Computer Graphics -Inverse Kinematics

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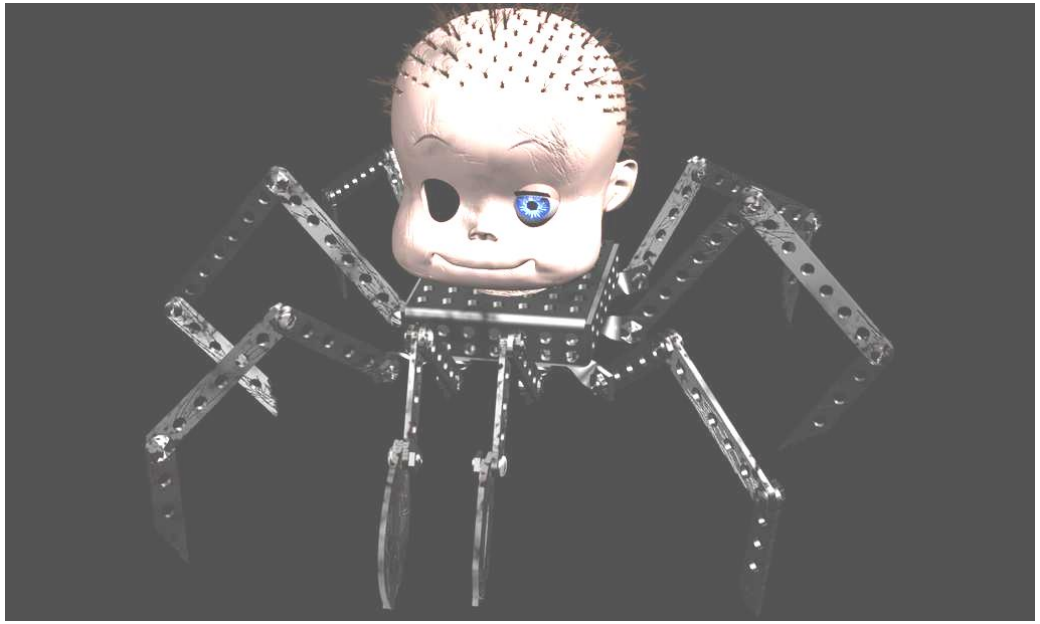
http://jjcao.github.io/ComputerGraphics/

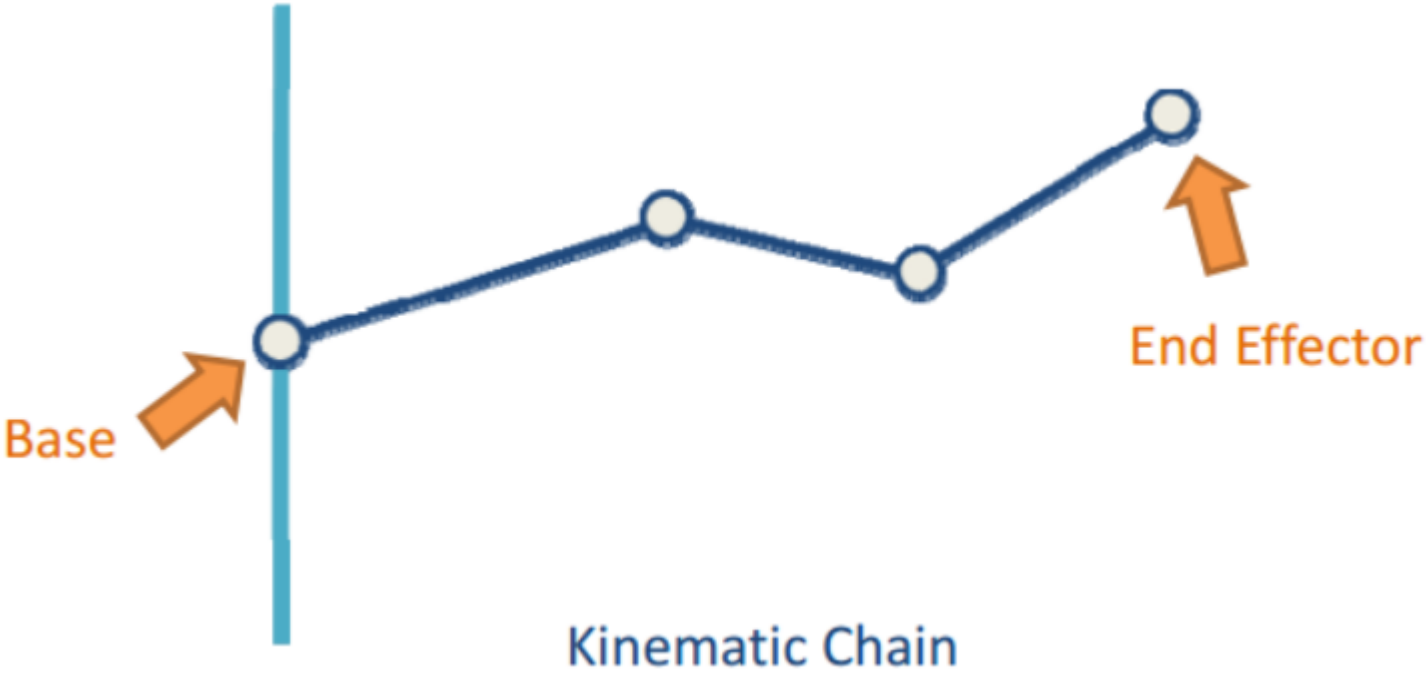


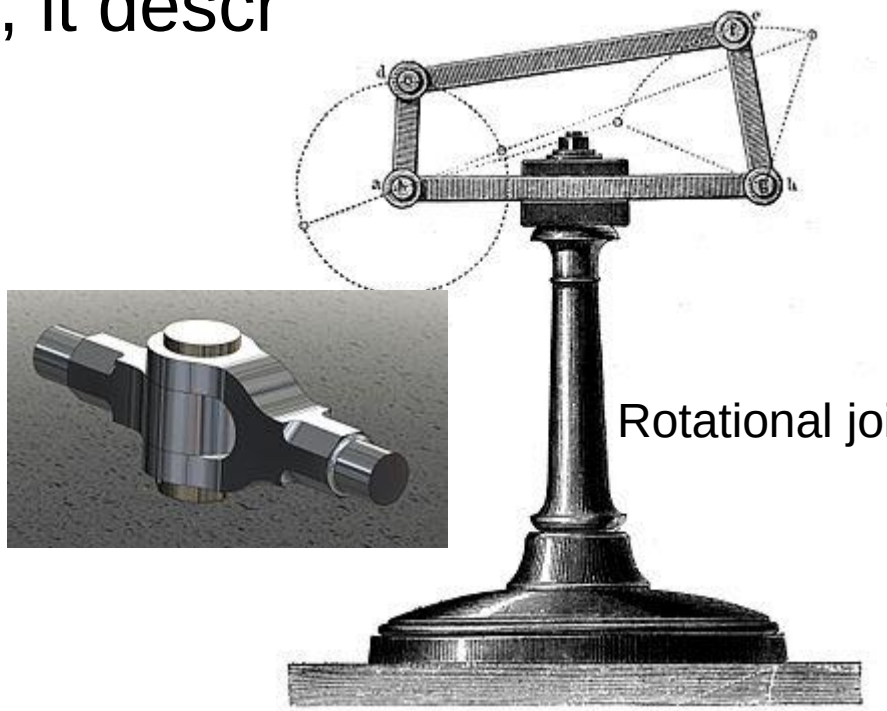
Overview

- Kinematics
- Forward Kinematics and Inverse Kinematics
- Jabobian
- Pseudoinverse of the Jacobian

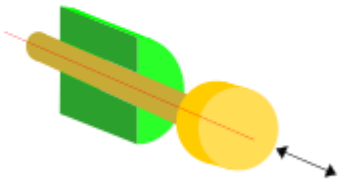
Vocabulary of Kinematics

- Kinematics is the study of how things move, it describes the motion of a hierarchical skeleton structure.
- Base and End Effector.





