Computer Graphics -Inverse Kinematics

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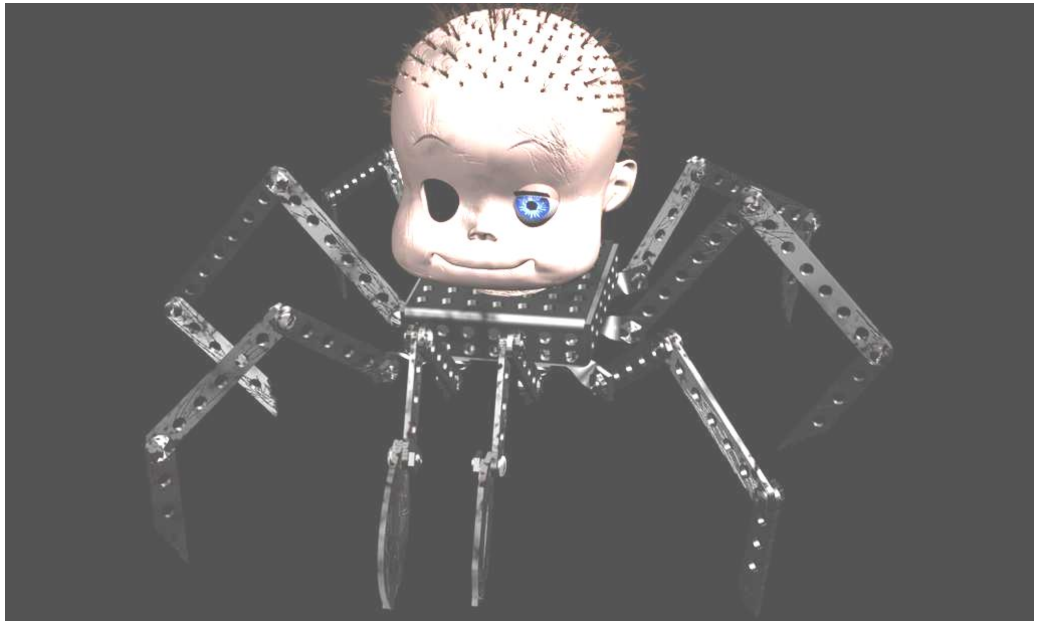
http://jjcao.github.io/ComputerGraphics/

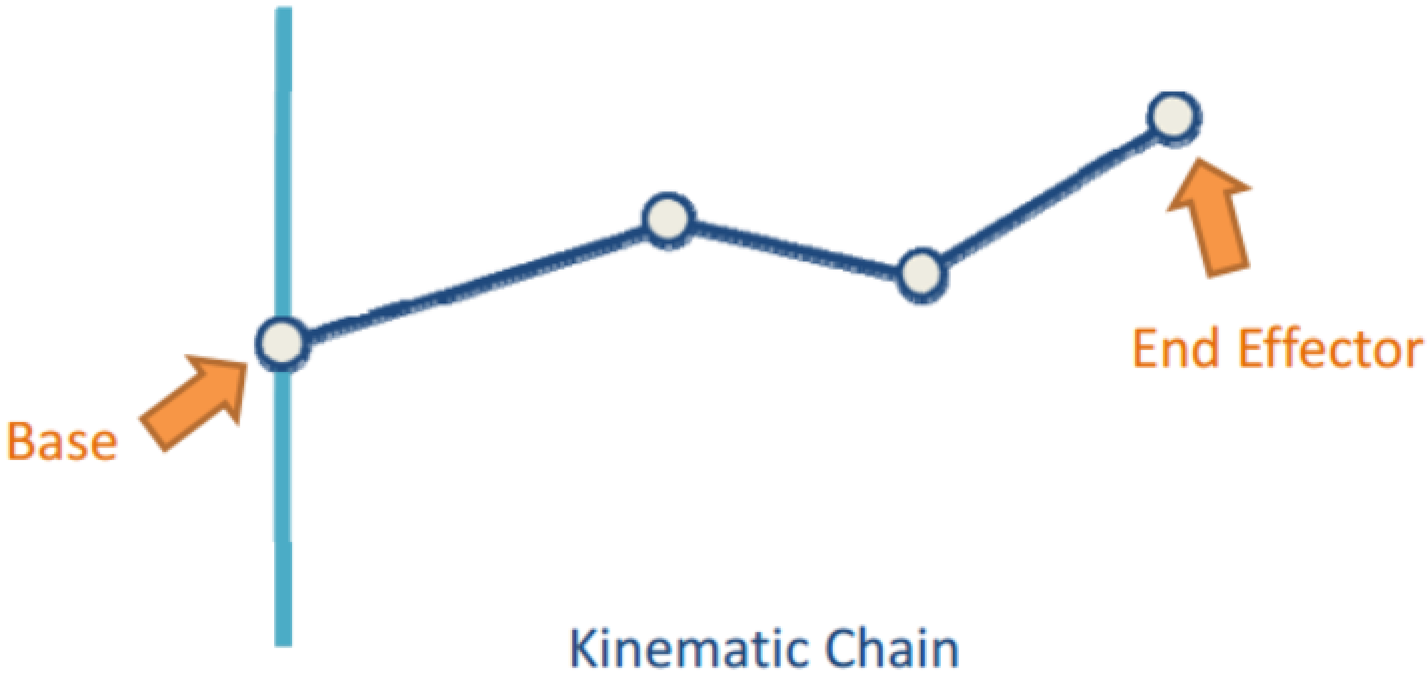


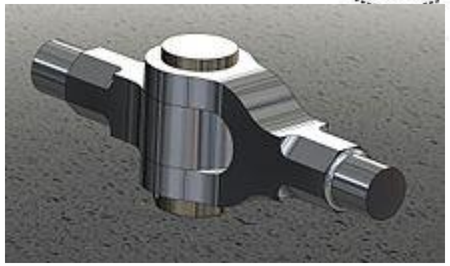
Overview

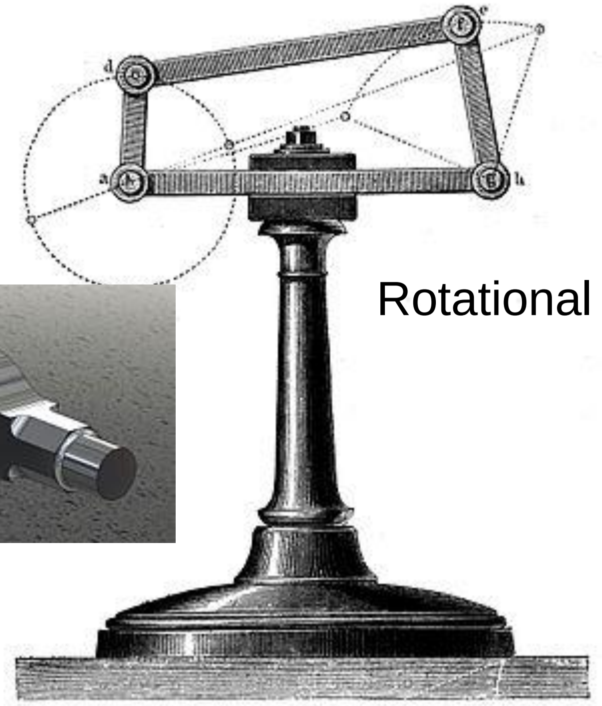
- Kinematics
- Forward Kinematics and Inverse Kinematics
- Jacobian
- Pseudoinverse of the Jacobian

Vocabulary of Kinematics

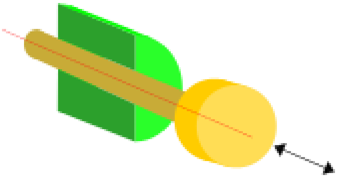
- Kinematics is the study of how things move, it describes the motion of a hierarchical skeleton structure.
- Base and End Effector.







