UNIVERSITY OF CAMBRIDGE
Microsoft
Je Hyeong Hong
Andrew W. Fitzgibbon

Secrets of Matrix Factorization:

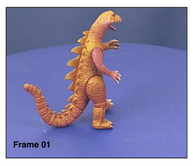
1. Matrix factorization with missing data

Given a noisy measurement matrix $\mathrm{M} \in \mathbb{R}^{m\times n}$, where some of the values are unobserved or weighted by another matrix $\mathrm{W} \in \mathbb{R}^{m\times n}$, we are interested in finding a set of low-rank matrices $\mathrm{U} \in \mathbb{R}^{m\times r}$ and $\mathrm{V} \in \mathbb{R}^{n\times r}$ that minimizes the error function

$$f(\mathrm{U},\mathrm{V}) = \|\boldsymbol{\varepsilon}(\mathrm{U},\mathrm{V})\|_2^2 = \|\mathrm{W} \odot (\mathrm{U}\mathrm{V}^\top - \mathrm{M})\|_F^2 + \mu(\|\mathrm{U}\|_F^2 + \|\mathrm{V}\|_F^2) \qquad (1.1)$$

where the operator $\odot$ is Hadamard or element-wise product, and the norms are Frobenius.

- This type of low-rank models appears in rigid and non-rigid structure-from-motion (SfM), photometric stereo and recommender systems.
- For computer vision problems, the regularization parameter μ is often set to 0.



(a) Rigid SfM [3]


(b) Non-rigid SfM [3]


(c) Photometric Stereo [4]

2. Known strategies in the literature

A. Block coordinate-descent (ALS)

- Fix U and optimize over V, and vice versa.
- Closed form updates for both U and V.

inputs: M, W, r, U, μ
$\widetilde{\mathrm{W}} \leftarrow \mathrm{diag}\big(\mathrm{vec}(\mathrm{W})\big)$ with zero-rows removed.
$\widetilde{\mathbf{m}} \leftarrow \widetilde{\mathrm{W}}\,\mathrm{vec}(\mathrm{M})$
repeat
  $\mathrm{U} \leftarrow \mathrm{unvec}\big(\widetilde{\mathrm{V}}^{-\mu}\widetilde{\mathbf{m}}\big)$
  $\mathrm{V} \leftarrow \mathrm{unvec}\big(\widetilde{\mathrm{U}}^{-\mu}\widetilde{\mathbf{m}}\big)$
until convergence
outputs: U, V

- $\mathrm{X}^{-\mu} \coloneqq (\mathrm{X}^\top\mathrm{X} + \mu\mathrm{I})^{-1}\mathrm{X}^\top$ (When $\mu = 0$, $\mathrm{X}^{-\mu} = \mathrm{X}^\dagger$.)
- $\widetilde{\mathrm{U}} \coloneqq \widetilde{\mathrm{W}}(\mathrm{I} \otimes \mathrm{U})$
- $\widetilde{\mathrm{V}} \coloneqq \widetilde{\mathrm{W}}(\mathrm{V} \otimes \mathrm{I})$
- $\mathrm{V}^*(\mathrm{U}) = \mathrm{unvec}\big(\widetilde{\mathrm{U}}^{-\mu}\widetilde{\mathbf{m}}\big)$

B. Joint optimization [3]

- Optimize over U and V simultaneously using a Newton-like solver.
- Bilinear structure → sparse Hessian (or JTJ) matrix.

inputs: M, W, r, U, V, μ, λ_0
$\lambda \leftarrow \lambda_0$
$\widetilde{\mathrm{W}} \leftarrow \mathrm{diag}\big(\mathrm{vec}(\mathrm{W})\big)$ with zero-rows removed.
$\widetilde{\mathbf{m}} \leftarrow \widetilde{\mathrm{W}}\,\mathrm{vec}(\mathrm{M})$
repeat
  $\mathbf{z} \leftarrow [\mathrm{vec}(\mathrm{U}) \quad ; \quad \mathrm{vec}(\mathrm{V}^\top)]$
  Compute the gradient $\mathbf{g} \coloneqq df(\mathrm{U},\mathrm{V})/d\mathbf{z}$.
  Compute the Hessian $\mathrm{H} \coloneqq d^2 f(\mathrm{U},\mathrm{V})/d\mathbf{z}^2$ or its approximation.
  repeat
    $\Delta\mathbf{z} \leftarrow (\mathrm{H} + \lambda\mathrm{I})^{-1}\mathbf{g}$
    Retrieve $\Delta\mathrm{U}$ and $\Delta\mathrm{V}$ from $\Delta\mathbf{z}$.
    $\lambda \leftarrow \lambda * 10$
  until $f(\mathrm{U}+\Delta\mathrm{U},\mathrm{V}+\Delta\mathrm{V}) < f(\mathrm{U},\mathrm{V})$
  $\mathrm{U} \leftarrow \mathrm{U} + \Delta\mathrm{U}$
  $\mathrm{V} \leftarrow \mathrm{V} + \Delta\mathrm{V}$
  $\lambda \leftarrow \lambda/100$
until convergence
outputs: U, V