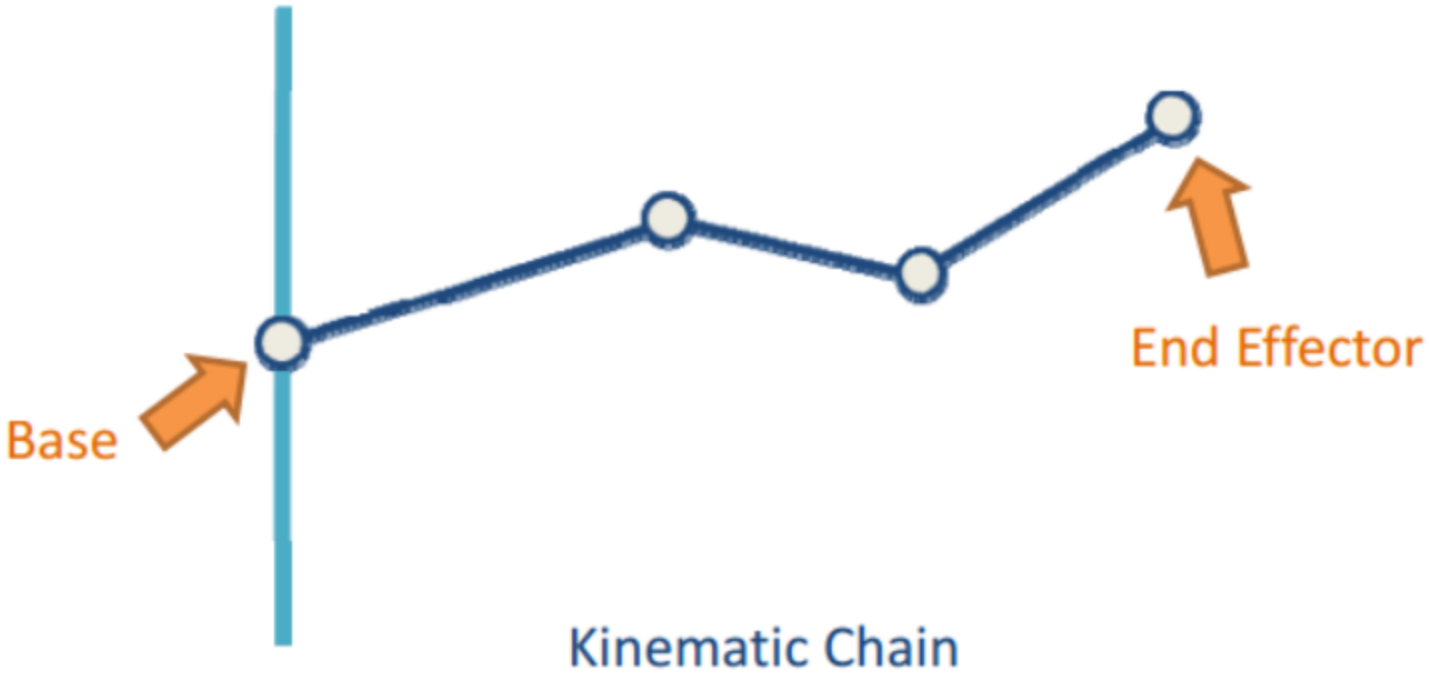
Rotational joint

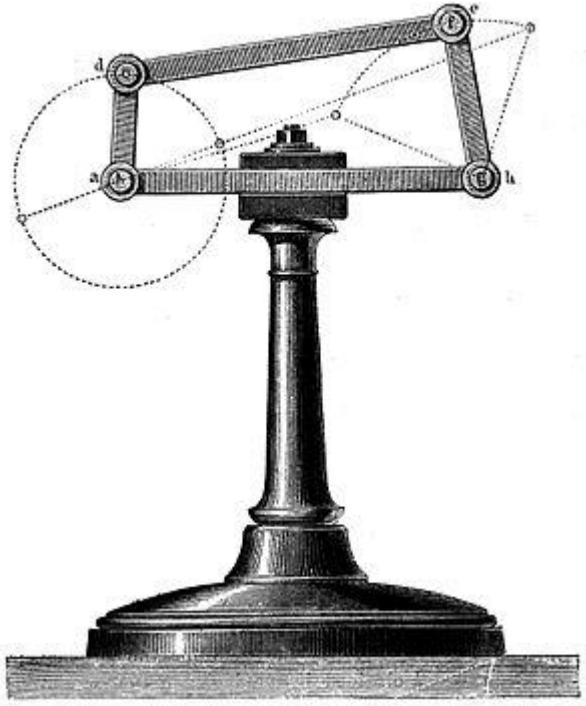


Translational/Prismatic joint

Kinematic Chains

- For today, we will limit our study to linear kinematic chains, rather than the more general hierarchies (i.e., stick with individual arms & legs rather than an entire body with multiple branching chains)





FK vs. IK

Forward Kinematics

Inverse Kinematics

Forward Kinematics

- We have joint DOF (Degrees of freedom) values:

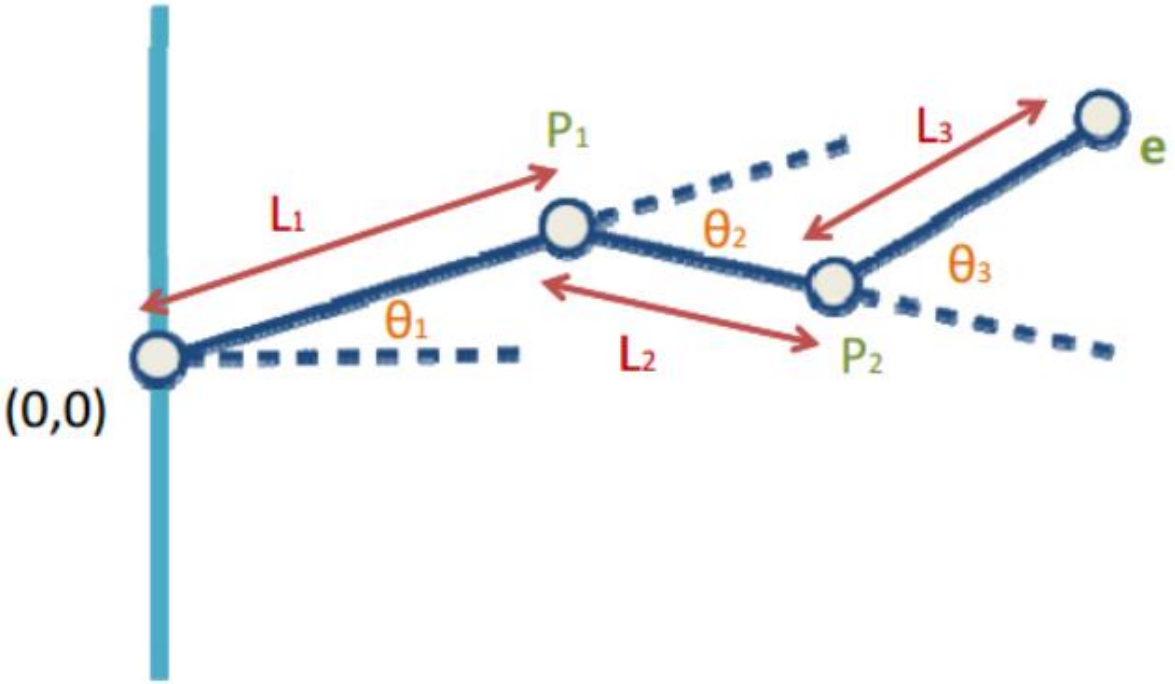
$$\boldsymbol{\theta} = [\theta_1 \ \theta_2 \ \cdots \ \theta_M]$$

- We want the end effector description in world space (N=3 in our case):

$$\mathbf{e} = [e_1 \ e_2 \ \cdots \ e_N]$$

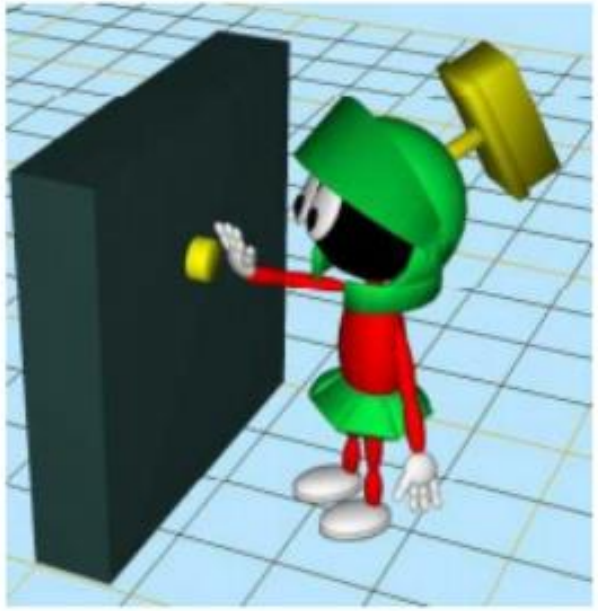
- FK gives us:

$$\mathbf{e} = \mathbf{f}(\boldsymbol{\theta})$$

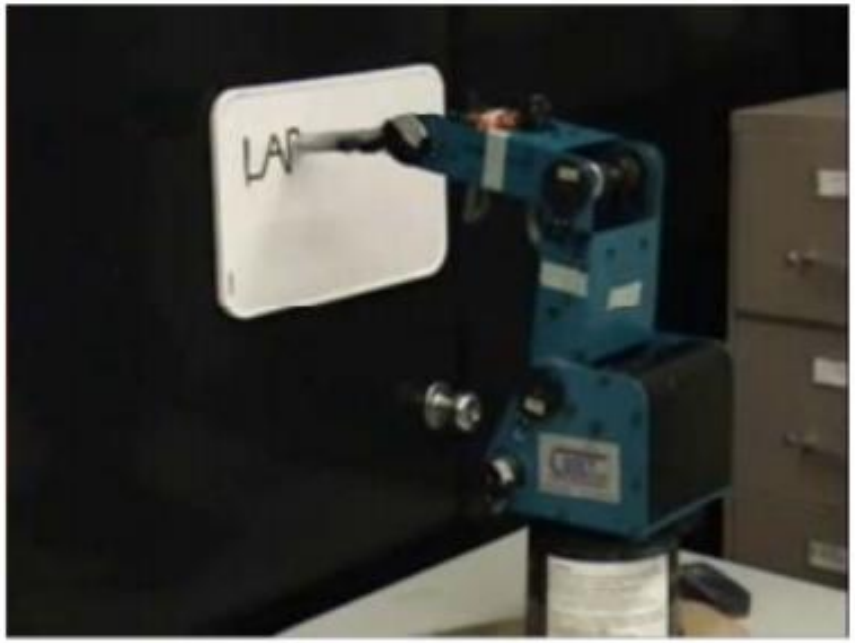


But Sometimes We Want the Opposite

- We want to know how the upper joints of the hierarchy would rotate if we want the end effector to reach some goal.