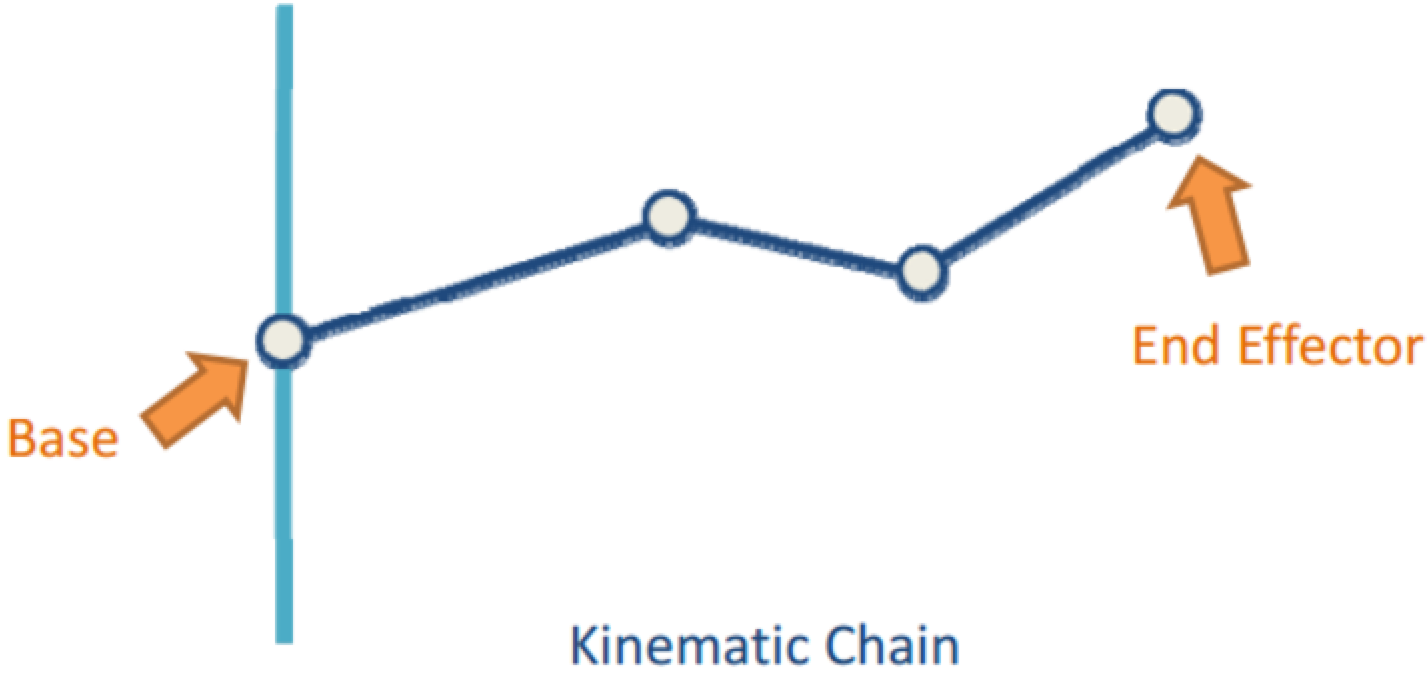
Rotational joint

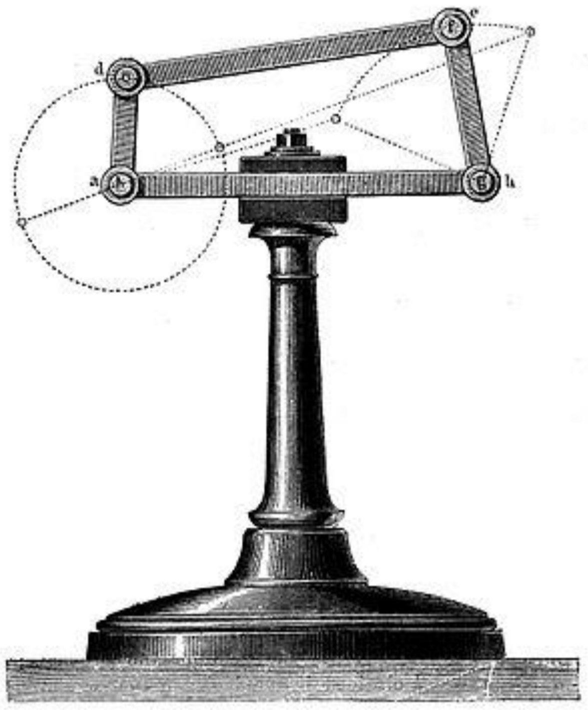


Translational/Prismatic joint

Kinematic Chains

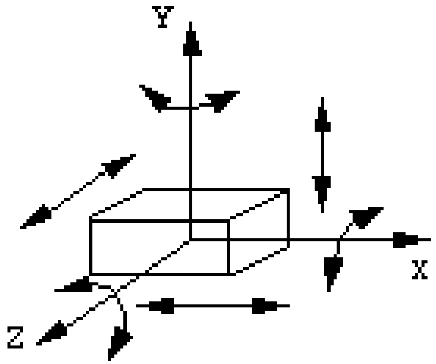
- For today, we will limit our study to linear kinematic chains, rather than the more general hierarchies (i.e., stick with individual arms & legs rather than an entire body with multiple branching chains)



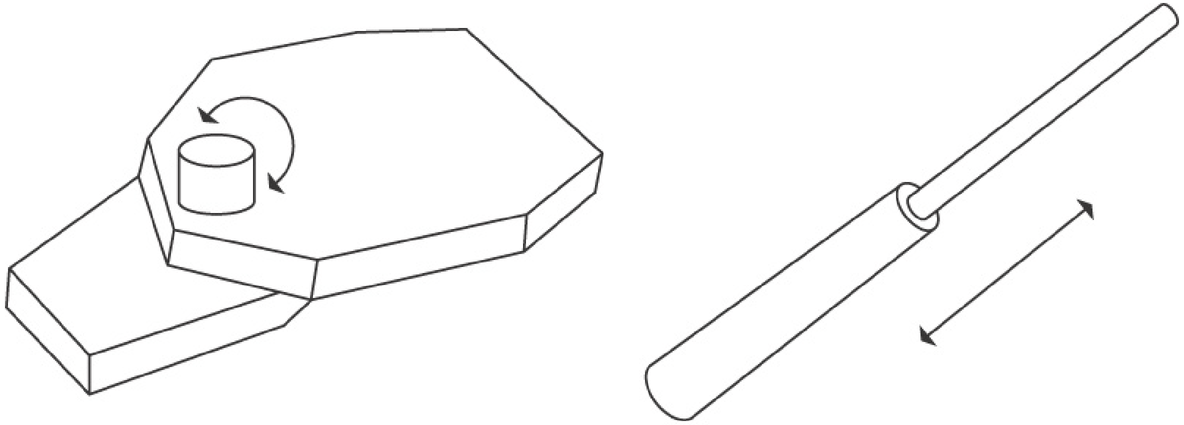


DOF (Degrees of freedom)

- An unrestrained rigid body in space has six degrees of freedom: three translating motions along the *x*, *y* and *z* axes and three rotary motions around the *x*, *y* and *z* axes respectively.