C. Variable projection (VarPro) [6]

- Replace V by its closed form minimizer $\mathrm{V}^*(\mathrm{U}) \coloneqq \mathrm{argmin}_\mathrm{V} f(\mathrm{U},\mathrm{V}) = \mathrm{unvec}\big(\widetilde{\mathrm{U}}^{-\mu}\widetilde{\mathbf{m}}\big)$.
- Optimize the reduced problem over U using a Newton-like solver.
- Nonlinear structure → denser Hessian (or JTJ) matrix.

inputs: M, W, r, U, μ, λ_0
$\lambda \leftarrow \lambda_0$
$\widetilde{\mathrm{W}} \leftarrow \mathrm{diag}\big(\mathrm{vec}(\mathrm{W})\big)$ with zero-rows removed.
$\widetilde{\mathbf{m}} \leftarrow \widetilde{\mathrm{W}}\,\mathrm{vec}(\mathrm{M})$
repeat
  Compute the gradient $\mathbf{g} \coloneqq df(\mathrm{U},\mathrm{V}^*(\mathrm{U}))/d\mathrm{vec}(\mathrm{U})$.
  Compute the Hessian $\mathrm{H} \coloneqq d^2 f(\mathrm{U},\mathrm{V}^*(\mathrm{U}))/d\mathrm{vec}(\mathrm{U})^2$ or its approximation.
  repeat
    $\Delta\mathrm{U} \leftarrow \mathrm{unvec}\big((\mathrm{H} + \lambda\mathrm{I})^{-1}\mathbf{g}\big)$
    $\lambda \leftarrow \lambda * 10$
  until $f\big(\mathrm{U}+\Delta\mathrm{U},\mathrm{V}^*(\mathrm{U}+\Delta\mathrm{U})\big) < f(\mathrm{U},\mathrm{V}^*(\mathrm{U}))$
  $\mathrm{U} \leftarrow \mathrm{U} + \Delta\mathrm{U}$
  $\mathrm{V} \leftarrow \mathrm{V}^*(\mathrm{U})$
  $\lambda \leftarrow \lambda/100$
until convergence
outputs: U, V

- $\mathrm{X}^{-\mu} \coloneqq (\mathrm{X}^\top\mathrm{X} + \mu\mathrm{I})^{-1}\mathrm{X}^\top$ (When $\mu = 0$, $\mathrm{X}^{-\mu} = \mathrm{X}^\dagger$.)
- $\widetilde{\mathrm{U}} \coloneqq \widetilde{\mathrm{W}}(\mathrm{I} \otimes \mathrm{U})$
- $\widetilde{\mathrm{V}} \coloneqq \widetilde{\mathrm{W}}(\mathrm{V} \otimes \mathrm{I})$
- $\mathrm{V}^*(\mathrm{U}) = \mathrm{unvec}\big(\widetilde{\mathrm{U}}^{-\mu}\widetilde{\mathbf{m}}\big)$

3. Random restarts

- Any algorithm started from a random starting point (i.e. random U and/or V) is improved by starting from multiple points.
- To find a strong optimum, just run from many random starting points. Report optimum that is seen twice and is lower than all the others. Hence,

A new meta algorithm, RUSSO-*X*

RUSSO-*X* is literally "run algorithm-*X* until seen second optimum", where this optimum is the best one observed so far. It is a simple meta-algorithm which wraps any existing low-rank matrix factorization algorithm and improves its success rate.