Animations

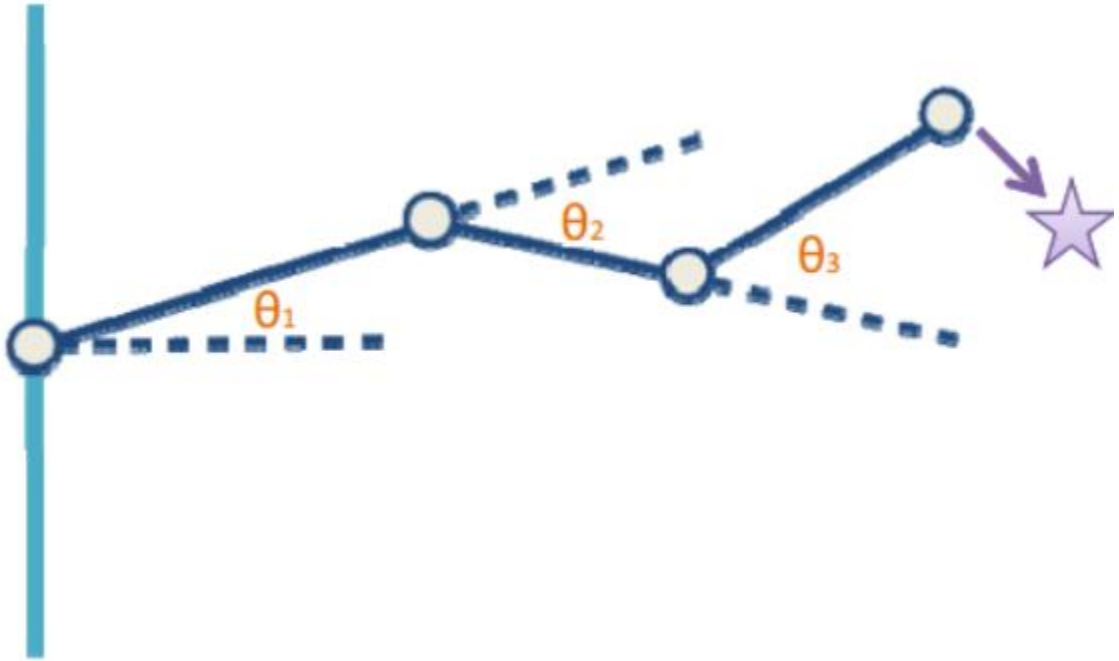


Robotics

Inverse Kinematics

- The goal of inverse kinematics is to compute the vector of joint DOFs that will cause the end effector to reach some desired goal state
- We have: $\mathbf{e} = [e_1\ e_2 \cdots e_N]$
- And we want:

$$\boldsymbol{\theta} = [\theta_1\ \theta_2 \cdots \theta_M]$$



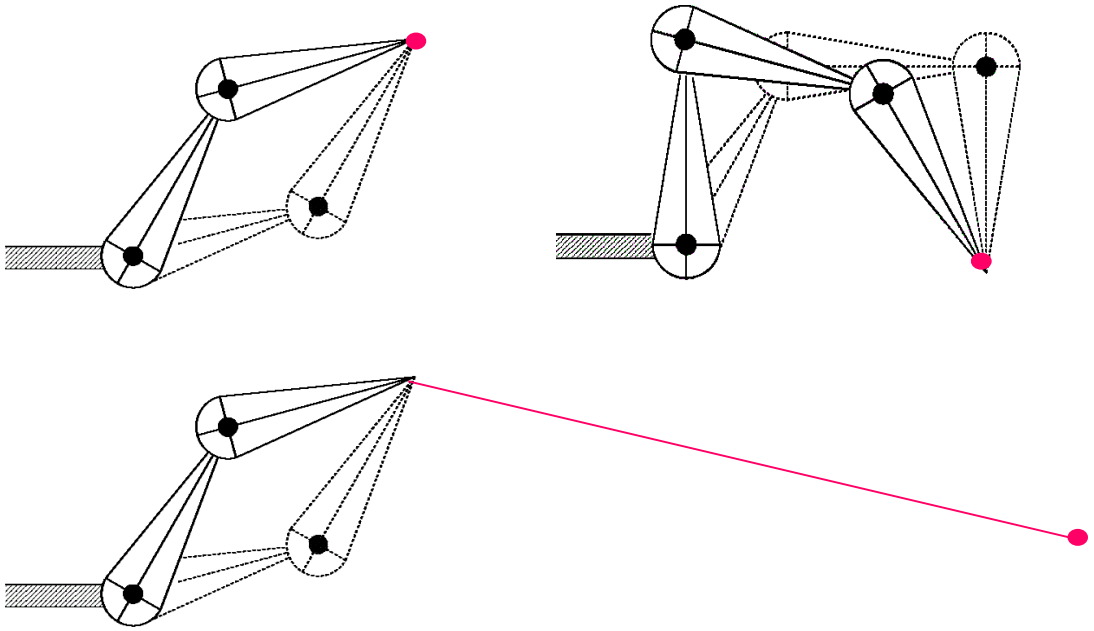
- We need:

$$\boldsymbol{\theta} = \mathbf{f}^{-1}(\mathbf{e})$$

f is a Multivariate nonlinear function

Inverse Kinematics Issues

- While FK is relatively easy to evaluate.
- IK is more challenging: several possible solutions, or sometimes maybe no solutions.



- Require Complex and Expensive computations to find a solution.
- As a result, there are many different approaches to solving IK problems

Analytical vs. Numerical Solutions

- One major way to classify IK solutions is into analytical and numerical methods
- Analytical methods attempt to mathematically solve an exact solution by directly inverting the forward kinematics equations. This is only possible on relatively simple chains.
- Numerical methods use approximation and iteration to converge on a solution. They tend to be more expensive, but far more general purpose.
- Today, we will examine a numerical IK technique based on Jacobian matrices

Numerical Solutions of IK

- Jacobian
- Cyclic Coordinate Descent (CCD)

Multivariate nonlinear root finding

- Want to solve f(θ)-X=0
- Taylor series expansion:
  $f(\theta+\Delta)=f(\theta)+f'(\theta)\,\Delta+f''(\theta)\,\Delta^2/2+\ldots$

- Given: current θ, f(θ) and target X
- How to find Δ such that f(θ+Δ)=X
  - Find Δ that gets closer
  - Then θ <- θ+Δ and repeat

Local Linearization

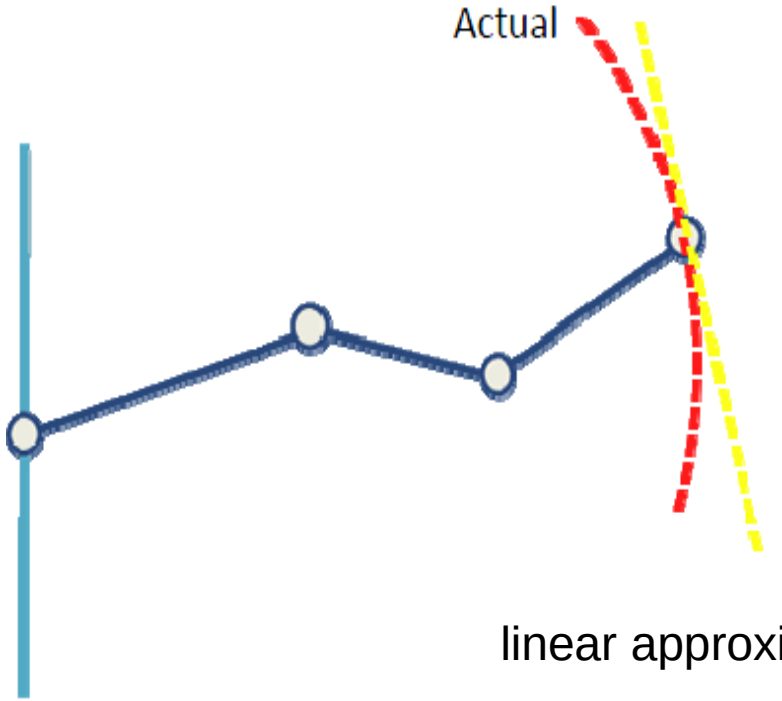
- Taylor series expansion:
  f(θ+Δ)=f(θ)+f'(θ) Δ+f''(θ) Δ^2/2+…
- Use first term of Taylor series:

  f(θ+Δ)= *f(θ)* + J*(θ)* *Δ*

- Jacobian matrix:

$$J(\mathrm{f},\theta)=\begin{bmatrix} \frac{\partial f_1}{\partial \theta_1} & \frac{\partial f_1}{\partial \theta_2} & \cdots & \frac{\partial f_1}{\partial \theta_N} \\ \frac{\partial f_2}{\partial \theta_1} & \frac{\partial f_2}{\partial \theta_2} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_M}{\partial \theta_1} & \cdots & \cdots & \frac{\partial f_M}{\partial \theta_N} \end{bmatrix}$$