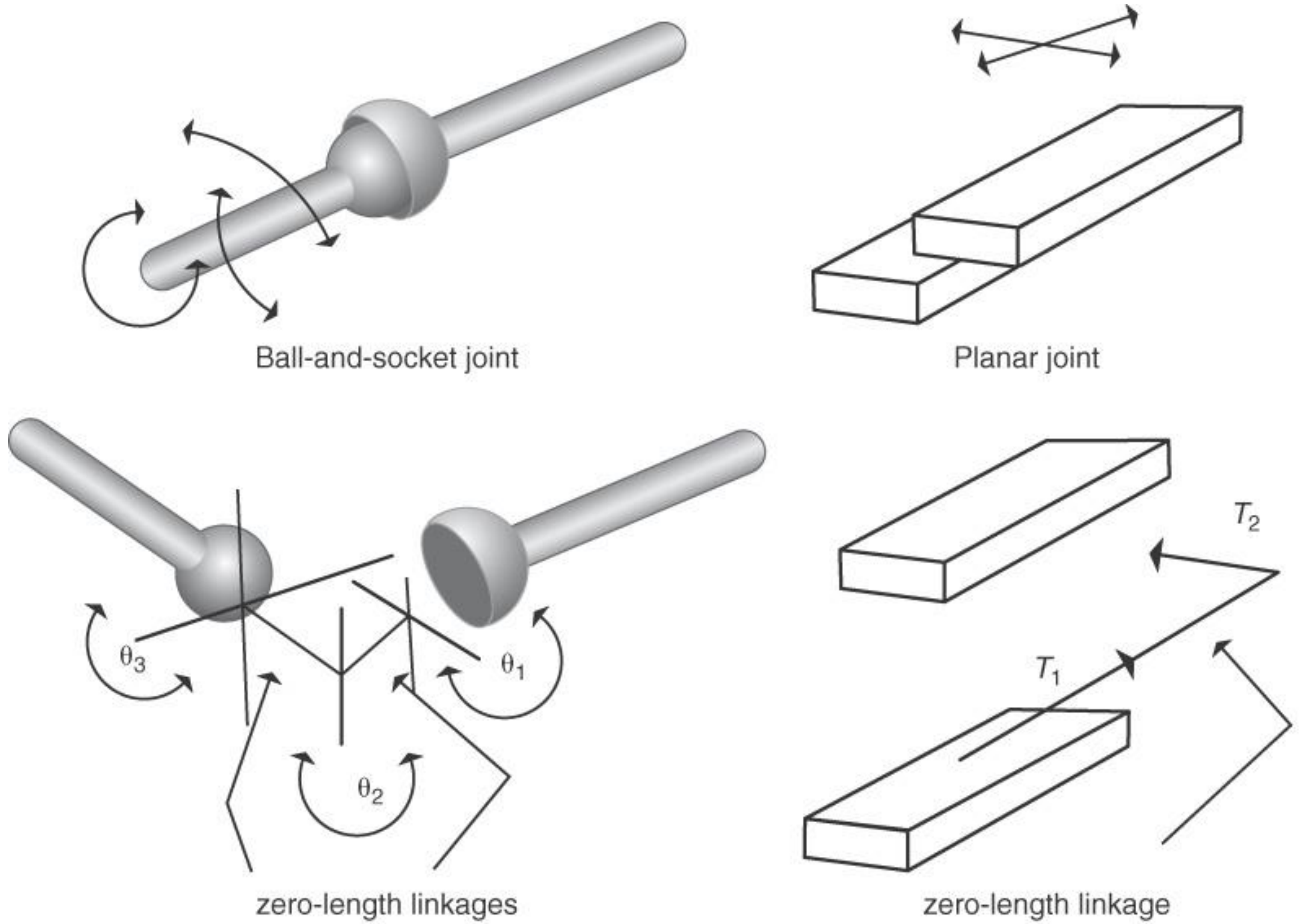


- Joints with 1 DOF



Revolute joint   Prismatic joint

DOF



Ball-and-socket joint modeled as 3 one-degree joints with zero-length links

Planar joint modeled as 2 one-degree prismatic joints with zero-length links

DOF of human joints

- Root: 3 translational DOF + 3 rotational DOF
- Most of the joints are Rotational joints
- Each joints has at most 3 DOF
    - Shoulder: 3 DOF
    - Wrist: 2 DOF
    - Knee: 1 DOF

Forward Kinematics

- We have joint DOF (Degrees of freedom) values:

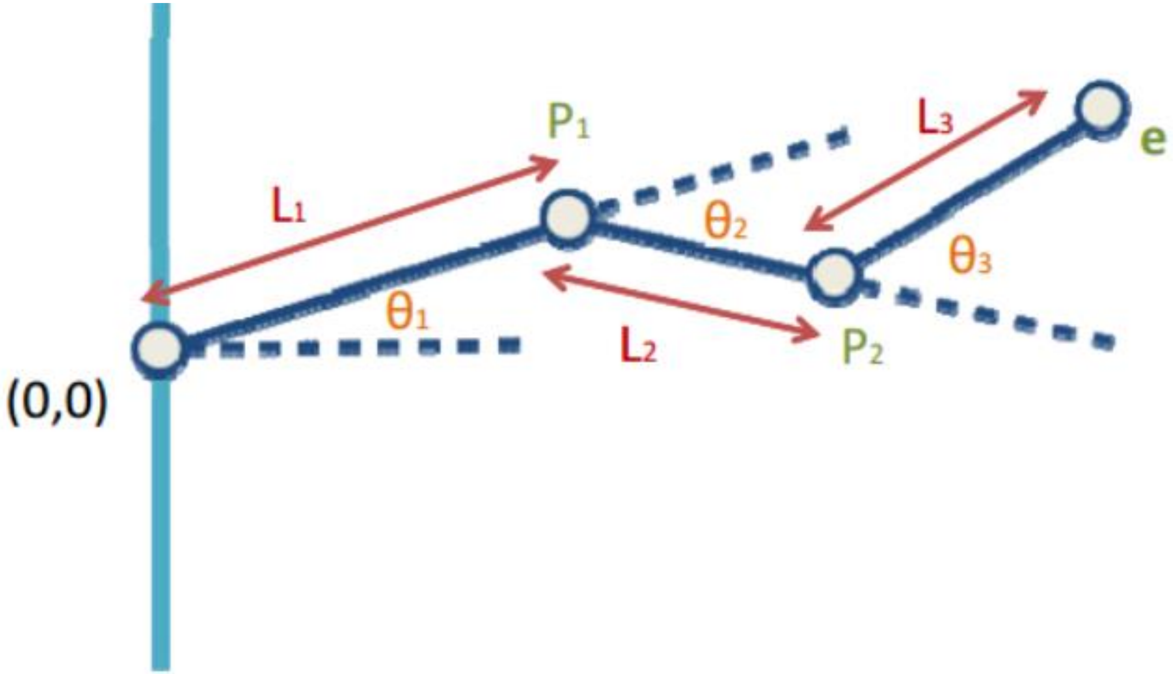
$$\boldsymbol{\theta} = [\theta_1 \ \theta_2 \ \cdots \ \theta_M]$$

- We want the end effector description in world space (N=3 in our case):

$$\mathbf{e} = [e_1 \ e_2 \ \cdots \ e_N]$$

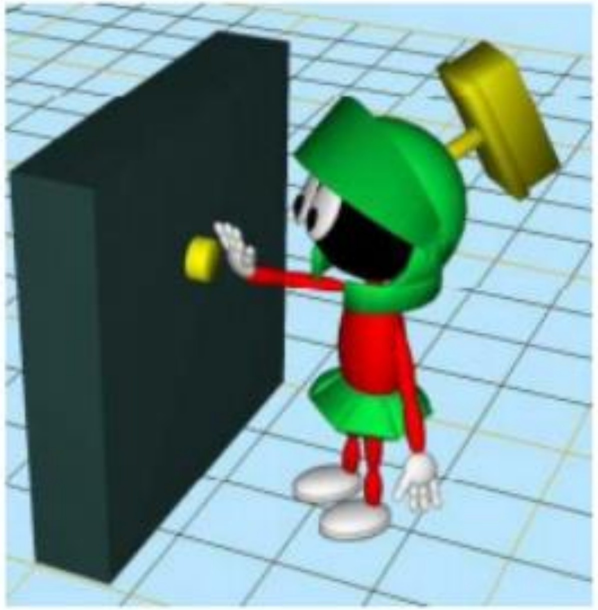
- FK gives us:

$$\mathbf{e} = \mathbf{f}(\boldsymbol{\theta})$$

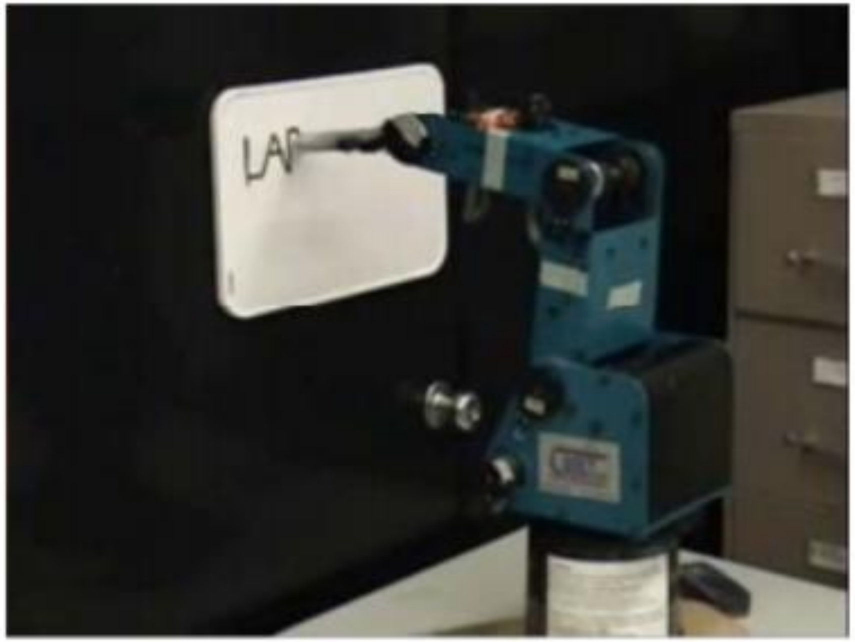


But Sometimes We Want the Opposite

- We want to know how the upper joints of the hierarchy would rotate if we want the end effector to reach some goal.