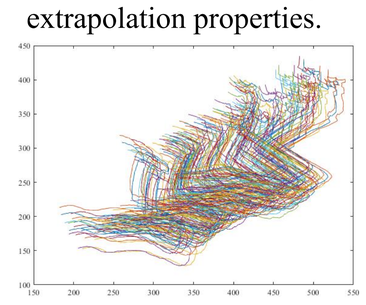
inputs: M, W, r, X, N
$k \leftarrow 0$
$f_{best} \leftarrow 0$
repeat
  $\mathrm{U} \leftarrow \mathrm{randn}(m, r)$ // m is the row size of M.
  $f_{current} \leftarrow \min_{\mathrm{U},\mathrm{V}} f(\mathrm{U},\mathrm{V})$ using algorithm-X
  if $|f_{current} - f_{best}| < 10^{-6}$ **do**
    break // output the RUSSO-X min. value.
  else if $f_{current} < f_{best}$ **do**
    $f_{best} \leftarrow f_{current}$ // update the current candidate for RUSSO-X min.
  end
  $k \leftarrow k + 1$
until $k = N$
output: f_{best}

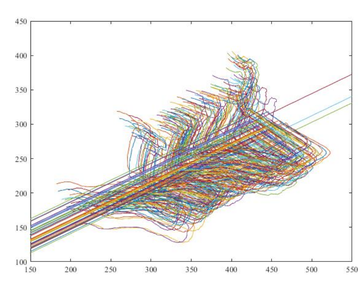
- An example of 10 runs from random starting points illustrates how RUSSO-*X* works in practice. (NB: RUSSO-*X* still returns the best optimum observed so far even when it is not observed twice.)

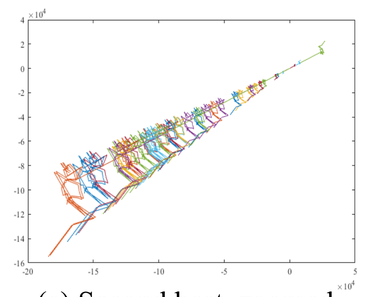
Run sequence	1	2	3	4	5	6	7	8	9	10	Success rate
Minimum	1.42	1.58	1.42	1.14	1.31	**1.02**	2.04	**1.02**	**1.02**	1.28	3 / 10
RUSSO-*X* min.	1.42	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.28	8 / 10

A side note: Why do we care about the best optimum?

The reconstruction of point trajectories (GIR in §8) shown below illustrates that a solution with function value just .06% above the optimum can have significantly worse extrapolation properties.


(a) Best known minimum (0.3228)


(b) Second best solution (0.3230)


(c) Second best, zoomed to image

4. Approximations

- All top matrix factorization algorithms in computer vision [5, 7, 9] benefit from the use of the **unregularized** VarPro, which is illustrated in §2.C.
- These just differ in how much approximation each makes in computing the Hessian. We use the RW*Y* convention where *Y* is the algorithm number mentioned in [10].

$$\frac{1}{2}\mathrm{H} = \underbrace{\widetilde{\mathrm{V}}^{*\top}\widetilde{\mathrm{V}}^*}_{0.5\,\mathrm{H}_{RW3}} - \underbrace{\widetilde{\mathrm{V}}^{*\top}\widetilde{\mathrm{U}}\widetilde{\mathrm{U}}^\dagger\widetilde{\mathrm{V}}^*}_{\mathrm{G}_{RW2}} - \underbrace{\mathrm{K}_{mr}^\top\mathrm{Z}^*\big(\widetilde{\mathrm{U}}^\top\widetilde{\mathrm{U}}\big)^{-1}\mathrm{Z}^{*\top}\mathrm{K}_{mr}}_{\mathrm{G}_{RW1}} + \underbrace{\mathrm{K}_{mr}^\top\mathrm{Z}^*\widetilde{\mathrm{U}}^\dagger\widetilde{\mathrm{V}}^*}_{\mathrm{G}_{FN}} + \underbrace{\widetilde{\mathrm{V}}^{*\top}\widetilde{\mathrm{U}}^{\dagger\top}\mathrm{Z}^{*\top}\mathrm{K}_{mr}}_{\mathrm{G}_{FN}^\top}$$

$$\frac{1}{2}\mathrm{H}_{RW2} = \frac{1}{2}\mathrm{H}_{RW3} - \mathrm{G}_{RW2}$$
$$\frac{1}{2}\mathrm{H}_{RW1} = \frac{1}{2}\mathrm{H}_{RW2} + \mathrm{G}_{RW1}$$
$$\frac{1}{2}\mathrm{H} = \frac{1}{2}\mathrm{H}_{RW2} - \mathrm{G}_{RW1} + \mathrm{G}_{FN} + \mathrm{G}_{FN}^\top$$

