Computer Graphics
-Introduction to Geometry & its Representations

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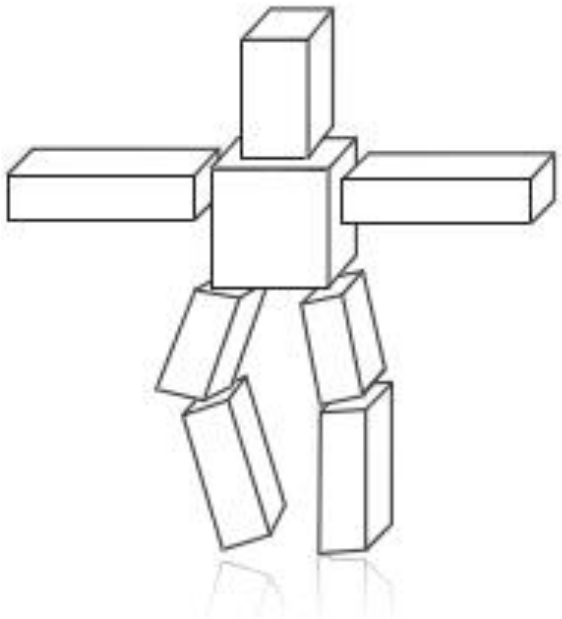
http://jjcao.github.io/ComputerGraphics/

Increasing the complexity of our models

Transformations



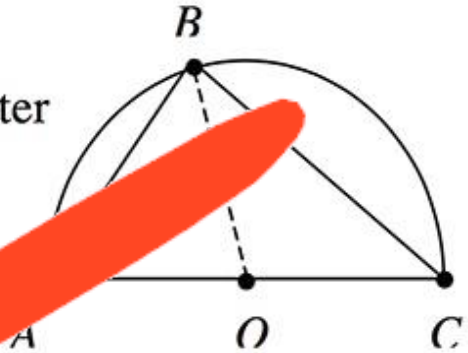
Geometry

Materials, lighting, ...

What is geometry?

THEOREM 9.5. Let ΔABC be inscribed in a semicircle with diameter $\overline{A}\ \overline{C}$.

Then $\angle ABC$ [illegible] angle.



Proof:

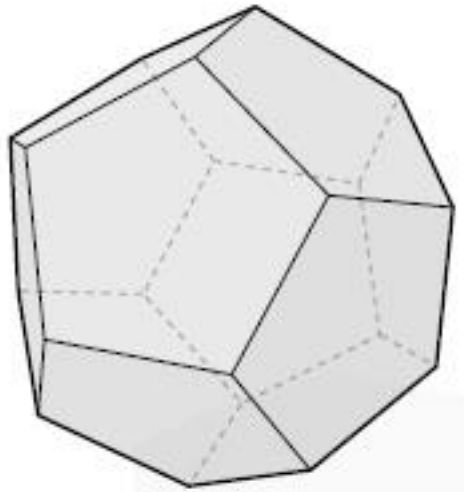
Statement	R[illegible]
1. Draw radius OB. Then $OB = OC = O$[illegible]	[illegible]iven
2. $m\angle OBC = m\angle BCA$ $m\angle OBA = m\angle BAC$	[illegible]osceles Triangle Theorem
3. $m\angle ABC = m\angle OBA + m$[illegible]	3. Angle [illegible] Postulate
4. $m\angle ABC + m\angle BCA$ [illegible]$BAC = 180$	4. The sum o[illegible]ngles of a triangle is 180
5. $m\angle ABC + m\angle$[illegible] $+ m\angle OBA = 180$	5. Substitution (line[illegible]
6. $2\, m\angle AB$[illegible]80	6. Substitution (line 3)
7. $m\angle A$[illegible]$C = 90$	7. Division Property of Equality
8. $\angle ABC$ is a right angle	8. Definition of Right Angle

What is geometry?

"Earth" "measure"

ge•o•m•et•ry /jēˈämətrē/ *n.*

1. The study of shapes, sizes, patterns, and positions.
2. The study of spaces where some quantity (lengths, angles, etc.) can be *measured*.



Plato: "...the earth is in appearance like one of those balls which have leather coverings in twelve pieces..."

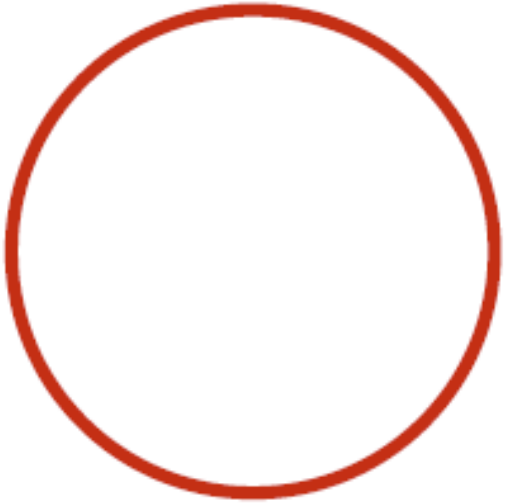
How can we describe geometry?

IMPLICIT

$$x^2+y^2=1$$

LINGUISTIC

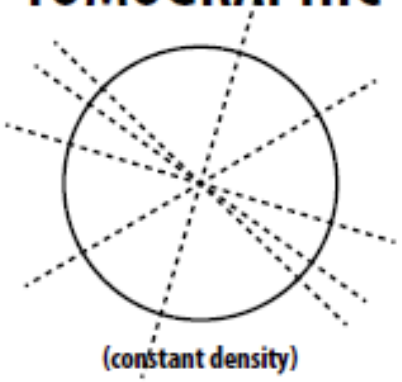
"unit circle"



EXPLICIT

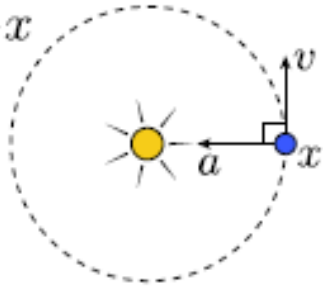
$$(\underbrace{\cos\theta}_{x}, \underbrace{\sin\theta}_{y})$$

TOMOGRAPHIC



DYNAMIC

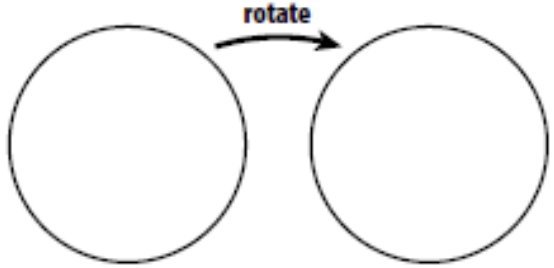
$$\frac{d^2}{dt^2}x = -x$$



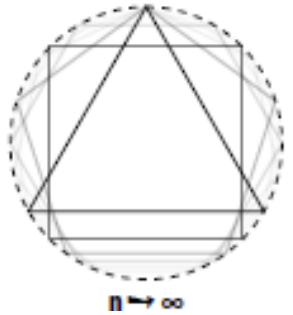
CURVATURE

$$\kappa = 1$$

SYMMETRIC

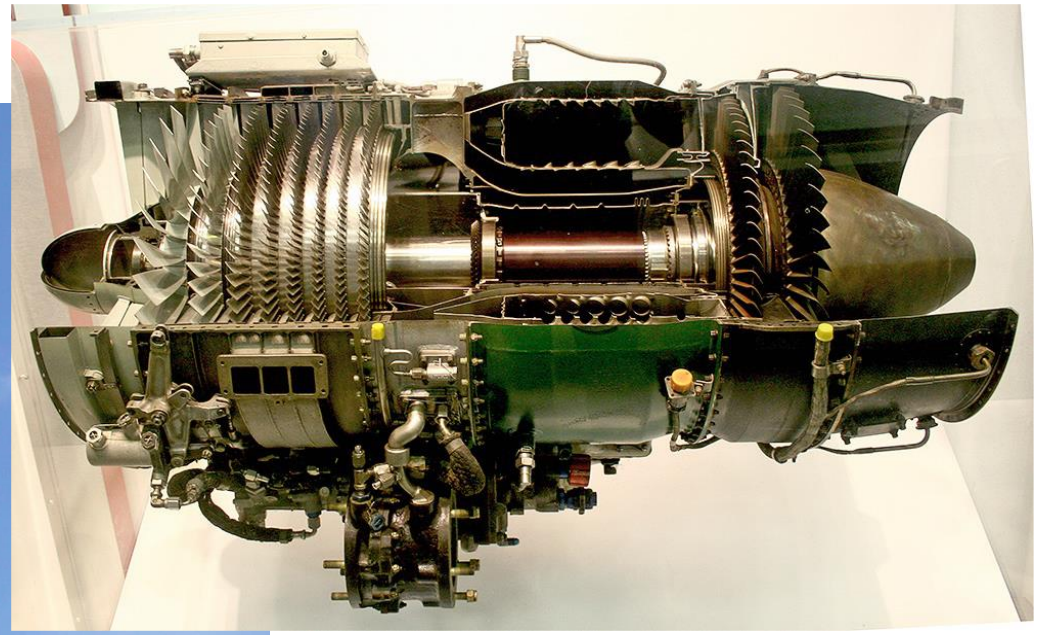


DISCRETE

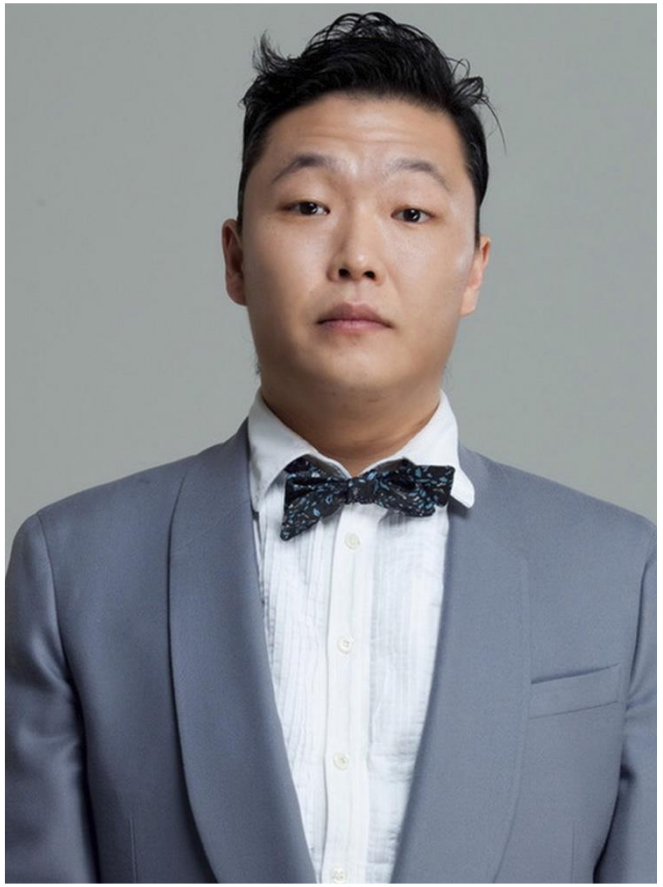


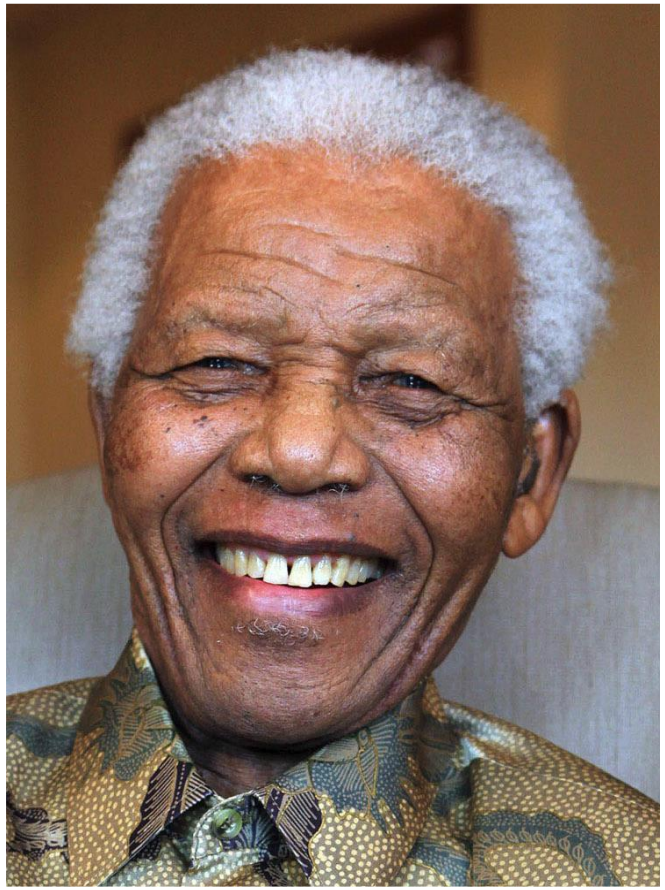
Given all these options, what's the best way to encode geometry on a computer?