Animations

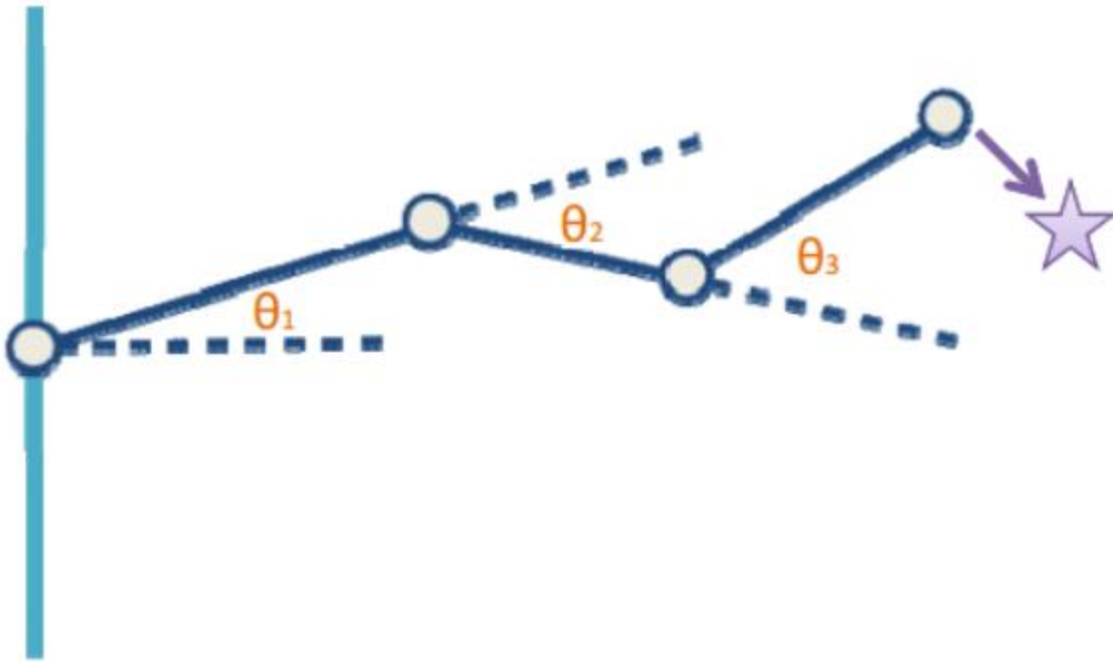


Robotics

Inverse Kinematics

- The goal of inverse kinematics is to compute the vector of joint DOFs that will cause the end effector to reach some desired goal state
- We have: $\mathbf{e} = [e_1 \; e_2 \; \cdots \; e_N]$
- And we want:

$$\boldsymbol{\theta} = [\theta_1 \; \theta_2 \; \cdots \; \theta_M]$$



- We need:

$$\boldsymbol{\theta} = \mathbf{f}^{-1}(\mathbf{e})$$

f is a Multivariate nonlinear function

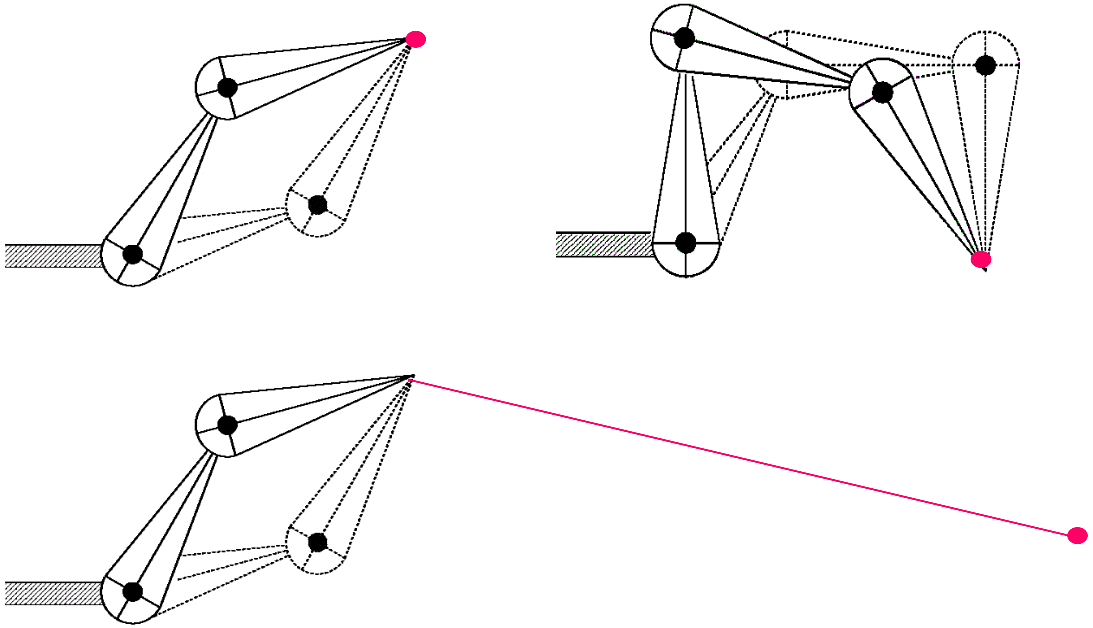
FK vs. IK

Forward Kinematics

Inverse Kinematics

Inverse Kinematics Issues

- While FK is relatively easy to evaluate.
- IK is more challenging: several possible solutions, or sometimes maybe no solutions.



- Require Complex and Expensive computations to find a solution.
- As a result, there are many different approaches to solving IK problems

Analytical vs. Numerical Solutions

- One major way to classify IK solutions is into analytical and numerical methods

- Analytical methods attempt to mathematically solve an exact solution by directly inverting the forward kinematics equations. This is only **possible on relatively simple** chains.

- Numerical methods use **approximation and iteration** to converge on a solution. They tend to be more expensive, but far more general purpose.

- Today, we will examine a numerical IK technique based on Jacobian matrices

Numerical Solutions of IK

- Jacobian
- Cyclic Coordinate Descent (CCD)

Multivariate nonlinear root finding

- Want to solve f(θ)-X=0 numerically

- Given: current θ, f(θ) and target X
- How to find Δ such that f(θ+Δ)=X
    - Find Δ that gets closer
    - Then θ <- θ+Δ and repeat

- Taylor series expansion:
  f(θ+Δ)=f(θ)+f'(θ) Δ+f''(θ) Δ^2/2+…

Local Linearization

- Taylor series expansion:

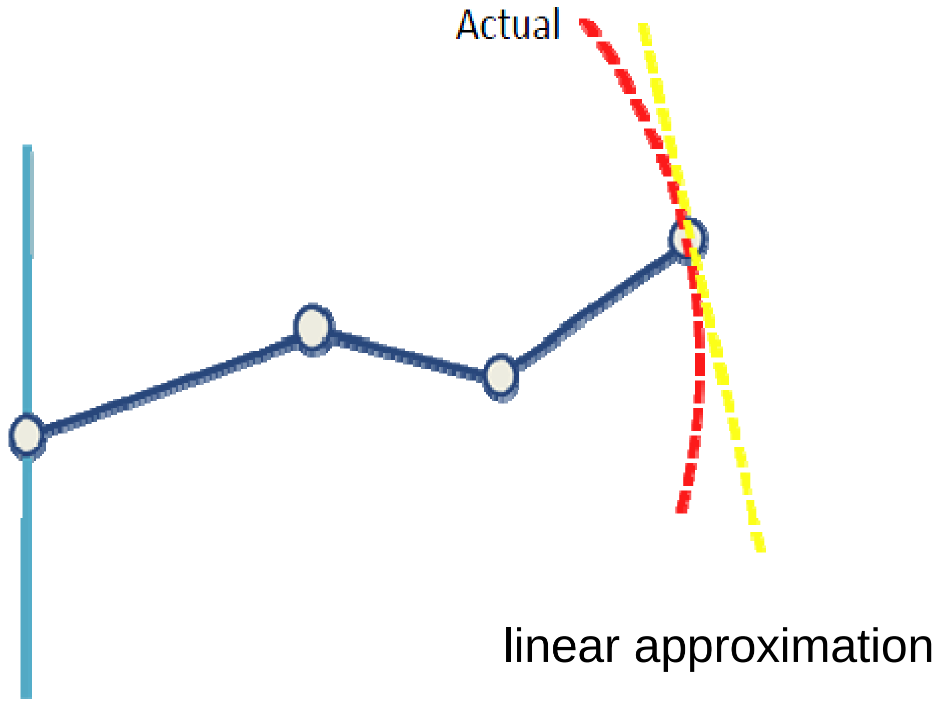
  f(θ+Δ)=f(θ)+f'(θ) Δ+f''(θ) Δ^2/2+…

- Use first term of Taylor series:

  f(θ+Δ)- *f(θ)* +=**J(*θ*) *Δ***

- Jacobian matrix:

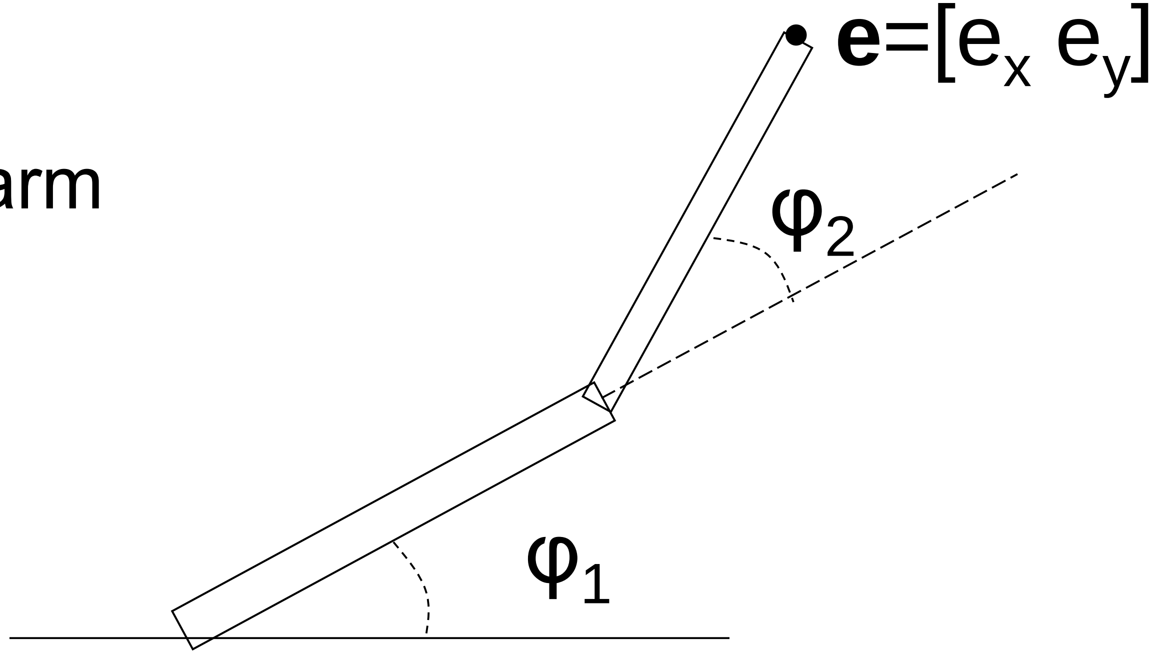
$$J(\mathbf{f},\theta) = \begin{bmatrix} \frac{\partial f_1}{\partial \theta_1} & \frac{\partial f_1}{\partial \theta_2} & \cdots & \frac{\partial f_1}{\partial \theta_N} \\ \frac{\partial f_2}{\partial \theta_1} & \frac{\partial f_2}{\partial \theta_2} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_M}{\partial \theta_1} & \cdots & \cdots & \frac{\partial f_M}{\partial \theta_N} \end{bmatrix}$$



  - Matrix of partial derivatives of entire system.

Jacobians

- Let's say we have a simple 2D robot arm with two 1-DOF rotational joints:



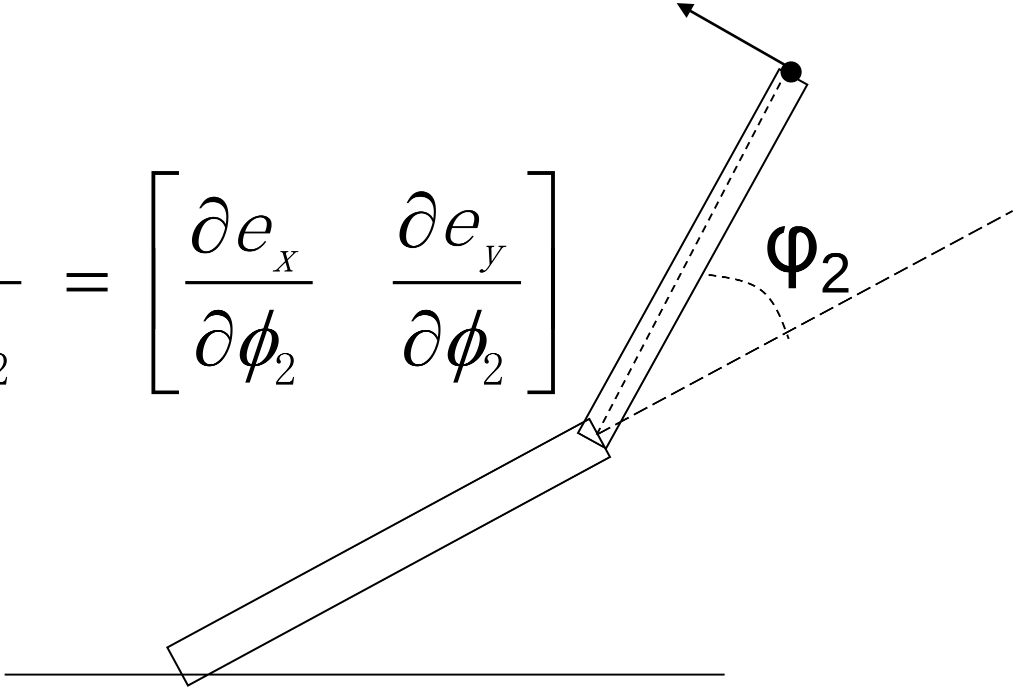
- The Jacobian matrix J(**e**,**Φ**) shows how each component of **e** varies with respect to each joint angle

$$J(\mathrm{e}, \Phi) = \begin{bmatrix} \dfrac{\partial e_x}{\partial \phi_1} & \dfrac{\partial e_x}{\partial \phi_2} \\ \dfrac{\partial e_y}{\partial \phi_1} & \dfrac{\partial e_y}{\partial \phi_2} \end{bmatrix}$$

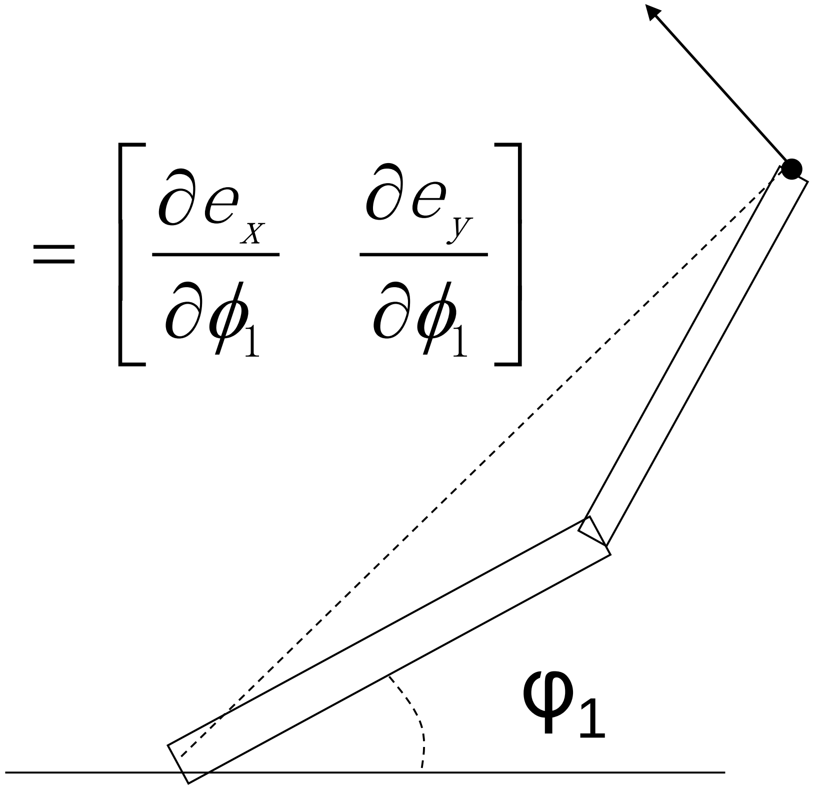
Jacobians

- Consider what would happen if we increased φ_2 by a small amount? What would happen to **e** ?
- What if we increased φ_1 by a small amount?

$$\frac{\partial \mathbf{e}}{\partial \phi_2} = \begin{bmatrix} \dfrac{\partial e_x}{\partial \phi_2} & \dfrac{\partial e_y}{\partial \phi_2} \end{bmatrix}$$