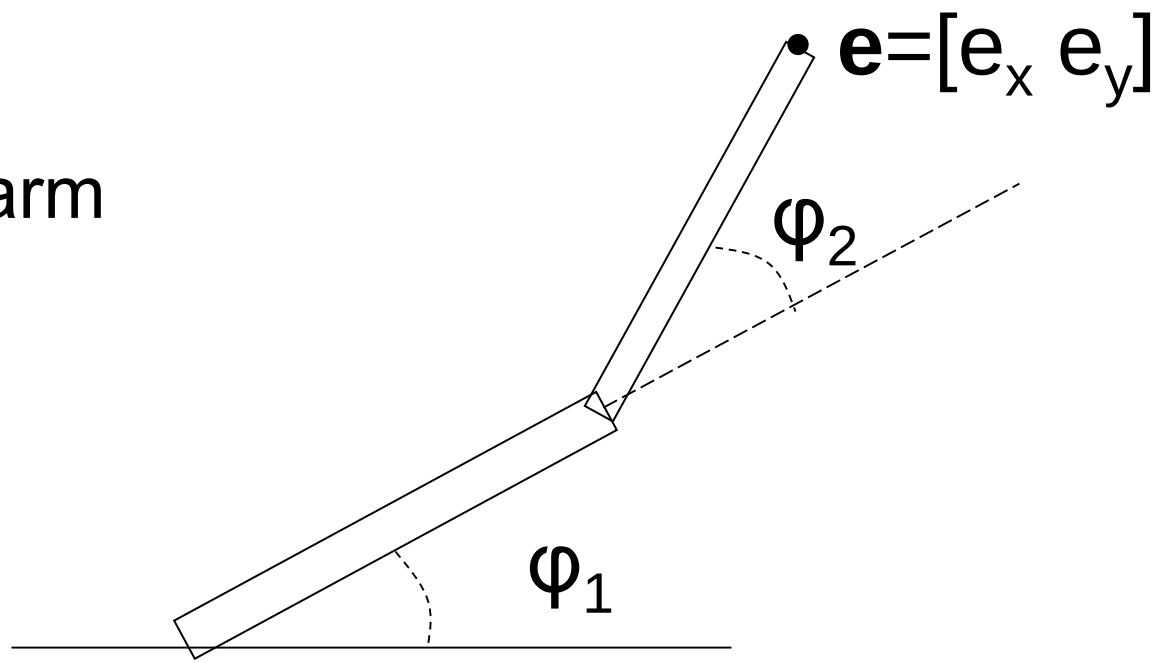
  - Matrix of partial derivatives of entire system.
  - .



linear approximation

Jacobians

- Let's say we have a simple 2D robot arm with two 1-DOF rotational joints:



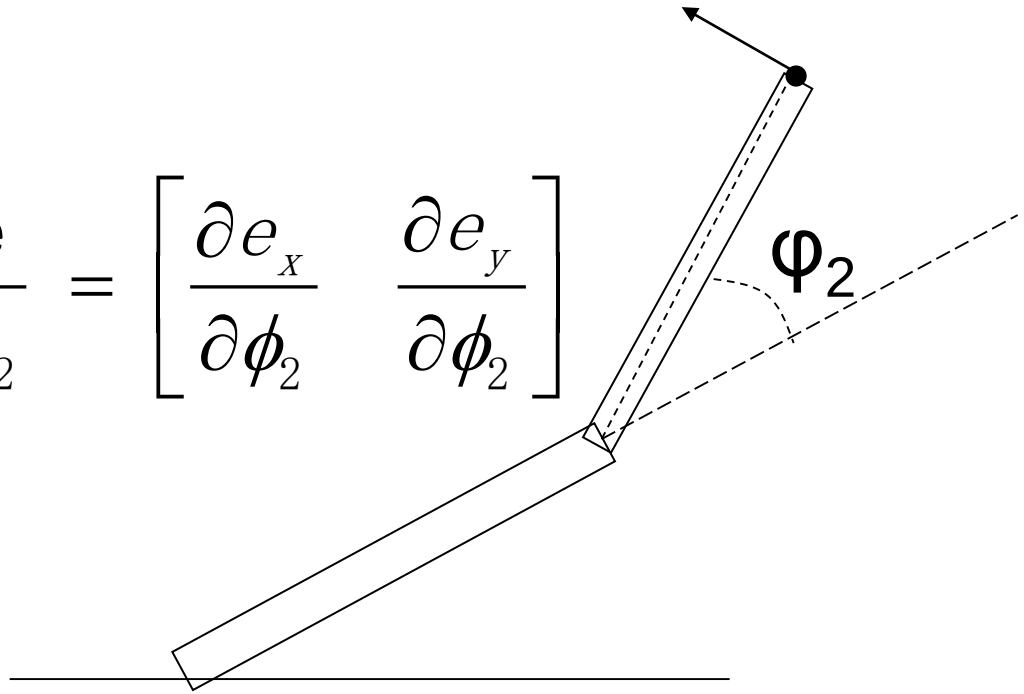
- The Jacobian matrix J(**e**,**Φ**) shows how each component of **e** varies with respect to each joint angle

$$J(\mathbf{e}, \Phi) = \begin{bmatrix} \frac{\partial e_x}{\partial \phi_1} & \frac{\partial e_x}{\partial \phi_2} \\ \frac{\partial e_y}{\partial \phi_1} & \frac{\partial e_y}{\partial \phi_2} \end{bmatrix}$$

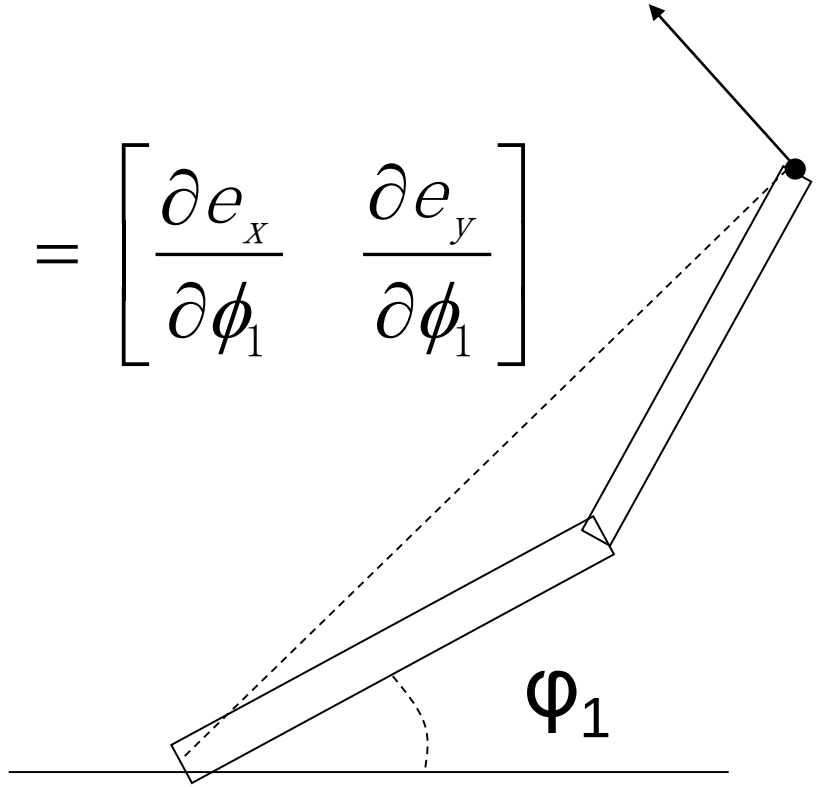
Jacobians

- Consider what would happen if we increased φ_2 by a small amount? What would happen to **e** ?
- What if we increased φ_1 by a small amount?