Examples of geometry



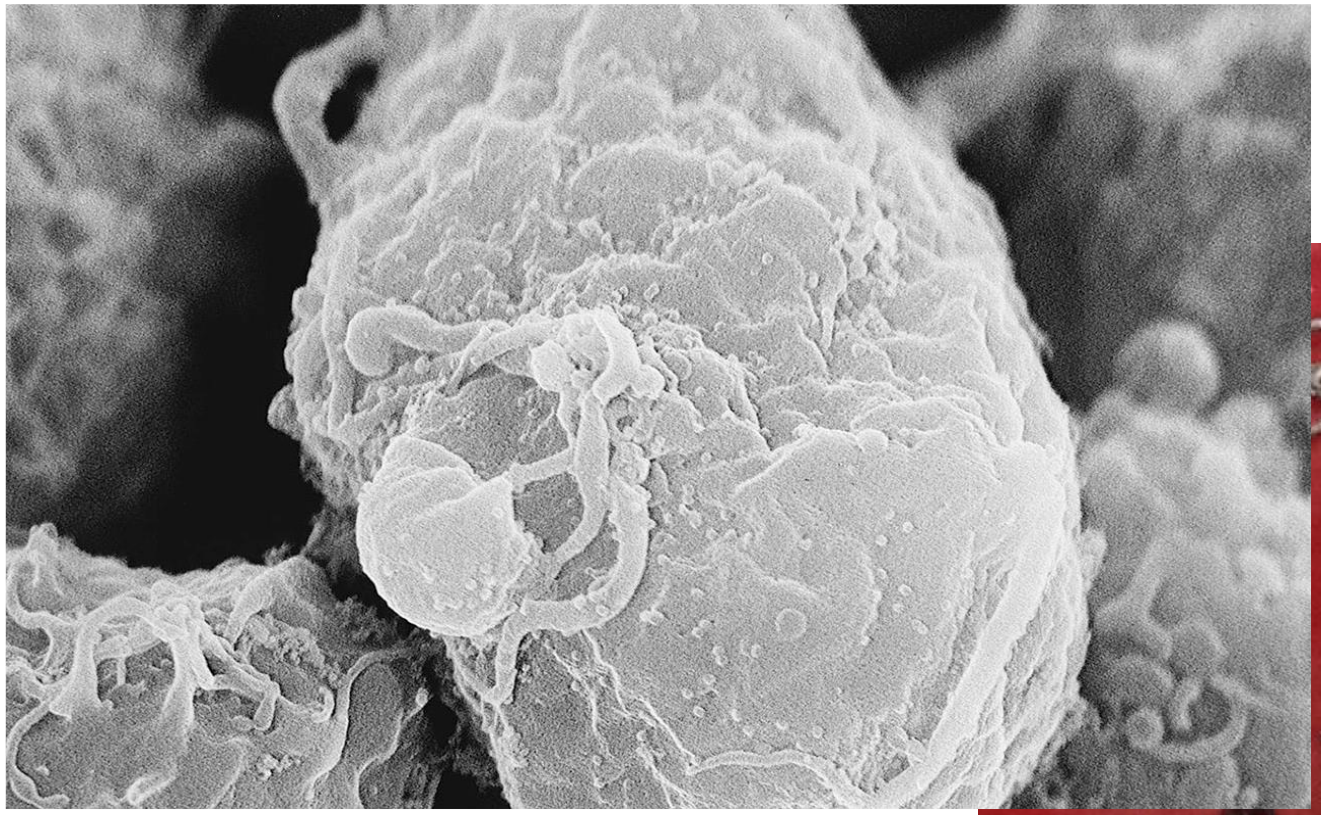
Examples of geometry

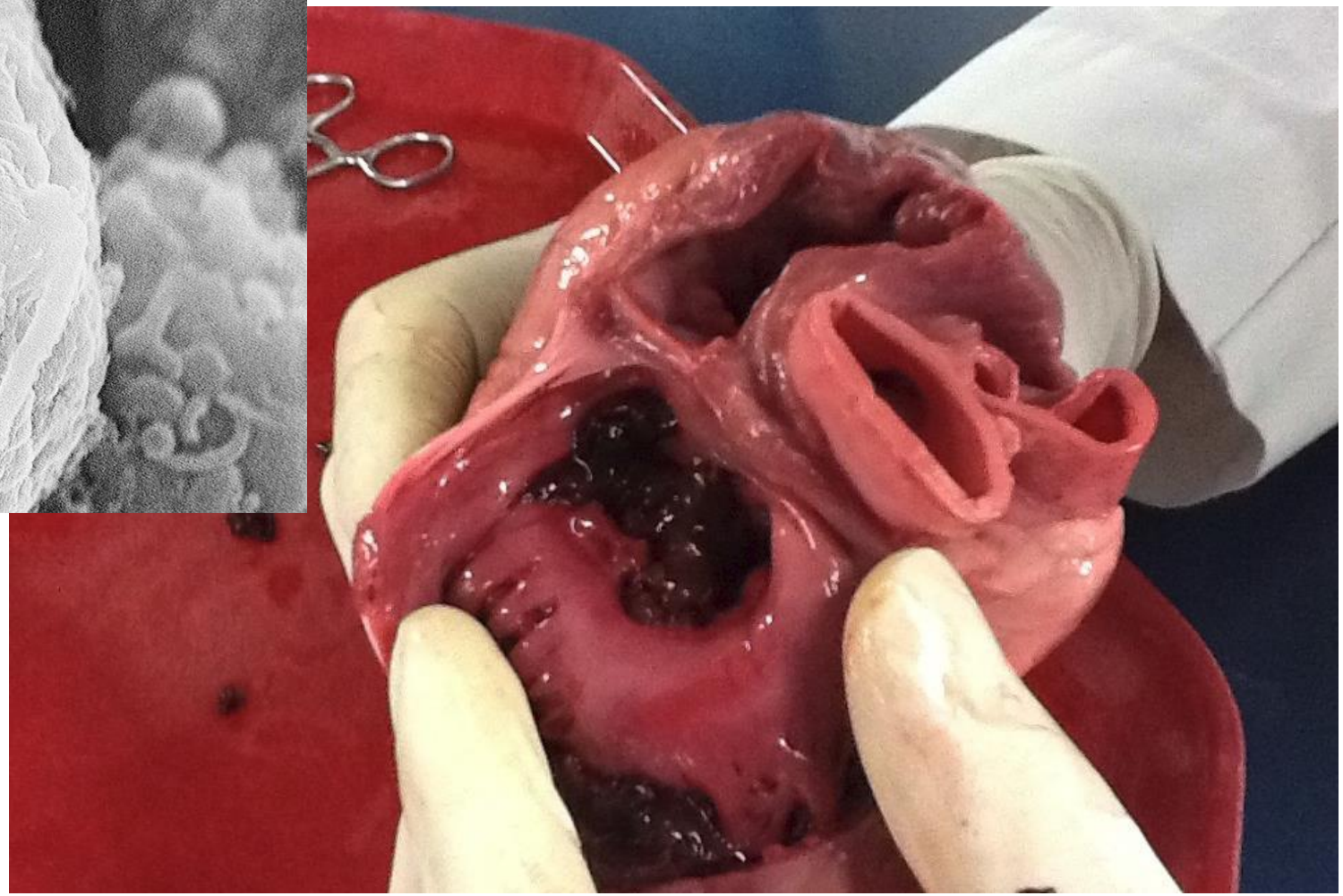




Examples of geometry

Examples of geometry

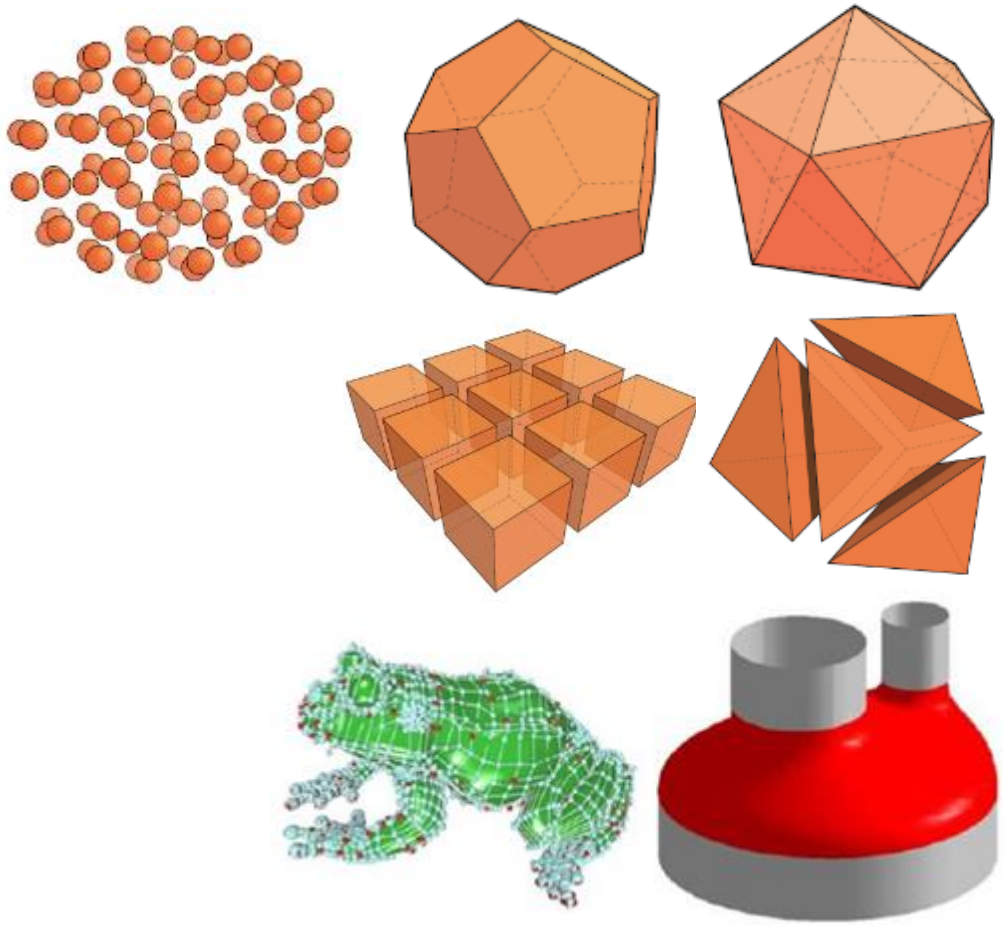




Examples of geometry

Many ways to digitally encode geometry

- EXPLICIT
  - point cloud
  - polygon mesh
  - subdivision, NURBS
  - L-systems
  - ...
- IMPLICIT
  - level set
  - algebraic surface
  - ...



- **Each choice best suited to a different task/type of geometry**

Modeling Complex Shapes

- An equation for a sphere is possible, but how about an equation for a rabbit?
- **Complexity is achieved using simple pieces**
  - polygons, parametric surfaces, or implicit surfaces
- High Resolution or Large scenes

- Goals
  - Model anything with arbitrary precision (in principle)
  - Easy to build and modify
  - Efficient computations (for rendering, collisions, etc.)
  - Easy to implement (a minor consideration...)

What do we need from shapes in Computer Graphics?

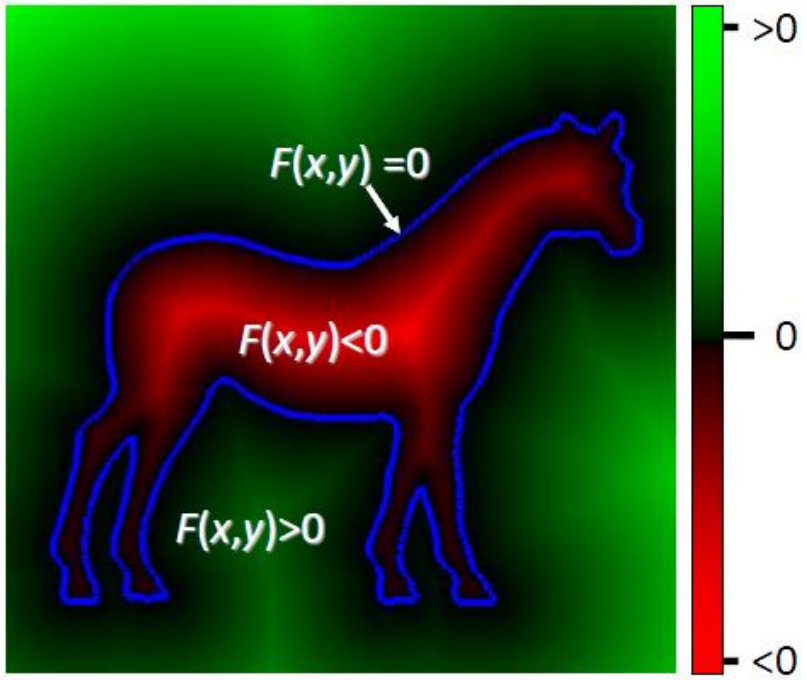
- Local control of shape for modeling
- Ability to model what we need
- Smoothness and continuity
- Ability to evaluate derivatives
- Ability to do collision detection
- Ease of rendering

- **No single technique solves all problems!**

“Implicit” Representations of Geometry

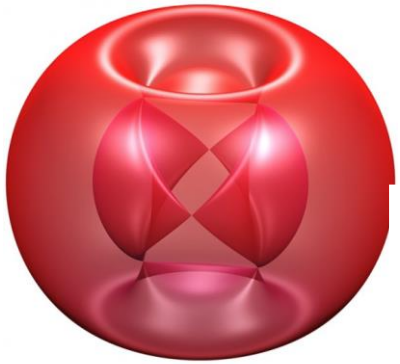
- Points aren’t known directly, but satisfy some relationship
  - E.g., unit sphere is all points x such that **x^2+y^2+z^2=1**
  - More generally, **f(x,y,z) = 0**
- Represent a surface as the zero set of a (regular) function defined in R^3.

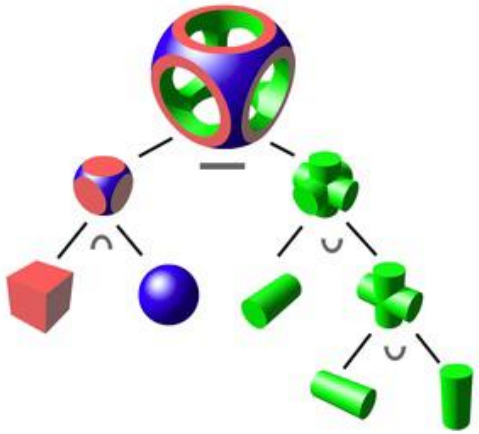
$$K = g^{-1}(0) = \{\mathbf{p} \in \mathrm{R}^3 : g(\mathbf{p}) = 0\}$$

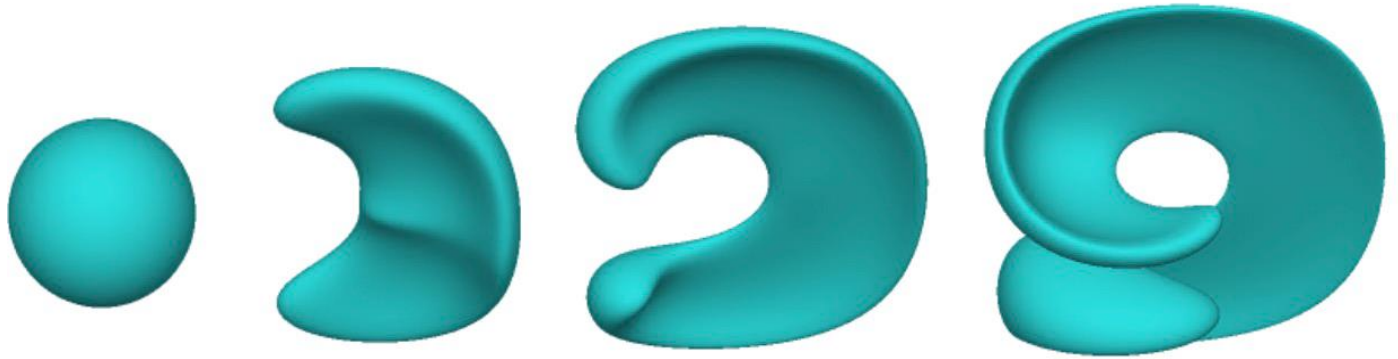


Many implicit representations in graphics

- algebraic surfaces
- constructive solid geometry
- level set methods
- blobby surfaces
- fractals
- ...







(Will see some of these a bit later.)

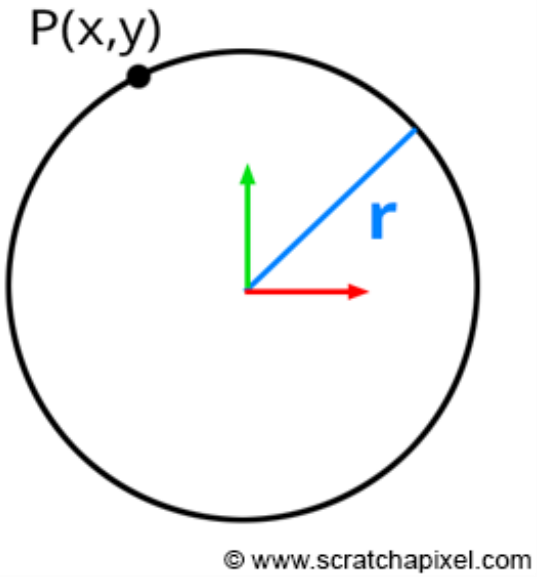
But first, let's play a game:

I'm thinking of an implicit surface f(x,y,z)=0.

Find *any* point on it.

Give up?

- **My function was** $g(x,y,z) = x^2 + y^2 + z^2 - r^2$



- **Implicit surfaces make some tasks hard (like sampling).**

Let’s play another game.

I have a surface f(x,y,z) = x^2 + y^2 + z^2 – 1

I want to see if a point is inside it.

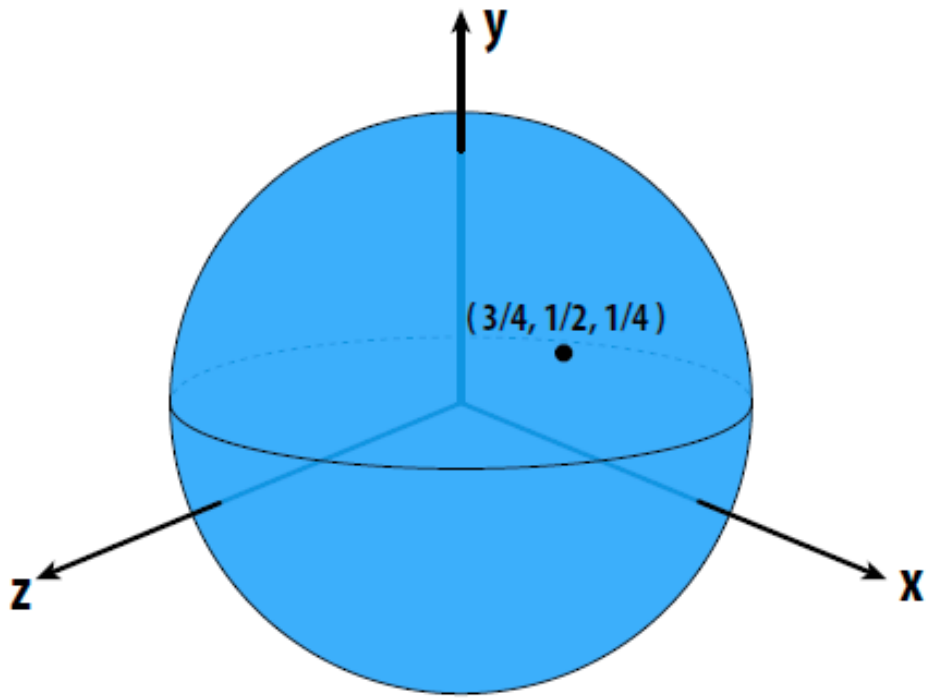
Check if this point is inside the unit sphere

How about the point (3/4, 1/2, 1/4)?

9/16 + 4/16 + 1/16 = 7/8

7/8 < 1

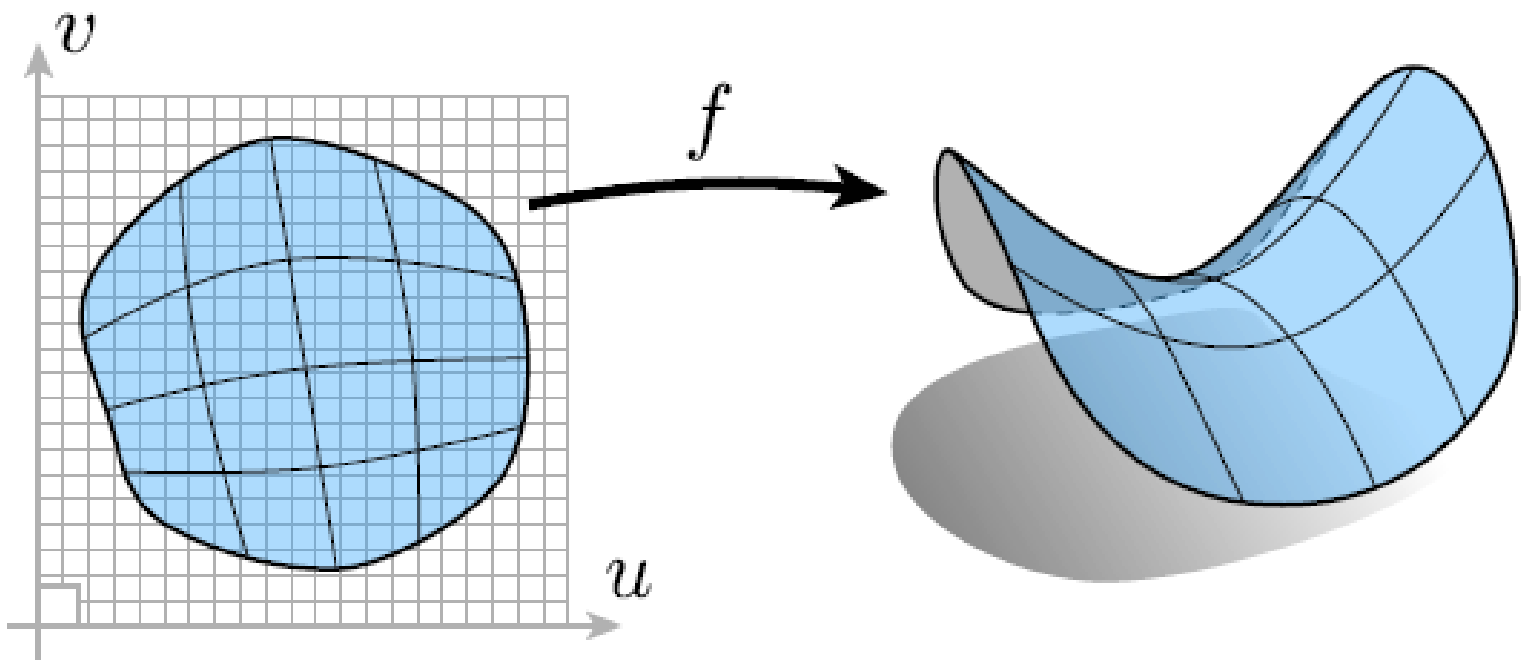
YES.



Implicit surfaces make other tasks easy (like inside/outside tests).