- $\mathrm{K}_{mr}\mathrm{vec}(\mathrm{U}) = \mathrm{vec}(\mathrm{U}^\top)$
- $\widetilde{\mathrm{U}} \coloneqq \widetilde{\mathrm{W}}(\mathrm{I} \otimes \mathrm{U})$
- $\mathrm{V}^*(\mathrm{U}) = \mathrm{unvec}\big(\widetilde{\mathrm{U}}^{-\mu}\widetilde{\mathbf{m}}\big)$
- $\widetilde{\mathrm{V}}^*(\mathrm{U}) \coloneqq \widetilde{\mathrm{W}}(\mathrm{V}^*(\mathrm{U}) \otimes \mathrm{I})$
- $\mathrm{Z}^*(\mathrm{U}) \coloneqq \big(\mathrm{W} \odot \mathrm{W} \odot (\mathrm{M} - \mathrm{U}\mathrm{V}^{*\top}(\mathrm{U}))\big) \otimes \mathrm{I}$

- The most influential term is $-\widetilde{\mathrm{V}}^*\widetilde{\mathrm{U}}\widetilde{\mathrm{U}}^\dagger\widetilde{\mathrm{V}}^*$, which brings accuracy from $O(0)$ to $O(100)\%$.

5. Manifold optimization

- As noted in [3], the unregularized matrix factorization problem have an innate gauge freedom. ($\mathrm{U}\mathrm{V}^\top = (\mathrm{U}\mathrm{A}^{-1})(\mathrm{V}\mathrm{A}^\top)^\top$ for any invertible matrix A.)
- For unregularized VarPro, this translates to U residing in the Grassmann manifold. i.e. U is a solution → entire $\mathrm{col}(\mathrm{U})$ is a solution space.
- To incorporate the manifold structure, we need to include the following steps in addition to the pseudocode illustrated in §2.C. (Background theory in [1].)

At each iteration,
1. $\mathbf{g} \leftarrow \mathrm{P}^\top\mathbf{g}$ // project the gradient and the Hessian to the tangent space of U.
2. $\mathrm{H} \leftarrow \mathrm{P}\mathrm{H}\mathrm{P}^\top$ // project the Hessian to the tangent space of U.
3. $\mathrm{U} \leftarrow \mathrm{unvec}\big(qf(\mathrm{U}+\Delta\mathrm{U})\big)$ // retract updated U back to the Grassmann manifold.

NB: For this problem, P can be $\mathrm{I} \otimes \mathrm{U}_\perp^\top \in \mathbb{R}^{(m-r)r\times mr}$ or $\mathrm{I} \otimes (I - \mathrm{U}\mathrm{U}^\top) \in \mathbb{R}^{mr\times mr}$.

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Project repository https://github.com/jhh37/matrix-factorization/

Approximations, Numerics, Manifold Optimization and Random Restarts

6. Unified analysis of algorithms

A. Alternation and VarPro algorithms for the unregularized problem ($\mu \leftarrow 0$)

ALS	VH_PF [8]	TW_WB [11]	TO_DW [9]	DRW1	DRW1P	DRW2	DRW2P	PG_CSF [7]	CH_LM_S [5]	CH_LM_S_GN [5]	CH_LM_S_RW2	CH_LM_M [5]	CH_LM_M_GN [5]	CH_LM_M_RW2	NB_RTRMC [2]	
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1:
		2	2	2	2	2	2	2	2	2	2	2	2	2	2	2:
				3	3					3			3			3:
									4			4			4	4:
					(5)		(5)								(5)	5:
												6	6	6		6:
					(7)		(7)					7	7	7	(7)	7:
					(8)		(8)					8	8	8	(8)	8:
			9		9		9									9:
		10	10	10	10	10	10	10	10	10	10	10	10	10	10	10:
11	11	11														11:
			12	12	12	12	12	12	12	12	12	12	12	12	12	12:
			13	13	13	13	13	13	13	13	13	13	13	13	13	13:
		14	14	14	14	14	14	14	14	14	14	14	14	14	14	14:
15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15:
	16				16											16:
17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17	17:
			18	18	18	18	18	18	18	18	18	18	18	18	18	18:

inputs: M, W, r, U, λ_0
$\lambda \leftarrow \lambda_0$
$\widetilde{\mathrm{W}} \leftarrow \mathrm{diag}\big(\mathrm{vec}(\mathrm{W})\big)$ with zero-rows removed.
$\widetilde{\mathbf{m}} \leftarrow \widetilde{\mathrm{W}}\,\mathrm{vec}(\mathrm{M})$
repeat
  $\mathbf{g} \leftarrow \mathrm{V}^*(\mathrm{U})^\top \mathrm{vec}(\mathrm{U}\mathrm{V}^*(\mathrm{U})^\top - \mathrm{M})$ // for all algorithms listed here.
  $\mathrm{H} \leftarrow \mathrm{V}^{*\top}\mathrm{V}^*$
  $\mathrm{H} \leftarrow \mathrm{H} - \widetilde{\mathrm{V}}^{*\top}\widetilde{\mathrm{U}}\widetilde{\mathrm{U}}^\dagger\widetilde{\mathrm{V}}^*$
  $\mathrm{H} \leftarrow \mathrm{H} + \mathrm{K}_{mr}^\top \mathrm{Z}^*\big(\widetilde{\mathrm{U}}^\top\widetilde{\mathrm{U}}\big)^{-1}\mathrm{Z}^{*\top}\mathrm{K}_{mr}$
  $\mathrm{H} \leftarrow \mathrm{H} - \mathrm{K}_{mr}^\top \mathrm{Z}^*\big(\widetilde{\mathrm{U}}^\top\widetilde{\mathrm{U}}\big)^{-1}\mathrm{Z}^{*\top}\mathrm{K}_{mr} + \mathrm{K}_{mr}^\top \mathrm{Z}^*\widetilde{\mathrm{U}}^\dagger\widetilde{\mathrm{V}}^* + \widetilde{\mathrm{V}}^{*\top}\widetilde{\mathrm{U}}^{\dagger\top}\mathrm{Z}^{*\top}\mathrm{K}_{mr}$
  $\mathrm{P} \leftarrow \mathrm{I} \otimes (\mathrm{I} - \mathrm{U}\mathrm{U}^\top) \in \mathbb{R}^{mr\times mr}$ // (5) is also a no-operation if 7 and 8 are.
  $\mathrm{P} \leftarrow \mathrm{I} \otimes \mathrm{U}_\perp^\top \in \mathbb{R}^{(m-r)r\times mr}$
  $\mathbf{g} \leftarrow \mathrm{P}\mathbf{g}$ // (7) is a no-operation since $\mathbf{g} = \mathrm{P}\mathbf{g}$.
  $\mathrm{H} \leftarrow \mathrm{P}\mathrm{H}\mathrm{P}^\top$ // (8) is a no-operation since $\mathrm{H} = \mathrm{P}\mathrm{H}$.
  $\mathrm{H} \leftarrow \mathrm{H} + \mathrm{I} \otimes \mathrm{U}\mathrm{U}^\top$ // relaxed constraint to promote $\mathrm{U}^\top\Delta\mathrm{U} = 0$.
  repeat
    $\Delta\mathrm{U} \leftarrow \mathrm{unvec}(\mathrm{H}^{-1}\mathbf{g})$
    $\Delta\mathrm{U} \leftarrow \mathrm{unvec}\big((\mathrm{H} + \lambda\mathrm{I})^{-1}\mathbf{g}\big)$
    $\lambda \leftarrow \lambda * 10$
  until $f\big(\mathrm{U} + \Delta\mathrm{U}, \mathrm{V}^*(\mathrm{U} + \Delta\mathrm{U})\big) < f(\mathrm{U}, \mathrm{V}^*(\mathrm{U}))$
  $\mathrm{U} \leftarrow \mathrm{U} + \Delta\mathrm{U}$
  $\mathrm{U} \leftarrow qf(\mathrm{U})$ // [U,~]=qr(U,0) in MATLAB.
  $\mathrm{V} \leftarrow \mathrm{unvec}\big(\widetilde{\mathrm{U}}^\dagger\widetilde{\mathbf{m}}\big)$
  $\lambda \leftarrow \lambda/100$
until convergence
outputs: U, V

- $\mathrm{K}_{mr}\mathrm{vec}(\mathrm{U}) = \mathrm{vec}(\mathrm{U}^\top)$
- $\widetilde{\mathrm{U}} \coloneqq \widetilde{\mathrm{W}}(\mathrm{I} \otimes \mathrm{U})$
- $\mathrm{V}^*(\mathrm{U}) = \mathrm{unvec}\big(\widetilde{\mathrm{U}}^\dagger\widetilde{\mathbf{m}}\big)$
- $\widetilde{\mathrm{V}}^*(\mathrm{U}) \coloneqq \widetilde{\mathrm{W}}(\mathrm{V}^*(\mathrm{U}) \otimes \mathrm{I})$
- $\mathrm{Z}^*(\mathrm{U}) \coloneqq \big(\mathrm{W} \odot \mathrm{W} \odot (\mathrm{M} - \mathrm{U}\mathrm{V}^{*\top}(\mathrm{U}))\big) \otimes \mathrm{I}$