$$\frac{\partial \mathbf{e}}{\partial \phi_2} = \begin{bmatrix} \frac{\partial e_x}{\partial \phi_2} & \frac{\partial e_y}{\partial \phi_2} \end{bmatrix}$$



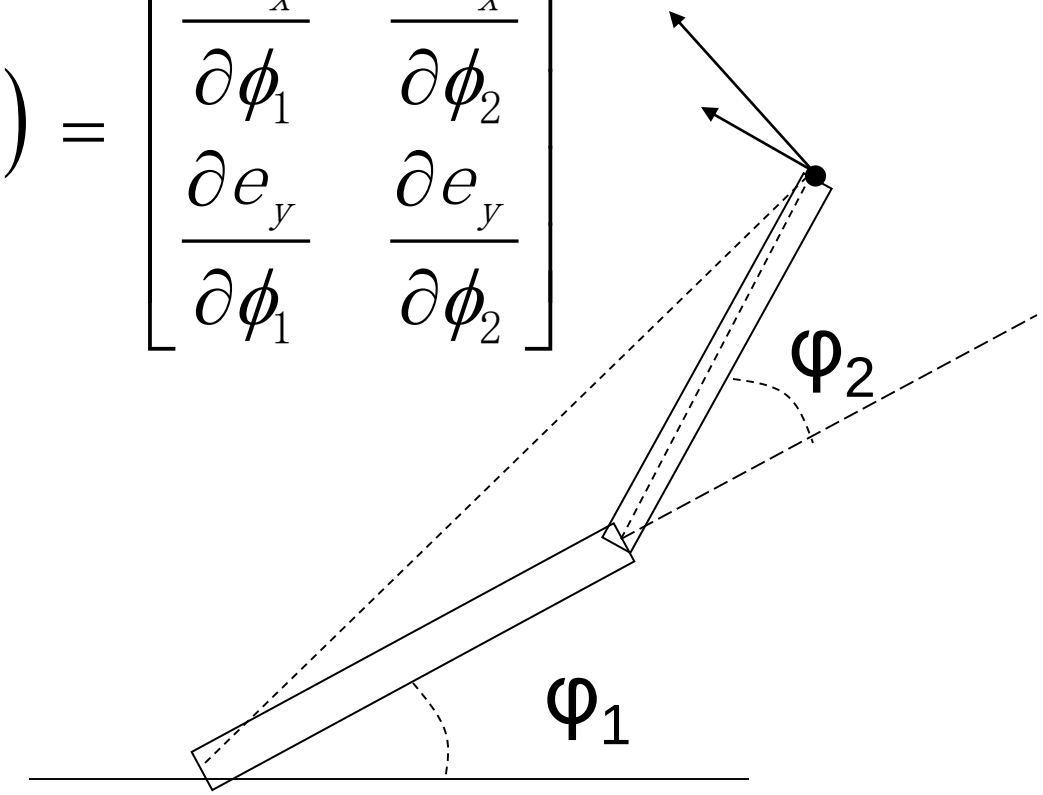
$$\frac{\partial \mathbf{e}}{\partial \phi_1} = \begin{bmatrix} \frac{\partial e_x}{\partial \phi_1} & \frac{\partial e_y}{\partial \phi_1} \end{bmatrix}$$



Jacobian for a 2D Robot Arm

- Defines how the end effector **e** changes relative to instantaneous changes of each joint angle

$$J(\mathrm{e}, \Phi) = \begin{bmatrix} \frac{\partial e_x}{\partial \phi_1} & \frac{\partial e_x}{\partial \phi_2} \\ \frac{\partial e_y}{\partial \phi_1} & \frac{\partial e_y}{\partial \phi_2} \end{bmatrix}$$

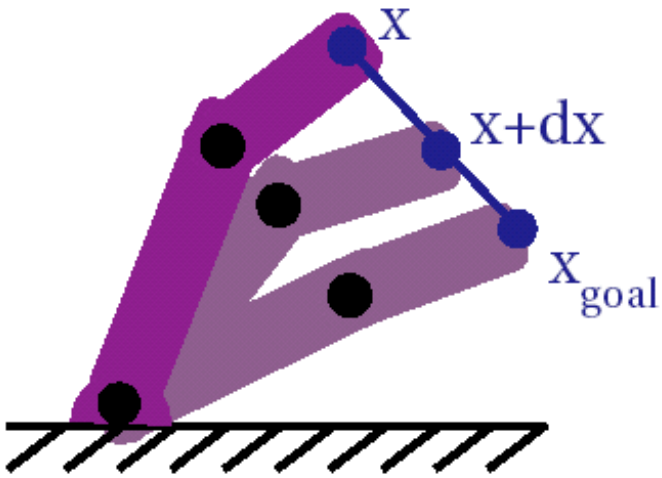


Solving IK—Incremental Changes

- ***θ:*** current set of joint DOFs;
- **e:** current end effector DOFs;
- **g:** goal DOFs that we want the end effector to reach
- Let ***E(θ) = g*** – *e(**θ**)*, error in the current pose:

$$\mathbf{J}\,\boldsymbol{\Delta} = \boldsymbol{E}$$

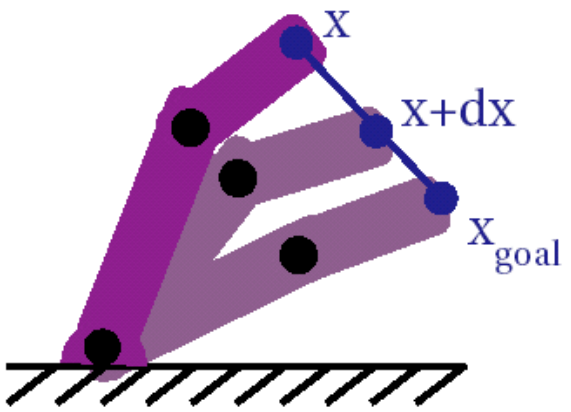
- solve for ***Δ***
- ***Δ*** moves end towards ***g***
  - Only valid for small ***Δ***
  - Take series of small steps
  - Recompute ***J(θ)*** and ***E(θ)*** at each step



- How to determine length of step?
  - Could try to find optimum size
  - know we're doing rotations:
    - keep less than ~2 degrees

Algorithm

```
solve()
  {
    start with previous θ;
     E = target - computeEndPoint();
    for(k=0; k<max && |E| > eps; k++){
      J = computeJacobian();
      solve J Δ = E;
      if (max(Δ)>2) Δ = 2Δ/max(Δ);
      θ = θ + Δ;
       E = target - computeEndPoint();
    }
  }
```



Problems

- How to invert J?
  - Pseudoinverse of Jacobian (Required)
  - Cheat by using transpose (Too easy, we don’t do that)
- How to compute J?
  - Numerically (Required)
  - Analytically (Extra Credit)

Inverting the Jacobian

- No guarantee it is invertible
  - Typically not a square matrix, in our case, 2 x N

$$\begin{bmatrix} \frac{\partial f_x}{\partial \theta_0} & \frac{\partial f_x}{\partial \theta_1} & \cdots & \frac{\partial f_x}{\partial \theta_N} \\ \frac{\partial f_y}{\partial \theta_0} & \frac{\partial f_y}{\partial \theta_1} & \cdots & \frac{\partial f_y}{\partial \theta_N} \end{bmatrix} \begin{bmatrix} D_0 \\ D_1 \\ \vdots \\ D_N \end{bmatrix} = \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

  - Singularities.
  - Even it’s invertible, as the pose vector changes, the properties of the matrix will change.

Solving $\boldsymbol{J}\,\Delta = E$: pseudo inverse

- Trick: $\boldsymbol{J}^T\boldsymbol{J}$ is square. So:

  $\boldsymbol{J}\,\Delta = E$

  $\boldsymbol{J}^T\boldsymbol{J}\,\Delta = \boldsymbol{J}^T E$

  $\Delta = (\boldsymbol{J}^T\boldsymbol{J})^{-1}\boldsymbol{J}^T E$

  $\Delta = \boldsymbol{J}^+ E$

- $\mathbf{J}^+=(\boldsymbol{J}^T\boldsymbol{J})^{-1}\boldsymbol{J}^T$ is the *pseudoinverse* of $\boldsymbol{J}$
  - Properties: $\boldsymbol{J}\boldsymbol{J}^+\boldsymbol{J}=\boldsymbol{J}$, $\boldsymbol{J}^+\boldsymbol{J}\boldsymbol{J}^+=\boldsymbol{J}^+$
  - same as $\boldsymbol{J}^{-1}$ when $\boldsymbol{J}$ is square and invertible
  - $\boldsymbol{J}$ is mxn => $\boldsymbol{J}^+$ is nxm
- How to compute pseudoinverse?
  - What if $(\boldsymbol{J}^T\boldsymbol{J})^{-1}$ is singular?