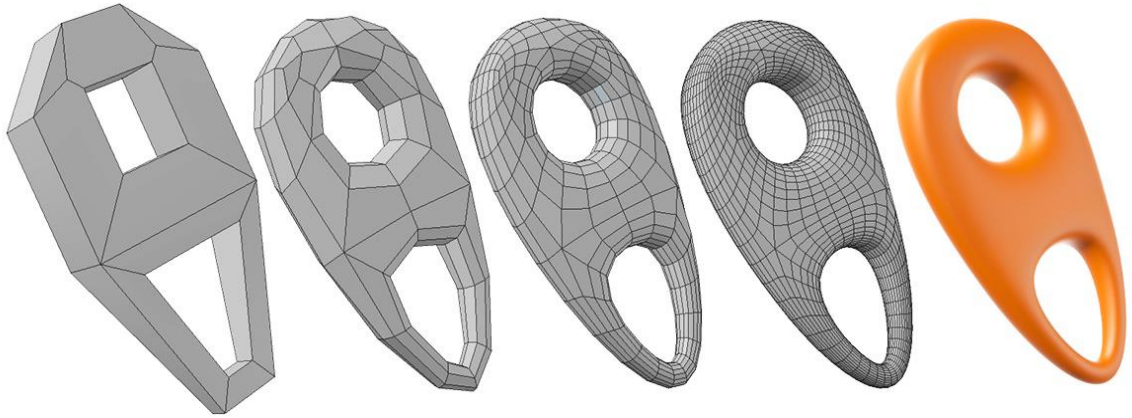
“Explicit” Representations of Geometry

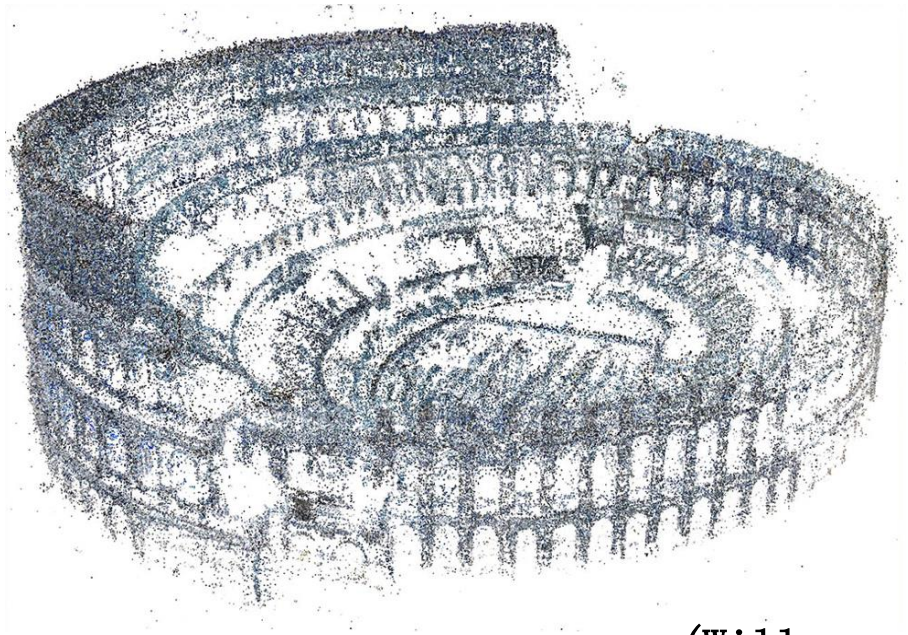
- All points are given directly
- E.g., points on sphere are $(\cos(u)\sin(v), \sin(u)\sin(v), \cos(v))$, for $0 \leq u < 2\pi$ and $0 \leq v \leq \pi$
- More generally: $f : \mathbb{R}^2 \to \mathbb{R}^3; (u, v) \mapsto (x, y, z)$

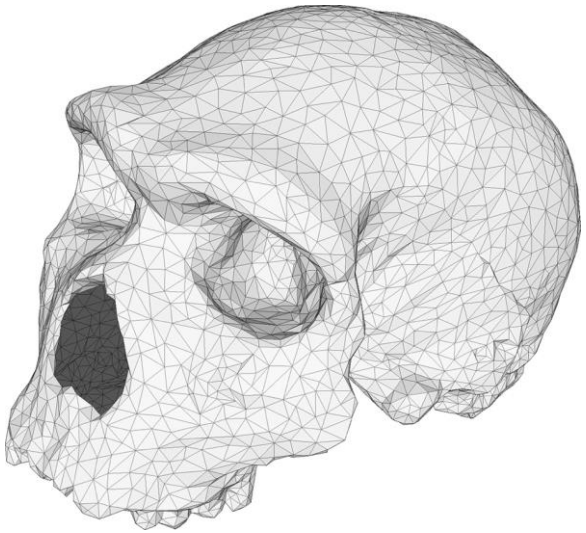


Many explicit representations in graphics

- triangle meshes
- polygon meshes
- subdivision surfaces
- NURBS
- point clouds
- ...







(Will see some of these a bit later.)

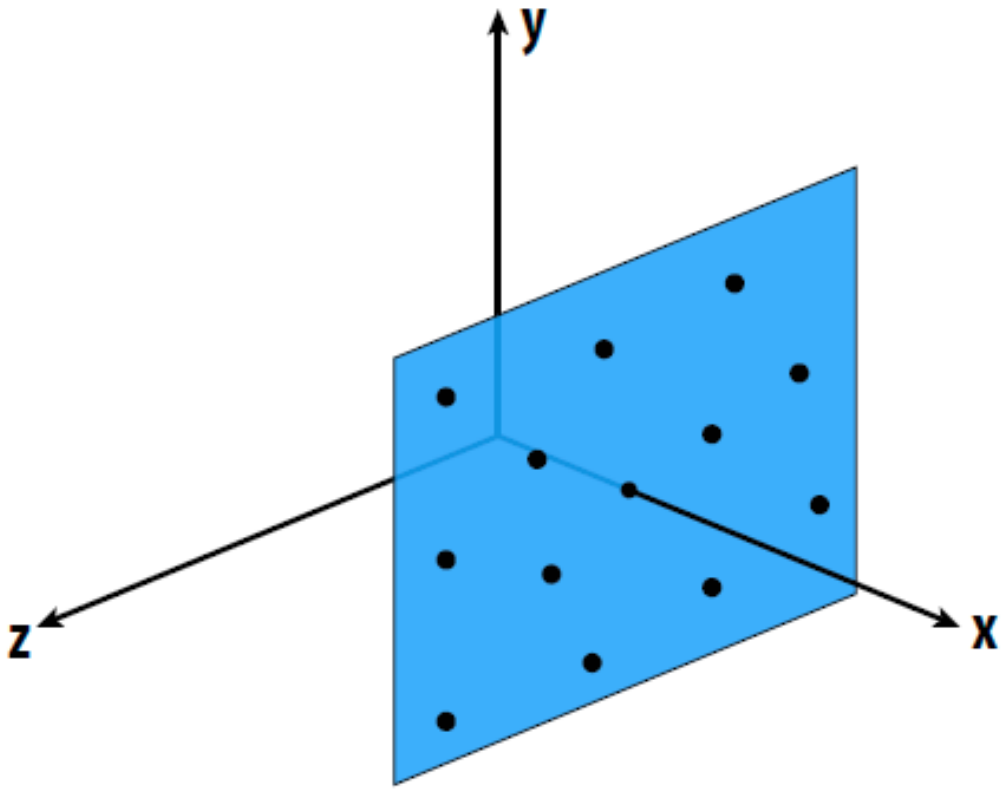
But first, let's play a game:

I'll give you an explicit surface.

You give me some points on it.

Sampling an explicit surface

- My surface is f(u, v) = (1.23, u, v).
- Just plug in any values (u,v)!



- Explicit surfaces make some tasks easy (like sampling).

Let’s play another game.

I have a new surface f(u,v).

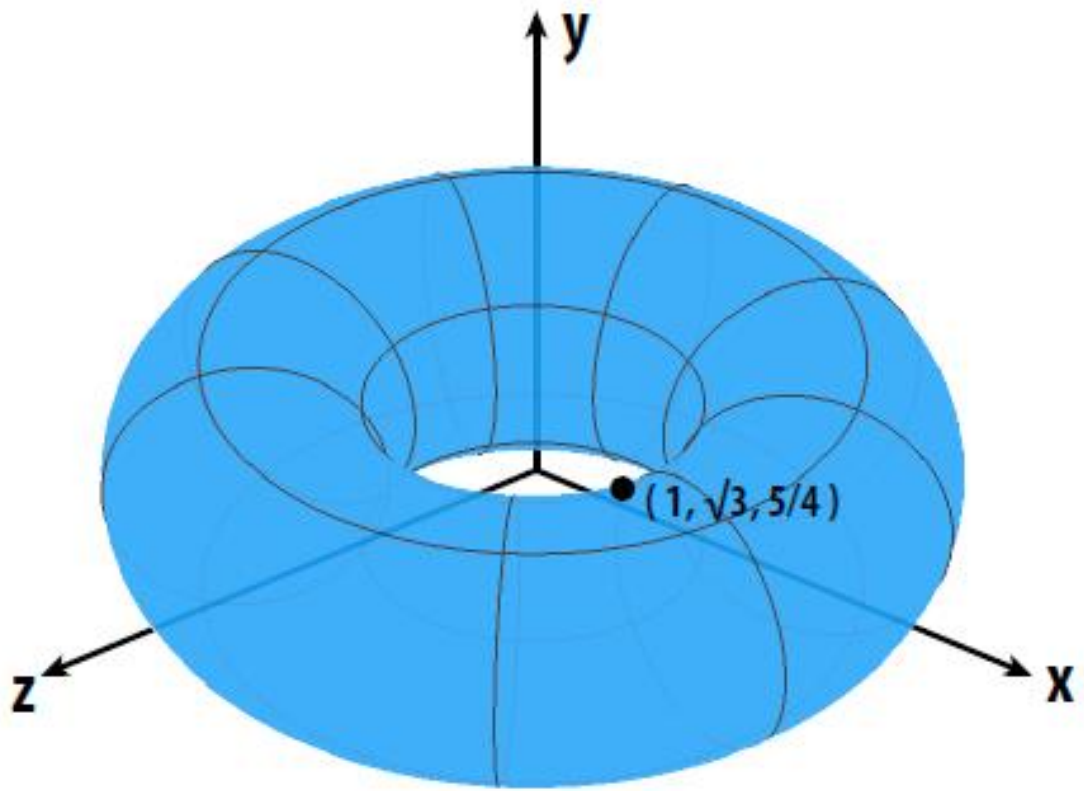
I want to see if a point is inside it.

Check if this point is inside the torus

My surface is f(u,v) = (2+cos(u))cos(v), 2+cos(u))sin(v), sin(u))

How about the point (1,√3,5/4)?

...NO!



Explicit surfaces make other tasks hard (like inside/outside tests).

CONCLUSION:

Some representations work better than others—depends on the task!

Different representations will also be better suited to different types of geometry.

Let’s take a look at some common representations used in computer graphics.

Algebraic Surfaces (Implicit)

- **Surface is zero set of a polynomial in x, y, z (“algebraic variety”)**
- **Examples:**

$x^2+y^2+z^2=1$

$(R-\sqrt{x^2+y^2})^2+z^2=r^2$

$(x^2+\frac{9y^2}{4}+z^2-1)^3=$

$x^2z^3+\frac{9y^2z^3}{80}$

- **What about more complicated shapes?**

- **Very hard to come up with polynomials!**

Gradient of Implicit Surfaces

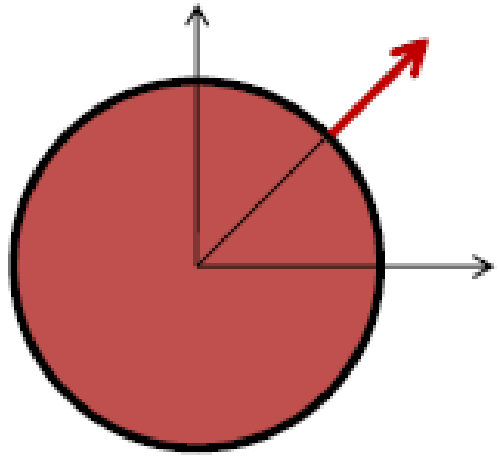
- The normal vector to the surface is given by the gradient of the (scalar) implicit function

$$\nabla g(x,y,z) = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)^{\mathrm{T}}$$

- Example

$$g(x,y,z) = x^2 + y^2 + z^2 - r^2$$

$$\nabla g(x,y,z) = (2x, 2y, 2z)^{\mathrm{T}}$$

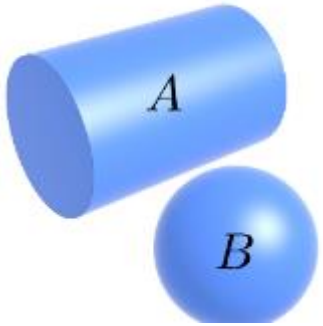


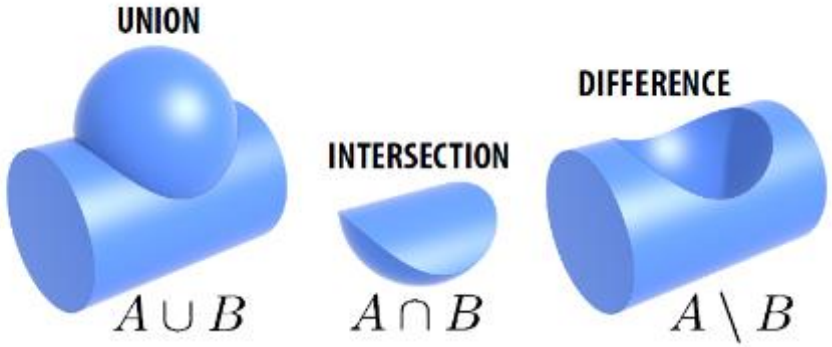
$$\nabla g(x,y,z) = (2,2,0)^{\mathrm{T}}$$

- Why is the condition that the function be regular (i.e. have non-vanishing derivative) necessary?
- How smooth is the surface?

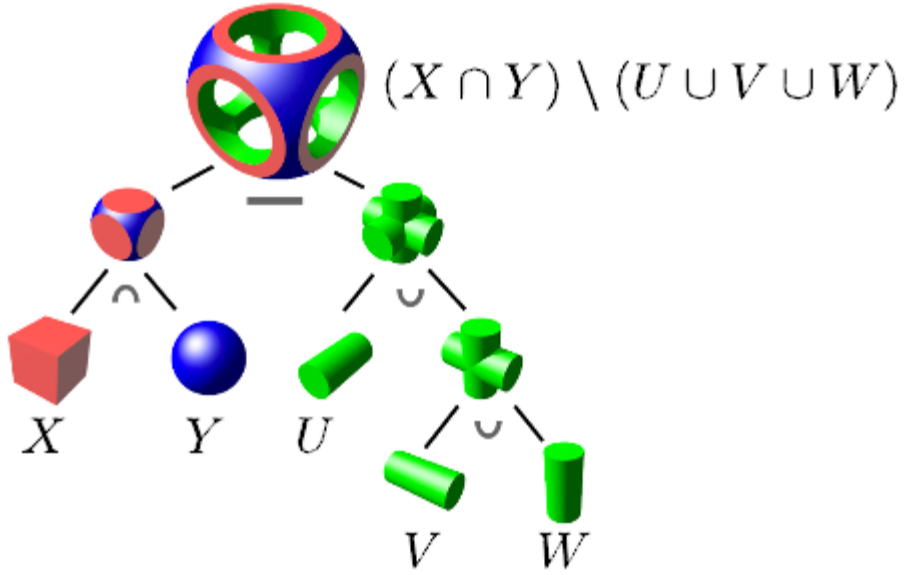
Constructive Solid Geometry (Implicit)

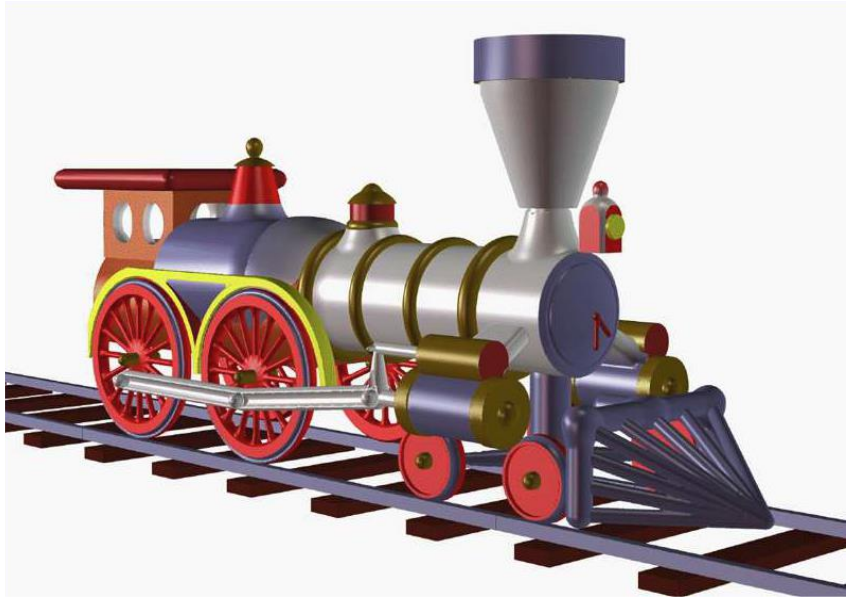
- **Build more complicated shapes via Boolean operations**
- **Basic operations:**





- **Then chain together expressions:**

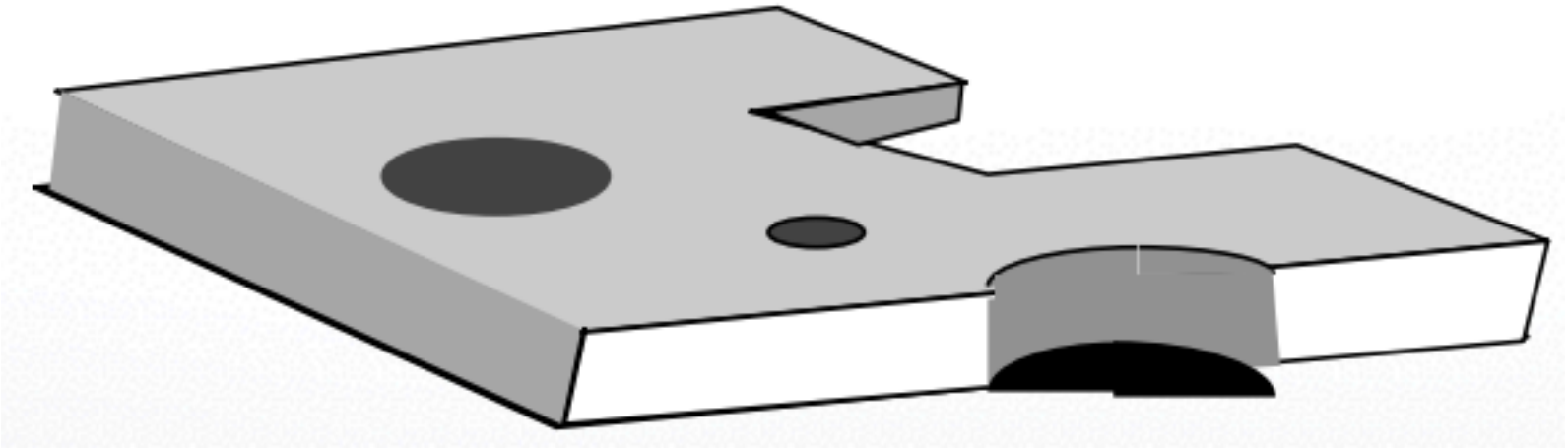




Constructive Solid Geometry (CSG)