B. Joint optimization algorithms

CE_LM	AB_DN [3]	
1	1	1:
	2	2:

inputs: M, W, r, U, V, μ, λ_0
$\lambda \leftarrow \lambda_0$
$\widetilde{\mathrm{W}} \leftarrow \mathrm{diag}\big(\mathrm{vec}(\mathrm{W})\big)$ with zero-rows removed.
$\widetilde{\mathbf{m}} \leftarrow \widetilde{\mathrm{W}}\,\mathrm{vec}(\mathrm{M})$
repeat
  $\mathbf{g} \leftarrow \begin{bmatrix} \widetilde{\mathrm{V}}^\top\mathrm{vec}(\mathrm{U}\mathrm{V}^\top - \mathrm{M}) + \mu\mathrm{vec}(\mathrm{U}) \\ \widetilde{\mathrm{U}}^\top\mathrm{vec}(\mathrm{U}\mathrm{V}^\top - \mathrm{M}) + \mu\mathrm{vec}(\mathrm{V}^\top) \end{bmatrix}$
  $\mathrm{H} \leftarrow \begin{bmatrix} \widetilde{\mathrm{V}}^\top\widetilde{\mathrm{V}} + \mu\mathrm{I} & \widetilde{\mathrm{V}}^\top\widetilde{\mathrm{U}} \\ \widetilde{\mathrm{U}}^\top\widetilde{\mathrm{V}} & \widetilde{\mathrm{U}}^\top\widetilde{\mathrm{U}} + \mu\mathrm{I} \end{bmatrix}$
  $\mathrm{H} \leftarrow \mathrm{H} + \begin{bmatrix} 0 & \mathrm{K}_{mr}^\top Z \\ \mathrm{Z}^\top\mathrm{K}_{mr} & 0 \end{bmatrix}$
  repeat
    $\Delta\mathbf{z} \leftarrow (\mathrm{H} + \lambda\mathrm{I})^{-1}\mathbf{g}$
    Retrieve ΔU and ΔV from $\Delta\mathbf{z}$.
    $\lambda \leftarrow \lambda * 10$
  until $f(\mathrm{U} + \Delta\mathrm{U}, \mathrm{V} + \Delta\mathrm{V}) < f(\mathrm{U}, \mathrm{V})$
  $\mathrm{U} \leftarrow \mathrm{U} + \Delta\mathrm{U}$
  $\mathrm{V} \leftarrow \mathrm{V} + \Delta\mathrm{V}$
  $\lambda \leftarrow \lambda/100$
until convergence
outputs: U, V

- $\mathrm{K}_{mr}\mathrm{vec}(\mathrm{U}) = \mathrm{vec}(\mathrm{U}^\top)$
- $\widetilde{\mathrm{U}} \coloneqq \widetilde{\mathrm{W}}(\mathrm{I} \otimes \mathrm{U})$
- $\widetilde{\mathrm{V}} \coloneqq \widetilde{\mathrm{W}}(\mathrm{V} \otimes \mathrm{I})$
- $\mathrm{Z} \coloneqq (\mathrm{W} \odot \mathrm{W} \odot (\mathrm{M} - \mathrm{U}\mathrm{V}^\top) \otimes \mathrm{I}$

7. Numerics

A. Numerical issues

- Mainly, computation of the Hessian (or its approximation) and its inversion.
- Use of the QR factorization ($\widetilde{\mathrm{U}} = \widetilde{\mathrm{U}}_Q\widetilde{\mathrm{U}}_R$) improves accuracy and performance.
- Cholesky decomposition is used for computing ΔU.

B. Profiler-guided optimization

- Standard Matlab tricks for code speedup.
- Mex files for some Kronecker product computations.