Singular Value Decomposition

- Any *m*x*n* matrix **A** can be expressed by SVD
  - **A** = **U S V**$^{\mathbf{T}}$
    - **U** is *m*xmin*(m,n)*, columns are orthogonal
    - **V** is *n*xmin*(m,n)*, columns are orthogonal
    - S is min*(m,n)*xmin*(m,n),* diagonal: *singular values*

$$A = \left(\vec{u}_1 \mid \vec{u}_2 \mid \cdots \mid \vec{u}_n \right) \begin{pmatrix} s_1 & 0 & 0 & 0 \\ 0 & s_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & s_n \end{pmatrix} \begin{pmatrix} \vec{a}_1 \\ \hline \vec{a}_2 \\ \hline \vdots \\ \hline \vec{a}_n \end{pmatrix}$$

  - unique up to sign and order of s_i values
    - canonical: positive, sorted largest to smallest
    - other properties: rank is # of non-zero values; determinant is product of all values, …

Pseudoinverse using SVD

- Given SVD, $\mathbf{A} = \mathbf{U}\,\mathbf{S}\,\mathbf{V}^T$
- pseudoinverse is easy: $\mathbf{A}^+ = \mathbf{V}\mathbf{S}^{-1}\mathbf{U}^T$

$$\begin{pmatrix} s_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & s_N \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{s_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{1}{s_N} \end{pmatrix}$$

- *singular*: some $s_i = 0$,
  ill-conditioned: some $s_i << s_0$
  - use 0 instead of $1/s_i$ for those ("truncated")
  - choose small threshold ε, test $s_i < \varepsilon\, s_0$

Solving $\mathbf{A}\,\boldsymbol{X} = \boldsymbol{B}$ using SVD

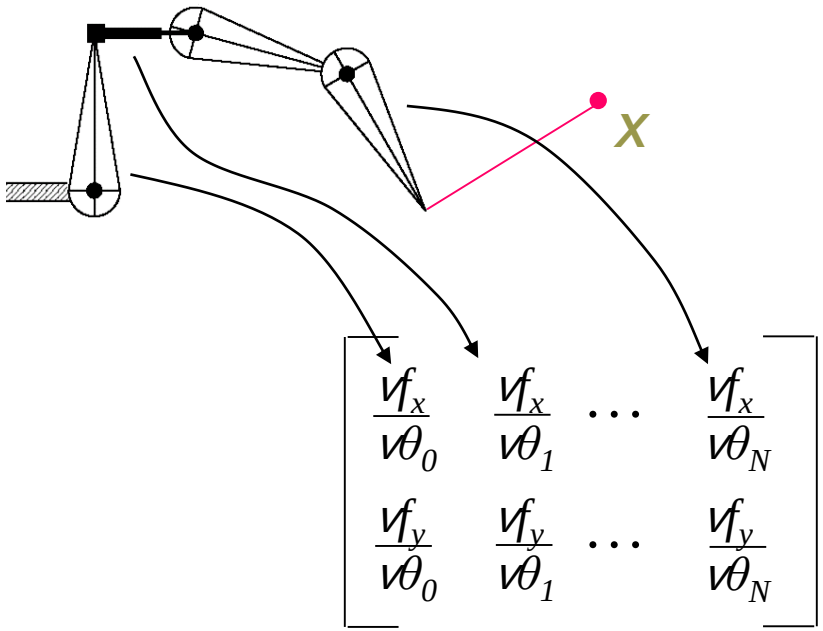
- Using truncated $\mathbf{A}^{+}\,\boldsymbol{B}$ gives *least-squares solution*:
  - If no solution, gives $\boldsymbol{X}$ that minimizes $||\mathbf{A}\boldsymbol{X}-\boldsymbol{B}||^2$
  - If many solutions, minimizes $||\boldsymbol{X}||^2$ such that $\mathbf{A}\boldsymbol{X}=\boldsymbol{B}$
  - Numerically stable for ill-conditioned matrices
- SVD has many other properties.
  - rank of **A** is # non-zero singular values, determinant is product of all singular values, …
  - known algorithm to compute it
- SVD is a powerful hammer!
  - slow $O(n^3)$; there are faster algorithms.
  - but SVD always works, is fast enough for us
  - hard to implement. some libraries have bugs (Java3D)

Back to IK

- Reminder: Let $\boldsymbol{E(\theta) = g} - e(\boldsymbol{\theta})$, error in the current pose

$$\mathbf{J}(\boldsymbol{\theta})\,\boldsymbol{\Delta} = \boldsymbol{E}$$

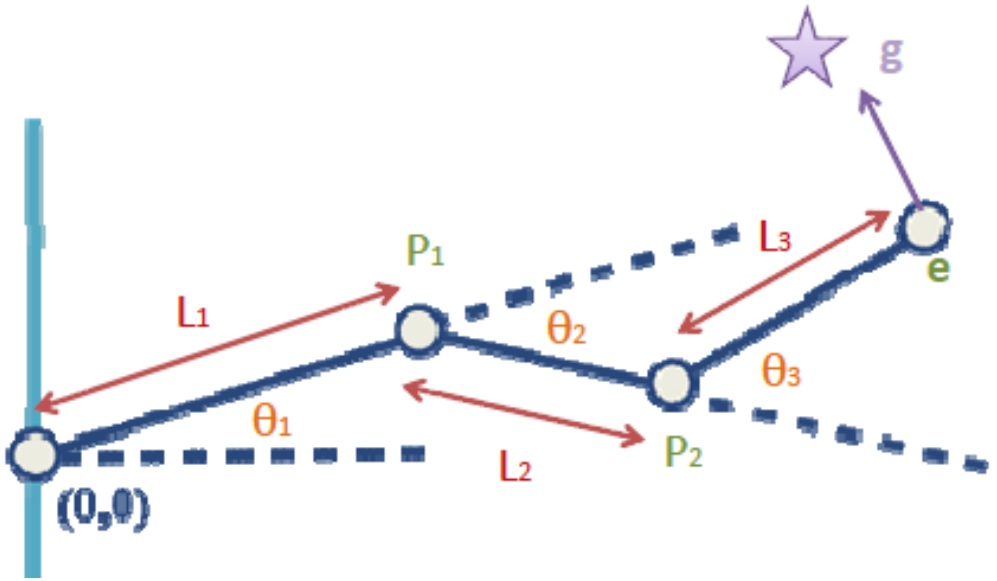
- solve for $\boldsymbol{\Delta}$
  - i^{th} column of **J** comes from link *i*



Computing the Jacobian columns

- For a rotational joint, the linear change in the end effector is the cross product of the axis of revolution and a vector from the joint to the end effector.

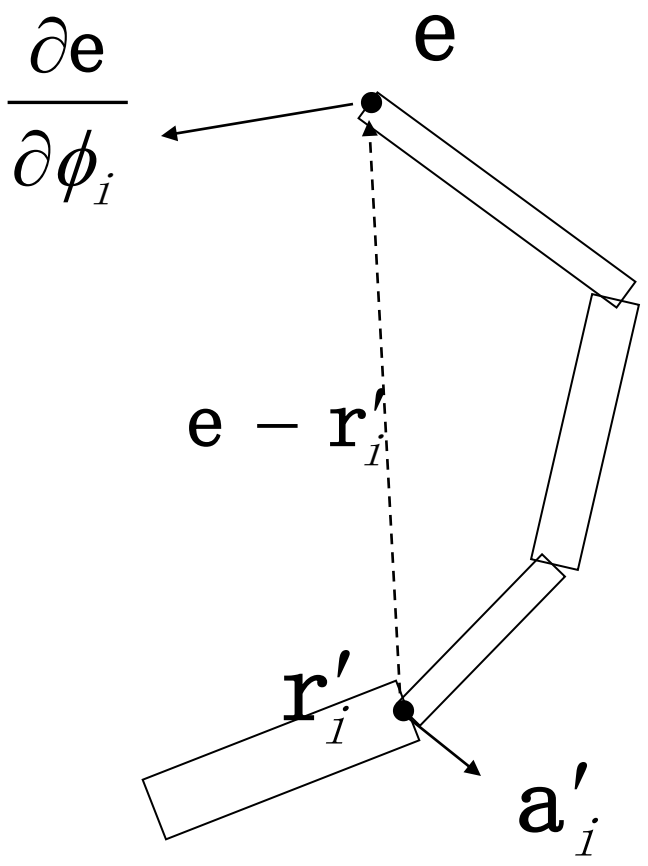
$$\frac{\partial \mathbf{e}}{\partial \theta_i} = \left[\frac{\partial e_x}{\partial \theta_i} \frac{\partial e_y}{\partial \theta_i} \frac{\partial e_z}{\partial \theta_i}\right]^T = \left(\mathbf{a}_i' \times (\mathbf{e} - \mathbf{r}_i')\right)$$



$$J = \begin{bmatrix} ((0,0,1)\times \mathbf{e})_x & ((0,0,1)\times(\mathbf{e}-\mathbf{P_1}))_x & ((0,0,1)\times(\mathbf{e}-\mathbf{P_2}))_x \\ ((0,0,1)\times \mathbf{e})_y & ((0,0,1)\times(\mathbf{e}-\mathbf{P_1}))_y & ((0,0,1)\times(\mathbf{e}-\mathbf{P_2}))_y \\ ((0,0,1)\times \mathbf{e})_z & ((0,0,1)\times(\mathbf{e}-\mathbf{P_1}))_z & ((0,0,1)\times(\mathbf{e}-\mathbf{P_2}))_z \end{bmatrix}$$

