the practice of carving up electoral districts in very careful ways so as to lead to outcomes that favor a particular political party. Recent court challenges to the practice have argued that through this calculated redistricting, large numbers of voters are being effectively (and intentionally) disenfranchised.

Computers, it turns out, have been implicated as the source of some of the “villainy” in the news coverage on this topic: Thanks to powerful software, gerrymandering has changed from an activity carried out by a bunch of people with maps, pencil, and paper into the industrial-strength process that it is today. Why is gerrymandering a computational problem? There are database issues involved in tracking voter demographics down to the level of individual streets and houses; and there are algorithmic issues involved in grouping voters into districts. Let’s think a bit about what these latter issues look like.

Suppose we have a set of  $n$  precincts  $P_1, P_2, \dots, P_n$ , each containing  $m$  registered voters. We’re supposed to divide these precincts into two districts, each consisting of  $n/2$  of the precincts. Now, for each precinct, we have information on how many voters are registered to each of two political parties. (Suppose, for simplicity, that every voter is registered to one of these two.) We’ll say that the set of precincts is susceptible to gerrymandering if it is possible to perform the division into two districts in such a way that the same party holds a majority in both districts.

Give an algorithm to determine whether a given set of precincts is susceptible to gerrymandering; the running time of your algorithm should be polynomial in  $n$  and  $m$ .

**Example.** Suppose we have  $n = 4$  precincts, and the following information on registered voters.

Precinct	1	2	3	4
Number registered for party A	55	43	60	47
Number registered for party B	45	57	40	53

This set of precincts is susceptible since, if we grouped precincts 1 and 4 into one district, and precincts 2 and 3 into the other, then party A would have a majority in both districts. (Presumably, the “we” who are doing the grouping here are members of party A.) This example is a quick illustration of the basic unfairness in gerrymandering: Although party A holds only a slim majority in the overall population (205 to 195), it ends up with a majority in not one but both districts.

## Problem 6

Recall the scheduling problem from Section 4.2 in which we sought to minimize the maximum lateness. There are  $n$  jobs, each with a deadline  $d_i$  and a required processing time  $t_i$ , and all jobs are available to be scheduled starting at time  $s$ . For a job  $i$  to be done, it needs to be assigned a period from  $s_i \geq s$  to  $f_i = s_i + t_i$ , and different jobs should be assigned nonoverlapping intervals. As usual, such an assignment of times will be called a schedule.

In this problem, we consider the same setup, but want to optimize a different objective. In particular, we consider the case in which each job must either be done by its deadline or not at all. We'll say that a subset  $J$  of the jobs is schedulable if there is a schedule for the jobs in  $J$  so that each of them finishes by its deadline. Your problem is to select a schedulable subset of maximum possible size and give a schedule for this subset that allows each job to finish by its deadline.

(a) Prove that there is an optimal solution  $J$  (i.e., a schedulable set of maximum size) in which the jobs in  $J$  are scheduled in increasing order of their deadlines.

(b) Assume that all deadlines  $d_i$  and required times  $t_i$  are integers. Give an algorithm to find an optimal solution. Your algorithm should run in time polynomial in the number of jobs  $n$ , and the maximum deadline  $D = \max_i d_i$ .