Algorithm	Successes on Din (Out of 20)		Runtime on Din (ms / iter)	
	Original	Modified	Original	Modified
CH_LM_S	4	4	380	140
CH_LM_M	1	6	369	143
CH_LM_M_GN	15	19	205	109

C. Removal of redundant computations

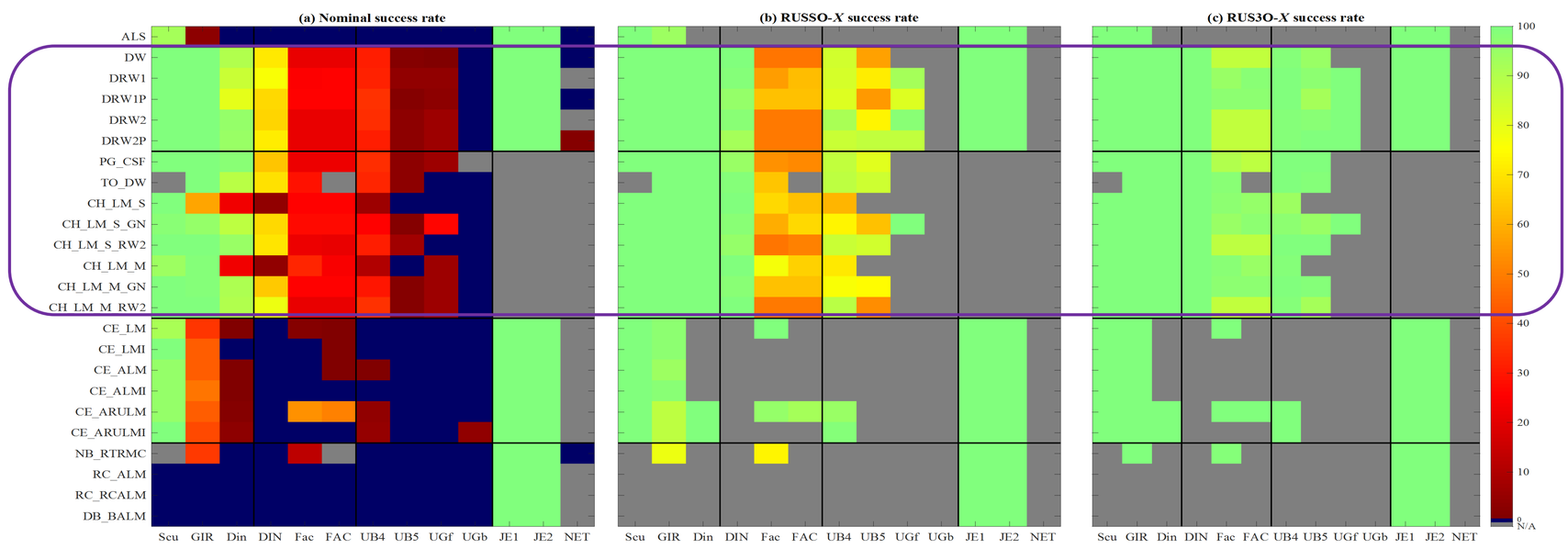
1. VarPro JTJ is of the form A $\otimes$ B where A and B are symmetric. i.e. internal symmetry in addition to the ordinary JTJ symmetry. (max. 4 fold speedup.)
2. $\widetilde{\mathrm{U}}_\mathrm{Q}$ is block-diagonal. Each sub-block $\widetilde{\mathrm{U}}_{\mathrm{j}\mathrm{Q}}$ is a function of the respective column of W; $\widetilde{\mathrm{U}}_{jQ} = qf\big(\widetilde{\mathrm{W}}_j\mathrm{U}\big)$ where $\widetilde{\mathrm{W}}_j = \mathrm{diag}(\mathrm{vec}\big(\mathrm{W}(:,j)\big)$. Hence, if $\mathrm{W}(:,i) = \mathrm{W}(:,j)$, $\widetilde{\mathrm{U}}_{iQ} = \widetilde{\mathrm{U}}_{jQ}$. i.e. redundant QR computations can be reduced depending on the dataset. (Up to 5 times faster for DIN)

8. A list of datasets used

ID	Nature	m	n	r	Fill (%)
Scu	Photometric stereo	46	16,301	3	66.5
GIR	Non-rigid SfM	166	240	6	69.8
Din	Rigid SfM	72	319	4	23.1
DIN	Rigid SfM	72	4,983	4	9.2
Fac	Photometric stereo	20	2,596	4	64.9
FAC	Photometric stereo	20	2,944	4	58.3
UB4	Non-rigid SfM	110	1,760	4	14.4
UB5	Non-rigid SfM	110	1,760	5	14.4
UGf	Non-rigid SfM	380	4,885	6	9.1
UGb	Non-rigid SfM	380	6,310	4	6.9
JE1	Recommender	100	24,983	7	72.5
JE2	Recommender	100	23,500	7	72.7
NET	Recommender	2,000	50,000	4	2.7

9. Comparison of success rates

When run from 20 to 100 different random starting points, the VarPro-based algorithms (row 2-14) perform best in terms of success rates. RUSSO-X and RUS3O-X improves it further. (RUSkO-X stands for "run algorithm-X until seen k^{th} optimum".)