Rotational DOFs

$$\frac{\partial \mathbf{e}}{\partial \phi_i} = \mathbf{a}'_i \times \left(\mathbf{e} - \mathbf{r}'_i\right)$$



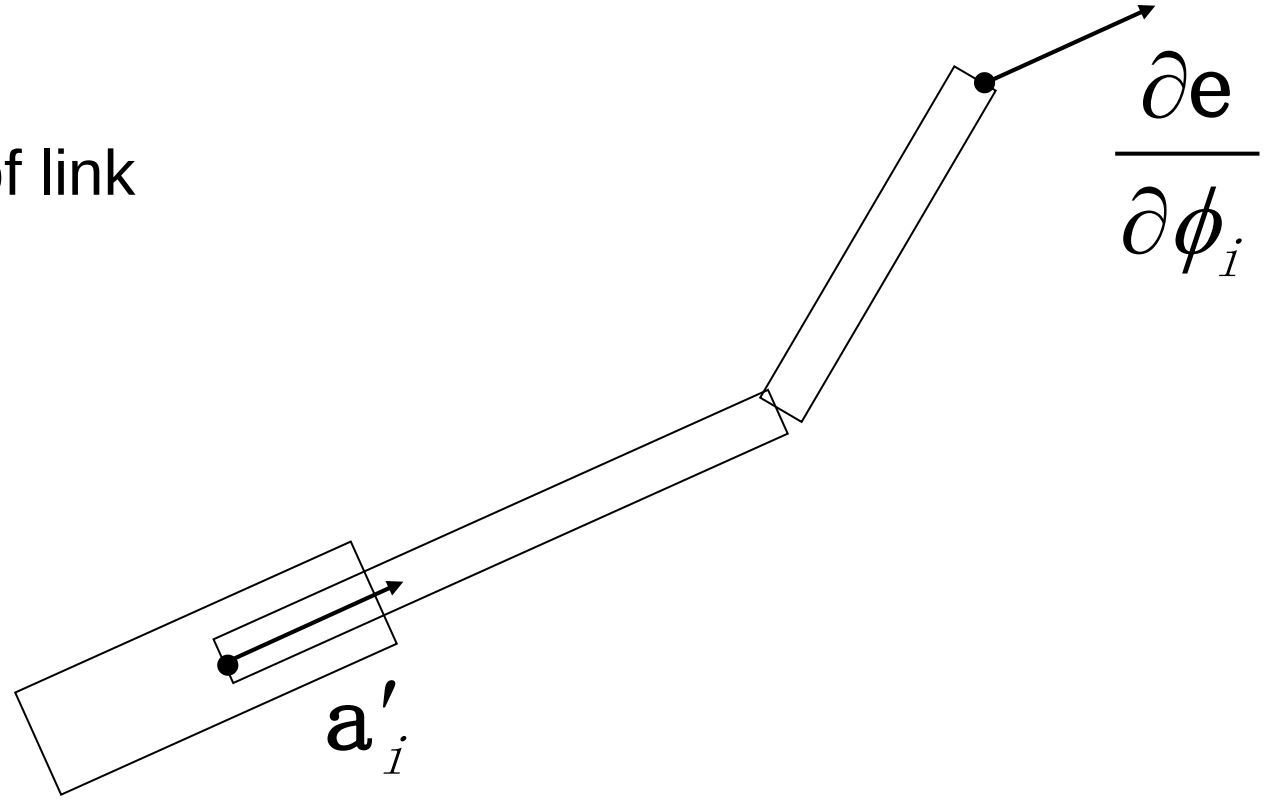
a’$_i$: unit length rotation axis in world space

r’$_i$: position of joint pivot in world space

e: end effector position in world space

Computing the Jacobian columns

- For a translational joint:
  - $\partial f(\boldsymbol{\theta})/\partial\theta_j$ = vector in direction of link



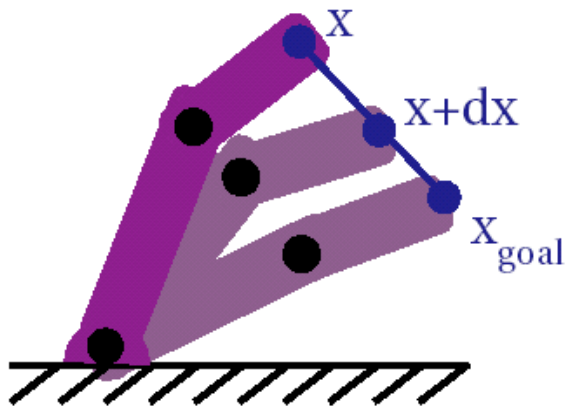
- Notes:
  - Remember to compute in world space!
  - I've assumed one degree of freedom per joint
  - If there are multiple DOFs per joint, refer to CSE169_12.ppt of CSE 169: Computer Animation @ UCSD winter 2004

Building the Jacobian

- To build the entire Jacobian matrix, we just loop through each DOF and compute a corresponding column in the matrix
- If we wanted, we could use more elaborate joint types (scaling, translation along a path, shearing…) and still compute an appropriate derivative
- If absolutely necessary, we could always resort to computing a numerical approximation to the derivative

IK Algorithm

```
solve()
  {
    Vector θ = getLinkParameters();
    Vector E = target - computeEndPoint();
    for(k=0; k<max && E.norm() > eps; k++){
      Matrix J = computeJacobian();
      Matrix J+ = J.pseudoinverse();
      Vector Δ = J+ E;
      if (max(Δ)>2) Δ *= 2/max(Δ);
      θ = θ + Δ;
      putLinkParameters(θ);
      E = target - computeEndPoint();
    }
  }
```



What’s left for IK?

- Joint limits
- Choosing desired configuration
- When to stop the iterations

Joint limits

- Each joint may have limited range.
- Modify algorithm:
  - After finding Δ, test each joint:

$$\boldsymbol{\theta min}_i < (\boldsymbol{\theta}+\Delta)_i < \boldsymbol{\theta max}_i$$

  - If it would go out of range
    - set column *i* of J to 0
    - claims “this parameter has no effect”
  - Recompute $\mathbf{J}^+$
    - Least-squares solution will make $\Delta_i z 0$
    - For robustness, you may want to force $\Delta_i = 0$
  - Find Δ, repeat

Choosing configuration

- Suppose you have a *homogeneous solution* $\boldsymbol{\delta}$:

$$\mathbf{J}\,\boldsymbol{\delta} = 0$$

If Δ solves $\boldsymbol{J}\,\Delta = E$, then ($\Delta$+ δ) does also:

$$\mathbf{J}\,(\boldsymbol{\Delta}+\boldsymbol{\delta}) = \mathbf{J}\,\boldsymbol{\Delta} + \mathbf{J}\,\boldsymbol{\delta} = \boldsymbol{E} + 0 = \mathbf{E}$$

- Given a desired change $\boldsymbol{C}$ to $\boldsymbol{\theta}$,
  - project into *null space* of **J** using $(\mathbf{J}^{+}\mathbf{J}-\boldsymbol{I})\,\boldsymbol{C}$:

$$\mathbf{J}\,[(\mathbf{J}^{+}\mathbf{J}-\boldsymbol{I})\,\boldsymbol{C}] = [\mathbf{J}\,(\mathbf{J}^{+}\mathbf{J}-\boldsymbol{I})]\,\boldsymbol{C} = (\mathbf{J}\mathbf{J}^{+}\mathbf{J}-\mathbf{J})\boldsymbol{C} = (\mathbf{J}-\mathbf{J})\boldsymbol{C} = 0$$

Choosing configuration

- Given preferred values $\boldsymbol{\theta}_{\boldsymbol{pref}}$
  - construct desired change $\boldsymbol{C}$:

$$C_i = \alpha_i(\boldsymbol{\theta} - \boldsymbol{\theta}_{\boldsymbol{pref}})_i$$

  - weights α_i give relative strengths
- Modify algorithm:
  - Construct $\boldsymbol{C}$
  - Use $\Delta = \mathbf{J}\boldsymbol{E} + (\mathbf{J}^{+}\mathbf{J} - \boldsymbol{I})\boldsymbol{C}$
- Null-space projection of $\boldsymbol{C}$ won't harm solution
- Solution will bias towards $\boldsymbol{\theta}_{\boldsymbol{pref}}$

Note on numerical algorithms

- Various algorithms for non-linear multidimensional root-finding…this one works for us
- Root-finding is related to *optimization*:
  - $F(\boldsymbol{\theta})=X \Leftrightarrow$ minimize $||F(\boldsymbol{\theta})-X||^2$
- Many computer animation problems are optimization problems
- Many algorithms have solving AX = B at their core.