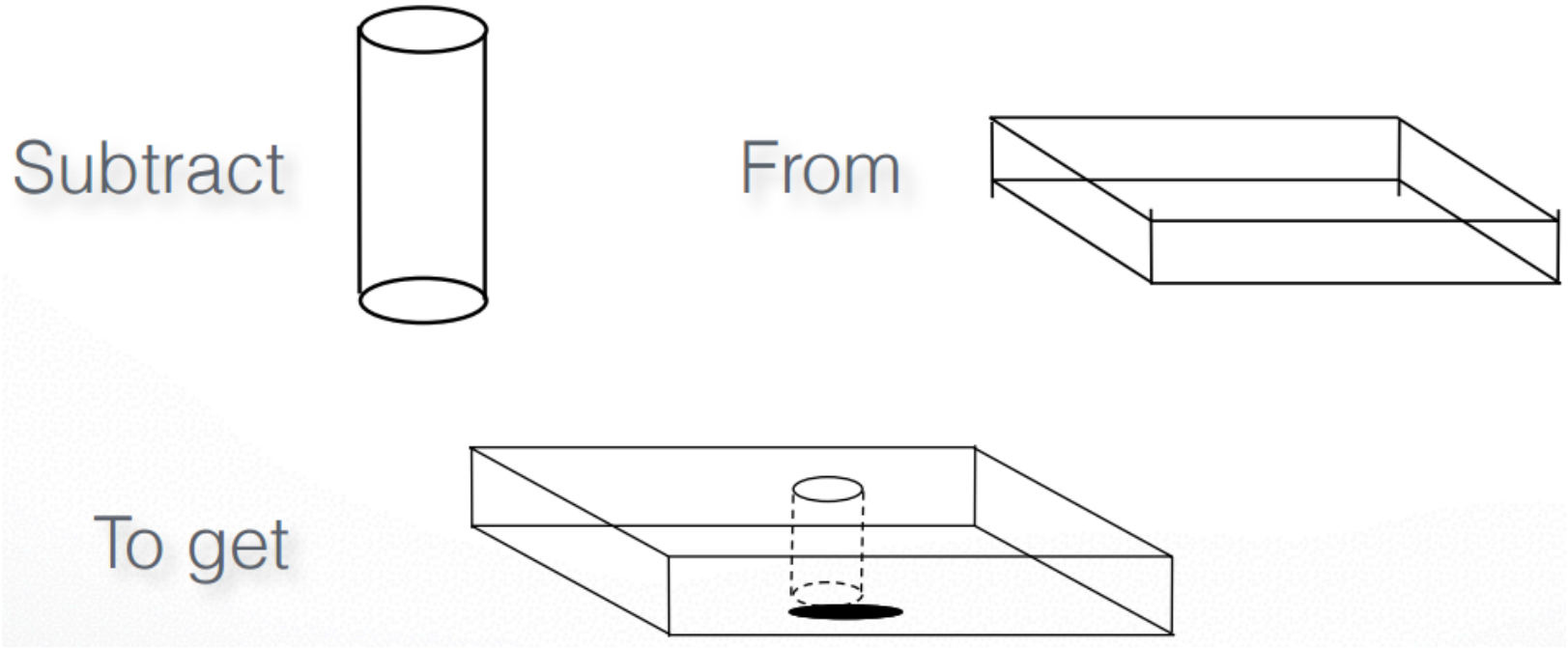
- Generate complex shapes with basic building blocks

- Machine an object - saw parts off, drill holes, glue pieces together
- This is sensible for objects that are actually made that way (human-made, particularly machined objects)



Negative Objects

- Use point-by-point boolean functions
  - remove a volume by using a negative object
  - e.g. drill a hole by subtracting a cylinder



- Inside(BLOCK-CYL) = Inside(BLOCK) And Not(Inside(CYL))

Set Operations

- UNION: Inside(A) || Inside(B)

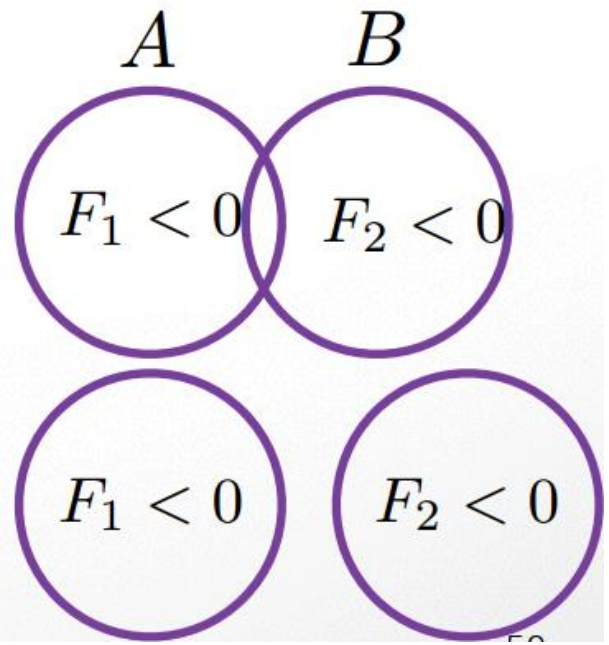
  Join A and B
- INTERSECTION: Inside(A) && Inside(B)

  Chop off any part of A that sticks out of B
- SUBTRACTION: Inside(A) && (! Inside(B))

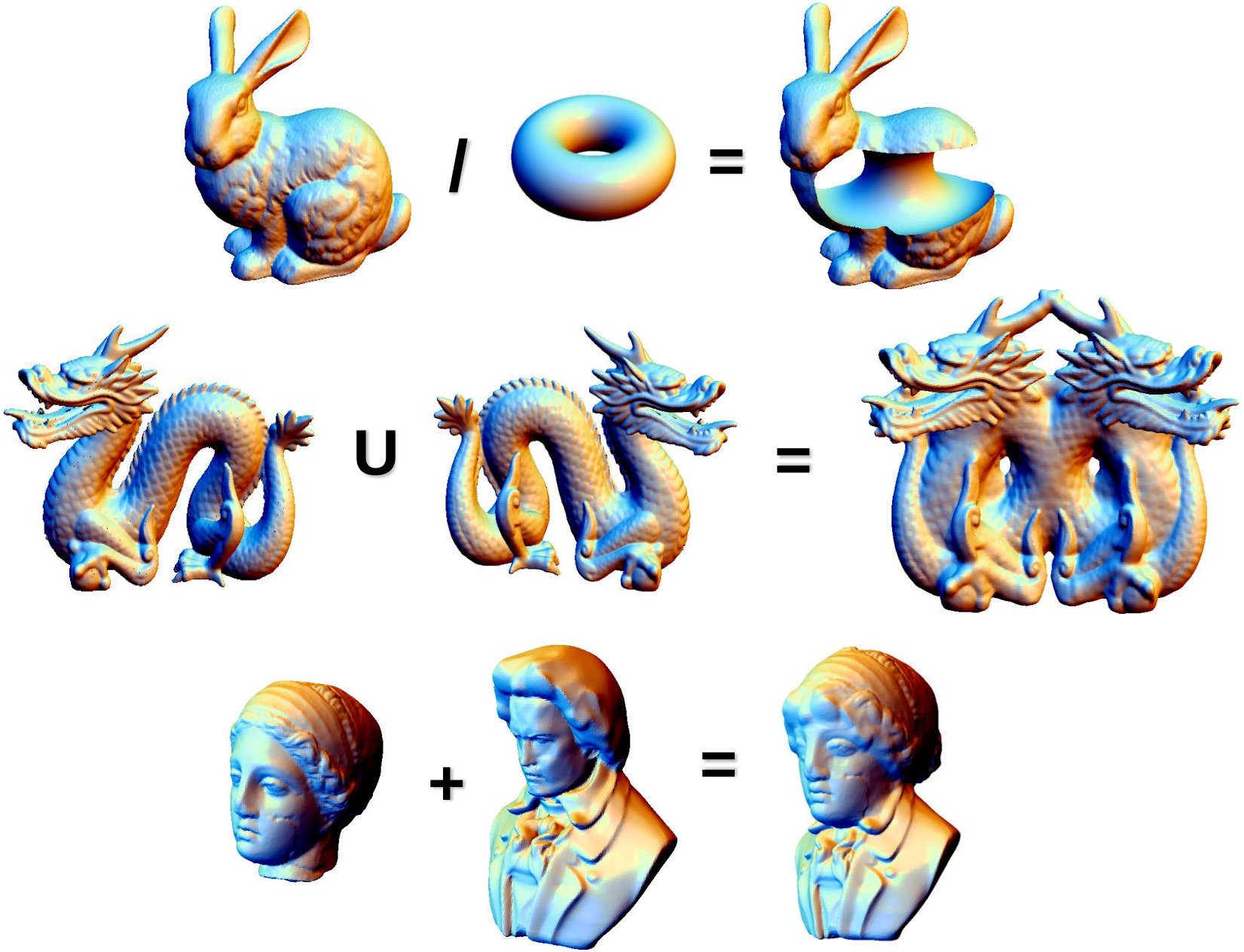
  Use B to Cut A
- Examples:
  - Use cylinders to drill holes
  - Use rectangular blocks to cut slots
  - Use half-spaces to cut planar faces
  - Use surfaces swept from curves as jigsaws, etc

Implicit Functions for Booleans

- Recall the implicit function for a solid: F(x,y,z)
- Boolean operations are replaced by arithmetic
  - MAX replaces And (intersection)
  - MIN replaces OR (union)
  - MINUS replaces NOT(unary subtraction)
- Thus
  - F(Intersect(A,B)) = MAX(F(A),F(B))
  - F(Union(A,B)) = MIN(F(A),F(B))
  - F(Subtract(A,B)) = MAX(F(A), -F(B))

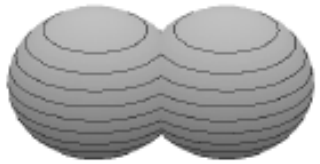


Shape Modeling with Implicits



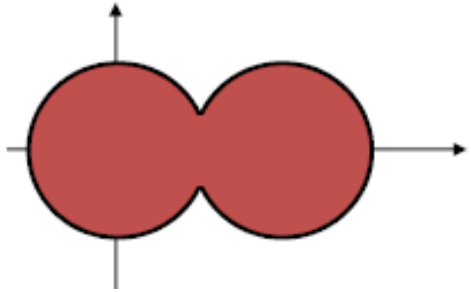
Implicit Surfaces -- Smooth set operation

- Standard operations: union and intersection



$$\bigcup_i g_i(\mathbf{p}) = \min g_i(\mathbf{p})$$

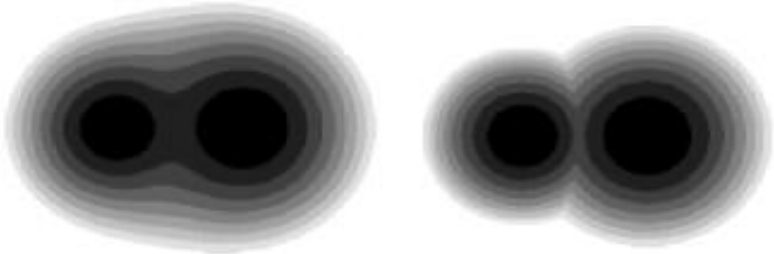
$$\bigcap_i g_i(\mathbf{p}) = \max g_i(\mathbf{p})$$



- In many cases, smooth blending is desired
  - Pasko and Savchenko [1994]

$$g \cup f = \frac{1}{1+\alpha}\left(g + f - \sqrt{g^2 + f^2 - 2\alpha g f}\right)$$

$$g \cap f = \frac{1}{1+\alpha}\left(g + f + \sqrt{g^2 + f^2 - 2\alpha g f}\right)$$



$\alpha = 0$ $\alpha = 1$

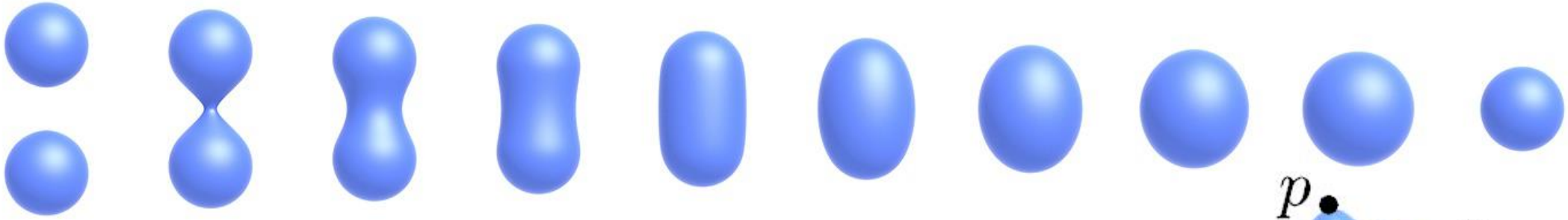
  - alpha

$$\lim_{\alpha \to 1} g \cup f = \frac{1}{2}\left(g + f - \sqrt{(g-f)^2}\right) = \frac{g+f}{2} - \frac{|g-f|}{2} = \min(g, f)$$

$$\lim_{\alpha \to 1} g \cap f = \frac{1}{2}\left(g + f + \sqrt{(g-f)^2}\right) = \frac{g+f}{2} + \frac{|g-f|}{2} = \max(g, f)$$

Blobby Surfaces (Implicit)

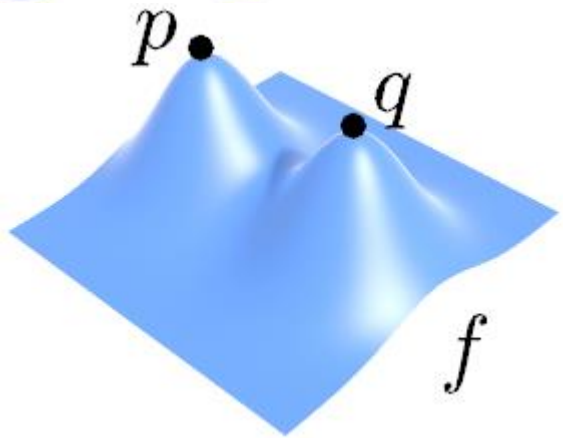
- **Instead of Booleans, gradually blend surfaces together:**

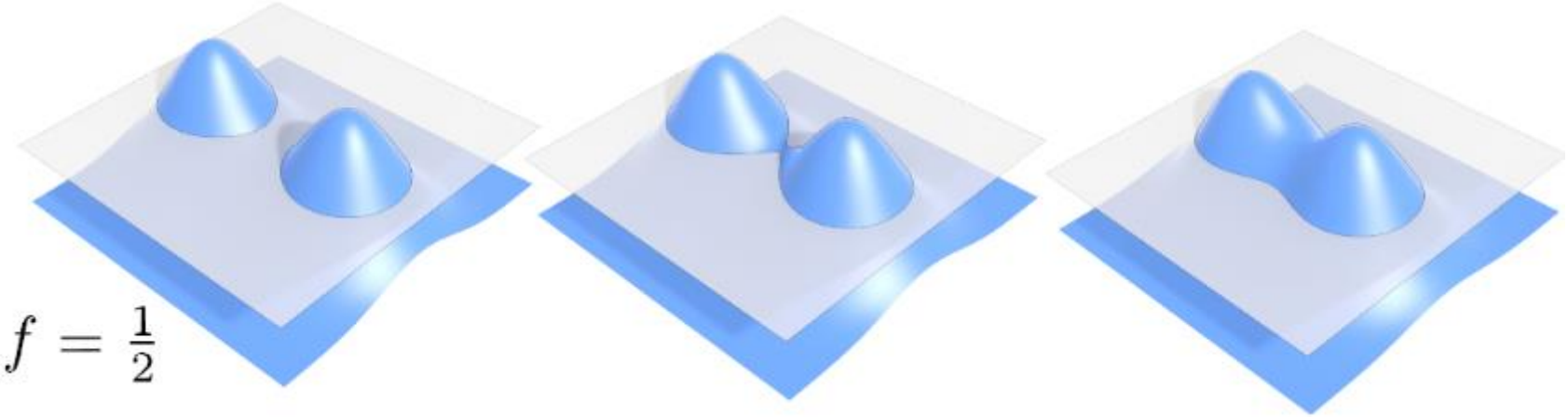


- **Easier to understand in 2D:**

$\phi_p(x) := e^{-|x-p|^2}$ **(Gaussian centered at p)**

$f := \phi_p + \phi_q$ **(Sum of Gaussians centered at different points)**





Shape Modeling with Implicits

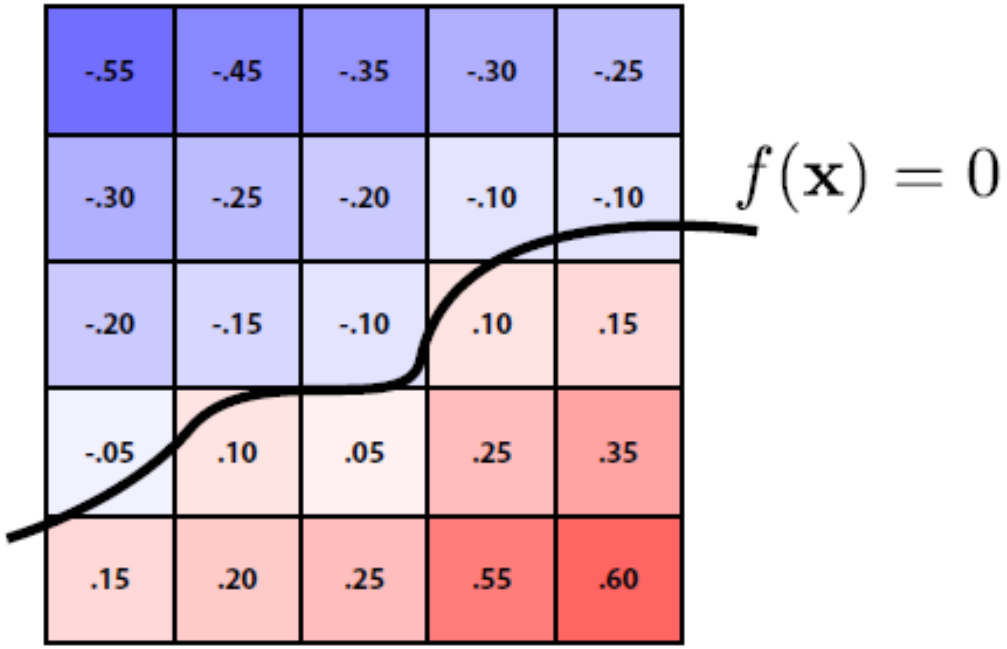
$$f(\boldsymbol{x}) = (1-t)\, f_1(\boldsymbol{x}) + t\, f_2(\boldsymbol{x})$$