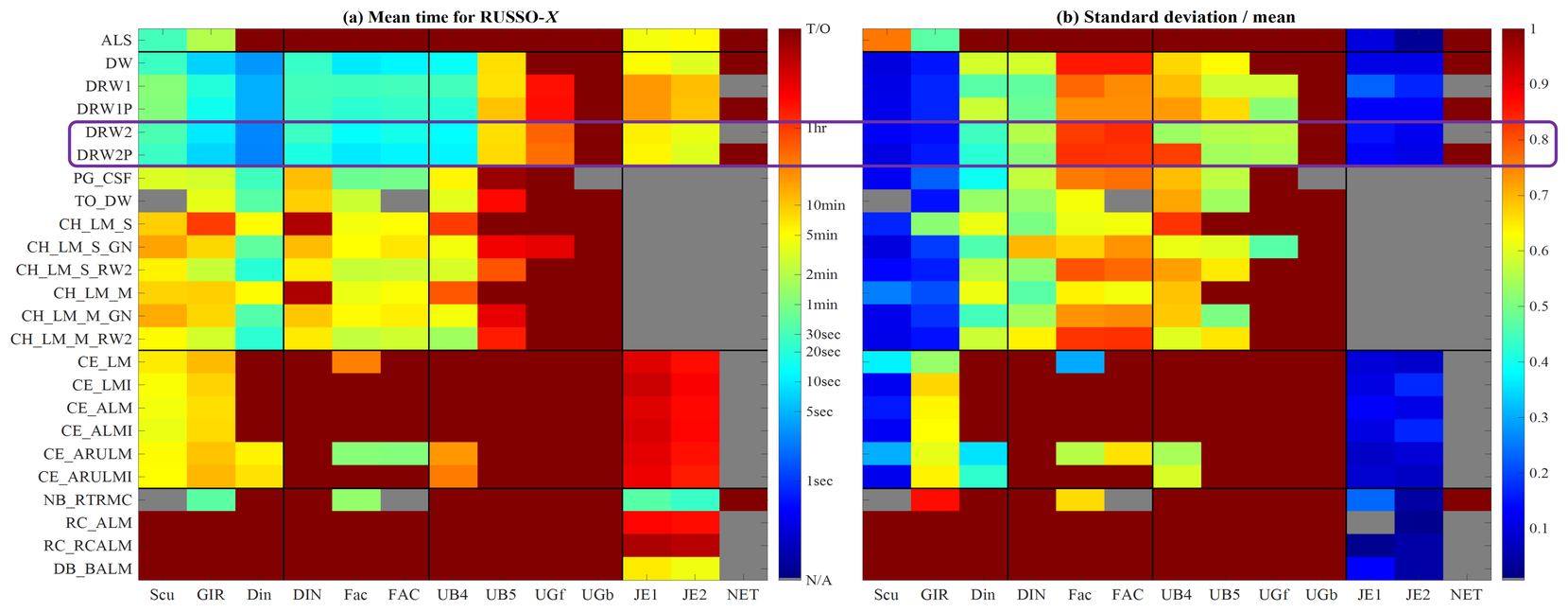


Conclusions

- **Approximations**: Recent top matrix factorization algorithms, which have been derived in a multitude of different ways, can be re-derived in a unified way such that similarities and differences are more easily observed.
- **Numerics**: Exploiting matrix structures and using block-QR decomposition improves the variable projection (VarPro) algorithms both in terms of speed and convergence.
- **Manifold optimization**: The Riemannian manifold optimization framework can be incorporated to give a slight more improvement.
- **Random restarts**: A simple meta-algorithm, RUSSO-X can be wrapped around any existing matrix factorization algorithm and improve its success rate.
- **Most importantly, VarPro is key to success** for matrix factorization, everything else is detail.

10. Mean time and standard deviation for RUSSO-X

All are run in single-threaded mode. Overall, the best performing family is the "DRW" set. DRW2 and DRW2P are the winners.



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