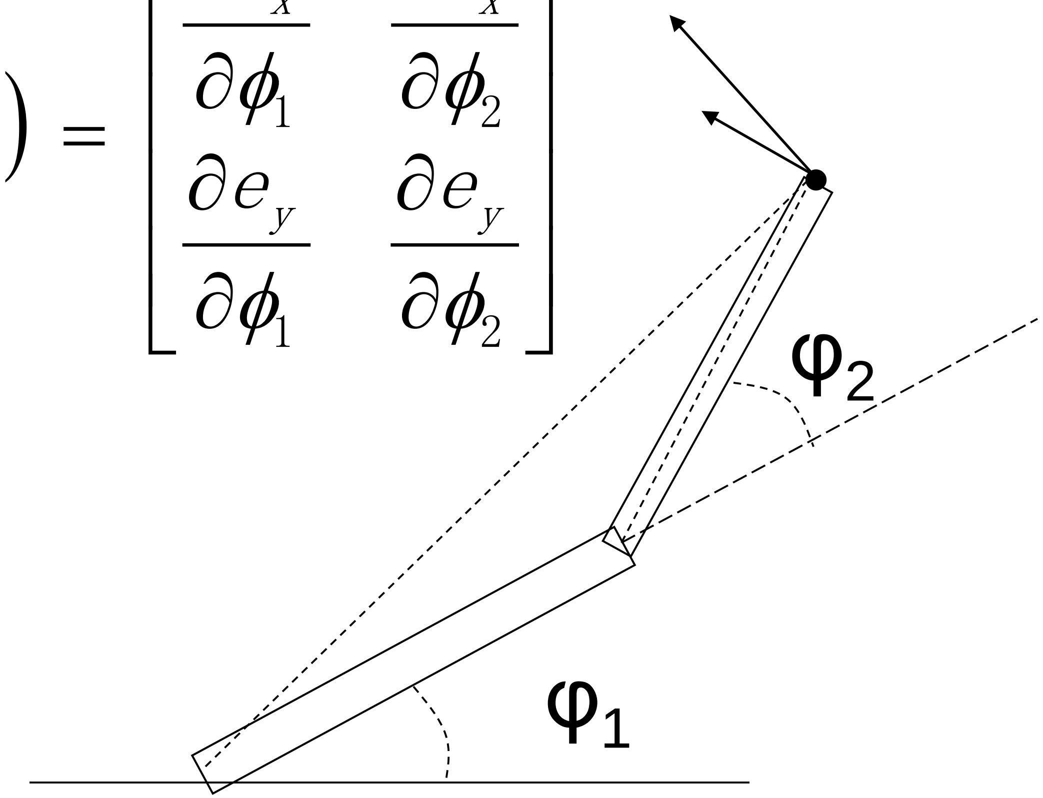
$$\frac{\partial \mathbf{e}}{\partial \phi_1} = \begin{bmatrix} \dfrac{\partial e_x}{\partial \phi_1} & \dfrac{\partial e_y}{\partial \phi_1} \end{bmatrix}$$



Jacobian for a 2D Robot Arm

- Defines how the end effector **e** changes relative to instantaneous changes of each joint angle

$$J(\mathrm{e}, \Phi) = \begin{bmatrix} \frac{\partial e_x}{\partial \phi_1} & \frac{\partial e_x}{\partial \phi_2} \\ \frac{\partial e_y}{\partial \phi_1} & \frac{\partial e_y}{\partial \phi_2} \end{bmatrix}$$

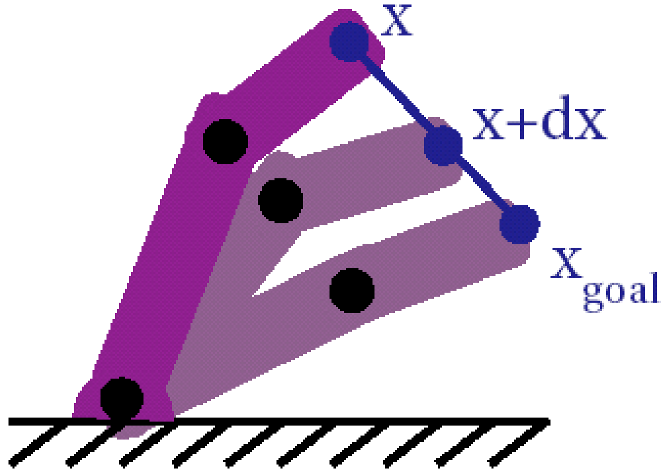


Solving IK—Incremental Changes

- ***θ:*** current set of joint DOFs;
- **e:** current end effector DOFs;
- **g:** goal DOFs that we want the end effector to reach
- Let $\boldsymbol{E(\theta) = g} - e(\boldsymbol{\theta})$, error in the current pose:

$$\mathbf{J}\,\boldsymbol{\Delta} = \boldsymbol{E}$$

- solve for $\boldsymbol{\Delta}$
- $\boldsymbol{\Delta}$ moves end towards $\boldsymbol{g}$
  - Only valid for small $\boldsymbol{\Delta}$
  - Take series of small steps
  - Recompute $\mathbf{J}(\boldsymbol{\theta})$ and $\boldsymbol{E}(\boldsymbol{\theta})$ at each step

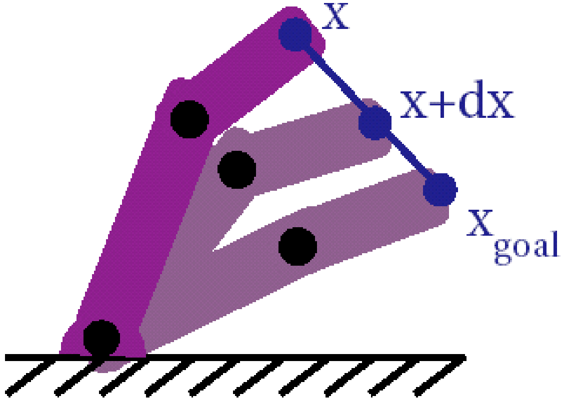


- How to determine length of step?
  - Could try to find optimum size
  - know we're doing rotations:
    - keep less than ~2 degrees

Algorithm

```
solve()
  {
    start with previous θ;
     E = target - computeEndPoint();
    for(k=0; k<max && |E| > eps; k++){
      J = computeJacobian();
      solve J Δ= E;
      if (max(Δ)>2) Δ= 2Δ/max(Δ);
      θ = θ + Δ;
       E = target - computeEndPoint();
    }
  }
```

$$\mathbf{e} = \mathbf{f}(\boldsymbol{\theta})$$



- **Inverse and Forward kinematics demo application**
- http://www.fit.vutbr.cz/~dobsik/projects/kinem_INV/kinem_INV.html

Problems

- How to invert J?
  - Pseudoinverse of Jacobian (Required)
  - Cheat by using transpose (Too easy, we don’t do that)
- How to compute J?
  - Numerically (Required)
  - Analytically (Extra Credit)

Inverting the Jacobian

- No guarantee it is invertible
  - Typically not a square matrix, in our case, 2 x N

$$
\begin{bmatrix}
\frac{\partial f_x}{\partial \theta_0} & \frac{\partial f_x}{\partial \theta_1} & \cdots & \frac{\partial f_x}{\partial \theta_N} \\
\frac{\partial f_y}{\partial \theta_0} & \frac{\partial f_y}{\partial \theta_1} & \cdots & \frac{\partial f_y}{\partial \theta_N}
\end{bmatrix}
\begin{bmatrix}
D_0 \\
D_1 \\
\vdots \\
D_N
\end{bmatrix}
=
\begin{bmatrix}
E_x \\
E_y
\end{bmatrix}
$$

  - Singularities.
  - Even it’s invertible, as the pose vector changes, the properties of the matrix will change.

Solving $\boldsymbol{J}\,\Delta = E$: pseudo inverse

- Trick: $\boldsymbol{J}^T\boldsymbol{J}$ is square. So:

  $$\boldsymbol{J}\,\Delta = E$$
  $$\boldsymbol{J}^T\boldsymbol{J}\,\Delta = \boldsymbol{J}^T E$$
  $$\Delta = (\boldsymbol{J}^T\boldsymbol{J})^{-1}\boldsymbol{J}^T E$$
  $$\Delta = \boldsymbol{J}^+ E$$

- $\boldsymbol{J}^+ = (\boldsymbol{J}^T\boldsymbol{J})^{-1}\boldsymbol{J}^T$ is the *pseudoinverse* of $\boldsymbol{J}$
  - Properties: $\boldsymbol{J}\boldsymbol{J}^+\boldsymbol{J}=\boldsymbol{J}$, $\boldsymbol{J}^+\boldsymbol{J}\boldsymbol{J}^+=\boldsymbol{J}^+$
  - same as $\boldsymbol{J}^{-1}$ when $\boldsymbol{J}$ is square and invertible
  - $\boldsymbol{J}$ is *mxn* => $\boldsymbol{J}^+$ is *nxm*

- How to compute pseudoinverse?
  - What if $(\boldsymbol{J}^T\boldsymbol{J})^{-1}$ is singular?