$f_1(\boldsymbol{x}) = 0$

$f_2(\boldsymbol{x}) = 0$

Level Set Methods (Implicit)

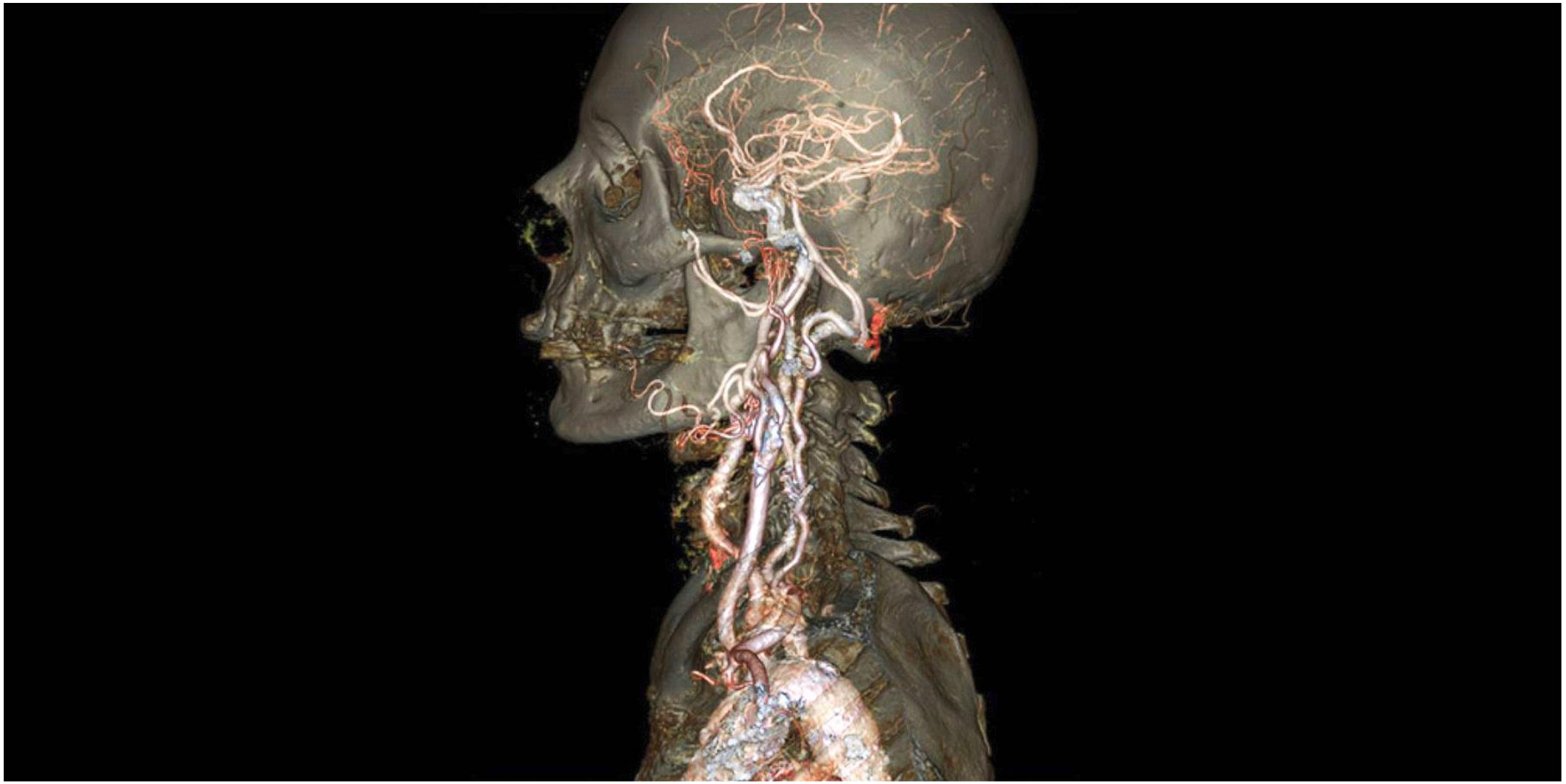
- Implicit surfaces have some nice features (e.g., merging/splitting)
- But, hard to describe complex shapes in closed form
- Alternative: store a grid of values approximating function



- Surface is found where *interpolated* values equal zero
- Provides much more explicit control over shape (like a texture)

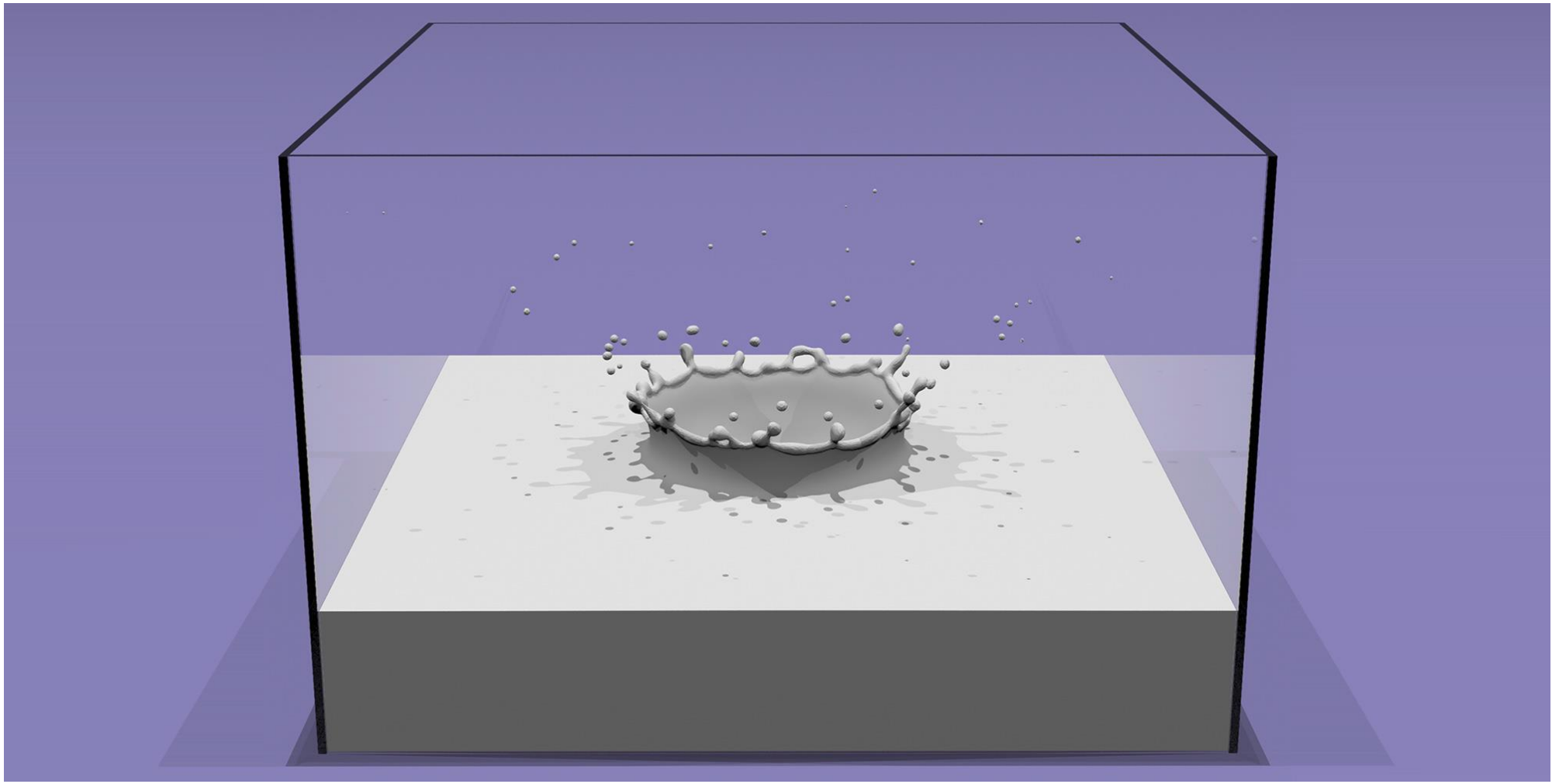
Level Sets from Medical Data (CT, MRI, etc.)

- **Level sets encode, e.g., constant tissue density**



Level Sets in Physical Simulation

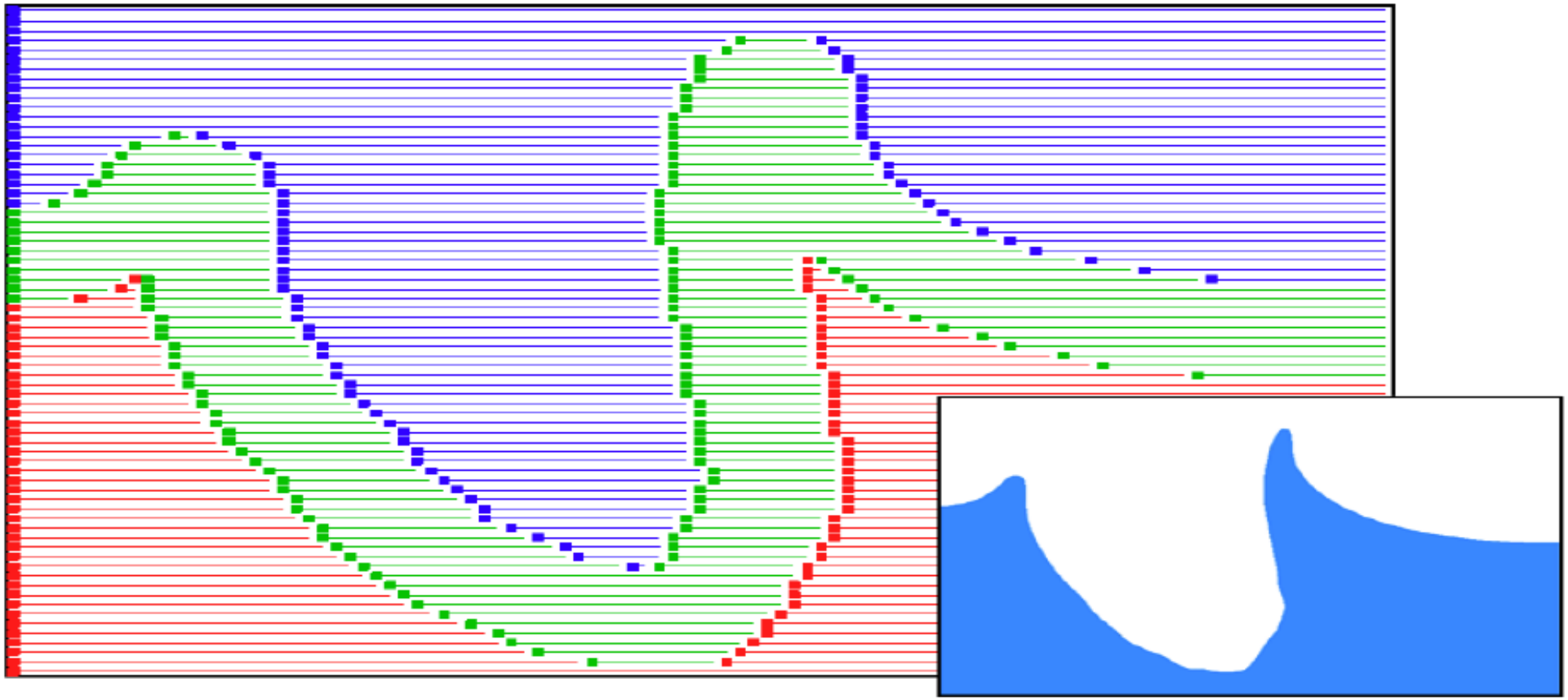
- **Level set encodes distance to air-liquid boundary**



See http://physbam.stanford.edu

Level Set Storage

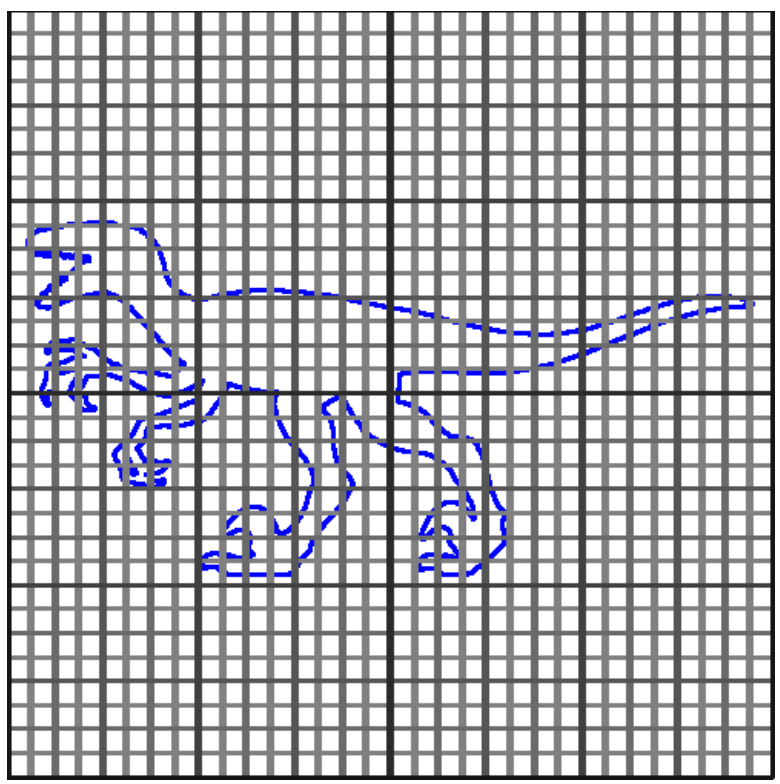
- **Drawback: storage for 2D surface is now O(n3)**
- **Can reduce cost by storing only a narrow band around surface:**

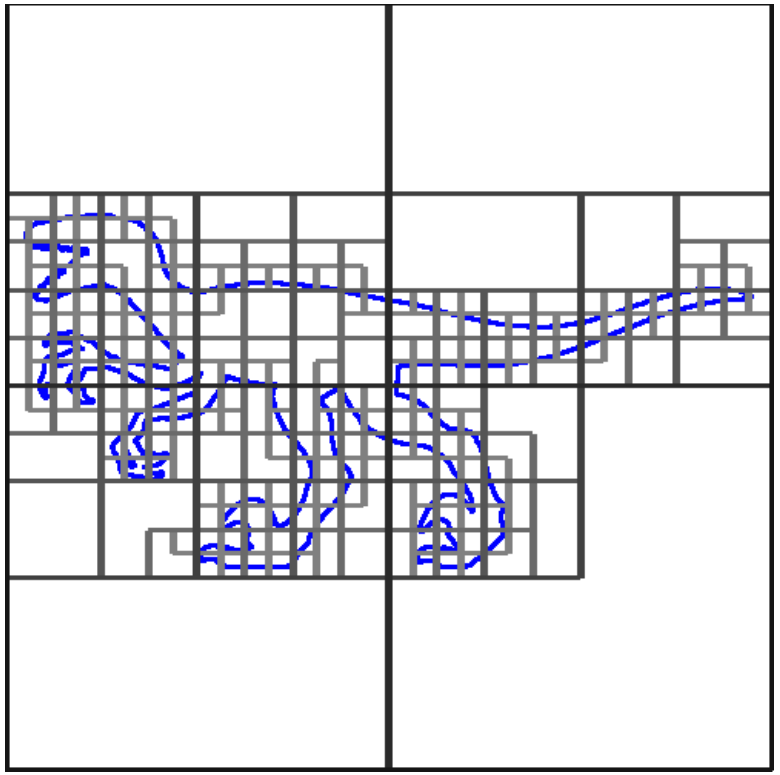


Implicit Representations

Adaptive Grids: Quadtree

In practice, we may only need a high-precision representation of the implicit function near the surface, so represent the function over an adaptive grid.

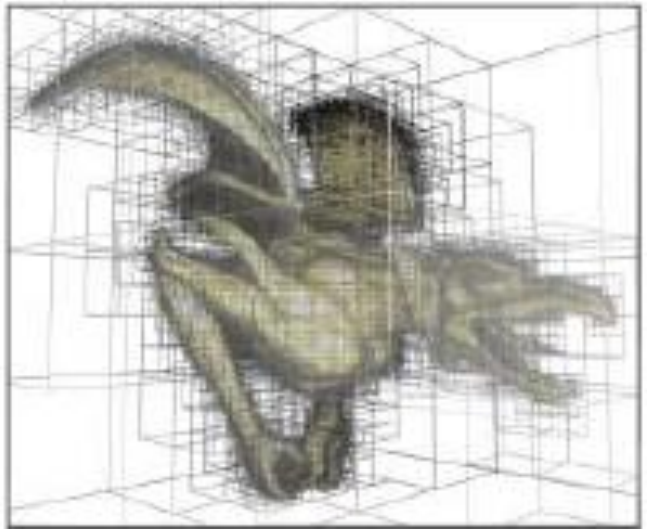


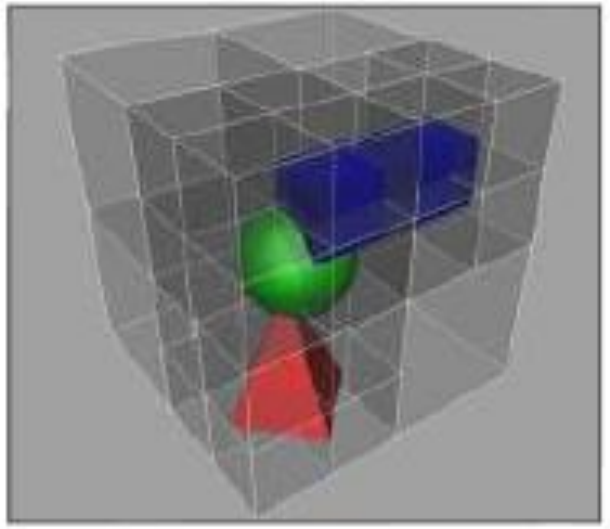


Octree

- A hierarchical tree build by sequential subdivision (8) of a occupied cells.
- Adaptive, i.e. only splits when too many points in cell
- Proximity search by (recursive) tree traversal and distance to neighboring cells
- Tree might not be balanced
- Widely used for complicated scenes that need faster processing and lower accuracy

E.g. Collision detection in real-time simulation or animation





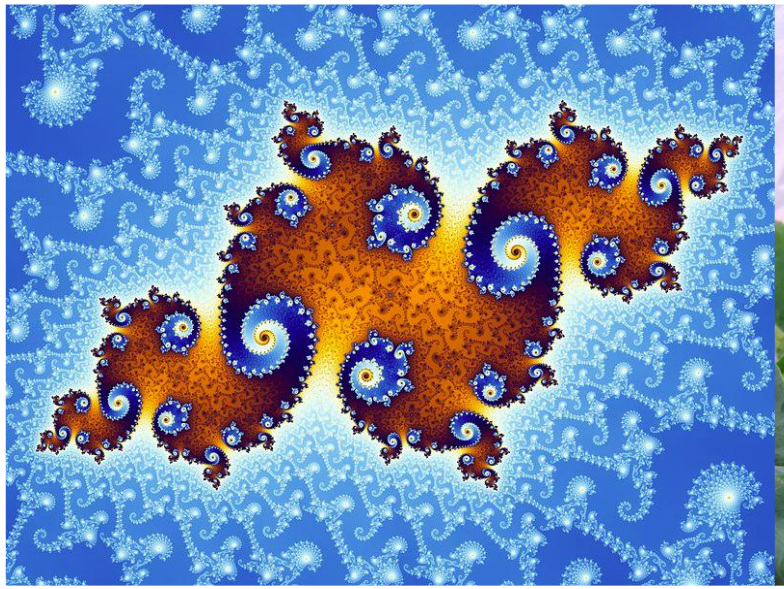
How to draw implicit surfaces?