When to Stop

- There are three main stopping conditions we should account for
  - Finding a successful solution (or close enough)
  - Getting stuck in a condition where we can’t improve (local minimum)
  - Taking too long (for interactive systems)
- All three of these are fairly easy to identify by monitoring the progress of **Φ**
- These rules are just coded into the while() statement for the controlling loop

Finding a Successful Solution

- We really just want to get close enough within some tolerance
- If we’re not in a big hurry, we can just iterate until we get within some floating point error range
- Alternately, we could choose to stop when we get within some tolerance measurable in pixels
- For example, we could position an end effector to 0.1 pixel accuracy
- This gives us a scheme that should look good and automatically adapt to spend more time when we are looking at the end effector up close (level-of-detail)

Local Minima

- If we get stuck in a local minimum, we have several options
  - Don’t worry about it and just accept it as the best we can do
  - Switch to a different algorithm (CCD…)
  - Randomize the pose vector slightly (or a lot) and try again
  - Send an error to whatever is controlling the end effector and tell it to try something else
- Basically, there are few options that are truly appealing, as they are likely to cause either an error in the solution or a possible discontinuity in the motion

Taking Too Long

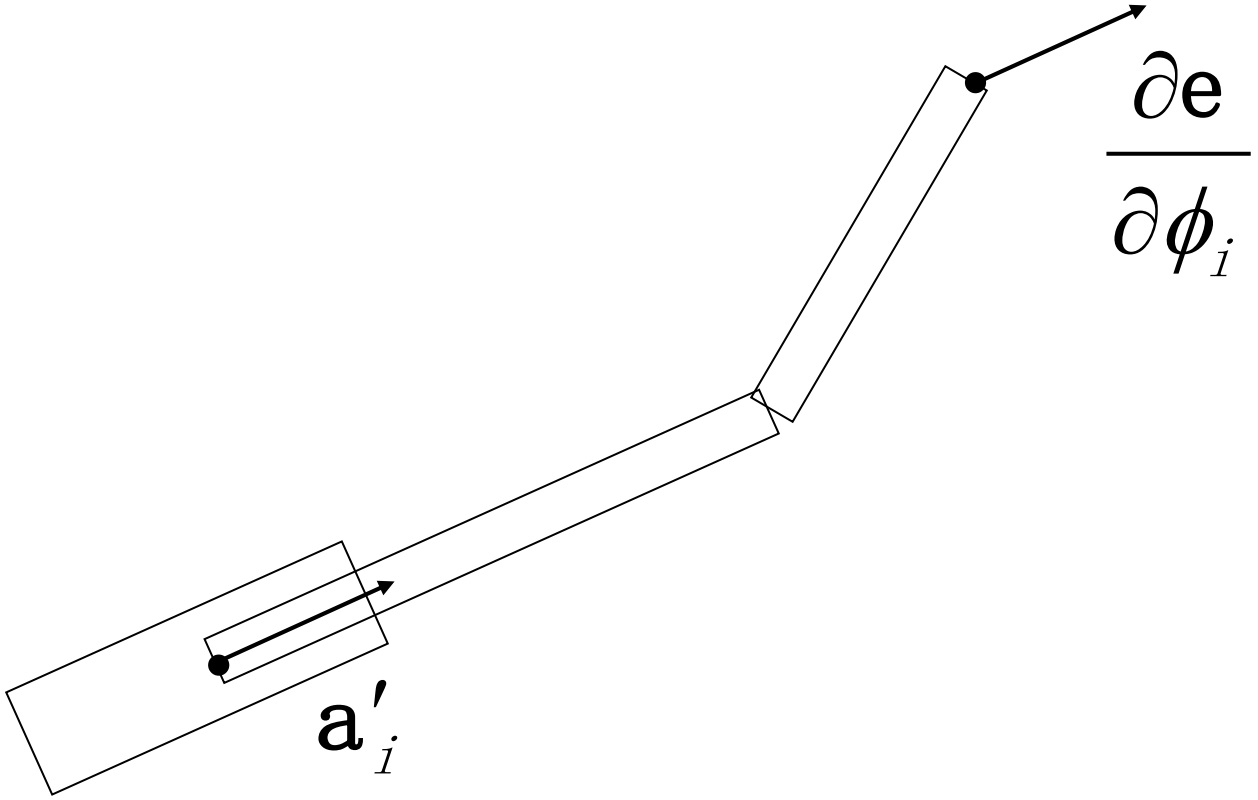
- In a time critical situation, we might just limit the iteration to a maximum number of steps
- Alternately, we could use internal timers to limit it to an actual time in seconds

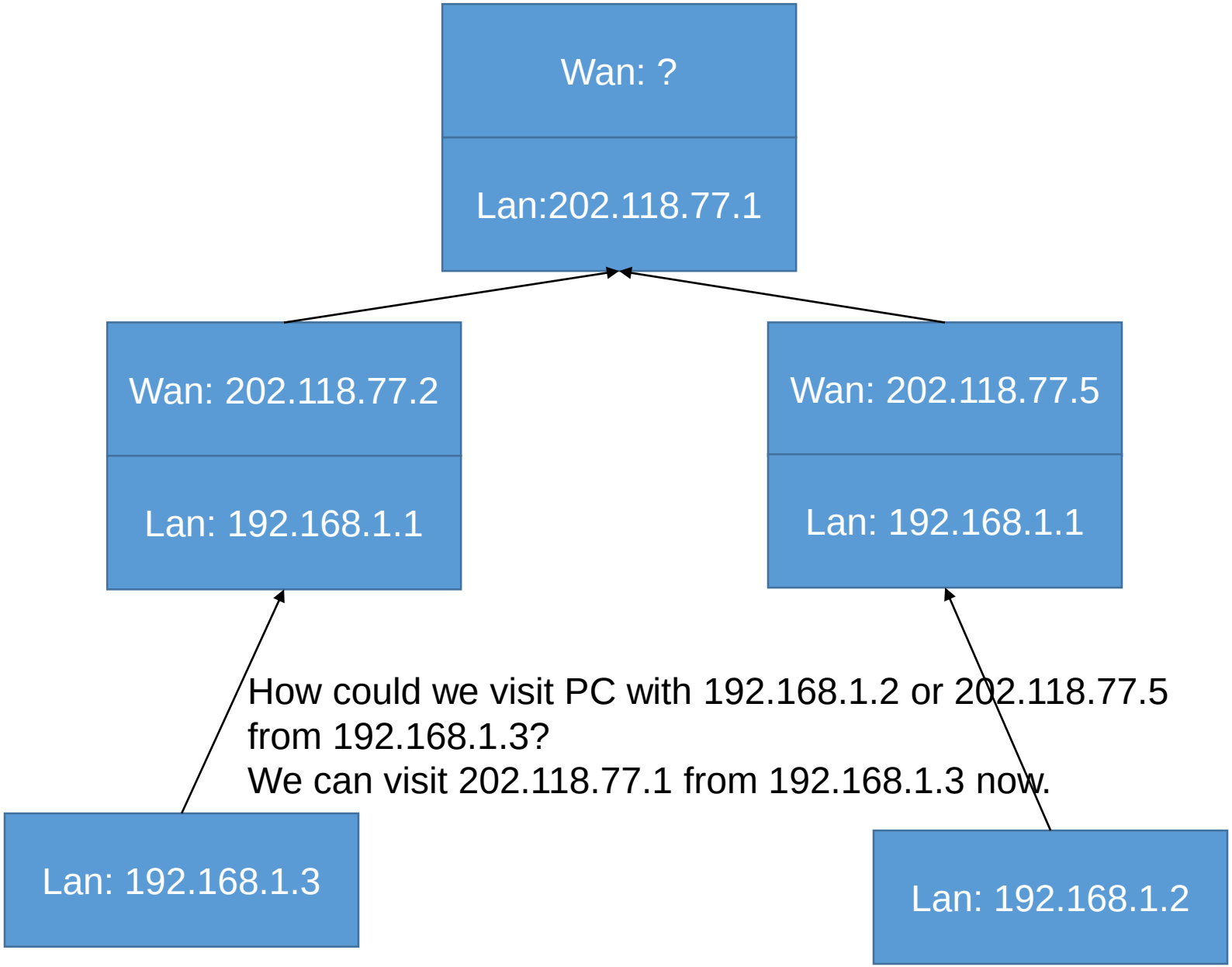
Iteration Stepping

- Step size
- Stability
- Performance

Thank you

Translational DOFs




Wan: ?
Lan:202.118.77.1
Wan: 202.118.77.2
Lan: 192.168.1.1
Wan: 202.118.77.5
Lan: 192.168.1.1
How could we visit PC with 192.168.1.2 or 202.118.77.5 from 192.168.1.3?
We can visit 202.118.77.1 from 192.168.1.3 now.
Lan: 192.168.1.3
Lan: 192.168.1.2