Singular Value Decomposition

- Any *mxn* matrix **A** can be expressed by SVD
  - **A** = **U S V**^{T}
    - **U** is *m*xmin*(m,n)*, columns are orthogonal
    - **V** is *n*xmin*(m,n)*, columns are orthogonal
    - S is min*(m,n)*xmin*(m,n)*, diagonal: *singular values*

$$A = \left(\vec{h}_1 \,\middle|\, \vec{h}_2 \,\middle|\, \cdots \,\middle|\, \vec{h}_N \right) \begin{pmatrix} s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & s_N \end{pmatrix} \begin{pmatrix} \vec{a}_1 \\ \hline \vec{a}_2 \\ \hline \vdots \\ \hline \vec{a}_N \end{pmatrix}$$

  - unique up to sign and order of s_i values
    - canonical: positive, sorted largest to smallest
    - other properties: rank is # of non-zero values; determinant is product of all values, …

Pseudoinverse using SVD

- Given SVD, $\mathbf{A} = \mathbf{U}\,\mathbf{S}\,\mathbf{V}^T$
- pseudoinverse is easy: $\mathbf{A}^+ = \mathbf{V}\mathbf{S}^{-1}\mathbf{U}^T$

$$\begin{pmatrix} s_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & s_N \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{s_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{1}{s_N} \end{pmatrix}$$

- *singular*: some $s_i = 0$,
- *ill-conditioned*: some $s_i << s_0$
  - use 0 instead of $1/s_i$ for those (“truncated”)
  - choose small threshold ε, test $s_i < \varepsilon\, s_0$

Solving **A** ***X*** = ***B*** using SVD

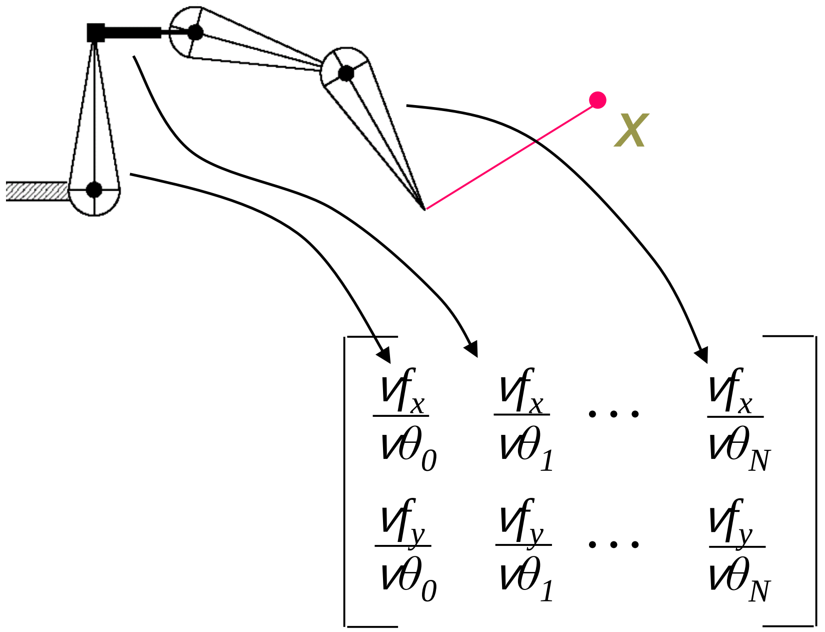
- Using truncated $\mathbf{A}^{+}$ ***B*** gives *least-squares solution*:
  - If no solution, gives ***X*** that minimizes $||\mathbf{A}\boldsymbol{X}\text{-}\boldsymbol{B}||^2$
  - If many solutions, minimizes $||\boldsymbol{X}||^2$ such that $\mathbf{A}\boldsymbol{X}=\boldsymbol{B}$
  - Numerically stable for ill-conditioned matrices
- SVD has many other properties.
  - rank of **A** is # non-zero singular values, determinant is product of all singular values, …
  - known algorithm to compute it
- SVD is a powerful hammer!
  - slow $O(n^3)$; there are faster algorithms.
  - but SVD always works, is fast enough for us
  - hard to implement. some libraries have bugs (Java3D)

Back to IK

- Reminder: Let $\boldsymbol{E}(\boldsymbol{\theta}) = \boldsymbol{g} - e(\boldsymbol{\theta})$, error in the current pose

$$\mathbf{J}(\boldsymbol{\theta})\,\boldsymbol{\Delta} = \boldsymbol{E}$$

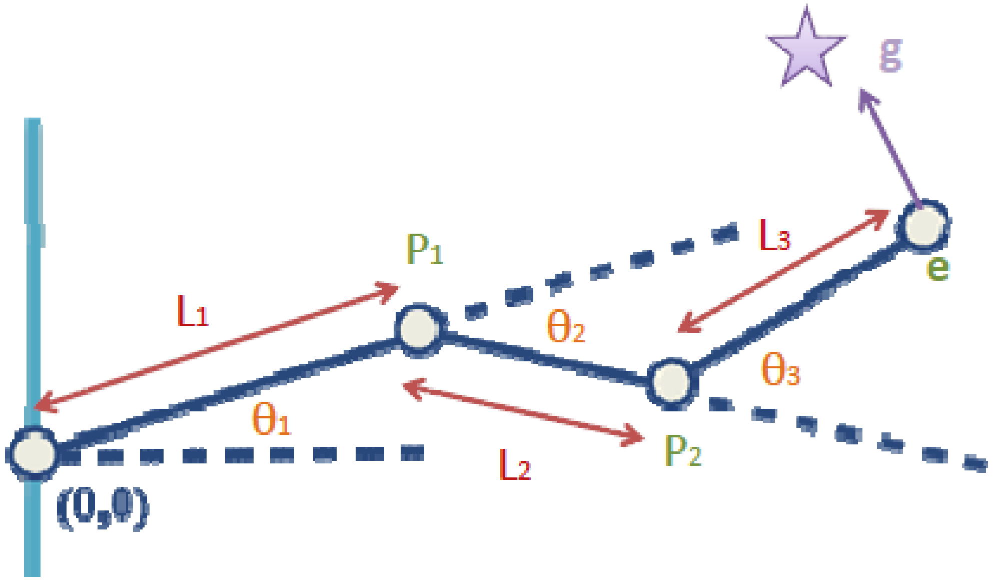
- solve for $\boldsymbol{\Delta}$
  - i^{th} column of **J** comes from link *i*



Computing the Jacobian columns

- For a rotational joint, the linear change in the end effector is the cross product of the axis of revolution and a vector from the joint to the end effector.

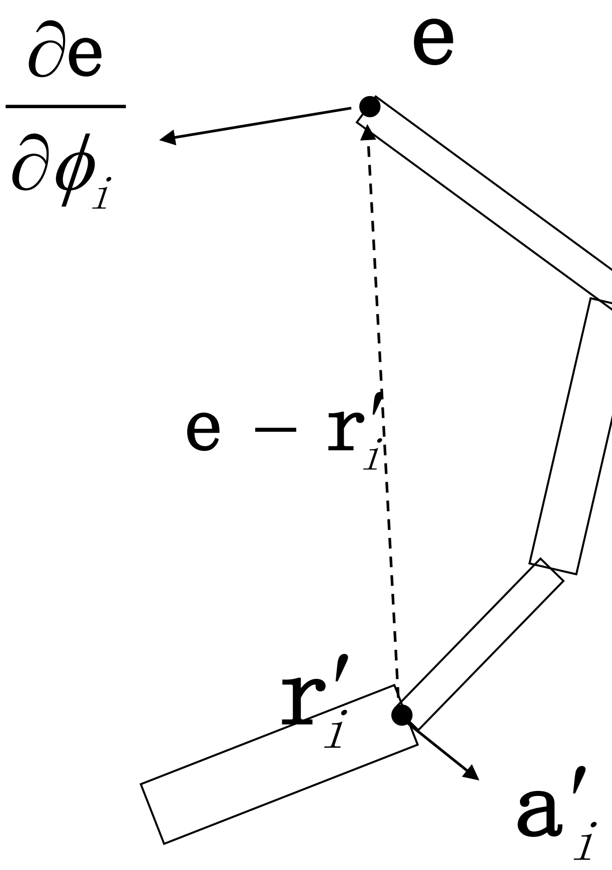
$$\frac{\partial \mathbf{e}}{\partial \theta_1} = \left[\frac{\partial e_x}{\partial \theta_1}\ \frac{\partial e_y}{\partial \theta_1}\ \frac{\partial e_z}{\partial \theta_1}\right]^T = \left(\mathbf{a}_1' \times (\mathbf{e} - \mathbf{r}_1')\right)$$



$$J = \begin{bmatrix} ((0,0,1)\times \mathbf{e})_x & ((0,0,1)\times(\mathbf{e}-\mathbf{P_1}))_x & ((0,0,1)\times(\mathbf{e}-\mathbf{P_2}))_x \\ ((0,0,1)\times \mathbf{e})_y & ((0,0,1)\times(\mathbf{e}-\mathbf{P_1}))_y & ((0,0,1)\times(\mathbf{e}-\mathbf{P_2}))_y \\ ((0,0,1)\times \mathbf{e})_z & ((0,0,1)\times(\mathbf{e}-\mathbf{P_1}))_z & ((0,0,1)\times(\mathbf{e}-\mathbf{P_2}))_z \end{bmatrix}$$