- It’s easy to ray trace implicit surfaces
  - because of that easy intersection test
- Volume Rendering can display them

- Convert to polygons: the Marching Cubes algorithm
  - Divide space into cubes
  - Evaluate implicit function at each cube vertex
  - Do root finding or linear interpolation along each edge
  - Polygonize on a cube-by-cube basis

Fractals (Implicit)

- No precise definition; exhibit self-similarity, detail at all scales
- New “language” for describing natural phenomena
- Hard to control shape!



Implicit Representations - Pros & Cons

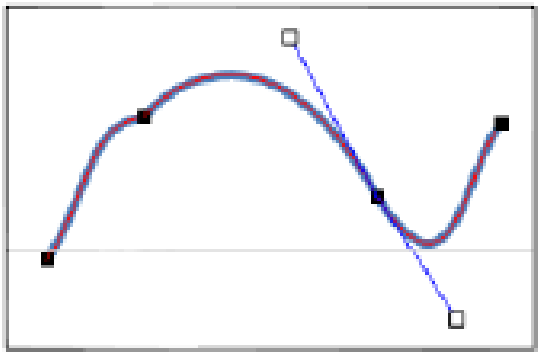
- **Pros:**
  - **description can be very compact (e.g., a polynomial)**
  - **easy to determine if a point is in our shape (just plug it in!)**
  - **other queries may also be easy (e.g., distance to surface)**
  - **for simple shapes, exact description/no sampling error**
  - **easy to handle changes in topology (e.g., fluid)**
- **Cons:**
  - **expensive to find all points in the shape (e.g., for drawing)**
  - ***very difficult to model complex shapes***

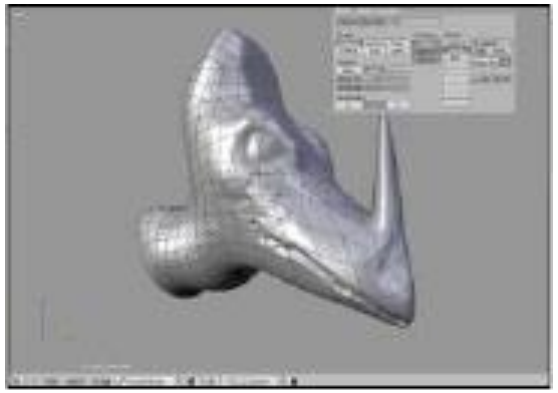
Shape representation

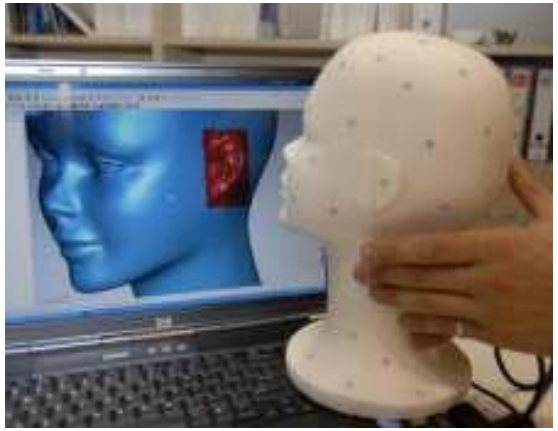
Where does the shape come from?

- **Modeling “by hand”**
  - Higher-level representations,
  - Amenable to modification, control

- **Acquired real-world objects**
  - Discrete sampling
  - Points, meshes



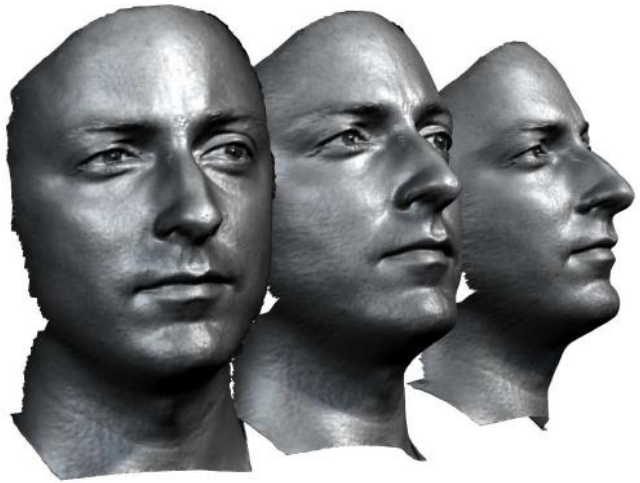




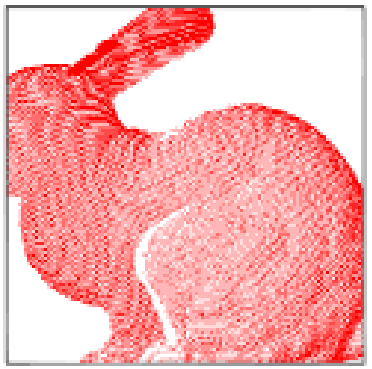
Shape Acquisition

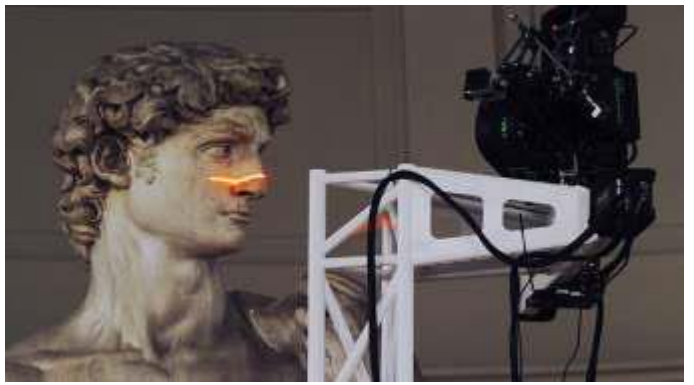
Sampling of real world objects

- Scanners
  - Laser
  - Depth imaging
- Properties & Operations
  - Potentially noisy, with outliers
  - Registration of multiple images
  - Non-uniform sampling, sparse, holes



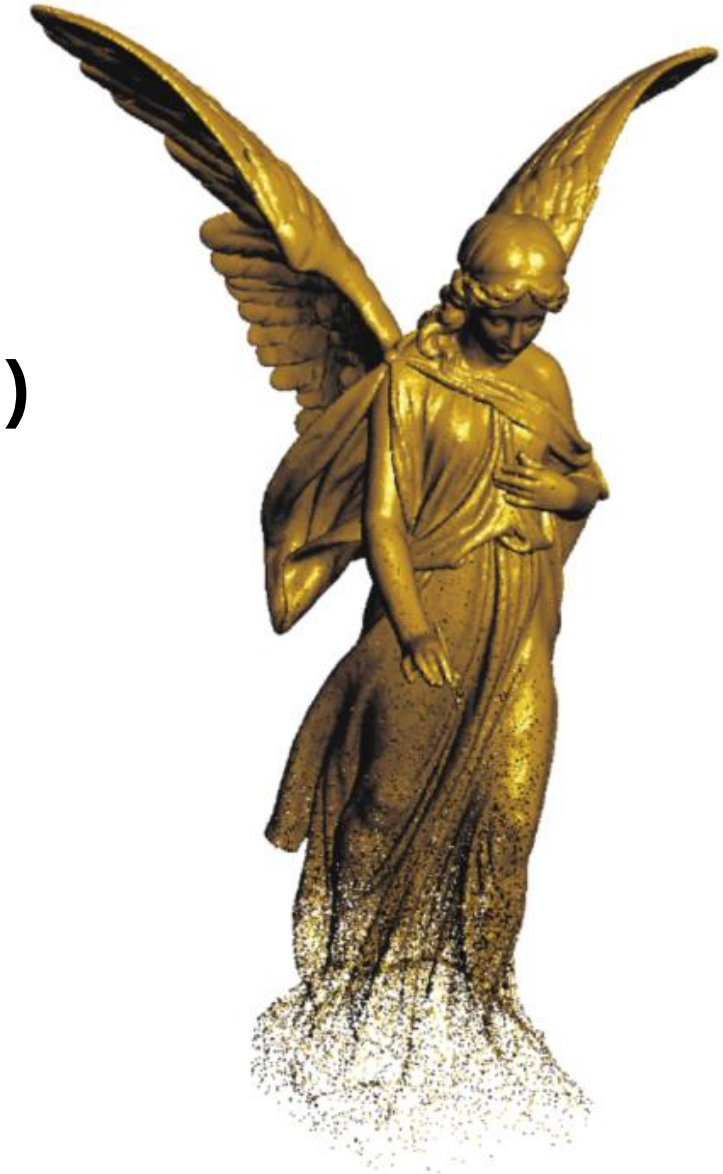
180 frames per second (From David Gu)

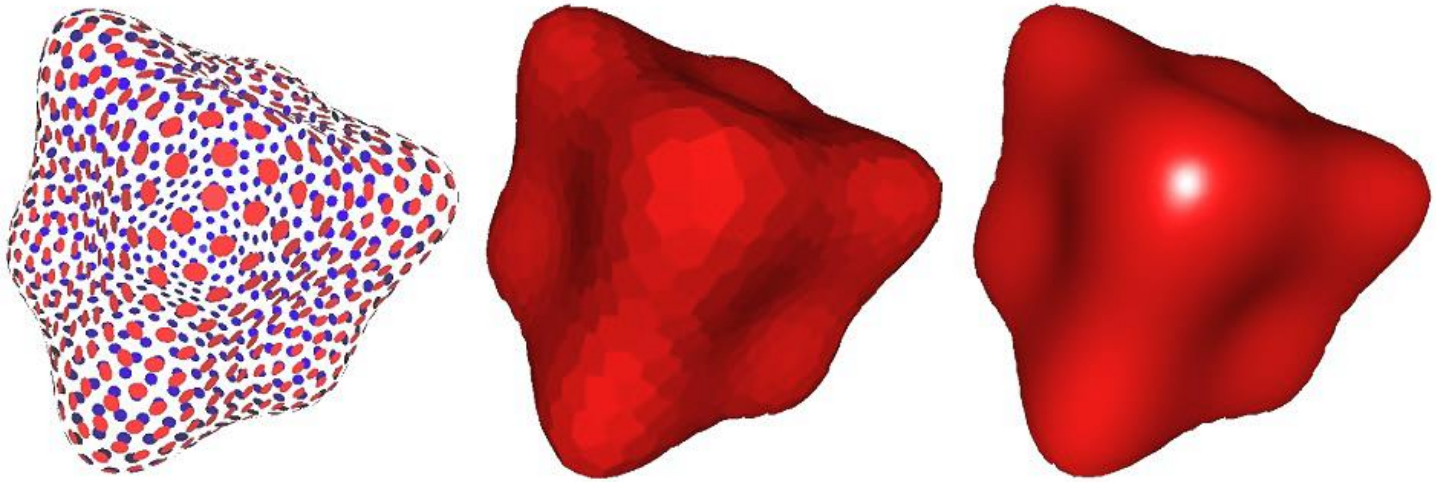




What about explicit representations?

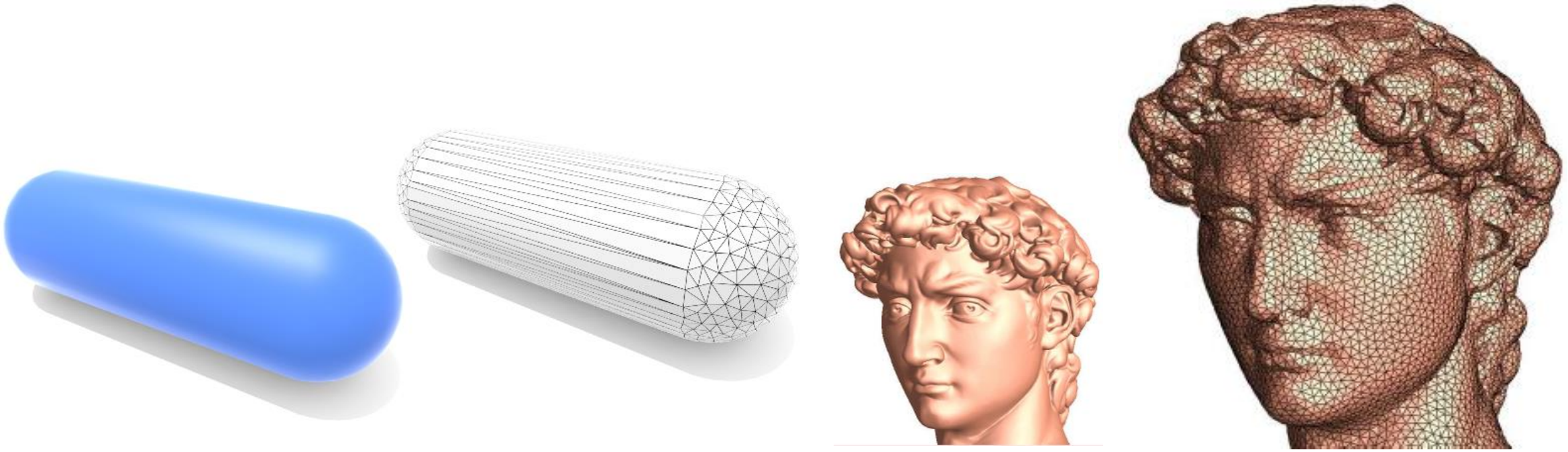
- **Easiest representation: list of points (x,y,z)**
- **Often augmented with *normals***
- **Easily represent any kind of geometry**
- **Useful for LARGE datasets (>>1 point/pixel)**
- **Difficult to draw in undersampled regions**
- **Hard to do processing / simulation**





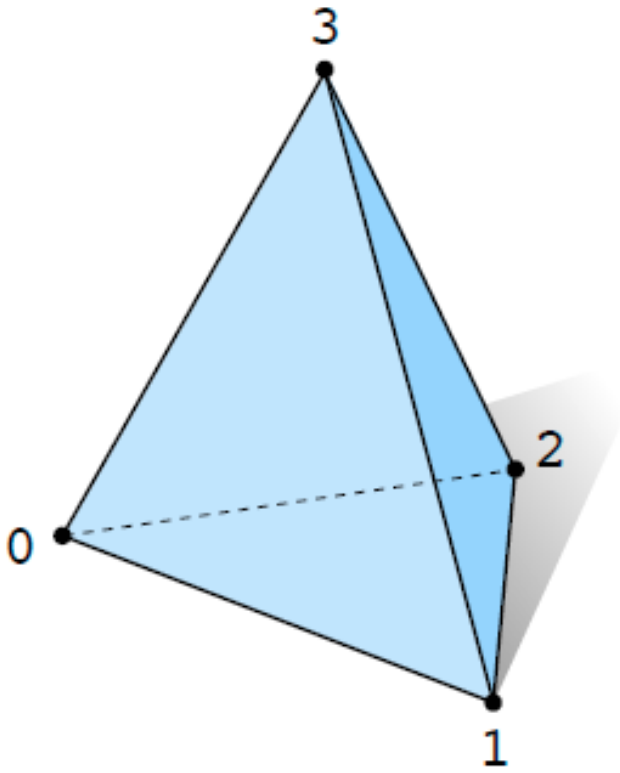
Polygon Mesh (Explicit)

- Store vertices *and* polygons (most often triangles or quads)
- Easier to do processing/simulation, adaptive sampling
- More complicated data structures
- Perhaps most common representation in graphics



Triangle Mesh (Explicit)

- Store vertices as triples of coordinates (x,y,z)
- Store triangles as triples of indices (i,j,k)
- E.g., tetrahedron:



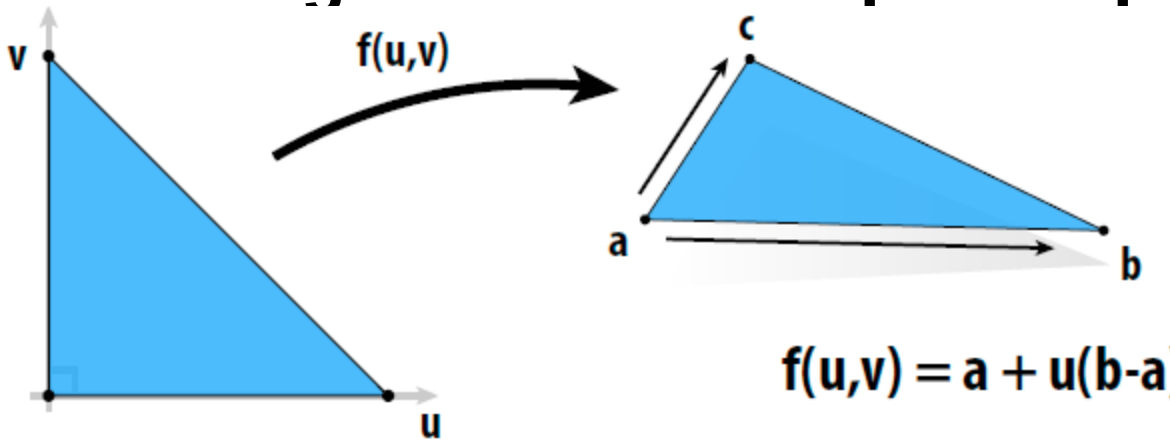
VERTICES

	x	y	z
0:	-1	-1	-1
1:	1	-1	1
2:	1	1	-1
3:	-1	1	1

TRIANGLES

i	j	k
0	2	1
0	3	2
3	0	1
3	1	2

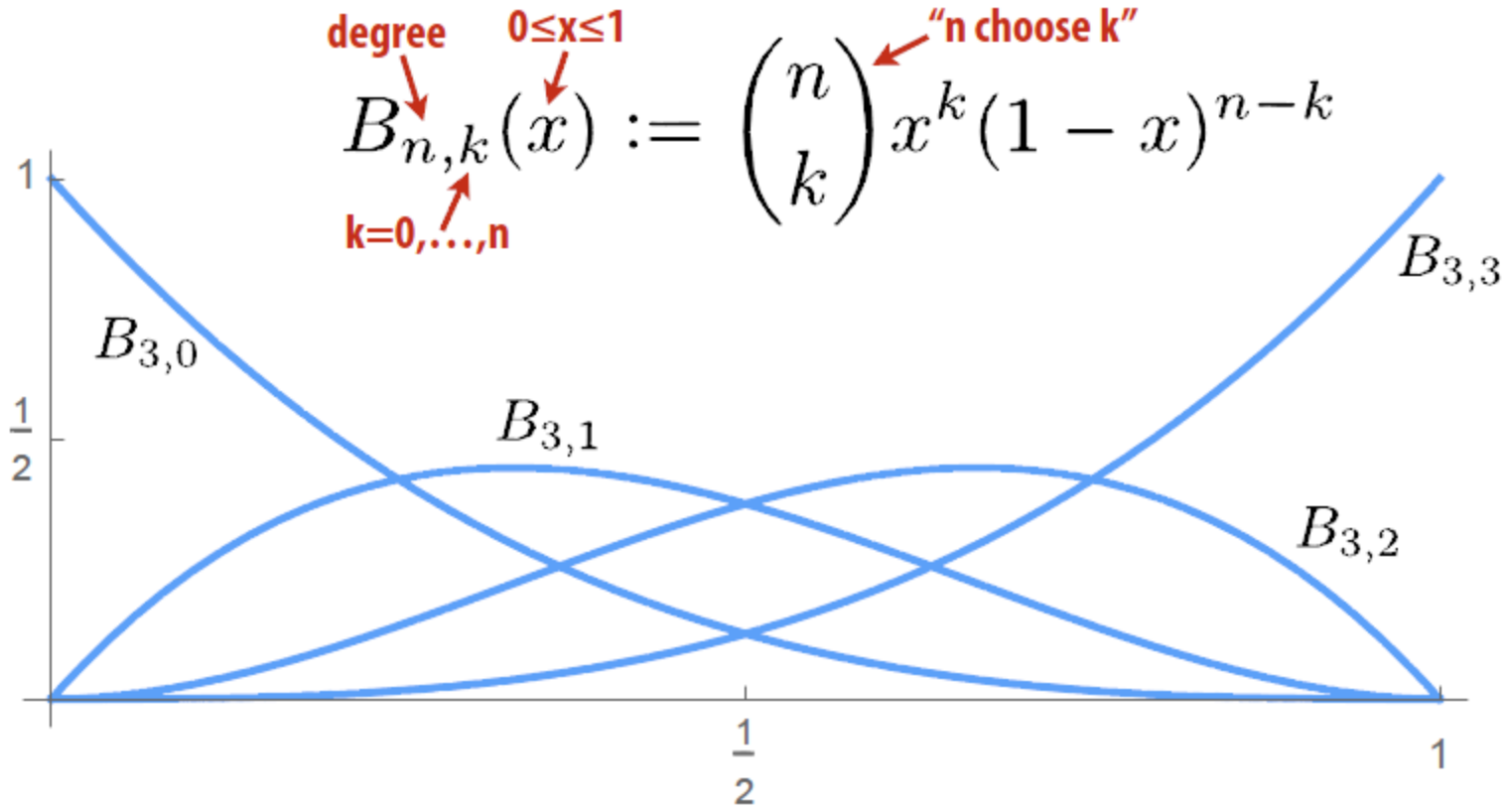
- Can think of triangle as *affine* map from plane into space:



$$f(u,v) = a + u(b-a) + v(c-a)$$

Bernstein Basis

- Why limit ourselves to just affine functions?
- More flexibility by using higher-order polynomials
- Instead of usual basis (1, x, x2, x3, ...), use Bernstein basis:



Bézier Curves (Explicit)

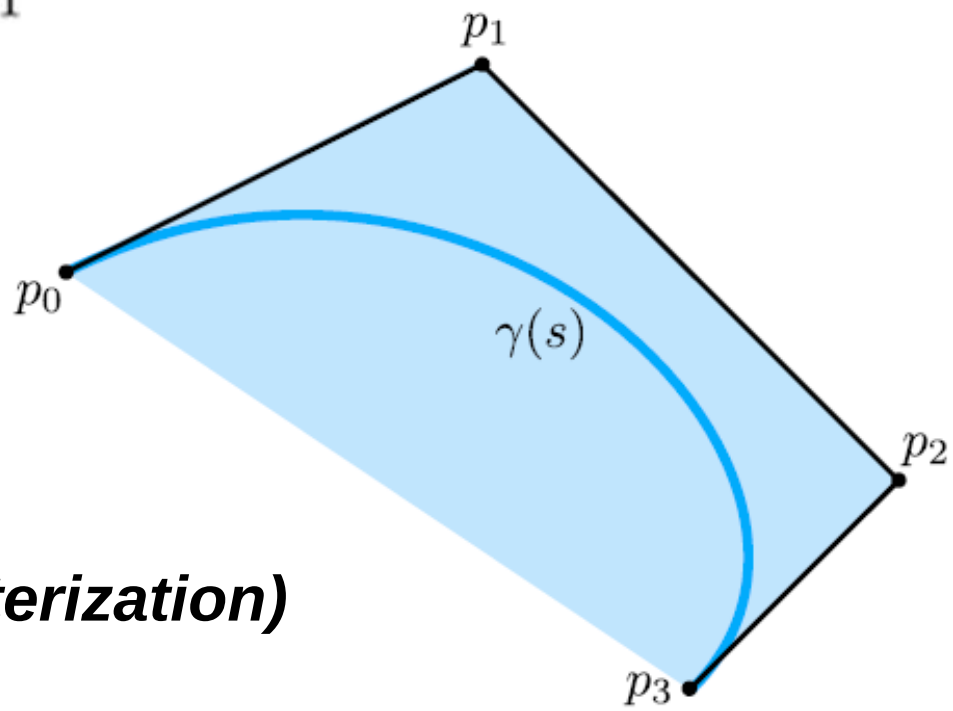
- **A Bézier curve is a curve expressed in the Bernstein basis:**

$$\gamma(s) := \sum_{k=0}^{n} B_{n,k}(s) p_k$$

control points

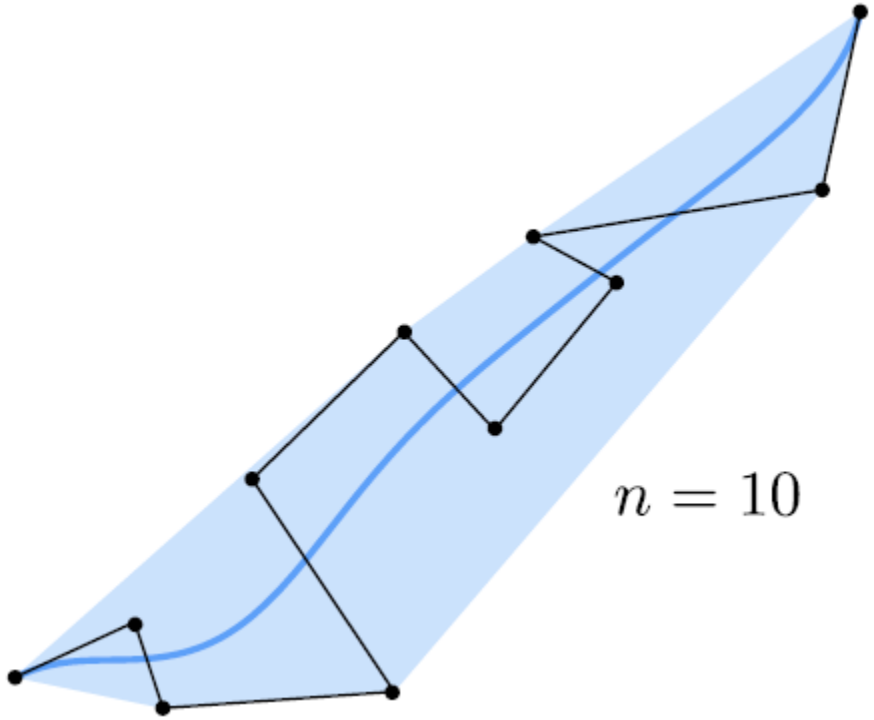
p_1

- **For n=1, just get a line segment!**
- **For n=3, get “cubic Bézier”:**
- **Important features:**
  1. **interpolates endpoints**
  2. **tangent to end segments**
  3. **contained in convex hull *(nice for rasterization)***



Higher-order Bézier Curves?

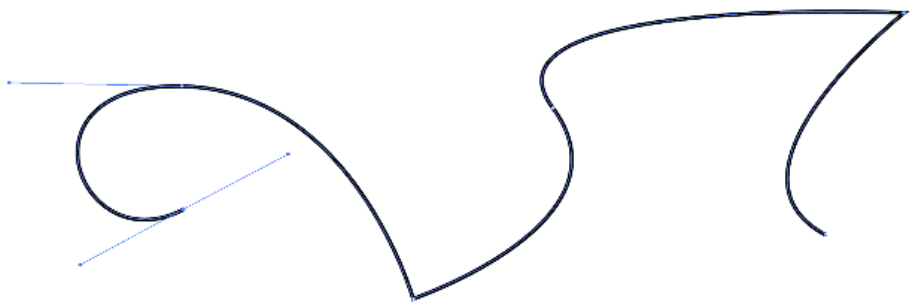
- What if we want a more interesting curve?
- High-degree Bernstein polynomials don’t interpolate well:

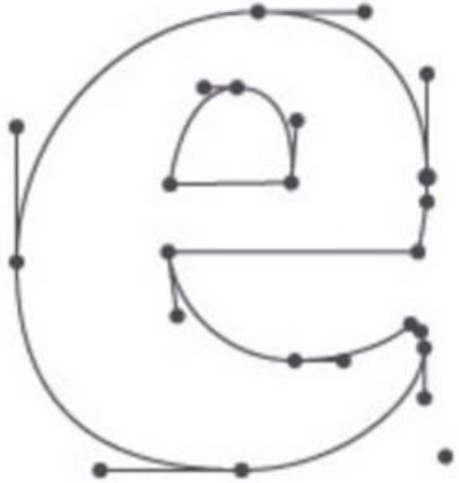


Very hard to control!

B-Spline Curves (Explicit)

- Instead, use many low-order Bézier curve (B-spline)
- Widely-used technique in software (Illustrator, Inkscape, etc.)





- Formally, piecewise Bézier curve:

B-spline

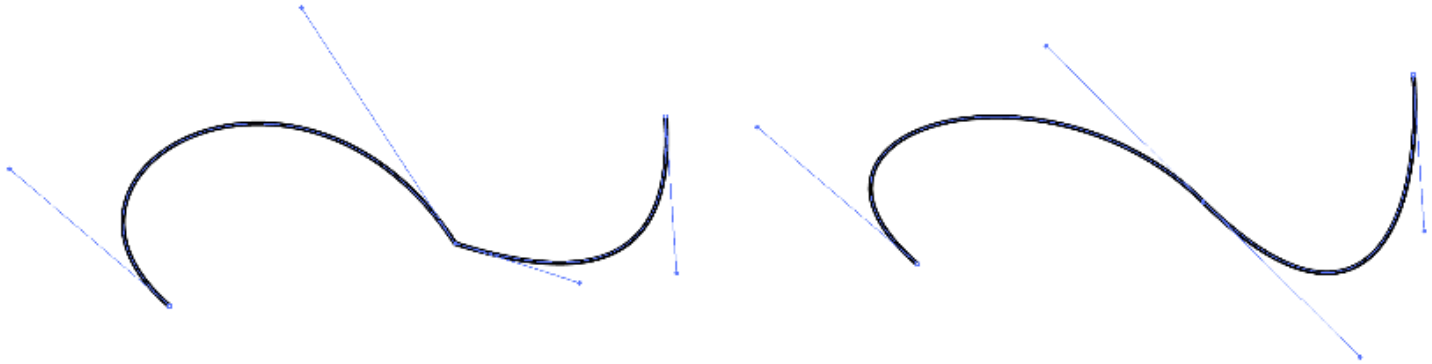
$$\gamma(u) := \gamma_i \left(\frac{u - u_i}{u_{i+1} - u_i} \right), \qquad u_i \le u < u_{i+1}$$

Bézier

- Location of ui parameters are called “knots”

B-Splines — tangent continuity

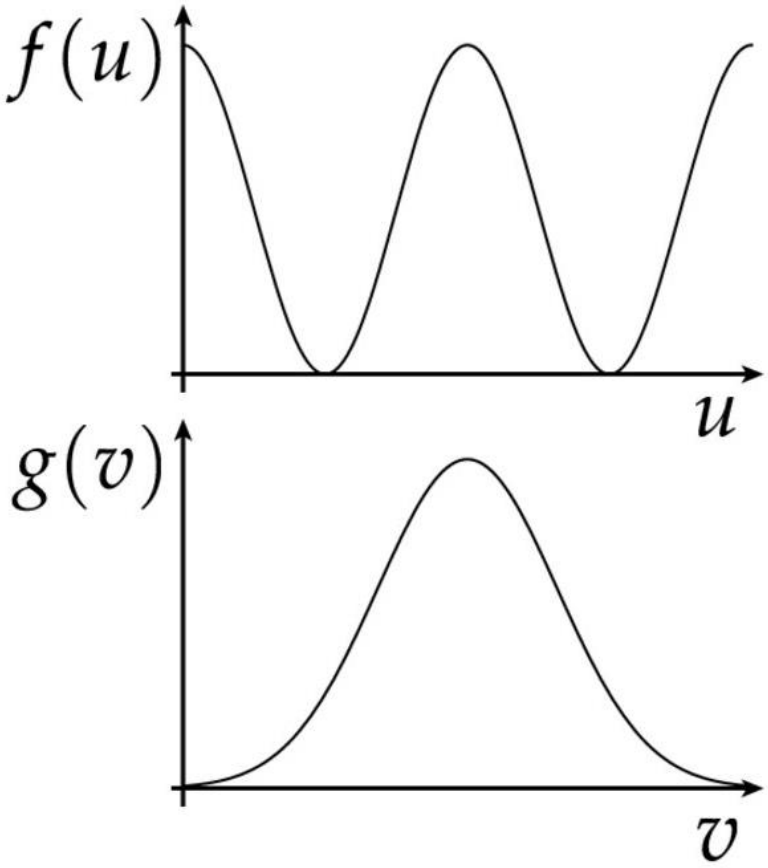
- To get “seamless” curves, want *tangents* to line up:

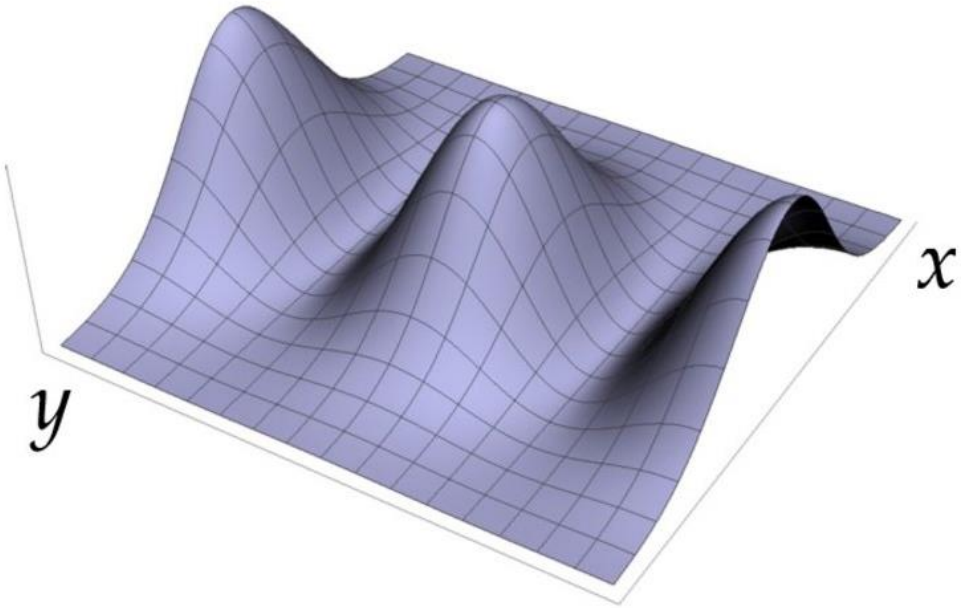


- Ok, but how?
- Each curve is cubic: u3p0 + 3u2(1-u)p1 + 3u(1-u)2p2 + (1-u)3p3
- Q: How many degrees of freedom in a single cubic Bézier
- Tangents are difference between first two & last two points
- Q: How many degrees of freedom per B-spline segment?
- Q: Could you do this with quadratic Bézier? Linear Bézier?