Rotational DOFs

$$\frac{\partial \mathbf{e}}{\partial \phi_i} = \mathbf{a}'_i \times \left(\mathbf{e} - \mathbf{r}'_i\right)$$



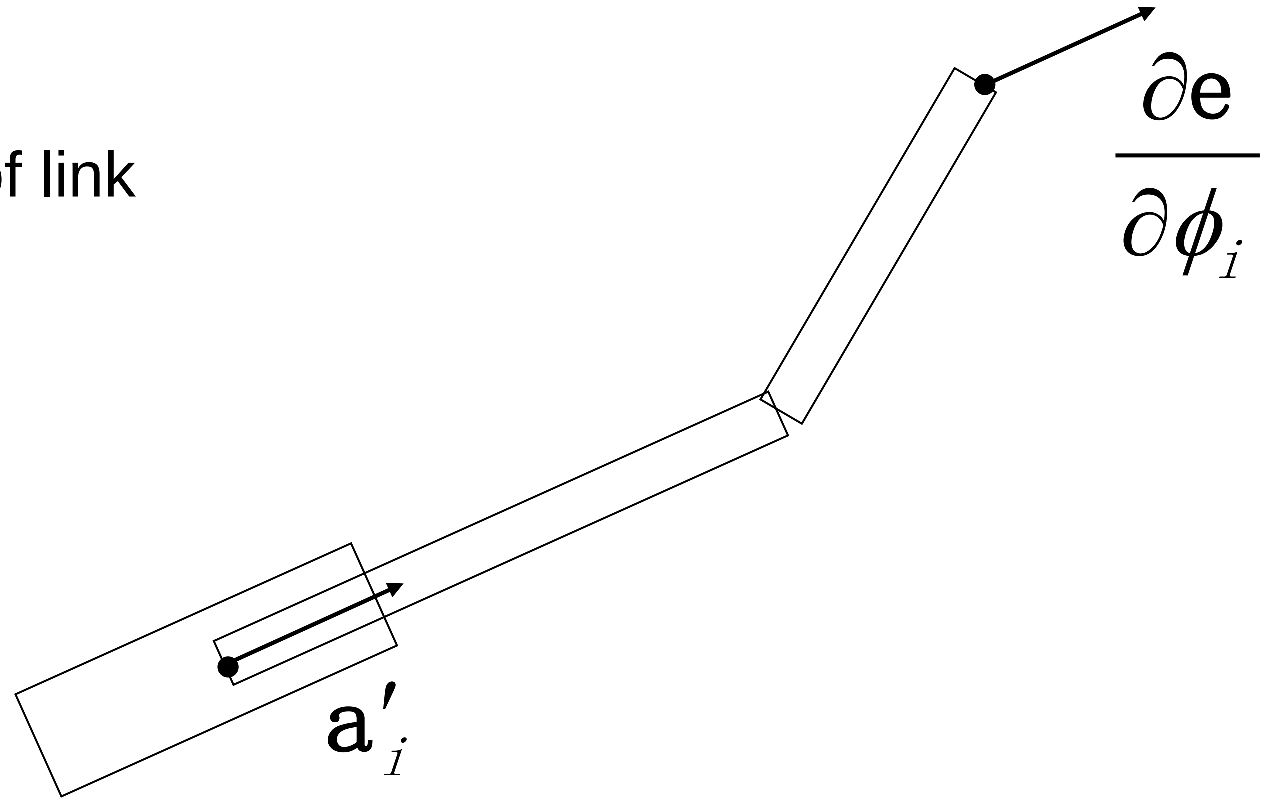
a'$_i$: unit length rotation axis in world space

r'$_i$: position of joint pivot in world space

e: end effector position in world space

Computing the Jacobian columns

- For a translational joint:
  - $\partial f(\boldsymbol{\theta})/\partial\theta_j$ = vector in direction of link



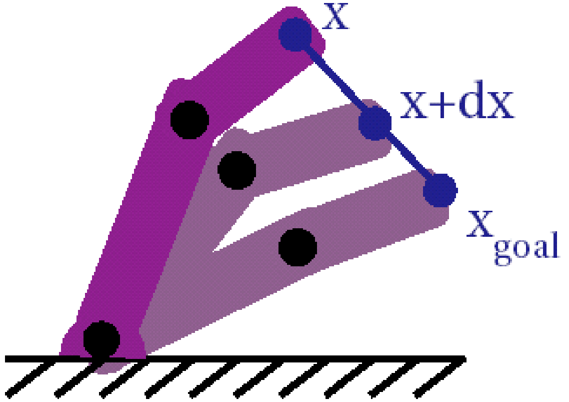
- Notes:
  - Remember to compute in world space!
  - I’ve assumed one degree of freedom per joint
  - If there are multiple DOFs per joint, refer to CSE169_12.ppt of CSE 169: Computer Animation @ UCSD winter 2004

Building the Jacobian

- To build the entire Jacobian matrix, we just loop through each DOF and compute a corresponding column in the matrix
- If we wanted, we could use more elaborate joint types (scaling, translation along a path, shearing…) and still compute an appropriate derivative
- If absolutely necessary, we could always resort to computing a numerical approximation to the derivative

IK Algorithm

```
solve()
  {
    Vector θ = getLinkParameters();
    Vector E = target - computeEndPoint();
    for(k=0; k<max && E.norm() > eps; k++){
      Matrix J = computeJacobian();
      Matrix J+ = J.pseudoinverse();
      Vector Δ = J+ E;
      if (max(Δ)>2) Δ *= 2/max(Δ);
      θ = θ + Δ;
      putLinkParameters(θ);
      E = target - computeEndPoint();
    }
  }
```



What’s left for IK?

- Joint limits
- When to stop the iterations

Joint limits

- Each joint may have limited range.
- Modify algorithm:
  - After finding *Δ,* test each joint:

    $$\boldsymbol{\theta min_i} < (\boldsymbol{\theta}+\Delta)_i < \boldsymbol{\theta max_i}$$

  - If it would go out of range
    - set column *i* of J to 0
    - claims "this parameter has no effect"
  - Recompute $\mathbf{J^+}$
    - Least-squares solution will make $\Delta_i\, z\, 0$
    - For robustness, you may want to force $\Delta_i = 0$
  - Find *Δ,* repeat

Note on numerical algorithms

- Various algorithms for non-linear multidimensional root-finding…this one works for us
- Root-finding is related to *optimization*:
  - $F(\boldsymbol{\theta})=X \Leftrightarrow$ minimize $||F(\boldsymbol{\theta})-X||^2$
- Many computer animation problems are optimization problems
- Many algorithms have solving AX = B at their core.

When to Stop

- There are three main stopping conditions we should account for
  - Finding a successful solution (or close enough)
  - Getting stuck in a condition where we can’t improve (local minimum)
  - Taking too long (for interactive systems)
- All three of these are fairly easy to identify by monitoring the progress of $\boldsymbol{\Phi}$
- These rules are just coded into the while() statement for the controlling loop

Finding a Successful Solution

- We really just want to get close enough within some tolerance
- If we’re not in a big hurry, we can just iterate until we get within some floating point error range
- Alternately, we could choose to stop when we get within some tolerance measurable in pixels
- For example, we could position an end effector to 0.1 pixel accuracy
- This gives us a scheme that should look good and automatically adapt to spend more time when we are looking at the end effector up close (level-of-detail)

Local Minima

- If we get stuck in a local minimum, we have several options
  - Don’t worry about it and just accept it as the best we can do
  - Switch to a different algorithm (CCD…)
  - Randomize the pose vector slightly (or a lot) and try again
  - Send an error to whatever is controlling the end effector and tell it to try something else
- Basically, there are few options that are truly appealing, as they are likely to cause either an error in the solution or a possible discontinuity in the motion

Taking Too Long

- In a time critical situation, we might just limit the iteration to a maximum number of steps
- Alternately, we could use internal timers to limit it to an actual time in seconds

Iteration Stepping

- Step size
- Stability
- Performance

Thank you