Tensor product

- Use a pair of curves to get a surface
- Value at any point (u,v) given by product of a curve f at u and a curve g at v

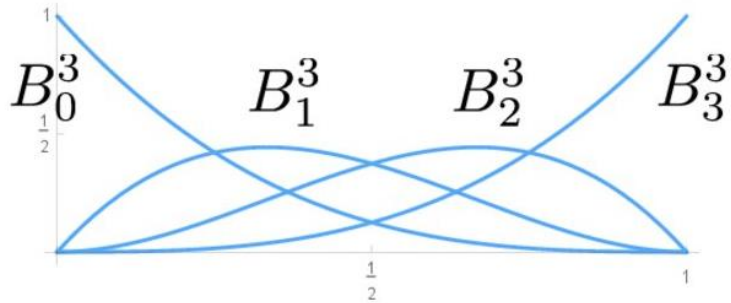




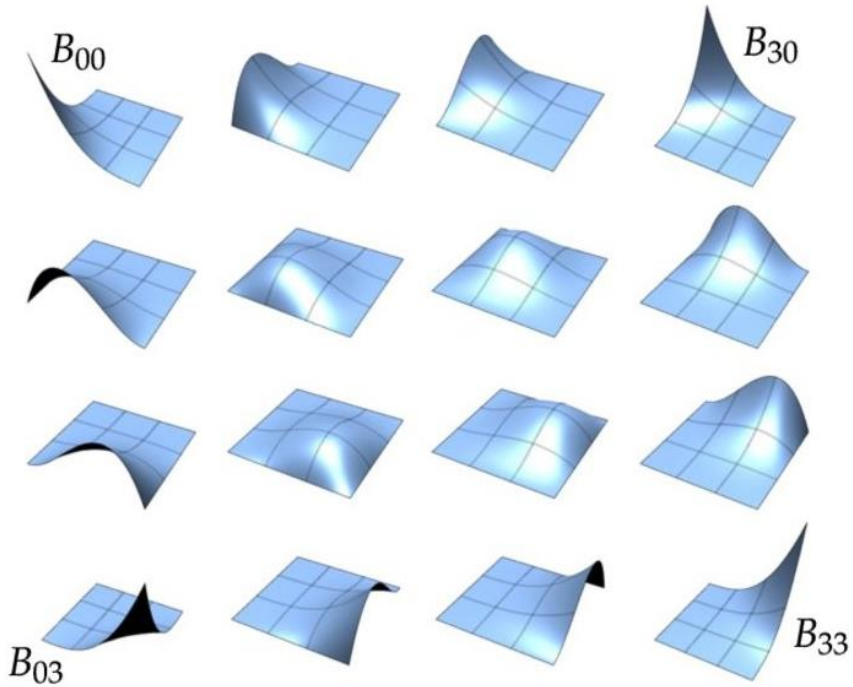
$$(f \otimes g)(u,v) := f(u)g(v)$$

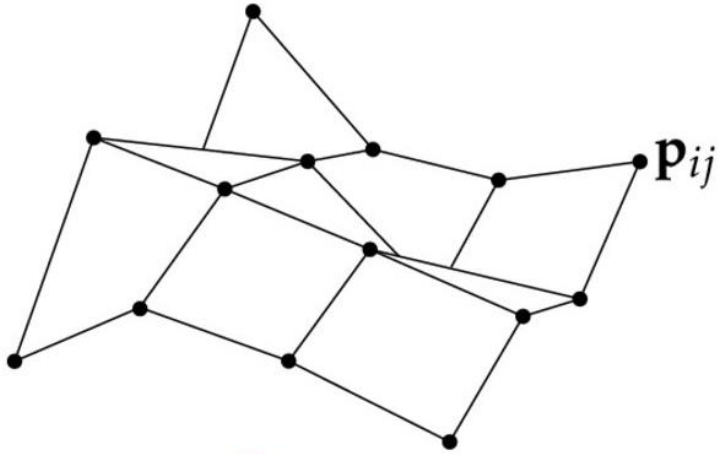
Bezier patches

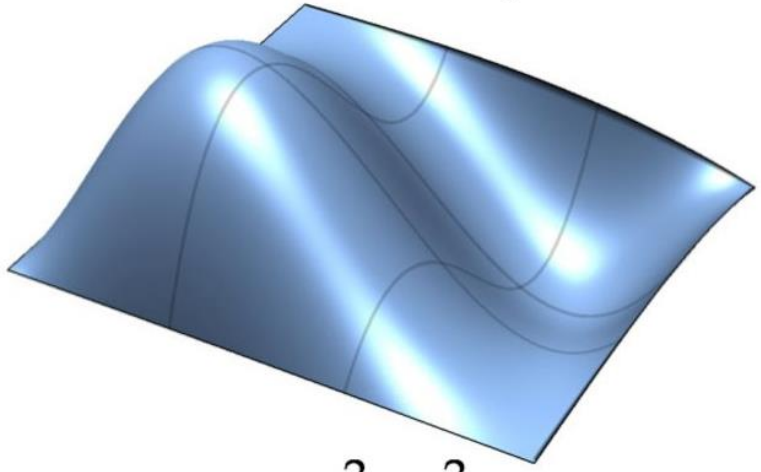
- Bezier patch is a sum of (tensor) product of Bernstein bases.



$$B^3_{i,j}(u,v) := B^3_i(u)B^3_j(v)$$



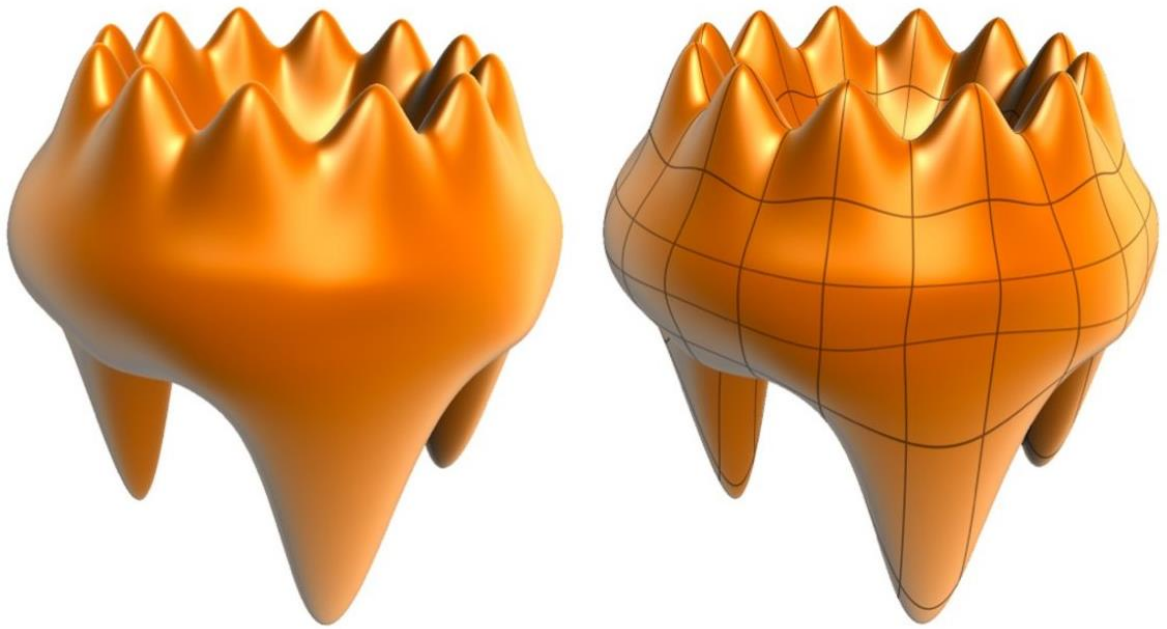




$$S(u,v) := \sum_{i=0}^{3}\sum_{j=0}^{3} B^3_{i,j}(u,v)\mathbf{p}_{ij}$$

Bezier surface

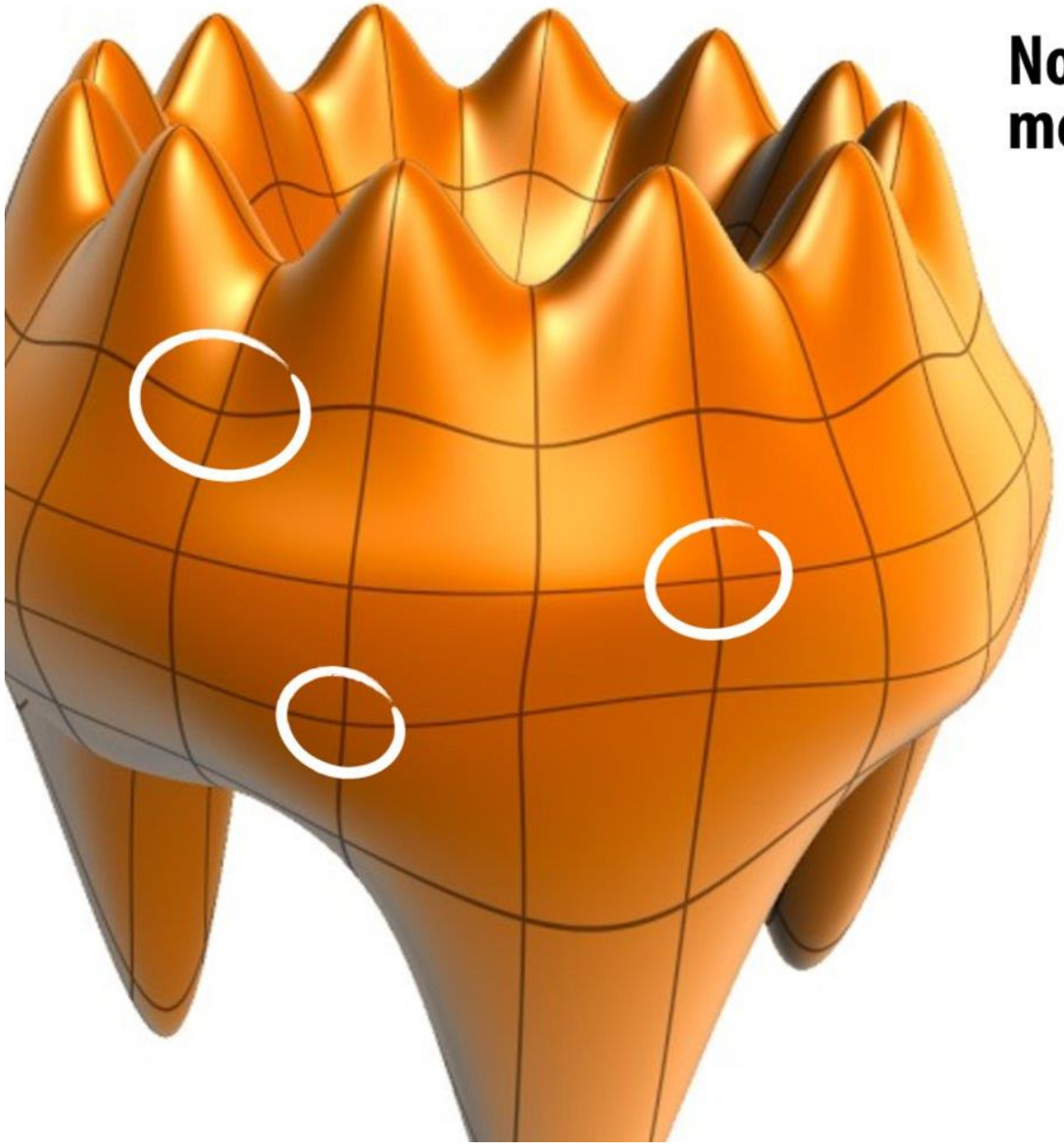
- Just as we connect Bezier curves, can connect Bezier patches to get a surface



- Very easy to draw: just dice each patch into a regular (u,v) grid!
- Q: Can we always get tangent continuity?

(Think: How many constraints? How many degrees of freedom?)

Bezier patches are too simple



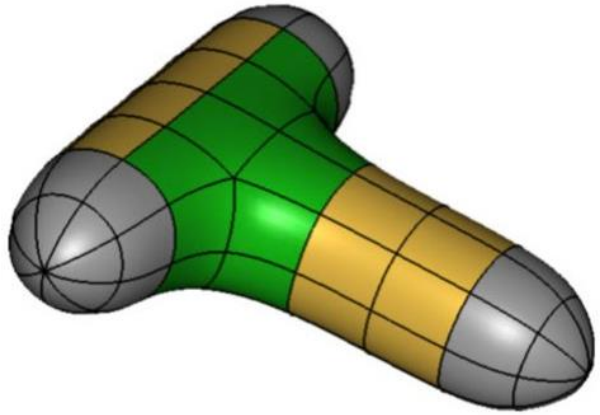
Notice that exactly four patches meet around *every* vertex!

In practice, far too constrained.

To make interesting shapes (with good continuity), we need patches that allow more interesting connectivity...

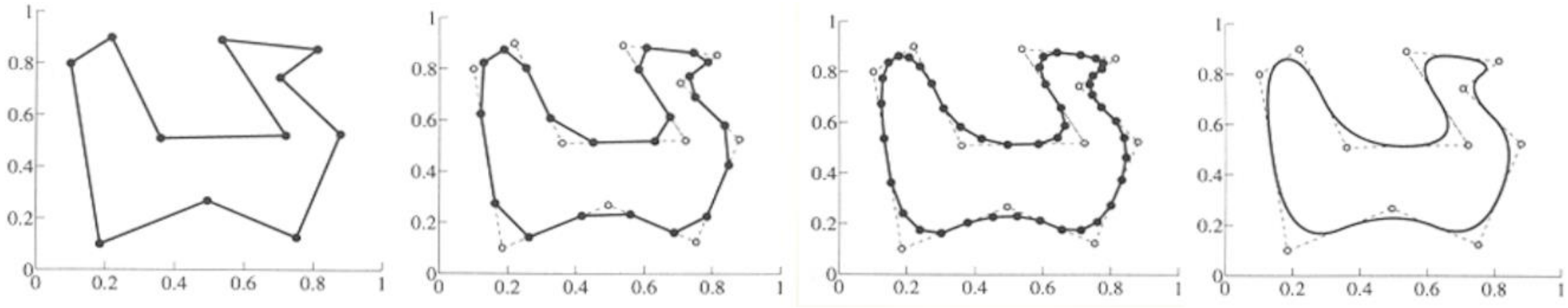
Spline patch schemes

- There are many alternatives
  - NURBS, Gregory, Pm, polar …
- Tradeoffs:
  - Degree of freedoms
  - Continuity
  - Difficulty of editing
  - Cost of evaluation
  - Generality
  - …
- As usual: pick the right tool for the job



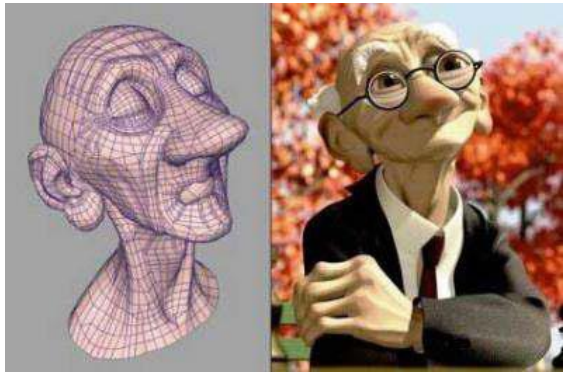
Subdivision (Explicit)

- **Alternative starting point for curves: *subdivision***
- **Start with control curve**
- **Insert new vertex at each edge midpoint**
- **Update vertex positions according to fixed rule**
- **For careful choice of averaging rule, yields smooth curve**
  - **Some subdivision schemes correspond to well-known spline schemes!**



Subdivision Surfaces (Explicit)

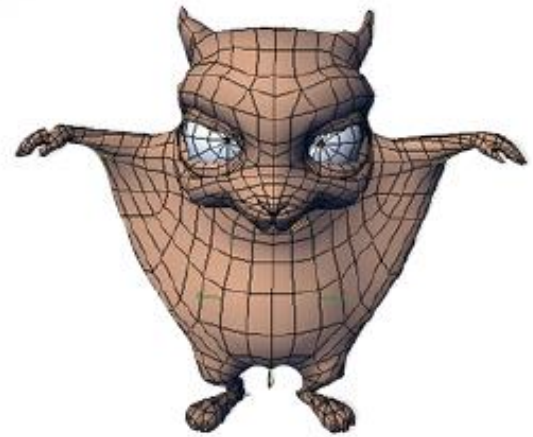
- Start with coarse polygon mesh (“control cage”)
- Subdivide each element
- Update vertices via local averaging
- Many possible rule:
  - Catmull-Clark (quads)
  - Loop (triangles)
  - ...



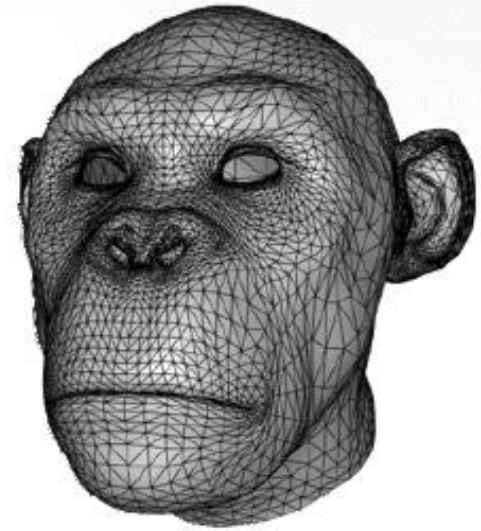
- Common issues:
  - interpolating or approximating?
  - continuity at vertices?
- Easier than splines for modeling; harder to guarantee continuity

Shape Representation

point based

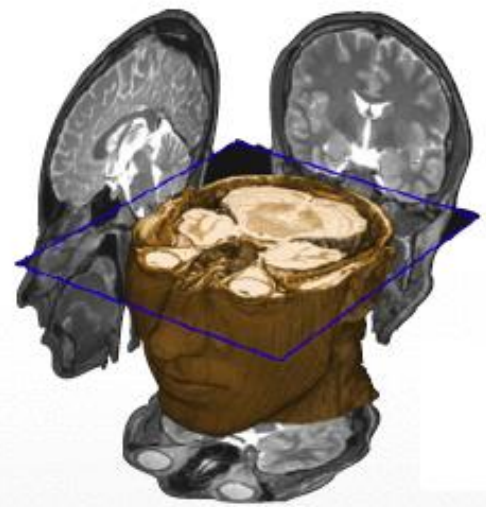


quad mesh

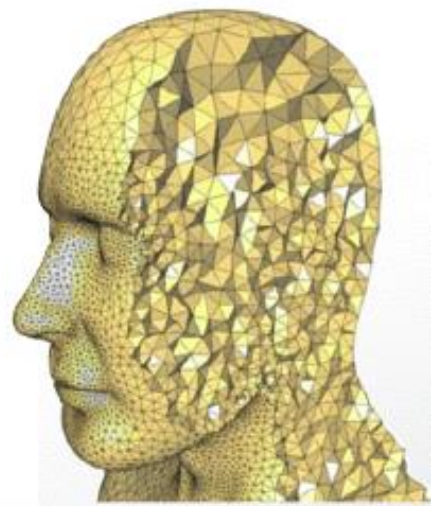


triangle mesh

implicit surfaces / particles



volumetric



tetrahedrons

Summary

- Implicit Surfaces
- Constructive Solid Geometry
- Parametric Surfaces
- Polygon Meshes

What do we need from shapes in Computer Graphics?

- Local control of shape for modeling
- Ability to model what we need
- Smoothness and continuity
- Ability to evaluate derivatives
- Ability to do collision detection
- Ease of rendering

No single technique solves all problems!

Resources

- read & display a mesh: jjcao_plot/eg_trisurf.m
- Read & display a huge point set (100k to 1 million points)
  - PC_processing_1.0
  - jjcao_code/tools/pcd_viewer